On cmplex plain

complex Random variable has 2 Real random variable

complx normal random variable

$$Z = Z_r + j Z_i$$

$N(0,1)$

$N(0,1)$ $\quad N\left(0,\frac{1}{2}\right) \quad N\left(0,\frac{1}{2}\right)$

$$\mathbb{E}[Z] = 0 \quad (0 + j0)$$

$$\mathbb{E}\left[|Z|^2\right] = 1/2 + 1/2 = 1$$

$$\hookrightarrow \mathbb{E}[Z\bar{Z}] = \mathbb{E}(Z_r^2 + Z_i^2) = \frac{1}{2} + \frac{1}{2} = 1$$

$$n(t) = n_r(t) + j n_i(t)$$

$\frac{N_0}{2} \qquad \frac{N_0}{2}$

Cmplex AWGN process

28.1.19

Recap

→ Complex Dimensions .....

- All PSK have 1 complex and 2 real dimensions, there is no "orthogonality"

$$\psi_1 e^{j\pi/4} = \frac{\psi_1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

→ ~~Cmplex n.~~

for $\quad z = Z_r + j Z_i$

$N\left(0,\frac{1}{2}\right) \quad N\left(0,\frac{1}{2}\right)$

$$C_z = \frac{1}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

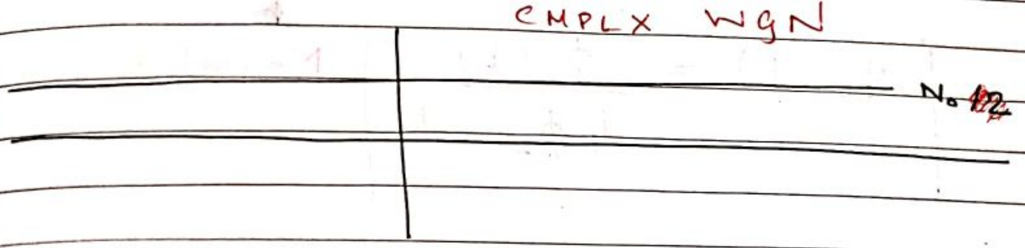
$$\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$$

And X, Y are jointly Gaussian ⇒

Scanned by CamScanner

X, Y are Independent

→ for Cmplex Noise



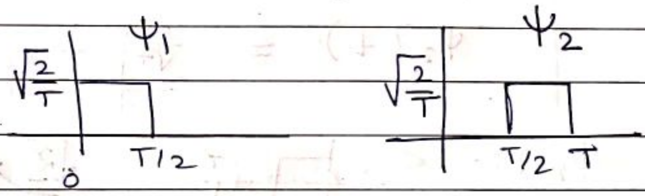
$N_0 = KT$

↳ Bolzman const J/K

Signal Space Concept

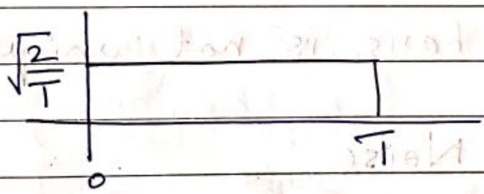
Ex 1)

→ Given the basis



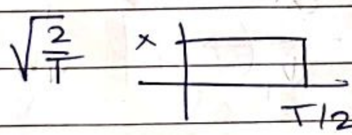
$S_i = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$

$S_1 =$



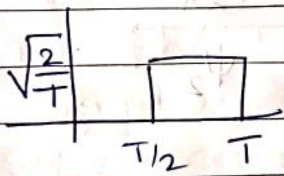
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$S_2 =$



$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$S_3 =$



$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

→ Algorithm to obtain an ~~bas~~ orthonormalization

Gramm - Schmidt

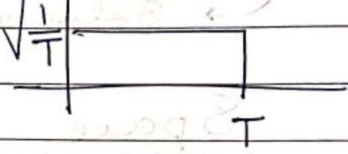
$$\psi_1(t) = \frac{s_1(t)}{||s_1(t)||} \qquad \sqrt{\int_{-\infty}^{\infty} ||s_1(t)||^2 \, dt}$$

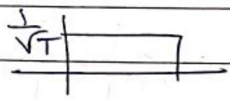
$$\phi_2(t) = s_2(t) - \langle \psi_1, s_2 \rangle \psi_1$$

$$\boxed{\psi_2(t) = \frac{\phi_2(t)}{\|\phi_2(t)\|}}$$ Normalize ....

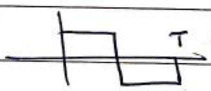
→ <u>Applied to ex 1</u>

G.S

$\psi_1(t) = \sqrt{\frac{1}{T}}$ 

$\psi_2(t) = \frac{1}{\sqrt{T}}$  $- \cancel{\frac{\sqrt{2}}{T}} \times \cancel{\frac{T}{2}}\ \frac{1}{\sqrt{2}} \times \sqrt{\frac{1}{T}}$

1

$\psi_2(t) =$  $\frac{\sqrt{\cancel{2}}}{T} \times \frac{T}{\cancel{2}}$

→ The final basis is not unique ....

<u>Effect of Noise</u>

→ $s(t) \xrightarrow[\psi_1]{\text{cor}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$s(t) \xrightarrow[\psi_2]{\text{cor}} \frac{1}{\sqrt{2}} \begin{bmatrix} 0.74 \\ 0.62 \end{bmatrix}$

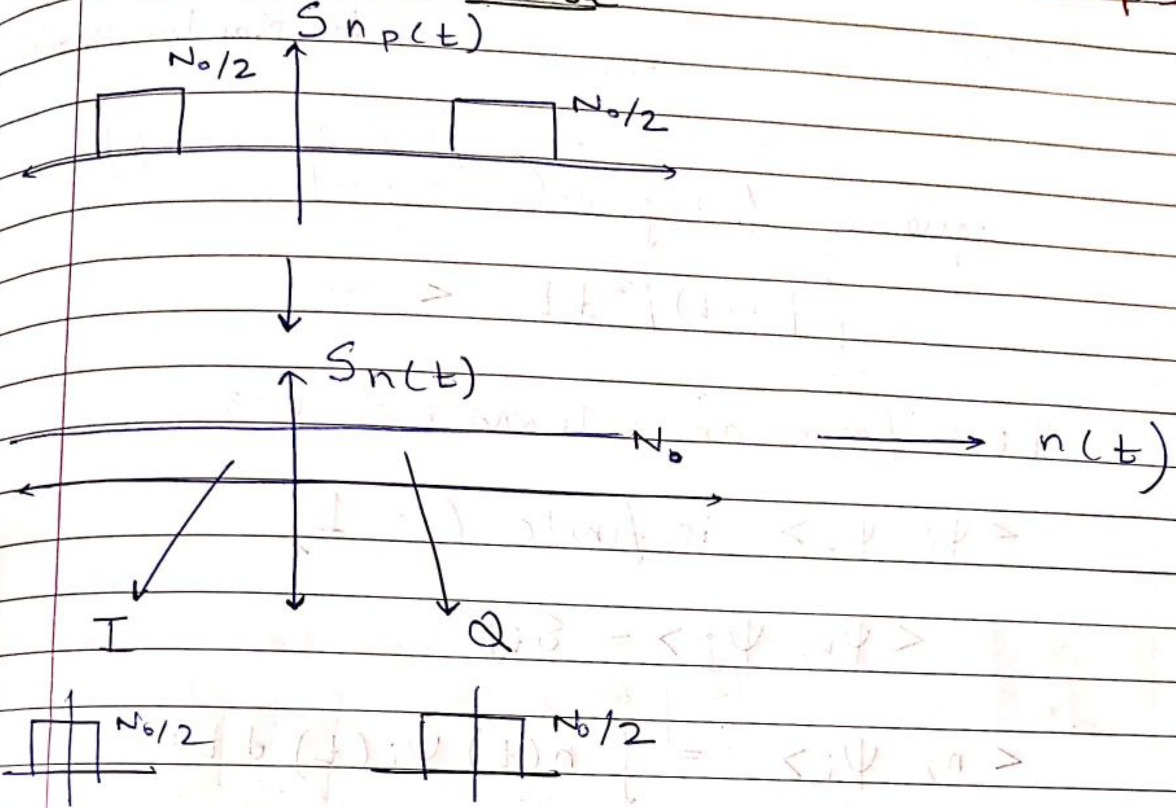
$r = s_i + n$

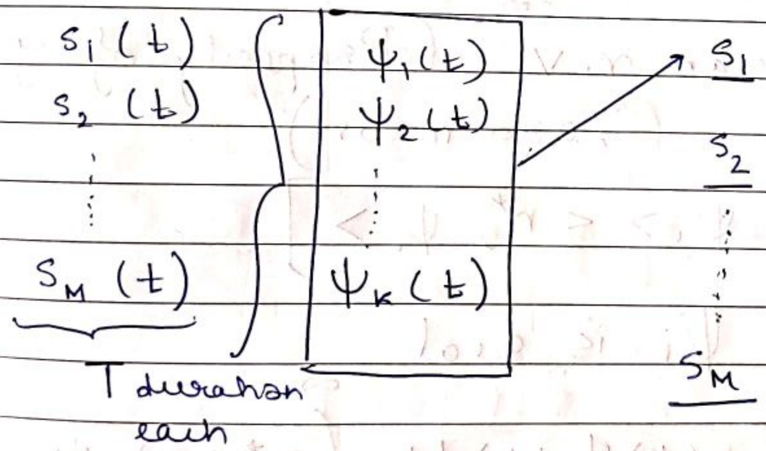
29.1.19

Detection Problem

– Chap 3

→ Pass-Band Noise





$$\underline{s}_i = s_i(t) \longrightarrow \oplus \longrightarrow y(t) = s_i(t) + n(t)$$

$$\uparrow n(t)$$

Find the most likely

$$j \in \{1, \ldots, M\}$$

Such that $y(t)$ is received when $s_j(t)$ is sent "Detection"

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$$I_{\langle n, u \rangle} = \int_{-\infty}^{\infty} n(\tau)\, u^*(\tau)\, d\tau$$
= A Random variable

where $u(t)$ is any finite Energy square Integrable signal

$$\int_{-\infty}^{\infty} |u(t)|^2 dt < \infty$$

→ ψ_i's form an orthonormal basis

$\langle \psi_i, \psi_i \rangle$ is finite (= 1)

$$\langle \psi_i, \psi_j \rangle = \delta_{ij}$$

→ $$\langle n, \psi_i \rangle = \int_{-\infty}^{\infty} n(t)\, \psi_i(t)\, dt$$

= Gaussian r.v (Property of GRP)
(Zero mean)

Variance?

$$\mathbb{E}\left[\langle n, \psi_i \rangle \langle n^*, \psi_i \rangle\right]$$

ψ_i is real

$$= \mathbb{E}\left[\int_{-\infty}^{\infty} n(t)\psi_i(t)\,dt \int_{-\infty}^{\infty} n^*(\tau)\, \psi_i(\tau)\,d\tau\right]$$

$$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} n(t)\, n^*(\tau)\, \psi_i(t)\, \psi_i(\tau)\, d\tau\, dt\right]$$

$$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} N_0\, \delta(t-\tau)\, \psi_i(t)\, \psi_i(\tau)\, dt\, d\tau\right]$$

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Since $\langle \psi_i, \psi_i \rangle = 1$

$E[\quad] = N_0 \times 1$ $\quad$ $\vec{N}$ is Noise vector in Sig. Space

$\Rightarrow$ variance σ^2 or N_0

$$n(t) \longrightarrow \boxed{\begin{matrix}\text{corr.} \\ \psi(t)\end{matrix}} \longrightarrow \begin{bmatrix} \langle n, \psi_1 \rangle \\ \langle n, \psi_2 \rangle \\ \vdots \\ \langle n, \psi_K \rangle \end{bmatrix} = N$$

$$\int_{-\infty}^{\infty} (\cdot)\, \psi_1(t)\, dt$$

$= \text{COV} = \sigma^2$ or N_0

K-dimensional jointly Gaussian r.v, mean 0, Cov ?
In fact, they are all IID

Proof

$$\mathbb{E}\left[\langle n, \psi_1 \rangle \langle n, \psi_2 \rangle\right]$$

$$= \mathbb{E}\left[\int_{-\infty}^{\infty} n(t)\, \psi_1(t)\, dt \int_{-\infty}^{\infty} n^*(\tau)\, \psi_2(\tau)\, d\tau\right]$$

$$= \mathbb{E}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{n(t)\, n^*(\tau)}_{\delta(t-\tau)}\, \psi_1(t)\, \psi_2(\tau)\, d\tau\right] = \langle \psi_1, \psi_2 \rangle = 0$$

$N_0 \rightarrow$ Noise per Complex Dimension

$\frac{N_0}{2} \rightarrow$ Noise per Real Dimension

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→ So what is going on?

- We are projecting Noise onto an orthonormal vector space

$$y(t) = s_i(t) + n(t)$$

$$\langle y, \psi_i \rangle = \underbrace{\langle s_i, \psi_i \rangle}_{\underline{s}_i} + \underbrace{\langle n, \psi_i \rangle}_{\underline{N}}$$

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_K \end{bmatrix}}_{\underline{y}} \begin{matrix} = \\ = \\ \\ = \end{matrix} \begin{bmatrix} \langle s_i, \psi_1 \rangle \\ \langle s_i, \psi_2 \rangle \\ \vdots \\ \langle s_i, \psi_K \rangle \end{bmatrix} + \begin{bmatrix} \langle n, \psi_1 \rangle \\ \langle n, \psi_2 \rangle \\ \vdots \\ \langle n, \psi_K \rangle \end{bmatrix}$$

→ We went from : Which Waveform was sent?
To → Which vector was sent

→ What Information have we lost? Was the Info useful?

$$\underline{y} = \underline{s_i} + \underline{N}$$

$$n(t) \longrightarrow \underline{N} = \begin{bmatrix} N(1) \\ \vdots \\ N(k) \end{bmatrix}$$

→ So we have lost the Component of Noise orthogonal to n(t) in the higher / Infinite dimensional vector space

Does this Info ~~lost~~ being lost create problems?

$$\underbrace{n^{\perp}(t)} = n(t) - \sum_{i=1}^{K} N[i]\,\psi_i(t)$$

$$y^{\perp}(t) = y(t) - \sum_{j=1}^{K} \langle y, \psi_j \rangle \psi_j(t)$$

$$= \cancel{s_i(t)} + n(t) - \sum_{j=1}^{K} \left(\cancel{\langle s_i, \psi_j \rangle} + \langle n, \psi_j \rangle \right) \psi_j(t)$$

$$y^{\perp}(t) = n^{\perp}(t)$$

$$\mathbb{E}\left[n^{\perp}(t)\, N[k] \right] = \mathbb{E}\left[n^{\perp} \langle n, \psi_k \rangle \right] \rightarrow 0$$

Hence, The Information lost has no part to play in $\underline{y} = \underline{s_i} + \underline{N}$

BPSK

31.1.19

Vector Space Rep. (Detection problem)

$$\left.\begin{matrix} s_1(t) \\ s_2(t) \\ \vdots \\ s_M(t) \end{matrix}\right\} \xrightarrow{s_i(t)} \oplus \longrightarrow s(t) = s_i(t) + n(t)$$

$n(t)$ (into $\oplus$)

$$s_i \longrightarrow \underline{r} = \underline{s_i} + \underline{N}$$

$$s_i(t) = s_{i1}\,\psi_1(t) + s_{i2}\,\psi_2(t) + \dots + s_{ik}\,\psi_k(t)$$

$$\underline{N} = \begin{bmatrix} \langle n, \psi_1 \rangle \\ \vdots \\ \langle n, \psi_k \rangle \end{bmatrix} \longrightarrow \mathcal{CN}(0, \sigma^2 I)$$

$$r(t) - \sum r_j \psi_j(t)$$

$$= n^{\perp}(t) = n(t) - \sum_{j=1}^{k} N[j]\,\psi_j(t)$$

→ $r(t)$ projected onto $\vec{\psi}$

$$= \mathbb{E}\left[n^{\perp}(t) N^*[k]\right] = \mathbb{E}\left[n^{\perp}(t) \int_{-\infty}^{\infty} n^*(\tau)\,\psi_k(\tau)\,d\tau\right]$$

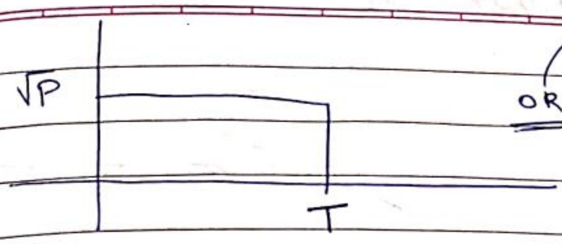
$$= \cancel{E\left[n(t)\int_{-\infty}^{\infty} n^*(\tau)\,\psi_k(\tau)\,d\tau\right]} - E\left[\sum_{j=1}^{k} N[j] N^*\right.$$

$$= E\left[\left(n(t) - \sum_{j=1}^{k} N[j]\,\psi_j(t)\right) \underbrace{\int_{-\infty}^{\infty} n^*(\tau)\,\psi_k(\tau)\,d\tau}_{N^*[k]}\right]$$

$$= E\left[n(t) N^*[k]\right] - \sum_{j=1}^{k} E\left[N[j] N^*[k]\right] \psi_j(t)$$

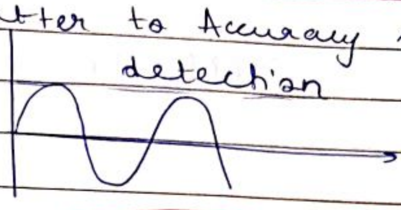
$$= \underbrace{E\left[n(t) N^*[k]\right]}_{} - \sigma^2 \psi_k(t)$$

ex **BPSK**

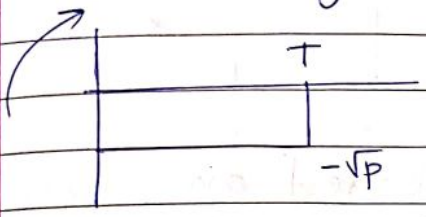


OR

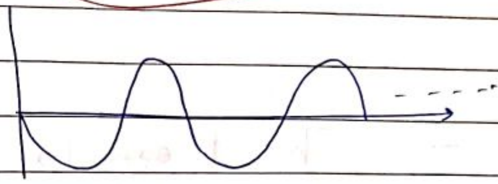
→ waveform does not matter to Accuracy of detection



→ The only difference is Bandwidth → Chap 2



OR



$$\vec{r} = \vec{s}_i + \vec{N} \rightarrow CN(0, \sigma^2 I)$$

$\sigma^2 = N_0$

→ Why?

Since ψ is taken to be dimensionless

Hypothesis Testing

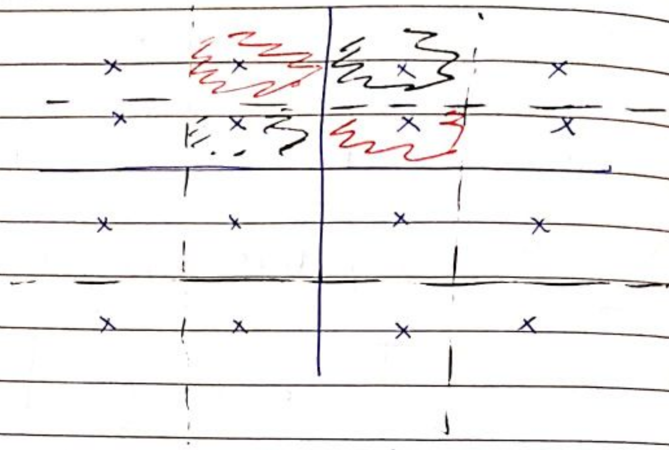
→ The maximum likelihood rule

$$f_{\vec{N}}(\vec{N}) = \underset{\substack{\text{Independent}\\ \text{of } s \text{ and } r}}{C} \exp\left(\frac{-(r-s_i)^H(r-s_i)}{N_0}\right)$$

$$\underset{i=1,\dots,M}{\arg\max}\; e^{-\frac{\|r-s_i\|^2}{N_0}} = \underset{i=1,2,\dots,M}{\arg\min} \|r-s_i\|^2$$

Decision Boundaries

QAM - 16



→ The boundaries are based on euclidean distance because of Gaussian Density

- Is the MLE Optimal? **No!**

ex Consider non-equiprobable s_i's

Minimize probability of error.

$s_1 \longrightarrow \pi(1)$ MPE rule

$$\arg\min_{1 \le i \le M} \|r - s_i\|^2 - 2\sigma^2 \log \pi(i)$$

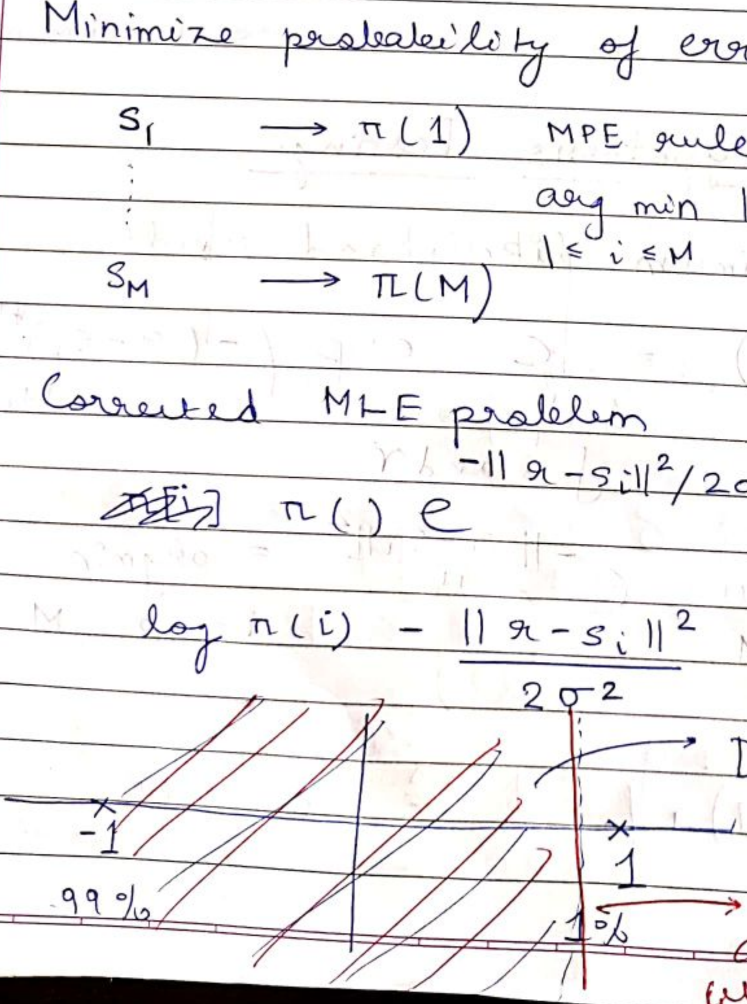
$\vdots$

$s_M \longrightarrow \pi(M)$

→ Corrected MLE problem

$$\pi(\,)\, e^{-\|r-s_i\|^2/2\sigma^2}$$

$$\log \pi(i) - \frac{\|r - s_i\|^2}{2\sigma^2} \quad \text{Optimal}$$



Decision boundary

If $\pi(i)$'s [prior] are skewed, this will be optimum boundary.

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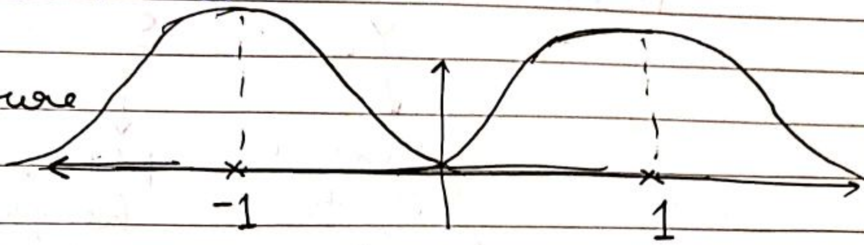
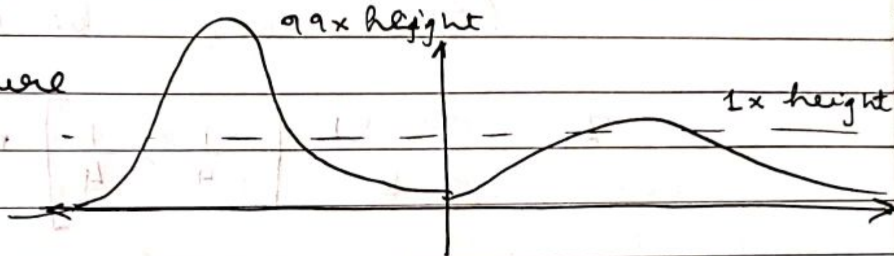
$$P(1 \text{ sent} \mid \vec{N}) = \frac{P(\vec{N} \mid 1 \text{ sent}) \cdot P(1 \text{ sent})}{}$$

$$= \pi(1)\, e^{\frac{-\|r - s_i\|^2}{N_0}}$$

→ The wrong picture



→ The Right picture



- For ~~non~~ equiprobable s_i's (Symbols)
  then Maximum Likelihood (Geometric)
  = Min. Prob. of error
  Since $\pi(i) = \frac{1}{M}$ (Independent)   ML ≡ MAP when $\pi(i) = \frac{1}{M}$

4.2.19

Recap: Receivers and Estimators

$x \in \mathcal{X} \rightarrow$ [$P_r(y|x)$] $\rightarrow y \in \mathcal{Y} \rightarrow$ [Detector] $\rightarrow \hat{x}(y)$

Case 1

Detection - $\mathcal{X}$ countable $\hat{x}(y) \in \{1, \ldots, M\}$
→ Like an MCQ

Case 2

Estimation - $\mathcal{X}$ real valued $\in \mathbb{R}^m$
→ Like a fill in the blank problem

Case 3

Soft Detection - $\hat{x}(y) = (\hat{x}, P_r(\hat{x}))$ Probability ~~measure~~
$\forall \hat{x} \in \mathcal{X}$

Q→ Estimating Student height (An estimation problem)

Model

$$Y_i = \theta + Z_i$$

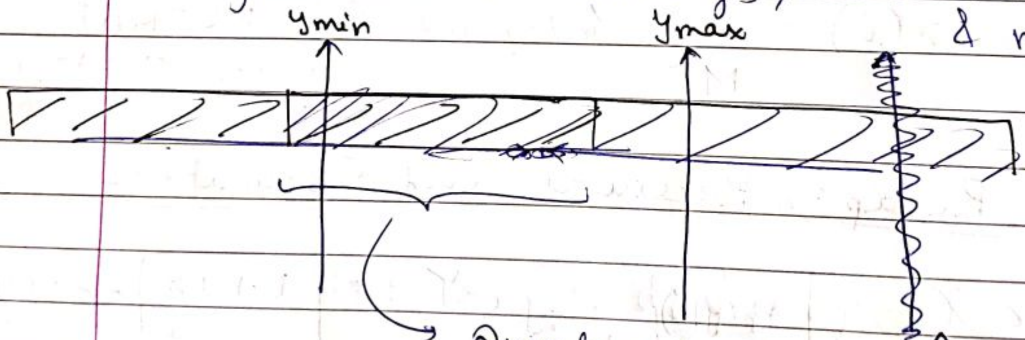
Y_i ↓ Student heights (Observable)

θ ↓ Mean height to be estimated

$$Z_i \sim U\left[-\frac{1}{4}, \frac{1}{4}\right] \text{ meters}$$

Problem : Estimate θ

Say we take Readings, and the min & max are



Overlap is where $\hat{\theta}$ lies

→ ~~Objective~~

Map Decoder

→ Probability of Correctness

$$Pr(C) \triangleq 1 - Pr(E)$$

$$\underset{\{1,\dots,n\}}{x} \longrightarrow \boxed{\qquad\qquad} \longrightarrow y \in \mathcal{Y}$$

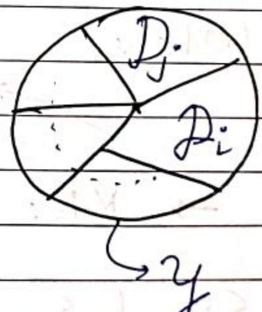
$$\hat{x}(y) = x$$

$$\Pr(C) = \sum_{i=1}^{M} \Pr(C, X=i)$$

$$= \sum_{i=1}^{M} \underbrace{\Pr(i)}_{\pi(i)} \Pr(C \mid X=i)$$

$$\Pr(C) = \sum_{i=1}^{M} \pi(i) \Pr(C \mid \cancel{X=}\, i)$$

$$= \sum_{i=1}^{M} \pi(i) \left\{ \int_{y \in D_i} \Pr(y \mid i)\, dy \right\}$$



$$= \sum_{i=1}^{M} \int_{y \in \mathcal{Y}} \pi(i) \Pr(y \mid i)\, \mathbb{1}_{\{y \in D_i\}}\, dy$$

$$= \int_{y \in \mathcal{Y}} \sum_{i=1}^{M} \pi(i) \Pr(y \mid i)\, \mathbb{1}_{\{y \in D_i\}}\, dy$$

$$\leq \int_{y \in \mathcal{Y}} \sum_{i=1}^{M} \left(\max_{1 \leq j \leq M} \pi(j) \Pr(y \mid j) \right) \mathbb{1}_{\{y \in D_i\}}\, dy$$

$$\int_{y \in \mathcal{Y}} \left(\max_{1 \le j \le m} \pi(j) \Pr(y|j) \right) \underbrace{\sum_{i=1}^{M} \mathbb{1}_{\{y \in D_i\}}}_{1} dy$$

$$\Pr(C) \le \int_{y \in \mathcal{Y}} \max \left(\pi(i) \Pr(y|i) dy \right)$$

$$= \sum_{i=1}^{m} \pi(i) \int_{y \in A_i} \Pr(y|i) dy$$

Where,

$$A_i = \{ y \in Y_i \quad \pi(i) \Pr(y|i) > \pi(j) \Pr(y|j) \} \quad \forall i \neq j$$

→ MAP : Maximum Aposteri Prob. Decoder

Signal Space

$$s(t) = \sum_{i \in I} d_i \phi_i(t)$$

$$\langle \phi_i(t), \phi_j(t) \rangle = \delta(i-j) \quad \text{" Orthonormal}$$

Examples

→ (i) Bandlimited Signals

$$\phi_m(t) = \text{sinc}\, \beta \left(t - \frac{m}{\beta} \right) \quad t \in \mathbb{R}$$

(ii) Time limited waveforms

5.2.19

ML Decision Rule / MAP protocol

$$\arg\min_i \|y - s_i\|^2 = \arg\min_i \left(\|y\|^2 + \|s_i\|^2 - 2\,\mathrm{Re}\langle \bar{y}, \bar{s}_i\rangle\right)$$

$$= \arg\max_i \left(\mathrm{Re}\langle y, s_i\rangle - \frac{\|s_i\|^2}{2}\right)$$

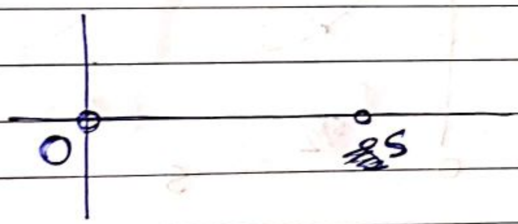
→ ML Decision rule for Gaussian Distribution

MEP (Minimum error prob. / MAP)

$$\mathrm{Re}(\langle y, s_i\rangle) - \frac{\|s_i\|^2}{2} + \underbrace{\sigma^2}_{\text{Noise Variance}} \log(\pi_i)$$

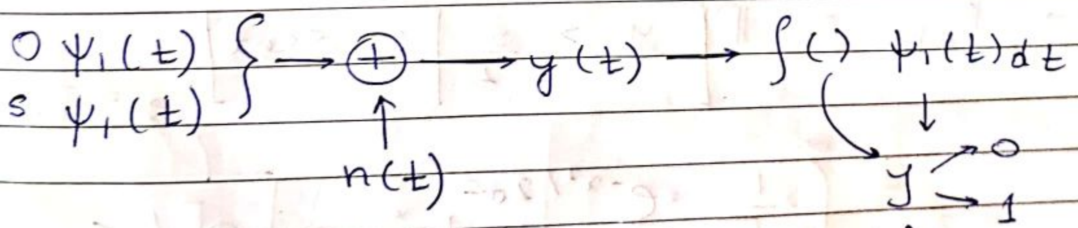
On-Off Keying

OOK + AWGN Channel



0 → 0

1 → S



For now, consider this 1-D problem

Probability of Error,

$$P_e = P_{e|0\text{ sent}} \times \underbrace{\frac{1}{2}}_{\pi(0)} + P_{e|1\text{ sent}} \times \underbrace{\frac{1}{2}}_{\pi(1)}$$

P(0 sent)

$H_1 : y(t) = s + n(t)$

$H_0 : y(t) = n(t)$

$\arg\max_i : \langle y, s_i \rangle - \frac{\|s_i\|^2}{2}$

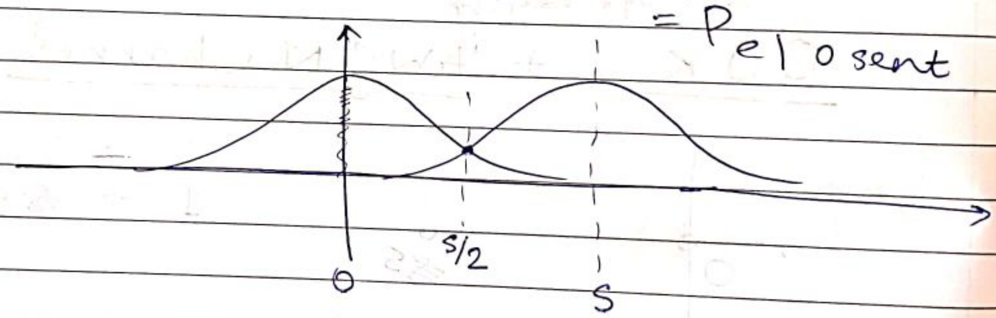
$0 \to 0 - 0 = 0$

$1 \to sy - \frac{s^2}{2}$ $\left(= 0, \Rightarrow \text{bound. @ } \frac{s}{2} \right)$

\# $\mathbb{P}_{e|1 \text{ sent}} = \mathbb{P}\left[z < \frac{\|s\|^2}{2} \,\middle|\, H_1 \right]$

$\to \mathbb{P}\left[sy < \frac{s^2}{2} \,\middle|\, H_1 \right] = \mathbb{P}\left[y < \frac{s}{2} \,\middle|\, 1 \text{ sent} \right]$

$= P_{e|0 \text{ sent}}$



$0 \to$ $\mathbb{P}_{e|0} = \mathbb{P}\left[y > \frac{s}{2} \,\middle|\, 0 \text{ sent} \right]$

$= \int_{s/2}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} dx = \boxed{Q\left(\frac{s}{2\sigma}\right)}$ $= P_e$ also

Where, $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-z^2/2} dz$

BER for OOK $= Q\left(\frac{s}{2\sigma}\right)$ Since 1 bit/Sy

BER $= P_e$

Bit Error Rate

$\lessgtr$

$$\frac{1}{2} P_{e|1} = \frac{1}{2} P_{e|0}$$

$\Rightarrow P_e = P_{e|1}$

Define

E_s : Energy per symbol

E_b : Energy per bit

for OOK Average $E_b = \frac{s^2}{2}$ $\left(\frac{1}{2} \times s^2 + \frac{1}{2} \times 0\right)$

$$E_b = \frac{s^2}{2} \quad , \quad S = \sqrt{2E_b}$$

with $E_b = \frac{s^2}{2}$

$$Q\left(\frac{s}{2\sigma}\right) = S\left(\sqrt{\frac{E_b}{2\sigma^2}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Binary Signalling

$s_1(t) \rightarrow s_1 \quad H_1 : y(t) = s_1(t) + n(t)$

$s_2(t) \rightarrow s_2 \quad H_2 : y(t) = s_2(t) + n(t)$

$$E_b = \frac{s_1^2 + s_2^2}{2}$$

$$\left[\langle y, s_1 \rangle - \frac{\|s_1\|^2}{2}\right] \underset{H_2}{\overset{H_1}{\gtrless}} \left[\langle y, s_2 \rangle - \frac{\|s_2\|^2}{2}\right]$$

$$\left[\langle y, s_1 - s_2 \rangle\right] \underset{H_2}{\overset{H_1}{\gtrless}} \left[\frac{|s_1|^2 - |s_2|^2}{2}\right]$$

$$\tilde{y}(t) = y(t) - s_2(t)$$

$$\tilde{y} = y - s_2$$

$H_1: \tilde{y}(t) = s_1(t) - s_2(t) + n(t)$

$H_0: \tilde{y}(t) = n(t)$

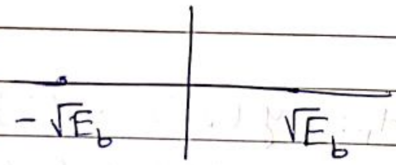
→ Constraint: $E_b = \frac{s_1^2 + s_2^2}{2} \le E_b$

$Q\left(\frac{|s_1 - s_2|}{2\sigma}\right) \rightarrow$ BER ~~minimize~~

maximize $|s_1 - s_2|$

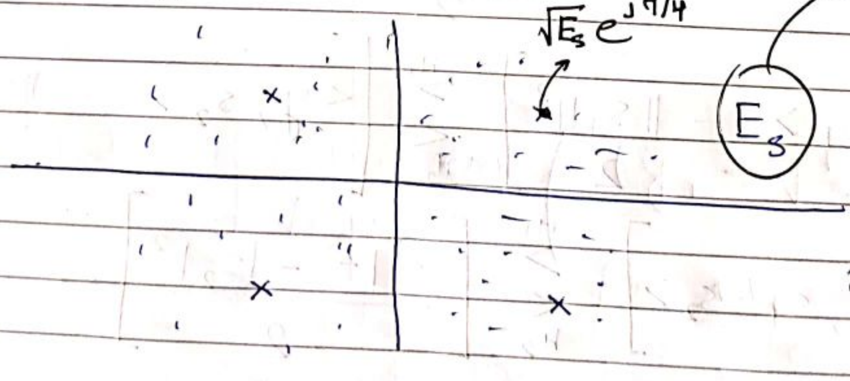
Subject to $\frac{s_1^2 + s_2^2}{2} \le E_b$ subject to

BER $|_{min}$ soln $= \boxed{-\sqrt{E_b}, \sqrt{E_b}}$



is optimal (BPSK)

→ Sending Multiple Bits



symbol energy

$E_s = 2E_b$

Bit Energy

→ Symbol Error Rate same for all 4

For QPSK $E_s = 2E_b$ → use this in calculations

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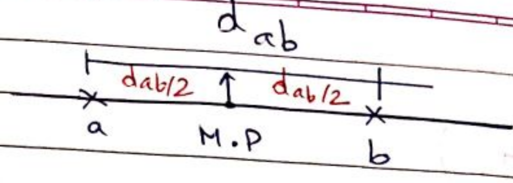
$\pi(i)$'s are equal



$$P_e = Q\left(\frac{d_{ab}}{2\sigma}\right)$$

eg $a, b = +\sqrt{E_b}, -\sqrt{E_b}$, $\sigma^2 = \frac{N_0}{2}$

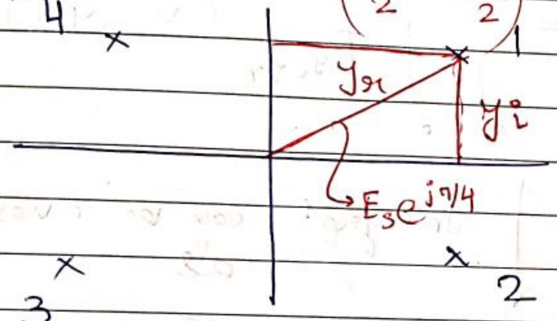
$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

$Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

$Q\left(\frac{\sqrt{E_b}}{\sigma}\right)$

$\sigma^2 = \frac{N_0}{2}$ $\sigma = \sqrt{\frac{N_0}{2}}$

$\left(\frac{E_s^2}{2} + \frac{E_s^2}{2}\right) = E_s^2$



$y = x + n$

complex Number as 2D vector

$$\begin{bmatrix} y_r \\ y_i \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{E_s}{2}} \\ \sqrt{\frac{E_s}{2}} \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

0 mean

each dim var $\left(\sigma^2 = \frac{N_0}{2}\right)$

$\begin{bmatrix} y_r \\ y_i \end{bmatrix} \longrightarrow$ jointly Gaussian

$\hookrightarrow$ jointly Gaussian

→ Mean changes, Cov same. (On Shift)

mean = $\begin{bmatrix} \sqrt{\frac{E_s}{2}} \\ \sqrt{\frac{E_s}{2}} \end{bmatrix}$ cov = $\begin{bmatrix} N_0/2 & 0 \\ 0 & N_0/2 \end{bmatrix}$

Distribution

$$f_{y_i y_r}(y_i, y_r) = \frac{1}{\frac{N_0}{2} \cdot 2\pi} \exp\left(\frac{-\left\| \vec{y} - \begin{pmatrix} \sqrt{\frac{E_s}{2}} \\ \sqrt{\frac{E_s}{2}} \end{pmatrix} \right\|^2}{2 \cdot N_0/2} \right)$$

mean

$$P_{c|1\,\text{sent}} = \int_0^\infty \int_0^\infty f_{y_i y_r}(y_i, y_r)\, dy_i\, dy_r$$

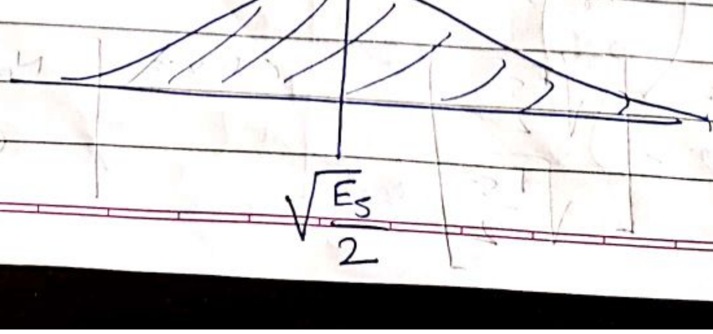
Correct

Integral can be evaluated as

$$\frac{1}{\cancel{\left(2\pi \cdot \frac{N_0}{2}\right)}} \underbrace{\int_0^\infty \frac{1}{\sqrt{2\pi}\sqrt{\frac{N_0}{2}}} \exp\left(\frac{-\left(y_i - \sqrt{\frac{E_s}{2}}\right)^2}{N_0} \right) dy_r}_{I} \underbrace{\int_0^\infty \exp\left(\frac{-\left(y_r - \sqrt{\frac{E_s}{2}}\right)^2}{N_0} \right) \times \frac{1}{\sqrt{2\pi}\sqrt{\frac{N_0}{2}}}\, dy_r}_{I}$$

$\Rightarrow$

$$\cancel{I}^2 \quad I = \int_0^\infty \frac{1}{\sqrt{\pi N_0}} \exp\left(\frac{-\left(y_r - \sqrt{\frac{E_s}{2}}\right)^2}{N_0} \right) d$$



Using $Q(-x) = 1 - Q(x)$

$$= \left(1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right)^2$$

$$P_{e|1\,sent} = 1 - P_{c|1\,sent}$$

$$= 1 - (\quad)^2$$

$$= \cancel{2Q\left(\sqrt{\frac{E_s}{N_0}}\right)^2\right) - Q^2}$$

$$= 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) - Q^2\left(\sqrt{\frac{E_s}{N_0}}\right)$$

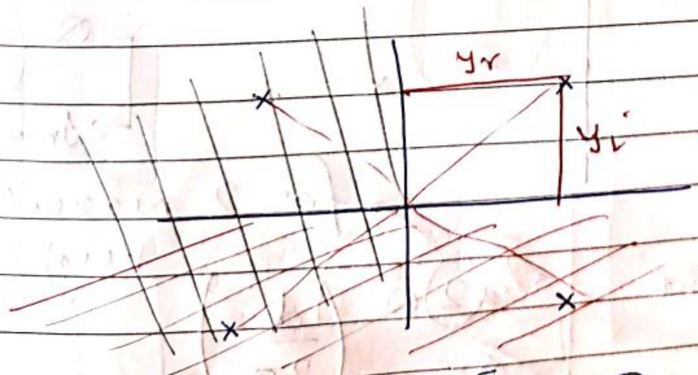
$$P_{total\ error} = P_e = P_{e|1\,sent}$$

$$\boxed{P_e = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q\left(\left(\sqrt{\frac{2E_b}{N_0}}\right)\right)^2}$$

Using $E_b = \frac{E_s}{2}$

Exact expression

Probability of Error using Union bound



$$P_{e|1\,sent} = \mathbb{P}[y_i < 0 \text{ or } y_r < 0]$$

$$\leq \mathbb{P}[y_i < 0] + \mathbb{P}[y_r < 0]$$

Given 1 sent

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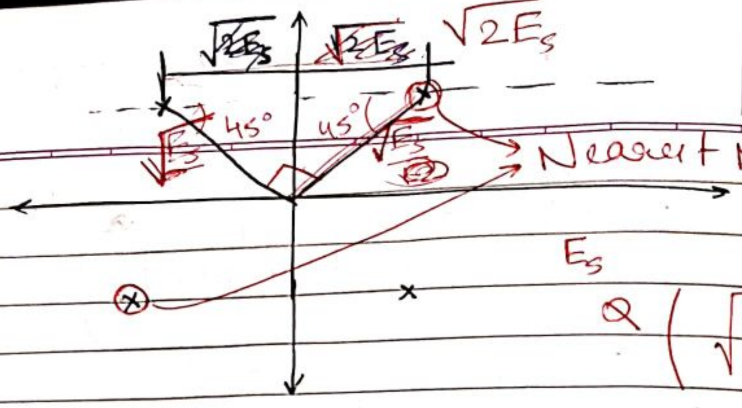


E_s

$$Q\left(\sqrt{\frac{E_s}{2}}\cdot\frac{\sqrt{2}}{\sqrt{N_0}}\right) \quad \leftarrow d/2$$

$\Rightarrow$ Error bound $= Q\left(\sqrt{\frac{2\sqrt{2}E_s}{N_0}}\right)$ $\rightarrow$ wrong ?

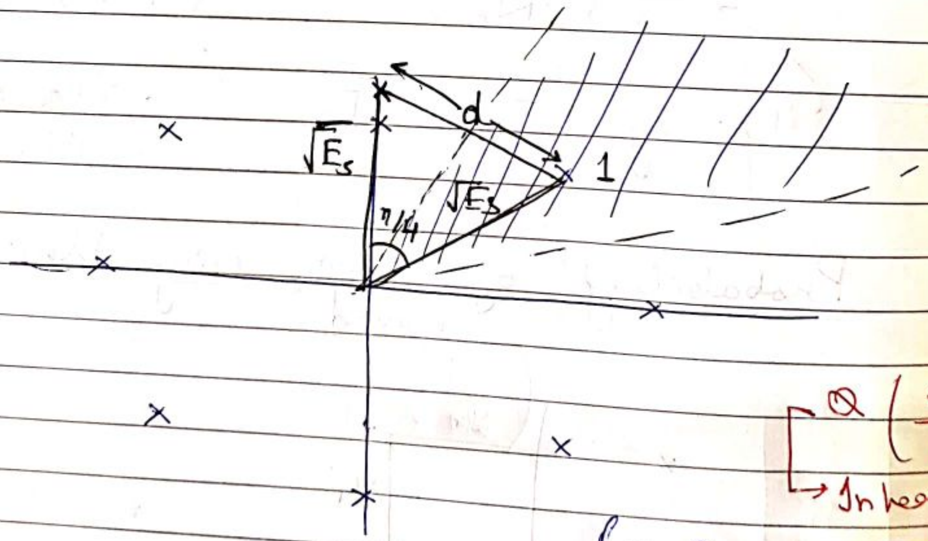
$\cancel{E_s =}\ E_b = \frac{E_s}{2}$

wrong ? $\leftarrow$ $Q\left(\sqrt{\frac{4\sqrt{2}E_b}{N_0}}\right)$

$E_s = 2E_b$

$$\boxed{2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)}$$

Approximation for 8-PSK

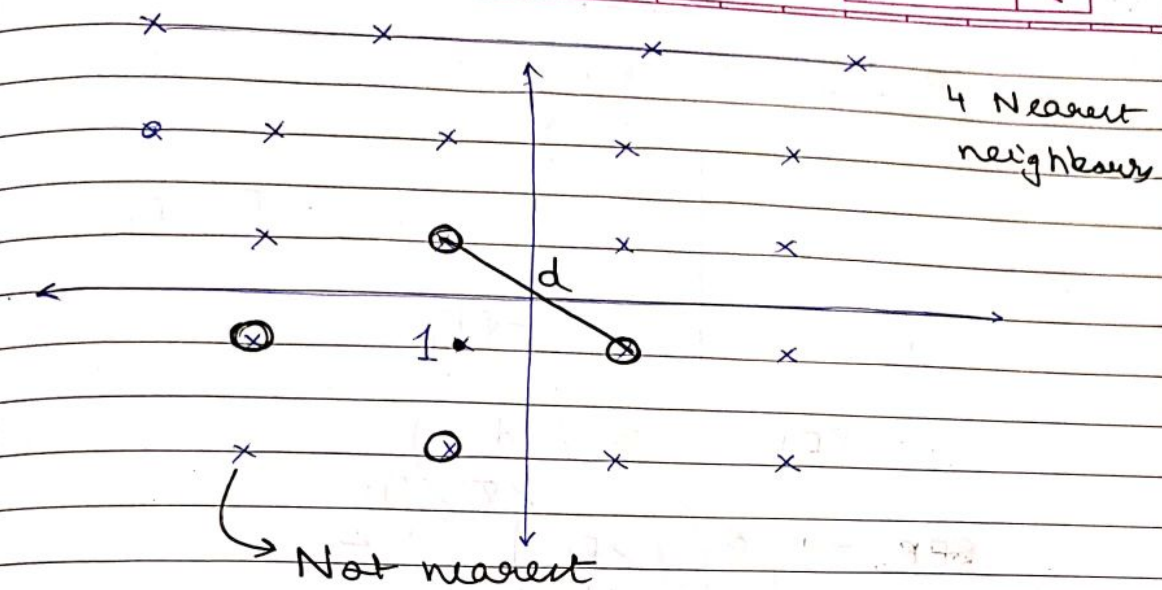


$Q\left(\frac{d/2}{\sigma}\right)$ $\rightarrow$ Inherited from BPSK

$P_{e|1\,sent} \le 2Q\left(\frac{d}{2\sigma}\right)$ — for 2 nearest neighbours

$\le$ → Union bounds

$\sigma^2 = \frac{N_0}{2}$, $\sigma = \sqrt{\frac{N_0}{2}}$



$$P_{e|1\,sent} = \underbrace{4}_{4NN} \times Q\left(\sqrt{\frac{2d}{N_0}}\right)$$

$$P_{e|i\,sent} = P_e \text{ (Symmetry)}$$

$$\Rightarrow P_e \le \underbrace{M}_{\text{No. of nearest neighbours.}} Q\left(\frac{d/2}{\sigma}\right)$$

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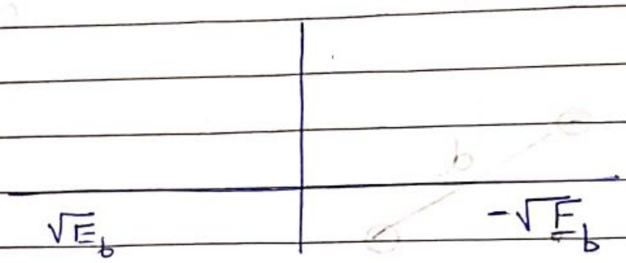
BER / SER

→ Binary Signalling

$E_s = E_b$



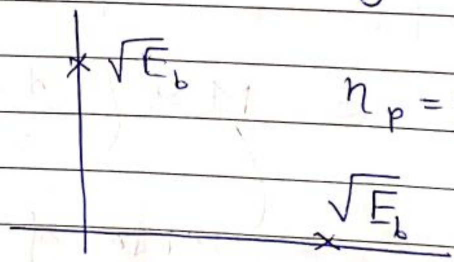
$$BER = Q\left(\frac{d}{2\sigma}\right)$$

$\sigma \rightarrow \sqrt{\frac{N_0}{2}}$

$$BER = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

$$Q\left(\sqrt{\frac{\eta_p E_b}{2N_0}}\right)$$

→ Orthogonal Signalling

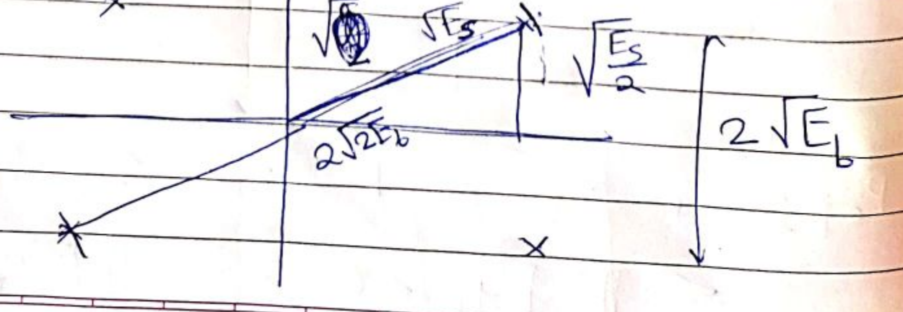


$\eta_p = 2$

$$BER = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

→ QPSK

$E_s = 2E_b$



$$SER = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q\left(\sqrt{\frac{2E_b}{N_0}}\right)^2$$

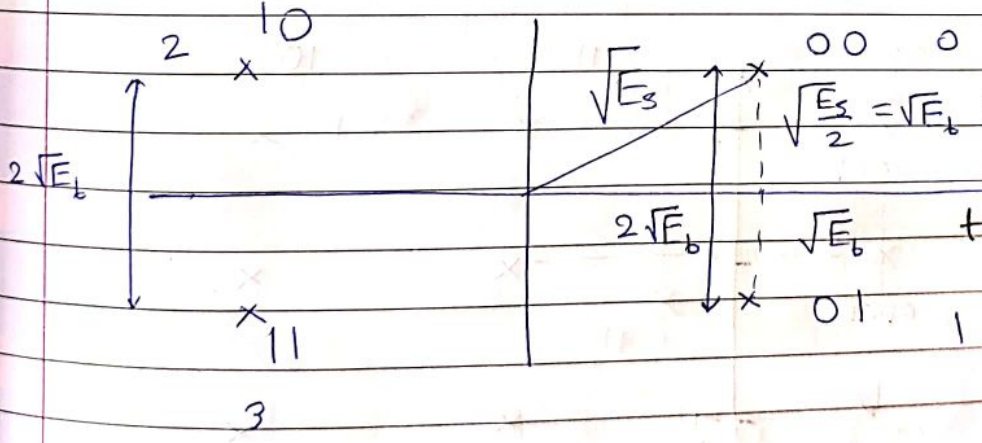
$$\leq 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Define

Efficiency / Metric $= \eta_p = \dfrac{d^2}{E_b}$

Gray Coding

→ A single symbol error should ideally cause only 1 bit error



In this coding real part corresponds to 1 bit im. to other

→ Can be viewed as 2 - BPSK Systems

$N_0 \rightarrow n_i + j n_q$ → project on any line, get $\frac{N_0}{2}$

n_i ↑ $\frac{N_0}{2}$, n_q ↑ $\frac{N_0}{2}$

N_0 ↓ 2-D Gaussian Noise

→ Error rate using Real/imaginary projection

$$Q\left(\frac{d}{2\sigma}\right) = Q\left(\frac{2\sqrt{E_b}}{2\sqrt{\frac{N_0}{2}}}\right) = \underbrace{Q\left(\sqrt{\frac{2E_b}{N_0}}\right)}_{\text{Same as BPS}}$$

→ Why the Same?

- Think in terms of power $E_s = 2E_b$ and distance going from $2\sqrt{E_b} \to \sqrt{2E_b}$

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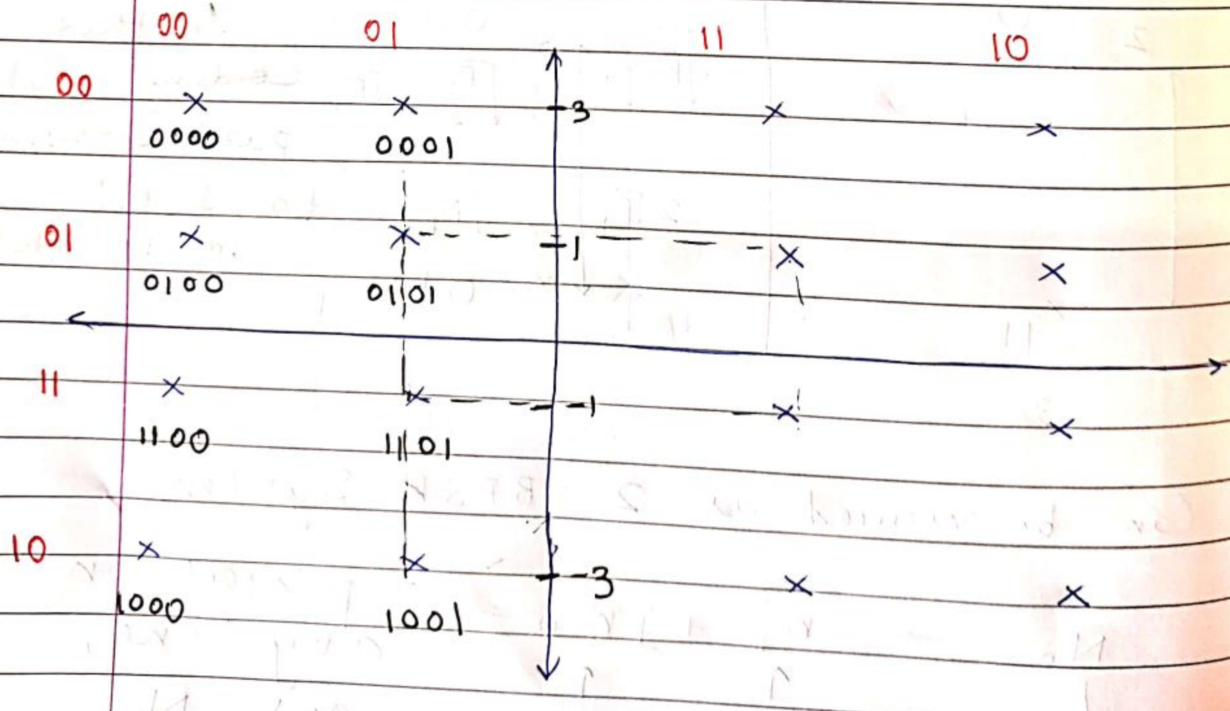
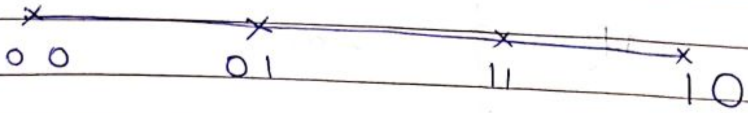
ex QAM-16 Gray Coding



Gray → Min BER for fixed SER

Consider QAM-4



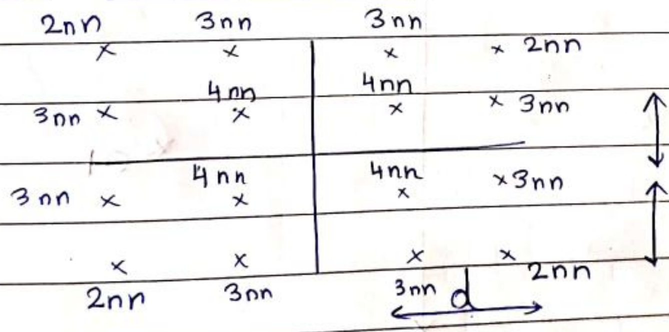
Average Energy (E_s)

$$E_s = \frac{1}{16}\left(12\times 3^2 + 4\times 1^2\right)$$

$$= \frac{\cancel{12}^{3}\times 9 + \cancel{4}1}{\cancel{16}4} = \frac{28}{4} = 7$$

↳ wrong

QAM-16 Union Bound

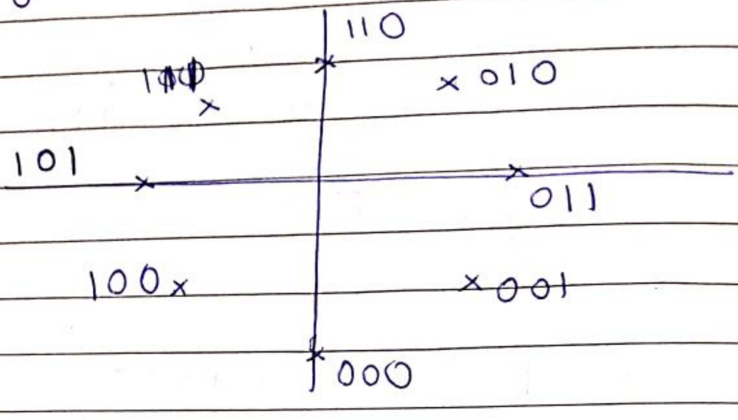


$d = \sqrt{\frac{2}{5}E_s}$

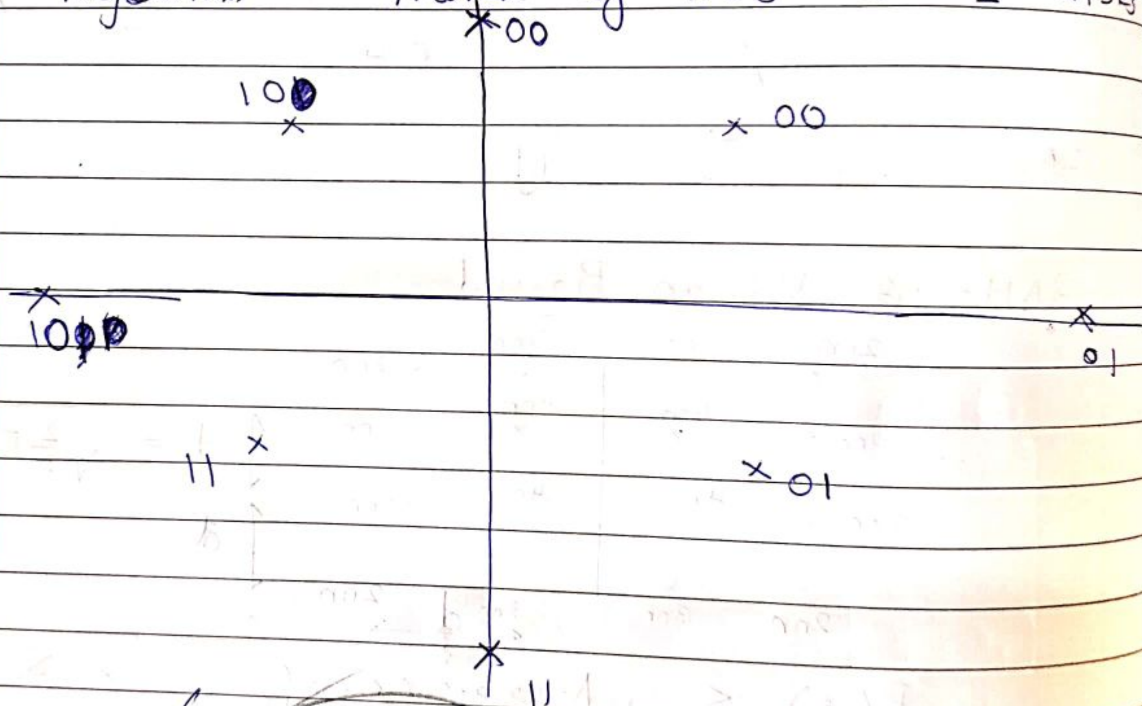
→ $P(e) \le$ Average no. of nearest neighbour $\times\, Q\left(\frac{d/2}{\sigma}\right)$

here $P(e) \le 3Q\left(\frac{d/2}{\sigma}\right)$

→ Gray code for 8 - PSK



→ Algorithm → Think of this as 2 - QPSKs

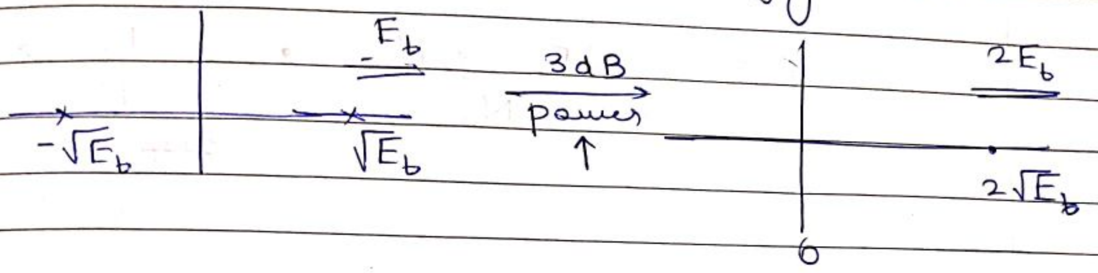


Do this.

$$10 \log_{10} 2 \rightarrow 0.3 \quad = 3\,dB$$

→ Rotation of Constellation has no effect on performance.

→ Shifting of Constellation gives same performance at higher energy cost



$$E_s = E_b \log_2 M$$

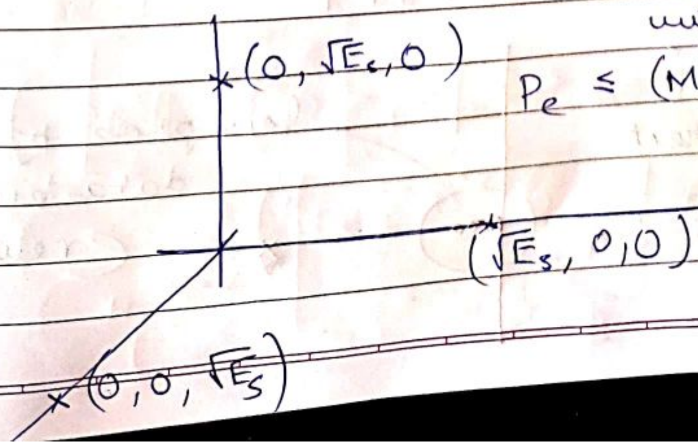
Orthogonal Signalling

$$\vec{s}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \sqrt{E_s} \qquad \vec{s}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \sqrt{E_s} \qquad \vec{s}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \sqrt{E_s}$$

$$\ldots\ldots \quad \vec{s}_M = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \sqrt{E_s}$$

$$\frac{\text{no. of bits}}{\text{dim}} = \frac{\log_2 M}{M} \qquad \frac{\log_2 M}{M} \xrightarrow[M \to \infty]{} 0$$

→ Union bound for Orthogonal Signalling



Union bound will give

$$P_e \leq (M-1)\, Q\left(\frac{\sqrt{2E_s}}{\sqrt{2N_0}}\right)$$

→ Here union bound is very loose, we exact expression

$$P_e = (M-1)\int_{-\infty}^{\infty} \left(\Phi(x)\right)^{m} \Phi(m-x)\, \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx$$

$$m = \sqrt{\frac{2E_b \log_2 M}{N_0}}, \quad \text{if } \frac{E_b}{N_0} > -1.6\,\text{dB}$$

~~SNR~~ BER ↓ as M ↑

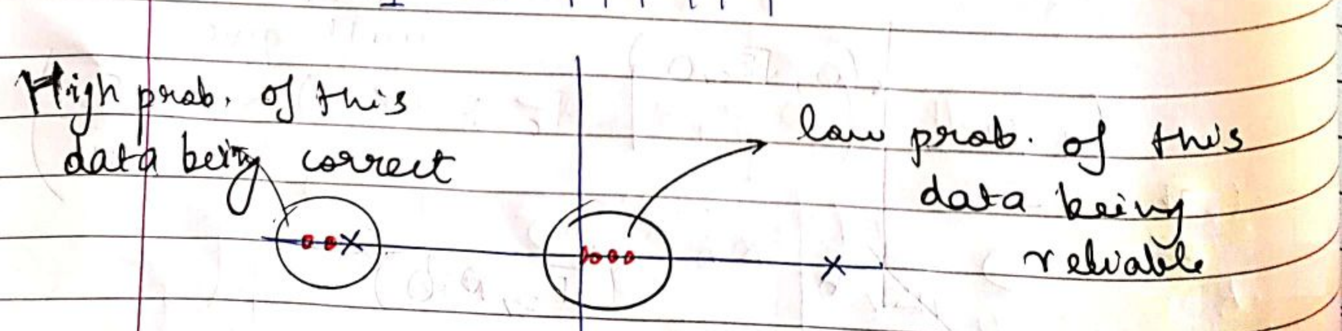
BER for orthogonal Signalling

- There is no Gray coding, all are equidistant
- To get BER consider all cases and no. of bits flipped
- or $\frac{M}{2}$, $\frac{M}{2} - 1$ reasoning per bit

Soft Decisions

→ ex, given redundancy such that, 6 times same bit sent

0 → 000000
1 → 111111



Hard Decision $\frac{4}{6}$ $\rightarrow +1$ $\Rightarrow +1$

Soft " $\rightarrow -1$

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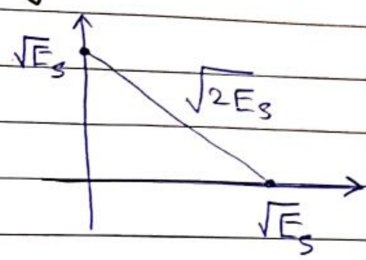
Orthogonal Signalling

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots\cdots\cdots \quad \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

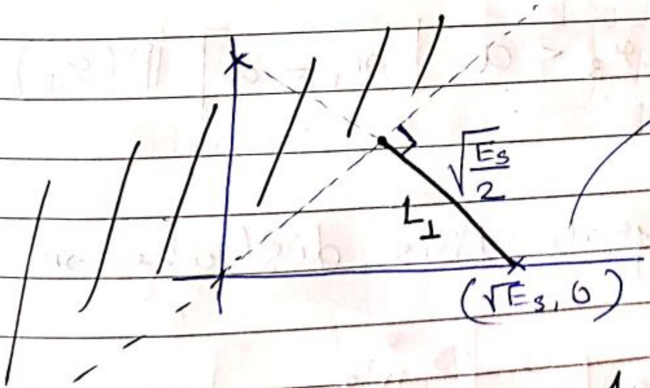


2-D Case : ~~B/E~~ SER $= Q\left(\frac{\sqrt{2E_s} \times \sqrt{2}}{2\sigma \sqrt{N_0}}\right)$

$\hookrightarrow \sigma^2 = \frac{N_0}{2}$

$$= Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

→ Sent $(\sqrt{E_s}, 0)$, What is received distribution :



Gaussian : mean : $\begin{bmatrix} \sqrt{E_s} \\ 0 \end{bmatrix}$

covariance : $\begin{bmatrix} N_0/2 & 0 \\ 0 & N_0/2 \end{bmatrix}$

$\hookrightarrow$ Instead of 2D Integral, integrate along L_1 from $\sqrt{\frac{E_s}{2}}$ to ∞

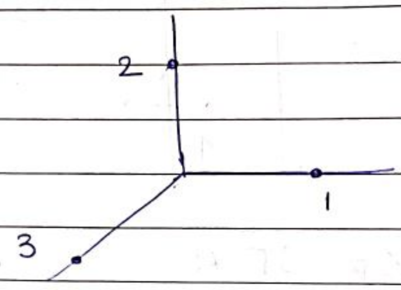
$$\int_{\sqrt{\frac{E_s}{2}}}^{\infty} \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi}\,\sigma}\, dx$$

→ Another approach,

Correct if $x > y$

<u>3-D</u>

Correct if $x > y$ & $x > z$



$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} \sqrt{E_s} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$\mathbb{P}\left[r_1 = \max_{1,2,3} r_i\right] =$$

$$\underbrace{\mathbb{P}\left[r_2 < a,\ r_3 < a \mid r_1 = a\right]}\ \mathbb{P}(r_1)$$

→ Integrate this distribution

$$= \mathbb{P}(r_2 < a \mid r_1 = a)^2\ \mathbb{P}(r_1)$$

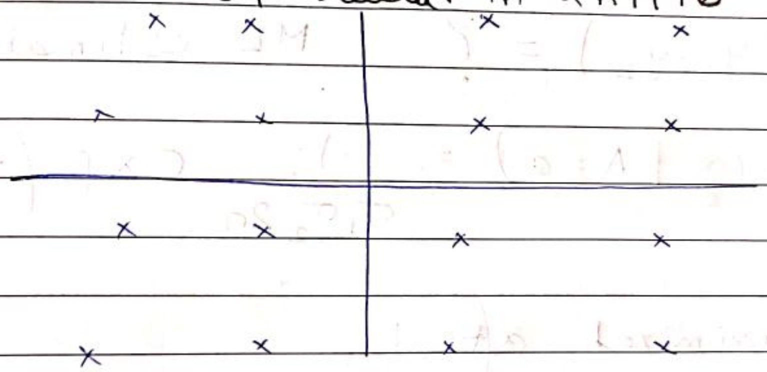
$$\int_{-\infty}^{\infty}\left[\underbrace{\int_{-\infty}^{y} \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}\, dx}_{\Phi}\right] \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\sqrt{E_s})^2}{2\sigma^2}}\, dy$$

Link - Budget Analysis

1. Data rate requirement
2. Bandwidth
3. Power
4. (N₀W)

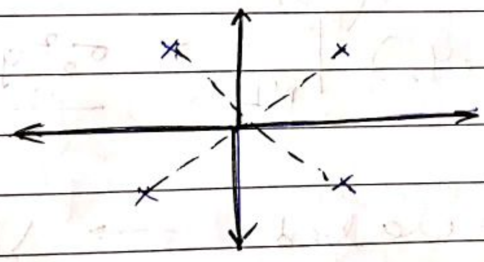
→ Difference b/w QPSK & QAM 16

complicated decision bound. in QAM16



• Channel also introduces amplitude variations

Vs



Chap 4 .... Synchronisation

→ Issues of equalization, synchronisation . . . .

18.2.19

$y_1 = A + w_1$

$y_2 = A + w_2$

w_1, w_2 are jointly Gaussian mean 0
covariance C_w (Invertible)

$$y_1 = A + n_1 \rightarrow \sigma_1^2$$
$$y_2 = A + n_2 \rightarrow \sigma_2^2$$
Independent

$\hat{A}(y_1, y_2) = ?$ ML estimate

$$f_Y(y \mid A = a) = \frac{1}{\sigma_1 \sigma_2 2\pi} \exp\left(\frac{-\left(y - \begin{bmatrix} a \\ a \end{bmatrix}\right)^T \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} \left(y - \begin{bmatrix} a \\ a \end{bmatrix}\right)}{2}\right)$$

Maximized at

$$\hat{A}(y_1, y_2)\Big|_{ML} = \frac{\sigma_2^2 y_1 + \sigma_1^2 y_2}{\sigma_1^2 + \sigma_2^2}$$

→ Reduce Correlated → Uncorr

$w_1, w_2 \rightarrow$ correlated

$$C_w = U \Lambda U^H$$

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$= (U \lambda^{1/2})(\lambda^{1/2} U^H)$$

$$= (U \lambda^{1/2})(U \lambda^{1/2})^H$$

Called square root of a Matrix

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If $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$(U\lambda)^{1/2}\, y \rightarrow (0, \lambda)$$

Pre multiply to get uncorrelated r.v.s

$$\ldots)^T \begin{pmatrix} \sigma_1^2 & 0 \\ & \sigma_2^2 \end{pmatrix} \left(y - \begin{bmatrix} a \\ a \end{bmatrix} \right) \Big) \Big)$$

Channel Modifies Signal

→ Consider a Real AWGN channel

real number

$$y = A x + w$$

• Get an optimal estimate of A given all the samples of y

→ We want to estimate para. θ

y depends on θ

$$\hat{\theta}_{ML}(y) = \arg\max_{\theta} P(y \mid \theta)$$

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Consider a new problem

$$y_1 = A + w_1$$
$$y_2 = A + w_2$$
$$\vdots$$

$$\hat{A}_{ML}(\bar{y}) = \arg\max_{A} p(\bar{y} \mid A)$$

If $w_1, w_2, \ldots$ are iid

$$\hat{A}_{ML}(\bar{y}) = \arg\max_{A} \prod_{i=1}^{n} p(y_i \mid A)$$

$$\boxed{\frac{\Sigma y_i}{n}} \quad = \arg\max_{A} \cancel{\frac{1}{\sqrt{}}} \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(y-a)^2}{\sigma^2}\right)$$

$$y = A\underset{\text{unknown}}{}b + w \qquad (\text{BPSK})$$

1) ~~b is~~ b is known, ~~to be~~ ~~be~~ ~~±1~~

$$\cancel{\hat{A}_{ML} = \hat{A}_{ML}\Big|_{\text{previous}}}$$

$$\hat{A}_{ML}\,(b \text{ known}) = \frac{y}{b}$$

↳ exactly

2) $$y = \underbrace{A}_{\text{unknown}}\overset{\pm 1\ (\text{unknown but lies in } \pm 1)}{b} + \underset{N(0,\sigma^2)}{w}$$

$$\hat{A}_{ML} = ?$$

$$p(y \mid A = a) = \frac{1}{2}\Big(p(y \mid A = a, b = +1) + p(y \mid A = a, b = -1)\Big)$$

$$p(y \mid A = a, b = +1)\,\tfrac{1}{2} + (\quad)\,\tfrac{1}{2}$$

$$\frac{1}{2\sigma\sqrt{2\pi}}\left(\exp\left(\frac{-(y-a)^2}{2\sigma^2}\right) + \exp\left(\frac{-(y+a)^2}{2\sigma^2}\right)\right)$$

~~Shifted w by <A>~~

Maximized for which a?

$$\frac{\cancel{-2}(y-a)}{\cancel{2\sigma^2}} \cdot \exp\left(-\frac{(y-a)^2}{2\sigma^2}\right) + \frac{\cancel{-2}(y+a)}{\cancel{2\sigma^2}}\exp\left(\frac{-(y+a)^2}{2\sigma^2}\right) = 0$$

→ Solve for a in terms of y

Q3)

$$y_1 = Ab_1 + w_1 \quad W(0, \sigma^2)$$
$$y_2 = Ab_2 + w_2 \quad W(0, \sigma^2)$$
$$\vdots$$
$$y_N = Ab_N + w_N$$

: } BPSK

$$\hat{A}_{ML} = \frac{1}{N}\sum_{i=1}^{N}\frac{y_i}{b_i}$$

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19.2.1

unknown

unknown

(~~Blind~~ estimation)

Q4) $y_1 = Ab_1 + w_1$

$y_2 = Ab_2 + w_2$

$\vdots$

$y_N = Ab_N + w_N$

$$\hat{\theta}_{ML}(\bar{y}) = \arg\max_{\theta} p(\bar{y}|\theta)$$

$$\arg\max_{\hat{A}_{ML}} \prod_{i=1}^{N} \exp\left(\frac{-(y_i - b_iA)^2}{2\sigma^2}\right) = \hat{A}_{ML}$$

$\rightarrow$ ~~argmax~~

↓ Take log (Strictly Increasing)

Minimizes this $\longrightarrow \sum_{i=1}^{N} \frac{(y_i - Ab_i)^2}{2\sigma^2}$

→ Substitute known y_i, b_i (after conditioning on b_i)

Get, $$\hat{A}_{ML} = \frac{\sum y_i \tanh\left(\frac{a y_i}{\sigma^2}\right)}{n}$$

19.2.19

$b[n] \in \{\pm 1\}$

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$y[0] = Ab[0] + w[0]$

$\vdots$

$y[K-1] = Ab[K-1] + w[K-1]$

$\hat{A}_{ML} = \frac{1}{K} \sum_{k=0}^{K-1} y[k]\, b[k]$ (or divide (squaring works because ± 1))

→ $b[k]$'s are unknown

$y[0] = b[0]A + w[0]$

~~$y[k+1] = b[$~~

$y[k-1] = b[k-1]A + w[k-1]$

$$\mathbb{P}[\vec{y} \mid A = a] = \sum_{b} \mathbb{P}(\vec{y} \mid \vec{b}, A = a) \frac{1}{2^K}$$

The tanh formula

$$a = \frac{1}{K} \sum_{k=0}^{K-1} y[k] \underbrace{\boxed{\tanh\left(\frac{a\, y[k]}{\sigma^2}\right)}}_{\approx b}$$

(since ÷ both sides g is same as x both sides ($b = \pm 1$))

$$\tanh(x) = \left(\frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}\right)$$

$$a = \tanh\left(\frac{y\, a}{\sigma^2}\right) \quad \approx a^2 b$$

(⇒ sign of a is linked to sign of b)

θ

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→ Earlier : write the pdf, estimate the unknown parameter.

$$Y \sim \mathcal{N}(\underbrace{\theta}_{\text{want this}}, \sigma^2)$$

Set A,
$$\mathbb{P}[Y \in A] = \int_{y \in A} f_Y(y)\,dy$$

$$\int_A \frac{f_Y(y)}{q(y)} \underbrace{q(y)}_{\text{Not depend on } \theta} dy$$

θ

Maximize this

Like expressing pdf of y in terms of another pdf

since q ind. of θ

$$\Rightarrow \underset{\theta}{\arg\max} \frac{f_Y(y)}{q(y)}$$

$$= \underset{\theta}{\arg\max}\, f_Y(y)$$

→ Requirement : $q(y) > 0$ whenever $f_Y(y) >$

$$\int_A L(y|\theta)\, q(y)\,dy$$

A Ratio of PDFs

Likelihood function (When q is also a pdf)

ex

$$\underset{\hat{\theta}}{\arg\max} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(y-\theta)^2}{2\sigma^2}\right)$$

$$= \underset{\hat{\theta}}{\arg\max} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(y-\theta)^2}{2\sigma^2}\right) \Big/ \exp($$

Likelihood & Log of LR Ratio

$$\text{Sign}\left(\log \left(\frac{p(y \mid +1)}{p(y \mid -1)} \right) \right)$$

$\log(\cdot) \rightarrow$ LLR

$$y[0] = A + w[0]$$
$$\vdots$$
$$y[k-1] = A + w[k-1]$$

$$\vec{\mu} = a \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \qquad \mathrm{Cov} = \sigma^2 I_{K \times K}$$