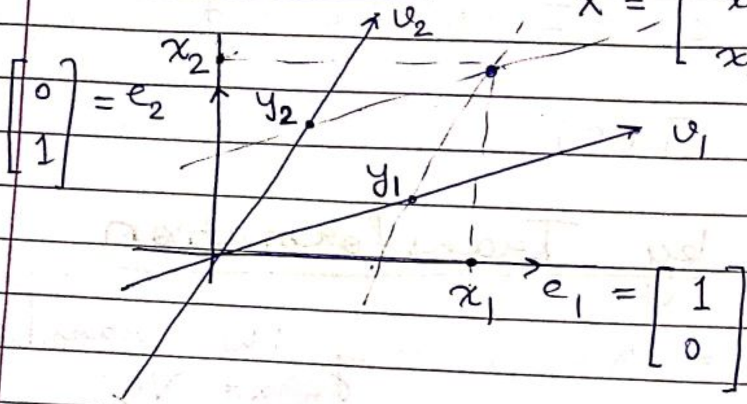
Solution of State Space Eqns by Coordinate transforms

$$\dot{x} = Ax + bu$$
$$w = Cx + du$$

Transformation



$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 e_1 + x_2 e_2$$
$$= y_1 v_1 + y_2 v_2$$
$$= \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$x = (Vy)$$

→ $\left.\begin{array}{l} \dot{x} = Ax + bu \\ w = Cx + du \end{array}\right\}$ original $\Phi_x(t)$

→ $V\dot{y} = AVy + bu$
$w = CVy + du$

$$\boxed{\Phi_y(t) = V^{-1}\Phi_x(t) V}$$

⇒ $\dot{y} = (V^{-1}AV)y + V^{-1}bu$

$w = (CV)y + du$

→ Can the transformation give us a simpler State transition Matrix

(A)

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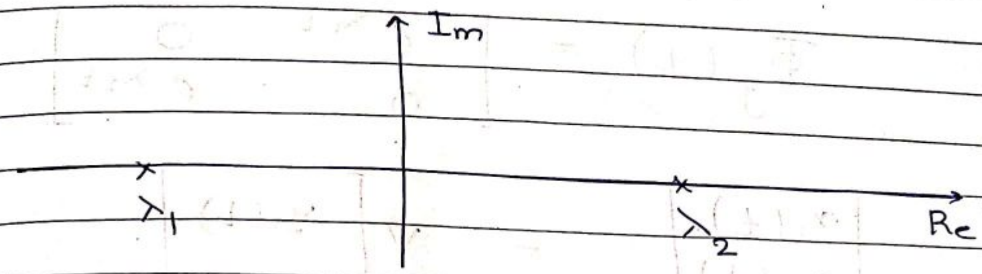
→ v_1, v_2 are eigenvector with distinct eigen values λ_1, λ_2
(Since distinct $\lambda \Rightarrow$ LI)



λ_1, λ_2 are eigen values of A
(Must be atleast complex conj. of each other)

(B)

$$Av_1 = \lambda_1 v_1 \qquad Av_2 = \lambda_2 v_2$$

$$x = y_1 v_1 + y_2 v_2$$

$$\dot{x} = \dot{y}_1 v_1 + \dot{y}_2 v_2 = A(y_1 v_1 + y_2 v_2) + bu$$

Say, $b = \beta_1 v_1 + \beta_2 v_2$

$$= y_1 \lambda_1 v_1 + y_2 \lambda_2 v_2 + \beta_1 u v_1 + \beta_2 u v_2$$

(components)

from: $\dot{y}_1 = \lambda_1 y_1 + \beta_1 u$

$\dot{y}_2 = \lambda_2 y_2 + \beta_2 u$

$$y_1(t) = e^{\lambda_1 t} y_1(0) + \int_0^t \beta_1 e^{\lambda_1 (t-\tau)} \, d\tau$$

$$y_2(t) = e^{\lambda_2 t} y(0) + \int_0^t \beta_2 e^{\lambda_2 (t-\tau)} \, d\tau$$

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→ We know the Initial conditions on the x variables, not necessarily the y variables.

$$\Phi_y(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = V \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

$$x = Vy$$
$$x(0) = Vy(0)$$

$$y = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} V^{-1} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \text{"}$$

$$\dot{x} = \left(V [\phi_y] V^{-1}\right) x + V\left[\int \text{"}\right]$$

ex

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad \lambda^2 + 3\lambda + 2 = 0$$
$$\lambda_1, \lambda_2 = -1, -2$$

$$v_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = (-2) \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\beta = -\alpha \qquad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\cancel{\alpha 2\beta}$ $\quad -2\beta$

$$\begin{matrix} \beta \\ -2\alpha - 3\beta \end{matrix} = (-1)\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{using } (\beta = -2\alpha)$$

$$V = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \qquad V^{-1} = \frac{1}{(-1)}\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$V A V^{-1} = [\vec{\lambda}]$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

For $u = 0$,

$$\dot{y}_1 = e^{-t} \qquad \dot{y}_2 = e^{-2t}$$

$$x(t) = \Phi_x(t)\,x(0) + \int_0^t \Phi_x(t-\tau)\,b\,u(\tau)\,d\tau$$

→ For zero State Solution $(x_0 = 0)$

$$w(t) = c\int_0^t \Phi_x(t-\tau)\,b\,u(\tau)\,d\tau$$

→ Step Response

$$c(t) = \mathcal{L}^{-1} \frac{1}{s(s^2+3s+2)}$$ L.T solution

$$w(t) = C\int_0^t \Phi_x(t-\tau)\, b\, d\tau$$ State space solution

---

$\dot{x} = Ax$ $\qquad$ $x = Vy$

$V\dot{y} = AVy$ $\qquad$ $\Phi_x(t) = \mathcal{L}^{-1}$

$$\dot{y} = (V^{-1}AV)y$$

Re-derive

$$\Phi_x(t) = \mathcal{L}^{-1}(sI - A)^{-1}$$

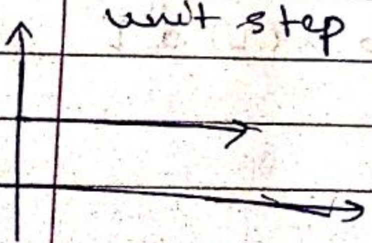
$$= \mathcal{L}^{-1}(sI - VBV^{-1})^{-1}$$

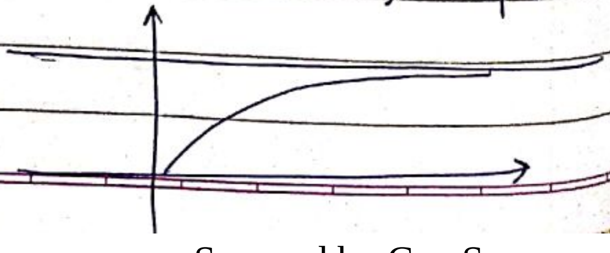
$$= V\mathcal{L}^{-1}(sI - B)^{-1}V^{-1}$$

$$B = V^{-1}AV$$

Diagram

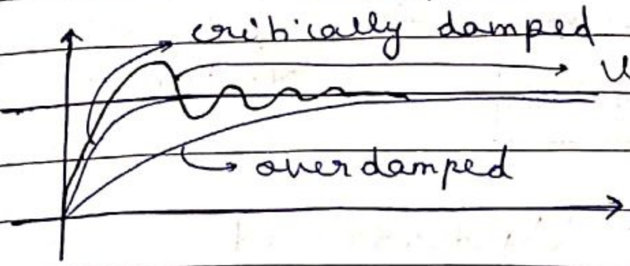
$u(t)$ (unit step) $\longrightarrow \boxed{\dfrac{1}{(s^2+3s+2)}} \longrightarrow c(t)$ (Step response an output corresponding to u(t) input.





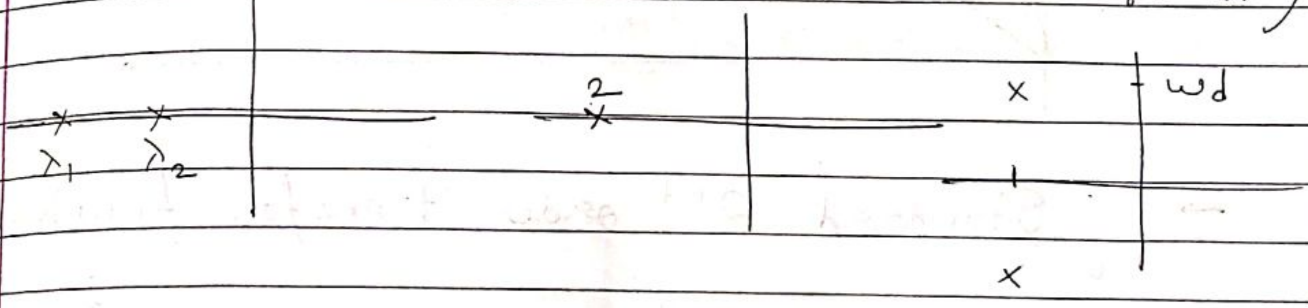
Big picture idea of C. S

→ Second order response to unit step



$$c(t) = K + c_1 e^{-t} + c_2 e^{-2t}$$

~~2nd ord~~ Location of Poles (Artificial Vs Real System)



→ Standard parameters of a 2nd order System

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\lambda_1, \lambda_2 = -\zeta\omega_n \pm \underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d}$$

ζ → Damping factor

ω_n → Natural frequency

ω_d → damped frequency

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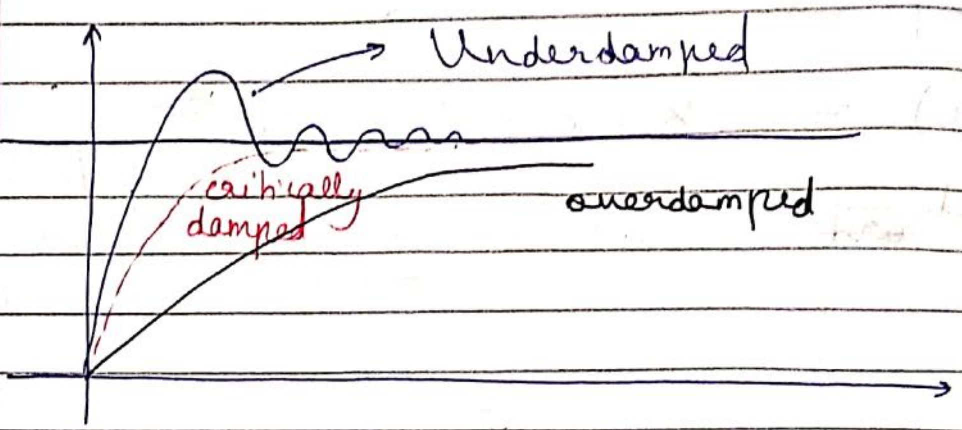
Second Order Response Model

Unit Step → System → Output



→ Standard 2nd order transfer function

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s-\lambda)(s-\bar{\lambda})}$$

$$\lambda_1, \lambda_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \quad \zeta < 1$$

$\sigma = -\zeta\omega_n$, ω_d : damped natural frequency

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$\lambda_1, \lambda_2 = -\underline{\underline{\sigma}} \pm j\omega_d$$

Exercise, Show that General form of the solution is,

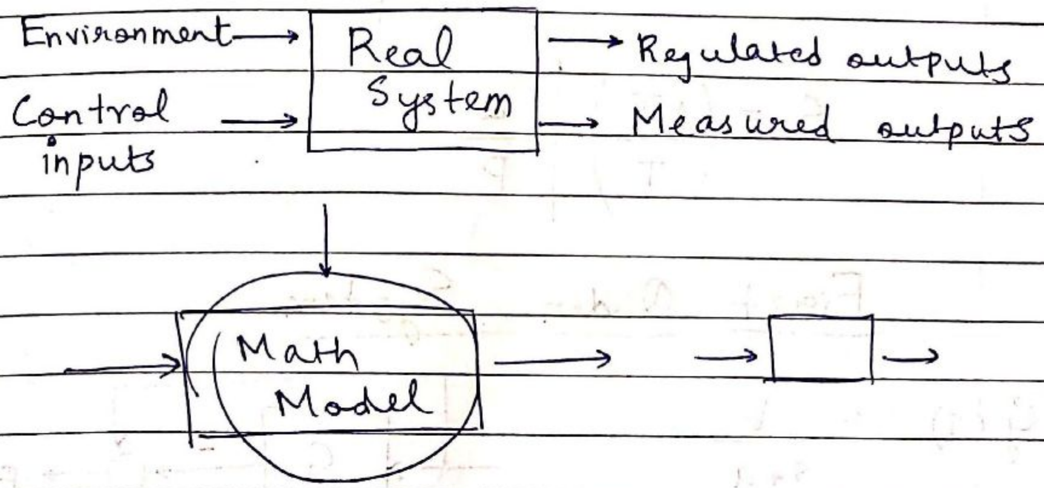
$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\sigma t} \cos(\omega_d t + \phi)$$

$$\phi = \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}$$

Qty's of Interest

- Time to peak T_p,
- Settling time T_s to settle in $\pm 2\%$ Band
- Rise time
- Steady State Error $e_{ss} = 1 - c(\infty)$

The Big Picture



Sensitivity

→ In the real world, parameters are not constant, this behaviour must be modelled.

$$\tilde{P}(s) = P(s) + \Delta(s)$$

% Change ~~ad~~ for the model $\dfrac{\Delta}{P(s)}$

→ The perturbations that we need to model here are inside the system.

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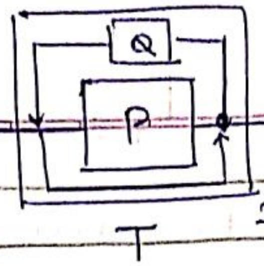


- After embedding P inside a network of controllers

T - Nominal system

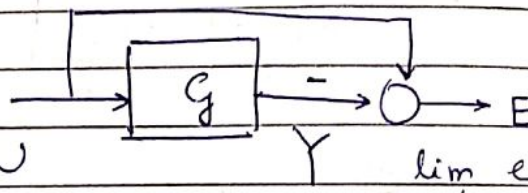
$\tilde{T}$ - Perturbed "

$$\Delta T = \tilde{T} - T \quad , \quad \frac{\Delta T}{T}$$

$$S = \left(\frac{\Delta T}{T}\right) \Bigg| \frac{\Delta}{P}$$

First Order System

$$G(s) = \frac{b}{s+a}$$



$$\lim_{t \to \infty} e(t) = E($$

$$E = \frac{1}{s} - Y = \frac{1}{s} - \frac{b}{s(s+a)}$$

$$= \frac{s + (a-b)}{s(s+a)}$$

$$e_{ss} = \frac{a-b}{a} = \frac{1 - b/a}{1}$$

$$\frac{\tilde{b}}{\tilde{a}} = \frac{b}{a} + \Delta$$

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$$\Delta e_{ss} = \tilde{e}_{ss} - e_{ss}$$

$$= \left(1 - \frac{\tilde{b}}{\tilde{a}}\right) - 1 + \frac{b}{a} = -\Delta$$

% var in e_{ss} without control

$$= \frac{\Delta e_{ss}}{e_{ss}} = \frac{-\Delta}{(1 - b/a)}$$

Scanned by CamScanner

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Bode Sensitivity

$$\text{inputs} \longrightarrow \boxed{\text{System}} \longrightarrow \text{outputs}$$

→ Given System Model

$$\boxed{P(s)}$$

P – model transfer func

$\tilde{P}$ – transfer function for actual system

→ $\Delta P \equiv$ Perturbation / uncertainity

p.u or % variation $= \dfrac{\Delta P}{\tilde{P}}$ → another T.F

→ Modify P by Encapsulating in T

$$\longrightarrow \boxed{\;\boxed{P}\;} \longrightarrow$$

(T) → Also has $\tilde{T}$ associated

p.u variation after control $= \dfrac{\Delta T}{\tilde{T}}$

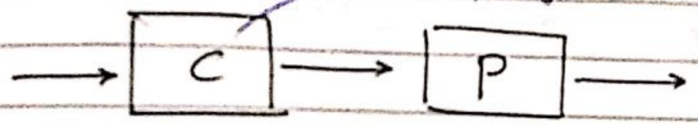
→ Sensitivity of T configuration,

$$\text{Sensitivity} \boxed{(S) = \left(\frac{\Delta T/\tilde{T}}{\Delta P/\tilde{P}}\right)}$$

→ No control ⇒ $(S = 1)$

→ $\underbrace{S < 1}_{\text{Ideal}}$ can be done only over a small b/w.

perfect system / known exactly



$T = CP$

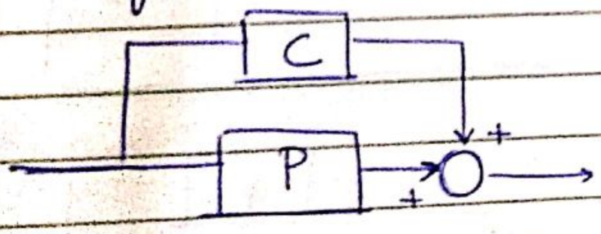
ex
C → Electronics
P → Plant

$\tilde{T} = C\tilde{P}$

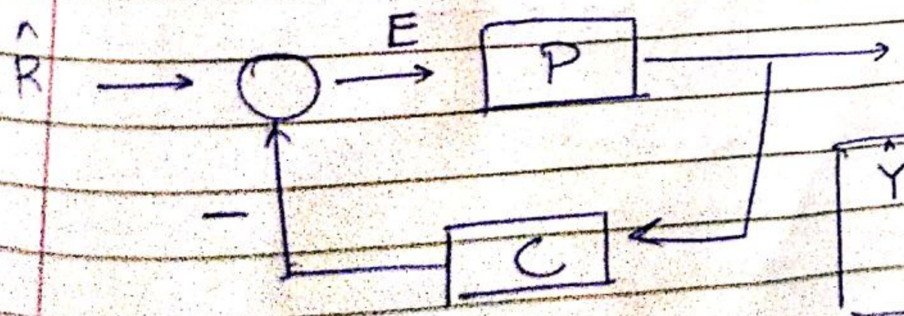
$\Delta T = C\Delta P$

$$S = \frac{C\Delta P / C\tilde{P}}{\Delta P/\tilde{P}} \quad \boxed{= 1}$$

→ Simple Cascade Control has no effect on Sensitivity



$T = P + C$



$$\left(\begin{aligned}\hat{Y} &= P\hat{E} \\ &= P(\hat{R} - C\hat{Y})\end{aligned}\right)$$

$\hat{Y} = T\hat{R}$

$$\boxed{\hat{Y} = \frac{P}{(1+PC)}\hat{R}}$$

$$\hat{Y} = \frac{P}{(1+PC)} \hat{R} \rightarrow \text{T.F of closed loop system}$$

$$\Delta T = \tilde{T} - T$$

$$P \rightarrow \tilde{P}$$

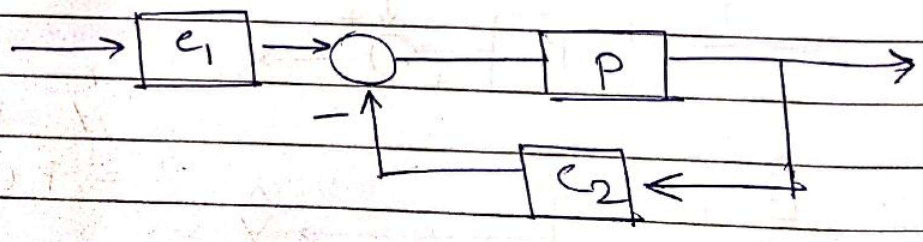
$$\frac{\tilde{P}}{(1+\tilde{P}C)} - \frac{P}{(1+PC)}$$

$$\tilde{P} + P\tilde{P}C - P - P\tilde{P}C = \frac{\Delta P}{(1+PC)(1+\tilde{P}C)}$$

$$S = \frac{\cancel{\dfrac{\Delta P}{(1+PC)(1+\tilde{P}C)}} \Big/ \cancel{\dfrac{\tilde{P}}{(1+\tilde{P}C)}}}{\cancel{P}\ \cancel{\Delta P / \tilde{P}}}$$

$$S = \boxed{\cancel{\tilde{P}} \left(\frac{\cancel{P}\ 1}{1+PC} \right)}$$

H.W



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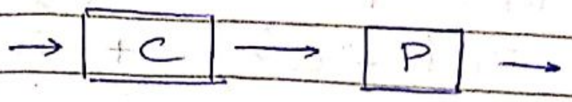
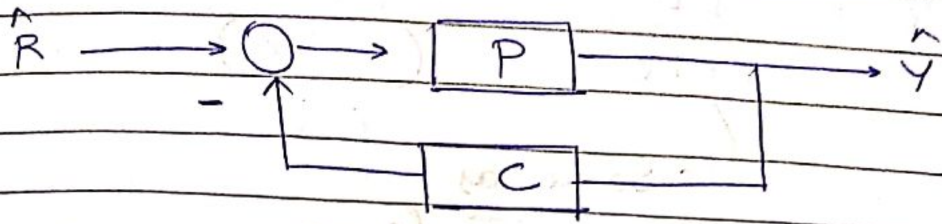
Sensitivity Function in feedback Control

1)



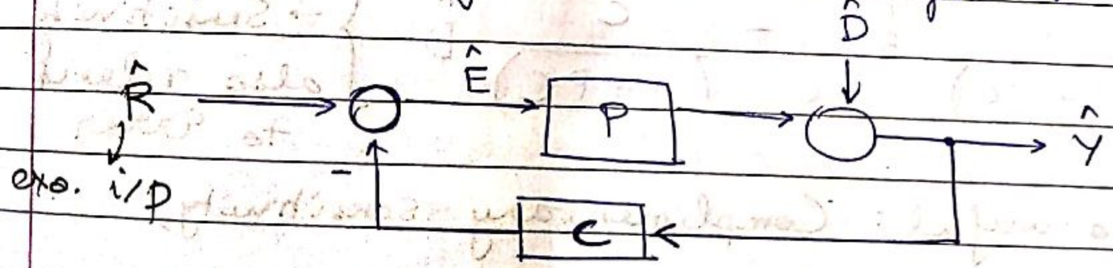
2)



$$\hat{Y} = \underbrace{\left(\frac{P}{1+PC}\right)}_{T} \cdot \hat{R}$$

$$S = \frac{\Delta T / \tilde{T}}{\Delta P / \tilde{P}} = \frac{1}{1+PC}$$

→ Properties of S in FB System



→ The Qty measured is always a disturbed version

$$\hat{Y} \rightarrow \hat{Y} + \hat{D}$$

$P - \tilde{P}$ → Uncertainity in System

$\hat{D}$ → External Disturbances – outside the Plant . . .

exogenous input

<u>Equations</u>,

$$\hat{Y} = P\hat{E} + \hat{D}$$
$$= P(\hat{R} - C\hat{Y}) + \hat{D}$$

$$\hat{Y} = \frac{P\hat{R}}{1+PC} + \frac{\hat{D}}{(1+PC)}$$

Same as effect of T from $\hat{R} \rightarrow \hat{Y}$

T is only from $\hat{R} \longrightarrow \hat{Y}$

$$\hat{E} = \hat{R} - C\hat{Y}$$

$$= \hat{R} - C\left(\frac{P}{1+PC}\hat{R} + \frac{1}{1+PC}\hat{D}\right)$$

$$\frac{1}{(1+PC)}\hat{R} - \frac{C}{(1+PC)}\hat{D}$$

→ Sensitivity also relates to Error

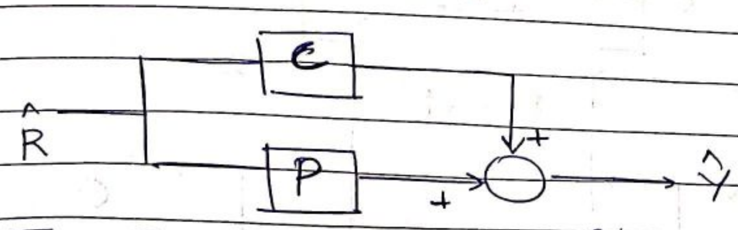
→ Also useful: Complimentary sensitivity,

$$S_c = \frac{PC}{1+PC} = 1 - S, \quad S = \frac{1}{1+PC}$$

1. Making sensitivity "small" is important in CSD

2. Feedback control scheme has scope for making Sensitivity small

ex



$T = P + C \qquad \tilde{T} = \tilde{P} + C$

$$\Delta T = \Delta P$$

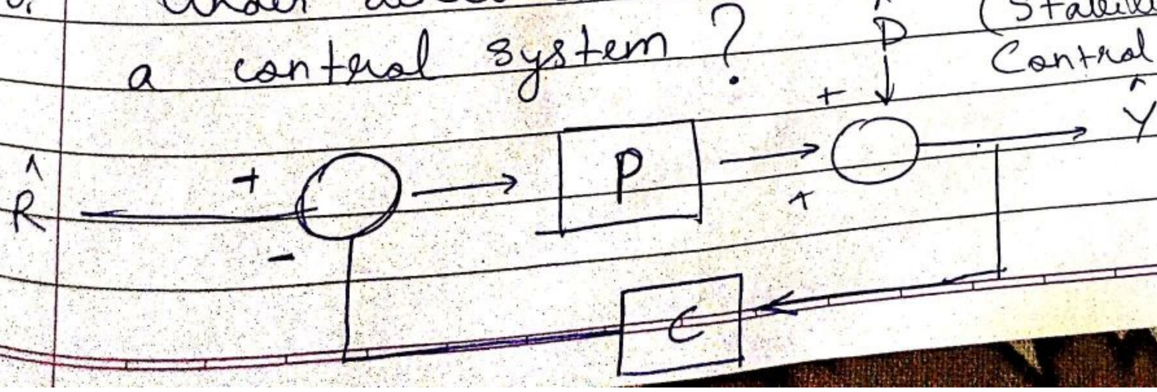
$$S = \frac{\Delta T / \tilde{T}}{\Delta P / \tilde{P}} = \frac{\overbrace{\Delta P}^{\Delta T}}{\underbrace{\tilde{P} + C}_{\tilde{T}}} \cdot \frac{\tilde{P}}{\Delta P} = \frac{\tilde{P}}{\tilde{P} + C}$$

3. Mathematical framework for handling S

* Ideas: Nbd (neighbourhood), distance, largeness smallness of transfer functions

4. What are possible limits to changing sensitivity

5. Under what constraints should we design a control system? (Stability of Control System)



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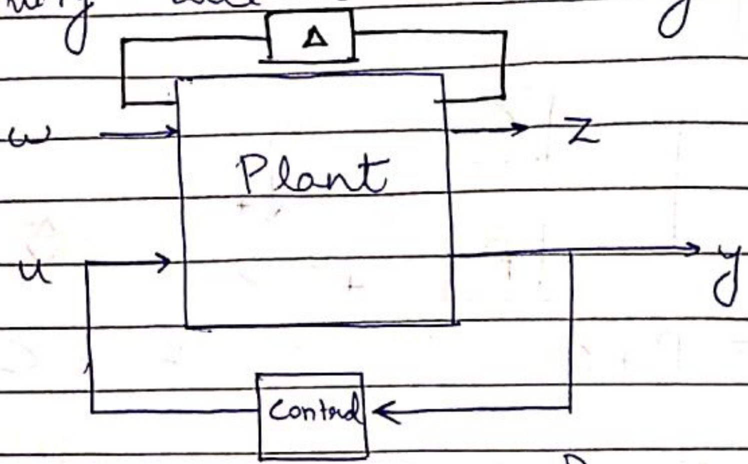
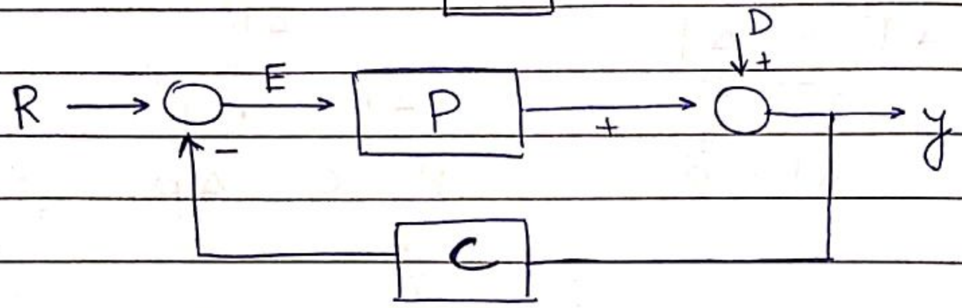
Control System Objectives

1. Desired performance against reference inputs, regulation etc.

2. Feedback control is necessary to reduce sensitivity due to uncertainity in the model





$$\hat{Y} = \left(\frac{1}{1+PC}\right)\hat{D} + \left(\frac{P}{1+PC}\right)\hat{R}$$

$$\hat{E} = \left(\frac{1}{1+PC}\right)\hat{R} + \left(\frac{C}{1+PC}\right)\hat{D}$$

Mathematical Stability Theory

1. $\boxed{\text{Constraints}}$ of feedback Control?
2. Limits to achievable performance?

1. What is a Stable System?
2. How to stabilize a given system?
3. Is there a characterization/classification of stable linear feedback systems

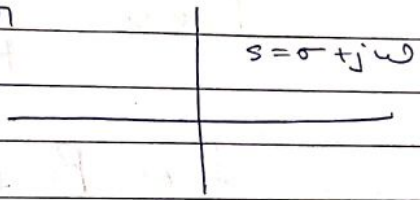
Constraints

(i) Consider only Laplace transformable signals

$$f(t) \in [0 \;\; \infty)$$

$$\hat{F}(s) = \int_0^\infty f(t)\, e^{-st}\, dt \quad , \quad \hat{F}(s) \text{ exists when } |\hat{F}(s)| < \infty$$

over



→ Also, $f(t)$ should be bounded,

$$|f(t)| \leq c e^{\sigma t}$$

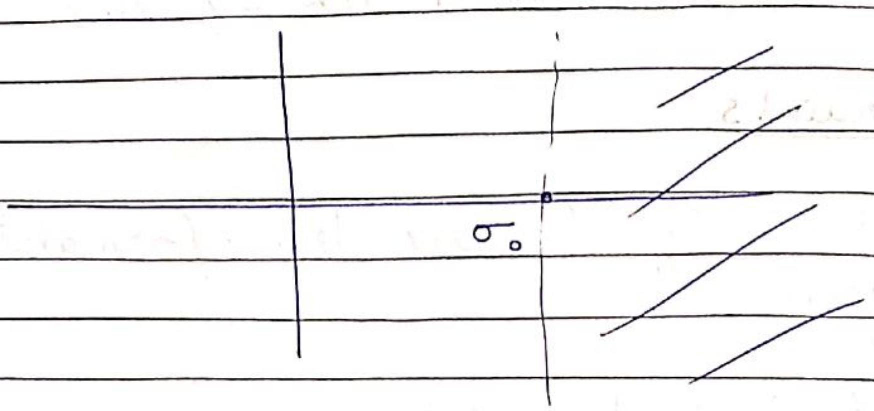
$$|\hat{F}(s)| = \left| \int_0^\infty f(t)\, e^{-\sigma t}\, e^{-j\omega t}\, dt \right|$$

$$\leq \int_0^\infty |f(t)|\, e^{-\sigma t}\, dt \leq \int_0^\infty c e^{(\sigma_0 - \sigma)t}\, dt$$

if $|f(t)| \leq ce^{\sigma_0 t} < \infty$ if $\sigma > \sigma_0$

→ If $|f(t)| \leq ce^{\sigma_0 t}$

then $|\hat{F}(s)| < \infty$ for all $\sigma = Re(s) > \sigma_0$



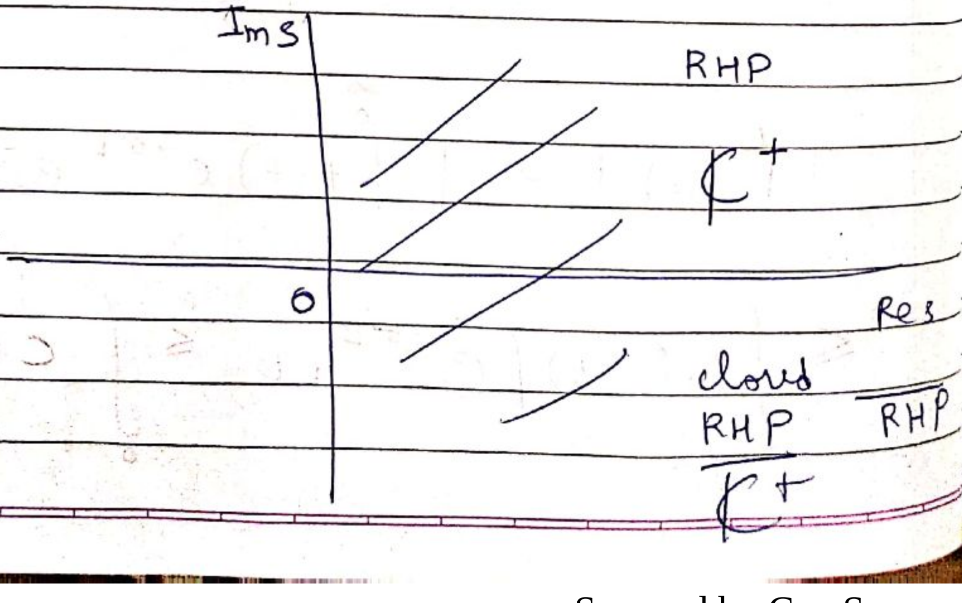
ROC of $\hat{F}(s)$

→ B is the set of all functions $f(t)$ s.t

$$|f(t)| \leq M < \infty$$

hence, ROC of any functions $f \in B$ in the R.H.P

$$RHP = \{ s \in \mathbb{C} \mid Re\ s > 0 \}$$



RHP

$\mathbb{C}^+$

closed RHP $\overline{RHP}$

$\overline{\mathbb{C}^+}$

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Stable Systems

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→ Bounded functions: (Recap)

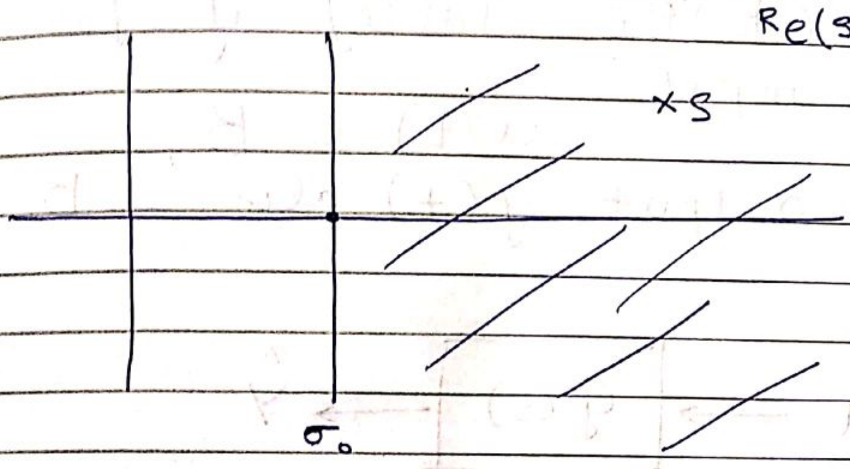
Uniformly bounded functions $f(t)$, $t \in [0, \infty)$ satisfy,

$$|f(t)| < M < \infty$$

$f(t)$ is Laplace Transformable if $\exists\ \sigma_0$ s.t

$$|f(t)| \leq ce^{\sigma_0 t}$$

When $f(t)$ is uniformly bounded ($<M$) then $\hat{F}(s)$ has no poles in $\mathbb{C}$ s.t $Re(s) > \sigma_0$



→ If $\hat{F}(s)$ has no poles in $\mathbb{C}^+$ (RHP) and those poles on $j\omega$ Axis are simple then $f(t)$ is uniformly bounded

(ex $\frac{1}{s} \rightarrow u(t)$: uniformly bounded
Simples pole

$\frac{1}{s^2} \rightarrow ramp(t)$: Not uniformly bounded

ex $\frac{1}{s^2}$ / $\frac{1}{(j\omega)^2}$ $\frac{s}{(s^2+\omega^2)}$

↳ Not uniformly bounded

→ Let B be the set,

B : the set of bounded functions of $t \in [0, \infty)$

which have Laplace Transforms satisfying

1. No poles in the $\mathbb{C}^+$ (RHP)
2. Poles on $j\omega$ Axis are simple

→ Then, an input - output linear system represented by a transfer function $G(s)$ is said to be "stable" (BIBO stable) if for every input

$$u(t) \in B,$$

the output $y(t)$ also $\in B$

$$\hat{u} \longrightarrow \boxed{G(s)} \longrightarrow \hat{y}$$

BIBO stable system

$$u(t) \in B \Rightarrow y(t) \in B$$

If $\mathcal{L}^{-1}(\hat{u}) \in B$

$\mathcal{L}^{-1}(\hat{u}\, G(s)) \in B$

To make seperate statements about Numerator & Den

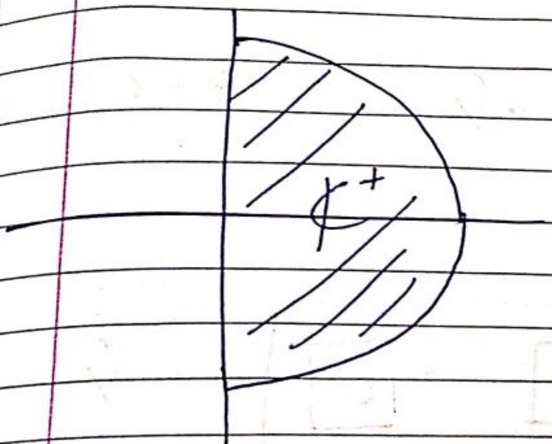
Thm If a system ~~G(s)~~ is represented by a transfer function $G(s)$ which has no pole-zero cancellation in the closed RHP ($\mathbb{C}^+$ or $\overline{RHP}$) then the system is BIBO stable iff $G(s)$ has no poles in $\mathbb{C}^+$

$$G(s) = \frac{q(s)}{p(s)}$$

$\Rightarrow$ eq. $p(s)$ has no roots in $\mathbb{C}^+$

BIBO Stable Systems are denoted as $\mathcal{S}$

$\mathcal{S}$ has no poles in $\overline{RHP}$ (or $\mathbb{C}^+$)



<u>Properties</u>

$G(s), H(s) \in \mathcal{S}$

then $G + H \in \mathcal{S}$

$GH \in \mathcal{S}$

all constants

$\mathbb{R} \in \mathcal{S}$

Same as

If $G \in \mathcal{S}$, when does $\frac{1}{G} \in \mathcal{S}$?

- $q(s)$ also has no roots in RHP
- $\deg(p(s)) = \deg(q(s))$

<u>Examples</u>

$$\frac{100(s-2)}{(s+1)(s+3)} \in \mathcal{S}$$

$$\frac{s^2+3}{(s+1)^3} \in \mathcal{S}$$

$$\frac{(s-1)(s+2)}{(s+1)(s-2)} \notin \mathcal{S}$$

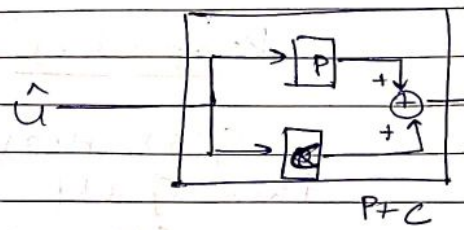
Hurwitz Polynomials

→ Polynomials $p(s)$ with no roots in the

(Note relation with $\mathcal{S}$ → denominator

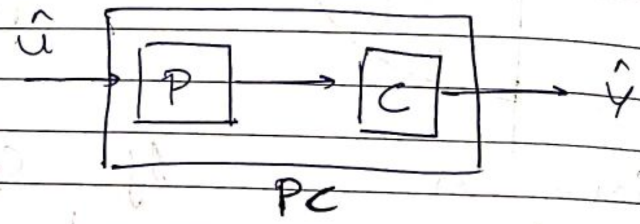
Connections of Systems

1) Sum.

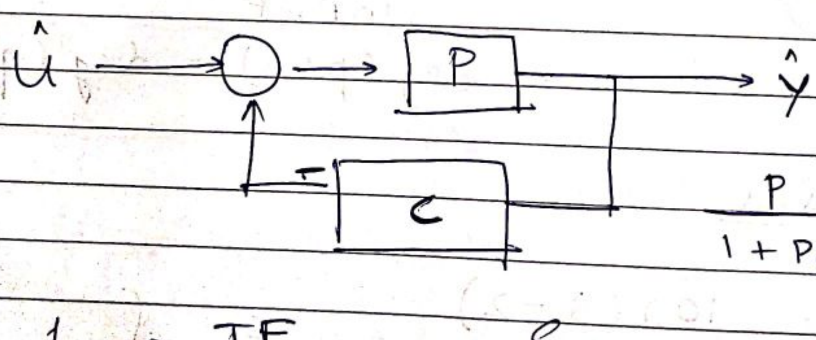


$(P+C)$

2) Cascade



3) Feedback Connection



$\frac{P}{1+PC}$ → transfer function

Open Loop TF

PC

Sensitivity ~~$\frac{1}{1+PG}$~~ $\frac{1}{1+PC}$

Complementary Sensitivity $\frac{PC}{1+PC}$

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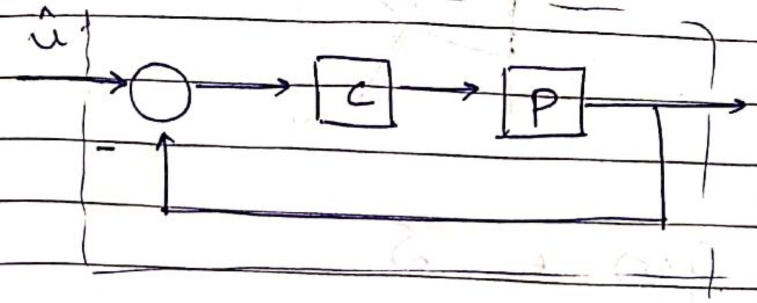
~~Alternately~~

→ Another form of feedback System

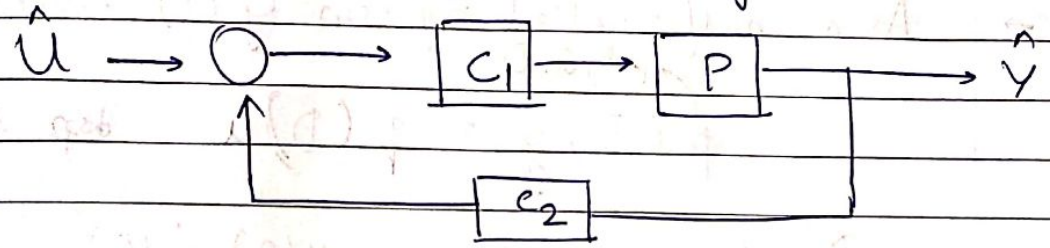


$$T = \frac{PC}{1+PC} = 1 - S$$

S → Sensitivity

OL TF PC

→ 2 - DOF feedback System,



Open Loop TF $(C_1 C_2 P)$

.2.19

Class of Stable Systems $\mathcal{S}$.

↳ The set of BIBO stable systems

Thm If $G(s) = \frac{q(s)}{p(s)}$

has no pole-zero cancellation in $\mathbb{C}^+$ then $G(s)$ represents a BIBO Stable system iff

$G(s) \in \mathcal{S}$ iff

$p(s)$ is a Hurwitz polynomial

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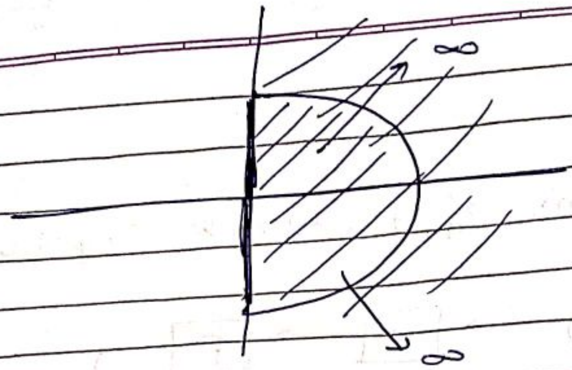


No poles in closed RHP

If $G(s) \in \mathcal{S}$

$$\hat{u} \longrightarrow \boxed{G(s)} \longrightarrow \hat{Y}$$

$u(t) \in B$ $y(t) \in B$

→ As a differential system $G(s)$ corresponds

$$\overset{\text{poly in } D}{p(D)}\, y = q(D)\, u \qquad \text{deg } p = d$$

$D = \frac{d}{dt}$ $u = 0$, $y(0), \dot{y}(0), \ldots, y^{(d-1)}$

Explanation Detour

$$G(s) = \frac{(s+1)}{(s^2+3s+5)}$$

Ratio of $\frac{\mathcal{L}(\text{out})}{\mathcal{L}(\text{in})}$ under zero initial conditions.

$$(s^2+3s+5)\,\hat{Y} = (s+1)\,\hat{U}$$

$$(D^2+3D+5)\,y = (D+1)\,u$$ → zero state + zero input

where $D = \frac{d}{dt}$

ham

→ Zero input response of $y(t)$ is a soln. of

$$(D^2 + 3D + 5)\, y(t) = 0$$

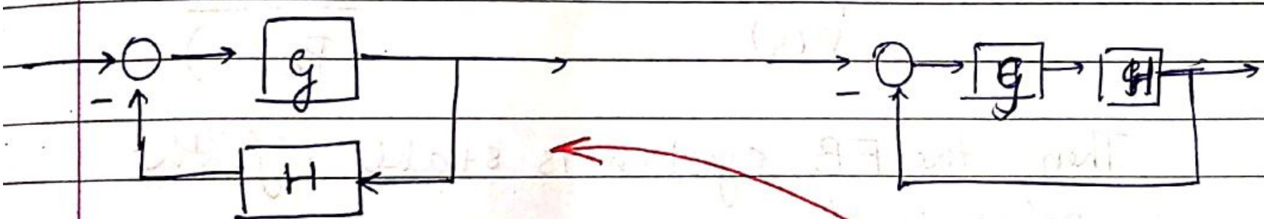
when $y(0) = y_0$ $\quad \dot{y}(0) = y_1$

→ **Internal Stability**

$G(s)$ represents internally stable system if $y(t) \to 0$ as $t \to \infty$ under 0 input.

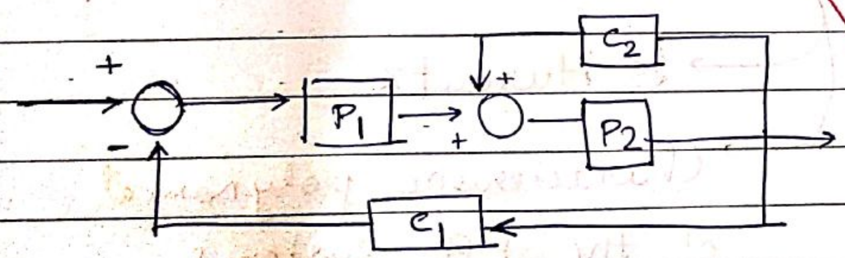
Thm $G(s)$ is Internally Stable iff $P(s)$ has no roots in $\overline{\mathbb{C}}^+$ (Same as BIBO)

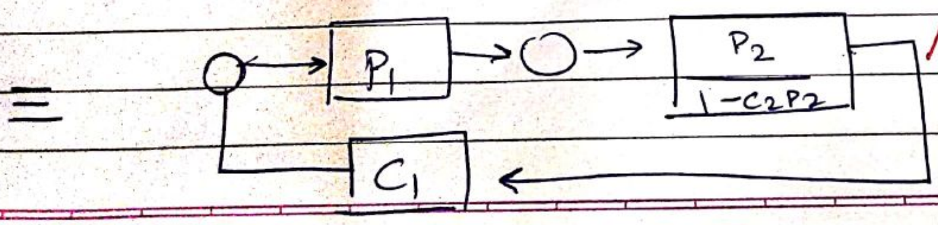
Feedback Configurations



ex Combinations of F.B Systems

Analyzing std. case is enough





Defn

→ Stability of the closed loop system

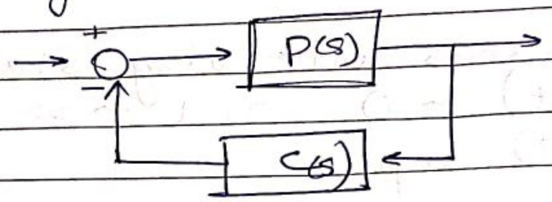
1. Input - Output Stability (BIBO)
2. Internal Stability   for a single block

→ The same only ~~when no~~ pole-zero cancellation.

How it is defined for F.B system? ..... later

→ Condition for stability of a closed loop system:

Thm



Let $P(s)$ and $C(s)$ be transfer functions each of which have no pole zero cancellation in $\mathbb{C}^+$

$$P(s) = \frac{q(s)}{p(s)} \qquad C(s) = \frac{q_c(s)}{p_c(s)}$$

Then the FB system is stable iff the polynomial

$$\psi(s) = p(s)\,p_c(s) + q(s)\,q_c(s)$$

→ Is Hurwitz

→ Characteristic polynomial of the FB systems