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→ Sampling Theorem

- Assumptions – Band-limited, Fourier transformable (Square integrability, Dirichlet conditions)
  There is a maximum value of rate of change.

- Given discrete-time samples, there are too many ~~...~~ signals that pass through them. (uncountably infinite)
- If we restrict ourselves only to sinusoids, even then all ☆ multiples of fundamentals would be acceptable.

eg – $A_0 \cos(\Omega_0 t + \phi_0)$ sampled at T_s

$$x(nT_s) = A_0 \cos(\Omega_0 nT + \phi_0)$$

$$= A_0 \cos\left[\pm(\Omega_0 nT + \phi_0 + 2\pi nk)\right] \quad \text{--- Ambiguity}$$

– Countably infinite no. of signals.

CP

$$x_{k+}(t) = A_0 \cos\left[\left(\frac{2\pi}{T_s}k + \Omega_0\right)t + \phi_0\right]$$

$$x_{k-}(t) = A_0 \cos\left[\left(\frac{2\pi}{T_s}k - \Omega_0\right)t - \phi_0\right]$$

Consider $S_N = x_0(t) + \sum_{k=1}^{N}\left[x_{k+}(t) + x_{k-}(t)\right]$

Interpret sampling theorem based on S_N

$$S_N = x_0(t) + \sum_{k=1}^{n} 2A_0 \cos(\Omega_0 t + \phi_0)\cos\left(\frac{2\pi k t}{T_s}\right) \quad \text{--- train of } \delta$$

$$= x_0(t)\left[1 + 2\,\frac{\sin\left(\frac{N}{2}\left(\frac{2\pi t}{T_s}\right)\right)\cos\left(\frac{N+1}{2}\,\frac{2\pi t}{T_s}\right)}{\sin\left(\frac{2\pi t}{2T_s}\right)}\right]$$

Defactorize numerator :- $= x_0(t)\,\dfrac{\sin\left(\frac{(2N+1)\pi t}{T_s}\right)}{\sin\left(\pi t/T_s\right)}$

At $t = nT_s$, use LHopital → $= (2N+1)\,x_0(nT_s)$

--- sampling

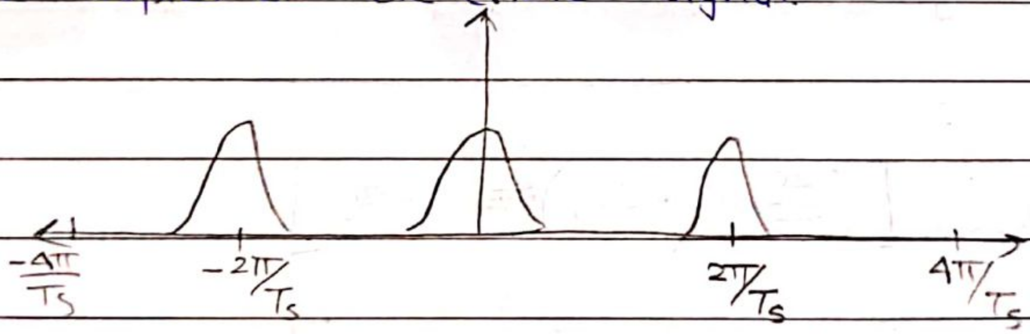
∴ Ideal sampling of sinusoid = Superposition of ~~ω~~ (infinitely) many periodically shifted fundamental sinusoid that add constructively at points of sampling and destructively everywhere else.

Delta function

Linear function

Sampling is linear :- Samples of $\alpha s_1 + \beta s_2 \rightarrow \alpha(\text{Sample of } s_1) + \beta(\text{Sample of } s_2)$

- Since a signal is summation of sinusoids, sampling makes periodic copies of the entire signal.



Discrete Time Signal Processing System

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Continuous $\rightarrow$ $\boxed{\text{ADC}}$ $\rightarrow$ Discrete $\rightarrow$ $\boxed{\text{System}}$ $\rightarrow$ Discrete $\rightarrow$ $\underbrace{\boxed{\text{DAC}}}_{\text{LPF of BW of signal}}$ $\rightarrow$ Continuous.

- CP :- Benefits of aliasing
- 'Discrete' v/s 'Digital' :- Digital is quantized.
  In this course, both are equal - Ignore quantization noise.

I Signals And Systems

- Signal $\triangleq$ Mapping from independent variable and dependent variable
  - Variables - Time, space, etc.
- System $\triangleq$ Mapping from signal to signal

Q Mapping from system to system = ?

- Signal Processing :- 1) Separating signals (as from noise)
  2) Modifying signals as required.
- Applications of Discrete Signal Processing - Audio, speech, biomedical, etc

Q More applications

Q Explain effects of 4 systems on audio :- $y[n] = \frac{1}{2}\left(x[n] \pm x[n-1]\right)$, 2

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* Discrete signal : $\mathbb{Z} \rightarrow \mathbb{C}$
- Output is a complex number because we can represent sinusoids as phasors.

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A] Properties of systems

1 Additivity

$x_{1,2}[n] \longrightarrow y_{1,2}[n]$ 'Multistatement'

Then $x_1[n] + x_2[n] \longrightarrow y_1[n] + y_2[n] \quad \forall\ x_1 \ \&\ x_2$

2 Homogeneity (Scaling)

$x[n] \longrightarrow y[n]$

Then $\alpha x[n] \longrightarrow \alpha y[n] \quad \forall\ x$ and $\forall\ \alpha \in \mathbb{C}$

3 Time (Shift) Invariance.

$x[n] \rightarrow y[n]$

Then $x[n - n_0] \longrightarrow y[n - n_0] \quad \forall\ x$ and $\forall\ n_0 \in \mathbb{Z}$

- System behaves the same all the time.

4 Linearity = Additivity + Homogeneity

III LTI SYSTEMS

The entire system is ~~described~~ completely characterized by unit impulse response 'h[n]'

* Arrow notation ≡ $x[n] = \{1, \underset{-1}{\overset{\uparrow}{5}}, 3, 0, 2\}$

$$= \delta[n+2] + 5\delta[n+1] + 3\delta[n] + 2\delta[n-2]$$

- $x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$

- $\delta[n] \longrightarrow h[n]$

$\therefore \delta[n-k] \longrightarrow h[n-k]$ .... Time invariance

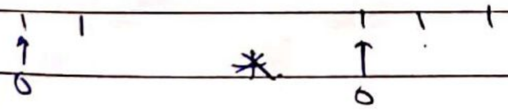
$\therefore x[k]\,\delta[n-k] \longrightarrow x[k]\,h[n-k]$ ..... Homogeneity

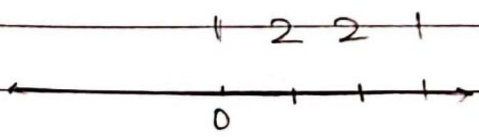
$\therefore \sum x[k]\,\delta[n-k] = x[n] \longrightarrow y[n] = \sum x[k]\,h[n-k]$ ..... Additivity

$$= [x * h][n].$$

A] → Convolution :- $y = x * h$, $x = x * \delta$.

$$y[n] = \sum x[k]\,\underbrace{h[n-k]}_{\text{Reflect, move ahead by } n}$$

Q. 



A Properties

1 Commutative :- $x_1 * x_2 = x_2 * x_1$

2 Associative :- $x_1 * (x_2 * x_3) = (x_1 * x_2) * x_3$

3 Distributive over addition :- $x_1 * (x_2 + x_3) = x_1 * x_2 + x_1 * x_3$

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Q Which systems will not have an impulse response?

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III PROPERTIES OF SYSTEMS (Part 2)

A] Causality

If x_1 and x_2 are such that $x_1[n] = x_2[n] \quad \forall\, n \leq n_0$,
then $y_1[n] = y_2[n] \quad \forall\, n \leq n_0$

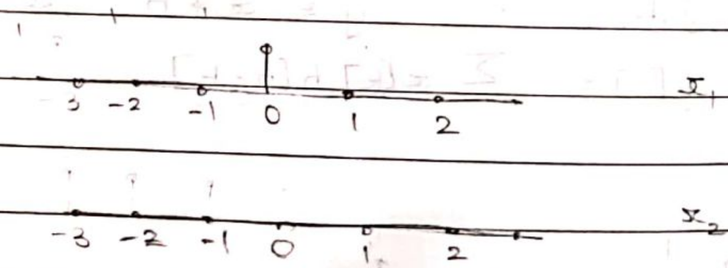
Theorem: LTI systems are causal iff $h[n] = 0 \quad \forall\, n < 0$.

Proof: $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=0}^{\infty} h[k]x[n-k]$

CP (only if) Assume $h[n] \neq 0 \quad \forall\, n < 0$

Counterexample

$h[-1] = 1$
and 0 otherwise



B] BIBO Stability

If $|x[n]| \leq M_x$ for some $0 \leq M_x \leq \infty$
then $|y[n]| \leq M_y$ $\quad M_y$

eg $y[n] = x^2[n]$ ✓
eg $\quad\;\; 1/x[n]$ ✗

Theorem: LTI system is BIBO stable iff $h[n]$ is absolutely summable

Proof: $y[n] = \sum h[k]\, x[n-k]$

$|y[n]| = |\sum (\quad)|$

$\leq \sum |h[k]|\,|x[n-k]|$

$\leq M_x \sum |h[k]|$

(only if) Suppose $\sum |h[k]| > M \quad \forall M \in \mathbb{R}$

Define $x[k] = \begin{cases} \dfrac{\overline{h[-k]}}{|h[-k]|} & \text{when } h[-k] \neq 0 \\ 0 & \text{when } h[-k] = 0 \end{cases}$

$\therefore x[k]$ is bounded by 1

$\rightarrow y[0] = \sum x[k]\, h[-k] = \sum |h[k]| \geq M \quad \forall M \in \mathbb{R}$

* Unit step $= u[n] = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$

CP $y[n] = 0.5 \{ x[n] \pm x[n - \frac{1}{2}] \}$

Consider $x[n] = A_0 \cos(\omega_0 n + \phi_0)$

Write $y[n]$ as $B_0 \cos(\omega_0 n + \alpha_0)$. Comment on B_0, α_0

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* Four simplest discrete time systems :- $y[n] = \frac{1}{2} \{ x[n] \pm x[n - \frac{1}{2}] \}$

- Linear, shift-invariant, stable, causal.

For such a system, Sampled sinusoid of frequency ω_0 $\xrightarrow{\text{system}}$ Sinusoid of frequency ω_0

- Normalized angular frequency $\triangleq 2\pi \dfrac{\text{Frequency}}{\text{Sampling frequency}}$ $\in [-\pi, \pi]$ at Nyquist condition,

$*$ Triangle input $\xrightarrow{?}$ Triangle input — Not always

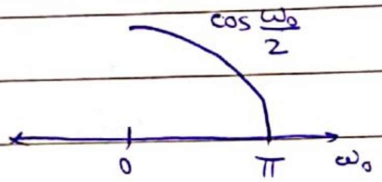
eg :- $y[n] = x[n] - x[n-1]$

1 $y[n] = \frac{1}{2}\left(x[n] + x[n-1]\right)$

$I/P = A_0 \cos(\omega_0 n + \phi_0)$

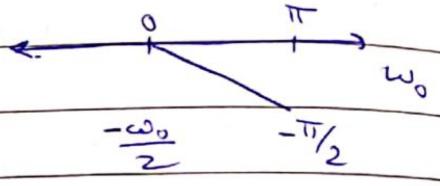
$O/P = A_0 \cos\left(\omega_0 n + \phi_0 - \frac{\omega_0}{2}\right) \cos\frac{\omega_0}{2}$

Amplitude change



LPF

Phase change



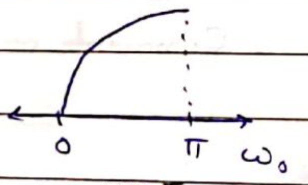
2 $y[n] = \frac{1}{2}\left(x[n] - x[n-1]\right)$

$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$

$= \frac{1}{2}\left(-2A_0 \sin\left(\omega_0 n + \phi_0 - \frac{\omega_0}{2}\right) \sin\frac{\omega_0}{2}\right)$

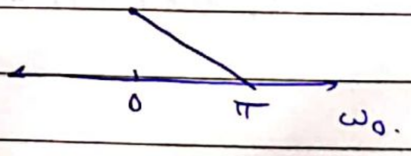
$\cos\left(\omega_0 n + \phi_0 - \frac{\omega_0}{2} + \frac{\pi}{2}\right)$

Amplitude change



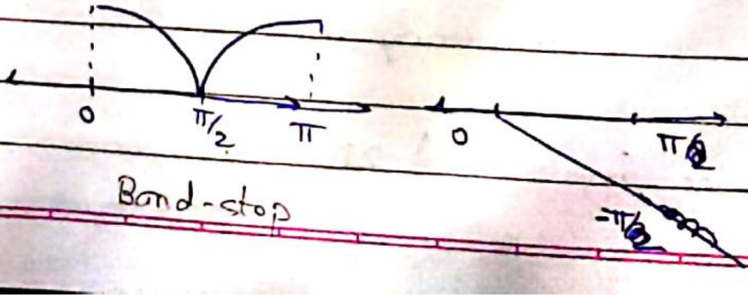
HPF

Phase change



3 $y[n] = \frac{1}{2}\left(x[n] + x[n-2]\right)$

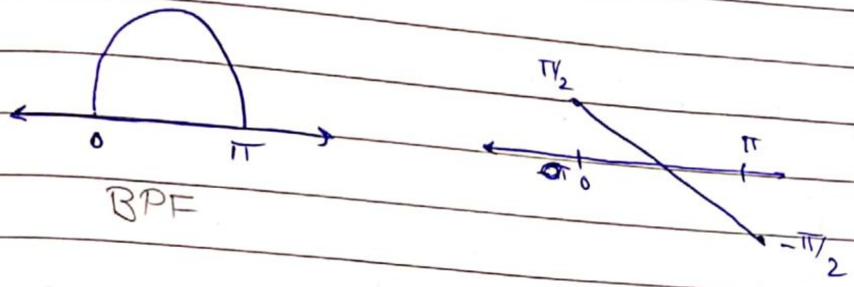
$= A_0 \cos(\omega_0 n + \phi_0 - \omega_0) \cos\omega_0$



Band-stop

4 $y[n] = \frac{1}{2}\left(x[n] - x[n-2]\right)$

$= -A_0 \sin(\omega_0 n + \phi_0 - \omega_0)\sin\omega_0$



- For all four filters, phase response is linear*.
  ∴ No time-delay distortion (No 'Dispersion')
- ~~Time~~ Group-delay is same for all frequencies.
- Phase response is pseudo-linear (affine) :- There is a shift by a constant.
- We cannot get such a linear* phase response in analog filters.

- Phasor description for sinusoids is preferred because like amplitude changes, even phase changes can be described by multiplication by a constant.

- $A_0 e^{j(\omega_0 n + \phi_0)} \xrightarrow[h[n]]{\text{Stable LSI}} (\ ?\ )\, A_0 e^{j(\omega_0 n + \phi_0)}$

$A_0 \sum h[k]\, e^{j(\omega_0(n-k)+\phi_0)}$

$\therefore\ ? = \boxed{\sum h[k]\, e^{-j\omega_0 k}}$

- Stability :- $\left|\sum h[k] e^{-j\omega_0 k}\right| \leq \sum |h[k]|$ ... finite

CP Is converse true?
Can the summation converge for unstable systems

- Frequency response $= H(e^{j\omega_0}) = \sum_{-\infty}^{\infty} h[k] e^{-j\omega_0 k}$

Q Find H for four systems

15/1 1 $y[n] = \frac{1}{2}(x[n] + x[n-1])$

- $h[n] = \underset{0}{\overset{1/2}{\uparrow}} \; 1/2$

- $H(e^{j\omega_0}) = \frac{1}{2} + \frac{1}{2}e^{-j\omega_0} = \frac{1}{2}e^{-j\frac{\omega_0}{2}}\left(2\cos\frac{\omega_0}{2}\right) = e^{-j\frac{\omega_0}{2}}\cos\frac{\omega_0}{2}$

- Input $= A_0\cos(\omega_0 n + \phi_0) = \frac{A_0}{2}\left(e^{j(\omega_0 n + \phi_0)} + e^{-j(\omega_0 n + \phi_0)}\right)$

- $\therefore$ Output $= \frac{A_0}{2}\left[H(e^{j\omega_0})e^{j(\omega_0 n + \phi_0)} + H(e^{-j\omega_0})e^{-j(\omega_0 n + \phi_0)}\right]$

Note that :- If $h[n]$ is real,

then $\overline{H(e^{-j\omega_0})} = \overline{\sum h[k] e^{-j(-\omega_0)k}}$

$= \sum h[k] e^{-j\omega_0 k}$

$= H(e^{j\omega_0})$

$\therefore$ Magnitude of frequency response is same, and phase is opposite for ω_0 and $-\omega_0$

$\therefore$ output $= \frac{A_0}{2}|H(e^{j\omega_0})|\left[e^{j(\omega_0 n + \phi_0 + \angle H(e^{j\omega_0}))} + e^{-j(\omega_0 n + \phi_0 + \angle H(e^{j\omega_0}))}\right]$

$= A_0 |H(e^{j\omega_0})| \cos(\omega_0 n + \phi_0 + \angle H(e^{j\omega_0}))$

For this case, $= A_0 \cos\frac{\omega_0}{2}\cos\left(\omega_0 n + \phi_0 - \frac{\omega_0}{2}\right)$

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IV DTFT

- A sequence is an infinite dimensional vector.

- Inner Product = $\langle x_1, x_2 \rangle = \sum_{k=-\infty}^{\infty} x_1[k]\overline{x_2[k]}$
  - Projection of x_1 on x_2

$\therefore \quad H(e^{j\omega_0}) = \sum_{-\infty}^{\infty} h[k]e^{-j\omega_0 k} = \langle h[k], e^{j\omega_0 k} \rangle$.

This is the projection of impulse response upon $e^{j\omega_0 k}$

* $H(e^{j\omega_0})$ is periodic (2π) $\Rightarrow H(e^{j\omega_0}) = H(e^{j(\omega_0+2\pi)})$

- Because of aliasing, ω_0 and $\omega_0 + 2\pi$ represent same frequency.

- Sequence $h[n]$ $\xrightarrow{\text{DTFT}}$ $H(e^{j\omega_0}) = \sum_{-\infty}^{\infty} h[k]e^{-j\omega_0 k}$

  for $-\pi \le \omega_0 \le \pi$.

- $x_0 \int_{-\pi}^{\pi} H(e^{j\omega_0})e^{j\omega_0 n}\, d\omega_0 = h[n]$

  $1/2\pi$

- Multiply the projection with the vector and sum over it.
- x_0 is a normalization constant since $h[n]$ might not be unit vector.

Proof LHS $= x_0 \int_{-\pi}^{\pi} \sum h[k]e^{-j\omega_0 k} e^{j\omega_0 n}\, d\omega_0$

$$= x_0 \sum_{-\infty}^{\infty} h[k] \underbrace{\int_{-\pi}^{\pi} e^{j\omega_0(n-k)} d\omega_0}_{2\pi\,\delta[n-k]}$$

$$= x_0\, 2\pi\, h[n]$$

$\therefore \quad = RHS$

with $x_0 = \dfrac{1}{2\pi}$.

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7/1 DTFT :- $H(e^{j\omega_0}) = \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega_0 k}$

.... Projection of $h[n]$ on $e^{j\omega_0 n}$

IDTFT :- $h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega$

→ $x[n] \longrightarrow \boxed{\begin{matrix} LSI \\ h[n] \end{matrix}} \longrightarrow y[n]$

$y[n] = (x * h)[n]$

DTFT of $y[n] = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \right) e^{-j\omega_0 n}$

Substitute $n-k = l$.

$$= \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]\, h[l]\, e^{-j\omega_0 (k+l)}$$

$$= \left(\sum_{l=-\infty}^{\infty} h[l]\, e^{-j\omega_0 l} \right) \left(\sum_{k=-\infty}^{\infty} x[k]\, e^{-j\omega_0 k} \right)$$

$$\boxed{Y(e^{j\omega_0}) = X(e^{j\omega_0})\, H(e^{j\omega_0})}$$

A] Properties of DTFT.

1 Convolution → Multiplication
2 Linearity.

3 Time Shifting

$x[n-n_0] \xrightarrow{DTFT} X(e^{j\omega_0})\, e^{-j\omega_0 n_0}$

$\sum_{n=-\infty}^{\infty} x[n-n_0]\, e^{-j\omega_0 n}$

$= \sum_{t=-\infty}^{\infty} x[t]\, e^{-j\omega_0 (t+n_0)}$ $\quad n - n_0 = t$

$= X(e^{j\omega_0})\, e^{-j\omega_0 n_0}$

Shift in time :- Linear change in phase.

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4 $-jn x[n] \xrightarrow{DTFT} \frac{dX(e^{j\omega})}{d\omega}$

$$X(e^{j\omega_0}) = \sum x[k] e^{-j\omega_0 k}$$

$$\frac{dX(e^{j\omega})}{d\omega} = -j \sum \underbrace{k x[k]} e^{-j\omega k}$$

Q. $\left(\frac{1}{2}\right)^n u[n] \xrightarrow{DTFT} ?$ $\quad \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$

GP

Q $n\left(\frac{1}{2}\right)^n u[n] \xrightarrow{DTFT} j \frac{d}{d\omega}\left(\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right) = \frac{\frac{1}{2}e^{-j\omega}}{(1 - \frac{1}{2}e^{-j\omega})^2}$

- Because DTFT is invertible, we say it retains all characteristics of the signal.
- DTFT is a mapping from one paradigm to another.

$x[n] \xrightarrow{h[n]} y[n] \Rightarrow X(e^{j\omega}) \xrightarrow{H(e^{j\omega})} Y(e^{j\omega})$

5 $x_1[n]\, x_2[n] \longrightarrow$ Periodic convolution of X_1 and X_2 (over one period)

$$\sum_{n=-\infty}^{\infty} x_1[n] x_2[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x_1[n]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} X_2(e^{j\lambda}) e^{j\lambda n} d\lambda\right) e^{-j\omega n}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_2(e^{j\lambda}) \left\{ \sum x_1[n] e^{j\lambda n} e^{-j\omega n} \right\} d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_2(e^{j\lambda}) \left\{ \sum x_1[n] e^{j(\lambda-\omega)n} \right\} d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_2(e^{j\lambda})\, X_1(e^{j(\omega-\lambda)})\, d\lambda$$

Absolute convolution is not infinite. $\Rightarrow$ Convolution over one period.

• Absolute

6 $x[-n] \longrightarrow X(e^{-j\omega})$

7 $\overline{x[-n]} \longrightarrow \overline{X(e^{j\omega})}$

8 $\overline{x[n]} \longrightarrow \overline{X(e^{-j\omega})}$

$$\Rightarrow x_1[n]\,\overline{x_2[n]} \longrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{X_2(e^{-j\lambda})}\, X_1(e^{j(\omega-\lambda)})\, d\lambda$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{X_2(e^{j\alpha})}\, X_1(e^{j(\omega+\alpha)})\, d\alpha.$$

For $\omega = 0$, we get inner product :-

$$\langle x_1[n], x_2[n] \rangle = \sum x_1[n]\,\overline{x_2}[n]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\alpha})\, \overline{X_2(e^{j\alpha})}\, d\alpha$$

$$= \checkmark$$

• Inner product remains the same in both paradigms

'Parseval's Theorem'.

Z ~~LAPLACE~~ TRANSFORM

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- Generalization of Fourier Transform is possible.

eg. $x[n] = 2^n u[n]$ has no DTFT, because not convergent.

$\therefore$ Multiply by r^{-n}, $r > 2$

Find DTFT of $x[n]r^{-n}$

$$= \frac{1}{1 - 2r^{-1}e^{-j\omega}}$$

- Frequency response of ~~re~~ LSI system $(h[n])$

- DTFT of $h[n]r^{-n}$

$$= \sum_{n=-\infty}^{\infty} h[n] r^{-n} e^{-j\omega n}.$$

$$= \sum_{n=\infty}^{\infty} h[n] (re^{j\omega})^{-n} = H(z)$$

Define $z = re^{j\omega}$ ... a complex variable.

- Output $= y[n] = \sum_{k=-\infty}^{\infty} x[k] r^{-k} h[n-k] r^{-(n-k)}$

$$= \sum_{k=-\infty}^{\infty} x[k] h[n-k] r^{-n}$$

$$= r^{-n} (x * h)[n]$$

~~Out~~ When x & h are multiplied by r^{-n}, so is output.

→ z-transform of $h[n] \triangleq \sum h[n] (re^{j\omega})^{-n} = \sum h[n] z^{-n}$

~~= Mapping from h[n] to~~

→ LSI system

$$x[n]r^{-n} \longrightarrow \boxed{h[n]r^{-n}} \longrightarrow y[n]r^{-n}.$$

$Y(z) = X(z) H(z)$ if these transforms exist for x, h, y for this z.

→ Convolution Property :-

$$x_1[n] * x_2[n] \xrightarrow[z]{} X_1(z)\, X_2(z)$$

provided we choose a 'z' so that ~~#~~ $X_1(z)$, $X_2(z)$ and Z-transform $x_1 * x_2$ exist for that Z.

- DTFT = Z-Transform at $r = 1$
- Existence of Z-transform depends only on magnitude of z (r), not phase.
- The Z-transform is valid only in a 'region of convergence'.

– In general, ROC is $R_1 \le |z| \le R_2$

$$0 \le R_1 \underset{<}{\overset{\text{or}}{\le}} r \underset{<}{\overset{\text{or}}{\le}} R_2 \le \infty$$

eg $x[n] = 2^n u[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} 2^n u[n]\, r^{-n} e^{-j\omega n}$$

$$= \sum u[n] \left(\frac{2}{r} e^{-j\omega}\right)^n$$

$$= \frac{1}{1 - 2z^{-1}} \quad \text{with ROC} \quad |z| > 2.$$

eg $x[n] = -2^n u[-n-1]$

$-n-1 \ge 0$

$n \le -1$

$$= \sum_{n=-\infty}^{\infty} -2^n u[-n-1]\, r^{-n} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} -\left(\frac{2}{r} e^{-j\omega}\right)^n$$

$$= \sum_{k=1}^{\infty} -\left(\frac{r}{2} e^{j\omega}\right)^k$$

$$= \frac{1}{1 - 2z^{-1}} \quad \text{with ROC} \quad |z| < 2$$

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- The same expression with different ROC corresponds to a different sequence

eg $\delta[n] \xrightarrow[z]{} \sum_{n=-\infty}^{\infty} \delta[n]\, z^{-n}.$

$= 1$ ROC :- $|z| > 0$

Equality if $0^0 = 1$

eg $\delta[n-1] \xrightarrow[z]{} \sum_{n=-\infty}^{\infty} \delta[n-1]\, z^{-n}.$

$= z^{-1}$ ROC :- $|z| > 0$

eg $\delta[n+1] \xrightarrow[z]{} z$ ROC :- $|z| \geq 0$, $z \ngtr \infty$.

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I PROPERTIES

1 Linearity

$x_1[n] \xrightarrow{z} X(z)$ .... R_1

$x_2[n] \xrightarrow{z} X_2(z)$ .... R_2

Then $\alpha x_1 + \beta x_2 \xrightarrow{z} \alpha X_1 + \beta X_2$ RoC : At least $R_1 \cap R_2$

eg $x_1[n] = \frac{1}{2^n} u[n]$

$x_2[n] = \frac{1}{4^n} u[n] - \frac{1}{2^n} u[n]$

$X_1(z) = \sum_{n=0}^{\infty} 2^{-n} z^{-n} = \frac{1}{1 - \frac{1}{2z}}$ with RoC :- $|z| > \frac{1}{2}$.

$X_2(z) = \frac{1}{1-\frac{1}{4z}} - \frac{1}{1-\frac{1}{2z}}$ with RoC : $|z| > \frac{1}{2}$

$|z| > 1$ ... $|z| > \frac{1}{}$

2 Time Shift

$x[n-n_0] \xrightarrow{Z} z^{-n_0} X(z)$ ...... R_x, except possibly the boundaries

$\sum x[n-n_0] z^{-n} = \sum_{l=-\infty}^{\infty} x[l] z^{-l} z^{-n_0} = z^{-n_0} X(z)$

3 Convolution

$(x_1 * x_2)[n] \xrightarrow{Z} X_1(\omega) X_2(\omega)$ ...... At least $R_1 \cap R_2$

4 Modulation

$\alpha^n x[n] \longrightarrow X\left(\frac{z}{\alpha}\right)$ .... αR_x .... $\frac{z}{\alpha} \in R_x$

α is a complex constant

$\sum_{n=-\infty}^{\infty} \alpha^n x[n] z^{-n} = \sum x[n] \left(\frac{z}{\alpha}\right)^{-n}$

5 Multiplication

$x_1[n]\, x_2[n] \longrightarrow$

6 Inversion (IZT)

$h[n] r^{-n} \xrightarrow{DTFT} H(z)$

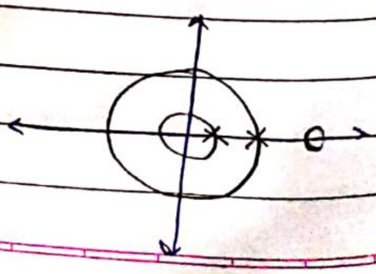
$$\therefore \quad h[n] = r^n \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} H(re^{j\omega})\, e^{j\omega n}\, d\omega \right)$$

Now $z = re^{j\omega} \Rightarrow dz = re^{j\omega} j\, d\omega$ at constant r (a circle)

$$\therefore \quad \boxed{h[n] = \frac{1}{j2\pi} \oint H(z)\, z^{n-1}\, dz}$$

any closed contour in the RoC.

eg $H(z) = \dfrac{1-3z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)}$



Possible RoC's :- $|z| < \frac{1}{2}$, $\frac{1}{2} < |z| < 2$, $|z| > 2$

eg $$H(z) = \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - 2z^{-1}\right)}$$

$$= \frac{-1}{3} \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{4}{3} \frac{1}{1 - 2z^{-1}}$$ .... Partial Fractions

* 1) $|z| > 2$:- $\frac{-1}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{4}{3} 2^n u[n]$.

2) $\frac{1}{2} < |z| < 2$:- $\frac{-1}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{4}{3}\left(-2^n u[-n-1]\right)$

3) $|z| < \frac{1}{2}$:- $\frac{-1}{3}\left(\frac{-1}{2^n} u[-n-1]\right) + \frac{4}{3}\left(-2^n u[-n-1]\right)$

7 Derivative in Z domain

$n x[n] \longrightarrow -z \frac{dX(z)}{dz}$ .... R_x, except possibly boundaries

$$\frac{d}{dz}\left(\sum_{-\infty}^{\infty} x[n] z^{-n}\right) = \sum -n x[n] z^{-n-1}$$