Design.

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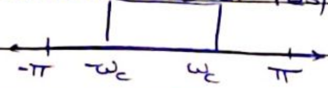
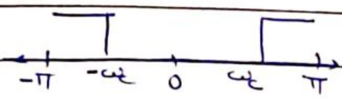
- Filter : For LSI systems with a frequency response.

→ Piecewise constant ideal responses (periodic (2π))

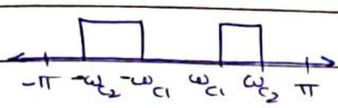
1 LPF



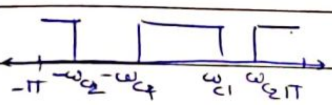
2 HPF



3 BPF



4 BSF



LPF $h[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_c}^{\omega_c}$

$= \frac{\sin(\omega_c n)}{\pi n}$ (take limit if $n = 0$)

HPF

$$HPF = 1 - LPF$$

IDTFT: $\underbrace{\frac{\sin(\pi n)}{\pi n}}_{\delta[n]} - \frac{\sin(\omega_c n)}{\pi n}$.

BPF $h[n] = \frac{\sin(\omega_{c_2} n) - \sin(\omega_{c_1} n)}{\pi n}$

BSF $h[n] = \delta[n] - h_{BPF}[n]$

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Analog ~~eq~~ filters cannot have linear phase

eg $h_{LPF}[n] = \begin{cases} \frac{\sin \omega_c n}{\pi n} & \text{for } n \neq 0 \\ \frac{\omega_c}{\pi} & \text{for } n = 0 \end{cases}$

Take $\omega_c = \pi/2$

n	0	±1	±2	±3	±4	±5
h	$1/2$	$1/\pi$	0	$\frac{-1}{3\pi}$	0	$\frac{1}{5\pi}$

– Disqualifications of piecewise constant ideal filter :-

1) Causality

$h[n]$ is non-zero for infinitely many $n < 0$

– If it were non-zero for finitely many $n < 0$, we could have time-shifted the output to make system causal.

Cause of non-causality :- Phase response = 0

2) Stability

$$\sum |h[n]| = \frac{1}{2} + \frac{2}{\pi}\left(1 + \underbrace{\frac{1}{3} + \frac{1}{5} \cdots}\right)$$

$$= \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13}$$

$$\geq \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}$$

$$= 1 + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \cdots$$

$\therefore \sum |h[n]|$ diverges

14/2 * 'Piecewise constant ideal responses' $\triangleq$ Ideal response which is constant on a finite no. of pieces of $]-\pi, \pi[$.

CP Prove: Cause of instability is discontinuity of ideal system

$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}\,d\omega$

$H(e^{j\omega_0}) = \sum h[n]e^{-j\omega_0 n}$

3) Ideal filters are irrational

We cannot express $\sum_{n=-\infty}^{\infty} h_{Ideal}[n]\,z^{-n}$ as a rational function of z.

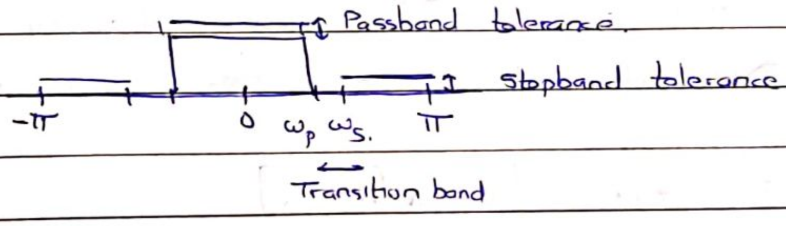
CP Prove: If this were rational, then the fact that it is constant over a non-trivial interval implies it is constant everywhere (analytic functions)

$\sum |n||h[n]| \geq \sum |n h[n]| \geq M \quad \forall M$

I Realizable Specifications With A Piecewise Constant Ideal Response

1. No discontinuities. There must be a transition band between passband and stopband.
2. There has to be tolerance in passband and stopband magnitude.
3. We must allow for a phase response.

eg LPF



Ex Find specs for your own filter

CP Find IDTFT of $H(e^{j\omega}) = j\omega$ ..... discontinuous at π & $-\pi$.

Does ~~dis~~ continuity guarantee stability?

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→ Analog Filters :- H(s) is

1. Rational in s.
2. Stable and causal. :- Poles must be in left-half plane

These properties need to be preserved.

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We will convert this to $T(z)$

1. $T(z)$ must be rational for realizability
2. Imaginary axis $(s = j\omega)$ should have a one-one mapping with unit circle $(|z| = 1)$

$*$ FT is at $s = j\omega$, DTFT is at $z = e^{j\omega}$

3. Left half is mapped to $|z| < 1$, right half is mapped to $|z| >$

4. $\rightarrow$ Tracing Y-axis and unit circle is monotonically increasing.

$(-\infty, \infty) \longrightarrow (-\pi, \pi)$

$\longrightarrow$ Consider differentiator :- $H(s) = s$

- $y[n] = x[n] - x[n-1]$

$T(z) = 1 - e^{-j\omega}$ (putting $z = e^{j\omega}$)

$\therefore$ Does not map imaginary axis to unit circle.

- $y[n] = x[n+\frac{1}{2}] - x[n-\frac{1}{2}]$

$T(z) = e^{j\omega/2} - e^{-j\omega/2} = 2j \sin\frac{\omega}{2}$

$\therefore$ ~~Does not trace~~ Maps imaginary axis to unit circle, but does not trace entire unit circle.

- We can try to get $T(z) = j\tan\left(\frac{\omega}{2}\right)$

$$= j\frac{\sin\left(\frac{\omega}{2}\right)}{\cos\left(\frac{\omega}{2}\right)}$$

$$T(z) = \frac{z^{1/2} - z^{-1/2}}{z^{1/2} + z^{-1/2}} = \frac{1 - z^{-1}}{1 + z^{-1}}$$

$$\therefore \quad \frac{y[n] + y[n-1]}{2} = \frac{x[n] - x[n-1]}{2}$$

$\therefore$ Average of current and first past output sample is proportional to the first difference of inputs.

$$\therefore \quad s \longleftarrow \frac{1-z^{-1}}{1+z^{-1}}$$

$$j\Omega = j\tan\left(\frac{\omega}{2}\right)$$

$$\therefore \quad \Omega = \tan\left(\frac{\omega}{2}\right)$$ ..... Monotonically increasing mapping

• If $s = \Sigma + j\Omega$, $\quad z = \dfrac{1+s}{1-s}$

$$= \frac{1+\Sigma+j\Omega}{1-\Sigma-j\Omega}$$

$$|z|^2 = \frac{(1+\Sigma)^2 + \Omega^2}{(1-\Sigma)^2 + \Omega^2}$$

$\therefore \quad \Sigma > 0 \equiv |z| > 1$ $\quad$:- Right half plane maps to exterior of unit circle.

$\therefore \quad \Sigma < 0 \equiv |z| < 1$

$$\therefore \quad \boxed{s = \frac{1-z^{-1}}{1+z^{-1}}}$$

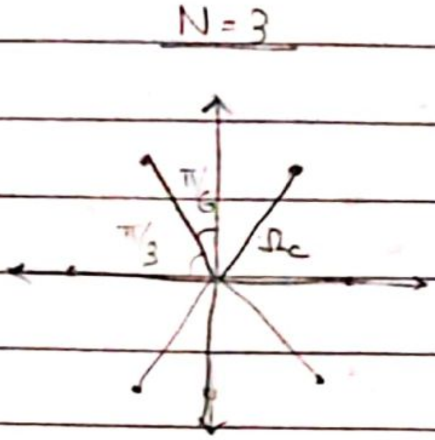
Theorem A bilinear transform like the one above maps

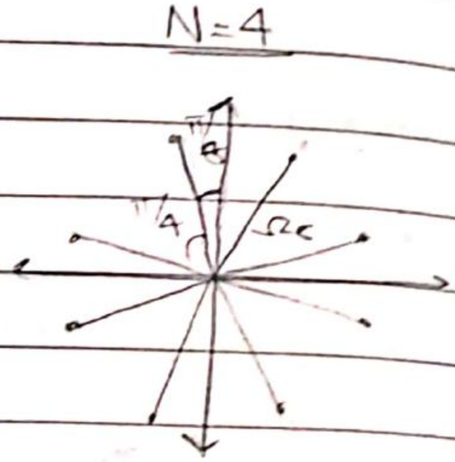
circles $\longrightarrow$ circles

straight lines $\longrightarrow$ straight lines

$$= e^{j\pi(2k+1)}$$

$$\therefore \quad s = j\Omega_c e^{j\frac{\pi}{2N}(2k+1)}$$





Now, all poles in LHP will go into $H_{analog}(s)$

RHP $H_{analog}(-s)$

$$\therefore \quad H_{analog}(s) = \frac{k_o}{\prod_{p=1}^{N}(s - s_p)}$$

For k_o, $H_{analog}(0) = 1 \Rightarrow k_o = \prod_{p=1}^{N}(-s_p)$

$$= \Omega_c^N$$

$$\therefore \quad H_{analog}(s) = \frac{\Omega_c^N}{\prod_{p=1}^{N}(s - s_p)}$$

For converting to digital filter : $s \rightarrow \frac{1 - z^{-1}}{1 + z^{-1}}$

Page No
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III CHEBYSCHEV FILTER.

Define $C_N(x) \triangleq \cos(N\cos^{-1}x)$

N	$C_N(x)$
0	1
1	x
2	$2x^2 - 1$
$\vdots$	$\vdots$

$\cos^{-1}x$ will be imaginary for $|x| > 1$.

Note :- $C_N(x) = \cosh(N\cosh^{-1}x)$

While the two expressions are equivalent, we use first one for $|x| < 1$ and second one for $|x| > 1$.

- $C_{N+1}(x) + C_{N-1}(x) = \cos((N+1)\cos^{-1}x) + \cos((N-1)\cos^{-1}x)$

$$= 2\cos(\cos^{-1}x)\cos(N\cos^{-1}x)$$

$$= 2x\, C_N(x)$$

- Properties :-
1) $C_N(x)$ is a polynomial
2) $C_N(x)$ is odd for odd N

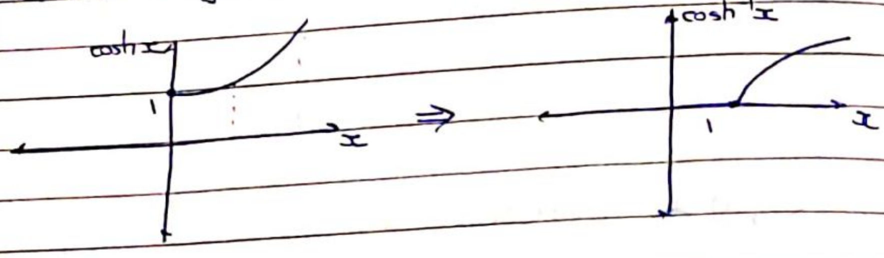
   even even
3) $C_N^2(x)$ is even always

→ Chebyschev Filter

$$H_{analog}(s)\, H_{analog}(-s) = \frac{1}{1 + \epsilon^2 C_N^2\left(\frac{\Omega}{\Omega_p}\right)} \quad \text{when } s = j\Omega$$

* $\cosh(N\cosh^{-1}x)$ for $x > 1$

Consider $y = \cosh^{-1}x \Rightarrow x = \dfrac{e^{y}+e^{-y}}{2}$



$\cosh(N\cosh^{-1}x)$

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- $|H(0)|$ depends on parity of N.

- $\dfrac{1}{1+\epsilon^2} \geq (1-\delta_1)^2$ ① and $\dfrac{1}{1+\epsilon^2 C_N^2\left(\frac{\Omega_s}{\Omega_p}\right)} \leq \delta_2^2$ ②

$\therefore \epsilon^2 \leq \dfrac{1}{(1-\delta_1)^2} - 1$ and $\epsilon^2 C_N^2\left(\dfrac{\Omega_s}{\Omega_p}\right) \geq \dfrac{1}{\delta_2^2} - 1$

$= D_1$ $\qquad = D_2$

$\therefore \epsilon \leq \sqrt{D_1}$ $\qquad \therefore C_N^2\left(\dfrac{\Omega_s}{\Omega_p}\right) \geq \dfrac{D_2}{\epsilon^2}$

$\Rightarrow \cosh^2\left(N\cosh^{-1}\left(\dfrac{\Omega_s}{\Omega_p}\right)\right) \geq \dfrac{D_2}{\epsilon^2}$

$\therefore$ Larger ϵ reduces N_{min} but increases δ_1

$\therefore$ Choose $\epsilon = \sqrt{D_1}$

$\therefore N \geq \dfrac{\cosh^{-1}\sqrt{D_2/D_1}}{\cosh^{-1}(\Omega_s/\Omega_p)}$

$\therefore \cosh\left(N\cosh^{-1}\left(\dfrac{\Omega_s}{\Omega_p}\right)\right) \geq \dfrac{\sqrt{D_2}}{\epsilon}$

($\because \sqrt{x}$ is monotonically increasing)

$\therefore N\cosh^{-1}\left(\dfrac{\Omega_s}{\Omega_p}\right) \geq \cosh^{-1}\dfrac{\sqrt{D_2}}{\epsilon}$

($\because \cosh^{-1}x$ is monotonically increasing)

$\therefore N \geq \dfrac{\cosh^{-1}\sqrt{D_2}/\epsilon}{\cosh^{-1}(\Omega_s/\Omega_p)}$

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Compare B & C phase response (group delay) for same specs
Band pass.

Theorem For the same specifications, $N_{min(Chebyschev)} \leq N_{min(Butterworth)}$

* However, Chebyschev has a worse (more non-linear) phase response than Butterworth.

→ Finding $H(s)$

$$H_{analog}(s)\, H_{analog}(-s) = \frac{1}{1+\epsilon^2 C_N^2\left(\frac{s}{j\Omega_p}\right)}$$

To find poles, $\epsilon^2 C_N^2\left(\frac{s}{j\Omega_p}\right) = -1$

$$\cos^2\left(N\cos^{-1}\frac{s}{j\Omega_p}\right) = \frac{-1}{\epsilon^2}$$

* $$\cos(A+jB) = \cos A \underbrace{\cos jB}_{\cosh B} - \sin A \underbrace{\sin jB}_{j\sinh B}.$$

~~Let~~ $$\cos\left(N\cos^{-1}\frac{s}{j\Omega_p}\right) = \pm\frac{j}{\epsilon}$$

Let s_k be the poles

Let $\cos^{-1}\left(\frac{s_k}{j\Omega_p}\right) = C_k + jD_k$

$$\therefore \frac{s_k}{j\Omega_p} = \cos C_k \cosh D_k - j\sin C_k \sinh D_k$$

$$\cos(NC_k + jND_k) = \frac{j}{\epsilon}$$

$$\therefore \cos NC_k \cosh ND_k - j\sin NC_k \sinh ND_k = \pm\frac{j}{\epsilon}$$

Comparing real and imaginary parts :-

$\cos NC_k \underbrace{\cosh ND_k}_{\geq 1} = 0$ and $\underbrace{\sin NC_k}_{=\pm 1} \sinh ND_k = \pm\frac{1}{\epsilon}$

$\therefore NC_k = (\text{odd})\frac{\pi}{2}$ $\therefore \sinh ND_k = \pm\frac{1}{\epsilon}$

P.T.O.

$\therefore \quad C_k = \frac{(2k+1)\pi}{2} \qquad D_k = \frac{1}{N}\sinh^{-1}(1/\epsilon)$

For all poles, we take any contiguous 2N values of k

- Obtain all poles

  $H_{analog}(s)$ contains only poles in LHP.

  Find constant by $H_{analog}(0)$.

CP Sketch poles of $H_{analog}(s)$ for N = 3, 4, 5, 6

Theorem Poles lie on an ellipse.

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Passband	Stop-band	Filter Type
Monotonic	Monotonic	Butterworth
Equiripple	M	Chebyschev.
M	E	Inverse Chebyschev
E	E	Jacobi / Elliptic.

IV Analog Frequency Transformations

eg Transform :- $H_{analog,\,BPF}(s) \xrightarrow[S_L \leftarrow F(s)]{} H_{analog,\,LPF}(S_L)$

- We transform bandpass specifications to auxilliary lowpass specifications, design the LPF, transform back to BPF

- The frequency transform [$s = j\Omega$ maps to $s_L = j\Omega_L$] must retain
① Stability with causality
② Rationality

$\therefore$ $F(s)$ is an LC immitance function
↳ admittance or impedance.

→ eg – LPF → HPF :- $F(s) = s_L = \frac{\Omega_p}{s}$ .... ~ C impedance or ~ L admittance.

$$s = j\Omega \Rightarrow s_L = -j\frac{\Omega_p}{\Omega} = j\Omega_L$$

$$\therefore \Omega_L = \frac{-\Omega_p}{\Omega}$$

Also, $F(-j\Omega) = -F(j\Omega)$
and $\frac{d\Omega_L}{d\Omega} = \frac{\Omega_p}{\Omega^2} > 0$

We observe only on positive frequencies. The result for negative images would just be a mirror image.

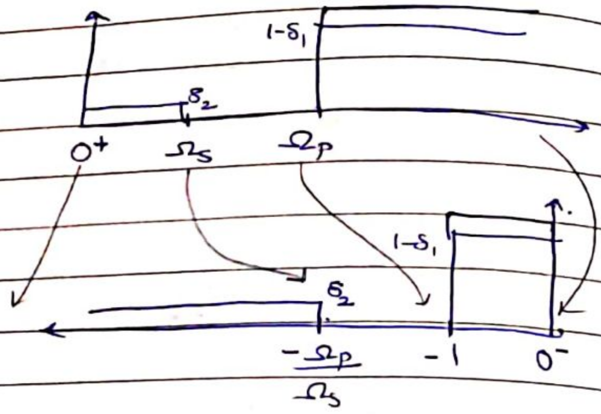
→ Critical Points :-

Ω	Ω_L
0^+	$-\infty$
Ω_p	-1
Ω_s	$-\Omega_p/\Omega_s$
$+\infty$	0^-

P.T.O.

HPF

Auxilliary LPF



* We ~~could~~ could not have taken $\Omega_L = \frac{\Omega_p}{\Omega} \Rightarrow s_L = \frac{-\Omega_p}{s}$

because stability would have been compromised.

- In F(s), LHP of s must be mapped to LHP of s_L
  RHP RHP.

→ BPF $\xrightarrow{F(s)}$ LPF

- F(s) cannot be a first order transformation.

Use

$$s_L = sL + \frac{1}{sC} = \frac{s^2LC+1}{sC}$$

$B = 1/L$, $\Omega_0 =$

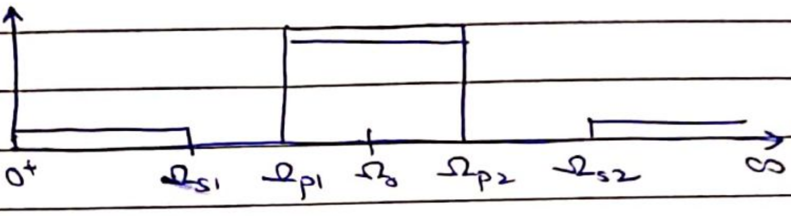
$$\Omega_L = \frac{\Omega^2 - \Omega_0}{B\,\Omega} = \frac{\Omega}{B} - \frac{\Omega_0^2}{B\,\Omega}$$

$$\frac{d\Omega_L}{d\Omega} = \frac{1}{B} + \frac{\Omega_0^2}{B\,\Omega^2}$$ ..... this mapping is also monotonically increasing.

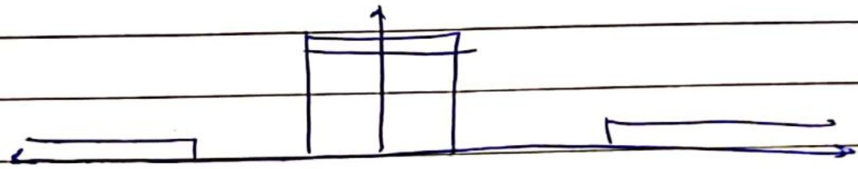
- Critical Points :

Ω	Ω_L
0^+	$-\infty$
Ω_{s1}	
Ω_{p1}	
Ω_0	0
Ω_{p2}	
Ω_{s2}	
$+\infty$	$+\infty$

BPF



LPF



- We constrain mapping of Ω_{p1} & Ω_{p2} (passband edges) to be $-1, +1$

$$\frac{\Omega_{p1}^2 - \Omega_0^2}{B\,\Omega_{p1}} = -1 \qquad \frac{\Omega_{p2}^2 - \Omega_0^2}{B\,\Omega_{p2}} = 1$$

Solve to get
$$\Omega_0^2 = \Omega_{p1}\Omega_{p2}$$
$$B = \Omega_{p2} - \Omega_{p1}$$

* Even in RLC BPF, the resonant frequency is the GM of the two half-power frequencies.