Algebraic, differential eqns : Element eqns

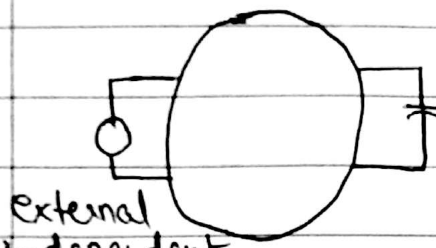
Initial conditions.

$\hookrightarrow \quad A_1 \ddot{x} + A_2 \dot{x} + A_3 x = f(t)$

A_1, A_2, A_3 are time dependent matrices (may not be invertible)

x is a matrix of variables.

State space equations :-



v, and $i = f(t)$ for $t \geq 0$

First order circuits :

$\frac{dy}{dt} = -ay + K \qquad t \geq 0 \quad$ given $y(0) = y_0$

General solution $y(t)$

$K = 0 \rightarrow$ homogeneous equation $\frac{dy}{dt} = -ay$

$\rightarrow \quad y = y_0 e^{-at}$

Particular solution $y_p(t)$

$\frac{dy_p(t)}{dt} = -a y_p(t) + K$

Any $z(t) = y(t) - y_p(t)$

Satisfies $\frac{dz}{dt} = -az$

Variation of parameters $\quad y_p(t) = e^{-at} c(t)$

$\frac{d}{dt}\left(e^{-at} c(t)\right) = -a\left(e^{-at} c(t)\right) + K$

$e^{-at} c'(t) = K \qquad c'(t) = K e^{at}$

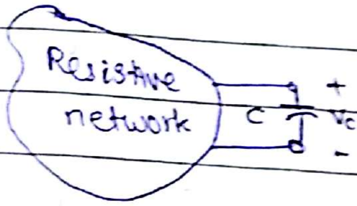
$c(t) = \int_0^t K e^{a\tau} d\tau$

$y_p(t) = e^{-at} \int_0^t K e^{a\tau} d\tau = \int_0^t K e^{-a(t-\tau)} d\tau$

Even if K was $K(t)$ $\quad y_p(t) = \int_0^t e^{-a(t-\tau)} K(\tau) d\tau$

Here, (general) $y(t) = \underbrace{e^{-at} y_0}_{\text{Zero input solution/response}} + \underbrace{\frac{k}{a}\left(1 - e^{-at}\right)}_{\text{Zero state solution/response}}$ $\frac{1}{a}$ = Time constant

e.g.

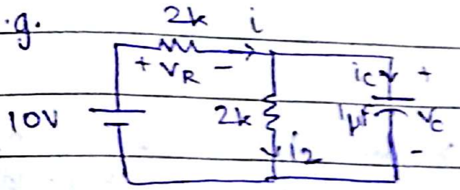


Only one energy storage element - first order circuit.

If there are 2 capacitors (or energy storage elements), such that initial state of one does not affect the initial state of another, it is a 2^{nd} order circuit, and so on...

→ Avoid all-capacitor loops and all-inductors cutsets.

e.g.



$V_R = 10 - V_c$

$i_2 = \frac{V_c}{2 \times 10^3} = 0.5 \times 10^{-3} V_c$

$10^{-6} \frac{dV_c}{dt} = i_c$

$10^{-6} \frac{dV_c}{dt} = i - 0.5 \times 10^{-3} V_c$

$10^{-6} \frac{dV_c}{dt} = \frac{V_R}{2 \times 10^3} - 0.5 \times 10^{-3} V_c$

$10^{-6} \frac{dV_c}{dt} = 0.5 \times 10^{-3} \left[10 - V_c - V_c \right]$

$\frac{dV_c}{dt} = -10^3 V_c + 5 \times 10^3$ $(\dot{y} = -ay + k)$

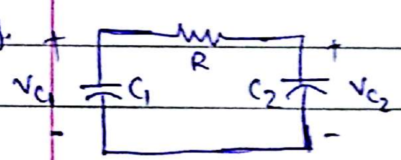
$V_c(t) = y_0 e^{-1000t} + 5\left(1 - e^{-1000t}\right)$

Given $i_c = 5mA$ at $t = 1ms$,

$10^{-6}\left(-1000 y_0 e^{-1000t} + 5000 e^{-1000t}\right) = 5 \times 10^{-3}$ $y_0 = \frac{5}{4} e$

$V_c(t) = \frac{5}{4} e^{-1000t+1} + 5\left(1 - e^{-1000t}\right)$

e.g.

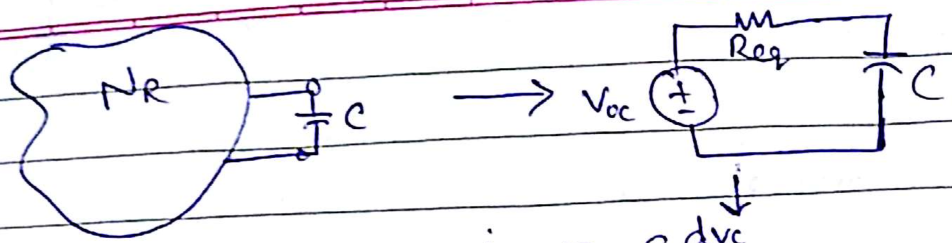


Given $V_{c_1}(0)$, $V_{c_2}(0)$

$C_2 \frac{dV_{c_2}}{dt} = \frac{1}{R}\left(V_{c_1} - V_{c_2}\right)$ (second order circuit)

$C_1 \frac{dV_{c_1}}{dt} = \frac{1}{R}\left(V_{c_2} - V_{c_1}\right)$

If $C_1 = C_2$,

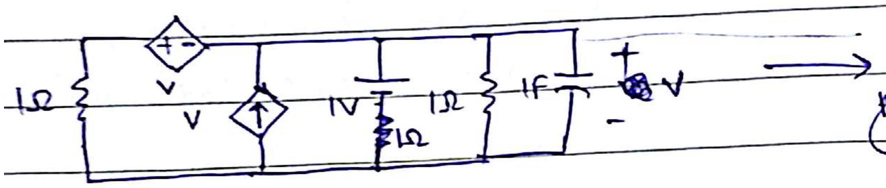


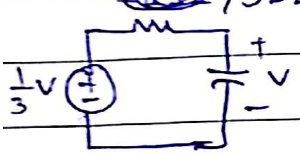
(Assuming Thevenin equivalent is solvable)

$$i_c = C\frac{dv_c}{dt}$$

$$C\frac{dV_c}{dt} = \frac{V_{oc} - V_c}{R_{eq}}$$

$$\frac{dV_c}{dt} = -\frac{V_c}{R_{eq}C} + \frac{V_{oc}}{R_{eq}C}$$

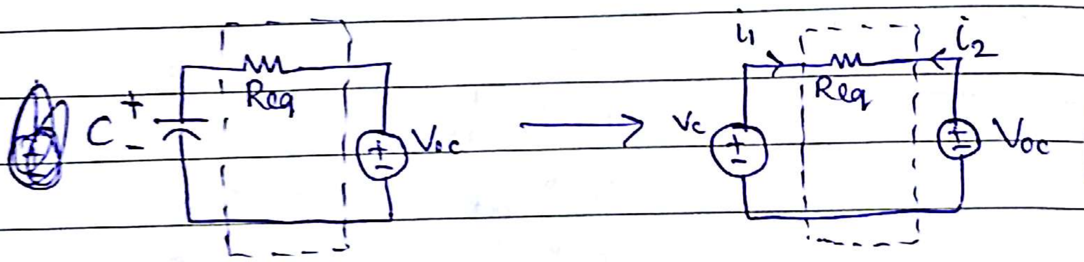




$$\frac{dv}{dt} = \frac{-v}{1/3} + \frac{1/3}{1/3}$$

$$= -3v + 1$$

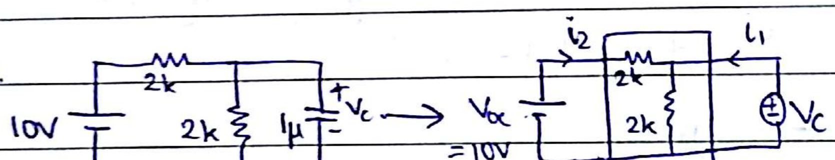
$$\tau = 1/3 \text{ s}$$



$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{R_{eq}} & -\frac{1}{R_{eq}} \\ -\frac{1}{R_{eq}} & \frac{1}{R_{eq}} \end{bmatrix} \begin{bmatrix} V_c \\ V_{oc} \end{bmatrix}$$

But $i_1 = -C\dfrac{dV_c}{dt} = \dfrac{V_c}{R_{eq}} - \dfrac{V_{oc}}{R_{eq}}$

e.g.

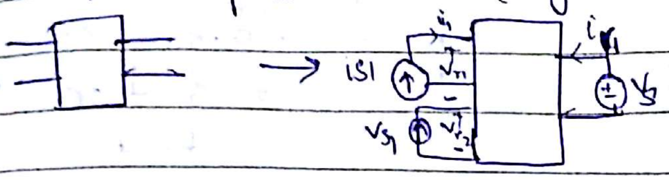


$i_1 = \dfrac{V_c}{1k\Omega} + \dfrac{V_{oc}}{2k\Omega} = \dfrac{V_c}{1k\Omega} + \dfrac{10V}{2k\Omega}$ (by enabling V_c and V_{oc} one by one

$$-10^{-6}\frac{dV_c}{dt} = 10^{-3}V_c + 5\times10^{-3}$$

$$\frac{dV_c}{dt} = 5000 - 1000V_c$$

Multi-port representation : (hybrid)



$$\begin{bmatrix} V_{r1} \\ V_{r2} \\ i_{r} \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} i_{s1} \\ i_{s2} \\ v_s \end{bmatrix}$$

Find hij by superposition.

Writing state space equations :-

i) Replace capacitors by voltage sources and inductors by current sources

ii) Take out these capacitors, inductors and independent sources as external sources across ports.

iv) Obtain hybrid representation for the Multiport network with the sources as independent and responses as dependent

iii) Index the sources, starting from capacitors, inductors then independent sources.

$$\begin{bmatrix} i_c \\ v_L \\ i_r \\ v_r \end{bmatrix} = \begin{bmatrix} & & \\ & H & \\ & & \end{bmatrix} \begin{bmatrix} v_c \\ i_L \\ v_s \\ i_s \end{bmatrix}$$

Calculate values of h by activating one source at a time.

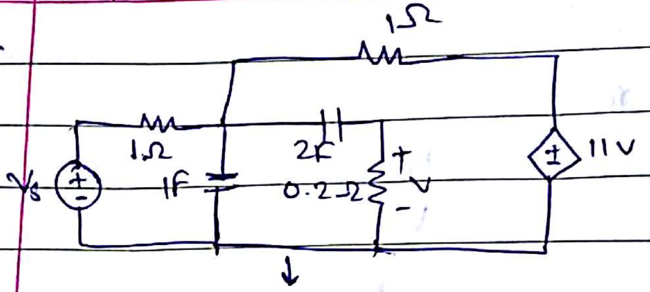
v) Replace i_c by $C\frac{dv_c}{dt}$ and v_L by $L\frac{di_L}{dt}$, divide by constants, getting state space equations:

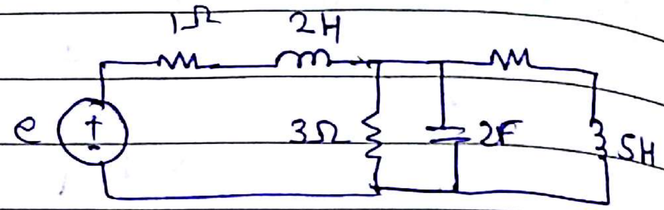
$$\frac{d\bar{x}}{dt} = A\bar{x} + B\bar{u}$$

$$H = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

or $$\bar{y} = C\bar{x} + D\bar{u}$$

$$x = \begin{bmatrix} v_c \\ i_L \end{bmatrix} \quad u = \begin{bmatrix} v_s \\ i_s \end{bmatrix} \quad y = \begin{bmatrix} i_r \\ v_r \end{bmatrix}$$





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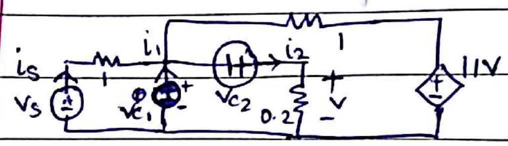
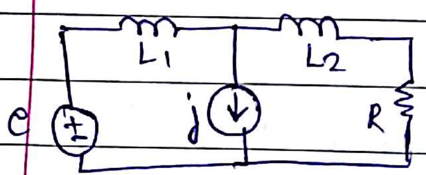
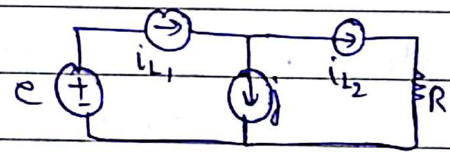


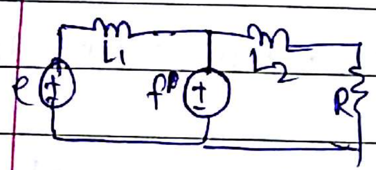
$$\begin{bmatrix} i_1\\ i_2\\ i_s\end{bmatrix} = \begin{bmatrix}-4&-6&-1\\5&5&0\\-1&0&1\end{bmatrix}\begin{bmatrix}v_{c_1}\\v_{c_2}\\v_s\end{bmatrix}$$



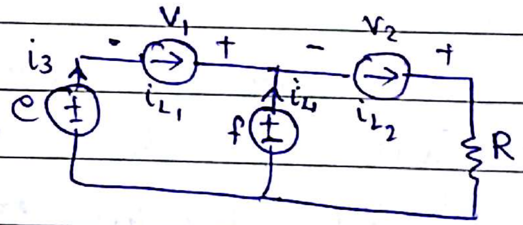
$\longrightarrow$



(Independent current sources forming a cutset. Not solvable)



$\longrightarrow$

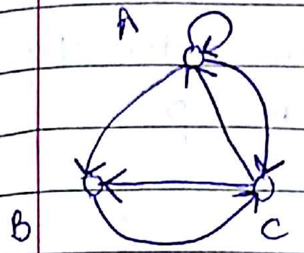


$$\begin{bmatrix} v_1\\ v_2\\ i_3\\ i_4\end{bmatrix} = \underbrace{\begin{bmatrix}0&0&-1&1\\0&R&0&-1\\1&0&0&0\\-1&1&0&0\end{bmatrix}}_{\text{Hybrid}}\begin{bmatrix}i_{L_1}\\i_{L_2}\\e\\f\end{bmatrix}$$

(non-singular) → solvable

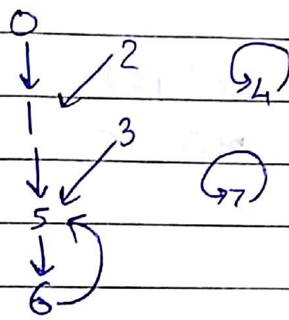
Networks (in general) :- for logic, computation
Modelling biological, chemical processes,
logistics and engineering (operational mgmt),
AI, digital circuits.

Boolean networks → Domain of variables ($\{0,1\}$)
→ Propositions
→ Laws



Current (k^{th}) state	$A(c)$	$B(c)$	$C(c)$
Next $(k+1)^{th}$ state	$A(n)$	$B(n)$	$C(n)$
Dependence	$f_A(A,C)$	$f(A,C)$	$f(B,A)$
e.g. =	$A \vee C$	$A \wedge C$	$\neg A \vee B$

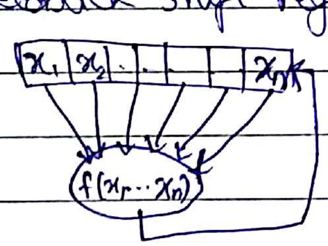
State space : set of all possible values of all states. no = 2^n.

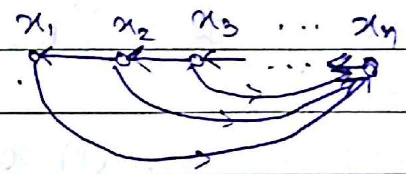


← Chains, periodic loops, fixed points. nothing other than these. permutations will have only fixed points or loops.

State $X = (x_1, x_2 \ldots x_n) \xrightarrow{F(X)} (x_2, x_3 \ldots f(x_1, x_2 \ldots x_n))$

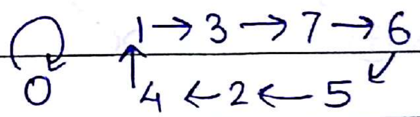
Feedback shift register :





e.g. $(x_1 \; x_2 \; x_3) \longrightarrow (x_2 \; x_3 \; x_1 \oplus x_3)$ ①
$\longrightarrow (x_2 \; x_3 \; x_1 \oplus x_2 x_3)$ ②

① $\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 3 & 4 & 7 & 1 & 2 & 5 & 6 \end{pmatrix}$



It is a permutation! fixedpoint orbit

② $(x_1 \; x_2 \; x_3) \rightarrow (x_2 \quad x_3 \quad x_1 \oplus x_2 x_3)$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 2 & 4 & 7 & 1 & 3 & 5 & 6 \end{pmatrix} \qquad 0 \circlearrowleft \qquad 1 \to 2 \to 4 \to 1 \qquad 3 \to 7 \to 6 \to 5 \to 3$$

→ ~~Finite state~~ Feedback shiftregister is a permutation iff
$$f(x_1, x_2 \ldots x_n) = x_1 \oplus g(x_2 \ldots x_n)$$

→ For modulo p algebra, any Multiple of prime factors of p (·a is not invertible, i.e. $\nexists$ b s.t. $ab = 1 \bmod p$

→ Rules of boolean algebra

Boolean algebra $B = \{B, 0, 1, +, \cdot, '\}$

i) B is closed under $+, \cdot, '$

ii) $x + y = y + x$, $xy = yx$

iii) $(x+y)+z = x+(y+z)$ $\quad (xy)z = x(yz)$

iv) $x + 0 = x \quad x + 1 = 1 \quad x0 = 0 \quad x1 = x$

v) $x + x' = 1$, $xx' = 0$

vi) $x(y+z) = xy + xz$

$(x+y)(x+z) = x + yz$

vii) $(x+y)' = x'y' \quad (xy)' = x' + y'$

Boolean Ring $B = \{B, 0, 1, \oplus, \cdot\}$ (No complement)

$x \oplus y = y \oplus z$

$x(y \oplus z) = xy \oplus xz$

$x \oplus x = 0$

Boolean Ring $\longleftrightarrow$ Boolean algebra

$x \oplus y \oplus xy = x + y$

$x \oplus y = xy' + x'y$

$x \oplus 1 = x'$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} a_1 & c_1 \\ b_1 & d_1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

eg

State space eq^{n}s continued :-

$$\dot{x} = Ax + Bu$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + B \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$s \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} - \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} + B \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = (sI - A)^{-1} \left(\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + B \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} \right)$$

Solving $\dot{x} = Ax$: (zero input)

Take $f(\lambda) = \det(A - \lambda I) = \lambda^2 - tr(A)\lambda + \det A$
(characteristic polynomial)

Roots of ch. polynomial (complex, in general) : eigen values of A.

$f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$ $\quad \lambda_1, \lambda_2 \in \mathbb{C}$ are

Case 1 : λ_1, λ_2 are distinct.

v_1, v_2 are eigen vectors of A for λ_1, λ_2 $\left(Av_1 = \lambda_1 v_1, \; Av_2 = \lambda_2 v_2 \right)$
(v_1, v_2 are linearly independent)

Then $\bar{x}(t) =$ $y_1(t)\bar{v}_1 + y_2(t)\bar{v}_2$

$\dot{x} = y_1'(t) v_1 + y_2'(t) v_2$ $\qquad Ax = A(y_1 v_1 + y_2 v_2) = \lambda_1 y_1 v_1 + \lambda_2 y_2 v_2$

$\dot{y}_1 v_1 + \dot{y}_2 v_2 = \lambda_1 y_1 v_1 + \lambda_2 y_2 v_2$

$\dot{y}_1 = \lambda_1 y_1$ and $\dot{y}_2 = \lambda_2 y_2$ $\qquad$ (from linear independence)

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

Solve y_1, y_2 using this and

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = y_1(t) V_1 + y_2(t) V_2$$

e.g. $A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$ $\quad f(\lambda) = \lambda^2 + 5\lambda + 6 = (\lambda+3)(\lambda+2)$

$\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} -3\alpha_1 \\ -3\alpha_2 \end{bmatrix}$ $\qquad \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} -2\alpha_1 \\ -2\alpha_2 \end{bmatrix}$

Take $\alpha_1 = 1$ to specify a unique eigen vector.

$[v_1 \;\; v_2] = \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix}$

Solve $\dot{y}_1 = -3y_1$, $\dot{y}_2 = -2y_2 \rightarrow y_1 = e^{-3t} y_1(0)$ $\quad y_2 = e^{-2t} y_2(0)$

Let $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Then $\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} y_1(0)e^{-3t} \\ y_2(0)e^{-2t} \end{bmatrix}$

Case 2 : $\lambda_1 = \lambda_2$

→ 2.1 : There are 2 L.I. eigenvectors v_1, v_2 (same as above)

→ 2.2. There is only one eigenvector.

e.g. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Cayley - Hamilton Theorem :

If $f(\lambda)$ is the char. polynomial $\Rightarrow f(A) = 0$

Proof : Use $M\,\mathrm{adj}(M) = (\det M) I$

$f(\lambda)I = \det(A-\lambda I)\, I = (A-\lambda I)\,\mathrm{adj}(A-\lambda I)$

$= A\,\mathrm{adj}(A-\lambda I) - \lambda I\,\mathrm{adj}(A-\lambda I)$

$\lambda = A$ gives $f(A)\, I = 0$

Since I is invertible, $f(A) = 0$

When $f(\lambda) = (\lambda - \lambda_1)^2 \longrightarrow f(A) = (A - \lambda_1 I)^2 = 0$

But $(A - \lambda_1 I) \neq 0$ because then there would be 2 linearly independent eigen vectors of λ_1.

Date

$(A-\lambda_1 I)^2 = 0 \quad , \quad (A-\lambda_1 I) \neq 0$

$\therefore \exists\ w \neq 0$ s.t. $(A-\lambda_1 I)w \neq 0$

Let $v = (A-\lambda_1 I)w \neq 0$

Then $(A-\lambda_1 I)v = (A-\lambda_1 I)^2 w = 0$

$\therefore$ v is an eigenvector of λ_1

$Aw = v + \lambda_1 w$ and $Av = \lambda_1 v$

$\rightarrow$ w and v are linearly independent :

$\alpha_1 w + \alpha_2 v = 0$

$\Rightarrow \alpha_1 Aw + \alpha_2 Av = 0$

$\Rightarrow \alpha_1 (v + \lambda_1 w) + \alpha_2 \lambda_1 v = 0$

Also, $\alpha_1 \lambda_1 w + \alpha_2 \lambda_1 v = 0$

$\Rightarrow \alpha_1 v = 0 \Rightarrow \alpha_1 = 0 \quad (v \neq 0)$

$\Rightarrow \alpha_2 v = 0 \Rightarrow \alpha_2 = 0$

So write $x = y_1 v + y_2 w$

$\dot{x} = \dot{y}_1 v + \dot{y}_2 w$

$Ax = y_1 Av + y_2 Aw$

$= y_1 \lambda_1 v + y_2 (v + \lambda_1 w)$

$\dot{y}_1 v + \dot{y}_2 w = (\lambda_1 y_1 + y_2) v + \lambda_1 y_2 w$

$\dot{y}_1 = \lambda_1 y_1 + y_2 \quad , \quad \dot{y}_2 = y_2 \lambda_1$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$y_2(t) = y_2(0) e^{\lambda_1 t}$

$\dot{y}_1 - \lambda_1 y_1 = y_2(0) e^{\lambda_1 t} \qquad y_1(t) = e^{\lambda_1 t} y_1(0) + \int_0^t e^{\lambda_1 (t-\tau)} y_2(0) e^{\lambda_1 \tau} d\tau$

$= e^{\lambda_1 t} y_1(0) + t e^{\lambda_1 t} y_2(0)$

$\therefore$ Solution $x = [y_1(0) + t y_2(0)] e^{\lambda_1 t} v + e^{\lambda_1 t} y_2(0) w$

e.g. $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ $\quad f(\lambda) = \lambda^2 + 2\lambda + 1$

$(\lambda_1 = -1)$

$$\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -a_1 \\ -a_2 \end{bmatrix} \qquad \begin{matrix} a_2 = -a_1 \\ -a_1 = a_2 \end{matrix}$$

$v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is the only L.I. eigenvector

$A - \lambda_1 I = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ $\quad$ Find w so that $(A - \lambda_1 I)w = v$

$w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$x = y_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + y_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Case 3 : $f(\lambda)$ has complex conjugate roots

$$f(\lambda) = (\lambda - \lambda_1)(\lambda - \overline{\lambda_1})$$

$$\lambda_1 = \sigma + j\omega \;,\; \overline{\lambda_1} = \sigma - j\omega \qquad (\omega \neq 0)$$

$\dot{x} = Ax$ is in a complex state space

$\exists$ L.I. eigenvectors for λ_1 and $\overline{\lambda_1}$.

$$Av = (\sigma + j\omega)v$$

A is real, $\therefore$ v must be complex.

$$v = v_r + jv_i$$

$$Av_r + jAv_i = (\sigma v_r - \omega v_i) + j(\sigma v_i + \omega v_r)$$

$$Av_r = \sigma v_r - \omega v_i \qquad Av_i = \sigma v_i + \omega v_r$$

v_r and v_i are linearly independent :

If $\alpha_1 v_r + \alpha_2 v_i = 0$ — (i)

$$\alpha_1 A v_r + \alpha_2 A v_i = 0$$

$$\alpha_1 \sigma v_r - \alpha_1 \omega v_i + \alpha_2 \sigma v_i + \alpha_2 \omega v_r = 0 \quad \text{— (ii)}$$

(ii) $- \sigma$(i) $\Rightarrow$ $\alpha_2 v_r - \alpha_1 v_i = 0$

Let $x = y_r v_r + y_i v_i$

$\dot{x} = \dot{y}_r v_r + \dot{y}_i v_i \qquad Ax = y_r A v_r + y_i A v_i$

$= y_r(\sigma v_r - \omega v_i) + y_i(\sigma v_i + \omega v_r)$

$$\begin{bmatrix} \dot{y}_r \\ \dot{y}_i \end{bmatrix} = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \begin{bmatrix} y_r \\ y_i \end{bmatrix}$$

Let $z = y_r + j y_i$

$\dot{z} = \dot{y}_r + j\dot{y}_i = \sigma y_r + \omega y_i - j\omega y_r + j\sigma y_i$

$= (\sigma - j\omega)(y_r + j y_i)$

$\dot{z} = (\sigma - j\omega) z$

$z(t) = e^{(\sigma - j\omega)t} z(0)$

$z(t) = e^{\sigma t}(\cos\omega t - j\sin\omega t) z(0) = y_r + j y_i$

$\cancel{y_r(t) = e^{\sigma t}\cos\omega t\, z(0) \quad y_i(t) = -e^{\sigma t}\sin\omega t}$

$= e^{\sigma t}(\cos\omega t - j\sin\omega t)(y_r(0) + j y_i(0)) = y_r + j y_i$

$y_r(t) = e^{\sigma t}\cos(\omega t - \theta) \qquad y_i(t) = e^{\sigma t}\sin(\omega t - \theta)$

$-\theta = \tan^{-1}\left(\frac{y_i(0)}{y_r(0)}\right)$

Boolean networks :-

$B_0 = \{0, 1\}$

$B = B_0 \cup \{a\} \qquad B_1 = \{0, 1, a, a'\}$

(a is an indeterminate) $\qquad$ (B_1 is closed in Boolean algebra)

$a + a' = 1 \qquad 1 + a = 1 + a' = 1$

$B_0 \subset B_1 \qquad aa' = 0$

Ring $B_1 = \{0, 1, a, 1 \oplus a\}$

$B_0(a) = B_1$

$B_0(x, y) = B_2$

Structure Theorem :

Any finite boolean algebra is isomorphic to the algebra of all subsets of a set S under the operations

$+ \leftrightarrow \cup \qquad \cdot \leftrightarrow \cap \qquad ' \leftrightarrow {}^c$

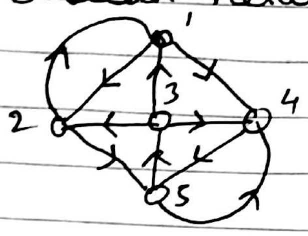
$a \leftrightarrow S_a \qquad b \leftrightarrow S_b \qquad 0 \leftrightarrow \phi \qquad 1 \leftrightarrow S \qquad (S_a, S_b \subseteq S$

F. M. Brown
Boolean Analysis

Inequality in B :

$$a \le b \iff S_a \subseteq S_b$$
$$\iff a+b=b$$
$$\iff ab=a \iff ab'=0$$

Boolean networks :



Linear system

$$X=(x_1\ x_2\ x_3\ x_4\ x_5)$$

$$F(X)=\begin{bmatrix} x_2\oplus x_3\\ x_1\oplus x_3\\ x_5\\ x_1\oplus x_3\oplus x_5\\ x_2\oplus x_4\end{bmatrix}$$

Monomial system

$$M(x)=\begin{bmatrix} x_2x_3\\ x_1x_3\\ x_3\\ x_1x_3x_5\\ x_2x_4\end{bmatrix}$$

Xnor system

$$G(x)=\begin{bmatrix} x_2'x_3'\\ x_1'x_3'\\ \vdots\\ \\ \end{bmatrix}=\begin{bmatrix} (x_2+x_3)'\\ (x_1+x_3)'\\ \vdots\\ \\ \end{bmatrix}$$

$$\langle x_1\ x_2\ x_3\rangle = x_1x_2\oplus x_2x_3\oplus x_3x_1$$

Fixed points (periodic points of period 1)

$x\ :\ F(x)=x$ period 2: $F^2(x)=F(F(x))=x$

Chain of length 1 : $x=F(y)$ but $y=F(z)$ has no solution

$$F(x)=x\ :\ \begin{bmatrix} x_2\oplus x_3\\ x_1\oplus x_3\\ x_5\\ x_1\oplus x_3\oplus x_5\\ x_2\oplus x_4\end{bmatrix}=\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{bmatrix}\ :$$

$(0,0,0,0,0)$
$\cancel{(0,0,0,0,0)}$
$(0\ 1\ 1\ 0\ 1)$
$(1\ 1\ 0\ 1\ 0)$
$(1\ 0\ 1\ 1\ 1)$

fixed points

$x_3=x_3$
$x_1=x_4$

$$\Downarrow$$

$$\begin{bmatrix} x_2\oplus x_3\oplus x_1\\ x_1\oplus x_3\oplus x_2\\ x_5\oplus x_3\\ x_1\oplus x_3\oplus x_4\oplus x_5\\ x_2\oplus x_4\oplus x_5\end{bmatrix}=0 \Rightarrow \begin{bmatrix} 1&1&1&0&0\\ \cancel{1}&\cancel{1}&\cancel{1}&\cancel{0}&\cancel{0}\\ 0&0&1&0&1\\ 1&0&1&1&1\\ 0&1&0&1&1\end{bmatrix} X=0$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} X = 0$$

(Adding rows)

$x_3 \oplus x_5 = 0$

$x_3 = x_5$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$x_3 \oplus x_5 = 0 \qquad x_2 \oplus x_4 \oplus x_5 = 0 \qquad x_1 \oplus x_4 = 0$

$x_3 = x_5 \;,\; x_1 = x_4 \;,\; x_2 = x_4 \oplus x_5$

Find chains of length 1 for fixed point 0
i.e. $F(y) = 0$ but $F(z) = y$ has no soln.

$x_2 = x_3 = x_1 = x_4 = 0$ or 1 $\quad f(z) = 0$
$x_5 = 0$ or

$\Rightarrow$ has soln.

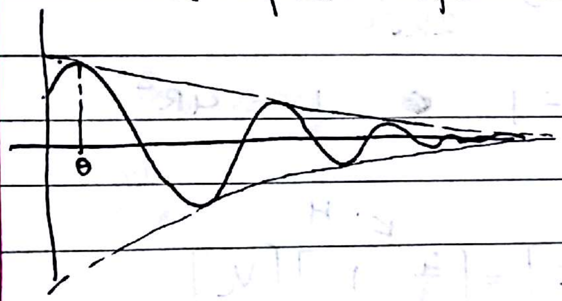
$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$x_1 = x_3' = x_2 = x_4 \qquad x_5 = 1 \qquad x_1 = x_3$

(No solution)

Chain of length 1 :
$(1,1,1,1,0) \rightarrow (0,0,0,0,0)$

Underdamped response :- $(\zeta < 1)$



$x(t) = e^{\sigma t} \cos(\omega t - \theta) \qquad (\sigma < 0)$

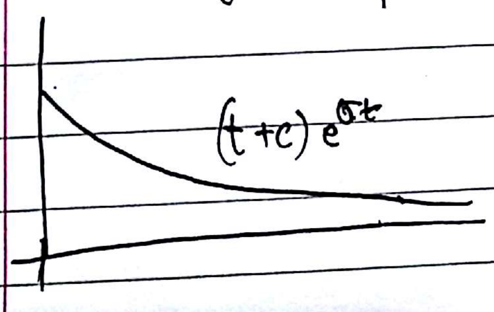
$\omega_n = \sqrt{\sigma^2 + \omega^2} \qquad \omega = \sqrt{\omega_n^2 - \sigma^2}$

$\frac{\sigma}{\omega_n} = \zeta$ (damping factor)

$\omega = \omega_n \sqrt{1 - \zeta^2}$

$f(\lambda) = (\lambda - \lambda_1)(\lambda - \bar{\lambda}_1) = \lambda^2 - 2\sigma\lambda + \sigma^2 + \omega^2$
$\qquad = \lambda^2 - 2\zeta\omega_n\lambda + \omega_n^2$

Critically damped :- $(\zeta = 1 \;,\; \omega = 0)$



- Let $\begin{bmatrix} i_c \\ v_L \end{bmatrix} = \underbrace{\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}}_{H} \begin{bmatrix} v_c \\ i_L \end{bmatrix}$

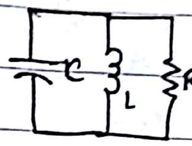
$\rightarrow \begin{bmatrix} dv_c/dt \\ dv_L/dt \end{bmatrix} = \begin{bmatrix} h_{11}/C & h_{12}/C \\ h_{21}/L & h_{22}/L \end{bmatrix} \begin{bmatrix} v_c \\ i_L \end{bmatrix}$

$$\dot{x} = Ax$$

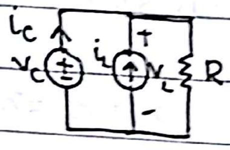
$\mathrm{tr}A = \frac{h_{11}}{C} + \frac{h_{22}}{L}$ $\qquad \det A = \frac{\det H}{LC}$

$\sigma = \frac{1}{2}\mathrm{tr}(A)$ $\qquad \omega_n = \sqrt{\frac{\det H}{LC}}$ $\qquad \zeta = \frac{1}{2}\mathrm{tr}(A)\sqrt{\frac{LC}{\det H}}$

$$f(\lambda) = \lambda^2 - \mathrm{tr}(A)\lambda + \det A$$
$$= \lambda^2 - 2\sigma\lambda + \omega_n^2 = \lambda^2 - 2\zeta\omega_n\lambda + \omega_n^2$$



$\longrightarrow$



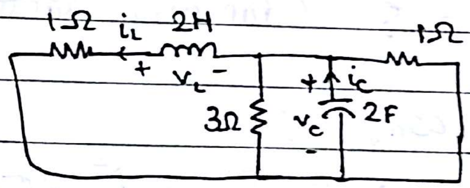
$\begin{bmatrix} i_c \\ v_L \end{bmatrix} = \underbrace{\begin{bmatrix} 1/R & -1 \\ 1 & 0 \end{bmatrix}}_{H} \begin{bmatrix} v_c \\ i_L \end{bmatrix}$

$f(\lambda) =$

$\lambda^2 - \frac{1}{RC}\lambda + \frac{1}{LC} = 0$ $\qquad A = \begin{bmatrix} \frac{1}{RC} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix}$

$\omega_n = \frac{1}{\sqrt{LC}}$, $\sigma = \frac{1}{2RC}$ $\qquad \zeta = \frac{1}{2R}\sqrt{\frac{L}{C}}$

Critically damped : $\frac{1}{2R}\sqrt{\frac{L}{C}} = 1$ $\qquad \frac{L}{C} = 4R^2$



$\begin{bmatrix} i_c \\ v_L \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{4}{3} & 1 \\ -1 & 1 \end{bmatrix}}_{H} \begin{bmatrix} v_c \\ i_L \end{bmatrix}$

$A = \begin{bmatrix} \frac{2}{3} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$f(\lambda) = \lambda^2 - \frac{7}{6}\lambda + \frac{7}{12}$

$\sigma = \frac{7}{12}$ $\qquad \omega_n = \sqrt{\frac{7}{12}}$ $\qquad \zeta = \sqrt{\frac{7}{12}}$

e. values $= \frac{7}{12} \pm j\frac{7}{12}$

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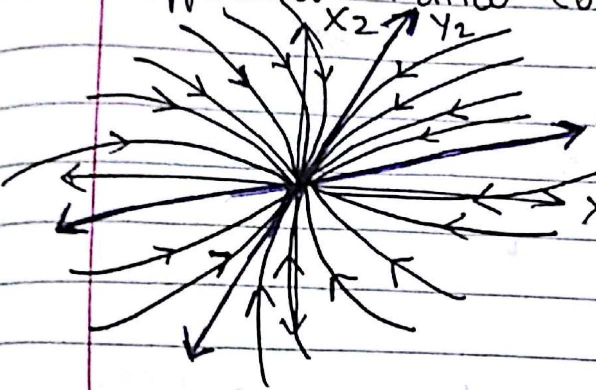
Plotting trajectories of $(y_1(t), y_2(t))$ w.r.t time for different initial conditions.



distinct real ~~positive~~ negative eigen values λ_1, λ_2

y_1, y_2 are L.I.

and $y_1(t) = e^{\lambda_1 t} y_1(0)$

$y_2(t) = e^{\lambda_2 t} y_2(0)$

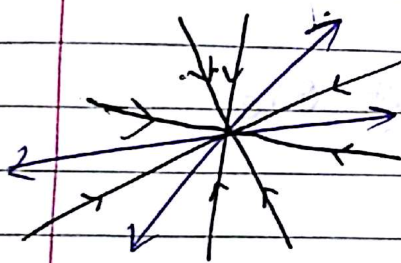
Let $|\lambda_1| > |\lambda_2|$

All these trajectories bend towards V_2 because $e^{\lambda_1 t}$ decays faster than $e^{\lambda_2 t}$.

$\lambda_1 = \lambda_2$ with L.I. eigenvectors

$y_1(t) = y_1(0)\, e^{\lambda t}$

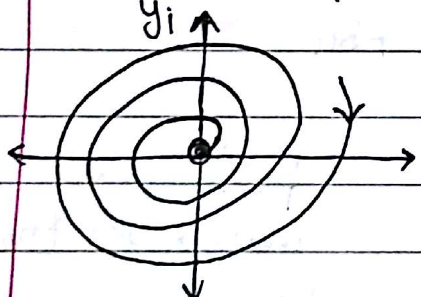
$y_2(t) = y_2(0)\, e^{\lambda t}$



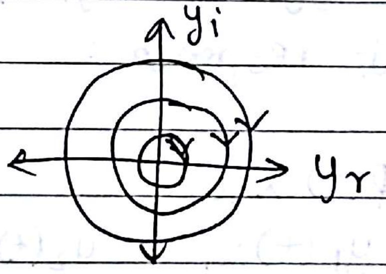
(Star)

Complex eigen values :

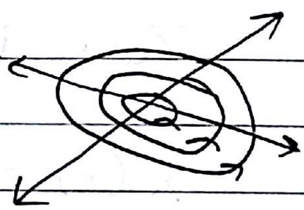
Underdamped case



$\sigma = 0$



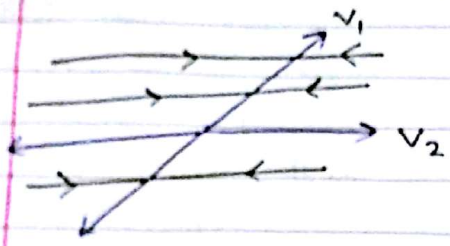
or

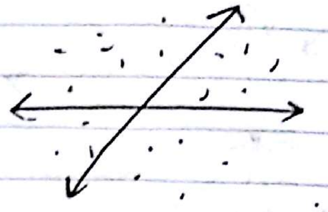


$\lambda_1 = 0 \quad \lambda_2 < 0$

$y_1(t) = y_1(0) \quad y_2(t) = y_2(0)e^{\lambda_2 t}$

$\lambda_1 = 0,\ \lambda_2 = 0,\ L.I.\ v_1 \& v_2$





$\lambda_1 > 0,\ \lambda_2 < 0$

Saddle

$y_r(t) = \cos(\omega t - \theta)\, e^{\sigma t} \qquad y_i(t) = \sin(\omega t - \theta)\, e^{\sigma t}$

$\theta = \tan^{-1}\left(\frac{y_i(0)}{y_r(0)}\right)$

$y_r(0) = R(0)\cos\theta \qquad y_i(0) = -R(0)\sin\theta$

$R(0)^2 = y_r(0)^2 + y_i(0)^2$

$x(t) = y_r(t) v_r + y_i(t) v_i$

Zero input response : $\dot{x} = Ax + Bu$

$x(t) = \phi(t)\, x(0)$

$x(t) = y_r(t) v_1 + y_2(t) v_2$

$= y_1(0) e^{\lambda_1 t} v_1 + y_2(0) e^{\lambda_2 t} v_2$

$\phi(t)$ = state transiti[on] matrix for the homogeneous equat[ion] $\dot{x} = Ax$

Let $T = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$

$\bar{x}(t) = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix}$

Put $t=0$, $\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = [v_1 \; v_2]^T \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$

Then $x(t) = [v_1 \; v_2] \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} [v_1 \; v_2]^{-1} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$

$$x(t) = T \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} T^{-1} x(0)$$

$$x(t) = \phi(t)\, x(0)$$

$$\dot{x}(t) = \dot{\phi}(t)\, x(0)$$

$$= A\,\phi(t)\, x(0)$$

$\dot{x}(t) = A x(t)$ and $\dot{\phi}(t) = A\phi(t)$

In laplace transform space,

$\dot{x} = Ax$ $\qquad L(f(t)) = \hat{f}(s)$

$\hookrightarrow A\hat{x}(s) = s\hat{x}(s) - x(0)$

$$x(0) = (sI - A)\,\hat{x}(s)$$

$$\hat{x}(s) = (sI - A)^{-1} x(0)$$

$$x(t) = L^{-1}\hat{x}(s) = L^{-1}(sI - A)^{-1} x(0)$$

$$\therefore \quad \phi(t) = L^{-1}(sI - A)^{-1}$$

e.g. $A = \begin{bmatrix} -G/C & 1/C \\ -1/L & 0 \end{bmatrix}$ $\qquad sI - A = \begin{bmatrix} s + G/C & -1/C \\ 1/L & s \end{bmatrix}$

$$(sI - A)^{-1} = \frac{1}{s(s + \frac{G}{C}) + \frac{1}{LC}} \begin{bmatrix} s & 1/C \\ -1/L & s + G/C \end{bmatrix}$$

Zero state response, general solution:

$$\dot{x} = Ax + bu, \quad y = cx + du$$

$$s\hat{x}(s) - x(0) = A\hat{x}(s) + b\hat{u}(s)$$

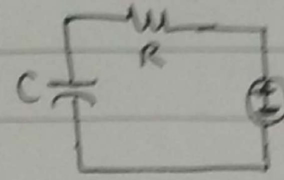
$$(sI - A)\hat{x}(s) = x(0) + b\hat{u}(s)$$

$$\hat{x}(s) = \underbrace{(sI-A)^{-1}x(0)}_{\text{zero input response}} + \underbrace{(sI-A)^{-1}b\hat{u}(s)}_{\text{zero state response}}$$

$$x(t) = \phi(t)x(0) + \int_0^t \phi(t-\tau)\,b\,u(\tau)\,d\tau$$

Zero state responses :-

e.g.



$u(t)$ = unit step function

$$\dot{v}_c = -\frac{1}{RC}v_c + \frac{1}{RC}u$$

$$sV_c(s) = -\frac{1}{RC}V_c(s) + \frac{1}{RC}\frac{1}{s}$$

Step response $= \dfrac{1/RC}{s(s+1/RC)}$ $\quad v_c(t) = (1 - e^{-t/RC})$

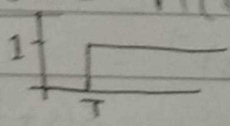
$\mathcal{L}$(response to input $u(t)$) = $\mathcal{L}$(Impulse response) $U(s)$

Impulse function : $u(t) = \delta(t)$ $\quad \mathcal{L}(\delta(t)) = 1$

Impulse response $= \dfrac{1/RC}{s+1/RC}$ $\quad v_c(t) = \dfrac{1}{RC}e^{-t/RC}$

$\mathcal{L}$(Impulse response) $= c(sI-A)^{-1}b + d = H(s)$

Delayed step function $u(t-T)$

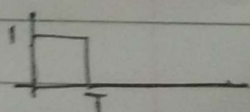


$\mathcal{L}$(Response to $u(t-T)$)

= $\mathcal{L}$(delayed response by T due to $U(t)$) $\rightarrow v_c(t) = 1 - e^{-(t-T)}$

$\mathcal{L}$(Response to pulse) $= H(s)\,\mathcal{L}(U(t) - U(t-T))$

$= H(s)\,\frac{1}{s}(1 - e^{-sT})$



Response to pulse $v_c(t) = (1 - e^{-t/RC}) - (1 - e^{-(t-T)/RC})$

$= e^{-t/RC}(e^{T/RC} - 1)$

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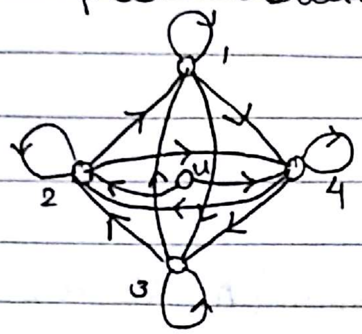
Linear Boolean networks :- Specified by a directed graph

Output node — Input node

Complete solution : = zero input + zero state



u = input node.

$x(k) = [x_1(k)\ x_2(k)\ x_3(k)\ x_4(k)]$ Current states

$u = u(k)$ current input

Next state $x(k+1) = F(x(k), u(k))$

$$x_1(k+1) = x_1 \oplus x_2 \oplus x_3$$
$$x_2(k+1) = x_2 \oplus x_3 \oplus x_4 \oplus u$$
$$x_3(k+1) = x_3 \oplus x_4 \oplus x_1$$
$$x_4(k+1) = x_1 \oplus x_2 \oplus u \oplus x_4$$

$$x(k+1) = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} x(k) \oplus \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} u(k) = Ax(k) \oplus bu(k)$$

Zero input response is the set of all trajectories of the autonomous network $x(k+1) = Ax(k)$

General solution : $x(0)$, input history $u(0), u(1) \cdots$

$$x(1) = Ax(0) + bu(0)$$
$$x(2) = A^2x(0) + Abu(0) + bu(1) \quad \cdots$$
$$x(3) = A^3x(0) + A^2bu(0) + Abu(1) + bu(2) \cdots$$

$$x(n) = \underbrace{A^n x(0)}_{\text{zero input}} + \underbrace{\sum_{i=0}^{n-1} A^{n-1-i}\, bu(i)}_{\text{zero state}}$$

Unit sample response :-

For $u(k)$: $u(0) = 1$, $u(k) = 0$ for $k \geq 1$

Reachability problem :

A state y is reachable from a state x at $k=0$ if there is n and an input sequence $u(0) \ldots u(n-1)$ such that $y = x(n)$

$$y = A^n x^0 + \sum_{i=0}^{n-1} A^{n-i-1} b u(i)$$

y is reachable from x $\Leftrightarrow$ $(y - A^n x) \in \underbrace{\text{span}\{b, Ab, A^2b \ldots A^{N-1}b\}}_{\text{Reachability subspace}}$

If it is a full state space, the whole graph is reachable (N = no. of nodes)

e.g. $b = [0\ 1\ 0\ 0]^T$ $\quad Ab = [1\ 0\ 0\ 0]^T$

$A^2b = [0\ 0\ 0\ 1]^T$ $\quad A^3b = [0\ 0\ 1\ 0]^T$ $\quad A^4b = [1\ 1\ 1\ 0]^T$

are linearly independent.

So $\text{span}\{b, Ab, A^2b, A^3b\} = \mathbb{R}^4$

So the network is reachable.

e.g. Consider $x(k+1) = Ax(k)$ $\quad A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Fixed points : $x = Ax$ $\quad (A \oplus I)x = 0$

$S = \{y \mid Ay = 0\}$

To find all y such that $Ay = x$: (x is a fixed point)

Let $x = Ay_1 = Ay_2$ Then $A(y_1 - y_2) = 0$ $\quad y_1 - y_2 \in S$

So i. Find one solution y s.t. $x = Ay$

ii. Write all vectors $\{y + S\}$

$$A \oplus I = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (A \oplus I)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0$$

$x_4 = 0, \; x_3 = 0, \; x_2 = x_5 = x_1 \Rightarrow [00000]^T$ and $[11001]^T$

$\hookrightarrow 0, 19$

$S = \{y \mid Ay = 0\} = \{[00000]^T, [00110]^T, [01010]^T,$

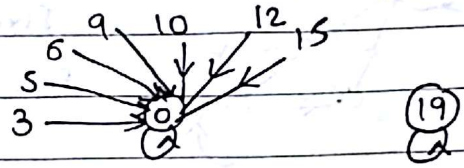
$x_5 = 0 \quad x_1 \oplus x_2 \oplus x_3 \oplus x_4 = 0 \qquad [01100]^T, [10010]^T,$

$[10100]^T, [11000]^T, [11110]^T\}$

$S = \{0, 12, 10, 6, 9, 5, 3, 15\}$

$\{y \mid Ay = 19\} = \{16, 28, 26, 22, 25, 21, 19, 31\}$

$Ax = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad x_5 = 1, \; x_1 \oplus x_2 \oplus x_3 \oplus x_4 = 0$



Solving quadratic equations :

Monomial system : $f = (x_1 x_2 x_3, \; x_2 x_3 x_4, \; x_1 x_3 x_4, \; x_1 x_2 x_4)$

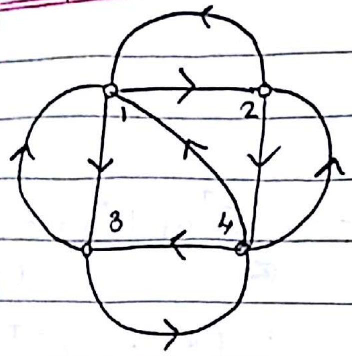
Other non linear : $F = (x_1 \oplus x_2 x_3, \; x_2 \oplus x_3 x_4, \; x_3 \oplus x_1 x_4, \; x_4 \oplus x_1 x_2)$

$\hookrightarrow$ Fixed points : $x_2 x_3 = 0, \; x_3 x_4 = 0, \; x_1 x_4 = 0, \; x_1 x_2 = 0$

Fixed points for monomial :

$x_1 \oplus x_1 x_2 x_3 = 0 \quad x_2 \oplus x_2 x_3 x_4 = 0 \quad x_3 \oplus x_1 x_3 x_4 = 0 \quad x_4 \oplus x_1 x_2 x_4 = 0$

Monomial system



$$F(x_1 x_2 x_3 x_4) = \begin{bmatrix} x_2 x_3 x_4 \\ x_1 x_4 \\ x_1 x_4 \\ x_2 x_3 \end{bmatrix}$$

Fixed points : 0000, 1111 (obvious)

$x_1 = x_2 x_3 x_4$

$x_2 = x_1 x_4$

$\cancel{x_3 = x_1 x_4}$

$x_4 = x_2 x_3$

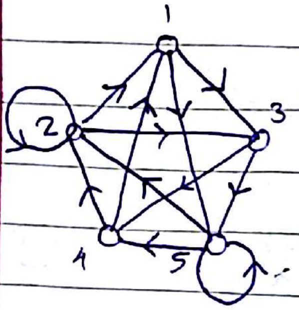
But only these 2 solutions

Fixed point of order 2 :

$x_1 = x_1 x_2 x_3 x_4 \qquad x_2 = x_2 x_3 x_4$

$x_3 = x_2 x_3 x_4 \qquad x_4 = x_1 x_4$

0000, 1111 (only 2 solutions)



Fixed points : 0000, 1111 (only 2 solutions)

$$F = \begin{bmatrix} x_2 x_4 \\ x_2 x_4 x_5 \\ x_1 x_2 \\ x_3 x_5 \\ x_1 x_3 x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$x_1 \oplus x_2 x_4 = 0 \Rightarrow x_1' \oplus x_2 x_4 = 1$

$\Rightarrow x_1 x_2 x_4 + x_1'(x_2' + x_4') = 1$

$\Rightarrow x_1 x_2 x_4 + x_1' x_2' + x_2' x_4' = 1$

$\Rightarrow x_1 = x_2 = x_4 = 1$ or $x_1 = x_2 = 0$ or $x_2 = x_4 = 0$

$x_1 x_2 x_4 = 111, 000, 001, 100$

Substitute in other equations and solve.

Chains of length one at 0 : $F(y) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

✓ $x_2 x_4 = 0$

$x_2 x_4 x_5 = 0$

✓ $x_1 x_2 = 0$

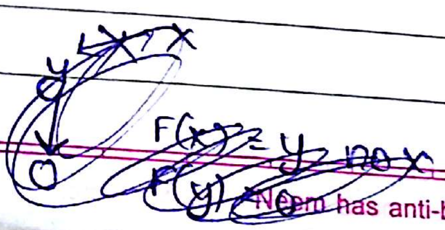
✓ $x_3 x_5 = 0$

$x_1 x_3 x_5 = 0$

$x_1 x_2 x_4 = 100, 101, 010, 001, 000$

$x_3 x_5 = 00, 01, 10$

14 solutions

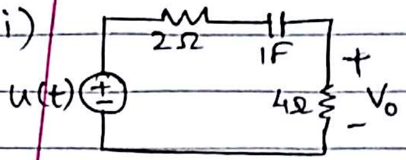


F(x) = y no x

F(y) = No

Examples for solving state space equations :-

i)



Find step and impulse response (zero state response)

$$u = 2\dot{V}_C + V_C + 4\dot{V}_C \qquad (4\dot{V}_C = V_o)$$

$$U(s) = 6sV(s) + V(s)$$

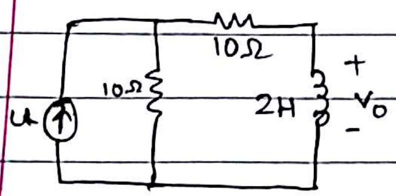
Impulse : $U(s) = 1 \qquad V(s) = \frac{1/6}{s + 1/6} \qquad V_c(t) = \frac{1}{6} e^{-t/6}$

$V_o(t) = \frac{-1}{9} e^{-t/6}$

Step : $U(s) = \frac{1}{s} \qquad V(s) = \frac{1/6}{s(s+1/6)}$

$V_c(t) = 1 - e^{-t/6} \qquad V_o(t) = \frac{4}{6} e^{-t/6}$

ii)



$$u = i_L + \frac{2\dot{i}_L + 10 i_L}{10} \qquad V_o = 2\dot{i}_L$$

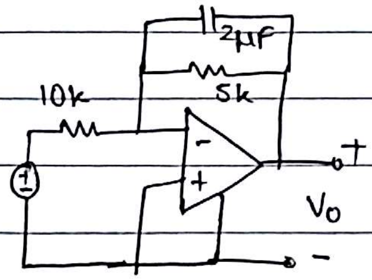
$$U(s) = 2I_L(s) + 0.2 s I_L(s)$$

Impulse : $I_L(s) = \frac{5}{(s+10)} \qquad i_L(t) = 5e^{-t/10}$

$V_o(t) = -e^{-t/10}$

Step : $I_L(s) = \frac{5}{s(s+10)} \qquad i_L(t) = \frac{1}{2}(1 - e^{-t/10}) \qquad V_o(t) = \frac{1}{10} e^{-t/10}$

iii)

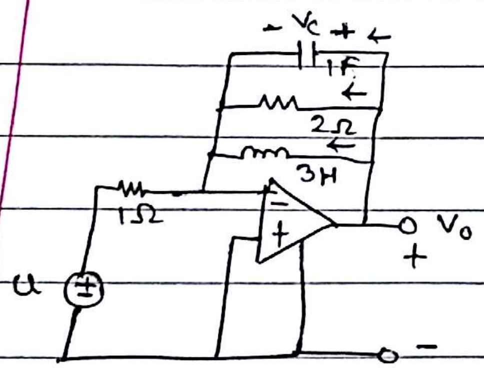


$$\frac{u}{10^4} = 2\cdot 10^{-6}\dot{V}_c + \frac{V_c}{5\times 10^3}$$

$\longrightarrow$ $U(s) = 0.02\dot{V}_c + 2V_c$

$V_o = -V_c$

iv)



$$\begin{bmatrix} i_c \\ V_L \end{bmatrix} = \begin{bmatrix} -1/2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} V_c \\ i_L \end{bmatrix}$$ (zero input)