EE 338: DSP

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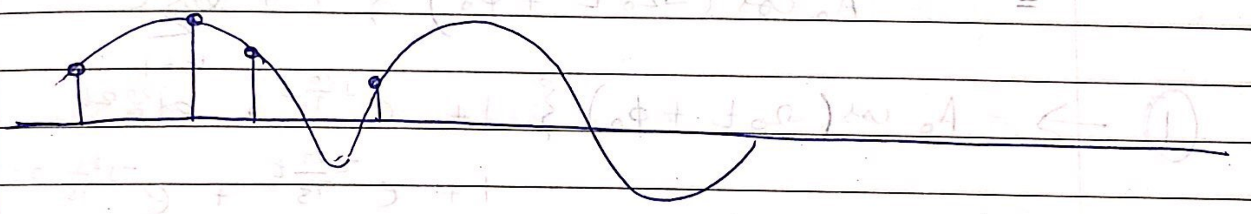
- Syllabus
  - → Discrete time Signals and Systems
  - → DT FT → Z - transform
  - → Digital Filters → FIR / IIR Design
  - → FFT → ~~Reala~~ Realization of DT Systems
  - → Polynomial Approximations

Sampling Theorem

- Continous time / variable ⟶ Discrete time / variable
- → Without losing any information ⇒ reconstructability
- → Fitting sinusoids onto samples



$$x(nT_s) = A_0 \cos(\Omega_0 n T_s + \phi_0)$$

$$= A_0 \cos \pm(\Omega_0 n T_s + \phi_0 + 2\pi n k)$$

even

→ Is the same as,

$$x(nT_s) = A_0 \cos\left\{\left(\frac{2\pi}{T_s}k + \Omega_0\right) nT_s + \phi_0\right\}$$

$$= A_0 \cos\left\{\left(\frac{2\pi}{T_s}k - \Omega_0\right) nT_s - \phi_0\right\}$$

Challenge prob- 1

Q→ $S_N = x_0(t) + \sum_{k=1}^{N} \{ x_{k+}(t) + x_{k-}(t)$

$x_{k+}(t) = A_0 \cos\left\{\left(\frac{2\pi}{T_s}k + \Omega_0\right) t + \phi_0\right\}$

$x_{k-}(t) = A_0 \cos\left\{\left(\frac{2\pi}{T_s}k - \Omega_0\right) t - \phi_0\right\}$

$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$2A_0 \cos\left(\frac{2\pi}{T_s}kt\right) \cos(\Omega_0 t + \phi_0)$

$\left(\frac{e^{j\frac{2\pi}{T_s}kt} + e^{-j\frac{2\pi}{T_s}kt}}{\cancel{2}}\right) \times \cancel{2}$

$+ A_0 \cos\{\Omega_0 t + \phi_0\}$

$= A_0 \cos(\Omega_0 t + \phi_0)\left\{1 + \sum_{k=1}^{N}\left(e^{j\frac{2\pi}{T_s}kt} + e^{-j\frac{2\pi}{T_s}kt}\right)\right.$

(1) → $= A_0 \cos(\Omega_0 t + \phi_0)\{1 + e^{j\frac{2\pi}{T_s}t} + e^{j\frac{2\pi}{T_s}2t} + \dots$

$1 + e^{-j\frac{2\pi}{T_s}t} + e^{-j\frac{2\pi}{T_s}\cdot 2t} + \dots\}$

-1

$= \frac{a((c\cdot r)^N - 1)}{(c\cdot r - 1)}$

$\frac{\left(e^{j\frac{2\pi}{T_s}t}\right)^{N+1} - 1}{\left(e^{j\frac{2\pi}{T_s}t} - 1\right)} + \frac{\left(e^{-j\frac{2\pi}{T_s}t}\right)^{N+1} - 1}{\left(e^{-j\frac{2\pi}{T_s}t} - 1\right)} - 1$

$\frac{\left(e^{j\frac{2\pi}{T_s}t}\right)^{N+1} - 1}{\left(e^{j\frac{2\pi}{T_s}t} - 1\right)} + \frac{\left(e^{-j\frac{2\pi}{T_s}t}\right)^{N} - e^{j\frac{2\pi}{T_s}t}}{\left(e^{j\frac{2\pi}{T_s}t} - 1\right)} - 1$

$f_{samp} > 2$

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$$\frac{e^{j\frac{2\pi}{T_s}t} - 1 + \left(e^{j\frac{2\pi}{T_s}t}\right)^{N+1} - \left(e^{-j\frac{2\pi}{T_s}t}\right)^{N}}{\left(e^{j\frac{2\pi}{T_s}t} - 1\right)} - 1$$

Sampling thm

Samples $\longrightarrow$ Curve

Curve $\longrightarrow$ Samples

Simplifying (1) (By pairing terms)

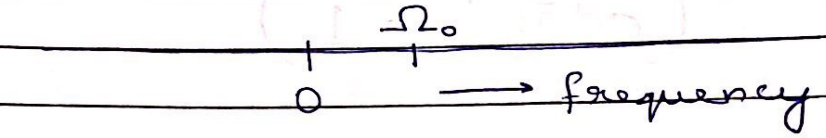
we get, $S_n = A_0 \cos(\phi_0 + \Omega_0 t) \dfrac{\sin\left(\frac{(n+1)\pi t}{T_s}\right)}{\sin\left(\frac{\pi t}{T_s}\right)} \cos\left(\frac{n\pi t}{T_s}\right)$

→ Redone : Sum evaluation

$$2A_0 \cos\left(\frac{2\pi}{T_s} k t\right) \cos(\Omega_0 t + \phi_0)$$

$+$

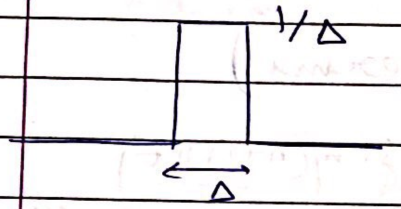
- Challenge problem 1 summary



$$\text{Sampling Rate} = \frac{1}{T_s}$$

$$\text{Internal} = T_s$$

→ What is an Impulse?



$\lim \Delta \to 0$

→

→ Sampling is a Linear process, so ~~we~~ in this problem we tried to see wh sinusoids come together to try an reconstruct a sinusoid at Ω_0

→ When we sample, there is an Ambiguit of $\frac{2\pi}{T_s} \times K$ in which sinusoids ~~oo~~ forms the signal, this is exactly the same as there being copies of original spectrum at $\frac{2\pi}{T_s}$ Internals

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Discrete time Signal Processing System

Continous Signal → [ADC] → Discrete Signal Input $x[n]$

$x[n]$ → [Discrete System] → output sequence $y[n]$

A sequence

$y[n]$ → [DAC] → continous o/p

ex speech/audio

Challenge Problem 2: What are the benefits of Aliasing?

Discrete time Vs Digital : Diff = Quantisation
Ignore Quantisation Noise (∞ bits)

Input Signal → [System] → Output Signal

Signal: Mapping
Independant variable → Dependent var.

Need not be the same (Oscilloscope)

System: Mapping b/w Signals

Question: What is a mapping b/w Systems called?
→ There can be different ways ~~of~~ of thinking about the same system
→ Transforms

Signal Processing

1. Seperating (Isolating)
   ex Signals & Noise

2. Modifying
   ex Graphic Equalizers
   Modifying signals in a desired way?

→ Applications of Discrete Signal Processing

1. Audio 2. Speech 3. Biomedical Signal
4. Seismic 5. Extra-Terristial (Mars)

~~Chal 3:~~ Fun ~~Exercise~~ : 1 Application of Discrete Signal Processing
On Moodle
Grps ← 11-20

Chall 3: Input 3 Octaves Vocal

सा रे , ..., सा , - - - -, . - . . .

Audio Signal → Record in Digital form

Record in Digital form → $\boxed{\text{Discrete Systems 1-4}}$ These signal were to be Interpreted as filters

$$y[n] = \frac{1}{2}[x[n] \pm x[n-1]]$$

or

$x[n-2]$]

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Discrete Signal = Sequence
Mapping

$$\mathbb{Z} \longrightarrow \mathbb{C}$$

System Mapping : Input sequence → Output sequence.

System Properties

- Prescribed results of experiments on the system.

1. Additivity

$$\begin{matrix} x_1[n] \\ x_2[n] \end{matrix} \longrightarrow \boxed{S} \longrightarrow \begin{matrix} y_1[n] \\ y_2[n] \end{matrix}$$

$$x_{1,2}[n] \longrightarrow y_{1,2}[n]$$

Multistatement

$$x_1[n] + x_2[n] \longrightarrow y_1[n] + y_2[n]$$

$\forall x_1, x_2$, for all n

2. Homogeneity or Scaling

$$x[n] \longrightarrow y[n]$$
$$\alpha\, x[n] \longrightarrow \alpha\, y[n]$$

$\forall \alpha \in \mathbb{C}$, $\forall x$

3. Shift Invariance or time invariance

$$x[n] \xrightarrow{\mathcal{S}} y[n]$$

$$\Rightarrow x[n-n_0] \xrightarrow{\mathcal{S}} y[n-n_0]$$

$$\forall x, \forall n_0 \in \mathbb{Z}$$

→ Implication is that a system does not evolve with time.

4. Linear Systems
   → Both Additive and Homogenous

Linear, Shift Invariant Systems

• Unit Impulse Sequence

$$\delta[n] = 1 \quad n = 0$$
$$= 0 \quad n \neq 0$$

Let, $\mathcal{S}$ be an LTI System

$$\delta[n] \longrightarrow \boxed{\mathcal{S}} \longrightarrow h[n]$$

Impulse Response

$$x[n] = 1 \quad \underset{\substack{\uparrow \\ -1}}{5} \quad 3 \quad 0 \quad 2$$

↳ Arrow notation, Assigns 1 index, Infer the others

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

$$\delta[n-k] \longrightarrow \boxed{\mathcal{S}} \longrightarrow h[n-k]$$

For fixed k ~~Shift Invariance~~

Reason: Shift Invariance

$$x[k]\,\delta[n-k] \longrightarrow \boxed{\mathcal{S}} \longrightarrow x[k]\,h[n-k]$$

Reason: Homogeneity

$$\underbrace{\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]}_{x[n]} \longrightarrow \boxed{\mathcal{S}} \longrightarrow \underbrace{\sum_{k=-\infty}^{\infty} x[k]\,h[n-k]}_{y[n]}$$

Reason: Additivity

An operation

Thm (From Above)

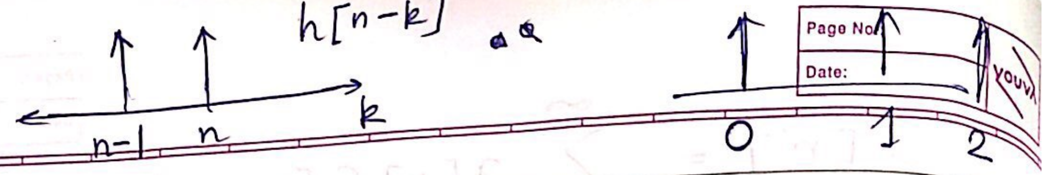
A Linear Shift-Invariant system is completely characterized by its impulse response

- Above operation is Convolution

$$x * h = y$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

- Analyze for a fixed n



Exercise

Convolve 1 1 (with 1 at 0)

with 1 1 1 (with 1 at 0)

$n=0 = 1$

$n = 1 = 2$

$n = 2$

$n = 3$

$n = 4$

0 1 1 1 0

1 1

0 1 2 2 1 0

-1 0 1 2 3 4

→ Convolution as an operation

1) Commutative

Yes,

$$x_1 * x_2 = x_2 * x_1$$

2) Associative

$$x_1 * (x_2 * x_3) = (x_1 * x_2) * x_3$$

3) Distributive ppt

$$x_1 * (x_2 + x_3) = x_1 * x_2 + x_1 * x_3$$

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→ Linear Shift-Invariant systems

- impulse response completely describes the System

- Output is convolution

Causality

$$x_{1,2}[n] \longrightarrow \boxed{S} \longrightarrow y_{1,2}[n]$$

Two inputs $x_1[n]$, $x_2[n]$ ~~$\forall n$~~
s.t $x_1[n] = x_2[n] \quad \forall n \le n_0$

Then, for a causal system
$$y_1[n] = y_2[n] \quad \forall n \le n_0 \quad \forall x_{1,2} \quad \forall n_0$$

Try understanding how this is the same as cause & effect.

Thm

LTI systems are causal iff the impulse response $h[n] = 0 \quad \forall n < 0$

Proof

If, $h[n] = 0 \quad \forall n < 0$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

put $n - k = l \Rightarrow$ ~~k~~ $n = k + l$
$\Rightarrow n \in -\infty, \infty$

$$y[n] = \sum_{l=-\infty}^{\infty} x[n-l]\, h[l]$$

$$= \sum_{l=0}^{\infty} x[n-l]\, h[l]$$

If ~~$x_1[n]$~~
$$x_1[n] = x_2[n] \quad \forall n \le n_0$$
$$n_0 - l \le n_0 - l$$

Proof : (only if)

Tutorial Exercise (first 4)

Bounded - Input , Bounded - output Stability

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Bounded input - Bounded output Stability (BIBO)

- Let $x[n]$, a bounded input be applied to S

$$|x[n]| \leq M_x \quad \text{for some } 0 \leq M_x < \infty$$

$$x[n] \rightarrow \boxed{S} \rightarrow y[n]$$

$\hookrightarrow$ Also bounded in a similar way

Examples

1) $y[n] = x^2[n]$ → BIBO stable
→ Bounded

2) $y[n] = \dfrac{1}{x[n]}$ → Not BIBO stable

3) $y[n] = \dfrac{1}{1+x[n]}$ ~~$y[n]$~~ → Not BIBO stable

4) $y[n] = e^{x[n]}$ → BIBO stable

→ Stability of Linear Shift Invariant system

Thm An LTI system is stable iff its impulse response is

Absolutely Summable

Proof If,

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]\,x[n-k]\right|$$

$$\leq \sum_{k=-\infty}^{\infty} |h[k]|\,|x[n-k]|$$

$$\leq M_x \sum_{k=-\infty}^{\infty} |h[k]|$$

Only if

Given $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$

$$\sum_{k=-\infty}^{\infty} |h[k]| = S > 0$$

then, $\sum_{k=-\infty}^{\infty} x[n-k]\,h[k] \leq \sum_{k=-\infty}^{\infty} |x[n-k]|\,|h[k]|$

$$\leq \sum_{k=-\infty}^{\infty}$$

→ Force system to reveal instability

ex) Unstable system

$$y[n] = \sum_{k=0}^{\infty} x[n-k]$$

$$h[n] = 1 \quad n \geq 0$$
$$= 0 \quad n < 0$$

$h[n] = u[n]$ ↳ unit step function

→ Next tutorial Exercise Grps 21-30

$$y[n] = 0.5 \left\{ x[n] \pm x[n-\tfrac{1}{2}] \right\}$$

$$x[n] = A_0 \cos(\omega_0 n + \phi_0)$$

↳ ~~Inputs~~

Write o/p in form

$$B_0 \cos(\omega_0 n + \alpha_0)$$

↳ comment on the B_0, α_0

7) 14.1.18

→ Four simplest discrete time LSI systems

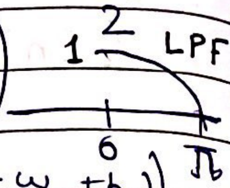
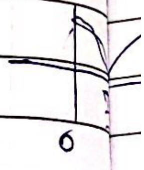
$$y[n] = \frac{1}{2} \{ x[n] \pm x[n-1,2] \}$$

Linear Shift Invariant stable systems

Input Sampled Sinusoid of frequency ω_0 ⟶ Output, also sinusoid of frequency ω_0

Normalized Angular frequency : $\dfrac{2\pi \times \text{Actual freq in Hz}}{\text{Sampling freq in}}$

↳ Nyquist ⇒ $(-\pi, \pi)$

System	Output when input =	Amp. Change	Phase Change
$y[n] = \frac{x[n]+x[n-1]}{2}$	$A_0\cos(\omega_0 n+\phi_0)$ $\frac{\cos\omega_0}{2} A_0 \cos\left(\omega_0 n + \phi_0 - \frac{\omega_0}{2}\right)$	$\cos\frac{\omega_0}{2}$ LPF 	
$\frac{x[n]-x[n-1]}{2}$	$\frac{A_0}{2}\left(\cos(\omega_0 n+\phi_0) - \cos(\omega_0 n-\omega_0+\phi_0)\right)$ $= A_0\sin\left(\frac{\omega_0}{2}\right)\cancel{\cos}\left(\omega_0 n + \phi_0 - \frac{\omega_0}{2} + \frac{\pi}{2} - \frac{\omega_0}{2}\right)$		
$\frac{x[n]+x[n-2]}{2}$	$\cancel{A_0\left(\cos(\omega_0 n+\phi_0-\omega_0)\right)\cos}$ $A_0\cos(\omega_0 n+\phi_0-\omega_0)\cos(\omega_0)$		
$\frac{x[n]-x[n-2]}{2}$	$A_0 \cos(\omega_0 n+\phi_0-\omega_0)\sin(\omega_0)$ $\frac{\pi}{2} + (\,)$		

→ All of the above are Filters (crude)

- Also, for all of them, the phase change is affine in the ω_0
  - ⇒ There is uniform time delay
  - ⇒ Called pseudo linear phase (fixed Group delay, varying phase delay)

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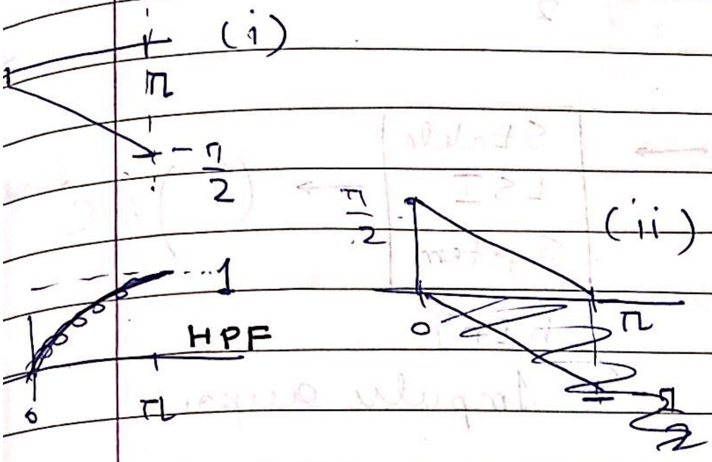
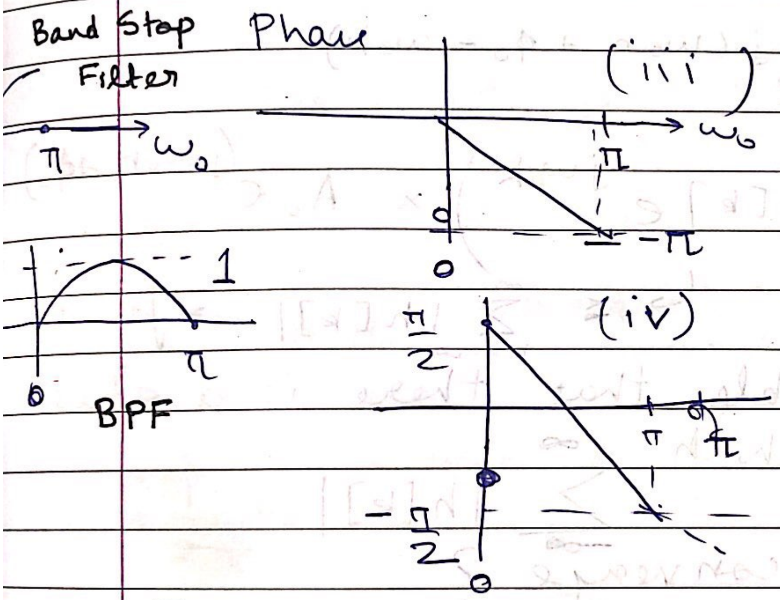




From this, Advantages of Digital filtering:

1. Robust
2. Versatile
3. New behaviours unattainable with analog

Complex Exponentials

$$A_0 \cos(\omega_0 n + \phi) = \frac{A_0}{2}\left[\, e \right.$$

$$\frac{A_0}{2} e^{j\phi} \left[e^{j\omega_0 n} + e \right.$$

$$A_0 e^{j(\omega_0 n + \phi_0)} \longrightarrow \boxed{\begin{array}{c}\text{Stable} \\ \text{LSI} \\ \text{System}\end{array}} \longrightarrow (\quad) A_0 e^{j(\ldots + \phi_0)}$$

h[n]

↳ Impulse response

$$\sum_{k=-\infty}^{\infty} h[k] A_0 e^{j(\omega_0 n + \phi_0 - \omega_0 k)}$$

$$= \left(\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k} \right) \times A_0 e^{(j\omega_0 n + \phi_0)}$$

$$\hookrightarrow \leq \sum_{k=-\infty}^{\infty} |h[k]| \rightarrow \text{finite}$$

→ Is it ~~to~~ possible that there is a stable system for which

$$\sum_{-\infty}^{\infty} |h[k]|$$

does not converge ?

Frequency Response

$$H(e^{j\omega_0}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k}$$

ex $\quad \frac{1}{2} \{ x[n] + x[n-1] \}$

Impulse Response

~~0~~ 1/2 $\quad$ 1/2

↑

◯

ex $\frac{1}{2}\{x[n] + x[n-1]\}$

$$h[n] = \underset{\substack{\uparrow \\ 0}}{1/2} \quad 1/2$$

$$H(e^{j\omega_0}) = \frac{1}{2} + \frac{1}{2}e^{-j\omega_0}$$

$$= \frac{1}{2}e^{-j\omega_0/2}\left(e^{j\omega_0/2} + e^{-j\omega_0/2}\right)$$

$$= e^{-j\omega_0/2}\cos\frac{\omega_0}{2}$$

$$A_0\cos(\omega_0 n + \phi_0) \longrightarrow \boxed{\begin{matrix}\text{LSI} \\ \text{Stable} \\ \text{system}\end{matrix}}^{\,h[n]} \longrightarrow$$

$$\frac{1}{2}A_0 e^{j(\omega_0 n + \phi_0)} \longrightarrow H(e^{j\omega_0})\frac{1}{2}A_0 e^{j(\omega_0 n + \phi_0)}$$

$$\frac{1}{2}A_0 e^{-j\omega_0 n - \phi_0} \longrightarrow H(e^{-j\omega_0})\frac{1}{2}A_0 e^{-j(\omega_0 n + \phi_0)}$$

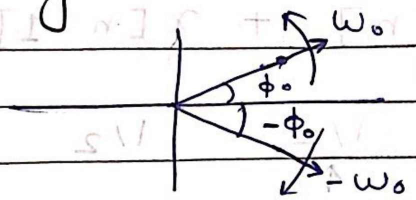
→ Properties of frequency Response,

$$H(e^{-j\omega_0}) = \sum_{k=-\infty}^{\infty} h[k]\, e^{-j(-\omega_0)k}$$

Let, $h[n]$ be real,

Property: $$\overline{H(e^{-j\omega_0})} = \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega_0 k} = H(e^{j\omega_0})$$

→ Visualizing a d-t cosine wave



→ Adding the Responses of +ve & -ve frequency

$$|H(e^{j\omega_0})|\ A_0\ e^{j(\omega_0 n + \phi_0 + \measuredangle H(e^{j\omega_0}))}$$
$$+\ |H(e^{-j\omega_0})|\ A_0\ e^{-j(\omega_0 n + \phi_0 + \measuredangle H(e^{j\omega_0}))}$$

$$= A_0\ |H(e^{j\omega_0})|\ \cos(\omega_0 n + \phi_0 + \measuredangle H(e^{j\omega_0}))$$

$$y[n] = (x[n] + x[n-1])/2$$

$$H(e^{j\omega_0}) = e^{-j\omega_0/2} \cos\frac{\omega_0}{2}$$

Expression for Frequency Response DTFT

$$H(e^{j\omega_0}) = \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega_0 k}$$

→ A sequence is an infinite dimentional vector $h[k]$

→ Coordinates are indexed by 'k'

Inner products

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→ Given sequences $x_1[k]$, $x_2[k]$

Inner product $= \langle x_1, x_2 \rangle$

$$= \sum_{k=-\infty}^{\infty} x_1[k]\, x_2^*[k]$$

$$H(e^{j\omega_0}) = \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega_0 k}$$

→ An Inner product of $h[k]$ and $e^{j\omega_0 k}$
= 'Projection' of $h[k]$ or the impulse response upon $e^{j\omega_0 k}$

$\omega_0 : -\pi \longrightarrow +\pi$

$$H\left(e^{j(\omega_0+2\pi)}\right) = H(e^{j\omega_0})$$

$$-\pi \le \omega_0 \le \pi$$

Sequence $h[n] \xrightarrow{DTFT} H(e^{j\omega_0})$

$$= \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega_0 k} \qquad -\pi \le \omega_0 \le \pi$$

ex DTFT does not always exist

$$h[n] = 2^n u[n]$$

$$\sum_{k=0}^{\infty} 2^k e^{-j\omega_0 k} \rightarrow \text{Does not converge}$$

$|C.R| > 1$

Reconstruction - Inverse DTFT

$$K_0 \int_{-\pi}^{\pi} H(e^{j\omega_0}) e^{j\omega_0 n} d\omega_0 = h[n]$$

$$LHS = K_0 \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k} e^{j\omega_0 n} d\omega_0$$

$$= K_0 \sum_{k=-\infty}^{\infty} \left(h[k] \underbrace{\int_{-\infty}^{\infty} e^{j\omega_0 (n-k)} d\omega_0}_{2\pi\delta[n-k]} \right)$$

$$= K_0 \cdot 2\pi \cdot \cancel{\delta[n-k]}\; h[n] \qquad = h[n]$$

$$\Rightarrow \boxed{K_0 = \frac{1}{2\pi}}$$

→ Expression for IDTFT

$$\boxed{h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega_0}) e^{j\omega_0 n} d\omega_0}$$

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→ Recap DTFT and Inverse DTFT

• DTFT : A projection of the sequence on $e^{j\omega n}$

Inverse DTFT :

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$$

$$x[n] \longrightarrow \boxed{\begin{matrix}\text{LSI}\\ h[n]\end{matrix}} \longrightarrow y[n]$$

→ Assume x, h, y all have a DTFT

$$y[n] = x * h[n]$$

DTFT of $y[n]$

$$= \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x[k]\, h[n-k]\right) e^{-j\omega n}$$

$n-k=\ell \Rightarrow$ for fixed k, $n\Big|_{-\infty}^{\infty} \Rightarrow \ell\Big|_{-\infty}^{\infty}$

$$= \sum_{\ell=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x[k]\, h[\ell]\, e^{-j\omega(k+\ell)}\right)$$

$$= \sum_{\ell=-\infty}^{\infty}\left(h[\ell]\, e^{-j\omega \ell} \sum_{k=-\infty}^{\infty} x[k]\, e^{-j\omega k}\right)$$

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$$\Rightarrow Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})$$

Properties of the DTFT

1. Convolution property <above>

2. Linear

$$x_{1,2}[n] \xrightarrow{DTFT} X_{1,2}(e^{j\omega})$$

$$\alpha x_1[n] + \beta \cancel{\beta} x_2[n] \xrightarrow{DTFT}$$

$$\alpha X_1(e^{j\omega}) + \beta X_2(e^{j\omega})$$

3. Time Shifting

$$x[n] \xrightarrow{DTFT} X(e^{j\omega})$$

$$x[n-n_0] \xrightarrow{DTFT} X(e^{j\omega}) \cdot e^{-j\omega n_0}$$

$$\sum_{\cancel{n}\, l=-\infty}^{\infty} x[\cancel{n-n_0}\; l]\, e^{-j\omega \cancel{n}(l+n_0)}$$

$$? \xrightarrow{DTFT} \cancel{X(e^{j\omega})}\ \frac{d X(e^{j\omega})}{d\omega}$$

$$\frac{d}{d\omega} \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= -(j\cancel{\omega}) \sum_{n=-\infty}^{\infty} n x[n] e^{-j\omega n}$$

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$\Rightarrow j \cancel{\frac{x[n]}{n}}$ Derivative in F.T

4) $\Rightarrow -jn\,x[n] \xrightarrow{DTFT} \dfrac{dX(e^{j\omega})}{d\omega}$

5) Frequency Shift

$$x[n]e^{j\omega_0} \xrightarrow{DTFT} X\left(e^{j(\omega-\omega_0)}\right)$$

$$= \cancel{\sum} \cancel{\frac{1}{2\pi}\int}$$

Q→ DTFT of (i) $\left(\frac{1}{2}\right)^n u[n]$

$$\sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n \cancel{u[n]}^{1} e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \left(\frac{e^{-j\omega}}{2}\right)^n$$

$$= \frac{1}{1-\left(\frac{e^{-j\omega}}{2}\right)}$$

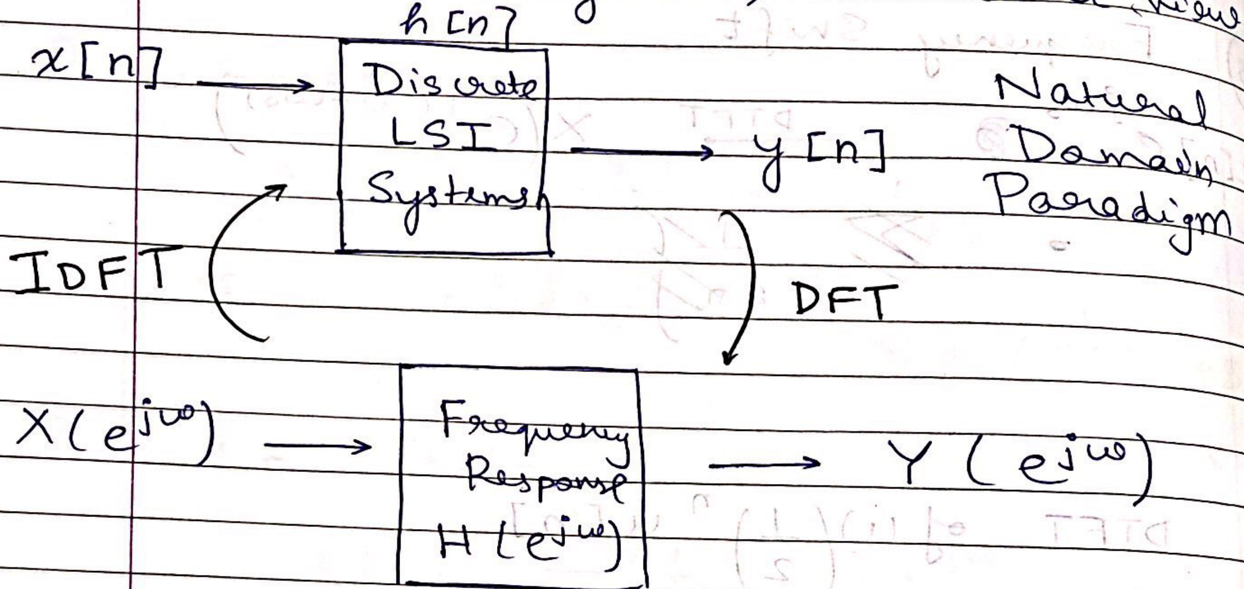
(ii) $n\left(\frac{1}{2}\right)^n u[n] \xrightarrow{DTFT} j \times \dfrac{d}{d\omega}\left(\dfrac{1}{1-\frac{e^{-j\omega}}{2}}\right)$

$$= \frac{-1}{\left(1-\frac{e^{-j\omega}}{2}\right)^2} \times \frac{j e^{-j\omega}}{2} \times j$$

$$= \frac{\cancel{1}}{\left(\cancel{1} + e^{-2j\omega} \cancel{- e^{-j\omega}}\right)} = \frac{\frac{1}{2}e^{-j\omega}}{\left(1-\frac{1}{2}e^{-j\omega}\right)^2}$$

→ The DTFT is a mapping of one "paradigm" to another

One view of sequences and discrete time LSI Systems, to another view



→ Given DFT of x_1, x_2 exist

$$x_{1,2}[n] \xrightarrow{DTFT} X_{1,2}(e^{j\omega})$$

$$x_1[n]\, x_2[n] \xrightarrow{DTFT}$$

$$\sum_{n=-\infty}^{\infty} x_1[n]\, x_2[n]\, e^{-j\omega n}$$

$$= \sum_n x_1[n] \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} X_2(e^{j\lambda n})\, e^{j\lambda n}\, d\lambda \right) e^{-j\omega n}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_n x_1[n]\, e^{-j\omega n} \right) X_2(e^{j\lambda n})\, e^{j\lambda n}\, d$$

Periodic ~~Convolution~~ Convolution

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$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cancel{X_1(\omega-\lambda)\, X_2} \; X_1(e^{j(\omega-\lambda)})\, X_2(e^{j\lambda})\, d\lambda$$

= periodic convolution of $X_1(\cdot)$ and X_2

= effectively a combination of one of these periodic DTFT's with one period of the other

More properties

$$x[n] \longrightarrow X(e^{j\omega})$$

$$x[-n] \longrightarrow X(e^{-j\omega})$$

$$\overline{\sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}} = \sum_{n=-\infty}^{\infty} x^*[n]\, e^{j\omega n} \qquad \sum_{\ell=-\infty}^{\infty} x'[\ell]\, e^{j\omega \ell}$$

$$\overline{x[-n]} \longrightarrow \overline{X(e^{j\omega})}$$

$$\overline{x[n]} \longrightarrow \overline{X(e^{-j\omega})}$$

$$\rightarrow \quad x_1[n]\, \overline{x_2[n]} \xrightarrow{DTFT} \frac{1}{2\pi}\int_{-\infty}^{\infty} \cancel{X_1(e^{j\omega})}\, X_1(e^{j\omega})\, X_2(e^{-j\omega})$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \cancel{X_1(e^{j\omega})\, X_2(e} \; X_1(e^{j(\omega-\lambda)})\, \overline{X_2(e^{-j\lambda})}\, d\lambda$$

$$= \frac{1}{\;}\int^{\pi} X_1(e^{j(\omega+\alpha)})\, \overline{X_2(e^{j\alpha})}\, d\alpha$$

Parseval's Theorem

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_2(e^{j\lambda})\, X_1(e^{j(\omega-\lambda)})\, d\lambda$$

$$\Big\downarrow \text{Put } \omega = 0$$

$$\sum_{n=-\infty}^{\infty} x_1[n]\, x_2^*[n]\, e^{-j\omega n} \xrightarrow{\omega=0} \sum_{n=-\infty}^{\infty} x_1[n]\, x_2^*[n]$$

→ Conclusion is that the Inner product remains the same in both the spaces

$$\sum_{n=-\infty}^{\infty} x_1[n]\, x_2^*[n] \xrightarrow{D.F.T} \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\omega})\, X_2^*(e^{j\omega})$$

→ The fact that Inner product remains same is parseval's theorem

21.1.19

→ Class test on Thu 24/1, open notes

→ Generalizing the DTFT to Z-transform

→ $x[n] = 2^n u[n]$

→ Control growth,

$$2^n u[n]\, r^{-n}$$

$r > 0$, $r > 2$

Scanned with CamScanner

→ Discrete time Fourier transform

DTFT of $x[n]\, r^{-n}$

$$\sum_{n=0}^{\infty} \left(\frac{2}{r}\right)^{n} e^{-j\omega n} = \frac{1}{1-\left(\frac{2}{r}\right)e^{-j\omega}}$$

$\longrightarrow \boxed{\text{DT LSI system } h[n]}$ $h[n]\, r^{-n}$

→ Frequency response of new discrete LSI system

$$h[n]\, r^{-n}$$

$$\sum_{n=-\infty}^{\infty} h[n]\, r^{-n} e^{-j\omega n}$$

<u>Z - transform Definition</u>

$$\sum_{n=-\infty}^{\infty} h[n]\, r^{-n} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} h[n] \left(r e^{j\omega}\right)^{-n}$$

$$Z = r e^{j\omega}$$: A complex variable

$x[n]\, r^{-n} \longrightarrow \boxed{\text{DT LSI sys } h[n]\, r^{-n}} \longrightarrow ?$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, r^{-k}\, h[n-k]\, r^{-(n-k)}$$

$$= r^{-n} \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]\, \cancel{r^{k}} \cdot \cancel{r^{-k}}$$

$$= \cancel{\vartheta}\ \cancel{r^{-n}}\ \cancel{y[n]\, r^{-n}}$$

$$= \boxed{\left(x * h[n]\right) r^{-n}}$$

→ Z - transform of $h[n]$ is given by

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] \left(re^{j\omega}\right)^{-n}$$

$$x[n]\, r^{-n} \longrightarrow \boxed{\begin{array}{c} \text{D-t} \\ \text{LSI system} \\ h[n]\, r^{-n} \end{array}} \longrightarrow y[n]\, r^{-n}$$

$X(z)$ $\quad$ $\hookrightarrow H(z)$ $\quad$ $y[n] = (x * h)[n]$

$$Y(z) = H(z)\, X(z)$$

→ Convolution property of Z - transform

$$x_{1,2}[n] \xrightarrow[\mathcal{Z}]{} X_{1,2}(z)$$

$$x_1[n] * x_2[n] \xrightarrow[\mathcal{Z}]{} X_1(z)\, X_2(z)$$

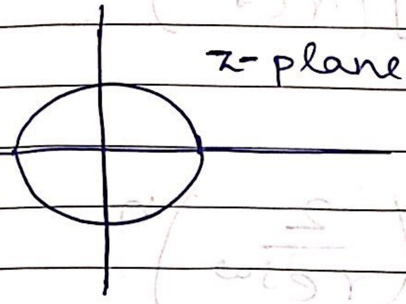
→ Holds, provided we choose a 'z' so that all of $x_{1,2}(z)$ and Z-trans of $x_1 * x_2$ exist for ~~all z~~ that z.

DTFT is the same as z-transform at $r = 1$

$r \to z = re^{j\omega}$ → Angle

↳ magnitude of comp. num

Region of Convergence of Z-transform



The convergence of z-transform depends only on the Magnitude of the complex number r

→ In General the R.O.C is of the form

$R_1 \leq |z| \leq R_2$ $\qquad \leq$: $\leq$ or $<$

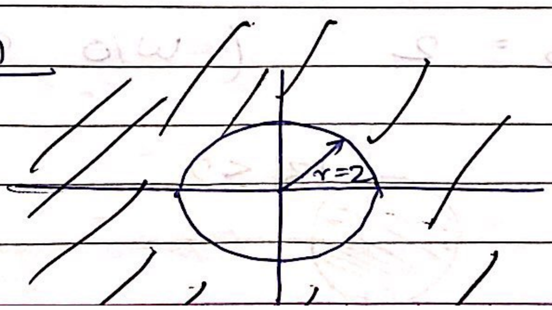
$0 \leq R_1 \leq r \leq R_2 \leq \infty$ → way of saying $\lim_{z \to \infty}$ exists

$x[n] = 2^n u[n]$

$X(z) = \dfrac{1}{1 - 2z^{-1}}$, $2 < |z| \leq \infty$

$0 < 2 < |z| \leq \infty$

Representation



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Ex 2) $x[n] = \cancel{-2^n u[n-1]} \quad -2^n u[-n-1]$

$u[-n-1] = 1$

$-n-1 \geq 0$

$n \leq -1$

Z-transform

$$\sum_{n=-\infty}^{\infty} -2^n u[-n-1] (re^{j\omega})^{-n}$$

$$-\sum_{n=-\infty}^{-1} \left(\frac{2}{re^{j\omega}}\right)^n$$

$$= -\sum_{1}^{\infty} \left(\frac{z}{2}\right)^n = \frac{-z/2}{1 - z/2}$$

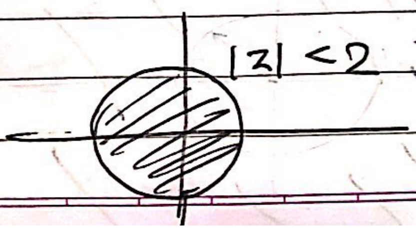
$$= \frac{-z}{2-z} = \frac{z}{z-2}$$

$$= \frac{1}{1-2z^{-1}} \quad , \quad |z| < 2$$

~~R.O~~ ROC $|z| < 2$

$R_1 = 0$ (with equality)

$R_2 = 2$ (w/o equality)



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$\delta[n]$ $\Rightarrow$ Z transform = 1

$$\sum_{n=-\infty}^{\infty} \delta[n] (re^{j\omega})^{-n}$$

Z - transform

1 : entire z - plane

$R_1 = 0$ (Included) $R_2 \to +\infty$ (Included)

↳ even though 0^0 is not defined, still Included

$\delta[n-1]$

$$\sum 0 \cdot (re^{j\omega})^{-1}$$

~~R.O.C~~

$\Rightarrow \frac{1}{re^{j\omega}}$ $r \neq 0$, $r \to \infty$ ✓

ROC $R_1 = 0$, $R_2 \Rightarrow +\infty$

$R_1 = 0$ → Not included; $R_2 \to +\infty$ → Included

Included just means limit exists

Z transform $= \frac{1}{Z}$

$\delta[n+1]$ $(re^{j\omega})^{-(-1)} = (re^{j\omega})$

~~$R \neq 0$~~ $r \in [0, \infty)$

Z - transform $= Z$ (R.O.C $\ni$ $R \neq \infty$)

22.1.19

Properties of the Z-transform

1. Linear

$$x_{1,2}[n] \xrightarrow{\mathcal{Z}} X_{1,2}(z)$$

$$\alpha x_1[n] + \beta x_2[n] \xrightarrow{\mathcal{Z}} \alpha X_1(z) + \beta X_2(z)$$

ROC : Atleast $R_1 \cap R_2$

ex

$$x_1[n] = \left(\frac{1}{2}\right)^n u[n]$$

$$x_2[n] = \left(\frac{1}{4}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[n]$$

$$\cancel{X_1(z) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u[k]}$$

→ Formula for z-transform

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \cancel{u[k]} \cancel{(re^{jw})^{-n}} z^{-n}$$

$$= \sum_{n=0}^{\infty} \cancel{(2re^{jw})^{-n}} (2z)^{-n}$$

$$= \cancel{\frac{1}{1-2re^{jw}}} \quad \frac{1}{1-\frac{1}{2z}}$$

$\cancel{\frac{1}{1-2z}}$ $\cancel{|z|<2}$ R.o.c $|z| > \frac{1}{2}$ R_1

$$X_2(z) = \frac{1}{1-\frac{1}{4z}} - \frac{1}{1-\frac{1}{2z}} \qquad |z| > \frac{1}{2} \quad R_2$$

$$X_1(z) + X_2(z) = \frac{1}{1-\frac{1}{4z}} \qquad |z| > \frac{1}{4}$$

$\Rightarrow$ R.O.C $\supset (R_1 \cap R_2)$

2. Time Shift property

$$x[n] \xrightarrow{\mathcal{Z}} X(z), \quad \mathbb{R}$$

stands for R.O.C of $X(z)$

$$x[n-n_0] \xrightarrow{\mathcal{Z}} \cancel{X(zz} \; z^{-n_0} X(z)$$

$\mathbb{R}_x$

$\mathbb{R}$ except at boundary points

3. Convolution property

$$x_{1,2}[n] \xrightarrow{\mathcal{Z}} X_{1,2}(z) \quad \mathbb{R}_{1,2}$$

$$x_1 * x_2 [n] \xrightarrow{\mathcal{Z}} X_1(z) X_2(z)$$

R.O.C Atleast $\mathbb{R}_1 \cap \mathbb{R}_2$

4. Modulation property

$$x[n] \xrightarrow{\mathcal{Z}} X(z), \quad \mathbb{R}_x$$

$$\alpha^n x[n] \xrightarrow{\mathcal{Z}} X(z\alpha^{-1})$$

$$(z\alpha^{-1}) \in \mathbb{R}_x$$

5. Inverse Z- transform

$$h[n] r^{-n} \xrightarrow{DTFT} H(z) \quad z = re^{j\omega}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} H(re^{j\omega})\, e^{j\omega n}\, d\omega = h[n]\, r^{-n}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} H(re^{j\omega})\, r^{n} e^{j\omega n}\, d\omega = h[n]$$

$$re^{j\omega} = z$$

$$dz = re^{j\omega}(j\,d\omega)$$

$$d\omega = \frac{1}{j} z^{-1}\, dz$$

$$h[n] = \frac{1}{2\pi j}\int H(z)\, z^{n} z^{-1}\, dz$$

Any closed r contour in the ROC

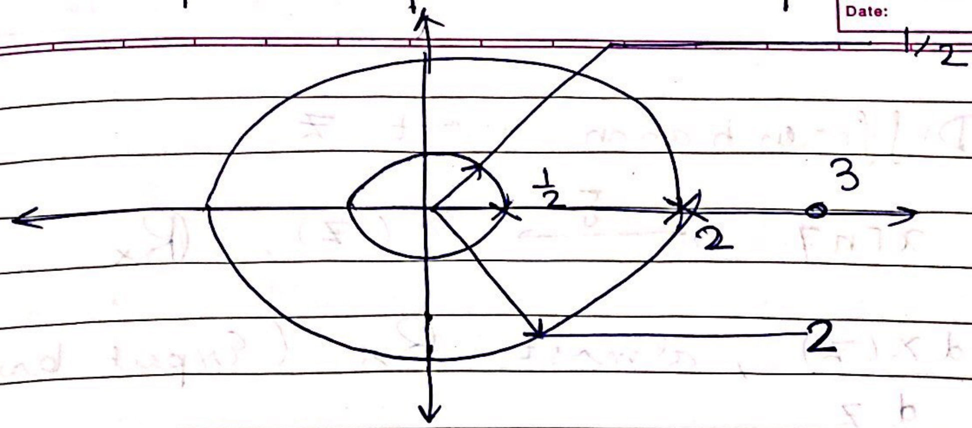
$$= \frac{1}{2\pi j}\oint H(z)\, z^{n-1}\, dz$$

any closed contour C in the ROC

→ To take Inverse, we don't use this formal inverse, instead inspection/ experience

$$H(z) = \frac{(1-3z^{-1})}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)}$$

Z-plane pole-zero plot



→ Possible Regions of Convergence are,

$\frac{1}{2} < |z| < 2$ $\qquad |z| < \frac{1}{2}$ $\qquad |z| > 2$

ex Invert $\frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)}$

$= \frac{A}{\cancel{\left(1-\frac{1}{2}z^{-1}\right)}} \left(\frac{1}{1-\frac{1}{2}z^{-1}} - \frac{4}{1-2z^{-1}} \right) \times \frac{-1}{3}$

$= 1 - 2z^{-1} - 4 + \frac{1}{\cancel{2}} 2 z^{-1}$

$= \frac{1}{3}\left(\frac{-1}{1-\frac{1}{2}z^{-1}} + \frac{4}{1-2z^{-1}} \right)$

$= \frac{-1}{3}\left(\frac{1}{1-\frac{1}{2}z^{-1}} \right) + \frac{4}{3}\left(\frac{1}{1-2z^{-1}} \right)$

(i) $|z| > 2$

$\frac{-1}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{4}{3} 2^n u[n]$

(ii) $\frac{1}{2} < |z| < 2$

$\frac{-1}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{4}{3}\left(-2^n\right) u[-n-1]$

(iii) $|z| < \frac{1}{2}$ $\quad \frac{-1}{3}\left(\left(\frac{-1}{2}\right)^n u[-n-1]\right) + \frac{4}{3}\left(-2^n u[-n-1]\right.$

6. Differentiation w.r.t z

$$x[n] \xrightarrow{\mathcal{Z}} X(z), \ R_x$$

$\dfrac{dX(z)}{dz}$, almost R_x (Expect boundary)

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \, z^{-n}$$

$$n x[n] \xrightarrow{\mathcal{Z}} -z \frac{d X(z)}{dz}$$

R.O.C almost R_x expect boundary

$$\frac{dX}{dz} = \sum_{n=-\infty}^{\infty} (-n x[n]) \, z^{-n} \cdot z^{-1}$$

GRPs 17 - End

Tut 4, 5, 6, 7

Balance difficulty

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28.1.19

Z - transform

- Linearity, Modulation, Convolution, Sequence Shift

→ Challenge problem

- Use the formal Inverse, to invert

$$\frac{1}{1-\alpha z^{-1}} \begin{cases} |z| > |\alpha| \\ \text{and} \\ |z| < |\alpha| \end{cases}$$

by Cauchy Integral Thm Groups 1 → 6

Z - transform of Product

→ $x[n]\, y[n] \xrightarrow{z\text{-transform}}$

$$\sum_{n=-\infty}^{\infty} x[n]\, y[n]\, z^{-n}$$

Replace x or y by its formal inverse

$$\sum_{n=-\infty}^{\infty} \underbrace{\frac{1}{2\pi j}\left(\oint_C X(z_1)\, z_1^{n-1}\, dz_1\right)}_{x[n]} y[n]\, z^{-n}$$

C : A closed contour in the R.O.C of X

→ Since it converges

$$\frac{1}{2\pi j} \oint_C X(z_1) \left(\sum_{n=-\infty}^{\infty} y[n]\, z_1^{n-1}\, z^{-n} \right) dz_1$$

$$= \frac{1}{2\pi j} \oint_C X(z) \left\{ \sum_{n=-\infty}^{\infty} y[n] \cancel{z_1} \frac{1}{\cancel{z_1}} \cdot \left(\frac{z}{z_1}\right)^{-n} \right\} z_1^{-1}\, dz_1$$

$Y\left(\frac{z}{z_1}\right)$

$$= \frac{1}{2\pi j} \oint_C X(z)\, Y\left(\frac{z}{z_1}\right) z_1^{-1}\, dz_1$$

C: belongs to ROC of $X(\cdot)$
but also $\left(\frac{z}{z_1}\right) \in$ ROC of $Y(\cdot)$

7. Z-transform of product sequence

$$x[n] \xrightarrow{\mathcal{Z}} X(z), \; R_x$$
$$y[n] \xrightarrow{\mathcal{Z}} Y(z), \; R_y$$

$$x[n]\,y[n] \xrightarrow{\mathcal{Z}} \left(\frac{1}{2\pi j} \oint_C X(z_1)\, Y\left(\frac{z}{z_1}\right) z_1^{-1}\, dz_1 \right)$$

$$z_1 \in R_x, \quad \frac{z}{z_1} \in R_y$$

Challenge Example Problem

$$x[n] = \alpha^n u[n]$$
$$y[n] = \beta^n u[n], \quad |\alpha|, |\beta| < 1$$

Obtain the Z transform of $x[n]\,y[n]$ using the formal expression

GRPs 7-12

Hint: # Cauchy's Integral theorem

→ Z transform tells us how we can realize a System

Rational z-transform

$X(z)$ = ration of two finite series in z

$$= \frac{N(z)}{D(z)}$$

N, D are 2 finite length series in z

ex

$$\frac{2z^{-1} + 3 + z + 2z^{2}}{4z^{-2} + 3z^{-1} + 5 + z + z^{2} + 2z^{3}}$$

Can always be written as

d z_1

$$z^{(\,)} \left(\frac{\text{poly}_1(z)}{\text{poly}_2(z)} \right)$$

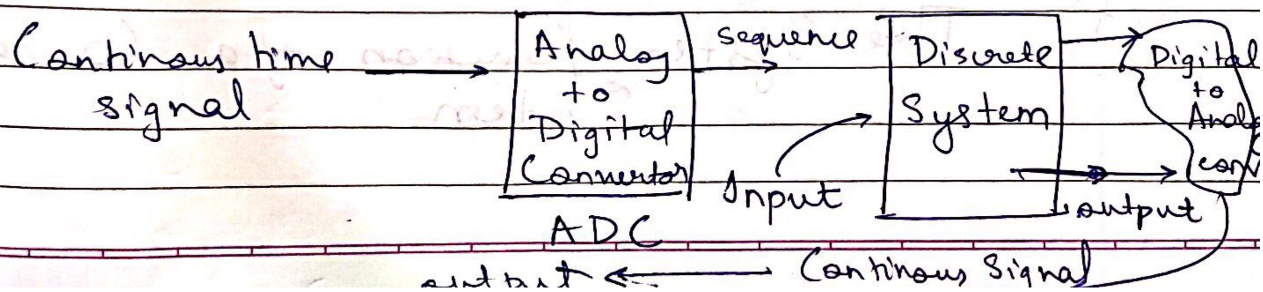
OR

$$z^{(\,)} \left(\frac{\text{poly in } z^{-1}}{\text{poly in } z^{-1}} \right)$$

→ Realization : Translating into a finite resource implementation

Resources ?

1. Unit sample delay
2. Two input Adders
3. Constant Multiplier



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Linear Shift Invariant Systems with a System function

→ System function : Z- transform of the Impulse response

$$x[n] \longrightarrow \boxed{\begin{matrix}\text{LSI} \\ \text{System} \\ \text{Impulse resp}\end{matrix}} \longrightarrow y[n]$$

i/p $h[n]$ o/p

→ There is common R

$$Y(z) = H(z)X(z)$$

$$\frac{Y(z)}{X(z)} = H(z)$$: Ratio of o/p Z trans to i/p Z trans

→ Ratio is independent of the input (System function)

↳ Only Defined for LSI

Expression + R.O.C

Constraints

- Have to be Stable (Can be compromise
- Have to be a Causal System

(P.T) The System function of a Causal System

Can always be written as

$$\underline{\text{polynomial in } z^{-1}}$$ → May have constants

$$\underbrace{1}_{\text{Constant Absorbed}} - \underbrace{(\text{polynomial in } z^{-1})}_{\text{No Constants Here}}$$

Example

$$\frac{1}{1-\alpha z^{-1}} \qquad |z| > |\alpha| \,, \quad |\alpha| < 1$$

↳ System function

⇒ $h[n]$ (Impulse response) $= \alpha^n u[n]$

→ Absolutely Summable ⇒ stable

$h[n] = 0 \quad \forall n < 0 \Rightarrow$ causal

Q) → Do all sequences have a z-transform

- Prove / give counter example Grps 13-15
- In the case of Bilateral z-transform, the following won't converge

(a) $a^{n^2} u[n]$

(b) $a^n \times 1$ (No $u[n]$ / limiting area ....)

→ Counter Example to show that the Z-transform ~~exists~~ does not exist

The Z-transform exists, provided

(1) Signal Starts at a finite time or stops (becomes identically 0) after a finite stop time (ex $u[-n-1]$)

(2) Is Asymptotically exponentially bounded

$$|x[n]| < \alpha^{|n|}, \quad \alpha > 1$$
$$\forall\, n \geq n_f$$

~~Some~~ examples which fail to have a Convergent Z-transform because (1) fails

Ex 1) $x[n] = a^{|n|} = a^{n} u[n] + a^{-n} u[-n-1]$

$$Z\left[x[n]\right] = \frac{1}{1-az^{-1}} + \frac{1}{1-a^{-1}z^{-1}} \quad R = |z| > |a| \cap |z| < \frac{1}{|a|}$$

Since $R = |z| > |a| \cap |z| < \frac{1}{|a|} \Rightarrow$ R.O.C $= \phi$ $\Rightarrow$ Z-transform does not exist for any z

Ex 2) Fails because of Exponential bound (2)

$x[n] = \alpha^{n^2} u[n] \qquad |\alpha| > 1$: power of α is [illegible]

$$X(z) = \sum x[n] z^{-n}, \quad \text{If it exists}$$
$$= \sum_{n=0}^{\infty} \alpha^{n^2} z^{-n}$$

Causal, rational system

→ System functions are only defined for LTI systems

→ System functions always have the following form :-

$$\left(\frac{\text{Polynomial in } z^{-1}}{1 - \sum_{k=1}^{N} a_k z^{-k}} \right)$$

→ Rational z-transform

$$= \frac{\text{finite length numerator series in } z \text{ (or } z^{-1})}{\text{finite length denominator series in } z \text{ (or } z^{-1})}$$

$$= z^D \frac{\text{Polynomial in } z^{-1}, N(z^{-1})}{\text{Polynomial in } z^{-1}, D(z^{-1})}$$

$D \in \mathbb{Z}$

$\frac{N(z^{-1})}{D(z^{-1})}$: Long Division

$$Q(z^{-1}) + \frac{R(z^{-1})}{D(z^{-1})}$$

where,

$$\deg R(z^{-1}) < \deg D(z^{-1})$$

$D \rightarrow$ Is an Integer

→ We can always take care of z^D as a Shift in the Final Stage

$$\mathcal{Z}^{-1}\{Q(z^{-1})\} = \text{finite length sequence}$$

factor $$D(z^{-1}) = \prod_{l=1}^{M} (1 - a_l z^{-1})^{M_l}$$

M distinct poles, l^{th} pole has multiplicity M_l

→ Fund. Thm. Algebra ⇒ Exactly n roots ↪ degree

↪ An existential proof ~~not a~~ constructive one ↪ Doesn't give the roots.

$$\frac{R(z^{-1})}{\prod_{l=1}^{M} (1 - a_l z^{-1})^{M_l}} \xrightarrow[\text{into partial fractions}]{\text{decompose}}$$

$$\sum_{l=1}^{M} \frac{\text{polynomial in } z^{-1} \text{ of degree } M_l - 1}{(1 - a_l z^{-1})^{M_l}}$$

→ For each of these terms look at the ROC
ROC is either to the interior or the exterior

Poly exp terms:
- ROC exterior ⟶ Right sided Inverse
- ROC Interior ⟶ left sided Inverse

GRPs 16 – 25

Atleast 2 different rational Z-transform with approp. ROCs, demonstrate their Inverse.

$$\text{Polyex} = l^{th}_{\text{term}} \underbrace{\left(\sum_{p=0}^{M_l - 1} l_p n^p\right) a_l^n}_{\text{exponential } i}$$

(polynomial in n)

$u[n] \rightarrow$ right sided

$u[-n-1] \rightarrow$ left sided

The l^{th} term in the partial fraction expansion of Z-transform corresponds to this poly x term

Q → Can their be Irrational Z-transform

$$x[n] = \frac{1}{n!} u[n] \quad , 0! = 1$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = e^{(1/z)} = \underline{\underline{e^{z^{-1}}}}$$

ROC: $|z| > 0$

→ A system with Irrational Z-transform is not realizable.

~~Kzzz~~

<u>Causal Rational Systems</u>

$$\text{System function} = \frac{\sum_{l=0}^{M} b_l z^{-l}}{1 - \sum_{k=1}^{N} a_k z^{-k}} = \frac{Y(z)}{X(z)}$$

$$Y(z) = \left(\sum_{k=1}^{N} a_k z^{-k}\right) Y(z) + \sum_{l=0}^{M} b_l z^{-l} X(z)$$

Invert the Z-transform

$$y[n] = \sum_{k=1}^{N} a_k\, y[n-k] + \sum_{\ell=0}^{M} b_\ell\, x[n-\ell]$$

A difference Equation

→ In particular, this one is a Linear, constant coefficient Difference Equation (LCCDE)

But what about?

1) Irrational Inputs 2) Irrational Systems

ex

$$x[n] = \frac{1}{n!} u[n] \qquad 0! = 1$$

$$n! = n(n-1)! \quad n \geq 1$$

$$h[n] = \frac{1}{n!} u[n]$$

$$\sum_{k=0}^{n} \frac{1}{k!} \cancel{u[k]} \, \frac{1}{(n-k)!} \cancel{u[n-k]}$$

$$= \frac{1}{n!} \sum_{k=0}^{n} {}^{n}C_{k}$$

$$y[n] = \boxed{\frac{2^n}{n!}} \quad n \geq 0 \Rightarrow \boxed{\frac{2^n}{n!} u[n]}$$

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Challenge GRPs 26 - 28
think of 3 other cases with ~~generalised~~ system/Input & find the o/p

Aliter

$$Y(z) = e^{2/z} = \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n \frac{1}{n!}$$

$$= \left(\sum_{n=0}^{\infty} \frac{2^n}{n!} \times \frac{1}{z^n} \right) = \frac{2^n}{n!} u[n]$$

→ What if Difference Equations only hold for limited amount of time?

30.1.19

Rational Systems : LSI systems ~~of~~ with Rational System function

→ System function of LSI need not be rational

→ Are systems with a ~~two~~ System function and LSI systems the same set?

→ For a system function to be defined it must be the Input independent ratio of

$$\frac{Z(\text{output})}{Z(\text{input})}$$

→ A z - transform or a system function in particular, is an expression in z along with an ROC

→ Properties other than Lin + Shift Inv.
- Causality
- Stability

→ By looking at the system function we should be able to Investigate the causality and stability of the system. (Since all Info. is captured)

→ Causality from System function

Impulse response zero for $n < 0$

System function $H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$

$h[n] = 0 \quad \forall \; n < 0$

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$$

$$= h[0] + h[1] z^{-1} + h[2] z^{-2} + \ldots\ldots$$

• Each Sample is finite

→ Definitely Converges as $|z| \to \infty$

→ In the z-plane, ∞ is an ever expanding contour (Circle or Anything)

→ For a causal system $|z| \to \infty$ must always lie in the ROC

ex

$$H(z) = \frac{1}{1 - \alpha z^{-1}} \qquad \text{ROC } |z| > |\alpha|$$

$h[n] = \alpha^n u[n] \Rightarrow$ Causal

ROC Includes $|z| \to \infty$

n	-1	0	1	2	3
$h[n]$	1	$\frac{1}{2}$	$\left(\frac{1}{2}\right)^2$	$\left(\frac{1}{2}\right)^3$	....

$$\frac{z}{1-\frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2}$$

$$z + 1 + \frac{1}{2z} + \frac{1}{\cancel{4z^2}(2z)^2} + \cdots\cdots$$

ROC

$$\frac{z}{1-\frac{1}{2}z^{-1}} \qquad \infty > |z| > \frac{1}{2}$$

→ Also System is not causal and the z-transfor reflects this fact

- Non-causal system ROC does not include $|z| \to \infty$

Challenge

- ~~Bu~~ So it is a necessary condition, but is having ∞ in the ROC also a sufficient condition to say that the associated system is causal.

Thm

- A rational system is causal iff the ROC of its System function includes the infinity of the z plane

- Rationality is NOT a requirement, for any System function.

Stability

→ ~~The~~ LSI System is stable iff its impulse response is Absolutely Summable.

$$H(z) = z^D \left\{ Q(z) + \frac{R(z)}{D(z)} \right\}$$

z^D : $n \leftarrow n + D$

⇒ Results in a time Shift

$Q(z) \xrightarrow{\mathcal{Z}^{-1}} q[n]$: finite length sequence

$$\frac{R(z)}{D(z)} = \sum_{\ell=1}^{M} \frac{\text{polynomial of degree } M_\ell - 1 \text{ in } z^{-1}}{\left(1 - a_\ell z^{-1}\right)^{M_\ell}}$$

$$\mathcal{Z}^{-1} = \sum_{\ell=1}^{M} \left(\begin{array}{c} \text{polynomial in } n \\ \text{of degree } M_\ell - 1 \end{array} \right) \cdot a_\ell^n$$

Right sided $u[n]$

Left sided $u[-n-1]$

Polyex terms

→ z^D only Shifts, Does not affect absolute sum.

→ $q[n]$ is a finite length sequence, does not affect summability

→ Stability $\longleftrightarrow$ Polyex terms

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Consider the l^{th} Polyx term, $M_l = 1$
For right sided sequences, absolutely summable if $|a_l| < 1$ R.O.C is to the exterior of $|z| = |a_l|$
For left sided sequences, absolutely summable if $|a_l| > 1$
ROC is to the Interior of $|z| = |a_l|$
For $|a_l| = 1$, cannot be absolutely summable

Even if $M_l \geq 1$, the exponential term will still dominate in Magnitude
Challenge Problem
Ratio test can be used to show that the exponential term dominates over the polynomial term.

Can we infer absolute summability combined expression from the Absolute summability of Individual terms.

Polyex terms with distinct exponential factors are linearly independent

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix}$$

For absolute summability, each polyex term must be absolutely summable.

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Thm

→ For rational systems, Stability is equivalent to the unit circle being in the Region of Convergence

• What can we say if a system is both Causal and Stable? .....

31.1.19

→ What happens to stability when $|a_l| = 1$

The polyex term cannot be absolutely summable

ROC's possible

ex $\quad H(z) = \dfrac{1}{1 - z^{-1}}$ $\quad |z| > 1$, $\quad |z| < 1$

→ Not BIBO stable ⇒ a bounded input can result in an unbounded output

→ ⇒ There exists atleast one sequence for which o/p is unbounded.

$R = |z| > 1, \quad h[n] = u[n]$

$x[n] = u[n]$

$$\sum_{k=0}^{n} 1 = (n+1)$$

$$\Rightarrow y[n] = (n+1)\,u[n]$$

(ii) Same system, $X(z) = \dfrac{1}{1 + z^{-1}}$ $\quad |z| > 1$

$x[n] = (-1)^n u[n]$

$$\Rightarrow \quad y[n] = \sum_{k=0}^{n} (-1)^n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \quad n > 0$$

2) $X(z) = \dfrac{1}{1-\alpha z^{-1}}$ ~~x[n]~~ Same $h[n]$

$|\alpha| = 1, \ \alpha \neq 1, \ |z| > 1$

$$x[n] = \alpha^n u[n]$$

$$\sum_{k=0}^{n} \alpha^k u[\cancel{k}]$$

$$\frac{(\alpha^{n+1} - 1)}{(\alpha - 1)} \rightarrow \text{finite since } \underline{\alpha \neq 1}$$

→ Such Systems are called "Marginally Stable"

$$H(z) = \frac{1}{(1-z^{-1})^2} \qquad |z| > 1$$

$h[n] = (n+1)u[n]$, even harder to find a bound. i/p giving a bound. o/p

Challenge 13 → 15,
Construct examples of bounded inputs for this system which would give

(a) bounded and
(b) Unbounded outputs

Marginally Stable ≡ Simple pole at $|z| = 1$

Stable & Causal and Rational System

→ Rational, Causal : 1) ROC includes infinity

→ Rational, Stable : 1) ROC includes $|z| = 1$ (unit circle)

2) Simply Connected Region (Any 2 points connected by an Internal contour)

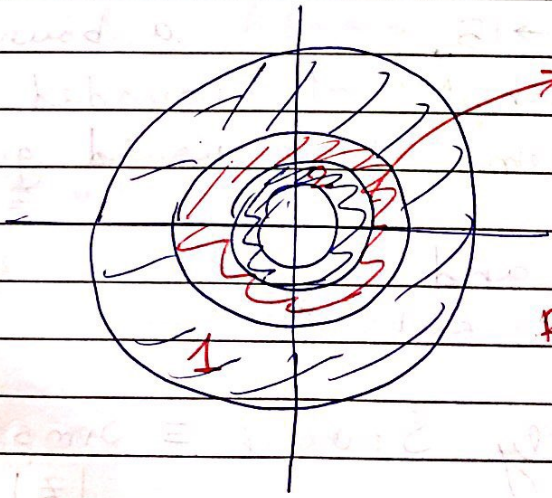
• Causal, Stable, Rational System

$$|z| \geq 1 \in ROC$$

Thm For a causal, stable rational system, all poles must lie inside the unit circle

Challenge 16 - 18

→ Can a rational expression in z correspond to an ROC which ~~a~~ is a union of two disjoint discs?



→ Excluded, hint prove that if 1 and 2 are in ROC, then Red also.

Till Now,

- → Inputs for all time were defined
- → LSI System
- → Output for all time,
  LCCDES

- But what if $\boxed{\text{we don't have}}$ an infinite time extent ?? → Tuesday

5.2.19

$\boxed{\text{Order}}$ for a rational system (Causal)

→ Degree of the denominator in the system function = order

ex

$$H(z) = \frac{1}{1-\alpha z^{-1}} \quad , \text{order} = 1$$

$$\frac{\beta_0 + \beta_1 z^{-1}}{(1-\alpha z^{-1})} \quad \text{also order } 1$$

LCCDE Representation

$$y[n] = \alpha y[n-1] + x[n]$$

Could hold $\forall n$ or it could hold for $n_1 \leq n \leq n_2$

Example : $n_1 = 0$, $n_2 = 1000$

→ Then in General we would require $y[n_1 - 1]$ to determine $y[n_1]$

$n = n_1, \ldots, n_2$

given, $x[n]$, $n = n_1, \ldots, n_2$

Proof (Constructive)

"Read off" the LCCDE for $n = n_1, \ldots$

$y[n_1] = \alpha y[n_1 - 1] + x[n_1]$

$\vdots$

$y[n_2] = \alpha y[n_2 - 1] + x[n_2]$

Challenge Problem : Generalize to N^{th} order LCCDES

Response from LCCDEs

$$y[n] = \alpha y[n-1] + x[n]$$

$n = n_1, \ldots, n_2$

Two parts in soln.
- zero-input response (homogenous)
- zero-state response (forced)

States $\equiv$ "Past Information"

ex $x[n] = 1$, $n = n_1, \ldots, n_2$

$y[n_1 - 1] = y_0$

$y[n] = \alpha y[n-1] + x[n]$

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Zero input solution

$$y[n] = \alpha y[n-1]$$

$$y[n] = \alpha y_0$$
$$1 = \alpha^2 y_0 \ldots$$

$$y[n] = \left(\alpha^{(n+1-n_1)} y_0\right) \quad \underline{\underline{n \geq n_1}}$$

→ Zero state response.

$$y[n] = x[n] + y[n-1] \quad \forall \quad n \geq n_1$$

→ Zero input response

$$y[n] = c_0 \alpha^n u[n-n_1]$$
$$c_0 \to \alpha^{-n_1}$$

→ Polyx terms caused by the system : $\underline{\underline{\alpha n}}$

• Zero ~~Input~~ State response

$$\cancel{y[n] =} \; x[n] = \beta^n \quad n = n_1, \ldots, n_2$$
$$y[n_1 - 1] = 0$$

(1) $\beta \neq \alpha$

$$y[n] = \alpha^{(n-n_1)} \left(\beta^{n_1} + \ldots + \beta^n\right)$$

$\cancel{y[0] = x[0]}$

$$y[n_1] = x[n_1] = \left(\beta^{n_1}\right)$$

$$y[n] = \alpha y[n-1] + \beta^n \quad n \neq n_1$$
$$n \geq n_1$$

$$\alpha^{(n-n_1)} \beta^n + \alpha^{(n+1-n_1)} \beta$$

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$y[n_1] = \beta^{n_1}$

$y[n_1+1] = \alpha\beta^{n_1} + \beta^{n_1+1}$

$y[n_1+2] = \alpha^2\beta^{n_1} + \alpha\beta^{n_1+1} + \beta^{n_1+2}$

$\cancel{y[n] = \sum_k \alpha^{(k-n_1)}\beta^{(n-2)}}$ $n \geq n_1$

$y[n] = \sum_{k=0}^{n} \beta^{(n_1-2+k)} \cdot \alpha^{(n-k-n_1)}$ $n \geq n_1$

$k = 1$

$\cancel{\alpha^2\beta^{n_1}}$ α

$= \left(\frac{\beta^n}{\alpha^{n-n_1}}\right) \sum_{k=0}^{n} \beta^k \cdot \alpha^{-k}$

$\frac{\left(\frac{\beta}{\alpha}\right)^{n+1} - 1}{\left(\frac{\beta}{\alpha} - 1\right)}$ $\left[\frac{\beta^{n+1} - \alpha^{n+1}}{(\beta-\alpha)} \times \frac{1}{\alpha}\cdot\alpha\right.$

(ii) $\alpha = \beta$

$\rightarrow$ $\cancel{\alpha^{2n-n_1}}$

$\alpha^{2n-n_1}\,(n+1)$

$\alpha \neq \beta$

→ Homogenous (natural soln.) and Forced (particular) solution.

$\alpha = \beta$

→ Resonant Solution

Challenge Solution : Illustrate examples of this for higher order systems.

$$\times \frac{1}{\beta} . \alpha^{-n} | \beta^{n} \Big]$$

27.2.19

Realization of a System Function

→ Realization : Translation into an implementation with finite preferably Generic Resources.

→ Generic Resources for Realizibility

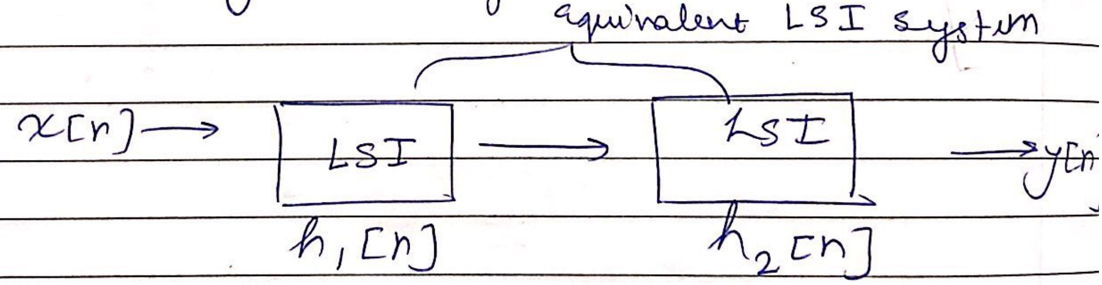
1. One Sample delay
2. Constant Multiplier
3. 2 - input Adder

ex

$$H(z) = \frac{2 + 3z^{-1}}{1 - \frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

↳ Because Causal

Cascade of LSI Systems



$$y[n] = \left((x * h_1) * h_2 \right)[n]$$

Using Associativity,

$$\left(x * (h_1 * h_2) \right)[n] = y[n]$$

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Signal Flow Graphs (SFG)

↳ Directed Graphs

$$x[n] \longrightarrow \boxed{LSI} \longrightarrow y[n]$$

$h_1 * h_2[n]$ ~~Each~~

↳ $H_1(z)\,H_2(z)$

$$H(z) = (2 + 3z^{-1}) \cdot \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)} \qquad |z| > \frac{1}{2}$$

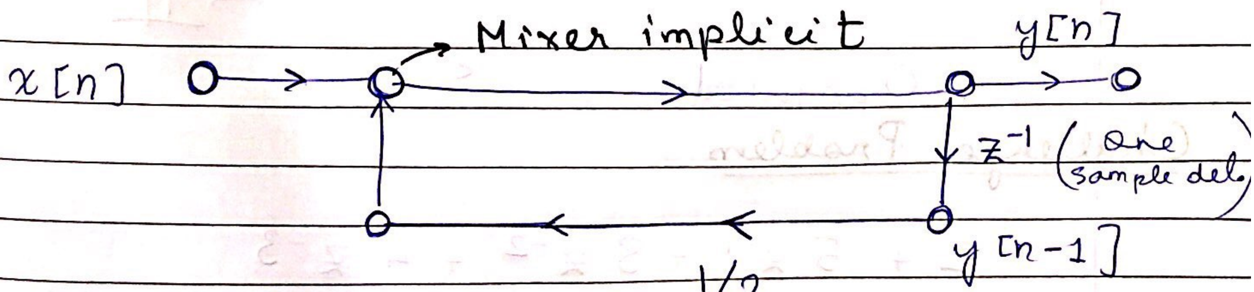
$$= \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)} (2 + 3z^{-1})$$

$$X(z) \longrightarrow \boxed{\frac{1}{1 - \frac{1}{2}z^{-1}}} \longrightarrow Y(z)$$

$$\left(1 - \frac{1}{2}z^{-1}\right) Y(z) = X(z)$$

$$y[n] - \frac{1}{2}y[n-1] = x[n]$$

$$x(t) \longrightarrow \boxed{A/D} \longrightarrow x[n]$$



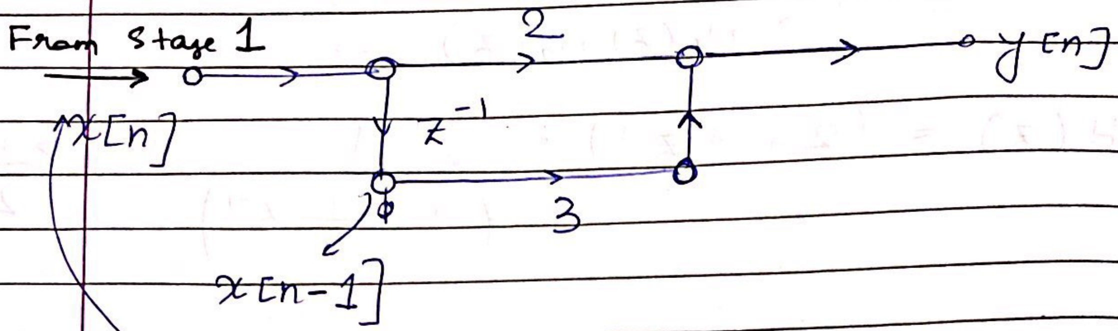
1/2
Constant Multiplier

Each edge in SFG is a multiplier

Numerator

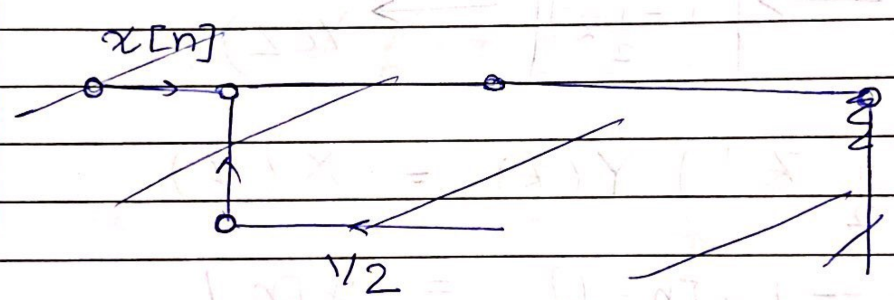
$$Y(z) = (2 + 3z^{-1})X(z)$$

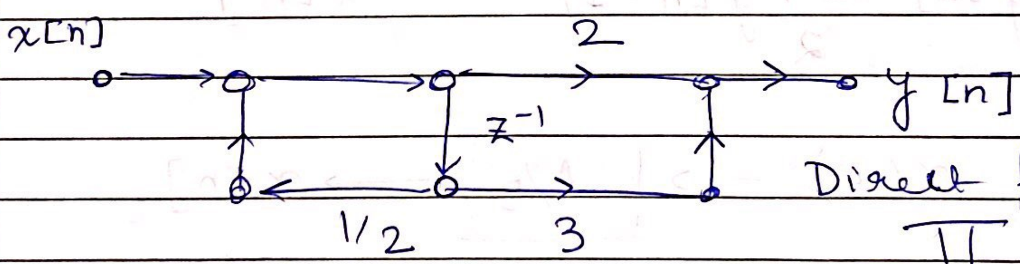
$$y[n] = 2x[n] + 3x[n-1]$$



Direct form I realization

→ Make More efficient.





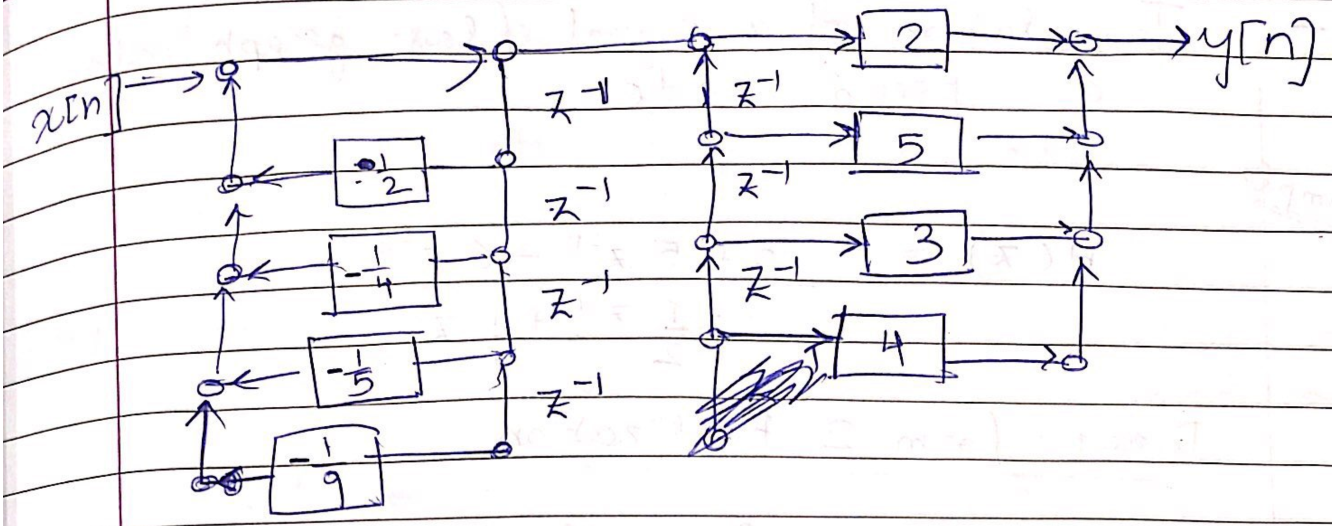
Direct form II

Optimized

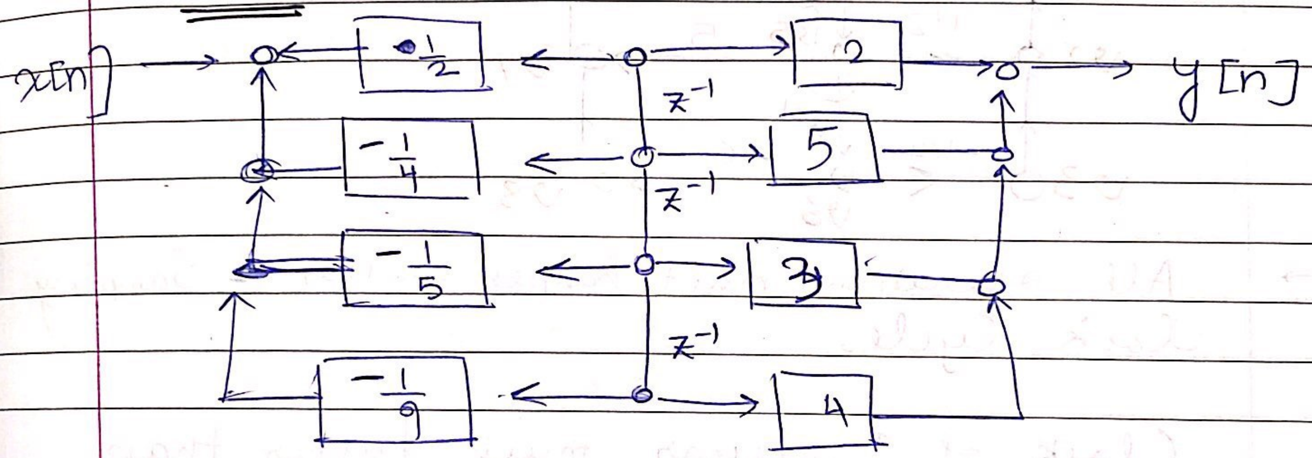
Challenge Problem

$$\frac{2 + 5z^{-1} + 3z^{-2} + 4z^{-3}}{1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{5}z^{-3} + \frac{1}{9}z^{-4}}$$

DF1



DF2



→ Challenge Problem

→ Is there some hidden challenge in implementing Direct form II ? In terms of programming not hardware.........

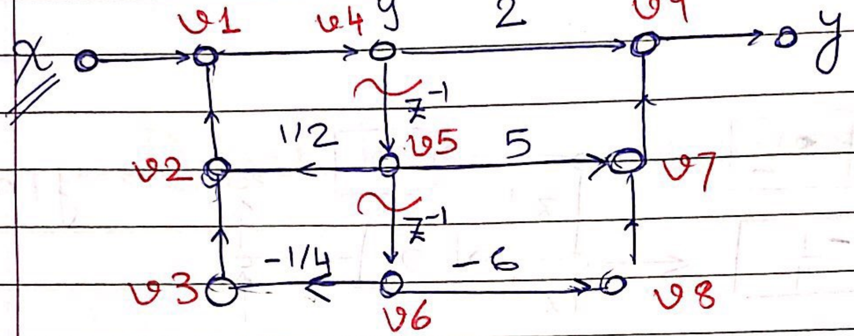
11.2.19

→ Translation of a signal flow graph into a pseudo code

Example

$$H(z) = \frac{2 + 5z^{-1} - 6z^{-2}}{1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}}$$

Direct - form 2 Realization



→ All operations must happen within 1 Sampling Clock Cycle.

Clock of Processor much faster than Sampling Clock

1) Assign names to all the nodes.

2) Cut the 1 sample delay branches

3) Ascertain whether the Signal flow graph is loop free
↳ If there is, this SFG is unrealizable

i.e an SFG has no delay free loops.

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4) Identify the source nodes in this Graph

→ Nodes with No incoming edges

~~x, v5~~ x, v_5, v_6

5) Propagate the sequence (sample) from the source nodes to the one-step forward nodes:

$v_1 \leftarrow x$

$v_7 \leftarrow 5v_5$

$v_2 \leftarrow \frac{1}{2}v_5$

$v_3 \leftarrow -\frac{1}{4}v_6$

$v_8 \leftarrow -6v_6$

6) Now propagate one step further

from $\frac{1}{2}v_5$

$v_2 \leftarrow v_2 + v_3$

$v_7 \leftarrow v_7 + v_8$

$v_1 \leftarrow v_1 + v_2$

$v_9 \leftarrow \cancel{v_9 +} v_7$

$v_4 \leftarrow v_1$

$v_9 \leftarrow v_9 + 2v_4$

$y \leftarrow v_9$

Challenge

Each group takes a different system function, progressively more complicated

→ Realize with an SFG and write pseudo code

7) Put back the Sample delays
Replace the one-sample delays, update beginning from the tail of each string

$v6 \leftarrow v5$ order
$v5 \leftarrow v4$ important

Questions (31-37)

1) How do we Initialise? ($v5, v6$)

2) Show how we can use the same strategy would work to program an LCCDE

$$H(z) = H_1(z)\, H_2(z) \ldots\ldots H_L(z)$$

(Rational)

$$\frac{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 - \frac{1}{5}z^{-1}\right)}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)} = \frac{\left(1 - \frac{1}{3}z^{-1}\right)}{\left(1 - \frac{1}{4}z^{-1}\right)} \cdot \frac{\left(1 - \frac{1}{5}z^{-1}\right)}{\left(1 - \frac{1}{2}z^{-1}\right)}$$

→ Or any other equivalent variant

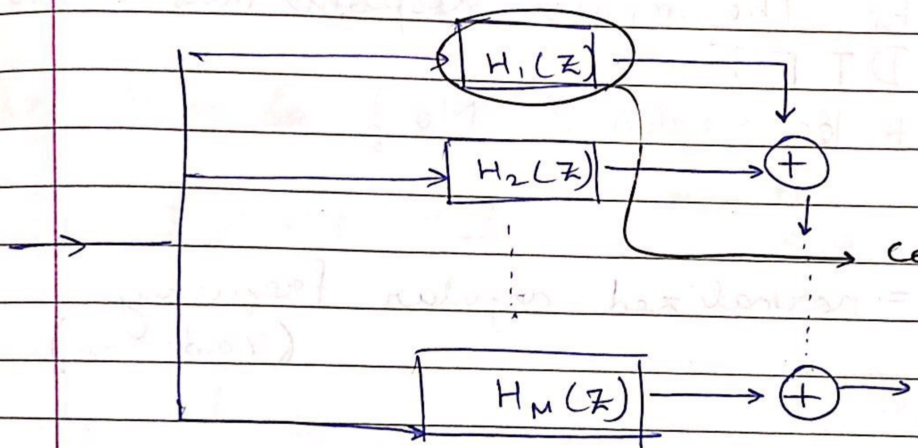
- Take above example, is it always possible to fuse delays?

- Show from the commutativity and associativity of convolution that the factors $H_1, \ldots, H_L$ can be reordered in any manner.

- Identify the possible pipelining and parallelism in your ~~ps~~ pseudo-code

$$\underbrace{H(z)}_{\text{rational}} = \underbrace{\sum_{m=1}^{M} H_m(z)}_{\text{partial fraction decomposition}}$$

Can lead to Parallel form decomposition



Cascade-Parallel form

could be in cascade

12.2.19

→ Deadline for app ass. 16 Feb

Analysis

1. Properties of systems
2. LSI systems
3. Transform / Change of Paradigm
   DTFT
   z - transform
4. Brief Introduc. to synthesis

→ Rational System Function

↓

Direct form I, II
Cascade
Cascade - Parallel Realizations.

Filter Design

Filter : An LSI System with a frequency response

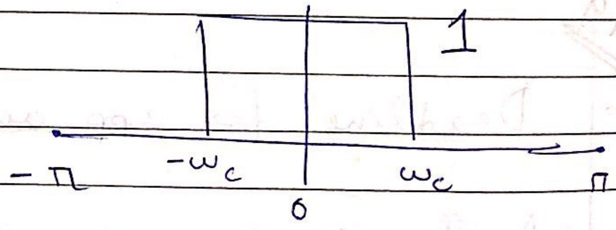
→ ~~Must the~~ The impulse response must have a DTFT

• Must it be stable? No!

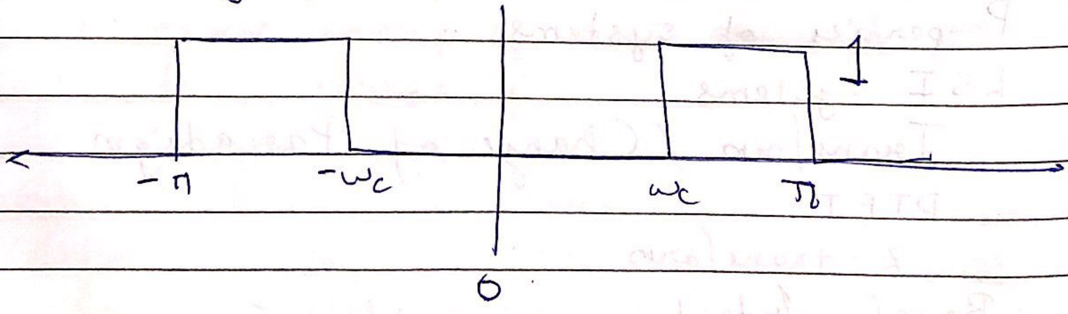
Let,
ω = normalized angular frequency (radians)

4 types

1) Ideal Low Pass



2) Ideal high pass filter



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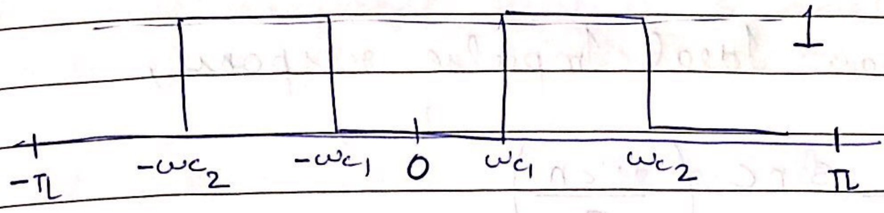
3) Band Pass filter

H.P.F (ω_{c_1}) − H.P.F (ω_{c_2})

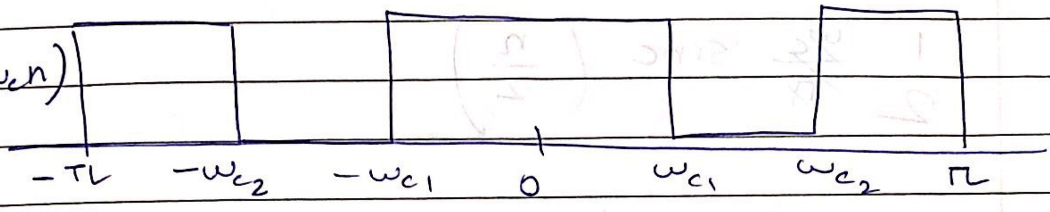


$$= \frac{2\omega_{c_2}}{\pi}\operatorname{sinc}\left(\frac{\omega_{c_2} n}{2}\right) - \frac{2\omega_{c_1}}{\pi}\operatorname{sinc}\left(\omega_{c_1} n\right)$$

4) Band-Stop (Band Reject)

L.P.F (ω_{c_1}) + H.P.F (ω_{c_2})

$\frac{2\omega_{c_1}}{\pi}\operatorname{sinc}(\omega_c n)$



Q→ Find h[n] for Ideal LPF

$$\text{IDTFT} = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega = \frac{1}{(jt)} e^{j\omega n}\Big|_{-\omega_c}^{\omega_c}$$

$$2\left(\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{(2jt)}\right) \qquad \frac{2\omega_c \sin(\omega_c t)}{\omega_c t}$$

$$2\omega_c \frac{\sin(\omega_c n)}{\omega_c n}$$

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{(\pi x)}$$

$$= \frac{2\omega_c}{} \frac{\sin\left(\pi \cdot \frac{\omega_c n}{\pi}\right)}{\pi\left(\frac{\omega_c n}{\pi}\right)} \qquad \frac{\omega_c}{\pi}\operatorname{sinc}\left(\frac{\omega_c n}{\pi}\right)$$

$$\boxed{\frac{\omega_c}{\pi}\operatorname{sinc}\left(\frac{\omega_c n}{\pi}\right)} = \frac{2\omega_c}{\pi}\operatorname{sinc}(\omega_c n)$$

2) $2\pi - \frac{2\omega_c}{\pi}\operatorname{sinc}(\omega_c n)$

Why is Filter Design a Challenge

Low pass Ideal Impulse response,

$$\frac{w_c}{\pi} \operatorname{sinc}\left(\frac{w_c n}{\pi}\right)$$

Let $w_c = \pi/2$

$$\frac{1}{2} \cancel{\frac{w_c}{\pi}} \operatorname{sinc}\left(\frac{n}{2}\right)$$

n	0	± 1	± 2	± 3	± 4	± 5
$h[n]$	$1/2$	$1/\pi$	0	$-1/3\pi$	0	$1/5\pi$

- Why is the Ideal filter unrealizable

Reasons

1) Causality

2) Just Causality can be handled using delays but this is Infinitely non-causal

3) the series $\sum \frac{1}{n}$ does not converge

2 proofs discussed

1) Integration < Summation & Int $\to \infty$ $\to \infty \Rightarrow$ Sum $\to \infty$

2) Wrote $\frac{1}{4}$ as $\frac{1}{4}, \frac{1}{8}+\frac{1}{8}, \frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}$ and so, on

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

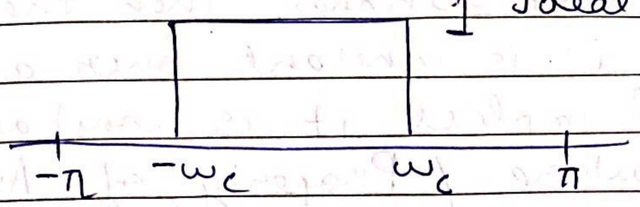
$\cancel{1 + \frac{1}{2}}\ \frac{1}{2} + \frac{1}{4} + \frac{1}{6}$

- Piecewise constant ideal responses:

An ideal response which is constant on each of a finite number of pieces of $]-\pi, \pi[$

The pieces cover the interval,

Ideal lowpass



- Disqualifications of the piecewise constant ideal filters

1. Infinitely Non-causal
2. Unstable

Challenge prob:

1) Show the instability for other examples, from among the piecewise constant filters.

2) ~~Char~~ Show that the cause of the instability is the discontinuity of the ideal response

freq. response

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3) Ideal filters are irrational
One cannot express

$$\sum_{n=-\infty}^{\infty} h_{ideal}[n]\, z^{-n}$$

as a rational function of z

Since,
If this were rational, then the fact that it is constant over a non-trivial interval implies it is constant everywhere (Property of Analytic functions)

Challenge : Prove that rational functions cannot be constant over an interval.

Q → Can a zero phase system ever be causal?

→ Keep in mind that,

→ A causal system must have non-zero phase response

→ Linear ~~Respo~~ Phase response is also acceptable.

→ Linear phase response is not Achievable in analog filters.

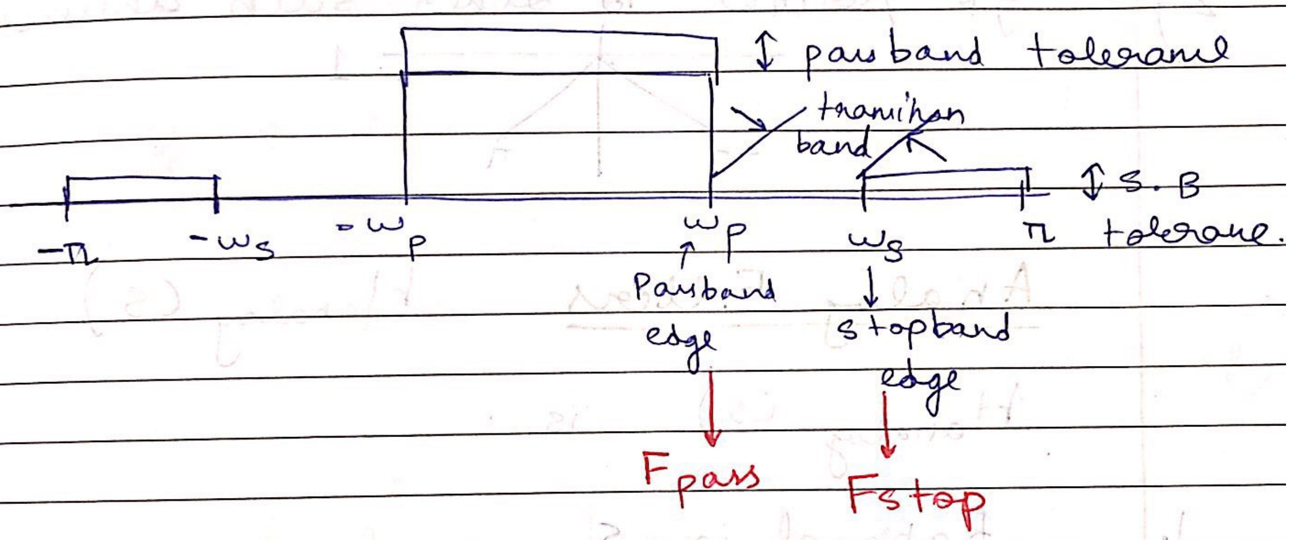
Realizable Specifications ~~with a piecewise~~

1. No discontinuities. We must have a transition band b/w passbands and stopbands

2. Tolerance in passband and stopband magnitude.

3. Allow a phase response.

Example LPF



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Challenge Problems

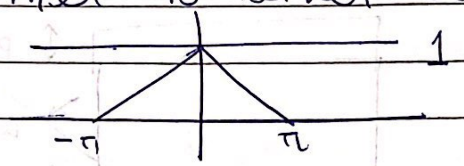
1) → Find the inverse DTFT of

$$H(e^{j\omega}) = j\omega$$

$$-\pi < \omega < \pi$$

Hence answer the question, does no discon. in $-\pi, \pi$ $\Rightarrow$ Stability

2) Go further to other such answers



Analog Filters $H_{analog}(s)$

$H_{analog}(s)$ is :

1. Rational in s
2. Stable and causal (poles on left half plane)
3. Right Sided R.O.C

→ These properties need to be preserved

$$T(z) \longrightarrow s$$

(i) $T(z)$ must be rational

(ii) Imag axis $s = j\Omega$ $\xleftrightarrow[\text{transform}]{\text{Invertible}}$ Unit circle $|z| = 1$

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(iii) Left half of s plane must go to the interior of $|z| = 1$ $(|z| < 1)$

Right half of s-plane must go to $|z| > 1$

Challenge Q) Irrational useful Analog filter

Q) What should be the Mapping b/w imaginary axis and the $|z| = 1$ circle

(ii) Should be a one to one mapping

Imag axis in s-plane $\longleftrightarrow$ $|z| = 1$

~~Sp~~ s-plane $\Omega : -\infty \to \infty$
$s = j\Omega$ $\omega : -\pi \to \pi$
(Strictly Increasing map)

Analogies from Analog

$$y[n] = x[n] - x[n-1]$$

1) $T(s) = s$ (Differentiator)

$$y[n] = x[n] - x[n-1]$$

$$s = 1 - z^{-1} = \underbrace{1 - e^{-j\omega}}$$

won't work

Can never be purely imaginary

→ Give 1/2 sample delay $(z^{-1/2})$ then possible but insensible.

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$$y[n] = x\left[n+\frac{1}{2}\right] - x\left[n-\frac{1}{2}\right]$$

$$s \longleftarrow z^{1/2} - z^{-1/2}$$

$$e^{j\omega} \longleftarrow e^{j\omega/2} - e^{-j\omega/2}$$
$$= 2j\sin\left(\frac{\omega}{2}\right)$$

$$\tan\left(\frac{\omega}{2}\right) = \frac{\sin\left(\frac{\omega}{2}\right)}{\cos\left(\frac{\omega}{2}\right)}$$

$$\frac{z^{1/2} - z^{-1/2}}{z^{1/2} + z^{-1/2}} = \left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

Average of o/ps

$$\frac{Y(z)}{X(z)} = \frac{1-z^{-1}}{1+z^{-1}} \qquad \frac{y[n] + y[n-1]}{2}$$

$$= \underbrace{\frac{x[n] - x[n-1]}{2}}_{\text{proportional to first derivative}}$$

* Does,
$$s \leftarrow \frac{1-z^{-1}}{1+z^{-1}}$$
Satisfy all the criteria?

- rational ✓
- $$z = e^{j\omega} \qquad \frac{1-e^{-j\omega}}{1+e^{-j\omega}}$$

$$\left(\frac{e^{j\omega/2} - e^{-j\omega/2}}{e^{j\omega/2} + e^{-j\omega/2}}\right) = \frac{2j\sin\left(\frac{\omega}{2}\right)}{2\cos\left(\frac{\omega}{2}\right)}$$

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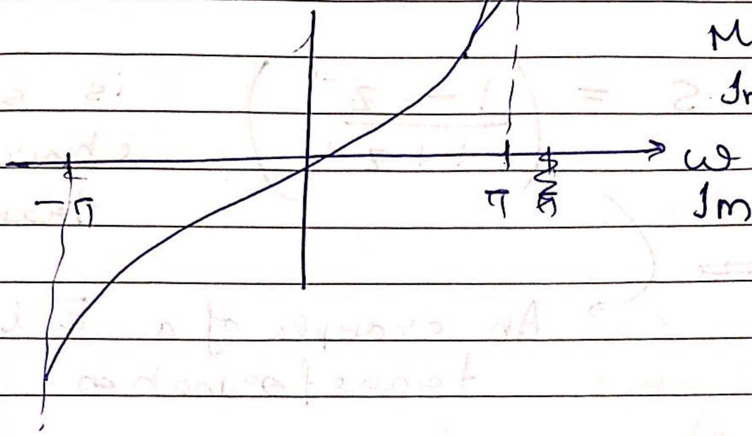
$$= j \tan\frac{\omega}{2} = j \cdot \Omega$$

$$\Omega = \tan\frac{\omega}{2} \quad (\text{our mapping})$$



$$s = \frac{1 - z^{-1}}{1 + z^{-1}}$$

$$(1 + z^{-1})\, s = (1 - z^{-1})$$

$$z^{-1}(s + 1) = (1 - s)$$

$$z^{-1} = \frac{(1-s)}{(1+s)} \Rightarrow z = \frac{(1+s)}{(1-s)}$$

$$s = \Sigma + j\Omega$$

$$z = \frac{(1 + \Sigma) + j\Omega}{(1 - \Sigma) - j\Omega}$$

$$|z|^2 = \frac{(1 + \Sigma)^2 + \Omega^2}{(1 - \Sigma)^2 + \Omega^2}$$

$\Sigma > 0$: Right half plane

$\Rightarrow |z|^2 > 1$ (exterior to the circle)

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Left half plane

$\Sigma < 0 \rightarrow |z|^2 < 1$ (Interior of unit circle)

$\Sigma = 0 \quad , \quad |z| = 1$

→ $S = \left(\dfrac{1 - z^{-1}}{1 + z^{-1}}\right)$ is our choice of transformation

~~Sstate~~ An example of a Bilinear transformation

→ A Bilinear transformation is of the form,

$$\overset{\sigma + j\omega}{s} = \frac{az + b}{cz + d}$$

Challenge Problem (13)

→ Prove that a bilinear transformation maps

circles/ straight lines → ~~circles~~/ straight lines/ circles

→ In particular illustrate for

$$s = \left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

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$H_{analog}(s) \quad | \quad s \rightarrow \frac{1 - z^{-1}}{1 + z^{-1}}$

Conversion of discrete filter specs to corresponding Analog filter specs.

$$s \leftarrow \frac{1 - z^{-1}}{1 + z^{-1}}$$

$$z = e^{j\omega}$$

$\Omega \rightarrow$ Analog sinusoidal frequency

$$\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} = j \tan \frac{\omega}{2}$$

$\omega \rightarrow$ Discrete time normalized angular frequency

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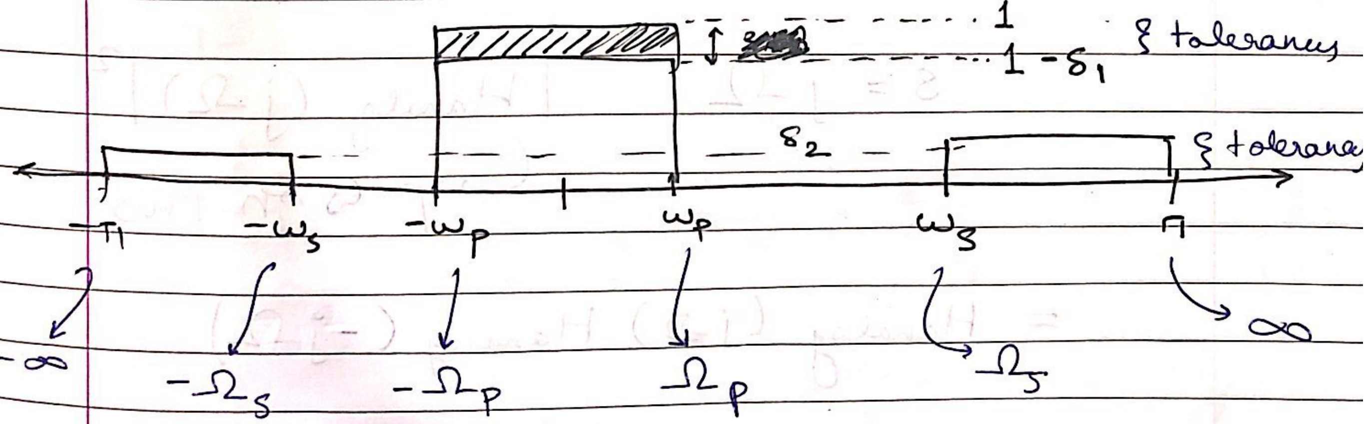
$$\Omega = \tan \frac{\omega}{2}$$

→ Explore other transformations from $s \leftrightarrow z$ and see if you can get one like this one.

Always valid — $s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$

Frequency Domain mapping

$$\Omega = \tan \frac{\omega}{2} \quad \text{(odd function)}$$



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Q) Why not do the following?

Analog Signal → Analog filter → sample → out

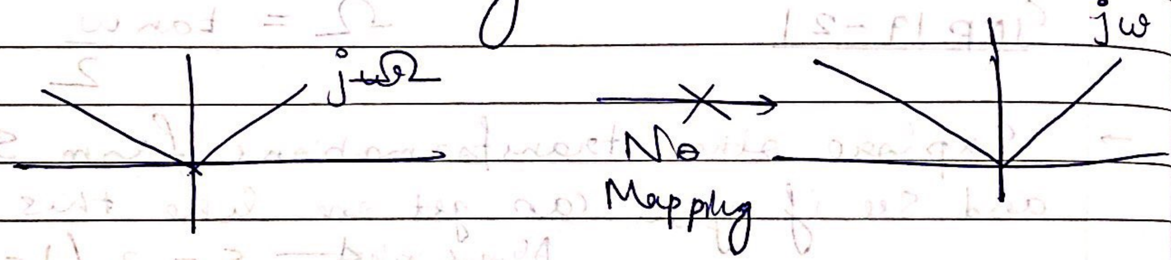
Before sampling consider a Non ideal

Vs

Analog signal → sample → Discrete filter → out

For LPF, HPF, BPF & BSF

→ The Dependent variable is carried forward directly



⇒ Stick to piece-wise const.

Lowpass Butterworth Filters

$$s = j\Omega \qquad |H_{analog}(j\Omega)|^2$$

↳ specs on this

$$= H_{analog}(j\Omega)\, H_{analog}(-j\Omega)$$

$j\Omega \rightarrow$

$= H_{analog}(s)\, H_{analog}(-s) \quad , \quad s = j\Omega$

Note this combination should be stable.

$$H_{analog}(s)\, H_{analog}(-s) = \frac{1}{(\; T(s) \;)}$$

(i) $T(s) = \frac{1}{s^{2N}}$; no pols on im.

(ii) ~~Te~~ $H_{analog}(s)\, H_{analog}(-s)$

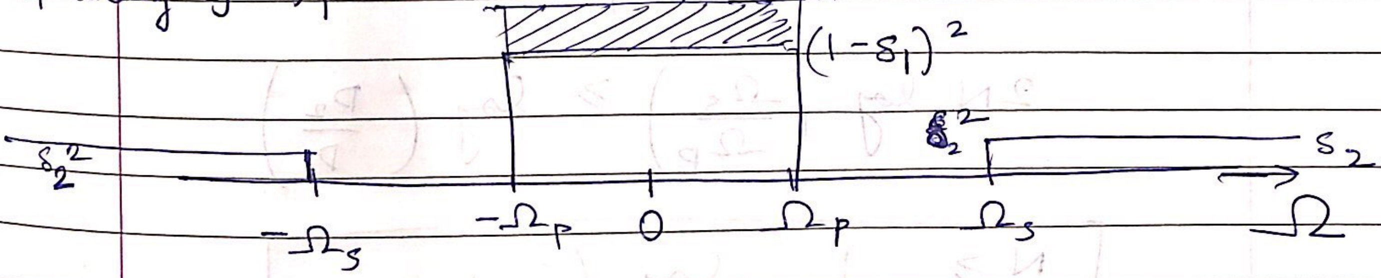
$$\downarrow s = j\Omega$$

$$|H_{analog}(j\Omega)|^2$$

$$\cancel{H_{ana}} \; H(s)H(-s) = \frac{1}{1 + \left(\frac{s}{j\Omega_c}\right)^{2N}}$$

$$|H_{analog}(j\Omega_c)|^2 = \frac{1}{2} \quad \left(\frac{1}{2} \text{ power frequency}\right)$$

$|H_{analog}(j\Omega)|^2$



$$\frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

Monotonically Decreases in Ω

$$\frac{1}{1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2N}} \geq (1-\delta_1)^2$$

pass

$$\frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}} \leq \delta_2^2$$

stop

$$\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} \leq \underbrace{\frac{1}{(1-\delta_1)^2} - 1}_{D_1}$$

$$\left(\frac{\Omega_c}{\Omega_p}\right)^{2N} \geq \frac{1}{D_1}$$

$$1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N} \geq \frac{1}{\delta_2^2}$$

$$\left(\frac{\Omega_s}{\Omega_c}\right)^{2N} \geq D_2$$

$$\left(\frac{\Omega_s}{\Omega_c}\right)^{2N} \geq \underbrace{\frac{1}{\delta_2^2} - 1}_{D_2}$$

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} \geq \frac{D_2}{D_1}$$

$$2N \log\left(\frac{\Omega_s}{\Omega_p}\right) \geq \log\left(\frac{D_2}{D_1}\right)$$

$$\boxed{N \geq \frac{\frac{1}{2}\log\left(\frac{D_2}{D_1}\right)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}}$$

Range on ω_c

$$\Omega_c \leq \frac{\Omega_s}{D_2^{1/2N}}$$

$$\Omega_c \geq \frac{\Omega_p}{D_1^{1/2N}}$$

$$\boxed{\frac{\Omega_p}{D_1^{1/2N}} \leq \Omega_c \leq \frac{\Omega_s}{D_2^{1/2N}}}$$