$$= \frac{1}{2\pi j} \oint_C X(z_1) \left(\sum y[n] \left(\frac{z}{z_1}\right)^{-n} \right) z_1^{-1}\, dz_1$$

$$= \frac{1}{2\pi j} \oint_C X(z_1)\, Y\left(\frac{z}{z_1}\right) z_1^{-1}\, dz_1$$

C : belongs within RoC of X $\Rightarrow z_1 \in$ RoC of X

Also, $\frac{z}{z_1}$ belongs to RoC of Y

CP $x[n] = \alpha^n u[n]$, $y[n] = \beta^n u[n]$, $|\alpha|, |\beta| < 1$.

Obtain ZT of $x[n]y[n]$ using above expression

II RATIONAL Z TRANSFORM

* Rational ZT $\triangleq \frac{N(z)}{D(z)}$ where N & D are finite length series in z or z^{-1}

- Can be written in two general forms :-

1) $z^{(\cdot)} \left(\frac{\text{Poly in } z}{\text{Poly in } z}\right)$ 2) $z^{(\cdot)} \left(\frac{\text{Poly in } z^{-1}}{\text{Poly in } z^{-1}}\right)$

- Realization $\triangleq$ Translation of ZT into practical system using finite resources.

- Resources :- 1) Unit sample delay
  2) Two input adders
  3) Constant multipliers

* 'Digital Signal Processor' is an 'Application specific integrated circuits (ASIC)'

→ LSI system :-

Ratio of Output ZT to Input ZT. $\left(\frac{Y(z)}{X(z)} = H(z)\right)$ is equal to 'system function' and independent of input.

System function requires RoC along with expression to be described.

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- Real systems must be causal

Theorem: A causal system function can always be written as :-

$$\frac{\text{Polynomial in } z^{-1}}{\text{Polynomial in } z^{-1}} \rightarrow \text{Denominator has constant} = 1$$

Proof: Exercise (CP)

eg $H(z) = \dfrac{1}{1 - \alpha z^{-1}}$ for $|\alpha| < 1$ and $|z| > |\alpha|$

$\therefore$ Impulse response $= h[n] = \alpha^n u[n]$

CP Do all sequences have a ZT? No - e^{n^2}

29/1 • Rational function $= z^{(\,)} \dfrac{N(z^{-1})}{D(z^{-1})}$ matter of concern

can be credited simply to a time-shifted input impulse

By long division, $\dfrac{N(z^{-1})}{D(z^{-1})} = Q(z^{-1}) + \dfrac{R(z^{-1})}{D(z^{-1})}$ where $\deg(R) < \deg(D)$

1 - Now, $IZT(Q)$ is a finite length sequence (sum of time shifted δ)

2 - Factorize $D(z^{-1})$:- $D(z^{-1}) = \prod_{l=1}^{M} (1 - a_l z^{-1})^{M_l}$ → Multiplicity of l^{th} po[le]

3 - Partial fractions :- Numerators could have degree at most $M_l - 1$

- RoC for every term could be in the interior of pole or in exterior of pole

↓ right-sided inverse ↓ left-sided inverse

$\underbrace{\text{right-sided inverse} \quad \text{left-sided inverse}}_{\text{Polyex terms}}$

↓Δ

(Polynomial in n)(Exponential in n)

l^{th} term :- $\left(\sum b_p n^p\right) a_l^n u[n]$ or $u[-n-1]$

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* Irrational ZT

$x[n] = \frac{1}{n!} u[n]$

An irrational system cannot be realized with finite resources.

$\therefore$ All LSI systems are not rational.

$$X(z) = \sum_{n=-\infty}^{\infty} \frac{u[n]}{n!} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}$$

$$= 1 + z^{-1} + \frac{z^{-1}}{2!} \cdots$$

$$= e^{1/z} \qquad |z| > 0$$

→ Causal system

$$\frac{Y(z)}{X(z)} = \frac{\sum_{l=0}^{M} b_l z^{-l}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

$$\therefore Y(z) = \sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{l=0}^{M} b_l z^{-l} X(z)$$

IZT :- $y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{l=0}^{M} b_l x[n-l]$

'Difference Equation' or LCCDE

- We now know how to find output given a rational $X(z)$

Q $x[n] = \frac{1}{n!} u[n]$ .... irrational

$h[n] = \frac{1}{n!} u[n]$

$X(z) = e^{1/z}$, $H(z) = e^{1/z}$

$Y(z) = e^{2/z}$

$y[n] = \frac{2^n}{n!} u[n]$

→ Causality :-

$h[n] = 0 \quad \forall\, n<0 \quad \Rightarrow \quad H(z) = \sum_{n=0}^{\infty} h[n]\, z^{-n}.$

$$= h[0] + h[1]z^{-1} + h[2]z^{-2} \ldots$$

Assuming $h[n]$ is finite $\forall\, n$, this $H(z)$ series definitely converges for $|z| \to \infty$.

eg $h[n] = \alpha^n u[n] \longleftrightarrow H(z) = \dfrac{1}{1-\alpha z^{-1}}$ for $|z| > |\alpha|$

Notice how $|z| \to \infty$ is in the RoC.

eg $H(z) = \dfrac{z}{1-\frac{1}{2}z^{-1}} \longleftrightarrow h[n] = \underset{-1}{1}\ \ \underset{0}{\tfrac{1}{2}}\ \ \underset{1}{\tfrac{1}{2^2}} \ldots$ ..... non-causal

RoC :- $\frac{1}{2} < |z| < \infty$ .... does not include $|z| = \infty$.

• If a system is causal, it must include $+\infty$ in RoC.

Theorem A rational system is causal iff RoC of its system function includes $+\infty$.

Could also hold for irrational system

B] → Stability

$\sum |h[n]| < \infty$.

Rational function :- $H(z) = z^D \left(Q(z) + \dfrac{R(z)}{D(z)} \right)$

$z^D \xrightarrow{IZT}$ Time shift; $Q(z) \xrightarrow{IZT}$ Finite length sequence

$\underbrace{\text{Time shift} \quad \text{Finite length sequence}}_{\text{Do not affect absolute summability}}$

$\dfrac{R(z)}{D(z)} = \sum_{l=1}^{M} \dfrac{\text{Polynomial of degree } M_l - 1 \text{ in } z^{-1}}{(1-a_l z^{-1})^{M_l}}$

$\downarrow$ IZT

$= \sum_{l=1}^{M} \begin{pmatrix}\text{Polynomial in } n \\ \text{of degree } M_l - 1\end{pmatrix} a_l^n u[n] \text{ or } a_l^n u[-n-1]$

Based on RoC of partial fraction, Polyex's IZT could be left-sided or right-sided

$$\frac{R(z)}{D(z)} \xrightarrow{IZT} \sum_{l=1}^{M} \left[\left(\begin{matrix}\text{Polynomial of degree} \\ M_l - 1\end{matrix}\right) a_l^n \, u[n] \text{ or } u[-n-1] \right]$$

Polyex term (left sided or right sided)

One polyex for each pole

Consider any one polyex term:-

- Right sided sequence :- Polyex converges (h[n] is absolutely summable)
- Left
- For $|a_l| = 1$, polyex does not converge.

• The polynomial part of polyex does not affect convergence, because exponential term is more potent.

* Ratio test.

• Deciding left-sided or right-sided :-

RoC is interior of pole $|a_l|$; RoC is exterior of pole $|a_l|$

• Polyex terms ~~are~~ with different 'a_l' are linearly independent

⇒ Sum of polyexes cannot become zero

⇒ For sum of polyexes to be finite, every polyex must be finite (be absolutely summable)

Theorem: For rational systems, stability is equivalent to the unit circle being in the RoC

31/1 → 'Marginally Stable'

• When $|a_l| = 1$, the polyex term diverges.

- No system with a pole on unit circle can be BIBO stable

eg $H(z) = \dfrac{1}{1 - z^{-1}}$ with RoC :- $|z| > 1$ or $|z| < 1$

Consider $|z| > 1 \longrightarrow h[n] = u[n]$

Consider $x[n] = u[n] \longrightarrow y[n] = (n+1)\,u[n]$ .. unbounded

- This is the only rational input for which output is unbounded.

CP Show that ~~it s~~ - above statement.

CP Generalize this to other systems with poles on unit circle.

CP $H(z) = \dfrac{1}{(1-z^{-1})^2}$ Construct bounded inputs that give bounded / unbounded outputs

Most will be unbounded

- Marginally stable ≡ 'Simple' pole on unit circle.
  ↳ Multiplicity 1

C] Causal and Stable Rational systems.

RoC includes both ∞ and unit circle.

Property RoC must be simply connected.
Any two points should be connectable by a contour within RoC.

Theorem All poles of a causal, stable rational system lie within unit circle.

CP Can a rational expression have an RoC which is a union of two disjoint annuli?

C] LCCDEs not valid $\forall n$.

3/2 • For a causal rational system,
'Order' $\triangleq$ Degree of denominator

eg $H(z) = \frac{\beta_0 + \beta_1 z^{-1}}{1 - \alpha z^{-1}}$ has order 1.

• LCCDE

eg $H(z) = \frac{1}{1-\alpha z^{-1}}$ $\Rightarrow$ $y[n] = \alpha y[n-1] + x[n]$

An LCCDE could hold $\forall n$ or for $n \in \{n_1, \dots n_2\}$

For $n_1 = 0$, $n_2 = 1000$ (say)
Given $x[n]$ for $n_1 \le n \le n_2$, we require $\underbrace{y[n_1 - 1]}_{\text{for 1st order systems}}$ to be able
to calculate $y[n]$ for $n_1 \le n \le n_2$

C'] Generalize to N^{th} order systems

• Consider $y[n] = \alpha y[n-1] + x[n]$.

The solution is made up of two components :- 1) Zero input response - $y[\cdot]$
2) Zero state response - $y[n\cdot]$
It is a linear combination (addition) of these components

eg $x[n] = 1$ $\forall$ $n_1 \le n \le n_2$
$y[n_1 - 1] = y_0$
$y[n] = \alpha y[n-1] + x[n]$

- Zero-input solution.
$y[n_1] = \alpha y_0$
$y[n] = \alpha^{n-n_1+1} y_0$ $\forall$ $n_1 \le n \le n_2$

Note :- Zero-input solution comes from polyex terms corresponding
to poles of system.

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- Zero-state solution

$y[n_1-1] = 0$, $x[n] = \beta^n$ for $n_1 \le n \le n_2$

- Case 1 :- $\beta \ne \alpha$

$$y[n_1] = x[n_1]$$

$$y[n] = \alpha y[n-1] + x[n]$$

$$y[n_1+i] = \sum_{j=0}^{i} \alpha^{i-j} x[n_1+j]$$

$$= \sum_{j=0}^{i} \alpha^{i-j} \beta^{n_1+j}$$

$$= \alpha^i \beta^{n_1} \sum_{j=0}^{i} \left(\frac{\beta}{\alpha}\right)^j$$

$$= \alpha^i \beta^{n_1} \frac{\left(\left(\frac{\beta}{\alpha}\right)^{i+1} - 1\right)}{\frac{\beta}{\alpha} - 1}$$

$$= \beta^{n_1} \left(\frac{\beta^{i+1} - \alpha^{i+1}}{\beta - \alpha}\right)$$

☆ Zero-state solution is of the form $c_1\alpha^n + c_2\beta^n$

- Case 2 :- $\beta = \alpha$

Solution is of the form $(c_0' + c_1' n)\alpha^n$

* If the LCCDE were valid for all time, we would have written

$$Y(z) = H(z)X(z) = \frac{1}{1-\alpha z^{-1}} \times \frac{1}{1-\beta z^{-1}}$$

For $\alpha \ne \beta$, we would have found y by partial fractions :- $c_1\alpha^n + c_2\beta^n$

$=$ derivative property :- $(c_0 + c_1 n)\alpha^n$

∴ Even when LCCDE is valid for only some time span, solution is of the same form, but with different coefficients.

7/2

→ Zero-input solution = 'Homogeneous (natural) solution'

~~For~~

Zero-state solution ≡ $\alpha \neq \beta$ = 'Forced (particular) solution' } Resem. form of input

$\alpha = \beta$ ≡ 'Resonant solution'

- The homogeneous solution is the same for any input. (property of system).

However, it may not be possible to find forced solution for all inputs.

III ~~REALI~~

Realization Of A System Function

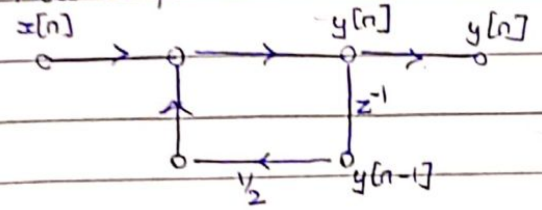
- Translation into an implementation with finite, preferrably $\underbrace{\text{generic resources}}$
  - One sample delay
  - Constant multiplier
  - Two-input adder.

eg $H(z) = \dfrac{2+3z^{-1}}{1-\frac{1}{2}z^{-1}}$ $\quad |z| > \frac{1}{2}$ (for causal)

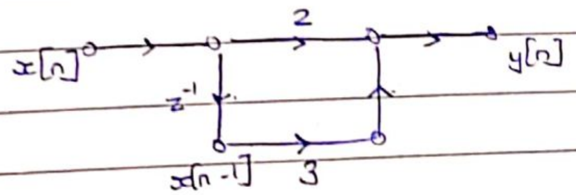
$= (2+3z^{-1})\left(\dfrac{1}{1-\frac{1}{2}z^{-1}}\right)$ ---- cascade

Consider $\dfrac{Y(z)}{X(z)} = \dfrac{1}{1-\frac{1}{2}z^{-1}} \Rightarrow y[n] = x[n] + \frac{1}{2}y[n-1]$

* Signal Flow Graph :-

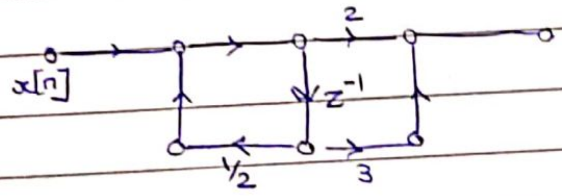


For $H(z) = 2+3z^{-1} \Rightarrow y[n] = 2x[n] + 3x[n-1]$



If we directly cascade, we would be redundantly using two z^{-1} blocks

$\therefore$ More efficient concatenation :-



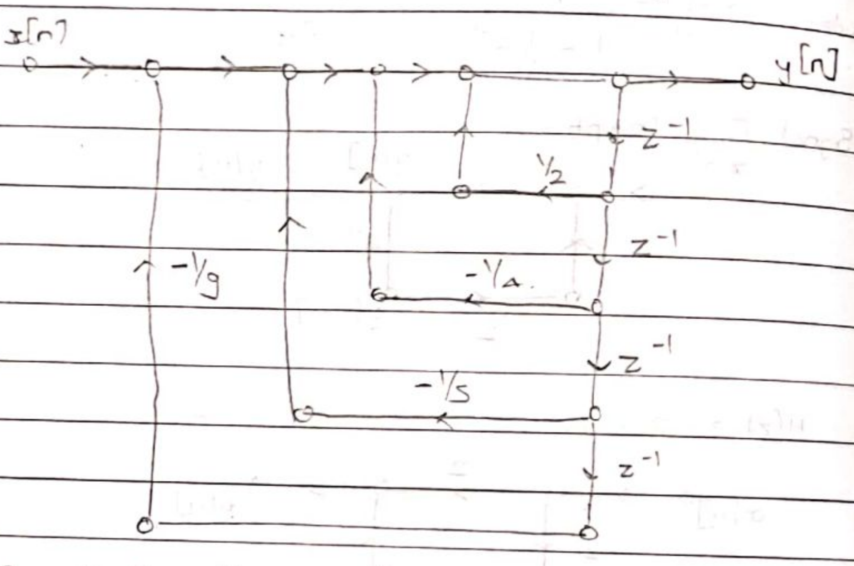
- If numerator has degree m and numerator has degree n, then $\max(m,n) \le$ No. of one sample delays $\le m+n$

eg $\dfrac{2 + 5z^{-1} + 3z^{-2} + 4z^{-3}}{1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{5}z^{-3} + \frac{1}{9}z^{-4}}$

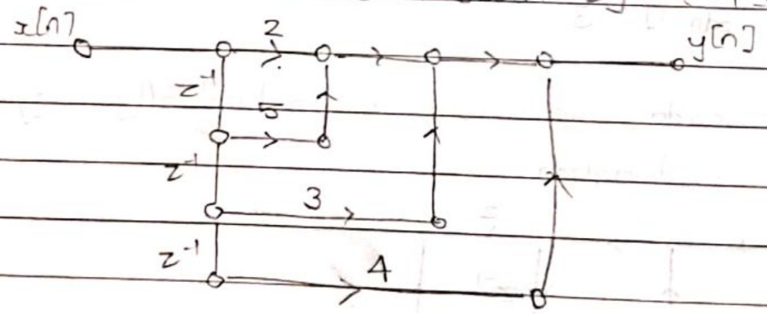
① Direct form 1

$H_1(z)$ = Denominator

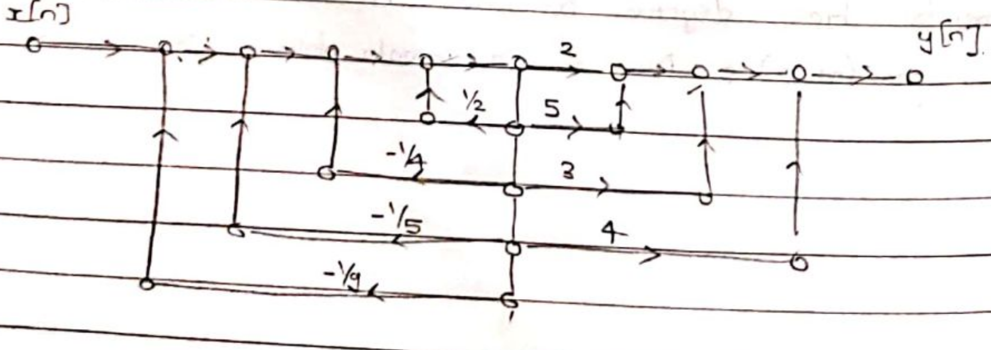
$y[n] = \frac{1}{2}y[n-1] - \frac{1}{4}y[n-2] - \frac{1}{5}y[n-3] - \frac{1}{9}y[n-4] + x[n]$



$y[n] = 2x[n] + 5x[n-1] + 3x[n-2] + 4x[n-3]$



② Direct form 2



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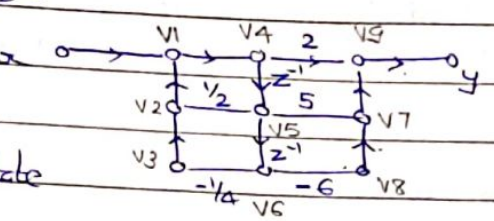
I TRANSLATION OF SIGNAL FLOW GRAPH INTO PSEUDO CODE.

eg $H(z) = \dfrac{2+5z^{-1}+6z^{-2}}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}$

Direct Form 2 :-



Assumption - Steady State

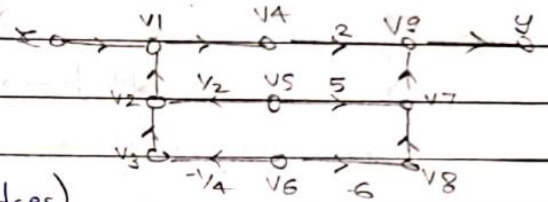
1 Cut one-sample delay branches

2 Ascertain whether signal flow graph is loop-free

directed

- Only unrealizable SFGs have loops

SFG should not have delay-free loops



3 Identify source nodes (with no incoming edges)

V5, V6, x

4 Propagate the sequence from source nodes to their one-step forward nodes

$V1 \leftarrow x$, $V7 \leftarrow 5V5$, $V2 \leftarrow \frac{V5}{2}$, $V3 \leftarrow -\frac{1}{4}V6$, $V8 \leftarrow -6V6$

5 Propagate one-step further at a time

$V2 \leftarrow V2+V3$, $V7 \leftarrow V7+V8$, $V1 \leftarrow V1+V2$, $V9 \leftarrow -V7$, $V4 \leftarrow V1$

$V9 \leftarrow V9+2V4$, $y \leftarrow V9$

- Need to respect order.

6 Put one-sample delays back. Update beginning from tail of each string

$V6 \leftarrow V5$, $V5 \leftarrow V4$

~~EP How do we initialize?~~

→ Realization as a product

$H(z) = H_1(z) \cdots H_L(z)$

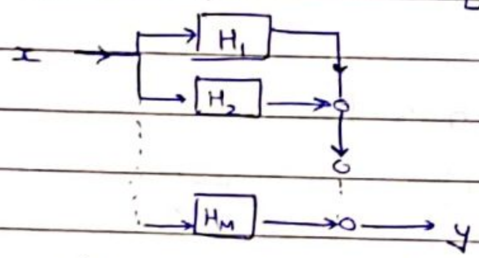
When you factorize numerator and denominator of $H(z)$, many such $\{H_1 \dots H_L\}$ can be formed.

Final signal flow graph is cascade of graphs of each facts.

eg $H(z) = \dfrac{(1-3z^{-1})(1-5z^{-1})}{(1-2z^{-1})(1-4z^{-1})} = (1-3z^{-1})\left(\dfrac{1-5z^{-1}}{1-2z^{-1}}\right)\left(\dfrac{1}{1-4z^{-1}}\right)$

→ Realization as sum

$H(z) = H_1(z) \dots + H_L(z)$ .... by partial fractions.



'Parallel Form'

or

'Cascade Parallel Form'

Each $H_i(z)$ could be in direct form or cascade form.

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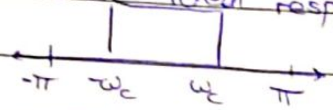
DESIGN.

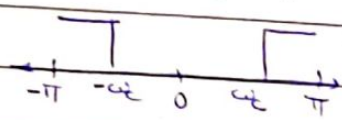
- Filter : For LSI systems with a frequency response.

→ Piecewise constant ideal responses (periodic (2π))

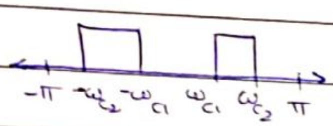
1 LPF



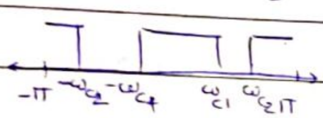
2 HPF



3 BPF



4 BSF



LPF $\quad h[n] = \dfrac{1}{2\pi}\displaystyle\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega = \dfrac{1}{2\pi}\left[\dfrac{e^{j\omega n}}{jn}\right]_{-\omega_c}^{\omega_c}$

$$= \frac{\sin(\omega_c n)}{\pi n} \quad \text{(take limit if } n = 0\text{)}$$

HPF

$$HPF = 1 - LPF$$

$$\text{IDTFT}: \underbrace{\frac{\sin(\pi n)}{\pi n}}_{\delta[n]} - \frac{\sin(\omega_c n)}{\pi n}.$$

BPF $\quad h[n] = \dfrac{\sin(\omega_{c_2} n) - \sin(\omega_{c_1} n)}{\pi n}$

BSF $\quad h[n] = \delta[n] - h_{BPF}[n]$

Analog eg filters cannot have linear phase

eg $h_{LPF}[n] = \begin{cases} \frac{\sin \omega_c n}{\pi n} & \text{for } n \neq 0 \\ \frac{\omega_c}{\pi} & \text{for } n = 0 \end{cases}$

Take $\omega_c = \pi/2$

n	0	± 1	± 2	± 3	± 4	± 5
h	$1/2$	$1/\pi$	0	$\frac{-1}{3\pi}$	0	$\frac{1}{5\pi}$

- Disqualifications of piecewise constant ideal filter :-
  1) Causality
     $h[n]$ is non-zero for infinitely many $n < 0$
     - If it were non-zero for finitely many $n < 0$, we could have time-shifted the output to make system causal.

Cause of non-causality :- Phase response $= 0$

2) Stability

$\sum |h[n]| = \frac{1}{2} + \frac{2}{\pi}\left(1 + \underbrace{\frac{1}{3} + \frac{1}{5} \dots}\right)$

$= \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} \dots$

$\geq \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} \dots$

$= 1 + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \dots$

$\therefore \sum |h[n]|$ diverges.

14/2 * 'Piecewise constant ideal responses' $\triangleq$ Ideal response which is constant on a finite no. of pieces of $]-\pi, \pi[$.

CP Prove: Cause of instability is discontinuity of ideal system

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}\,d\omega$$

$$H(e^{j\omega_0}) = \sum h[n]e^{-j\omega_0 n}$$

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3) Ideal filters are irrational

We cannot express $\sum_{n=-\infty}^{\infty} h_{ideal}[n]\,z^{-n}$ as a rational function of z.

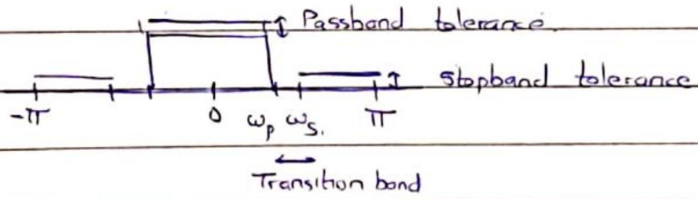
CP Prove: If this were rational, then the fact that it is constant over a non-trivial interval implies it is constant everywhere (analytic functions)

$\sum |n||h[n]| \geq \sum |nh[n]| \geq M \quad \forall M$

I REALIZABLE SPECIFICATIONS WITH A PIECEWISE CONSTANT IDEAL RESPONSE

1 No discontinuities. There must be a transition band between passband and stopband.
2 There has to be tolerance in passband and stopband magnitude.
3 We must allow for a phase response.

eg LPF



Ex Find specs for your own filter