C.P Grp (5)

→ We use the following facts to derive the inverse z-transform of $\frac{1}{1-\alpha z^{-1}}$ formula:

(1) Cauchy's integral formula:

$$\frac{1}{2\pi j}\oint_C \frac{f(z)}{z-z_0}\,dz = f(z_0)$$ if C is in the region where $f(z)$ is holomorphic and z_0 is in the interior of C.

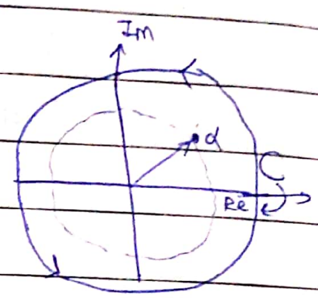
(Further, $\frac{n!}{2\pi j}\oint \frac{f(z)}{(z-z_0)^{n+1}}dz = f^n(z_0)$)

(2) Cauchy's integral theorem

$$\oint_C f(z)\,dz = 0$$ if f(z) is holomorphic in its simply connected domain

Case (I) : ROC is $|z| > |\alpha|$

If we integrate over C as shown,



→ for $n \geq 0$,

$$x[n] = \frac{1}{2\pi j}\oint_C \frac{1}{1-\frac{\alpha}{z}} z^{n-1}\,dz$$

$$= \frac{1}{2\pi j}\oint_C \frac{z^n}{z-\alpha}\,dz$$

$= \alpha^n$ .... from fact (1)

→ for $n < 0$

$$x[n] = \frac{1}{2\pi j}\oint_C \frac{1}{z^m(z-\alpha)}\,dz$$ ... where $m = -n$

$$= \frac{1}{2\pi j} \times \frac{1}{\alpha}\left[\oint_C \frac{dz}{z^{m-1}(z-\alpha)} - \underbrace{\oint_C \frac{1}{z^m}\,dz}\right] \ldots (1)$$

this integral = 0 from Cauchy integral formula for $f^n(z_0)$ with $f(z) = 1\ \forall z$

$$\therefore x[n] = \frac{1}{2\pi j} \times \frac{1}{\alpha^{2}} \left[\oint \frac{dz}{z^{m-1}(z-\alpha)} \right]$$

$$= \frac{1}{2\pi j} \times \frac{1}{\alpha^{m-1}} \left[\oint \frac{dz}{z(z-\alpha)} \right]$$

.... continuing to reduce the eq^n integral ~~as in~~ ①.

Eq^n ② .....
$$= \frac{1}{2\pi j} \times \frac{1}{\alpha^{m}} \left(\oint \frac{dz}{z-\alpha} - \oint \frac{dz}{z} \right)$$

$$= \frac{1}{\alpha^{m}} (1 - 1) \quad \text{.... (take } f(z) = 1\text{) in fact 1.}$$

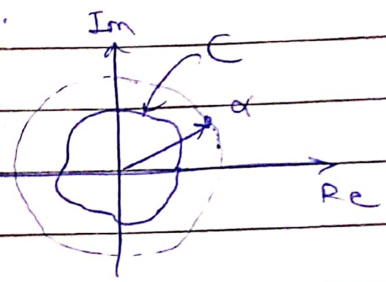
$$= 0$$

$$\therefore x[n] = \alpha^{n} \quad n \geq 0$$
$$= 0 \quad n < 0$$

$$\therefore \boxed{x[n] = \alpha^{n} u[n]}$$

Case ②: ROC is $|z| < |\alpha|$.

We integrate over the C as shown ⟶



→ for $n \geq 0$

$$x[n] = \frac{1}{2\pi j} \oint_C \frac{z^{n}}{z-\alpha} dz$$

$$= 0$$ ..... since $\frac{z^n}{z-\alpha}$ is halomorphic for $|z| < |\alpha|$ and using fact ②.

→ for $n < 0$,

$$x[n] = \frac{1}{2\pi j} \times \frac{1}{\alpha^{m}} \left(\oint_C \frac{dz}{z-\alpha} - \oint \frac{dz}{z} \right)$$

⟶ from Eq^n ②

$$= \cancel{Z}\ \frac{1}{\alpha^{m}} \left(0 - 1 \right)$$

from fact ①

from fact ②

$$= -\alpha^{-m} = -\alpha^{n}$$

$\therefore x[n] = -\alpha^n \quad n < 0$

$= 0 \quad n \geq 0$

$\therefore \boxed{x[n] = -\alpha^n u[-n-1]}$

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