Reading - Chapter 2 upto Pg 41

3/1/19

* Circuit-switched :- Direct connection between every sender and receiver
× Packet-switched

* Inter-symbol interference - Fundamental limit because of medium. Medium acts as LPF and causes distortion of short-duration change in signal.

* GIF, PNG, JPG are ways of compression of bit-streams

→ Notations

- Vectors :- $\underline{r}, \underline{s}$

- Inner product :- $\langle \underline{s}, \underline{r} \rangle \triangleq z\, \underline{s}^H \underline{r}$ where $\underline{s}^H = \overline{\underline{s}^T}$

eg :- $\underline{s} = \begin{bmatrix} 1 \\ j \end{bmatrix}$ $\underline{r} = \begin{bmatrix} j \\ -1 \end{bmatrix}$, $\langle \underline{s}, \underline{r} \rangle = \begin{bmatrix} 1 & -j \end{bmatrix} \begin{bmatrix} j \\ -1 \end{bmatrix} = 2j$

- Cauchy Schwarz :- $|\langle \underline{s}, \underline{r} \rangle| \leq \|\underline{s}\| \|\underline{r}\|$ → equality if $s = kr$

where $\|\underline{s}\| = \sqrt{\langle \underline{s}, \underline{s} \rangle}$

- Indicator :- $I_{[-0.5, 0.5]}(t) = \text{rect}(t)$

- If $s_1(t)$ and $s_2(t)$ are square-integrable, then they form a normed vector space (can define an inner product)

~~Norm~~

Inner product $= \langle s_1, s_2 \rangle = \int_{-\infty}^{\infty} s_1(t)\, s_2^*(t)\, dt = \int_{-\infty}^{\infty} S_1(f)\, S_2^*(f)\, df$

- Parseval :- $s_1 = s_2$

- ~~Autocorrelation function~~ Proof :- $s_1(t)\, s_2(t) \longrightarrow S_1(f) * S_2(f)$

$s_1(t)\, s_2^*(t) \longrightarrow S_1(f) * S_2^*(-f)$

$= S_1(f) * S_2^*(f)$

... assuming ~~real~~

For LHS, put $f = 0$ in $S_1(f) * S_2^*(f)$

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Symmetry :- $Re(X)$, $|X|$ are even $\quad s(f) = s^*(-f)$
$Im(X)$, $\angle X$ are odd

→ Baseband signal :- $S(f) \approx 0$ for $|f| > W$
↳ because time-limited cannot be band-limited

→ Passband signal :- $S(f) \approx 0$ for $|f - f_c| > W$

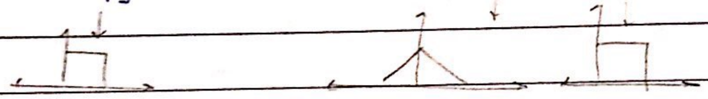
I Complex Baseband Representation.

* Practical analysis of ~~a~~ passband signal is mostly done by treating it as baseband signal.

- Baseband $s_s(t)$, $s_c(t)$ → Passband :- $\sqrt{2}\, s_c(t) \cos 2\pi f_c t - \sqrt{2}\, s_s(t) \sin 2\pi f_c t$
  Quadrature In-phase. $s_p(t)$ ↳ won't require halving

eg $I_{[a,b]}(x) = \begin{cases} 1 & \text{for } x \in [a,b] \\ 0 \end{cases}$

Consider $\sqrt{2}\, I_{[0,1]}(t) \cos 2\pi f_c t - \sqrt{2}(1 - |t|)\, I_{[-1,1]}(t) \sin 2\pi f_c t$



Almost band-limited when compared to f_c

- Complex baseband equivalent :- $s(t) = s_c(t) + j\, s_s(t)$
  → $s_p(t) = \sqrt{2}\, Re\left[s(t)\, e^{j 2\pi f_c t} \right]$
  BW of $s(t)$ also considers negative frequ

- Envelope :- $e(t) = \sqrt{s_c^2(t) + s_s^2(t)}$
  Phase :- $\theta(t) = \tan^{-1}\left(\frac{s_s(t)}{s_c(t)} \right)$

- Define $x_c(t) = s_c(t) \cos 2\pi f_c t$ and $x_s(t) = s_s(t) \sin 2\pi f_c t$
  such that $s_p(t) = \sqrt{2}\,(x_c(t) - x_s(t))$

Then, $x_c(t)$ and $x_s(t)$ are orthogonal :- $\langle x_c, x_s \rangle = 0$
Proof in book :- $\langle x_c, x_s \rangle = \int X_c X_s^* \, df$.

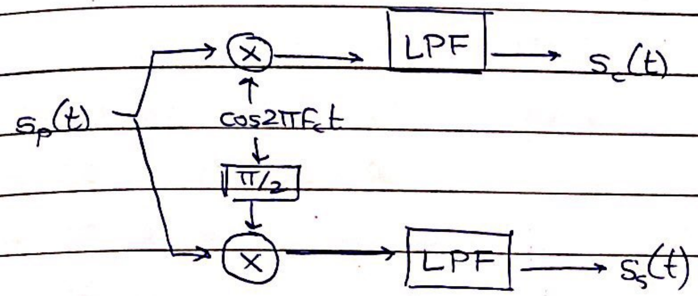
- $y(t) = \frac{1}{\sqrt{2}} (x(t) * h(t))$
- $\langle u_p, v_p \rangle = \langle u_c, v_c \rangle + \langle u_s, v_s \rangle = Re(\langle u, v \rangle)$

- $\|s_p\|^2 = \|s\|^2$ because of $\sqrt{2}$

  $\hookrightarrow \langle s, s \rangle = \langle s_c, s_c \rangle + \langle s_s, s_s \rangle \quad \because 1 \,\&\, j$ are orthogonal

  $= \|s_c\|^2 + \|s_s\|^2$

- $S(f) = \sqrt{2}\, S_p^+(f + f_c)$ and $S_p(f) = \frac{S(f-f_c) + S^*(-f-f_c)}{\sqrt{2}}$ *



8/1

II Power Spectral Density

- To check frequency spectrum, bandwidth :- Deterministic signal $\rightarrow$ FT

  Random $\rightarrow$ PSD.

- PSD at f_0 = Energy in $\left(f_0 - \frac{\Delta f}{2}, f_0 + \frac{\Delta f}{2}\right)$

* Consider time-limited $s_{T_0}(t) = s(t)\, I_{[-\frac{T_0}{2}, \frac{T_0}{2}]} \xleftrightarrow{FT} S_{T_0}(f)$

$$ESD = |S_{T_0}(f)|^2$$

$$PSD = \frac{1}{T_0} |S_{T_0}(f)|^2 \quad \text{for } T_0 \rightarrow \infty$$

$\rightarrow$ Autocorrelation :- $\hat{R}_s(\tau) = \frac{1}{T_0} \int_{-\infty}^{\infty} s_{T_0}(u)\, s_{T_0}^*(u-\tau)\, du.$

$$\bar{R}_s(\tau) = \lim_{T_0 \rightarrow \infty} \hat{R}_s(\tau)$$

→ Matched Filter :- $s_{MF}(t) = s^*(-t) \Rightarrow S_{MF}(f) = S^*(f)$

- Inner product :- $\langle x, s \rangle = (x * s_{MF})(0)$
- Auto correlation :- $R_{ss}(\tau) = (s * s_{MF})(\tau)$
- $s(t - t_0) * s_{MF}(t)$ is maximum at $t = t_0$

→ WSS :- 1) $E(x(t))$ is independent of t

2) $E(x(t_1)\, x^*(t_2))$ depends only on τ.

eg Consider $b[n] = \begin{cases} 1 & \text{wp } 1/2 \\ -1 & \text{wp } 1/2 \end{cases}$ iid for all n

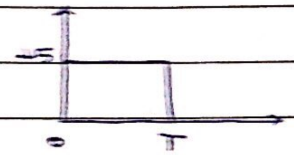
This is WSS

→ $\mu_b = 0$

→ $b[n]\, b[n+\tau] = \begin{cases} 1 & \text{wp } 1/2 \\ -1 & \text{wp } 1/2 \end{cases}$ if $\tau \neq 0$

and $= 1$ wp 1 if $\tau = 0$

* Define $p(t)$ as



Then transmission of bits $\equiv \underbrace{\sum_{n=-\infty}^{\infty} b[n]\, p(t - nT)}_{s(t)}$ .... continuous RP.

"Linear Modulation"

TFT :- BW required to transmit $s(t)$

Method :- Find ACF

☆ - Is $s(t)$ WSS? → Yes ☆ No? Wide-sense Cyclostationary

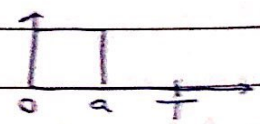
→ $\mu_s(t) = 0$

→ If t and $t+\tau$ lie in same period T, $s(t)s(t+\tau) = 1$

don't lie $E(s(t)\, s(t+\tau)) = 0$

→ depends on t

- We could have chosen a pulse differently



~~But~~ This would work for $a < T$, but take more BW than $a = T$.

This wouldn't work for $a > T$, due to inter-symbol interference.

III SAMPLING THEOREM

"Any signal $s(t)$ band-limited to $\left[\frac{-W}{2}, \frac{W}{2}\right]$ can be described completely by its samples $1/W$ time apart."

$$s(t) = \sum_{n=-\infty}^{\infty} s\left(\frac{n}{W}\right) p\left(t - \frac{n}{W}\right)$$

where $p(t) = \text{sinc}(Wt)$

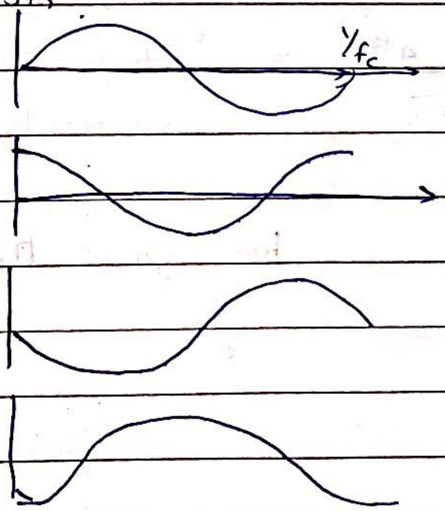
- Characteristics of $p(t) = \text{sinc}(Wt)$
  - It is band-limited
  - It is zero for multiples of $T_s = 1/W$ other than zero. Hence, at $t = \frac{k}{W}$, value of above summation will be $s\left(\frac{k}{W}\right)$

    $\therefore$ No inter-symbol interference :)

  - Since $p(t)$ is band-limited $\left[\frac{-W}{2}, \frac{W}{2}\right]$, so is $s(t)$, which is just a summation of such $p(t)$'s.

* In time interval T_0, we can send 'WT_0' worth of information
- If W is higher $\Rightarrow$ higher bit rate, but ~~to~~ higher bandwidth

- Transmission of Q-PSK



		Symbol	Phase
00	0	$-j$	~~90~~ -90
01	1	1	0
10	2	j	90
11	3	-1	180

Nyquist - Any signal band-limited to W is completely characterized by a minimum of W samples in one second (or WT_0 samples in T_0 seconds)

$\therefore WT_0$ = 'Degrees of freedom'

P.T.O.

	$\cos 2\pi f_c t$ Component	$\sin 2\pi f_c t$ Component	Symbol (Finite set of values of complex baseband signal)
00	0	1	$-j$
01	1	0	1
10	0	-1	j
11	-1	0	-1

Note :- $\mathrm{Re}\{\text{Complex baseband} \times e^{j2\pi f_c t}\}$ = correct waveform (real)

→ Bandwidth Efficiency (bits/(s)(Hz))

Say sequence $s\left(\frac{n}{W}\right)$ takes one of M values ~~(M=4 for QPSK)~~.

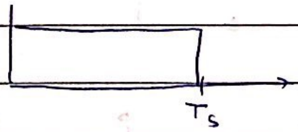
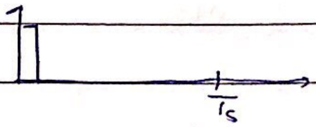
• $BE = \eta_B = \dfrac{\log_2 M}{D}$ — $\log_2 M$ → no. of bits; D → No. of dimensions (WT_0)

eg Consider sending 10 symbols in T_0 period by QPSK

2 bits 2 2

M = No. of possible combinations $= 4^{10}$

$\therefore \eta_B = \dfrac{\log_2 4^{10}}{WT_0}$ = 'Bits per second normalized to BW' = 'Spectral efficiency'

eg  has higher η_B than 

• Memoryless

• If $b[n]$ are real → 'Pulse amplitude modulation' (PAM)

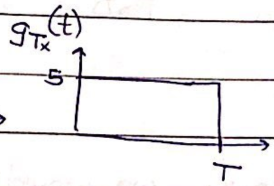
• $g_{Tx}(t) \rightarrow g_{Tx}(at) \Rightarrow G_{Tx}(f) \rightarrow G_{Tx}(f/a)$, Symbol rate : $\frac{1}{T} \rightarrow \frac{a}{T}$, BW :- $B \rightarrow aB$

$\therefore$ 'Normalized BW' $\triangleq BT$

IV → Linear Modulation

$$u(t) = \sum_{n} b[n]\, g_{Tx}(t - nT)$$

symbols from constellation
$(\{1, j, -1, -j\}$ for QPSK)



eg Example of non-linear :- $u(t) = \begin{cases} 1 & \text{if bit toggles} \\ 0 & \text{if bit is same as previous} \end{cases}$

This is not linear :- Waveform is not obtained by multiplication of symbols with pulse.

* Summary

Bit-stream → Symbols → Complex baseband signal → Passband signal

Bit-stream $\{0, 1, ...\}$; Symbols (Finite set of complex numbers); Complex baseband signal ($p(t)$, rate $= 1/T$, W); Passband signal (at f_c)

- Linear Modulation
  $s(t) = \sum b[n]\, p(t - nT)$
- $s(nT) = b[n]$
- W depends on choice of $p(t)$

$$u(t) = \sum_{n=-\infty}^{\infty} b[n]\, g_{Tx}(t - nT)$$

- TFT :- BW used to transmit $u(t)$.

- $u(t)$ is a random process because $b[n]$ are random variables.

P.T.O.

• Because of some headers, 'Information Rate' $\lesssim$ Bit rate

Theorem If i) All $b[n]$ are uncorrelated

ii) $E[b[n]] = 0$

Then :- PSD of $u(t) = \frac{1}{T} E(|b[n]|^2) |G_{Tx}(f)|^2$

$\rightarrow \propto |G_{Tx}(f)|^2$

• Ideally we would use $g_{Tx}(t)$ as sinc, so that $G_{Tx}(f)$ is a perfect rect, and because sinc is zero at non-zero integers.

– Not practical because sinc is prone to sampling errors.

• Practical BW $\triangleq$ 99% energy containment.

$$\frac{\int_{-W}^{W} |P(f)|^2 df}{\int_{-\infty}^{\infty} |P(f)|^2 df} \leq 0.99$$

Practical BW for $u(t)$ = Practical BW for $g_{Tx}(t)$

15/1 • sinc t decays as $1/t$. But we want faster decay :(

– Slow decay increases ISI

– If you want better decay $(1/t^2)$, use $\text{sinc}^2 t$

– $\downarrow$ ISI :)

– $\uparrow$ BW :(

A] ISI - free condition.

$$\rightarrow b[n] \rightarrow \boxed{g_{Tx}(t)} \rightarrow \boxed{g_c(t)} \rightarrow \boxed{g_{Rx}(t)} \rightarrow$$

$b[n]$ at rate $1/T$ (Channel) Substitute $t = nT$ to obtain $z[n]$ (Sampling)

$\Sigma b[n] \delta(t-nT)$ $\Sigma b[n] g_{Tx}(t-nT)$ $z(t) = \Sigma b[n] (g_{Tx} * g_c * g_{Rx})(t-nT)$

We want :- $z[n] = b[n]$ ..... 'ISI avoidance'

– All written signals are complex baseband representation.

– Let $(g_{Tx} * g_c * g_{Rx})(t) = x(t)$

At $t = nT$, we want $z(nT) = z[n] = b[n]$ ....

P.T.O.

$\therefore$ Condition on $x(mT) = \begin{cases} 1 & \text{for } m = 0 \\ 0 & \text{for } m \neq 0 \end{cases}$ 'δ_{m0}'.

... like sinc(t) or rect(t) or more

Then $z[n] = z(nT) = b[n]$ since $z(t) = \sum b[n]\, x(t - nT)$

- In practice, received $z[n]$ is not exactly $b[n]$. Received signal must be 'equalized'.

Equivalently, $x(t) \sum \delta(t - nT) = \delta(t)$

Fourier Transform :- $X(f) * \frac{1}{T} \sum \delta(f - \frac{n}{T}) = 1$.

$$\therefore \frac{1}{T} \sum X(f - \tfrac{n}{T}) = 1.$$

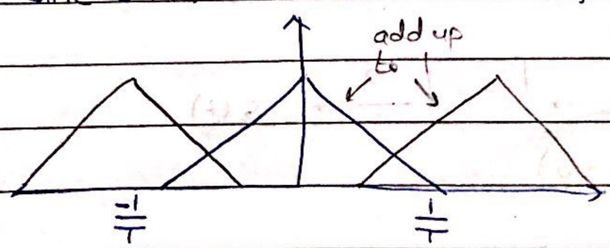
- An $x(t)$ satisfying above condition and having minimum BW is sinc(t)
- Minimum BW = $\left[\frac{-1}{2T}, \frac{1}{2T}\right]$ goes from 1 at m = 0 to 0 at m = 1 slowest

If you replicate $X(f)$ at periods $1/T$, we get flat response, like $\delta(t) \longrightarrow 1$

- Linear modulation using sinc takes up all degrees of freedom

A Excess BW.

- $x(t) = \text{sinc}^2(t/T)$ ✓ Satisfies condition.

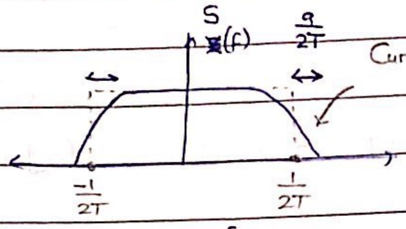


Reading - Chap 2 + Exercise Qs

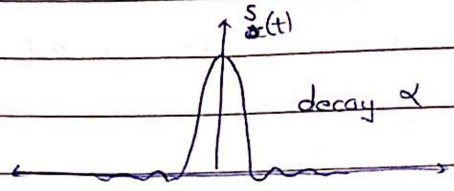
RCP
$$s(f) = \begin{cases} T & -\frac{(1-a)}{2T} \le f \le \frac{1-a}{2T} \\ \frac{T}{2}\left(1 - \sin\left(\frac{\pi T}{a}\left(|f| - \frac{1}{2T}\right)\right)\right) & \text{for } \frac{1-a}{2T} \le |f| \le \frac{1+a}{2T} \end{cases}$$

→ Raised Cosine Pulse.

Family of curves :-



Time domain :-



decay $\propto 1/t^{\alpha}$ where $\alpha > 1$

Equation :- $s(t) = \dfrac{\text{sinc}(t/T)\cos\left(\frac{\pi a t}{T}\right)}{1 - \left(\frac{2at}{T}\right)^2}$ ✓ Satisfies ISI avoidance condition.

For small T, s(t) decays as $\sim 1/t^3$

7/1 $a \to$ 'Excess bandwidth' :- New BW $= \frac{1+a}{2T}$

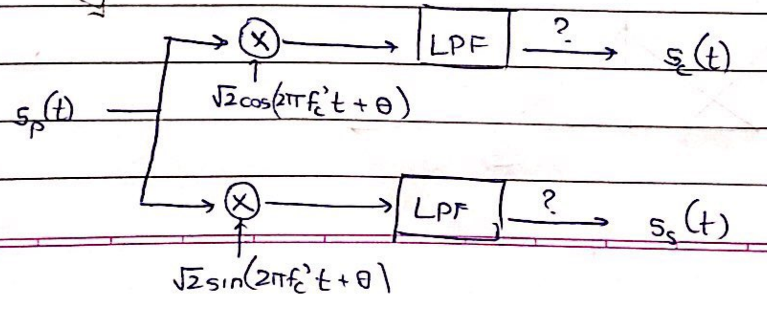
For $a=0$, $s(t) = \text{sinc}(t/T)$

Increasing a increases rate of decay of waveform, but also the BW.

→ Root Raised Cosine Pulse

- Fourier domain :- $\sqrt{s(f)}$ $\because s(f)$ is real and positive
- Choose g_{Tx} and g_{Rx} as RRCP

B] Coherent v/s Non-coherent Demodulation.



Because of θ :- $s_c(t)$ is multiplied by $\cos\theta$ and also have some $s_s(t)$ component. :(

Because of f_c' :- Causes beats :(

* M-ary FSK.

$$s_i(t) = \cos(2\pi f_i t)\, I_{[0,T]} \qquad i = 1, 2 \ldots M$$

- Demodulation :- - Multiply received signal by all $\cos 2\pi f_i t$ and LPF.
  - Conclude the symbol that gives highest time-average after multiplication
  - $\overline{\cos 2\pi f_i t \times \cos 2\pi f_j t} = \begin{cases} 1/2 & \text{for } i=j \\ 0 & \text{otherwise.} \end{cases}$

This type of averaging is unaffected by phase :)

↳ multiplication by $\cos(2\pi f_i t + \theta)$

- Coherent :- Equal f_c and phase

Non-coherent :- Equal f_c.

- ~~Receive~~

→ Non-coherent demodulation :-

- Received signal $= y_p(t) = s_p(t) + \text{Noise}$.

- Possible choices of $s(t) =$ (eg) $I_{[0,T]}, -I_{[0,T]}, j\, I_{[0,T]}, -j\, I_{[0,T]}$ .... 'QPSK'

- To guess which symbol was sent :-

For all choices of $s(t) = s_c(t) + j s_s(t)$,

calculate $\langle y_p, s_p \rangle = \langle y_c, s_c \rangle + \langle y_s, s_s \rangle$ ..... Real ✓

Conclude the symbol that maximizes above expression.

☆ Verify

Reason :- $\langle y_p, s_p \rangle$ is maximum when $y_p = \alpha s_p$ (collinear)

P.T.O

:) • In non-coherent, we don't care about phase offset.

Even if $y(t) = s(t)e^{j\theta}$,

$\langle y, s\rangle$ will still be maximum when $y = ks$

:(• Problem :- Because $|\langle y, s\rangle| = \|s\|^2$ when maximum,

you cannot distinguish ~~but~~ sign of symbol.

Verify from book that $|\langle y, s\rangle|$

1/1

• Coherent modulation (eg - PSK) - Info is stored in phase (each symbol is sent via a particular phase)

Demodulation :- Find i that maximizes $Re(\langle y, s_i\rangle)$

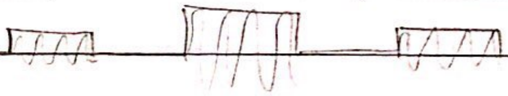
• Non-coherent modulation (eg - FSK) - Each symbol is sent via a particular frequency/amplitude.

- We don't care about phase of modulated signal.

Demodulation :- Find i that maximizes $|\langle y, s_i\rangle|$.

* FSK need not be non-coherent and may embed phase information.

eg 2 - Non-coherent ASK (PAM)



~~Dem~~

Demodulation - Envelope detection (no info in phase)

* Coherent FSK (eg - 2-FSK (1 bit))

$s_1(t) = \cos 2\pi f_1 t \; I_{[0,T]}$

$s_2(t) = \cos 2\pi f_2 t \; I_{[0,T]}$

} Start with zero phase

- Coherent because phase difference between s_1 & s_2 is not unknown

- $y(t) = s(t) + \text{noise}$

- To find which was sent, maximise $\int_0^T y(t)\, s_i(t)\, dt$.

- We want 'orthogonal' FSK :- If s_1 was sent, inner product with s_2 should give zero.

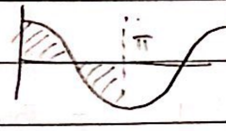
$$\int_0^T \cos 2\pi f_1 t \cos 2\pi f_2 t \, dt = 0 \quad \ldots \text{Find } \min |f_1 - f_2| \text{ such that this holds.}$$

$$\int_0^T \cos 2\pi(f_1+f_2)t + \cos 2\pi(f_1-f_2)t \, dt = 0$$

Condition :- $2\pi(f_1+f_2)T = m\pi$

Condition :- $2\pi(f_1-f_2)_{min} T = \pi$

$$\boxed{|f_1-f_2|_{min} = \frac{1}{2T}}$$



$*$ Non-coherent FSK :- (eg - 2-FSK)

$$s_1(t) = \cos(2\pi f_1 t + \phi_1) I_{[0,T]}$$
$$s_2(t) = \cos(2\pi f_2 t + \phi_2) I_{[0,T]}$$

- Cannot simply maximize $\int_0^T y(t) s_i(t) \, dt$.
- Find $\int_0^T \cos(2\pi f_1 t + \phi) \cos 2\pi f_1 t \, dt = \sim \cos\phi$

and $\int_0^T \cos(2\pi f_1 t + \phi) \sin 2\pi f_1 t \, dt = \sim \sin\phi$ } Square and add.

Equivalently :- Maximize $\langle y_p, s_p \rangle = \langle y_c, s_c \rangle + \langle y_s, s_s \rangle$.

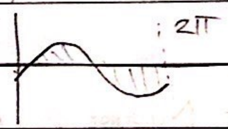
- We want orthogonal FSK :-

$\int_0^T \cos(2\pi f_1 t + \phi) \cos 2\pi f_2 t \, dt$ should be zero.

Condition :- $2\pi(f_1+f_2)T = m\pi$ ✓

Condition : $2\pi(f_1-f_2)_{min} T = 2\pi$

$$\boxed{|f_1-f_2|_{min} = \frac{1}{T}}$$



	Coherent	Non-coherent
→ No. of complex dimensions = Time-BW product	$M/2$	M
→ BW efficiency = $\frac{\log_2(\text{Real dimensions})}{M \text{ Complex dimension}}$ (bits/complex dimension)	$\frac{\log_2 2M}{M}$	$\frac{\log_2 M}{M}$

* PSK :- $s_i(t) = e^{j\phi_i} I_{[0,T]}$

eg – For QPSK, $\phi_i \in \{ \pi/4, 3\pi/4, -3\pi/4, -\pi/4 \}$

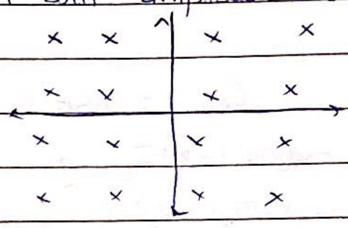
- Info is purely in phase.

* FSK

* ASK = PAM.

* QAM – Info is in both amplitude and phase.

eg – 16-QAM



* FSK, ASK could be coherent or non-coherent
PSK, QAM are coherent.

~~Differential Modulation~~.

* Consider M-FSK.

Verify

Bandwidth efficiency = $\frac{\log_2 M}{M}$ → bits per symbol (bits)

As $M \to \infty$, $\eta_B \to 0$

∴ It is inefficient to pack in too many frequencies.

* Nyquist :- Minimum required BW for linear modulation = symbol rate = $1/$

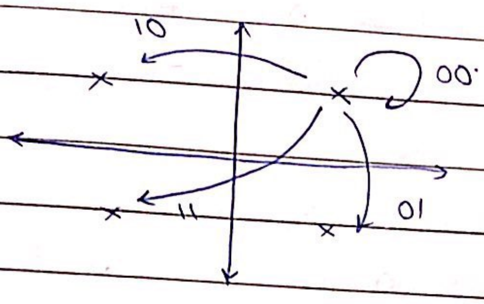
$$B_{min} = 1/T_s$$

$$\therefore \eta_B = \frac{1}{M} \times \frac{\text{Bit rate}}{B_{min}}$$

c] Differential Modulation.

eg QPSK

☆ Verify



Data stream :- 00 11 01 1 0 11

- Start with 00. (Ask receiver to ignore first symbol)

$$\underbrace{00}_{e^{j\frac{\pi}{4}}}\ \underline{00}_{\frac{\pi}{4}}\ \underline{11}_{\frac{-3\pi}{4}}\ 011011$$

Always start with this

- At the receiver, $y[0] = he^{j\pi/4}$. Assuming $g_c(t) = h\delta(t)$

$y[1] = he^{j\pi/4}$.

Find $y[1]y[0]^* \rightarrow = |h|^2$

$\angle y[1]y[0]^* = 0 \Rightarrow$ Move 0 phase from initial

Now $y[2] = he^{j3\pi/4}$

Find $y[2]y[1]^* = |h|^2 e^{j\pi} \Rightarrow$ Move π phase from where you were

- Benefit :- Phase added by channel is immaterial.
It becomes $|h|^2$.

☆ Wrong

• 'Send as is. Determine difference at receiver'.

22/1 → Differential BPSK

☆ Verify

1 Send '+1'

2 If you want to send 1 → Previous

0 → Toggle.

P.T.O.

Standard non-coherent BPSK has $r_b = \frac{1}{2}$ bits/complex dimension

DBPSK 1 bits/complex dimension

(because it uses info from two symbols while sending one symbol)

Receiver finds phase change between consecutive symbols

$\hookrightarrow 0 : +1$

$\pi : -1$

- Benefits of DBSK :-

eg- Bitstream 0 1 1 0 1 0 1

Send +1 -1 -1 -1 +1 +1 -1 -1

1 Phase added by channel is removed, by doing $\angle(y[n]y^*[n-1])$

2 This works even if channel conditions vary with time $\equiv h(t)$

- We assume $h(t)$ does not change much over two consecutive samples.

* If you have passband BW limit of 'W', then the maximum data rate you can use is W sps. (because pulse needs to decay fast enough to be zero at next integral point — best done by sinc)

$\Rightarrow$ This minimum sampling time $- T = \frac{1}{W}$

* $\boxed{\text{Basis}}$:-

Consider QPSK at $\frac{1}{T}$ sps.

1) $s_1(t) = e^{j\pi/4} I_{[0,T]}$

2) $j^{3\pi/4}$

3) $j^{-\pi/4}$

4) $j^{-3\pi/4}$

- Basis vectors for all 4 symbols :- $\varphi_1(t) = \frac{I_{[0,T]}}{\sqrt{T}}$

'Orthonormal'

$\varphi_2(t) = j\frac{I_{[0,T]}}{\sqrt{T}}$ → to normalize

- Orthogonality :- $\mathrm{Re}\left(\int_{-\infty}^{\infty} \varphi_1 \varphi_2 \, dt\right) = 0$

$\rightarrow \quad s_1(t) = \cos\frac{\pi}{4}\,\varphi_1(t) + j\sin\frac{\pi}{4}\,\varphi_2(t)$

- Alternate orthonormal basis :- $\psi_1(t) = s_1(t)$, $\psi_2(t) = s_2(t)$

23/1

★ • k basis vectors $\Rightarrow$ k complex dimensions $\equiv$ $2k$ real dimensions
for FSK

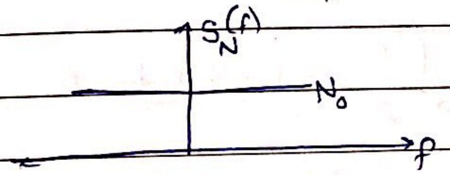
This is because real part and imaginary part are orthogonal
ie - In passband form, $\cos 2\pi f_c t$ and $\sin 2\pi f_c t$ are orthogonal.

• All M-PSK have 1 complex dimension, 2 real dimensions

→ Additive White Gaussian Noise.

$$\to \oplus \to \boxed{g_{Rx}} \to$$
$$\uparrow n(t)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$$



$$R_N(\tau) = \delta(\tau)$$

• Jointly Gaussian :- $\alpha_1 X_1 + \alpha_2 X_2$ is Gaussian $\forall\ \alpha_1, \alpha_2$

$X \sim N(0,1)$, $Y = \alpha X$ where $\alpha = \begin{cases} 1 & \text{wp } 1/2 \\ -1 & \text{wp } 1/2 \end{cases}$

X, Y are not JG because $X+Y$ is zero wp $1/2$ (not zero)

✱ BPSK :- $0 \to -\sqrt{P}$, $1 \to \sqrt{P}$ where P = Power.

Pulse for modulation chosen as $\frac{I_{[0,T]}}{\sqrt{T}}$

$$s_1(t) = \sqrt{P}\,\psi_1(t), \quad s_0(t) = -\sqrt{P}\,\psi_1(t)$$

$$s(t) \longrightarrow \oplus \longrightarrow r(t)$$
$$\uparrow n(t) : \text{with PSD } N_0$$

We can equivalently assume noise to be baseband, or passband.

$$r(t) = s_?(t) + n(t)$$

Finding $\langle r, \psi_1 \rangle$:- $r(t) \rightarrow \otimes \rightarrow \boxed{\int_0^T} \rightarrow$ with $\psi_1^*(t)$ into $\otimes$

$$= \int_0^T s_?(t)\,dt + \underbrace{\int_0^T n(t)\,dt}_{\text{'RV'}}$$

$$\rightarrow E\left[\int_0^T n(t)\,dt\right] = 0$$

$$\rightarrow Var\left[\int_0^T n(t)\,dt\right]$$

$$= E\left[\left(\int_0^T n(t_1)\,dt\right)^2\right]$$

$$= E\left[\left(\int_0^T n(t_1)\,dt_1\right)\left(\int_0^T n(t_2)\,dt_2\right)\right]$$

$$= E\left[\int_0^T\int_0^T n(t_1)\,n(t_2)\,dt_1\,dt_2\right]$$

$$= \int_0^T\int_0^T E[n(t_1)\,n(t_2)]\,dt_1\,dt_2$$

$$= \int_0^T\int_0^T R_n(t_1, t_2)\,dt_1\,dt_2$$

$$= \int_0^T\int_0^T N_0\,\delta(t_1 - t_2)\,dt_1\,dt_2$$

$$= N_0 T$$

We don't lose information because ~~$E(\langle r, \psi_1 \rangle)$~~ $E(RV) = 0$

* QPSK :- 1 complex dimension.

$$\underset{\text{Complex}}{s_1(t)} \longrightarrow \oplus \longrightarrow r(t) \text{ Complex}$$

$n(t)$ complex $\uparrow$ into $\oplus$: PSD N_0

$\text{Re}(n(t))$ and $\text{Im}(n(t))$ are independent and each has PSD $\frac{N_0}{2}$

* Complex normal random variable :- $Z = Z_r + jZ_i$

$Z_r, Z_i \to N(0, \frac{1}{2})$

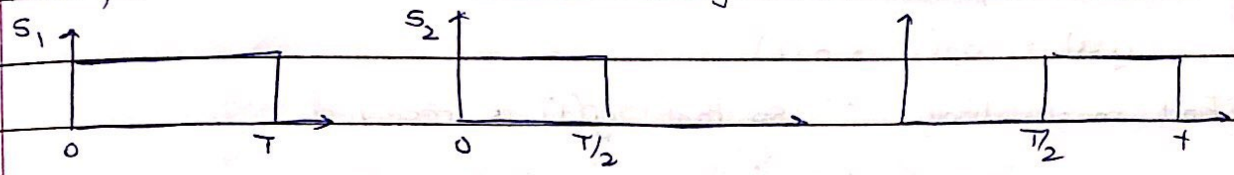
- For Gaussian, uncorrelated $\Rightarrow$ independent.

* $N_0 = kT$

$k \to$ Boltzmann's constant

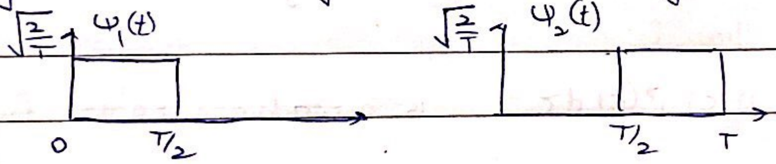
$T \to$ Temperature.

* Example :- Possible choices for symbols



Instead of directly checking the received signals, we check after resolving all three symbols into orthogonal componets.

Basis :-



* Bases can be found out by Gramm-Schmidt orthogonalization process.

① $\psi_1(t) = \dfrac{s_1(t)}{\|s_1(t)\|}$ $\quad \|s_1(t)\| \longrightarrow \sqrt{\int_{-\infty}^{\infty} |s(t)|^2 dt}$

② $\phi_2(t) = s_2(t) - \langle \psi_1, s_2 \rangle \psi_1$

and $\psi_2(t) = \dfrac{\phi_2(t)}{\|\phi_2(t)\|}$

③ and so on.

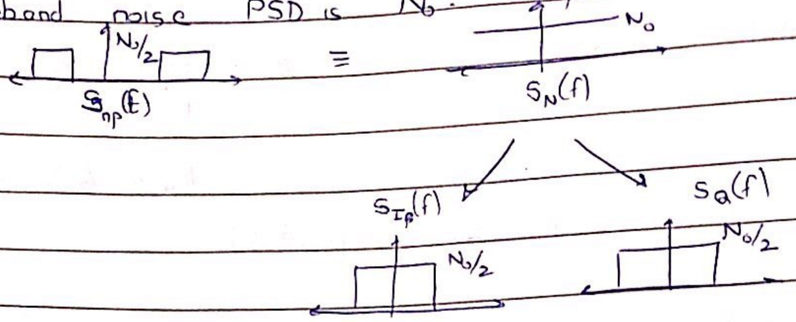
- Such a basis is not the only possible basis

20/1

* Baseband noise PSD is N_0. In passband it is $N_0/2$

N_0 = noise per complex dimension

$\frac{N_0}{2}$ = noise per real dimension



* M-ary FSK :- M basis vectors (one for each frequency)

→ Detection problem:-

$$y(t) = s_i(t) + n(t)$$

Find most likely 'i' so that $y(t)$ is received.

Property If, ~~the~~ we resolve $y(t)$ in the same basis vectors that we use for $s_i(t)$, then we will not lose any info due to noise.

- $\langle n, u \rangle = \int_{-\infty}^{\infty} n(\tau) u^*(\tau) \, d\tau$ is a random variable, for any $u(t)$

- Basis vectors ψ_i are such that $\langle \psi_i, \psi_j \rangle = \delta_{ij}$

- Consider $\langle n, u \rangle = \int_{-\infty}^{\infty} n(\tau) \psi_i^*(\tau) \, d\tau \leq \int |n| \int |\psi_i| < \infty$ so finite

→ Property :- $\langle n, \psi_i \rangle$ is Gaussian $(0, N_0)$

↳ Because linear operation on Gaussian 'r'

$$Var(\langle n, \psi_i \rangle) = E\left[\langle n, \psi_i \rangle \langle n^*, \psi_i^* \rangle\right] \qquad \psi_i \text{ is real}$$

$$= E\left[\int_{-\infty}^{\infty} n(t)\psi_i(t)\,dt \int_{-\infty}^{\infty} n^*(\tau)\psi_i(\tau)\,d\tau\right]$$

$$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{n(t)\, n^*(\tau)}_{R_N(\tau - t)} \psi_i(t)\psi_i(\tau)\, dt\, d\tau\right]$$

$$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} N_0\, \delta(t-\tau)\, \underbrace{\psi^2}_{=1}\, dt\, d\tau\right]$$

$$= N_0$$

- $\left[\langle n, \psi_1 \rangle \;\; \langle n, \psi_2 \rangle \; \ldots \; \langle n, \psi_k \rangle \right]^T$ are jointly Gaussian. $(0, N_0 I)$
- All elements have zero mean.
- Claim:- Covariance matrix is $N_0 \times$ identity matrix $\Leftrightarrow$ All elements are iid.

Uncorrelated $\Rightarrow$ independent

Proof:- $E\left[\langle n, \psi_i \rangle \langle n^*, \psi_j^* \rangle \right]$

$$= E\left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} N_0 \, \delta(t-\tau) \;\; \psi_i \psi_j \;\; dt \, d\tau \right]$$

$$= 0 \quad \text{(orthogonality)}$$

→ Detection (continued)

$$y(t) = s_i(t) + n(t)$$

Since $\langle y, \psi_j \rangle = \langle s_i, \psi_j \rangle + \langle n, \psi_j \rangle$

$\therefore \quad \underline{y}(t) = \underline{s_i}(t) + \underline{N}(t)$

Changed problem :- Detecting which i will give vectory of y in basis ψ.

!! Does doing this (inner product with ψ_i) make us lose valuable info?

We have converted $n(t) \longrightarrow N = \left[N(1), \; N(2) \; \ldots \; N(k) \right]^T$

'Loss of information' $= n^{\perp}(t) = n(t) - \sum_{i=1}^{k} N(i) \, \psi_i(t)$

Loss in received signal $= y^{\perp}(t) = y(t) - \sum_{i=1}^{k} \langle y, \psi_i \rangle \psi_i(t)$

$$= s(t) + n(t) - \sum \left(\langle s_i, \psi_i \rangle + \langle n, \psi_i \rangle \right) \psi_i(t)$$

$$= n^{\perp}(t) \qquad \qquad = s(t) \text{ (because } s(t) \text{ lies in the space of } \psi_i\text{'s)}$$

$y^{\perp}(t)$ could contain some info about $s(t)$ if $n^{\perp}(t)$ is correlated with $N(k)$, as the correlation info could be used to reduce noise.

But, $n^{\perp}(t)$ and $N(k) = \langle n, \psi_k \rangle$ are uncorrelated. (PTO)

$\therefore$ No valuable info for detection is lost.

→ $\therefore$ Detection via vectors is equivalent to detection normally

31/1

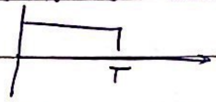
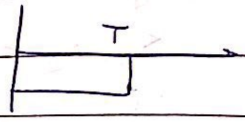
$$E\left[n^{\perp}(t)\ N^*[k]\right] = E\left[\left(n(t) - \sum_{j=1}^{k} N[j]\ \psi_j(t)\right) N^*[k]\right]$$

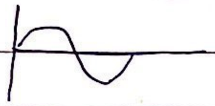
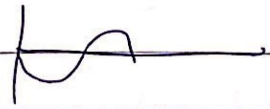
$$= E\left[n(t)\, N^*[k]\right] - \sum_{j=1}^{k} E\left[N[j]\, N^*[k]\right]\psi_j(t)$$

$$\vdots$$

$$= 0 \quad \ldots\ldots \quad n^{\perp} \text{ and } N[k] \text{ are uncorrelated.}$$

→ Performance is independent of waveforms used.

eg – BPSK :-  and 

or BPSK :-  and 

∴ We will care only about vector representation, not actual waveforms

V HYPOTHESIS TESTING

$$\underline{r} = \underline{s_i} + \boxed{N}$$

↳ Complex Noise ~ Gaussian $(0, \sigma^2 I)$ ↓ N_0

→ Maximum Likelihood Rule

$$f_{\underline{N}}(N) = c\ \exp\left(\frac{-(\underline{r} - \underline{s_i})^H(\underline{r} - \underline{s_i})}{N_0}\right) \quad \ldots \; ⊠$$

Guess that $s_i(t)$ which maximizes this $f_{\underline{N}}(N)$

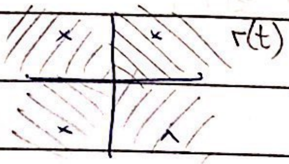
P.T.O.

Maximum likelihood decision $= \underset{i \in \{1 \dots M\}}{\text{argmax}} \; e^{-\frac{\|\underline{r} - \underline{s}_i\|^2}{N_0}}$

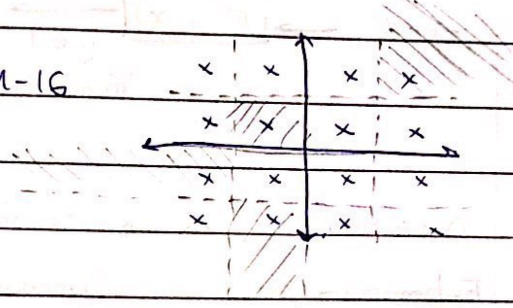
$= \underset{i}{\text{argmin}} \; \|\underline{r} - \underline{s}_i\|^2$

- Choose the s_i that is closest to the r (Euclidean distance)

eg QPSK

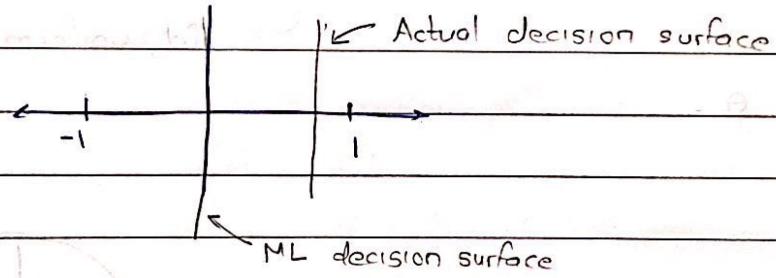


QAM-16



- ML Rule may not always give least error
  - This happens when each of the symbols are not equiprobably sent

eg :- -1 with 99% probability
$+1$ 1%



☆ - $P_c = \sum \pi(i) \, P_{c|i}$ ---- probability of making correct decision

$= \sum \pi(i) \, e^{-\|r - s_i\|^2 / 2\sigma^2}$

$\ln P_c = \ln \pi(i) - \dfrac{\|r - s_i\|^2}{2\sigma^2}$

$= \mathbb{P}(1 \text{ is sent} \mid \underline{r})$ $\qquad = \mathbb{P}(\underline{r} \mid 1 \text{ is sent})$

- $\mathbb{P}(1 \text{ is sent} \mid \underline{N}) = \underbrace{\mathbb{P}(\underline{N} \mid 1 \text{ is sent})} \times \dfrac{\mathbb{P}(1 \text{ is sent})}{\mathbb{P}(\underline{N})}$

Probability that you guess correctly :-

Maximize $\mathbb{P}(1 \text{ is sent} \mid \underline{N})$ $\qquad e^{-\frac{\|\underline{r} - \underline{s}_i\|^2}{N_0}}$

$\Rightarrow$ Maximize $\mathbb{P}(\underline{N} \mid 1 \text{ is sent}) \, \mathbb{P}(1 \text{ is sent})$

∴ Error ~~can~~ is be ~~estimated~~ least when ML is considered with prior info.

Not Appaiah

4/2

$$x \in X \longrightarrow \boxed{P(y|x)} \xrightarrow[y \in Y]{} \boxed{\text{Detector}} \longrightarrow \hat{x}(y)$$

Transmitted

Case 1 | Detection :- X is countably, ~~finite~~ ∴ $\hat{x}(y) \in \{1 \ldots M\}$

Case 2 | Estimation :- uncountable : $\hat{x}(y) \in \mathbb{R}^m$

Case 3 | Soft detection :- $\hat{x}(y) \equiv (x, P(\hat{x}))$ - Assign confidence to each prediction.

→ Estimating class heights.

Measured heights :- $Y_i, i \in \{1, \ldots 135)$

Model : $Y_i = \theta + Z_i$

↳ iid, uniform $\left(\frac{-1}{4}, \frac{1}{4}\right)$

Estimate θ.

→ MAP

Y

Objective : $P(\hat{x}(y) \neq x)$ is minimum.

– For this object, MAP is best decoder.

$D_i = \{y \in Y : \hat{x}(y) = i\}$

$P(\text{Correct}) = 1 - P(\text{Error})$

$$P(C) = \sum_{i=1}^{M} P(C, x = i)$$

$$= \sum_{1}^{M} \pi_i \, P(C \mid x = i)$$

$$= \sum \pi_i \, P(y \in D_i \mid x = i)$$

$$= \sum \pi_i \int_{y \in D_i} P(y \mid x = i) \, dy.$$

$$= \sum_{i=1}^{3} \int_{y \in Y} \pi_i \, \mathbb{P}(y|x=i) \, \mathbb{1}_{\{y \in D_i\}} \, dy$$

$$\leq \int_{y \in Y} \sum_{i=1}^{3} \left(\max_i \pi_i \, \mathbb{P}(y|i) \right) \mathbb{1}_{\{y \in D_i\}} \, dy.$$

$$= \int_{y \in Y} \left(\max_i \pi_i \, \mathbb{P}(y|i) \right) \underbrace{\sum_{i=1}^{3} \mathbb{1}_{\{y \in D_i\}}}_{=1 \ \forall y} \, dy$$

$$\cancel{\mathbb{P}(c)} \ \cancel{\leq} = \sum_{i=1}^{3} \left(\pi_i \int_{y \in A_i} \mathbb{P}(y|i) \, dy \right)$$

where $A_i = \{ y \in Y \mid \pi_i \mathbb{P}(y|i) > \pi_j \mathbb{P}(y|j) \quad \forall j \neq i \}$

'MAP Decoder'

$\therefore$ P(Correct) is maximum for these A_i

$\rightarrow$ Monotonicity in noise

Model 1 Inband AWGN : $y_1(t) = x(t) + z(t)$

Model 2 Wideband AWGN : $y_2(t) = x(t) + z(t) + z^{\perp}(t)$

(Out of band) $\hookrightarrow$ independent noise added (out of signal space)

Model 2 gives worse prediction than Model 1.

$\rightarrow$ Signal Space

$s(t) = \sum d_i \phi_i(t)$

$\hookrightarrow$ orthonormal bases

- Band-limited waveform space
  Basis = sinc functions

- Time-limited waveform space
  Basis = exponential functions

→ Receive Filtering

$$y(t) = s(t) + z(t)$$

Write $y(t) =$

Theorem 3.A.3, Madhow

→ MLE decision rule :- $\underset{1 \le m \le M}{\arg\max} \; Re\{\langle r(t), s_m(t)\rangle\} - \frac{1}{2}\|s_m(t)\|^2$

5/2

→ $\boxed{\text{ML Rule :}}$

Choose $i = \boxed{\arg\min_i \|y - s_i\|^2}$ --- Geometrically closest

$$= \arg\min_i \left(\cancel{\|y\|^2} + \|s_i\|^2 - 2\,Re\langle y, s_i\rangle\right)$$

$$= \boxed{\arg\max_i \left(Re\langle y, s_i\rangle - \frac{\|s_i\|^2}{2}\right)}$$

→ $\boxed{\text{Minimum Error Probability rule}}$

Choose $i = \boxed{\arg\max_i \left(Re\langle y, s_i\rangle - \frac{\|s_i\|^2}{2} + \sigma^2 \log(\pi_i)\right)}$

noise variance $\left(\frac{N_0}{2}\right)$

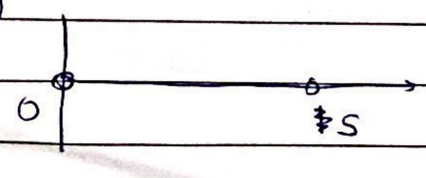
eg For QPSK, $\|s_i\|^2$ is same

~~∴ Answer for ML :- Geometrically~~

eg On-Off Keying (OOK)

$0 \rightarrow 0$

$1 \rightarrow s\,\varphi_1(t)$



Dimension of signal space = 1, basis = $\{\varphi_1(t)\}$

Received signal $y(t) = s(t) + n(t)$

Obtain 'y' by projecting $y(t)$ onto $\underrightarrow{\psi}(t)$ ~~(which is~~

Here, since $\underrightarrow{\psi}(t)$ is one-dimensional, $\underline{y} = \int y(t)\,\psi_1(t)\,dt$

We want to guess $s(t)$ from y.

- If $s_i(t)$ are equiprobable.

$$P_{error} = P_{error|0\,sent}\,\underbrace{P_{0\,sent}}_{1/2} + P_{error|1\,sent}\,\underbrace{P_{1\,sent}}_{1/2}$$

Hypothesis 0 :- $H_0 \equiv y(t) = n(t)$

1 :- $H_1 \equiv y(t) = s + n(t)$

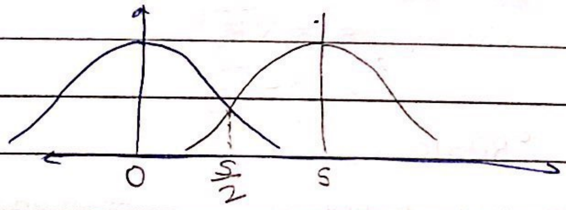
$\underline{y} = y\,\psi_1$

$s_i = 0$ or $s\,\psi_1$

- ~~If 0 was sent, received~~, ~~For H_0~~
  - If 0 was sent, $Re\langle \underline{y}, s_i\rangle - \|s_i\|^2/2 = 0 - 0 = 0$
  - 1 $\quad sy - \frac{s^2}{2}$

• We guess 1 if $sy > \frac{s^2}{2} \Rightarrow y > \frac{s}{2}$

0 < <



• $P_{error|0\,sent} = \mathbb{P}\left(y > \frac{s}{2} \,\middle|\, 0 \text{ was sent}\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_{s/2}^{\infty} e^{-x^2/2\sigma^2}\,dx$

$= Q\left(s/2\sigma\right)$

$= Q\left(\frac{\sqrt{2E_b}}{2\sigma}\right)$

$= \mathbb{P}_{error|1\,sent}$

✱ $\underline{E_s}$ = Energy per symbol $= \underbrace{n}_{\text{bits in a symbol}} E_b$

$\underline{E_b}$ = Energy per bit.

$0 \to 0$, $s \to s^2 \Rightarrow E_b = \frac{s^2}{2}$

* Binary Signaling

$0 \rightarrow s_1$ $\qquad H_1 \equiv y(t) = s_1 + n(t)$

$1 \rightarrow s_2$ $\qquad H_2 \equiv y(t) = s_2 + n(t)$

$$\therefore \quad E_b = \frac{s_1^2 + s_2^2}{2}$$

- Decision :-

$$\langle y, s_1 \rangle - \frac{\|s_1\|^2}{2} \underset{H_2}{\overset{H_1}{\gtrless}} \langle y, s_2 \rangle - \frac{\|s_2\|^2}{2}$$

ie $\rightarrow$ $$\langle y, s_1 - s_2 \rangle \underset{H_2}{\overset{H_1}{\gtrless}} \frac{|s_1|^2 - |s_2|^2}{2}$$

- Mathematical simplification

Define $\tilde{y}(t) = y(t) - s_2(t)$

$$\therefore \quad H_1 \equiv \tilde{y}(t) = s_1(t) - s_2(t) + n(t)$$
$$H_2 \equiv \tilde{y}(t) = n(t)$$

This is just like OOK

$$\therefore \quad \mathbb{P}_{error | s_2 \text{ is sent}} = Q\left(\frac{|s_1 - s_2|}{2\sigma}\right) = \mathbb{P}_{error | s_1 \text{ is sent}}$$

- To minimize this, maximize $|s_1 - s_2|$

$\therefore$ Under E_b constraint,

choose $s_1 = \sqrt{E_b}$

$s_2 = -\sqrt{E_b}$

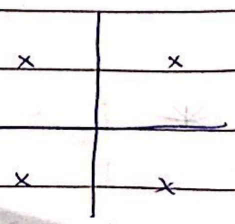
'BPSK'

This is the best binary signaling under energy constraint for equiprobable symbols.

QPSK

$$E_s = 2E_b = \frac{s_1^2 + s_2^2 + s_3^2 + s_4^2}{4}$$

For farthest spacing, $\|s\| = \sqrt{E_s}$



7/2 If symbol ~~has 2 bits~~ is QPSK,

$$\underbrace{\begin{bmatrix} y_r \\ y_i \end{bmatrix}}_{\text{1 symbol}} = \begin{bmatrix} \sqrt{E_s/2} \\ \sqrt{E_s/2} \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

$n_1 \to G(0, N_0/2)$, $n_2 \to G(0, N_0/2)$ independent

$$\sqrt{\frac{E_s}{2}}\, e^{j\pi/4}$$

$\therefore \begin{bmatrix} y_r \\ y_i \end{bmatrix}$:- Jointly Gaussian : Mean $= \begin{bmatrix} \sqrt{E_s/2} \\ \sqrt{E_s/2} \end{bmatrix}$, Covariance $= \begin{bmatrix} N_0/2 & 0 \\ 0 & N_0/2 \end{bmatrix}$

$\therefore$ Distribution :- $f_{\underline{y}}(y_r, y_i) = \dfrac{1}{(\sqrt{2\pi\sigma^2})^2}\, e^{-\frac{(y_r-\mu_r)\Sigma^{-1}(y_i-\mu_i)}{2}}$

$$= \frac{1}{2\pi\left(\frac{N_0}{2}\right)}\, e^{-\frac{\|\underline{y}-\underline{\mu}\|^2}{2\left(\frac{N_0}{2}\right)}} \qquad \Sigma^{-1} \Rightarrow \frac{1}{N_0/2}$$

Integrate Gaussian over correct area

$\therefore P_{\text{correct} \mid 1 \text{ is sent}} = \displaystyle\int_0^\infty\!\!\int_0^\infty f_{y_r y_i}(y_r, y_i)\, dy_r\, dy_i$

$$= \int_0^\infty \frac{1}{\sqrt{\pi N_0}}\, e^{-\frac{\left(y_r - \sqrt{\frac{E_s}{2}}\right)^2}{N_0}}\, dy_r \int_0^\infty \frac{1}{\sqrt{\pi N_0}}\, e^{-\frac{\left(y_i - \sqrt{\frac{E_s}{2}}\right)^2}{N_0}}\, dy_i$$

$$= \left(\int_0^\infty \frac{1}{\sqrt{\pi N_0}}\, e^{-\frac{\left(y - \sqrt{\frac{E_s}{2}}\right)^2}{N_0}}\, dy\right)^2$$

$$= Q^2\left(-\sqrt{\frac{E_s}{\cancel{2}N_0}}\right)$$

$$= \left(1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right)^2$$

$\therefore P_{\text{error} \mid 1 \text{ is sent}} = 1 - P_{\text{correct} \mid 1 \text{ is sent}}$

$$= 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) - \underbrace{Q^2\left(\sqrt{\frac{E_s}{N_0}}\right)}_{\text{Quite small}}$$

- For higher number of symbols (eg - ~~QA~~ 16-QAM), it is cumbersome to ~~fo~~ detect.

$\therefore$ Use union bound.

Scanned with CamScanner

→ Union Bound Approximation

eg QPSK

$$P_{error \mid 1 \text{ is sent}} = P(y_r < 0 \quad \text{OR} \quad y_i < 0)$$

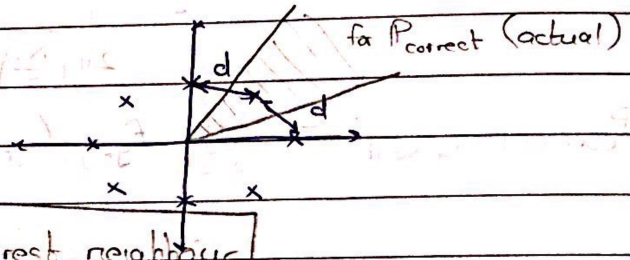
$$\leq P(y_r < 0) + P(y_i < 0) = 1 - Q\left(-\sqrt{\frac{E_s}{N_0}}\right) + 1 - Q(-$$

$\because$ $P(y_r \,\&\, y_i < 0)$ is counted twice

$$= 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) = 2Q\left(\frac{d}{2\sigma}\right) \qquad d = \sqrt{2E_s}$$

eg 8-PSK

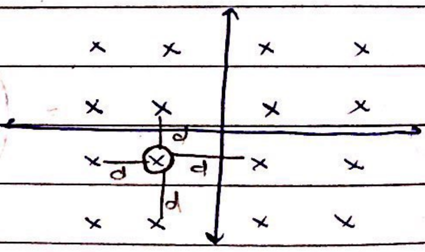
$$P_{error \mid 1 \text{ is sent}} \leq 2Q\left(\frac{d}{2\sigma}\right)$$



One $\boxed{Q\left(\frac{d}{2\sigma}\right) \text{ for each nearest neighbour}}$

eg 16-QAM

$$P_{error \mid 1 \text{ is sent}} \leq 4Q\left(\frac{d}{2\sigma}\right)$$



$$\sigma = \sqrt{\frac{N_0}{2}}$$