Tutorial-3

1. Show that if $f(x,y) = u(x,y) + iv(x,y)$ is analytic, the real & imaginary parts of $f(z)$ are harmonic. Do they satisfy the Cauchy-Riemann equations?

2. Show that the harmonic function $u = \frac{1}{2}\log\left\{\frac{(x+1)^2+y^2}{(x-1)^2+y^2}\right\}$ has a harmonic conjugate in the ring $2 < x^2+y^2 < 3$.

3. Prove that $|e^z| = e^{\operatorname{Re} z}$

4. Prove that the exponential function exp never attains the value 0 although it

attains all other complex values infinitely often.

5. Find the image of the horizontal strip

$\mathbb{R} \times (-\pi, \pi)$ under the exponential function.

Find the image of the horizontal strip

$\mathbb{R} \times (\alpha, \alpha - 2\pi)$ under the exp function.

6. A polynomial is a power series. What is its

radius of convergence? Does it attain all

complex values? How many times does it

attain the value?

7. Prove the following identities:

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

$$\cos z = \cos x \cosh y - i \sin x \sinh y,$$

and use them to show that the function $\sin z$ maps horizontal lines to ellipses & vertical lines to hyperbolae.

Show that:

$$|\sin z|^2 = \sin^2 x + i \sinh^2 y$$

$$|\cos z|^2 = \cos^2 x + \sinh^2 y.$$

8 Prove the identities:

(i) $-4\sin^3 z + 3\sin z = \sin 3z$

(ii) $4\sinh^3 + 3\sinh z = \sinh 3z$

(iii) $4\cosh^3 z - 3\cosh z = \cosh 3z$.

9. Does $\sin z$ attain every complex value? How often does it attain a given value). what happens to vertical strips of width 2π.

10. The function $1/z$ is expressed as a power series around the polynomial $z_0 = 4i + \frac{\pi}{2}$. Explain why the radius convergence cannot exceed $2\sqrt{64+\pi^2}$.

11. Determine the largest domain which

$$\frac{e^z}{\sin z + \cos z}$$ is analytic

12. Discuss the convergence & find the sum of the following series. Let $\theta \in \mathbb{R}$.

(i) $1 + \cos\theta + \frac{1}{2!}\cos 2\theta + \cdots$

(ii) $\sin\theta + \frac{\sin 2\theta}{2!} + \cdots$

(iii) $1 + \frac{1}{4!}\cos 4\theta$

(iv) $\frac{1}{4!}\sin 4\theta + \frac{1}{8!}\sin 8\theta + \cdots$

13) Mobius map:

Show that the functions $T(z) = z + a$,

$R(z) = e^{i\theta} z$ & $M(z) = c \cdot z$, for $c \in \mathbb{R}, c > 0$, map circles & straight lines to

circles or lines. Write $f(z) = \dfrac{az+b}{cz+d}$

as a composite of the above functions &

$f(z) = \dfrac{1}{z}$. Show that f takes circles to

lines to circles or lines.

10) Show that $\tan(z)$ fails to taken on

the values i & $-i$.