→ Union Bound Approximation

eg QPSK

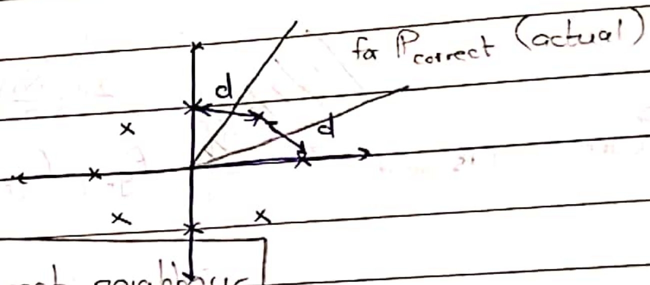
$P_{error|1 \text{ is sent}} = P(y_r < 0 \text{ OR } y_i < 0) = 1 - Q\left(-\sqrt{\frac{E_s}{N_0}}\right) + 1 - Q(\ldots)$

$\leq P(y_r < 0) + P(y_i < 0)$

$\therefore P(y_r \,\&\, y_i < 0)$ is counted twice

$= 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) = 2Q\left(\frac{d}{2\sigma}\right)$ $\quad d = \sqrt{2E_s}$

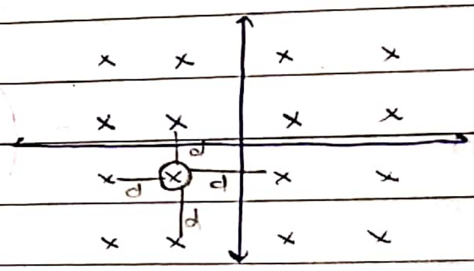
eg 8-PSK



$P_{error|1 \text{ is sent}} \leq 2Q\left(\frac{d}{2\sigma}\right)$

Ans: $\boxed{Q\left(\frac{d}{2\sigma}\right) \text{ for each nearest neighbour}}$

eg 16-QAM

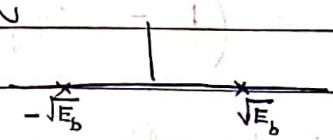


$P_{error|1 \text{ is sent}} \leq 4Q\left(\frac{d}{2\sigma}\right)$

$\sigma = \sqrt{\frac{N_0}{2}}$

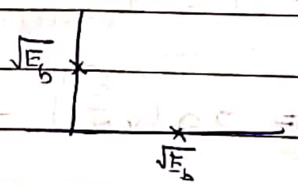
11/2 III Bit Level Demodulation

eg Binary Signalling



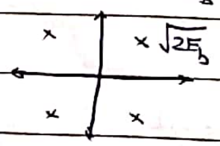
$BER = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

Orthogonal Signalling



$BER = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$

QPSK



$SER = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right)$

$\leq 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

eg ~~Ex~~ For QPSK, average probability of bit error $= \frac{1}{2}(P(\text{bit 1 is wrong}) + P(\text{bit 2 is wrong}))$

- Lexicographic bit mapping, $P(\text{LSB is wrong} \mid 00 \text{ is sent})$ ..... PTO

$\rightarrow$ $\eta_r \triangleq \frac{d^2}{E_b}$

If all symbols are scaled up by A,

$E_b \rightarrow A^2 E_b$

$d \rightarrow Ad$

$\therefore$ η_r is scale invariant.

For constant E_b/N_0, performance is better for higher η_r

$\rightarrow$ Bit Mapping

- Arbit coding for QPSK

11 ×	× 00
10 ×	× 01

There will be 2 bit errors on horizontal direction misprediction.

Better - Gray Coding for QPSK. : Neighbors differ by one bit.

10 ×	× 00
11 ×	× 01

(d between 00 and 01)

MSB is determined by real part

LSB imaginary part.

For one adjacent neighbour, $Q\left(\frac{d}{2\sigma}\right) = Q\left(\frac{2\sqrt{E_b}}{2\sqrt{\frac{N_0}{2}}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

.... same expression as BPSK

(because the vertical/horizontal distance can be thought of as BPSK).

☆ $\therefore$ For ~~QPS~~ same E_b and P_e, QPSK gives twice as much bit rate as BPSK

- Similarly, 16-QAM can be thought of as two separate PAM-4.

11/2 • Bit mapping aims to reduce E_b. Does not affect E_s

- 16-QAM

	00	01	11	10
00	×	× (3)	×	×
01	×	× (1)	×	×
11	×	× (-1)	×	× 1110
10	×	× (-3)	×	×

PAM-4 × PAM-4.

Normalize scale to obtain unit E_s

$E_s = \frac{4}{16}(9+9) + \frac{8}{16}(1+9) + \frac{4}{16}(1+1) = 10$

$\therefore$ Divide all lengths by $\sqrt{10}$

$= \mathbb{P}(\text{ML decision is } 01 \text{ or } 11 \mid 00 \text{ is sent})$

$= \mathbb{P}\left(N_c < -\frac{d}{2}, N_s > -\frac{d}{2}\right) + \mathbb{P}\left(N_c > -\frac{d}{2}, N_s < -\frac{d}{2}\right)$

$= \sim Q \sim$

This probability of wrong LSB is twice as much as in gray coding (approx

* Alternatively, find Energy of one PAM-4. Multiply by 2.

Union Bound :: $\mathbb{P}_e \leq \underbrace{\left(\text{No. of nearest neighbours}\right)}_{\substack{\text{average} = 3 \\ \text{(equiprobable case)}}} Q\left(\frac{d}{2\sigma}\right)$

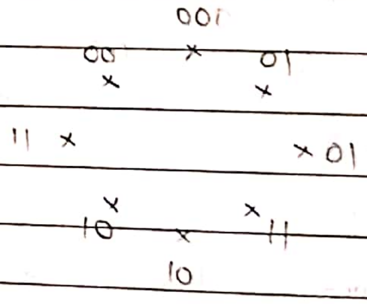
$$\mathbb{P}_e \leq 3Q\left(\frac{d}{2\sigma}\right)$$

$d = \frac{2}{\sqrt{10}} \times \sqrt{E_s}$ (for BPSK it was $\sqrt{E_s}$)

$\therefore$ QAM-16 is less immune to noise.

- Gray code for ~~QPSK~~ 8-PSK

Complete — Consider 8-PSK as 2 QPSKs



→ Rotation of Constellation

- No change in performance.

• Shifting changes performance — Changes energy (E_b) requirement.

→ Orthogonal Signalling

- Poorer BW efficiency $= \frac{\log_2 M}{M}$

$s_1 = [1 \ 0 \ 0 \ \dots]^T \sqrt{E_s}$

$s_2 = [0 \ 1 \ 0 \ \dots]^T \sqrt{E_s}$

$\vdots$

no. of other symbols with error in given bit

$\mathbb{P}(\text{bit error}) = \frac{M/2}{M-1} \, \mathbb{P}(\text{symbol error})$

no. of other symbols

Nearest neighbour approximation:- $\mathbb{P}(i^{th} \text{ bit is wrong} \mid \vec{b} \text{ was sent}) \approx N(\vec{b}, i)\, Q\left(\frac{d}{2\sigma}\right)$

→ N = No. of nearest neighbors of $\vec{b}$ that differ in i^{th} bit (≤ 1 for gray)

→ d = nearest neighbour distance

→ $\mathbb{P}(i^{th} \text{ bit is wrong}) = \frac{1}{2^n} \sum_{\vec{b}} N(\vec{b}, i)\, Q\left(\frac{d}{2\sigma}\right)$ and $\mathbb{P}(\text{bit error})$ (average) $= \frac{1}{n} \sum_i \mathbb{P}(i^{th} \text{ bit is wrong})$

Every s_i has $M-1$ nearest neighbours, each $\sqrt{2E_s}$ away.

Union bound :- $\mathbb{P}_e \leq (M-1)\, Q\left(\frac{\sqrt{2E_s}}{\sqrt{2N_0}}\right)$

- This bound is very loose for higher M. ⇒ not useful
  - Use exact error instead

- Decision boundary (ML)

  For any point on $\mathbb{R}^M$, project it onto each of the axes and detect the symbol for whose axis the projection is closest to $\sqrt{E_s}$

- Exact $\mathbb{P}_e = (M-1) \int_{-\infty}^{\infty} (\Phi(x))^{M-2}\, \Phi(m-x) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$

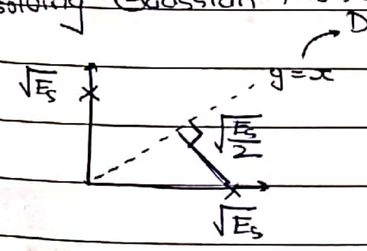
  where $m = \sqrt{\frac{2E_b \log_2 M}{N_0}}$

  * For orthogonal signalling, bit error can be made arbitrarily small for $\frac{E_b}{N_0} > -1.6$ dB

'Resolving Gaussian Noise'

14/2

eg



For that symbol, $Y \sim$ Mean $\begin{bmatrix} \sqrt{E_s} \\ 0 \end{bmatrix}$

Covariance $\begin{bmatrix} N_0/2 & 0 \\ 0 & N_0/2 \end{bmatrix}$

For calculating error, we resolve the (circularly symmetric) noise along perpendicular to $y=x$.

This resolved component has variance $\frac{N_0}{2}$

$$\mathbb{P}_{error} = \int_{\sqrt{\frac{E_s}{2}}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} dx$$

which is same as we would have obtained by double integral on x, y

eg 3D orthogonal signalling

$$s_1 = \begin{bmatrix} \sqrt{E_s} \\ 0 \\ 0 \end{bmatrix}, \quad y_1 \sim \text{Mean} \begin{bmatrix} \sqrt{E_s} \\ 0 \\ 0 \end{bmatrix} \quad \text{Covariance} = \frac{N_0}{2} I_3$$

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} \sqrt{E_s} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$P_c = \mathbb{P}(r_1 > r_2, \; r_1 > r_3)$$

$$= \mathbb{P}(r_2 < a, \; r_3 < a \mid r_1 = a) \, \mathbb{P}(r_1 = a)$$

$$= \mathbb{P}^2(r_2 < a \mid r_1 = a) \, \mathbb{P}(r_1 = a)$$

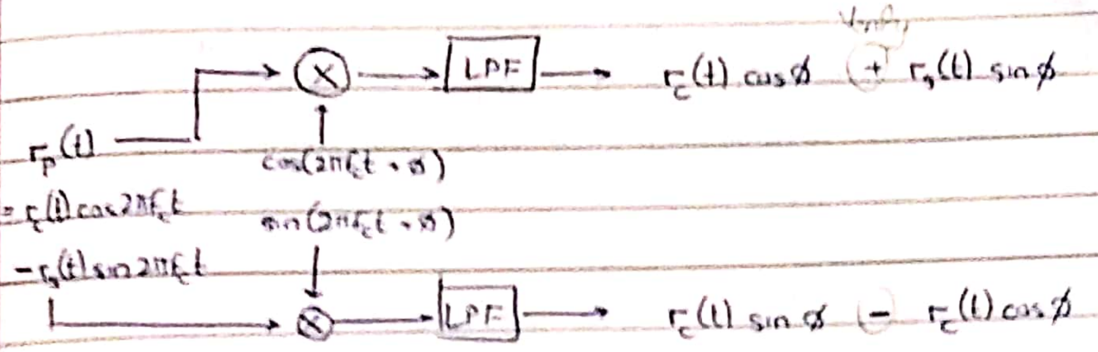
$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} dx \right)^{\overset{M-1}{2}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(a - \sqrt{E_s})^2}{2\sigma^2}} da$$

* Training sequence

- Transmitter periodically sends predecided sequence for training

P.T.O. Chap 4

SYNCHRONIZATION AND NON-COHERENT COMMUNICATION



- If f_c is also indefinite :- $r_c(t)\cos(2\pi\,\Delta f_c\,\tau) - r_s(t)\sin(2\pi\Delta f_c\,\tau)$

13/3

→ Consider $y_1 = A + n_1$ (σ_1^2) independent
$y_2 = A + n_2$ (σ_2^2)

ML estimate :- $$f_x(y \mid A=a) = \frac{1}{\sigma_1\sigma_2\, 2\pi} e^{-\frac{\left(y - \begin{bmatrix} a \\ a\end{bmatrix}\right)^T \begin{bmatrix}\sigma_1^2 & 0 \\ 0 & \sigma_2^2\end{bmatrix}^{-1}\left(y - \begin{bmatrix} a \\ a\end{bmatrix}\right)}{2}}$$

Answer : $$\boxed{\hat{A}(y_1, y_2) = \frac{\sigma_2^2 y_1 + \sigma_1^2 y_2}{\sigma_1^2 + \sigma_2^2}}$$

→ Consider $y_1 = A + \omega_1$ not independent
$y_2 = A + \omega_2$

ω_1, ω_2 are jointly Gaussian : Mean 0, Covariance C_ω
- C_ω is invertible.

- Can diagonalize C_ω :- $C_\omega = U \Lambda U^H$ (Λ: diagonal matrix)

$$= U \Lambda^{1/2} \Lambda^{1/2} U^H$$

$$= (U\Lambda^{1/2})(U\Lambda^{1/2})^H$$

Transmitted complex baseband :- $u(t) = \sum b[n] g_{Tx}(t - nT)$

Received :- $y(t) = \underset{\substack{\uparrow \\ \text{unknown}}}{A} e^{j(2\pi \Delta f t + \theta)} u(t - \tau) + n(t)$

Estimated by training sequence

★ For this case, multiply $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ by $(U \Lambda^{1/2})^H$

Then it becomes equivalent to previous case.

→ Like above case, but C_ω is not invertible

eg - For $\omega_1 = \omega_2$ ⇒ $C_\omega = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Then $y_1 = y_2$ and $\hat{A} = y_1 = y_2$

→ PAM - 4   -3   -1   1   3

$y = Ax + \omega$

A is unknown. Need to estimate A.

- Send training sequence
- If not possible, optimum estimation : Chapter 4.

---

* If y depends on θ,

$$\hat{\theta}_{ML}(y) = \arg\max_{\theta} P(y \mid \theta)$$

- Measured $y_1 = A + \omega_1$

$y_2 = A + \omega_2$

$\vdots$

A = unknown constant.

↓

∴ MLE not MAP

If $\omega_1, \omega_2 \ldots$ are iid,

$$\hat{A}_{ML}(y) = \arg\max_{A} P(y_1 \mid A) P(y_2 \mid A) \ldots$$

$$= \arg\max_{A} e^{\frac{-(y_1 - A)^2 - (y_2 - A)^2 - (y_3 - A)^2 \ldots}{\ldots}}$$

$$= \boxed{\frac{\sum y_i}{n}}$$

eg - BPSK :- $y = Ab + \omega$ ... single measurement

b is known $\in \{-1, 1\}$

'Training'

$\therefore \boxed{\hat{A}_{ML} = \frac{y}{b}}$

$\therefore$ If b is known, this becomes equivalent to previous case.

eg Single BPSK measurement :- $y = Ab + \omega$

measured (y) unknown (A) only distribution is known (b): ± 1 w.p. $\frac{1}{2}$

$$\hat{A}_{ML} = \arg\max_A \mathbb{P}(y \mid A = a) = \mathbb{P}(y \mid A=a, b=1)\,\mathbb{P}(b=1)$$

$$+ \mathbb{P}(y \mid A=a, b=-1)\,\mathbb{P}(b=-1)$$

$$= \frac{1}{2}\left[\mathbb{P}(y \mid A=a, b=1) + \mathbb{P}(y \mid A=a, b=-1)\right]$$

$$= \frac{1}{2\sqrt{2\pi}\,\sigma}\left[e^{-\frac{(y-a)^2}{2\sigma^2}} + e^{-\frac{(y+a)^2}{2\sigma^2}}\right]$$

To maximize, differentiate wrt a

$$= \frac{1}{2\sqrt{2\pi}\,\sigma}\, e^{-\frac{a^2}{2\sigma^2}} \cosh\left(\frac{ay}{\sigma^2}\right) e^{-y^2/2\sigma^2}$$

- If SNR is high ($\omega \ll Ab$), even one measured y can be enough to estimate A.
  Otherwise, we need to repeat this over multiple samples.

eg Measured :- $y_1 = Ab_1 + \omega_1$ All ω_i are iid

$y_2 = Ab_2 + \omega_2$ A is unknown

$\vdots$ All b_i are known.

Estimate :- $\boxed{\hat{A}_{ML}(y_1 \cdots y_n) = \frac{\sum \frac{y_i}{b_i}}{n}}$

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eg Blind Training :- Above case, but b_i are unknown.

$$\hat{A}_{ML}(y_1 \dots y_n) = \underset{\theta}{\arg\max}\ \mathbb{P}(y_1 \dots y_n \mid \theta)$$
$$= \underset{\theta}{\arg\max}\ \ln \mathbb{P}(y_1 \dots y_n \mid \theta)$$
$$= \underset{\theta}{\arg\min}\ \sum \frac{(y_i - Ab_i)^2}{2\sigma^2}$$

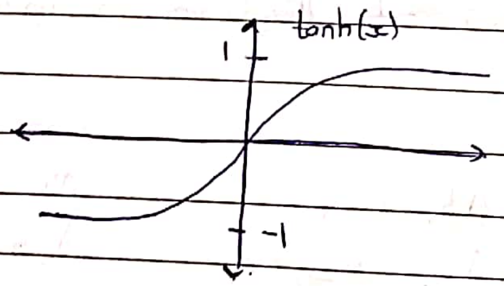
Substitute known y_i, b_i (after conditioning on b_i)

$$\hat{A}_{ML} = \boxed{\frac{\sum y_i \tanh\left(\frac{a y_i}{\sigma^2}\right)}{n} = a}$$

19/2 $\mathbb{P}(y \mid A = a) = \sum_{\underline{b}} \mathbb{P}(y \mid \underline{b}, A = a) \times \underbrace{\mathbb{P}(\underline{b})}_{\frac{1}{2^k}}$

☆ Unique solution exists because summation of concave functions is concave

* But this does not hold for Gaussian, which is not entirely concave or convex.



- When $\sigma^2 \to \infty$, $a = 0$. ($w[k]$ have infinite variance)
- $a\,y[k]$ has same sign as $b[k]$ (roughly)

Comparing with $\frac{1}{n} \sum \frac{y_i}{b_i}$, $\left|\tanh\left(\frac{a y_i}{\sigma^2}\right)\right| < 1 = |b_i|$

- When $\sigma^2 \to 0$, ignore $w[k]$ $\Rightarrow y = ab$

$$a = \tanh\left(\frac{ay}{\sigma^2}\right) = \begin{cases} 1 & \text{if } b > 0 \\ -1 & b < 0 \end{cases}$$

$a^2 b$

For low noise, even poor initial guess of a using very few samples still gives accurate answer of a in one iteration.

* $Y \sim N(\theta, \sigma^2)$ — TFT

To find probability over set A :- $\int_A f_Y(y)\,dy$ .... maximize ~~this~~ $f_Y(y)$

∴ ~~Can also maximize~~ :- $= \int_A \frac{f_Y(y)}{q(y)} q(y)\,dy$

where $q(y) \neq 0$ anywhere over A.

and $q(y)$ does not depend on θ

If $q(y) > 0$ whenever $f_Y(y) > 0$, then MLE ~~as~~ answer could also be obtained by maximizing $\frac{f_Y(y)}{q(y)}$

- If $q(y)$ is also a pdf,

$$= \int L(y|\theta)\, q(y)\, dy$$

= 'Likelihood function' .... maximize this.

$$= \frac{f_Y(y)}{q(y)}$$

eg $y = r + \omega$

TFT : θ,

Maximize $\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\theta)^2}{2\sigma^2}} = f_Y(y)$

Use $q(y) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-y^2/2\sigma^2}$

∴ Equivalently maximize $e^{\frac{2\theta y - \theta^2}{2\sigma^2}}$

P.TO.

Same can be done in multi-dimensional case

$$y[0] = A + w[0]$$
$$\vdots$$
$$y[k-1] = A + w[k-1]$$

$$f_y(y) \propto e^{-(\underline{y} - A)^T C_y^{-1} (\underline{y} - A\xi)}$$

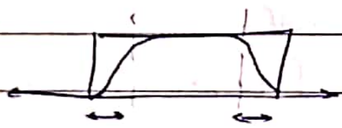
Take $q(y) \propto e^{-\underline{y}^T C_y^{-1} y}$

21/2

→ Cost analysis Link budget analysis

* QAM-64 will have higher (~~6 times~~) bit rate than QAM-16, but has a much worse SNR (noise immunity)

* Deciding BW - need some tolerance (cannot occupy entire BW)



eg - 1Mbps QAM-256

Symbol rate $= \frac{1}{8}$ MSPS.

$BW = \frac{1}{8}$ MHz

Practical ~~F~~ BW $= \frac{1}{8}(1+a)$ MHz

a → Excess.

→ Based on upper bound of acceptable BER, we can find $\frac{E_s}{N_0}$ required at receiver.

'Receiver sensitivity'

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eg Channel total loss = 20 dB
If power required at receiver = 1 mW
then transmitter = 100 mW.

* Parameters :-
~~Bit~~ Symbol rate, BW, Constellation, SNR, SER or BER.

$\propto \frac{E_s}{N_0}$