MA 205 Complex Analysis: Gamma Function and Harmonic Functions

September 5, 2017

Now, let's introduce ourselves to a very important function; the Gamma function. It's defined as:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt,$$

for $\text{Re}(z) > 0$.

Real Integrals

Now, let's introduce ourselves to a very important function; the Gamma function. It's defined as:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt,$$

for $\mathrm{Re}(z) > 0$. One checks that this integral exists and defines a holomorphic function in the right half plane. $\Gamma(1) = 1$.

Integration by parts in the right half plane gives:

$$\Gamma(z+1) = z\Gamma(z).$$

Real Integrals

Integration by parts in the right half plane gives:

$$\Gamma(z+1) = z\Gamma(z).$$

Thus,

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1)\ldots\Gamma(1) = n!.$$

The Gamma function interpolates the factorial function!

Real Integrals

We would like to define the Gamma function throughout $\mathbb{C}$. Right now, it's defined only on the right half plane.

We would like to define the Gamma function throughout $\mathbb{C}$. Right now, it's defined only on the right half plane. For this, use the identity

$$\Gamma(z+1) = z\Gamma(z).$$

Real Integrals

We would like to define the Gamma function throughout $\mathbb{C}$. Right now, it's defined only on the right half plane. For this, use the identity

$$\Gamma(z+1) = z\Gamma(z).$$

Plug in $z = 0$, and we get:

$$0 \cdot \Gamma(0) = 1.$$

Real Integrals

We would like to define the Gamma function throughout $\mathbb{C}$. Right now, it's defined only on the right half plane. For this, use the identity

$$\Gamma(z+1) = z\Gamma(z).$$

Plug in $z = 0$, and we get:

$$0 \cdot \Gamma(0) = 1.$$

Thus 0 must be a pole for the extended Gamma function.

We would like to define the Gamma function throughout $\mathbb{C}$. Right now, it's defined only on the right half plane. For this, use the identity

$$\Gamma(z+1) = z\Gamma(z).$$

Plug in $z = 0$, and we get:

$$0 \cdot \Gamma(0) = 1.$$

Thus 0 must be a pole for the extended Gamma function. Similarly, all negative integers are also poles. And these are the only poles.

We would like to define the Gamma function throughout $\mathbb{C}$. Right now, it's defined only on the right half plane. For this, use the identity

$$\Gamma(z+1) = z\Gamma(z).$$

Plug in $z = 0$, and we get:

negative

$$0 \cdot \Gamma(0) = 1.$$

Thus 0 must be a pole for the extended Gamma function. Similarly, all negative integers are also poles. And these are the only poles.

Exercise: Check that these poles are simple and

$$\mathrm{Res}(\Gamma; -n) = \frac{(-1)^n}{n!}.$$

Real Integrals

What is $\Gamma(x)\Gamma(y)$?

$$\Gamma(x)\Gamma(y) = \int_0^\infty \int_0^\infty e^{-u-v} u^{x-1} v^{y-1} dudv.$$

Put

$$u = zt; \ v = z(1-t),$$

and apply the change of variables formula from MA 105.

Real Integrals

What is $\Gamma(x)\Gamma(y)$?

$$\Gamma(x)\Gamma(y) = \int_0^\infty \int_0^\infty e^{-u-v} u^{x-1} v^{y-1} du dv.$$

Put

$$u = zt; \;\; v = z(1-t),$$

and apply the change of variables formula from MA 105. Check:

$$\Gamma(x)\Gamma(y) = \Gamma(x+y)B(x,y),$$

where

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt.$$

What is $\Gamma(x)\Gamma(y)$?

$$\Gamma(x)\Gamma(y) = \int_0^\infty \int_0^\infty e^{-u-v} u^{x-1} v^{y-1} du dv.$$

Put

$$u = zt; \;\; v = z(1-t),$$

and apply the change of variables formula from MA 105. Check:

$$\Gamma(x)\Gamma(y) = \Gamma(x+y)B(x,y),$$

ye ho kya raha hai

where

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt.$$

Put $t = \frac{s}{s+1}$ to get:

x+y=1

$$B(1-c,c) = \int_0^\infty \frac{s^{-c}}{1+s} ds,$$

for $0 < c < 1$.

Real Integrals

Thus, for $0 < x < 1$,

$$\Gamma(x)\Gamma(1-x) = \int_0^\infty \frac{t^{-x}}{1+t} dt = \frac{\pi}{\sin \pi x}.$$

Thus, for $0 < x < 1$,

$$\Gamma(x)\Gamma(1-x) = \int_0^\infty \frac{t^{-x}}{1+t}dt = \frac{\pi}{\sin \pi x}.$$

In particular,

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}.$$

By identity theorem,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

for all $z \in \mathbb{C}$

Thus, for $0 < x < 1$,

$$\Gamma(x)\Gamma(1-x) = \int_0^\infty \frac{t^{-x}}{1+t}dt = \frac{\pi}{\sin \pi x}.$$

In particular,

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}.$$

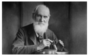
By identity theorem,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

for all $z \in \mathbb{C}$ domain minus pts of pole

Introduce meromorphic functions. Why is the identity theorem valid for meromorphic functions?

Quote



"A mathematician is one to whom $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ is as obvious as that twice two makes four is to you. " - Lord Kelvin

Harmonic Functions

We now revisit a topic we studied at the beginning of the course, namely harmonic functions. Recall that a function $u(x, y)$ of real variables is said to be harmonic if it is twice differentiable and $u_{xx} + u_{yy} = 0$. It turns out that harmonic functions share many properties similar to holomorphic functions. We'll see some of them.

Harmonic Functions

We now revisit a topic we studied at the beginning of the course, namely harmonic functions. Recall that a function $u(x, y)$ of real variables is said to be harmonic if it is twice differentiable and $u_{xx} + u_{yy} = 0$. It turns out that harmonic functions share many properties similar to holomorphic functions. We'll see some of them.

Recall that if u is a harmonic function, then a harmonic conjugate of u is another harmonic function v such that $u + iv$ is holomorphic. We saw some examples of computing harmonic conjugates and that time I commented that if the domain is "nice", then a harmonic conjugate always exists. The mathematical notion that replaces "nice" is simply connectedness.

Harmonic Functions

We now revisit a topic we studied at the beginning of the course, namely harmonic functions. Recall that a function $u(x, y)$ of real variables is said to be harmonic if it is twice differentiable and $u_{xx} + u_{yy} = 0$. It turns out that harmonic functions share many properties similar to holomorphic functions. We'll see some of them.

Recall that if u is a harmonic function, then a harmonic conjugate of u is another harmonic function v such that $u + iv$ is holomorphic. We saw some examples of computing harmonic conjugates and that time I commented that if the domain is "nice", then a harmonic conjugate always exists. The mathematical notion that replaces "nice" is simply connectedness.
Theorem: Let U be a simply-connected domain in $\mathbb{C}$ and let u be a harmonic fuction on U. Then u admits exactly one harmonic conjugate up to a constant.

Proof: Let's dismiss the uniqueness first. Suppose u has a harmonic conjugate v. Let $f(z) = u + iv$. By CR equations, v_x and v_y are determined and hence v is determined upto a constant. To prove existence, let $g(z) = u_x - iu_y$. Then by CR equations, $g(z)$ is holomorphic. Now fix $z_0 \in U$, and define f to be the anti-derivative of g:

Proof: Let's dismiss the uniqueness first. Suppose u has a harmonic conjugate v. Let $f(z) = u + iv$. By CR equations, v_x and v_y are determined and hence v is determined upto a constant. To prove existence, let $g(z) = u_x - iu_y$. Then by CR equations, $g(z)$ is holomorphic. Now fix $z_0 \in U$, and define f to be the anti-derivative of g:
$f(z) = u(z_0) + \int_{z_0}^{z} g(z)dz$ with the integral being along a path in U connecting z_0 and z. As U is simply-connected, this function is well-defined. By construction, f is holomorphic and
$f' = g = u_x - iu_y$.

Proof: Let's dismiss the uniqueness first. Suppose u has a harmonic conjugate v. Let $f(z) = u + iv$. By CR equations, v_x and v_y are determined and hence v is determined upto a constant. To prove existence, let $g(z) = u_x - iu_y$. Then by CR equations, $g(z)$ is holomorphic. Now fix $z_0 \in U$, and define f to be the anti-derivative of g:
$f(z) = u(z_0) + \int_{z_0}^{z} g(z)dz$ with the integral being along a path in U connecting z_0 and z. As U is simply-connected, this function is well-defined. By construction, f is holomorphic and $f' = g = u_x - iu_y$. Letting $\tilde{u} = Re(f)$, by the Cauchy-Riemann for f , $f' = \tilde{u}_x - i\tilde{u}_y$. On comparing the two equations, one concludes that $\tilde{u}_x = u_x$ and $\tilde{u}_y = u_y$.

Proof: Let's dismiss the uniqueness first. Suppose u has a harmonic conjugate v. Let $f(z) = u + iv$. By CR equations, v_x and v_y are determined and hence v is determined upto a constant. To prove existence, let $g(z) = u_x - iu_y$. Then by CR equations, $g(z)$ is holomorphic. Now fix $z_0 \in U$, and define f to be the anti-derivative of g:
$f(z) = u(z_0) + \int_{z_0}^{z} g(z)dz$ with the integral being along a path in U connecting z_0 and z. As U is simply-connected, this function is well-defined. By construction, f is holomorphic and $f' = g = u_x - iu_y$. Letting $\tilde{u} = Re(f)$, by the Cauchy-Riemann for f , $f' = \tilde{u}_x - i\tilde{u}_y$. On comparing the two equations, one concludes that $\tilde{u}_x = u_x$ and $\tilde{u}_y = u_y$.Therefore $u - \tilde{u}$ is a constant. Since u and $\tilde{u}$ are equal at z_0 , they are equal throughout.

Corollary: Harmonic functions are infinitely differentiable. (Why ?)

Corollary: Harmonic functions are infinitely differentiable. (Why ?)
Let us now derive an expression analogues to the CIF.

<u>Corollary</u>: Harmonic functions are infinitely differentiable. (Why ?)
Let us now derive an expression analogues to the CIF.

<u>**Theorem:**</u> (Mean-Value Property): Let u be a harmonic function on a disc of radius R. Then for any $r < R$, we have,

$$u(w) = \frac{1}{2\pi} \int_0^{2\pi} u(w + re^{i\theta}) d\theta.$$

In particular, u does not attain its maximum at any interior point unless it is constant.

Mean Value Property

Corollary: Harmonic functions are infinitely differentiable. (Why ?)
Let us now derive an expression analogues to the CIF.
Theorem: (Mean-Value Property): Let u be a harmonic function on a disc of radius R. Then for any $r < R$, we have,

$$u(w) = \frac{1}{2\pi}\int_0^{2\pi} u(w + re^{i\theta})d\theta.$$

In particular, u does not attain its maximum at any interior point unless it is constant.
Proof: Can assume $w = 0$ without loss of generality. Since u is harmonic and the domain is simply-connected, there exists a holomorphic function $f(z)$ such that $Re(f) = u$. By Cauchy's integral formula,

$$f(0) = \frac{1}{2\pi i}\int_{|z|=r} \frac{f(z)}{z} dz$$

Hence

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) d\theta$$

Taking real parts on both sides, we get the desired result.

Identity Principle

Hence

$$f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})d\theta$$

Taking real parts on both sides, we get the desired result.
Note that this in particular implies that if u is a harmonic function on a domain Ω and $z \in \Omega$. Suppose $u(z) \geq u(w)$ for all w in a neighborhood of z, then u is a constant. By considering $-u$ instead of u, the same holds for $\geq$ replaced by $\leq$.

Hence

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) d\theta$$

Taking real parts on both sides, we get the desired result.
Note that this in particular implies that if u is a harmonic function on a domain Ω and $z \in \Omega$. Suppose $u(z) \geq u(w)$ for all w in a neighborhood of z, then u is a constant. By considering $-u$ instead of u, the same holds for $\geq$ replaced by $\leq$.

Identity Principle: Let u be a harmonic function on a domain $\Omega \in \mathbb{C}$. If $u = 0$ on a non-empty open subset $U \subseteq \Omega$, then $u = 0$ throughout Ω.

Hence

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) d\theta$$

Taking real parts on both sides, we get the desired result.
Note that this in particular implies that if u is a harmonic function on a domain Ω and $z \in \Omega$. Suppose $u(z) \geq u(w)$ for all w in a neighborhood of z, then u is a constant. By considering $-u$ instead of u, the same holds for $\geq$ replaced by $\leq$.

Identity Principle: Let u be a harmonic function on a domain $\Omega \in \mathbb{C}$. If $u = 0$ on a non-empty open subset $U \subseteq \Omega$, then $u = 0$ throughout Ω.

Proof: Set $f = u_x - iu_y$. Then as before, f is holomorphic on Ω. Since $u = 0$ on U then so is f. Hence, by the Identity Principle for the holomorphic functions $f = 0$ on Ω, and consequently, $u_x = x_y = 0$ on Ω. Therefore u is constant on Ω, and as it is zero on U, it must be zero on Ω.

Identity Theorem

Remark: Recall that the identity theorem for holomorphic functions is stronger; namely if a holomorphic function vanishes on a set of points having a limit point, then it is identically zero. This is not true for harmonic functions. The function $Re(z)$ vanishes identically on imaginary axis but is non-zero elsewhere.
However the following stronger identity theorem holds:

Identity Theorem

Remark: Recall that the identity theorem for holomorphic functions is stronger; namely if a holomorphic function vanishes on a set of points having a limit point, then it is identically zero. This is not true for harmonic functions. The function $Re(z)$ vanishes identically on imaginary axis but is non-zero elsewhere.
However the following stronger identity theorem holds:

Theorem: Let Ω be a domain in $\mathbb{C}$ and u_1 and u_2 are two harmonic functions that extend continuously to the boundary $\partial\Omega$ of Ω. If $u_1 = u_2$ on $\partial\Omega$ then these two functions are equal throughout Ω.

Dirichlet Problem

A very important problem in Mathematics and Mathematical Physics is the Dirichlet Problem. It asks if Ω is a domain with boundary $\partial\Omega$, and f is a continuous real function on the boundary, does there exists a function u on $\bar{\Omega}$ which is harmonic on Ω and equals f on the boundary. The problem has a positive answer if $\partial\Omega$ is "sufficiently smooth ". (the precise condition is more technical). Many mathematicians have contributed to the solution - Green, Gauss, Kelvin, Dirichlet (who solved it for the ball), Riemann, Poincare, Hilbert ....

Dirichlet Problem

A very important problem in Mathematics and Mathematical Physics is the Dirichlet Problem. It asks if Ω is a domain with boundary $\partial\Omega$, and f is a continuous real function on the boundary, does there exists a function u on $\bar{\Omega}$ which is harmonic on Ω and equals f on the boundary. The problem has a positive answer if $\partial\Omega$ is "sufficiently smooth ". (the precise condition is more technical). Many mathematicians have contributed to the solution - Green, Gauss, Kelvin, Dirichlet (who solved it for the ball), Riemann, Poincare, Hilbert .... Riemann gave a physical argument roughly as follows: any charge distribution on the boundary should, by the laws of electrostatics, determine an electrical potential as solution. I confess I don't understand this ! In case of the disc in $\mathbb{C}$, the solution can be simply given as :

Dirichlet Problem

A very important problem in Mathematics and Mathematical Physics is the Dirichlet Problem. It asks if Ω is a domain with boundary $\partial\Omega$, and f is a continuous real function on the boundary, does there exists a function u on $\bar{\Omega}$ which is harmonic on Ω and equals f on the boundary. The problem has a positive answer if $\partial\Omega$ is "sufficiently smooth ". (the precise condition is more technical). Many mathematicians have contributed to the solution - Green, Gauss, Kelvin, Dirichlet (who solved it for the ball), Riemann, Poincare, Hilbert .... Riemann gave a physical argument roughly as follows: any charge distribution on the boundary should, by the laws of electrostatics, determine an electrical potential as solution. I confess I don't understand this ! In case of the disc in $\mathbb{C}$, the solution can be simply given as :

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\psi})\frac{(1-|z|^2)}{|z-e^{i\psi}|^2} \textit{ on } \mathbb{D}$$

and equal to $f(z)$ on $\partial\mathbb{D}$.

Exercise

Exercise: Show that the function $\log(\sqrt{X^2+Y^2})$ on $\mathbb{C}^*$ admits no harmonic conjugate.