- Generalization of Fourier Transform is possible.

eg. • $x[n] = 2^n u[n]$ has no DTFT, because not convergent.

∴ Multiply by r^{-n}, $r > 2$

Find DTFT of $x[n]r^{-n}$

$$= \frac{1}{1 - 2r^{-1}e^{-j\omega}}$$

• Frequency response of ~~ne~~ LSI system $(h[n])$

- DTFT of $h[n]r^{-n}$

$$= \sum_{n=-\infty}^{\infty} h[n] r^{-n} e^{-j\omega n}.$$

$$= \sum_{n=-\infty}^{\infty} h[n] \left(re^{j\omega}\right)^{-n} = H(z)$$

Define $z = re^{j\omega}$ .... a complex variable.

- Output = $y[n]$ $$= \sum_{k=-\infty}^{\infty} x[k] r^{-k} h[n-k] r^{-(n-k)}$$

$$= \sum_{k=-\infty}^{\infty} x[k] h[n-k] r^{-n}$$

$$= r^{-n} (x * h)[n]$$

~~Outy~~ When x & h are multiplied by r^{-n}, so is output.

→ Z-transform of $h[n] \triangleq \sum h[n] \left(re^{j\omega}\right)^{-n} = \sum h[n] z^{-n}$

~~= Mapping from h[n] to~~

→ LSI system

$x[n]r^{-n} \longrightarrow \boxed{h[n]r^{-n}} \longrightarrow y[n]r^{-n}.$

$Y(z) = X(z)H(z)$ if these transforms exist for x, h, y for this z.

→ Convolution Property :-

$$x_1[n] * x_2[n] \xrightarrow[Z]{} X_1(z)\,X_2(z)$$

provided we choose a 'z' so that ~~all~~ $X_1(z)$, $X_2(z)$ and Z-transform of $x_1 * x_2$ exist for that z.

- DTFT = Z-Transform at $r = 1$

- Existence of Z-transform depends only on magnitude of z (r), not the phase.

- The Z-transform is valid only in a 'region of convergence'.

  – In general, ROC is $R_1 \leq |z| \leq R_2$

$$0 \leq R_1 \underset{\text{or } <}{\leq} r \underset{\text{or } <}{\leq} R_2 \leq \infty$$

eg $x[n] = 2^n u[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} 2^n u[n]\, r^{-n} \cdot e^{-j\omega n}$$

$$= \sum u[n] \left(\frac{2}{r} e^{-j\omega}\right)^n$$

$$= \frac{1}{1 - 2z^{-1}} \quad \text{with ROC} \quad |z| > 2.$$

eg $x[n] = -2^n u[-n-1]$ $\qquad -n-1 \geq 0$, $n \leq -1$

$$= \sum_{n=-\infty}^{\infty} -2^n u[-n-1]\, r^{-n} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} - \left(\frac{2}{r} e^{-j\omega}\right)^n$$

$$= \sum_{k=1}^{\infty} - \left(\frac{r}{2} e^{j\omega}\right)^k$$

$$= \frac{1}{1 - 2z^{-1}} \quad \text{with ROC} \quad |z| < 2$$

- The same expression with different ROC corresponds to a different sequence

eg $\delta[n] \xrightarrow{z} \sum_{n=-\infty}^{\infty} \delta[n] z^{-n}$.

$= 1$ ROC :- $|z| \geq 0$

Equality if $0^0 = 1$

eg $\delta[n-1] \xrightarrow{z} \sum_{n=-\infty}^{\infty} \delta[n-1] z^{-n}$.

$= z^{-1}$ ROC :- $|z| > 0$

eg $\delta[n+1] \xrightarrow{z} z$ ROC :- $|z| \geq 0$, $z \neq \infty$.

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I PROPERTIES

1 Linearity

$x_1[n] \xrightarrow{z} X_1(z)$ .... R_1

$x_2[n] \xrightarrow{z} X_2(z)$ .... R_2

Then $\alpha x_1 + \beta x_2 \xrightarrow{z} \alpha X_1 + \beta X_2$ RoC : At least $R_1 \cap R_2$

eg $x_1[n] = \frac{1}{2^n} u[n]$

$x_2[n] = \frac{1}{4^n} u[n] - \frac{1}{2^n} u[n]$

$X_1(z) = \sum_{n=0}^{\infty} 2^{-n} z^{-n} = \frac{1}{1 - \frac{1}{2z}}$ with RoC :- $|z| > \frac{1}{2}$.

$X_2(z) = \frac{1}{1 - \frac{1}{4z}} - \frac{1}{1 - \frac{1}{2z}}$ with RoC :- $|z| > \frac{1}{2}$

- $X_1(z) + X_2(z)$ has RoC $|z| > \frac{1}{4}$ $\supset$ $|z| > \frac{1}{2}$

2 Time Shift.

$x[n-n_0] \xrightarrow{Z} z^{-n_0} X(z)$ ...... R_x, except possibly the boundary

$\sum x[n-n_0] z^{-n} = \sum_{l=-\infty}^{\infty} x[l] z^{-l} z^{-n_0} = z^{-n_0} X(z)$

3 Convolution

$(x_1 * x_2)[n] \xrightarrow{Z} X_1(\omega) X_2(\omega)$ ...... At least $R_1 \cap R_2$

4 Modulation

$\alpha^n x[n] \longrightarrow X\left(\frac{z}{\alpha}\right)$ .... αR_x .... $\frac{z}{\alpha} \in R_x$

α is a complex constant

$\sum_{n=-\infty}^{\infty} \alpha^n x[n] z^{-n} = \sum x[n] \left(\frac{z}{\alpha}\right)^{-n}$

5 Multiplication

$x_1[n]\, x_2[n] \longrightarrow$

Later

6 Inversion (IZT)

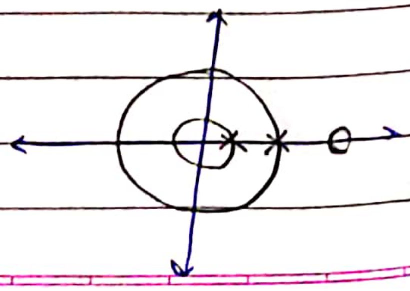
$h[n] r^{-n} \xrightarrow{DTFT} H(z)$

$\therefore \quad h[n] = \underline{r^n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} H(re^{j\omega})\, \underline{e^{j\omega n}}\, d\omega \right)$

Now $z = re^{j\omega} \Rightarrow dz = re^{j\omega} j\, d\omega$ at constant r (a circle)

$$\therefore \quad \boxed{h[n] = \frac{1}{j2\pi} \oint_{\substack{\text{any closed contour} \\ \text{in the RoC.}}} H(z) z^{n-1} dz}$$

eg $H(z) = \dfrac{1-3z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)}$



$= \frac{-1}{3} \cdot \frac{1}{1-\frac{1}{2}z^{-1}} \quad \frac{3}{} \quad 1-2z^{-1}$

1) $|z| > 2$:- $\frac{-1}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{4}{3} 2^n u[n]$.

2) $\frac{1}{2} < |z| < 2$:- $\frac{-1}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{4}{3}\left(-2^n u[-n-1]\right)$

3) $|z| < \frac{1}{2}$:- $\frac{-1}{3}\left(\frac{-1}{2^n} u[-n-1]\right) + \frac{4}{3}\left(-2^n u[-n-1]\right)$

7 Derivative in Z domain

$n x[n] \longrightarrow -z \frac{dX(z)}{dz}$ .... ∄ R_x, except possibly boun

$\frac{d}{dz}\left(\sum_{-\infty}^{\infty} x[n] z^{-n}\right) = \sum -n x[n] z^{-n-1}$

• CP:-

Use IZT formula to invert $\frac{1}{1-\alpha z^{-1}}$ for $|z| > |\alpha|$ and $|z| <$
using Cauchy integral theorem.
(for every value of n)

5 Multiplication (Contd)

$x[n]y[n] \longrightarrow \sum_{n=-\infty}^{\infty} x[n]\, y[n]\, z^{-n}$.

$= \sum_{n=-\infty}^{\infty}\left(\frac{1}{j2\pi}\oint X(z_1) z_1^{n-1} dz_1\right) y[n]\, z^{-n}$.

$= \frac{1}{2\pi j}\oint X(z_1)\left(\sum y[n]\, z^{-n} z_1^{n-1}\right) dz_1$

$$= \frac{1}{2\pi j} \oint_C X(z_1) \left(\sum y[n] \left(\frac{z}{z_1}\right)^{-n} \right) z_1^{-1} dz_1$$

$$= \frac{1}{2\pi j} \oint_C X(z_1)\, Y\left(\frac{z}{z_1}\right) z_1^{-1} dz_1$$

C : belongs within RoC of X $\Rightarrow z_1 \in$ RoC of X

Also, $\frac{z}{z_1}$ belongs to RoC of Y

$x[n] = \alpha^n u[n]$, $y[n] = \beta^n u[n]$, $|\alpha|, |\beta| < 1$.

Obtain ZT of $x[n]y[n]$ using above expression

RATIONAL Z TRANSFORM

Rational ZT $\triangleq \frac{N(z)}{D(z)}$ where N & D are finite length series in z

- Can be written in two general forms :-

1) $z^{(\,)} \left(\frac{\text{Poly in } z}{\text{Poly in } z} \right)$ 2) $z^{(\,)} \left(\frac{\text{Poly in } z^{-1}}{\text{Poly in } z^{-1}} \right)$

Realization $\triangleq$ Translation of ZT into practical system using finite resources

- Resources :- 1) Unit sample delay
  2) Two input adders
  3) Constant multipliers

* 'Digital Signal Processor' is an 'Application specific integrated circuits (ASIC)'

Polynomial in z^{-1} → Denominator has constant = 1

Proof Exercise (CP)

eg $H(z) = \dfrac{1}{1-\alpha z^{-1}}$ for $|\alpha| < 1$ and $|z| > |\alpha|$.

∴ Impulse response $= h[n] = \alpha^n u[n]$

CP Do all sequences have a ZT? No - e^{n^2}

29/1 • Rational function $= z^{(\ )} \dfrac{N(z^{-1})}{D(z^{-1})}$

can be credited simply to a time-shifted input impulse

By long division, $\dfrac{N(z^{-1})}{D(z^{-1})} = Q(z^{-1}) + \dfrac{R(z^{-1})}{D(z^{-1})}$ where $\deg(R) < \deg(D)$

1 Now, IZT(Q) is a finite length sequence (sum of time shifted δ)

2 - Factorize $D(z^{-1})$:- $D(z^{-1}) = \prod_{l=1}^{M} (1 - a_l z^{-1})^{M_l}$ Multiplicity of l^{th} pole

3 - Partial fractions :- Numerators could have degree at most $M_l - 1$

- RoC for every term could be in the interior of pole or in exterior of pole

right-sided inverse left-sided inverse

Polyex terms

↓Δ

(Polynomial in n)(Exponential in n)

l^{th} term :- $\left(\sum b_p n^p\right) a_l^n \, u[n]$ or $u[-n-1]$

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* Irrational ZT

$$x[n] = \frac{1}{n!} u[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} \frac{u[n]}{n!} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}$$

$$= 1 + z^{-1} + \frac{z^{-1}}{2!} \ldots$$

$$= e^{1/z} \qquad |z| > 0$$

An irrational system cannot be realized with finite resources

$\therefore$ All LSI systems are not rational.

→ Causal system

$$\frac{Y(z)}{X(z)} = \frac{\sum_{l=0}^{M} b_l z^{-l}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

$$\therefore Y(z) = \sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{l=0}^{M} b_l z^{-l} X(z)$$

IZT:- $$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{l=0}^{M} b_l x[n-l]$$

'Difference Equation' or LCCDE

- We now know how to find output given a rational $X(z)$

Q $x[n] = \frac{1}{n!} u[n]$ ..... irrational

$h[n] = \frac{1}{n!} u[n]$

$X(z) = e^{1/z}$, $H(z) = e^{1/z}$

$Y(z) = e^{2/z}$

$$y[n] = \frac{2^n}{n!} u[n]$$

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→ Causality :-

$h[n] = 0 \quad \forall n < 0 \quad \Longrightarrow \quad H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}.$

$= h[0] + h[1] z^{-1} + h[2] z^{-2} \ldots$

Assuming $h[n]$ is finite $\forall n$, this $H(z)$ series definitely converges for $|z| \to \infty$.

eg $h[n] = \alpha^n u[n] \longleftrightarrow H(z) = \frac{1}{1-\alpha z^{-1}}$ for $|z| > |\alpha|$

Notice how $|z| \to \infty$ is in the RoC.

eg $H(z) = \frac{z}{1-\frac{1}{2}z^{-1}} \longleftrightarrow h[n] = \underset{-1}{\overset{1}{\uparrow}} \quad \underset{\cancel{2}0}{\overset{1/2}{\uparrow}} \quad \underset{1}{\overset{1/2^2}{\uparrow}} \ldots$ ..... non-causal

n

RoC :- $\frac{1}{2} < |z| < \infty$ .... does not include $|z| = \infty$.

- If a system is causal, it must include $+\infty$ in RoC

Theorem A rational system is causal iff RoC of its system function includes $+\infty$.

Could also hold for irrational system

→ Stability

$\sum |h[n]| < \infty$.

Rational function :- $H(z) = z^D \left(Q(z) + \frac{R(z)}{D(z)} \right)$

$z^D \xrightarrow{IZT}$ Time shift, $Q(z) \xrightarrow{IZT}$ Finite length sequence

$\underbrace{\text{Time shift} \quad \text{Finite length sequence}}_{\text{Do not affect absolute summability}}$

$\frac{R(z)}{D(z)} = \sum_{l=1}^{M} \frac{\text{Polynomial of degree } M_l - 1 \text{ in } z^{-1}}{(1-a_l z^{-1})^{M_l}}$

$\downarrow IZT$ ~~IZT~~

$= \sum_{l=1}^{M} \left(\begin{array}{c}\text{Polynomial in } n \\ \text{of degree } M_l - 1\end{array} \right) q_l^n u[n]$

Based on RoC of partial fraction,

Polyex's IZT could be left-sided or right-sided

$$\frac{R(z)}{D(z)} \xrightarrow{IZT} \sum_{l=1}^{M} \underbrace{\left[\left(\begin{matrix}\text{Polynomial of degree} \\ M_l - 1\end{matrix} \right) a_l^n \; u[n] \text{ or } u[-n-1] \right]}_{\text{Polyex term}}$$

(left sided or right sided)

One polyex for fi...

Consider any one polyex term:-

- Right sided sequence :- Polyex converges ($h[n]$ is absolutely summable) i...
- Left
- For $|a_l| = 1$, polyex does not converge.

- The polynomial part of polyex does not affect convergence, beca... exponential term is more potent.

* Ratio test.

- Deciding left-sided or right-sided :-

RoC is interior of pole $|a_l|$ RoC is exterior of pole $|a_l|$

- Polyex terms ~~are~~ with different 'a_l' are linearly independent
  - $\Rightarrow$ Sum of polyexes cannot become zero
  - $\Rightarrow$ For sum of polyexes to be finite, every polyex must be fini... (be absolutely summable)

Theorem: For rational systems, stability is equivalent to the unit circle bei... in the RoC