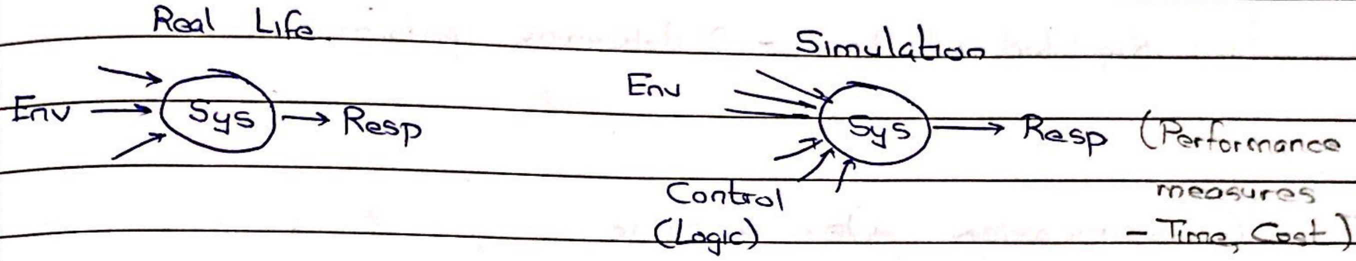
3/1/19

- System $\triangleq$ Combination of interdependent blocks that function differently than each independently does.
- system responds to 'environment' and 'controls' through 'responses' — 'Cause' / 'Effect'
- Causes: Uncontrolled / Choices

  'Instruments of pleasure' xD

A] Model of a system



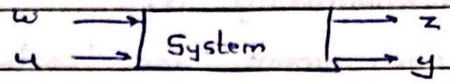
- Control logic is mostly based on feedback from performance measures
  - It may not be feedback-based, if the number of combinations of possible environmental perturbations and performance measures is finite. Control choices are directly predecided.
- System gives feedback to computer logic, and changes in control logic are physically implemented by an actuator within the system.
  - The actuator is a system by itself.

* Control by interaction

The 'compensator' works together with device to produce correct output of device.

- Applicability of simulation in real world.

I Classification Of Inputs And Outputs



known / unknown

w :- Exogenous inputs – reference values, approximate disturbances, etc
– cannot be controlled *Verify

u :- Controlled inputs –

y :- Response measurements – can be measured

z :- Regulated outputs – → determine performance

II Classification Of Systems

1 Memoryless – react to inputs instantaneously

2 With memory – react to present and past inputs.

8/1 3 Causal – state $x(t)$ depends only on present or past inputs

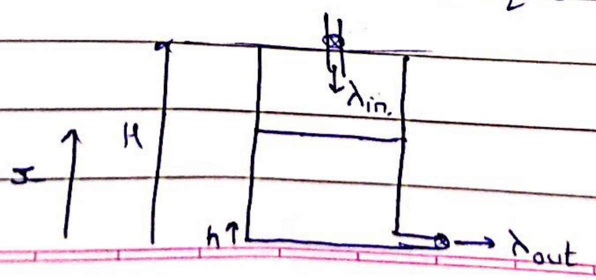
- We consider all systems to be causal.

* Transfer function : $R(s) \triangleq \dfrac{q(s)}{p(s)}$ $p(s) \neq 0$ polynomial.

- Causal :- degree of $q \leq$ degree of p.
  'Proper transfer functions'

III Examples Of Systems

A] Dose Tank / Flush Tank / Liquid level system.



λ = flow rate

$\dot{x} \propto \lambda_{in} - \lambda_{out}$

Take $\dot{x} = \lambda_{in} - \lambda_{out}$

$\lambda_{out} = \phi(x(t)) \approx \rho\, x(t)$ or $\lambda_{out} = 0$

$\lambda_{in} \in \{0, \lambda_{in}\}$

- State space model :- State $= x(t)$

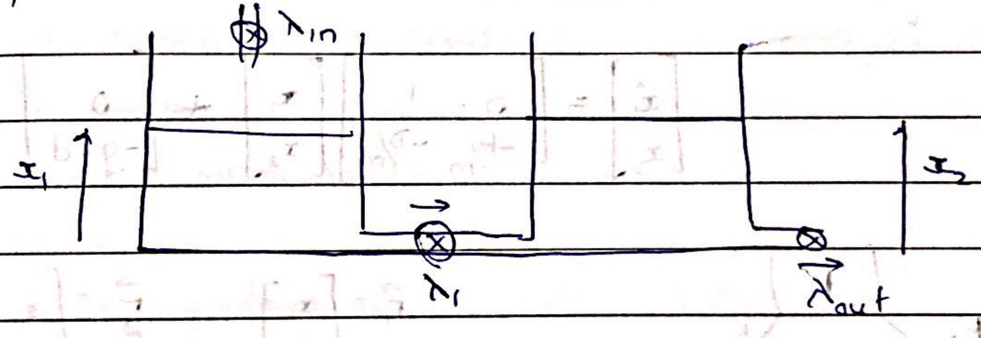
  Control input $= \lambda_{in}$

  External input $\in \{0(\text{OFF}), 1(\text{ON})\}$ for λ_{out}

  $h \leq x \leq H$.

→ Applications :- Inventory controls, warehouse control, bank account with spending, communication networks.

B] Coupled Tank



$\dot{x}_1 = \lambda_{in} - \lambda_1$

Use $\lambda_1 = \phi(x_1 - x_2) \approx c(x_1 - x_2)$

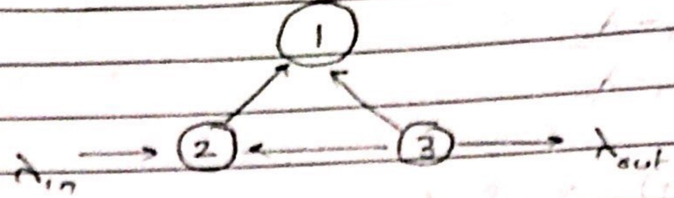
$\Rightarrow \dot{x}_1 = -c(x_1 - x_2) + \lambda_{in}$

$\dot{x}_2 = \lambda_1 - \lambda_{out}$

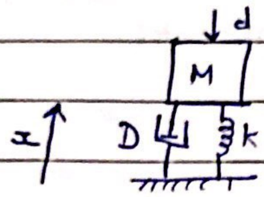
Use $\lambda_{out} \approx \rho x_2$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -c & c \\ c & -c\rho \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \lambda_{in}$$

HW 3 tanks



C] Suspension System.

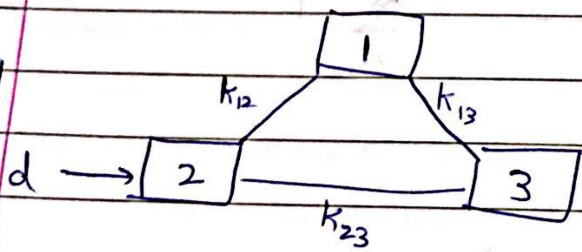


$$M\ddot{x} = -Mg - D\dot{x} - Kx + d$$

If $x_1 = x$ and $x_2 = \dot{x}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -D/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -g+d \end{bmatrix}$$

eg



$$\vec{r}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad \vec{r}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \quad \vec{r}_3 = \begin{bmatrix} x_3 \\ y_3 \end{bmatrix}$$

$$\ddot{\vec{r}}_1 = -k_{12}\vec{r}_1$$

$$\vec{r}_3 = -k_{13}(\vec{r}_3 - \vec{r}_1)$$

$$\vec{r}_2 = d$$

- In general, state equations are of the form:- (x = state vector)

$$\dot{x} = Ax + B_1 u + B_2 \omega$$

$$y = C_1 x + D_1 u + D_2 \omega$$

$$z = C_2 x + D_3 u + D_4 \omega$$

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D] Circuits

- R, L, C, VCVS, VCCS, CCVS, CCCS, transformer, gyrator, independent sources

Energy storing elements

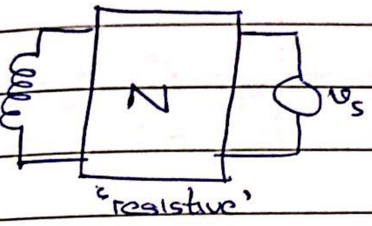
Resistive elements

- Static circuit :- × L, C

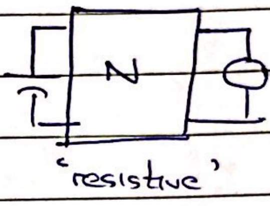
Dynamic ✓

- $i_C = C\dfrac{dv_C}{dt}$, $v_L = L\dfrac{di_L}{dt}$. → State variables

→ First order circuits have first order state space equations.



or



- Assume currents and directions, find voltages and polarities, or vice versa.

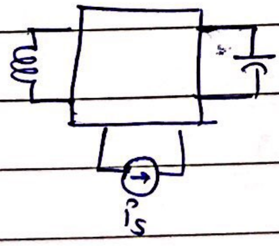
Write v_L = Linear function (i_L, v_{source})

i_C $\qquad\qquad\qquad\qquad v_C$

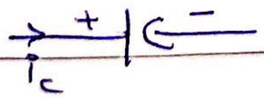
State space equations come from $C\dfrac{dv_C}{dt}$ and $L\dfrac{di_L}{dt}$

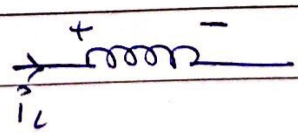
→ Second order circuits have two energy storing elements

eg -



Find $v_L = LF(i_L, v_C, i_s)$
$i_C = LF(i_L, v_C, i_s)$
} Use superpos

* 



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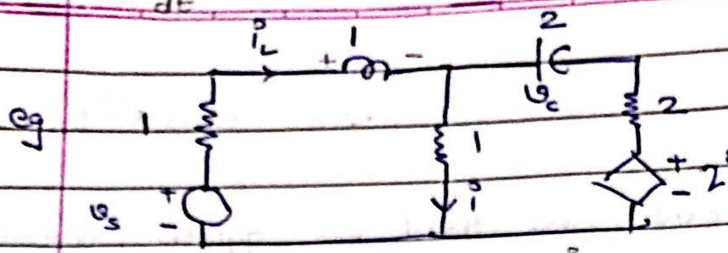
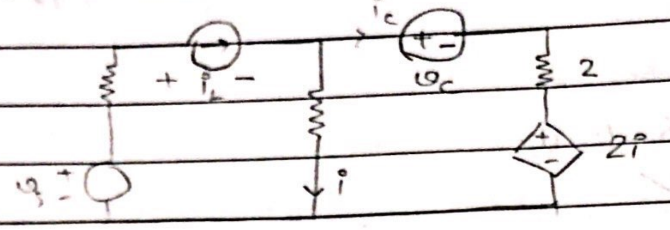
$$\frac{dv_c}{dt} = -\frac{1}{2}(v_c + i_L)$$

$$\frac{di_L}{dt} = v_s - v_c - 3i_L$$



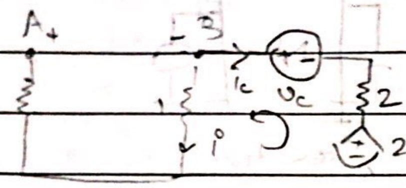
Treat i_L, v_c as s



Find v_L and i_c as functions by superposition

$\longrightarrow$ $v_L =$

1)



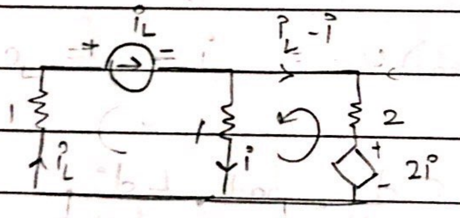
$\longrightarrow$ $v_c + i_c - 2i_c + 2i_c = 0$

$\boxed{i_c = -v_c}$

$\longrightarrow$ $v_A - i_c = v_B$

$\boxed{v_L = i_c = -v_c}$

2)



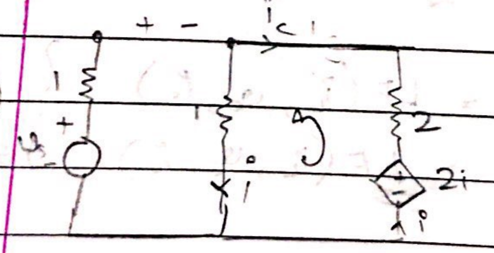
$2i + 2i_L - 2i - i = 0$

$i = 2i_L$

$\longrightarrow$ $\boxed{i_c = -i_L}$

$\longrightarrow$ $v_L = -2i_L - i_L = \boxed{-3i_L}$

3)

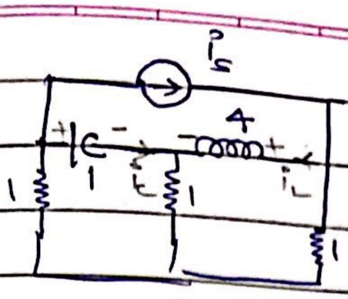


$i = 0$

$\boxed{v_L = v_s}$

$\boxed{i_c = 0}$

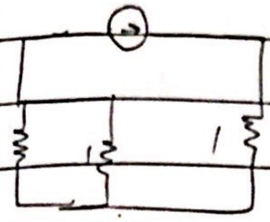
Q2



$$\frac{di_L}{dt} = \frac{1}{4}\left(\frac{3i_s}{2} - \frac{3i_L}{2} + \frac{v_c}{2}\right)$$

$$\frac{dv_c}{dt} = \frac{1}{2}\left(-i_s - v_c - i_L\right)$$

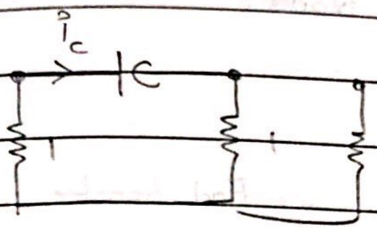
1



$$v_L = \frac{3}{2} i_s$$

$$i_c = -\frac{i_s}{2}$$

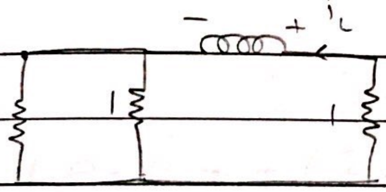
2



$$v_L = \frac{v_c}{2}$$

$$i_c = \frac{-v_c}{2}$$

3



$$v_L = -\frac{3i_L}{2}$$

$$i_c = -\frac{i_L}{2}$$

16/1 # Modeling of LTI Systems

Time domain :- LCCDE

Frequency domain :- Transfer functions (Rational function of s)

$$G(s) = \frac{q(s)}{p(s)}$$

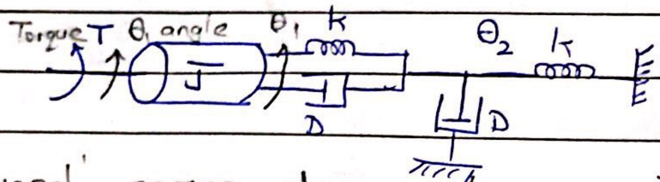
$p(s) \rightarrow$ higher degree

- Effect of LTI system on sinusoids :- Amplitude, phase.

$$A\cos\omega t \longrightarrow B\cos(\omega t + \phi)$$

- $p(s)$ should not have roots in right half plane.

eg



Find transfer function T to θ_2

'Torsional' spring, damper, etc

J = Moment of inertia

Time domain — Frequency domain

θ_2, T — $\theta_2(s)$, $T(s)$

or $\hat{\theta}_2$, or $\hat{T}$

Torque balance 1 :- $T = J\ddot{\theta}_1 + D(\dot{\theta}_1 - \dot{\theta}_2) + k(\theta_1 - \theta_2)$

~~Torque balance 2~~ $\hat{T} = Js^2\hat{\theta}_1 + (Ds + k)(\hat{\theta}_1 - \hat{\theta}_2)$

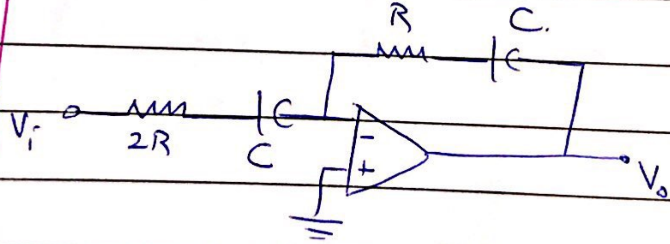
Torque balance 2 :- $k(\theta_2 - \theta_1) + D(\dot{\theta}_2 - \dot{\theta}_1) + D\dot{\theta}_2 + k\theta_2 = 0$

$$2(Ds + k)\hat{\theta}_2 - (Ds + k)\hat{\theta}_1 = 0$$

$$\Rightarrow \hat{\theta}_1 = 2\hat{\theta}_2$$

Answer :- $$\frac{\hat{T}}{\hat{\theta}_2} = \frac{1}{2Js^2 + Ds + k}$$

eg



$$\frac{V_o}{V_{in}} = \frac{-\left(R + \frac{1}{sC}\right)}{2R + \frac{1}{sC}}$$

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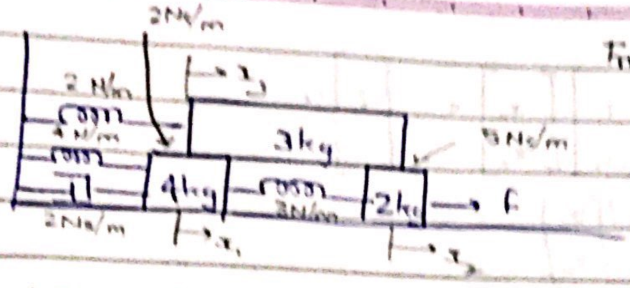
Q



Friction like damping

Find equations

Find $\frac{\hat{X}_2}{\hat{F}}$

For 1 : $4\ddot{x}_1 + 2\dot{x}_1 + 4x_1 + 2(\dot{x}_1 - \dot{x}_3) + 3(x_1 - x_2) = 0$

2 : $2\ddot{x}_2 + 5(\dot{x}_2 - \dot{x}_3) + 3(x_2 - x_1) = f$

3 : $2\ddot{x}_3 + 2x_3 + 2(\dot{x}_1 - \dot{x}_1) + 5(\dot{x}_3 - \dot{x}_2) = 0$

Zero initial conditions :- $x_1(0) = x_2(0) = x_3(0) = 0$

Laplace 1 : $(4s^2 + 4s + 7)\hat{X}_1 - 3\hat{X}_2 - 2s\hat{X}_3 = 0$

$(2s^2 + 5s + 3)\hat{X}_2 - 3\hat{X}_1 - 5s\hat{X}_3 = \hat{F}$

$(3s^2 + 7s + 2)\hat{X}_3 - 5s\hat{X}_2 - 2s\hat{X}_1 = 0$

Matrix :
$$\underbrace{\begin{bmatrix} 4s^2+4s+7 & -3 & -2s \\ -3 & 2s^2+5s+3 & -5s \\ -2s & -5s & 3s^2+7s+2 \end{bmatrix}}_{\text{'}P(s)\text{'}} \begin{bmatrix} \hat{X}_1 \\ \hat{X}_2 \\ \hat{X}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{F} \\ 0 \end{bmatrix}$$

Cramer's rule :- Unique solution for $\hat{X}_1, \hat{X}_2, \hat{X}_3$ can be found if $\det P(s) \neq 0$

* $P^{-1}(s) = \dfrac{\operatorname{adj} P(s)}{\det P(s)}$

→ 2^{nd} order state space ~~eq~~ system
↳ 2 state variables.

- State space form only has first order derivative
- If a system has x_1'', then choose state variables as x_1 (for x_1, x_1') and $x_2 = x_1'$ (for x_1' and x_1'')

eg $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$

Output $y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

This is a state space system because :-

1) Derivatives are at most 1^{st} order

2) Output is a linear function of state variables.

Equations :- $s\hat{X}_1 = \hat{X}_2$

$s\hat{X}_2 = -6\hat{X}_1 - 5\hat{X}_2 + \hat{U}$

$\hat{Y} = \hat{X}_1$

Solve to get transfer function :- $\dfrac{\hat{Y}}{\hat{U}} = \dfrac{1}{s^2+5s+6}$

$\equiv$

$\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$

→ ABCD form :- $\dot{x} = Ax + Bu$

$y = Cx + Du$

$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -6 & -5 & 1 \\ \hline 1 & 0 & 0 \end{array}\right]$

Zero initial conditions:

$X(0) = 0$

If not, then $\dot{y} \rightarrow s\hat{Y} -$

Laplace :- $\hat{X} = A\hat{X}_1 + B\hat{U}$

$s\hat{Y} = C\hat{X} + D\hat{U}$

Solution :- $\hat{X} = (sI - A)^{-1} B\hat{U}$

$\hat{Y} = (C(sI - A)^{-1}B + D)\hat{U}$

P.T.O.

$$sI - A = \begin{bmatrix} s & -1 \\ 6 & s+5 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{\mathrm{adj}(sI - A)}{|sI - A|} \qquad |sI-A| \rightarrow s^2 + 5s + 6$$

$$\mathrm{adj}(sI - A) = \begin{bmatrix} s+5 & -6 \\ 1 & s \end{bmatrix}^T = \begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix}$$

$$\hat{y} = \frac{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{s^2 + 5s + 6} = \frac{1}{s^2 + 5s + 6} \checkmark$$

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eg $\dot{x} = -ax + bu \Rightarrow \dfrac{\hat{X}}{\hat{U}} = \dfrac{b}{s+a}$

- First order state space system have denominator of degree 1.

eg $\dot{y} = -ay + bu.$

$$s\hat{Y} - y(0) = -a\hat{Y} + b\hat{U}$$

$$\hat{Y} = \frac{1}{s+a} y_0 + \frac{b}{s+a} \hat{U}$$

$$\therefore y(t) = y_0 e^{-at} + L^{-1}\left(\frac{b}{s+a} \hat{U}\right)$$

- $L^{-1}\left(F(s)\,G(s)\right) = \int_0^{\infty} f(t-\tau)\,g(\tau)\,d\tau$

$$= \int_0^{t} f(t-\tau)\,g(\tau)\,d\tau$$ (for causal systems to bound $g(\tau)$ to within $(0,t)$)

A. We assume that $f(t) = 0 \;\; \forall \; t < 0$

$$y(t) = y_0 e^{-at} + b\int_0^t e^{-a(t-\tau)} u(\tau)\,d\tau$$

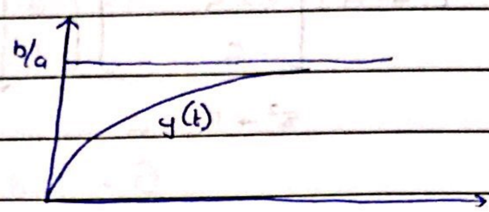
→ Response parameters

eg $\dot{y} = -ay + bu$. with $u(t) =$ ~~Step~~ Unit step

$$\hat{u}(s) = \frac{1}{s}.$$

$$\hat{y} = \frac{b}{s+a} \times \frac{1}{s} = \frac{b/a}{s} + \frac{b/-a}{s+a}$$

$$y(t) = \frac{b}{a}\left(1 - e^{-at}\right).$$



1. Time constant $= \frac{1}{a}$.

2. Rise time $\triangleq$ Time taken to rise from 10% to 90% $= T_r$

3. Settling time $\triangleq$ Time taken to reach 2% of settling value $= T_s$

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$$\dot{x} = Ax + Bu$$
$$y = Cx + Du.$$

where $x : n \times 1$, $u : m \times 1$, $y : p \times 1$

$\therefore$ $A : n \times n$, $B : n \times m$, $C = p \times n$, $D : p \times m$

- Simplified single-input single-output systems.

$$\dot{x} = Ax + bu \qquad \text{with } x(0) = x_0$$
$$y = \cancel{Cx}\ cx + du.$$

Represented as $\left[\begin{array}{c|c} A & b \\ \hline c & d \end{array}\right]$

ution

$$\dot{x} = Ax + bu$$
$$s\hat{X} - x_0 = A\hat{x} + b\hat{U}$$
$$\hat{X} = (sI - A)^{-1}(X_0 + b\hat{U})$$

Substitute in $\hat{Y} = c\hat{X} + d\hat{U}$

III STATE TRANSITION MATRIX → Element-wise Laplace inverse

→ Define $\Phi(t) = L^{-1}((sI - A)^{-1})$

$\therefore$ $\Phi(t)$ is a solution of $\dot{z} = Az$ with $z(0) = I$

$$s\hat{z} - I = A\hat{z}$$

$$\therefore \hat{z} = (sI - A)^{-1}$$

$$\Phi(t) \sim e^{t}$$

- Properties :-

1) $\Phi(0) = 1$
2) $\dfrac{d\Phi(t)}{dt} = A\Phi(t)$
3) $\Phi(t+s) = \Phi(t)\Phi(s)$
4) $\Phi(-t) = \Phi(t)^{-1}$
5) $\det \Phi(t) =$

eg $A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 6 & s+5 \end{bmatrix}^{-1}$$

$$= \frac{1}{s^2+5s+6} \begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix}$$

$$L^{-1}((sI-A)^{-1}) = L^{-1}\left(\frac{1}{(s+2)(s+3)} \begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix}\right)$$

$$= L^{-1}\left(\frac{[M]}{s+2} + \frac{[N]}{s+3}\right)$$

$$= e^{-2t} M + e^{-3t} N.$$

$$= \Phi(t)$$

→ Matrix exponential function (e^A)

$$= I + A + \frac{A^2}{2!} \ldots$$

Define $M_n = I + A + \frac{A^2}{2!} \ldots \frac{A^n}{n!}$

Define $\|M\| = \max_{n} \cancel{(|m_{ij}|)} \; (\Sigma |m_{ij}|)$ = maximum absolute element.

- Satisfies triangle inequality :- $\|A+B\| \leq \|A\| + \|B\|$
- Satisfies Schwarz inequality :- $\|AB\| \leq \|A\| \, \|B\|$

Using the inequalities, $\|M_n\| \leq 1 + \|A\| + \frac{\|A\|^2}{2!} \ldots \frac{\|A\|^n}{n!}$

$$\therefore \lim_{n \to \infty} \|M_n\| \leq e^{\|A\|}$$

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Definition State Transition Matrix (STM) Φ

General solution of homogeneous equation $\dot{x} = Ax$, with $x(0) = x_0$

$$x(t) = \Phi(t) x_0$$

$$\hookrightarrow \triangleq STM = L^{-1}(sI - A)^{-1}$$

- $\Phi(t)$ depends only on A, not on x_0

$$s\hat{X} - X_0 = A\hat{X}$$

$$\therefore \hat{X} = (sI - A)^{-1} X_0$$

- $x(t) = \Phi(t) x_0$ is unique solution of the differential equation

- $\Phi(t)$ is the unique solution of $\dot{z} = Az$ with $z(0) = I$

This unique solution must be $M(t) = I + At + \frac{A^2 t^2}{2!} \ldots$

$M(0) = I$, $\dot{M}(t) = A\,M(t)$

$$\therefore \Phi(t) = e^{At}$$

- Transition from t_0 to t_2 with $t_1 \in (t_0, t_2)$:-

$$x(t_2) = \Phi(t_2 - t_1)\,\Phi(t_1 - t_0)\,x(t_0)$$
$$= \Phi(t_2 - t_0)\,x(t_0)$$

$\therefore \quad \Phi(s+t) = \Phi(s)\,\Phi(t)$

Is $e^{A(t+s)} = e^{At} e^{As}$?

Theorem $\quad e^{A+B} = e^{A} e^{B} \quad$ iff $\quad AB = BA$

→ Properties of STM

1. $\Phi(0) = I$
2. $\Phi(t+s) = \Phi(t)\,\Phi(s)$
3. $\Phi(-t) = \Phi(t)^{-1}$
4. $\det \Phi(t) \neq 0 \quad \forall t$
5. $\dot{\Phi}(t) = A\,\Phi(t)$
6. $\det \Phi(t) = e^{(\text{trace } A)\, t}$

→ Effect of transformation (from x variable to y)

$\dot{x} = Ax$ with $x_0 = x(t_0)$

$x = Ty$ and $y = T^{-1}x$ (where T is a non-singular operation)

$$\frac{d}{dt}(Ty) = ATy$$

$$T\dot{y} = ATy$$

$$\therefore \quad \dot{y} = (T^{-1}AT)\,y$$

~~STM~~: 'A → $T^{-1}AT$'

- Given $\Phi(t) = L^{-1}(sI - A)^{-1}$, what is $\psi(t) = L^{-1}(sI - B)^{-1}$ where $B = T^{-1}AT$?

$$\psi(t) = T^{-1}\,\Phi(t)\,T$$

Write $\psi(t) = I + B + \frac{B^2}{2!} \cdots = T^{-1}\left(I + A + \frac{A^2}{2!} \cdots\right)T = T^{-1}\Phi(t)\,T$

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IV ~~SISO~~ ~~SYSTEMS~~ STATE SPACE FROM TRANSFER FUNCTION

B is a column vector, C row } for SISO

→ $\dot{x} = Ax + Bu$

$y = Cx + Du$

$x(t) = \Phi(t)\,x_0 + \int_{t_0}^{t} \Phi(t-\tau)\,B\,u(\tau)\,d\tau$ $\quad$ $\Phi(t) = e^{-at}$ for non-matrix

$y(t) = C\,\Phi(t)\,x_0 + \int_{t_0}^{t} C\,\Phi(t-\tau)\,B\,u(\tau)\,d\tau + D\,u(t)$

Transfer matrix :- $T(s) = C(sI - A)^{-1}B + D$

$L^{-1}\,T(s) = C\,\Phi(t)\,B$

$Y(s) = T(s)\,U(s)$ $\quad$ If $U(s)$ is scalar and not a matrix then $T(s) = \dfrac{\hat{Y}(s)}{\hat{U}(s)}$

→ State space from transfer function

1 $\quad \dfrac{\hat{y}}{\hat{u}} = \dfrac{b_0}{s^2 + a_1 s + a_2}$

$\ddot{y} + a_1\dot{y} + a_2 y = b_0 u$

State variables :- $x_1 = y$, $x_2 = \dot{y}$,

$\dot{x}_2 = -a_2 x_1 - a_1 x_2 + b_0 u$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\therefore \quad \frac{b_0}{s^2 + a_1 s + a_2} \longleftrightarrow \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -a_2 & -a_1 & b_0 \\ \hline 1 & 0 & 0 \end{array}\right]$$

2 $\quad \dfrac{\hat{Y}}{\hat{U}} = \dfrac{b_0 s + b_1}{s^2 + a_1 s + a_2}$

Consider $\quad \hat{y} \leftarrow \boxed{b_0 s + b_1} \xleftarrow{\hat{Y}_1 = \hat{X}} \boxed{\dfrac{1}{s^2 + a_1 s + a_2}} \leftarrow \hat{U}$

$x_1 = y_1 \qquad x_2 = \dot{x}_1$

$\dot{x}_1 = x_2$

$\dot{x}_2 = -a_2 x_1 - a_1 x_2 + u$

$y = b_0 x_2 + b_1 x_1$

$\therefore \dfrac{\hat{y}}{\hat{u}} = \dfrac{b_0 s + b_1}{s^2 + a_1 s + a_2}$ and $\dfrac{\hat{x}}{\hat{u}} = \dfrac{1}{s^2 + a_1 s + a_2}$ and $\hat{y} = (b_0 s + b_1)\hat{x}$

$$\longleftrightarrow \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -a_2 & -a_1 & 1 \\ \hline b_1 & b_0 & 0 \end{array}\right]$$

3 $\dfrac{\hat{y}}{\hat{u}} = \dfrac{b_0 s^2 + b_1 s + b_2}{s^2 + a_1 s + a_2} = b_0 + \dfrac{(b_1 - b_0 a_1)s + (b_2 - b_0 a_2)}{s^2 + a_1 s + a_2}$

$= b_0 + ②$

$\therefore \dot{x} = Ax + bu$

$y = cx + b_0 u$

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ -a_2 & -a_1 & 1 \\ \hline b_1 - b_0 a_1 & b_2 - b_0 a_2 & b_0 \end{array}\right]$$

4 $\dfrac{\hat{y}}{\hat{u}} = \dfrac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3} \longleftrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -a_3 & -a_2 & -a_1 & 1 \\ \hline \cdot & & & b_0 \end{array}\right]$

HW

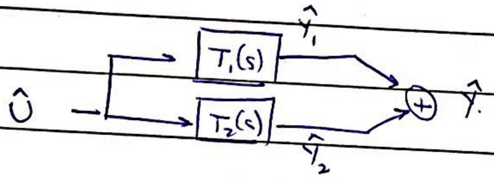
eg $\dfrac{\hat{y}}{\hat{u}} = \dfrac{s^3 + 2s + 1}{s^3 + 3s^2 + 3s + 1} = 1 + \dfrac{-3s^2 - s}{s^3 + 3s^2 + 3s + 1}$

$A + 3 = 0$

$$\hat{u} \longrightarrow \boxed{\dfrac{1}{s^3 + 3s^2 + 3s + 1}} \xrightarrow{\hat{x}} \boxed{s^3 + 2s + 1} \longrightarrow \hat{y}$$

IV Block Diagram Calculus In State Space

A] Sum of two systems :-



$$T(s) = T_1(s) + T_2(s)$$

If $T_1(s) \longrightarrow \left[\begin{array}{c|c} A_1 & b_1 \\ \hline c_1 & d_1 \end{array}\right]$ with x_1 and y_1

and $T_2(s) \longrightarrow \left[\begin{array}{c|c} A_2 & b_2 \\ \hline c_2 & d_2 \end{array}\right]$ with x_2 and y_2

Then we can take $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $y = y_1 + y_2$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 x_1 + b_1 u \\ A_2 x_2 + b_2 u \end{bmatrix}$$

$$y = c_1 x + c_2 x_2 + (d_1 + d_2) u$$

This gives $\left[\begin{array}{cc|c} A_1 & 0 & b_1 \\ 0 & A_2 & b_2 \\ \hline c_1 & c_2 & d_1 + d_2 \end{array}\right]$

Then $T(s) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} (sI - A_1)^{-1} & 0 \\ 0 & (sI - A_2)^{-1} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + d_1 + d_2$

comes out to be $T_1(s) + T_2(s)$

29/1

B] Product of two systems

$$\hat{u} \longrightarrow \underset{\frac{b_1}{s+a_1}}{\boxed{T_1(s)}} \xrightarrow{\hat{y}_1} \underset{\frac{b_2}{s+a_2}}{\boxed{T_2(s)}} \longrightarrow \hat{y}$$

T_1 : $\dot{x}_1 = -a_1 x_1 + b_1 u$; $y_1 = x_1$

T_2 : $\dot{x}_2 = -a_2 x_2 + b_2 y_1$; $y = x_2$

~~After cascading~~
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -a_1 & 0 \\ b_2 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

eg →
$$\hat{u} \longrightarrow \underset{x_1 \;\; x_2}{\boxed{\frac{b_0 s + b_1}{s^2 + a_1 s + a_2}}} \xrightarrow{y} \underset{x_3}{\boxed{\frac{c}{s+a}}} \longrightarrow \hat{y}$$

$\dot{x}_1 = x_2$

$\dot{x}_2 = -a_2 x_1 - a_1 x_2 + u$

$y_1 = b_1 x_1 + b_0 x_2$

$\dot{x}_3 = -a x_3 + c y_1$

$= b_1 c x_1 + b_0 c x_2 - a x_3$

$y = x_3$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -a_2 & -a_1 & 0 \\ b_1 c & b_0 c & -a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \quad , \quad y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

V INVERSE OF A SYSTEM

* We cannot write ABCD form if transfer function has deg(N) > deg(D)

* $T(s) \longleftrightarrow \left[\begin{array}{c|c} A & b \\ \hline c & d \end{array}\right]$ for SISO and $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ for MIMO

where $T(s) = \dfrac{b_0 s^2 + b_1 s + b_2}{s^2 + a_1 s + a_2} = c(sI - A)^{-1} b + d$

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Let $T(s)^{-1} = \dfrac{s^2 + a_1 s + a_2}{b_0 s^2 + b_1 s + b_2}$

TFT:- Formula for $T(s)^{-1}$, when $T(s)^{-1}$ exists.

- If $T(s)^{-1}$ exists, both N and D of both T & T^{-1} must have same degrees ~~and~~ (otherwise one of them would become improper rational function)

$\therefore$ ~~Let $T(s) = s^2$~~

- Since $T(s) = c(sI - A)^{-1} b + d = \dfrac{b_0 s^2 + b_1 s + b_2}{s^2 + a_1 s + a_2}$

$\therefore \quad d = b_0 = T(\infty)$

- Now, $T^{-1}(s)\, T(s) = I \quad \forall s$

$\therefore \quad T^{-1}(-\infty)\, D = I$

- For T, $\dot{x} = Ax + Bu$

$y = Cx + Du$

Assuming D^{-1} exists, $D^{-1} y = D^{-1} C x + u$

$\therefore \quad u = -D^{-1} C x + D^{-1} y$

$\therefore$ Equations for $T^{-1}(s)$:- $\dot{x} = Ax + B(-D^{-1} C x + D^{-1} y)$

y is I/p, u is I/p $\quad u = -D^{-1} C x + D^{-1} y$

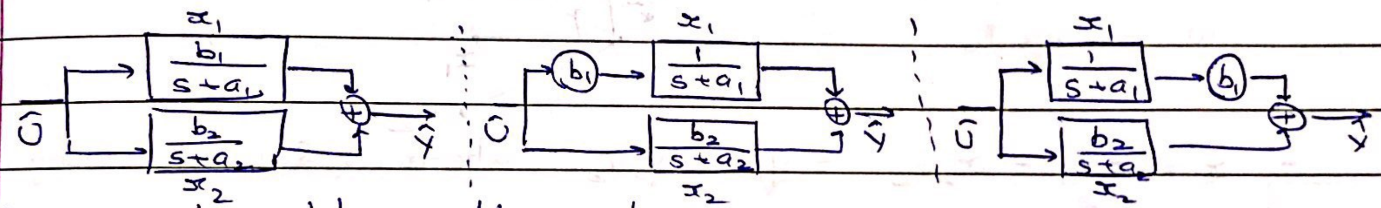
$$T^{-1}(s) \longleftrightarrow \left[\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right]$$

eg $T(s) = \dfrac{2s^2 + 3s + 1}{s^2 + 2s + 1}$

$\ddot{y} + 2\dot{y} + y = 2\ddot{u} + 3\dot{u} + 1$

VI Pole Zero Cancellation

If we don't use a minimal set of state variables (there is redundancy), then final transfer function obtained by solving from ABCD form will have a pole-zero cancellation.

eg


TF is a two-state-variable system

$$\frac{b_1}{s+a_1} + \frac{b_2}{s+a_2} = \frac{b_1 s + b_1 a_2 + b_2 s + b_2 a_1}{(s+a_1)(s+a_2)}$$

Consider $b_1 \to 0$: '$s+a_1$' is a pole-zero cancellation.

~~This~~

For $b_1 \neq 0$, all three systems behave equivalently.

For $b_1 \to 0$, in system 2 → input is not reaching system

System output is reaching output

3 → उल्टा

→ When $T(s)$ has no pole-zero cancellation, then

No. of state variables = Degree of denominator

↓

= Order of system

Converse - If state space model has minimal order, then there is no pole-zero cancellation in the formula $T(s) = C(sI - A)^{-1}b + d$

A] Test for minimality of state space representing a transfer function.
From the system of vectors,

$$\text{rank}[b, Ab, A^2b \dots A^{n-1}b] = n$$

$$\text{rank}[C^T, A^TC^T, A^{T^2}C^T \dots A^{T^{n-1}}C^T] = n$$

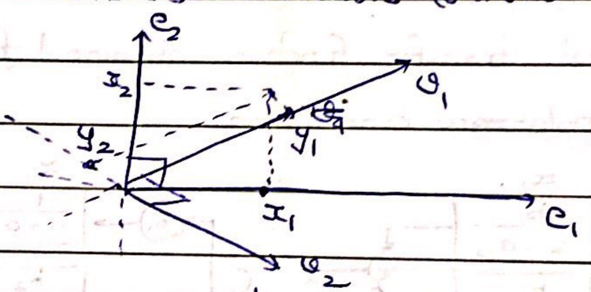
- Set of matrices of rank $< n \subseteq n \times n$ matrices

VII

- Vector $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \sum x_i e_i$
  ↳ ~~on basis~~ unit vectors (orthonormal)

$\dot{x} = Ax + bu$

$\omega = Cx + du$



- $x = x_1e_1 + x_2e_2 = y_1v_1 + y_2v_2$ where v_1 & v_2 are linearly independent

- $X = Vy$ $\quad V = [v_1 \; v_2]$, $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ --- new co-ordinates
- $\dot{x} = V\dot{y}$
- $x = Ax + bu$
  $= AVy + bu$

$W = Cx + du$
$= CVy + du$

$$\left[\begin{array}{c|c} V^{-1}AV & V^{-1}b \\ \hline CV & d \end{array}\right]$$

- $\dot{y} = (V^{-1}AV)y + (V^{-1}b)u$, $\omega = CVY + du$

$y = (s - V^{-1}AV)^{-1} V^{-1} bu$

- $G(s) = cV(sI - V^{-1}AV)^{-1}V^{-1}b + d$

$= c(sI - A)^{-1}b + d$

$G(s) = T(s)$

$(AB)^{-1} = B^{-1}A^{-1}$

$(V(sI - V^{-1}AV)V^{-1})^{-1}$

$= (sI - A)^{-1}$

VII Solution of State Space Equations by Coordinate Transformations

- Solution by Transformation

$$\dot{x} = Ax + bu$$

Transform $x = Vy$ (assumption $x \in$ eigenvector space $\subset \mathbb{R}^n$)

$A v_1 = \lambda_1 v_1$, $A v_2 = \lambda_2 v_2$

$x = y_1 v_1 + y_2 v_2$ v_1 & v_2 are eigenvectors

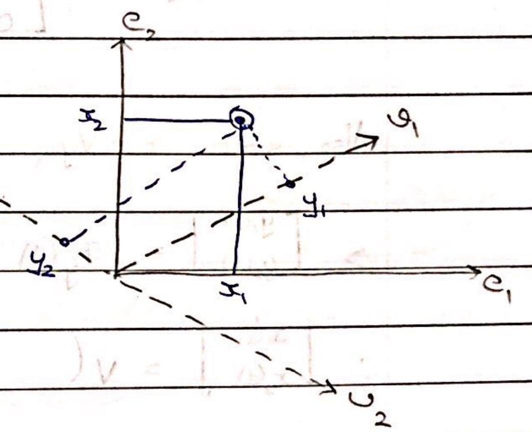
$$\dot{x} = A(y_1 v_1 + y_2 v_2) + bu$$
$$= y_1 \lambda_1 v_1 + y_2 \lambda_2 v_2 + bu$$

4/2 $\dot{x} = Ax + bu$

$\omega = Cx + du$

↳ output

→ Transformation :-



Initially, $X \equiv \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 e_1 + x_2 e_2$

New $X \equiv \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = y_1 v_1 + y_2 v_2$

$\therefore$ $X = Vy = [v_1 \;\; v_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$\therefore$ y_1, y_2 are new state variables

(v_1, v_2 are linearly independent like e_1, e_2)

- New equations : $V\dot{y} = AVy + bu$

$\omega = CVy + du$

- $\Phi_x = V \Phi_y V^{-1}$

★ Eigen values $\triangleq$ Solutions of $p(\lambda) = \det(Av - \lambda v)$

Choose v_1, v_2 to be eigenvectors of A.

Assume A is non-singular $\Rightarrow$ v_1, v_2 are linearly independent

$$Av_1 = \lambda_1 v_1 \quad \text{and} \quad Av_2 = \lambda_2 v_2.$$

$$x = y_1 v_1 + y_2 v_2$$

$$\therefore \dot{x} = \dot{y}_1 v_1 + \dot{y}_2 v_2 = Ax + bu$$

$$= A(y_1 v_1 + y_2 v_2) + bu$$

$$= y_1 \lambda_1 v_1 + y_2 \lambda_2 v_2 + \underbrace{(\beta_1 v_1 + \beta_2 v_2)}_{=b} u$$

Projection of $\vec{b}$ in v_1 & v_2 space

$$\therefore \dot{y}_1 \cancel{v_1} = y_1 \lambda_1 \cancel{v_1} + \beta_1 u \cancel{v_1}$$

$$\dot{y}_2 = y_2 \lambda_2 + \beta_2 u$$

$$y_1(t) = e^{\lambda_1 t} y_1(0) + \int_0^t \beta_1 e^{\lambda_1 (t-\tau)} u(\tau)\, d\tau$$

$$y_2(t) = e^{\lambda_2 t} y_2(0) + \int_0^t \beta_2 e^{\lambda_2 (t-\tau)} u(\tau)\, d\tau.$$

$$\therefore \Phi_y(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix}$$

zero initial condition

Also, since $x = Vy$, write $\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = V^{-1} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$

$$\vec{x}(0) \qquad \vec{y}(0) = V^{-1} [\vec{x}(0)]$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \underbrace{\Phi_y(t)\ V^{-1} x(0)}_{y(0)} + \underbrace{\ldots}$$

$$\therefore \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{V(\qquad)} + V(\qquad)$$

$$V \Phi_y V^{-1} = \Phi_x$$

eg $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

- Eigenvalues :- $\lambda^2 + 3\lambda + 2 \Longrightarrow \lambda_1 = -1,\ \lambda_2 = -2$
- Eigenvectors :- Consider $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

P.T.O

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = (-1)\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(-2) \qquad v_2 \qquad \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\therefore V = [v_1 \; v_2] = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$$

$$\therefore V^{-1} = \frac{1}{(-1)}\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

* Common eigenvalue condition :- $AV = V\beta$

or $V^{-1}AV = \beta$ → diagonal matrix of all eigenvalues.

- For $u = 0$, $y_1 = e^{-t}$ and $y_2 = e^{-2t}$

OR

- In general, $x(t) = \Phi_x(t)\,x(0) + \int \Phi_x(t-\tau)\,b\,u(\tau)\,d\tau$

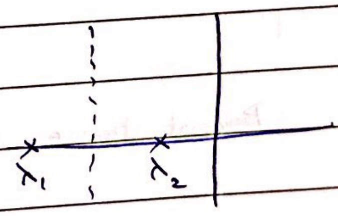
- Zero-state solution $(x(0) = \vec{0})$

☆ • Step response :- $c(t) = L^{-1}\left(\dfrac{1}{s(s^2+3s+2)}\right)$ .... output response

$$\omega(t) = C\int_0^t \Phi_x(t-\tau)\,b\,d\tau$$

* Second order response to unit step.

$$c(t) = k + c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

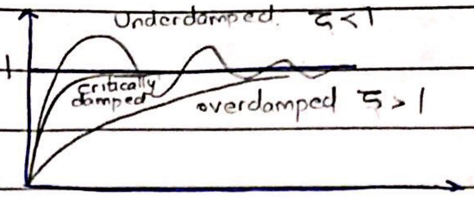


As λ_1 and λ_2 both approach vertical line, $\left(\frac{\lambda_1+\lambda_2}{2}\right)$, rise time ↓

Overdamped ⟶ Critically damped.

P.T.O.

→



Standard Second Order Response Model

Transfer function :- $\dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ → ω_n Natural frequency, ζ → Damping factor

Eigenvalues :- $\lambda_1, \lambda_2 = -\underbrace{\zeta\omega_n}_{\sigma} \pm j\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d}$ .... roots of denominator

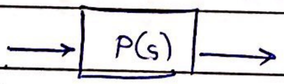
ω_d → Damped frequency

Q $c(t) = 1 - \dfrac{1}{\sqrt{1-\zeta^2}} e^{-\sigma t} \cos(\omega_d t - \phi)$ where $\phi = \tan^{-1} \dfrac{\zeta}{\sqrt{1-\zeta^2}}$

Prove that step response of general second order system is this

- Properties – 'Performance Parameters'
  - Time to peak (T_p) : First root of $c'(t) = 0$
  - Settling time : $\pm 2\%$ band
  - Rise Time
  - Steady state error $= 1 - c(\infty)$

→ Sensitivity

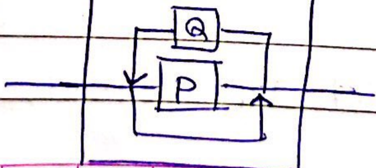


$\tilde{P}(s) = P(s) + \Delta(s)$

Actual, Modelled, Perturbation

Percent change $\sim \dfrac{\Delta(s)}{\tilde{P}(s)}$ (per unit change)

- For correction, system P is embedded inside a network of controllers



After embedding, $T \equiv$ Nominal system

$\tilde{T} \equiv$ Perturbed system

$\Delta T = \tilde{T} - T \equiv$ Perturbation

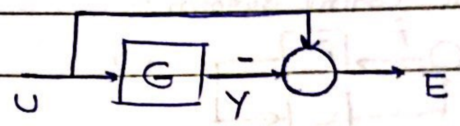
Percent change $\sim \dfrac{\Delta T}{\tilde{T}}$

We want : $\dfrac{\Delta T/\tilde{T}}{\Delta/\tilde{P}} < 1$

'Sensitivity'

$\rightarrow$ First order system

$$G(s) = \frac{b}{s+a}$$



$$E = \frac{1}{s} - Y = \frac{1}{s} - \frac{b}{s(s+a)} = \frac{s+(a-b)}{s(s+a)}$$

Steady state error $= e_{ss} = \dfrac{a-b}{a} = 1 - \dfrac{b}{a}$ (Final Value Theorem $(sE)\big|_{s=0}$)

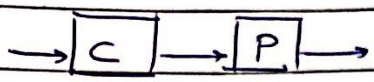
$$\frac{\tilde{b}}{\tilde{a}} = \frac{b}{a} + \Delta$$

$$\Delta e_{ss} = \tilde{e}_{ss} - e_{ss} = 1 - \frac{\tilde{b}}{\tilde{a}} - \left(1 - \frac{b}{a}\right) = \frac{b}{a} - \frac{\tilde{b}}{\tilde{a}} = -\Delta$$

Percent variation in e_{ss} without any control $= \dfrac{\Delta e_{ss}}{e_{ss}} = \dfrac{-\Delta}{1 - \frac{b}{a}}$.

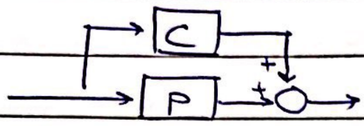
11/2 Sensitivity (S) = 100% in absence of control.

- If $S < 1$ over some frequencies, it must be > 1 over some other frequencies, to preserve Bode integral.

1 

$T = CP$, $\tilde{T} = C\tilde{P}$

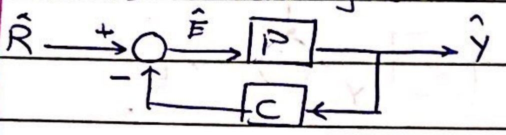
$$S = \frac{\Delta T/\tilde{T}}{\Delta P/\tilde{P}} = \frac{C\Delta P/C\tilde{P}}{\Delta P/\tilde{P}} = 1$$ ..... no effect on sensitivity.

2 

$T = P + C$, $\tilde{T} = \tilde{P} + C$

$S =$

3 Feedback Control System :-



$\hat{Y} = T\hat{R}$

$\hat{Y} = P\hat{E} = P(\hat{R} - C\hat{Y})$

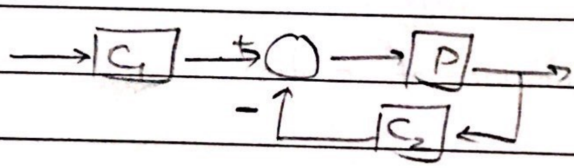
$$\Rightarrow T = \frac{P}{1+PC}$$

$$\Delta T = \tilde{T} - T = \frac{\tilde{P}}{1+\tilde{P}C} - \frac{P}{1+PC} = \frac{\tilde{P} - P}{(1+\tilde{P}C)(1+PC)}$$

$$= \frac{\Delta P}{(1+\tilde{P}C)(1+PC)}$$

$$\therefore \frac{\Delta T}{\tilde{T}} = \frac{\Delta P}{(1+\tilde{P}C)(1+PC)} \times \frac{1+\tilde{P}C}{\tilde{P}}$$

$$\therefore S = \frac{1}{1+PC}$$

HW 

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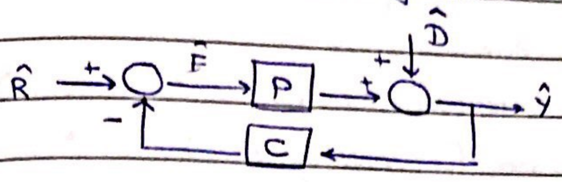
→ Properties of S in FB system :-



$\Delta P = \tilde{P} - P$ is ~~difference~~ ~~betwee~~ because of issues within model

$\hat{D}$, OTOH, is external disturbance

- Disturbed output is being read ($\hat{Y}$)

$$\hat{Y} = P\hat{E} + \hat{D}$$
$$= P(\hat{R} - C\hat{Y}) + \hat{D}$$
$$\hat{Y} = \frac{P}{1+PC}\hat{R} + \frac{1}{1+PC}\hat{D}$$

$= T \triangleq$ Transfer function from $\hat{R}$ to $\hat{Y}$

∴ In FB system, Transfer function from $\hat{D}$ to $\hat{Y}$ = Sensitivity.

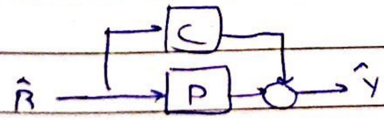
$$\hat{E} = \hat{R} - C\hat{Y}$$
$$= \left(1 - \frac{PC}{1+PC}\right)\hat{R} + \left(\frac{-C}{1+PC}\right)\hat{D}$$
$$= \left(\frac{1}{1+PC}\right)\hat{R} + \left(\frac{-C}{1+PC}\right)\hat{D}$$

∴ In FB system, Transfer function from $\hat{R}$ to $\hat{E}$ = Sensitivity

- Complementary Sensitivity Function $= S_c \triangleq 1 - S = \dfrac{PC}{1+PC}$

- Making S small is important in control system design, which can be done by FB system.

→ Forward Control



$T = P + C$, $\tilde{T} = \tilde{P} + C$

$\Delta T = \Delta P$

$$S = \frac{\Delta P / (\tilde{P} + C)}{\Delta P / \tilde{P}} = \frac{\tilde{P}}{\tilde{P} + C}$$

→ Mathematical framework for handling S.

1 Making sensitivity *small* is important to Control Systems Design
2 Feedback control scheme has scope for making sensitivity *small*
3 Mathematical framework for handling S
(Nbd, distance, large/smallness of transfer function)
4 Possible limits to changing S.
5 Constraints in designing control systems : stability.

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Happy Valentine's

Laplace Transform : $f(t) \in [0, \infty)$

$$\hat{F}(s) = \int_0^\infty f(t) e^{-st} dt$$

$f(t)$ is bounded by an exponential

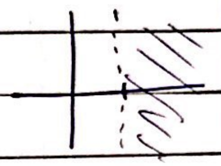
$\hat{F}(s)$ exists when $|\hat{F}(s)| < \infty$

$$|\hat{F}| = \left| \int_0^\infty f(t) e^{-st} dt \right| \leq \int |f(t)| e^{-\sigma t} dt$$

$$\leq \int C e^{(\sigma_0 - \sigma) t} dt$$

where $e^{\sigma_0 t}$ is the function bounding $f(t)$

$\therefore$ If $|f(t)| \leq C e^{\sigma_0 t}$, the $\hat{F}(s)$ exists $\forall \; \sigma > \sigma_0$



- Let $B \equiv$ ~~func~~ set of all bounded functions s.t. $f(t) \leq M < \infty$ for some M.
  'Uniformly bounded functions'

$B \subset$ Set of functions bounded by exponentials.

$\therefore$ Since $\sigma_0 = 0$, the RoC of any function in $B \equiv Re(s) > 0$.
'Right Half Plane'

$*$ $Re(s) \geq 0$: Closed RHP

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Theorem

- If $\hat{F}(s)$ has no poles in RHP and the poles on imaginary axis are simple, then $f(t)$ is uniformly bounded

  - If poles on imaginary axis are not simple,

$$\hat{F}(s) \propto \frac{1}{(s^2 + \omega^2)^M} \quad M > 1.$$

Then the inverse will be multiplied by $t, t^2, t^3 \ldots$, which makes $f(t)$ uniformly unbounded.

→ Let B be the set of bounded functions of $t \in [0, \infty)$ which have Laplace Transforms satisfying

1) No poles in RHP
2) Poles on imaginary axis are simple.

- An input-output linear system represented by a transfer function $G(s)$ is said to be BIBO stable if, for every input $u(t) \in B$, output $y(t)$ also $\in B$

$$\hat{u} \longrightarrow \boxed{G(s)} \longrightarrow \hat{y}$$

$$u(t) \in B \Rightarrow y(t) \in B$$

Theorem If a system is represented by transfer function $G(s)$ which has no pole-zero cancellation in the closed RHP ($\overline{RHP}$) (numerator and denominator don't have common roots lying in $\overline{RHP}$) then the system is BIBO stable iff $G(s)$ has no poles in $\overline{RHP}$

* $\mathcal{S}$:= Set of all BIBO stable systems

- If $G(s), H(s) \in \mathcal{S}$, then $G+H \in \mathcal{S}$ and $GH \in \mathcal{S}$
- Constants $\in \mathcal{S}$
- $\frac{1}{G(s)} \in \mathcal{S}$, only if all zeroes of $G(s)$ also lie in LHP and numerator and denominator have same degree.

→ Hurwitz Polynomials $\triangleq$ Polynomials with no root in $\overline{RHP}$.

- P, C can be in cascade, parallel or feedback

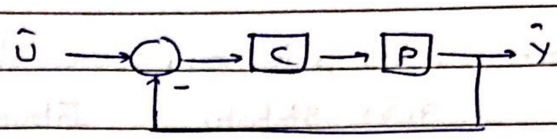
PC ↙ $\quad P+C$ ↙ $\quad$ ↘ $\frac{P}{1+PC}$

$PC \triangleq$ Open-loop transfer function

$\frac{1}{1+PC} \triangleq$ Sensitivity

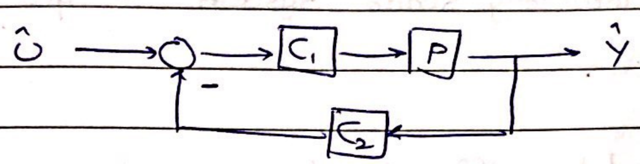
* Open loop transfer function $\triangleq$ Transfer function from one point in the loop back to the same point in the loop

- Feedback of cascade :-



$$T = \frac{PC}{1+PC} = 1 - S \quad , \quad OLTF = PC$$

- 2 degree of freedom feedback system.



$OLTF = C_1 P C_2$

- Feedback

—— x ——

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$$\hat{u} \longrightarrow \boxed{G(s)} \longrightarrow \hat{y}$$

$u(t) \in B \qquad\qquad y(t) \in B$

As a differential system $G(s)$ corresponds to:

$$p(D)\, y = q(D)\, u$$

$$\left(D = \frac{d}{dt} \ , \quad \text{All initial conditions zero } u(0) = 0 \right)$$

eg $G(s) = \dfrac{s+1}{s^2 + 3s + 5}$

$\Rightarrow (D^2 + 3D + 5)\, y(t) = (D+1)\, u(t)$ under zero initial condition

Now, zero-input response :-

$(D^2 + 3D + 5)\, y(t) = 0$ when $y(0) = y_0$, $\dot{y}(0) = y_1$

P.T.O.

- $\not\exists$ 'Internal Stability' :- Stability for zero input.
  - Depends only on denominator polynomial of $G(s)$.

- If there is no pole zero cancellation in RHP,

  BIBO stability = Internal Stability.

→ Internal Stability :-

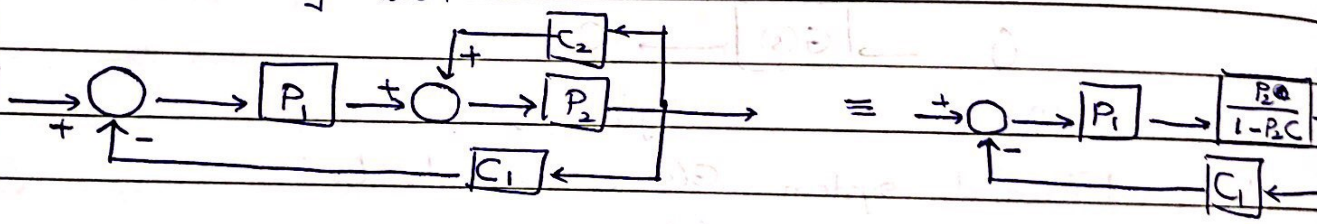
$G(s)$ represents an internally stable system if $y(t) \to 0$ for $t \to \infty$ under $u(t) = 0$.

Theorem $G(s)$ is internally stable iff $p(s)$ has no roots in RHP.

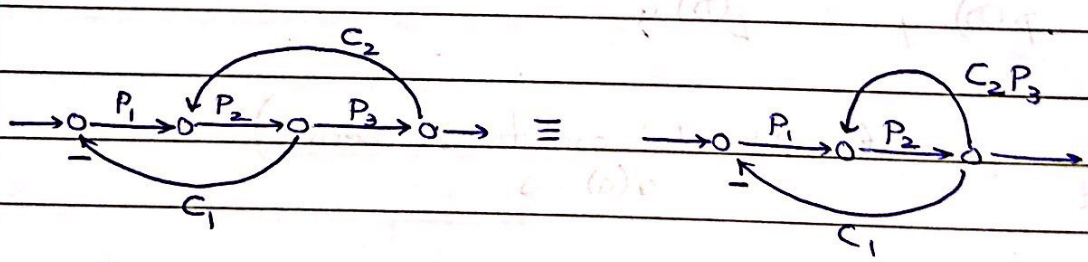
—×—

* Feedback Configuration

eg

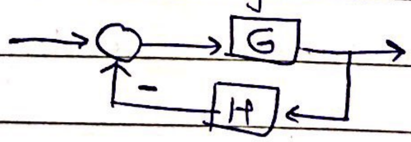


eg



Same as previous example.

∴ Every feedback configuration can be simplified to



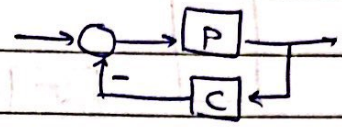
→ Stability of closed loop system :-

1) Input - output stability (BIBO)

2) Internal stability

Theorem: Let $P(s)$ and $C(s)$ be transfer functions, each of which have no pole-zero cancellation in RHP.

$P(s) = \frac{q(s)}{p(s)}$ and $C(s) = \frac{q_c(s)}{p_c \cancel{q}(s)}$



Then the FB system is stable iff

'Characteristic Polynomial' $\triangleq \psi(s) \triangleq p(s)p_c(s) + q(s)q_c(s)$

is Hurwitz.

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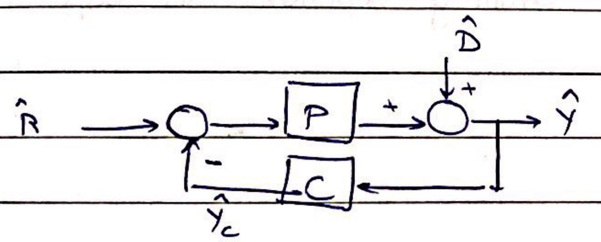
* While calculating sensitivity, if final expression contains $\tilde{P}$, we ~~kot~~ take $\Delta P \to 0$, $\tilde{P} \to P$.

Q



T = TF from $\hat{D}$ to $\hat{Y}_c$

Find sensitivity

$T = \frac{C}{1+PC}$, $\tilde{T} = \frac{C}{1+\tilde{P}C}$

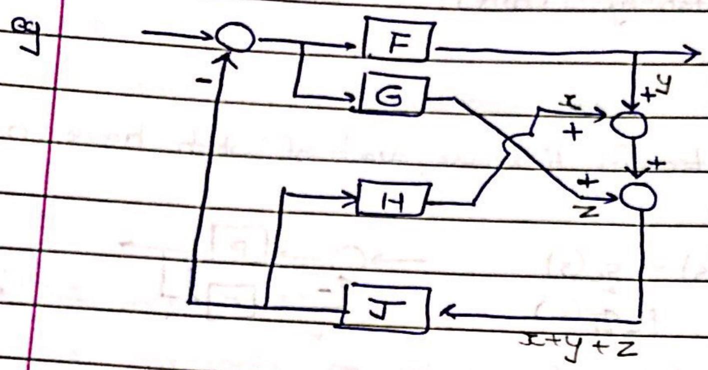
$\Delta T = \tilde{T} - T =$

$\frac{\Delta T}{\tilde{T}} = \frac{-C\Delta P}{1+PC}$

$S = \frac{\Delta T / \tilde{T}}{\Delta P / \tilde{P}} = \frac{-C\tilde{P}}{1+PC} \approx \frac{-CP}{1+PC}$, taking $\tilde{P} \to P$.

→ Finding characteristic polynomial.

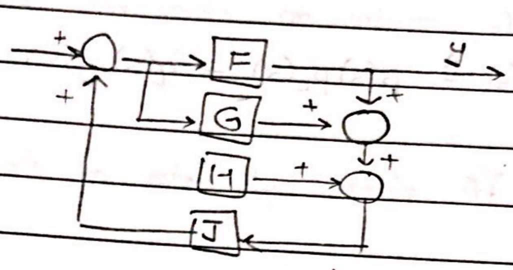
eg



First, convert to standard feedback form

Actual output = y

①



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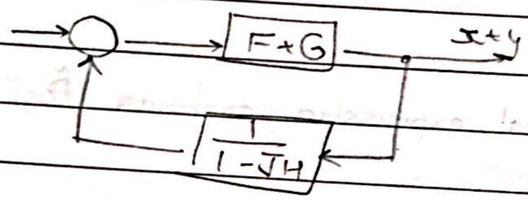

For (closed loop) stability purposes, we don't need actual output to be represent.

Stability depends on loop.

→ Routh Hurwitz Test.

eg $p(s) = s(s+1)(s+3) + k(s+2)$

$= s^3 + 4s^2 + (3+k)s + 2k$ ---- denominator of TF.

s^3	1	$3+k$	---- Alternate coefficients
s^2	4	$2k$	---- Alternate coefficients
s	$\frac{-(1)(2k)+(4)(3+k)}{4}$	0	
1	$\frac{-(4)(0)+(2k)\left(\frac{(1)(2k)-(4)(3+k)}{4}\right)}{\frac{(1)(2k)-(4)(3+k)}{4}} = 2k$	0	

Number of roots (poles) in RHP = No. of sign changes in first column

$\therefore$ For stability, $1 > 0$

$4 > 0$

$$\frac{(1)(2k) + (4)(3+k)}{4} > 0$$

$\sim\sim > 0$

eg $p(s) = s^4 + 14s^3 + 53s^2 + 82s + 300$

s^4	1	53	300
s^3	14	82	0
s^2	$\frac{(14)(53) - 82}{14}$	$\frac{(14)(300) - (1)(0)}{14}$	0
s			
1			

eg $p(s) = s^3 + 2s^2 + (k-3)s + 2k$

Issues 1) An element in first column becomes zero

2) An entire row becomes zero.

eg -

1	2	3
1	2	3
→ 0	0	0

1) Zero in first column

s^5	1	4	3
s^4	1	2	1
s^3	$\cancel{0}$ 2	2	0
s^2	1	1	0
s	0	0	
1			

2) An entire row becomes zero

s^6	1	2	3
s^5	1	2	3
s^4	$\cancel{0}$ 5	$\cancel{0}$ 6	$\cancel{0}$ 3
s^3	1	3	
s^2	-3	1	
s	$10/3$	0	
1	1		

→ Take Auxilliary polynomial

$A(s) = s^5 + 2s^3 + 3s$

(from second row)

$\frac{dA(s)}{ds} = 5s^4 + 6s^2 + 3$

~~No. of sig~~

- If an entire row becomes zero,

$\Rightarrow$ $A(s)$ is a factor of $p(s)$

$p(s) = A(s)\, p_1(s)$.

- No. of sign changes above row s^4 = No. of roots of $p_1(s)$ in RHP.
- No. of sign changes below s^4 = No. of roots of $A(s)$ in RHP