→ Receive Filtering

$$y(t) = s(t) + z(t)$$

Write $y(t) =$

Theorem 3.A.3, Madhow

→ MLE decision rule :- $\underset{1 \le m \le M}{\arg\max} \; Re\{ \langle r(t), s_m(t) \rangle \} - \frac{1}{2}\|s_m(t)\|^2$

5/2

→ $\boxed{\text{ML Rule :-}}$

Choose $i = \boxed{\arg\min_i \|y - s_i\|^2}$ --- Geometrically closest

$$= \arg\min_i \left(\cancel{\|y\|^2} + \|s_i\|^2 - 2\,Re\langle y, s_i \rangle \right)$$

$$= \boxed{\arg\max_i \left(Re\langle y, s_i \rangle - \frac{\|s_i\|^2}{2} \right)}$$

→ $\boxed{\text{Minimum Error Probability rule}}$

Choose $i = \boxed{\arg\max_i \left(Re\langle y, s_i \rangle - \frac{\|s_i\|^2}{2} + \sigma^2 \log(\pi_i) \right)}$

σ^2 → noise variance $\left(\frac{N_0}{2}\right)$

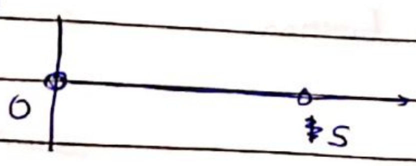
eg For QPSK, $\|s_i\|^2$ is same

~~∴ Answer for ML :- Geometrically~~

eg On-Off Keying (OOK)

$0 \rightarrow 0$

$1 \rightarrow s\,\varphi_1(t)$



Dimension of signal space = 1, basis = $\{\varphi_1(t)\}$

Received signal $y(t) = s(t) + n(t)$

Obtain 'y' by projecting $y(t)$ onto $\underrightarrow{\psi}(t)$ (~~which is~~

Here, since $\underrightarrow{\psi}(t)$ is one-dimensional, $y = \int y(t)\, \psi_1(t)\, dt$

We want to guess $s(t)$ from y.

- If $s_i(t)$ are equiprobable.

$$P_{error} = P_{error|0\ sent} \underbrace{P_{0\ sent}}_{1/2} + P_{error|1\ sent} \underbrace{P_{1\ sent}}_{1/2}$$

Hypothesis 0 :- $H_0 \equiv y(t) = n(t)$

1 :- $H_1 \equiv y(t) = s\cancel{(t)} + n(t)$

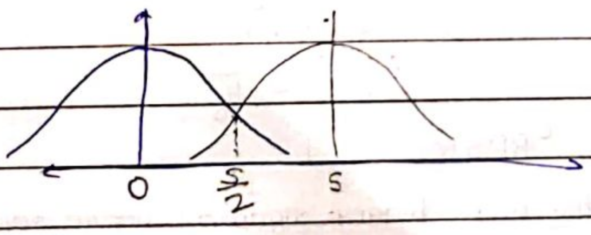
$\underrightarrow{y} = y\psi_1$

$s_i = 0$ or $s\psi_1$

- ~~If 0 was sent, received,~~ ~~For H_0~~
  - If 0 was sent, $\mathrm{Re}\langle y, s_i\rangle - \|s_i\|^2/2 = 0 - 0 = 0$
  - 1 $\quad sy - \frac{s^2}{2}$

• We guess 1 if $sy > \frac{s^2}{2} \Rightarrow y > \frac{s}{2}$

0 < <



• $P_{error|0\ sent} = \mathbb{P}\left(y > \frac{s}{2} \,\middle|\, 0 \text{ was sent}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_{s/2}^{\infty} e^{-x^2/2\sigma^2}\, dx$

$= Q\left(s/2\sigma\right)$

$= Q\left(\frac{\sqrt{2E_b}}{2\sigma}\right)$

$= \mathbb{P}_{error|1\ sent}$

$*$ E_s = Energy per symbol $= \overset{\text{bits in a symbol}}{n} E_b$

E_b = Energy per bit.

$0 \to 0$, $s \to s^2 \Rightarrow E_b = \frac{s^2}{2}$

* Binary Signaling

$0 \rightarrow s_1 \qquad H_1 \equiv y(t) = s_1 + n(t)$

$1 \rightarrow s_2 \qquad H_2 \equiv y(t) = s_2 + n(t)$

$$\therefore \quad E_b = \frac{s_1^2 + s_2^2}{2}$$

– Decision :-

$$\langle y, s_1 \rangle - \frac{\|s_1\|^2}{2} \underset{H_2}{\overset{H_1}{\gtrless}} \langle y, s_2 \rangle - \frac{\|s_2\|^2}{2}$$

ie $\rightarrow$ $$\langle y, s_1 - s_2 \rangle \underset{H_2}{\overset{H_1}{\gtrless}} \frac{|s_1|^2 - |s_2|^2}{2}$$

– Mathematical simplification

Define $\tilde{y}(t) = y(t) - s_2(t)$

$$\therefore \quad H_1 \equiv \tilde{y}(t) = s_1(t) - s_2(t) + n(t)$$

$$H_2 \equiv \tilde{y}(t) = n(t)$$

This is just like OOK

$$\therefore \quad \mathbb{P}_{error / s_2 \text{ is sent}} = Q\left(\frac{|s_1 - s_2|}{2\sigma}\right) = \mathbb{P}_{error / s_1 \text{ is sent}}$$

– To minimize this, maximize $|s_1 - s_2|$

$\therefore$ Under E_b constraint,

choose $s_1 = \sqrt{E_b}$

$s_2 = -\sqrt{E_b}$

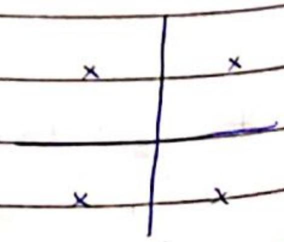
'BPSK'

This is the best binary signaling under energy constraint for equiprobable symbols.

eg QPSK

$$E_s = 2E_b = \frac{s_1^2 + s_2^2 + s_3^2 + s_4^2}{4}$$

For farthest spacing, $\|s\| = \sqrt{E_s}$



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7/2 If symbol ~~has 2-bits~~ is QPSK, $\underbrace{\begin{bmatrix} y_r \\ y_i \end{bmatrix}}_{1\ \text{symbol}} = \underbrace{\begin{bmatrix} \sqrt{E_s/2} \\ \sqrt{E_s/2} \end{bmatrix}}_{\sqrt{\frac{E_s}{2}} e^{j\pi/4}} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$ $n_1 \to G(0, N_0/2)$, $n_2 \to G(0, N_0/2)$ independent

$\therefore \begin{bmatrix} y_r \\ y_i \end{bmatrix}$:- Jointly Gaussian : Mean $= \begin{bmatrix} \sqrt{E_s/2} \\ \sqrt{E_s/2} \end{bmatrix}$, Covariance $= \begin{bmatrix} N_0/2 & 0 \\ 0 & N_0/2 \end{bmatrix}$

$\therefore$ Distribution :- $f_{\underline{y}}(y_r, y_i) = \dfrac{1}{(\sqrt{2\pi\sigma^2})^2} e^{\frac{-(y_r-\mu_r)\Sigma^{-1}(y_i-\mu_i)}{2}}$

$$= \frac{1}{2\pi\left(\frac{N_0}{2}\right)} e^{-\frac{\|\underline{y}-\underline{\mu}\|^2}{2\left(\frac{N_0}{2}\right)}} \qquad \Sigma^{-1} \Rightarrow \frac{1}{N_0/2}$$

Integrate Gaussian over correct area $\therefore P_{correct \mid 1 \text{ is sent}} = \displaystyle\int_0^\infty\int_0^\infty f_{y_r y_i}(y_r, y_i)\, dy_r\, dy_i$

$$= \int_0^\infty \frac{1}{\sqrt{\pi N_0}} e^{-\frac{\left(y_r - \sqrt{\frac{E_s}{2}}\right)^2}{N_0}} dy_r \int_0^\infty \frac{1}{\sqrt{\pi N_0}} e^{-\frac{\left(y_i - \sqrt{\frac{E_s}{2}}\right)^2}{N_0}} dy_i$$

$$= \left(\int_0^\infty \frac{1}{\sqrt{\pi N_0}} e^{-\frac{\left(y - \sqrt{\frac{E_s}{2}}\right)^2}{N_0}} dy \right)^2$$

$$= Q^2\left(-\sqrt{\frac{E_s}{2N_0}}\right)$$

$$= \left(1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right)^2$$

$\therefore P_{error \mid 1 \text{ is sent}} = 1 - P_{correct \mid 1 \text{ is sent}}$

$$= 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) - \underbrace{Q^2\left(\sqrt{\frac{E_s}{N_0}}\right)}_{\text{Quite small}}$$

- For higher number of symbols (eg - ~~8A~~ 16-QAM), it is cumbersome to ~~fi~~ detect.

$\therefore$ Use union bound.

→ Union Bound Approximation

eg QPSK

$P_{error|1 \text{ is sent}} = P(y_r < 0 \ \text{OR} \ y_i < 0)$

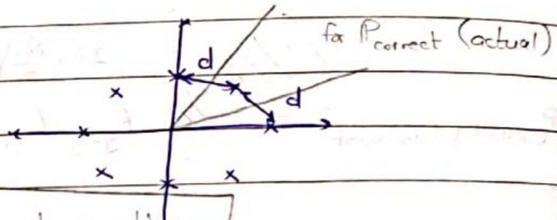
$\leq P(y_r < 0) + P(y_i < 0) = 1 - Q\left(-\sqrt{\frac{E_s}{N_0}}\right) + 1 - \ldots$

$\because P(y_r \ \& \ y_i < 0)$ is counted twice

$= 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) = 2Q\left(\frac{d}{2\sigma}\right)$ $\quad d = \sqrt{2E_s}$

eg 8-PSK

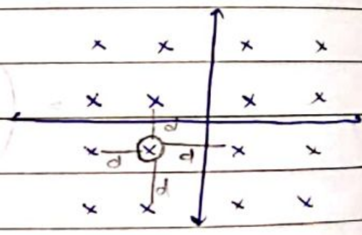
$P_{error|1 \text{ is sent}} \leq 2Q\left(\frac{d}{2\sigma}\right)$



$\boxed{\text{One } Q\left(\frac{d}{2\sigma}\right) \text{ for each nearest neighbour}}$

eg 16-QAM

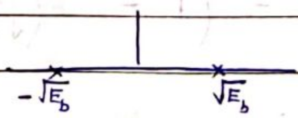
$P_{error|1 \text{ is sent}} \leq 4Q\left(\frac{d}{2\sigma}\right)$



$\sigma = \sqrt{\frac{N_0}{2}}$

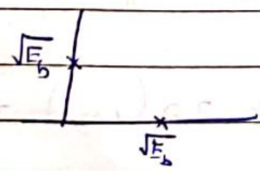
11/2

eg Binary Signalling



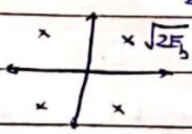
$BER = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

Orthogonal Signalling



$BER = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$

QPSK



$SER = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\ldots}\right)$

$\leq 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

→ $\eta_r \triangleq \dfrac{d^2}{E_b}$

→ Bit Mapping

- Arbit coding for QPSK

11 ×	× 00
10 ×	× 01

There will be 2 bit errors on horizontal direction misprediction.

Better - Gray Coding for QPSK : Neighbors differ by one bit.

10 ×	× 00
11 ×	× 01

d

MSB is determined by real part

LSB imaginary part.

For one adjacent neighbour, $Q\left(\dfrac{d}{2\sigma}\right) = Q\left(\dfrac{2\sqrt{E_b}}{2\sqrt{\frac{N_0}{2}}}\right) = Q\left(\sqrt{\dfrac{2E_b}{N_0}}\right)$

.... same expression as BPSK

(because the vertical/horizontal distance can be thought of as BPSK)

☆ ∴ For ~~QPS~~ same E_b and P_e, QPSK gives twice as much bit rate as BPSK

- Similarly, 16-QAM can be thought of as two separate PAM-4

11/2 • Bit mapping aims to reduce E_b. Does not affect E_s

- 16-QAM

	00	01		11	10
00	×	×	3	×	×
01	×	×	1	×	×
11	×	×	-1	×	×1110
10	×	×	-3	×	×

PAM-4 × PAM-4.

Normalize scale to obtain unit E_s

$$E_s = \frac{4}{16}(9+9) + \frac{8}{16}(1+9) + \frac{4}{16}(1+1) = 10$$

∴ Divide all lengths by $\sqrt{10}$

* Alternatively, find Energy of one PAM-4. Multiply by

Union Bound : $\mathbb{P}_e \leq \underbrace{\left(\text{No. of nearest neighbours}\right)}_{\substack{\text{average} = 3 \\ \text{(equiprobable case)}}} Q\left(\frac{d}{2\sigma}\right)$

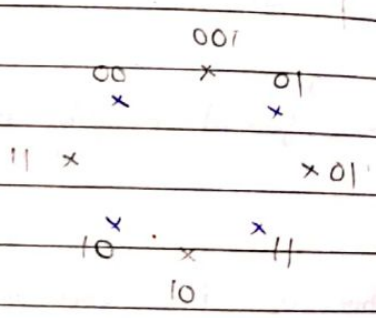
$$\mathbb{P}_e \leq 3Q\left(\frac{d}{2\sigma}\right)$$

$d = \frac{2}{\sqrt{10}} \times \sqrt{E_s}$ (for BPSK it was $\sqrt{E_s}$)

$\therefore$ QAM-16 is less immune to noise.

- Gray code for ~~QPSK~~ 8-PSK

Complete Consider 8-PSK as 2 QPSKs



→ Rotation of Constellation

- No change in performance.

- Shifting changes performance - Changes energy (E_b) requirement.

→ Orthogonal Signalling

- Poorer BW efficiency $= \frac{\log_2 M}{M}$

$s_1 = [1 \ 0 \ 0 \ \ldots]^T \sqrt{E_s}$

$s_2 = [0 \ 1 \ 0 \ \ldots]^T \sqrt{E_s}$

$\vdots$

Every s_i has $M-1$ nearest neighbours, each $\sqrt{2E_s}$ away.

Union bound :- $\mathbb{P}_e \leq (M-1)\, Q\left(\frac{\sqrt{2E_s}}{\sqrt{2N_0}}\right)$

- This bound is very loose for higher M. $\Rightarrow$ not useful
  - Use exact error instead

- Decision boundary (ML)

  For any point on $\mathbb{R}^M$, project it onto each of the axes and detect the symbol for whose axis the projection is closest to $\sqrt{E_s}$

- Exact $\mathbb{P}_e = (M-1)\int_{-\infty}^{\infty} (\Phi(x))^{M-2}\, \Phi(m-x)\, \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx$

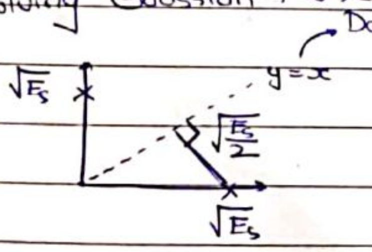
  where $m = \sqrt{\frac{2E_b \log_2 M}{N_0}}$

  * For orthogonal signalling, bit error can be made arbitrarily small for $\frac{E_b}{N_0} > -1.6$ dB

"Resolving Gaussian Noise ~~only~~

14/2

eg



For that symbol, $Y \sim$ Mean $\begin{bmatrix}\sqrt{E_s}\\ 0\end{bmatrix}$

Covariance $\begin{bmatrix}N_0/2 & 0\\ 0 & N_0/2\end{bmatrix}$

For calculating error, we resolve the (circularly symmetric) noise along perpendicular to $y=x$.

This resolved component has variance $\frac{N_0}{2}$

$$\mathbb{P}_{error} = \int_{\sqrt{\frac{E_s}{2}}}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-x^2/2\sigma}\, dx$$

which is same as we would have obtained by double integral on x, y

eg 3D orthogonal signalling

$$s_1 = \begin{bmatrix} \sqrt{E_s} \\ 0 \\ 0 \end{bmatrix}, \quad y_1 \sim \text{Mean} \begin{bmatrix} \sqrt{E_s} \\ 0 \\ 0 \end{bmatrix} \quad \text{Covariance: } \frac{N_0}{2} I_3$$

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} \sqrt{E_s} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$P_c = \mathbb{P}(r_1 > r_2, \; r_1 > r_3)$$

$$= \mathbb{P}(r_2 < a, \; r_3 < a \mid r_1 = a) \, \mathbb{P}(r_1 = a)$$

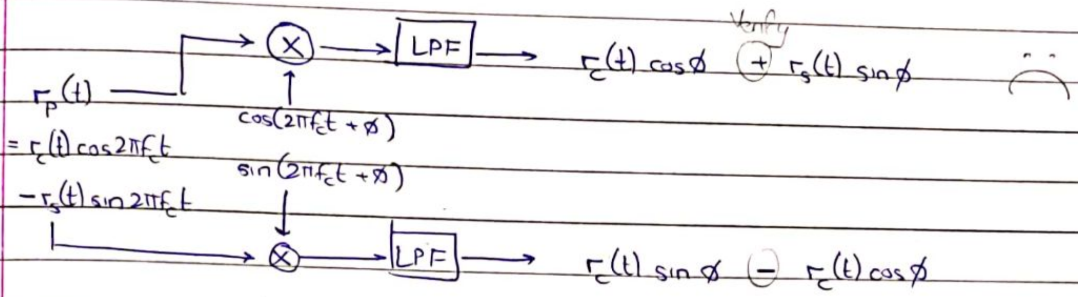
$$= \mathbb{P}^2(r_2 < a \mid r_1 = a) \, \mathbb{P}(r_1 = a)$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} dx \right)^{\overset{M-1}{2}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(a - \sqrt{E_s})^2}{2\sigma^2}} da$$

* Training sequence

- Transmitter periodically sends predecided sequence for training

Chap 4



- If f_c is also indefinite :- $r_c(t) \cos(2\pi \Delta f_c \tau) - r_s(t) \sin(2\pi \Delta f \tau)$