MA-207 Differential Equations II
S1 - Lecture 5

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Recall: Consider $x^2y'' + xB(x)y' + C(x)y = 0$

$$B(x) = \sum_{i\geq 0} b_i x^i \quad \text{and} \quad C(x) = \sum_{i\geq 0} c_i x^i$$

analytic at 0. Put

$$y(x,r) = x^r \sum_{n\geq 0} a_n(r)x^n, \ a_0(r) = 1$$

Coefficient of x^r gives "Indicial equation"

$$I(r) := r(r-1) + b_0 r + c_0 = 0$$

For $n \geq 1$, the coefficient of x^{n+r} gives

$$a_n(r)I(n+r) + \sum_{i=0}^{n-1} b_{n-i}\,(i+r)a_i(r) + \sum_{i=0}^{n-1} c_{n-i}a_i(r) = 0$$

$$a_n(r) = -\frac{\sum_{i=0}^{n-1} b_{n-i}\,(i+r)a_i(r) + \sum_{i=0}^{n-1} c_{n-i}a_i(r)}{I(n+r)}$$

is a rational function of r.

- Let $r_1 \geq r_2$ be real roots of $I(r) = 0$. Then

$$y_1(x) = y(x, r_1) = \sum_{n \geq 0} a_n(r_1)x^{n+r_1}$$

is the first solution.

- If $r_1 - r_2 \notin \mathbb{Z}$, then second solution is

$$y_2(x) = y(x, r_2) = \sum_{n \geq 0} a_n(r_2)x^{n+r_2}$$

- If $r_1 = r_2$. Then the second solution is

$$y_2(x) = \frac{\partial}{\partial r}\left(x^r \sum_0^\infty a_n(r)x^n\right)\Big|_{r=r_1}$$

$$= y_1(x)\ln x + x^{r_1}\sum_{n\geq 1} a_n'(r_1)x^n$$

Example. Let us solve Bessel equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0, \qquad p \geq 0$$

Frobenius solution

$$y(x,r) = x^r \sum_{n\geq 0} a_n(r)x^n, \quad a_0(r) = 1$$

exists on $(0, \infty)$. Indicial equation (coefficient of x^r)

$$I(r) = r(r-1) + r - p^2 = r^2 - p^2 = (r-p)(r+p)$$

Roots are $r_1 = p$, $r_2 = -p$. Coefficient of x^{n+r} is

$$I(n+r)a_n(r) + a_{n-2}(r) = 0$$

$$a_{-1}(r) = 0 \implies a_{2n+1}(r) = 0, \quad n \geq 0$$

$$I(n+r)a_n(r) + a_{n-2}(r) = 0$$

$$\implies a_{2n}(r) = -\frac{a_{2n-2}(r)}{I(2n+r)} = \frac{(-1)^n}{\prod_{i=1}^n I(2i+r)}$$

• Assume $r_1 - r_2 = p - (-p) = 2p \notin \mathbb{Z}$, we get two Frobenius solutions for r_1 and r_2.

$I(2i+p) = (2i+p+p)(2i+p-p) = 2^2(i+p)i$,

$I(2i-p) = (2i-p+p)(2i-p-p) = 2^2 i(i-p)$

$$a_{2n}(p) = \frac{(-1)^n}{\prod_{i=1}^n 2^2 i(i+p)} = \frac{(-1)^n}{2^{2n}n!(1+p)\dots(n+p)}$$

$$a_{2n}(-p) = \frac{(-1)^n}{\prod_{i=1}^n 2^2 i(i-p)} = \frac{(-1)^n}{2^{2n}n!(1-p)\dots(n-p)}$$

First solution is $(r = p)$

$$y_1(x) = y(x,p) = \sum_{n\geq 0} \frac{(-1)^n}{2^{2n} n!(1+p)\dots(n+p)} x^{2n+p}$$

The second solution is $(r = -p)$

$$y_2(x) = y(x,-p) = \sum_{n\geq 0} \frac{(-1)^n}{2^{2n} n!(1-p)\dots(n-p)} x^{2n-p}$$

Next we consider repeated root case, i.e.
$p = -p \implies p = 0.$

Assume $p = 0$. Then Bessel equation is

$$x^2y'' + xy' + x^2y = 0$$

$I(r) = r(r-1) + r = r^2$ has repeated roots $0, 0$.

$$a_{2n}(r) = -\frac{a_{2n-2}(r)}{I(2n+r)} = -\frac{a_{2n-2}(r)}{(2n+r)^2} = \frac{(-1)^n}{\prod_{i=1}^n (2i+r)^2}$$

$$a_{2n}(0) = \frac{(-1)^n}{\prod_{i=1}^n (2i)^2} = \frac{(-1)^n}{2^{2n}(n!)^2}$$

So Frobenius solution is

$$y_1(x) = y(x,0) = \sum_{n\geq 0} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n}$$

The second solution is

$$y_2(x) = y_1(x)\ln x + \sum_{n\geq 1} a'_{2n}(0)x^n$$

$$a_{2n}(r) = \frac{(-1)^n}{\prod_{i=1}^{n}(2i+r)^2} = \frac{(-1)^n}{[(r+2)(r+4)\dots(r+2n)]^2}$$

$$a_{2n}(r)' = -2a_{2n}(r)\left(\frac{1}{r+2} + \frac{1}{r+4} + \cdots + \frac{1}{r+2n}\right)$$

$$a'_{2n}(0) = \frac{(-1)^{n+1}}{2^{2n}(n!)^2}H_n, \qquad H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

The second solution is

$$y_2(x) = y_1(x)\ln x - \sum_{n\geq 1} \frac{(-1)^n H_n}{2^{2n}(n!)^2}x^{2n}$$

Second solution: $r_1 - r_2 \in \mathbb{Z}$

Consider ODE $x^2y'' + xB(x)y' + B(x)y = 0$

$B(x) = \sum_{i\geq 0} b_i x^i$ and $B(x) = \sum_{i\geq 0} c_i x^i$ analytic at 0.

Then $x = 0$ is a regular singular point.

Assume $I(r) = r(r-1) + b_0 r + c_0 = 0$
has roots $r_1 > r_2$ with $r_1 - r_2 = N$ an integer.

First solution is given by

$$y_1(x) = y(x, r_1) = x^{r_1}\sum_{n=0}^{\infty} a_n(r_1)x^n, \qquad a_0 = 1$$

where $a_n(r)$ $(n \geq 1)$ are determined inductively by

$$I(n+r)a_n(r)+\sum_{i=0}^{n-1} b_{n-i}(i+r)a_i(r)+\sum_{i=0}^{n-1} c_{n-i}a_i(r) = 0$$

This can be solved since $I(n+r_1) \neq 0$ for $n \geq 1$.
Each $a_n(r)$ is a rational function in r and

$$a_n(r)\prod_{i=1}^{n} I(i+r)$$

is a polynomial in r. The polynomial

$$\prod_{i=1}^{n} I(i+r) = \prod_{i=1}^{n}(i+r-r_1)(i+r-r_2)$$

evaluated at r_2 vanishes if and only if $n \geq N$.
For $n \geq N$ it vanishes at r_2 to order exactly 1.

Thus if we define $A_n(r) := a_n(r)(r-r_2)$
Then $A_n(r)$ is analytic at r_2 for every $n \geq 0$.
Consider the function of two variables

$$\psi(x,r) := (r-r_2)y(x,r) = \sum_{n\geq 0} A_n(r)x^{n+r}$$

$$Ly(x,r) = I(r)x^r \implies L\psi(x,r) = (r-r_2)I(r)x^r.$$

$$\frac{\partial}{\partial r}(L\psi(x,r)) = \frac{\partial}{\partial r}((r-r_2)I(r)x^r)$$

$$\implies L\left(\frac{\partial}{\partial r}(\psi(x,r)\right) = I(r)x^r + (r-r_2)\left(I(r)x^r\right)'$$

RHS vanishes at $r = r_2$. Hence

Second solution: $r_1 - r_2 \in \mathbb{Z}$

The second solution is

$$y_2(x) = \frac{\partial}{\partial r}(\psi(x, r_2)\big|_{r=r_2} = \frac{\partial}{\partial r}\left(\sum_{n\geq 0} A_n(r)x^{n+r}\right)\big|_{r=r_2}$$

$$= \sum_{n\geq 0} A_n'(r_2)x^{n+r_2} + \sum_{n\geq 0} A_n(r_2)x^{n+r_2}\log x$$

Since $A_n(r_2) = 0$ for $n \leq N-1$.

$$y_2(x) = \sum_{n\geq 0} A_n'(r_2)x^{n+r_2} + \sum_{n\geq N} A_n(r_2)x^{n+r_2}\log x$$

Example

Consider the ODE $\quad xy'' - (4+x)y' + 2y = 0$

Multiplying with x, we get $x = 0$ is a regular singular point.

Indicial equation for $x^2y'' - x(4+x)y' + 2xy = 0$ is

$$I(r) = r(r-1) - 4r + 0 = r(r-5) = 0$$

with the roots $r_1 = 5, r_2 = 0$ and $N = r_1 - r_2 = 5$.

Put $\quad y(x, r) = x^r \sum_{n=0}^{\infty} a_n(r)x^n, \quad a_0(r) = 1,$

the coefficient of x^{n+r} for $n \geq 1$ gives

$$I(n+r)a_n(r) - (n+r-1)a_{n-1}(r) + 2a_{n-1}(r) = 0$$

For $n \geq 1$, $\quad a_n(r) = \dfrac{(n+r-3)}{(n+r)(n+r-5)} a_{n-1}$

$$= \frac{(n+r-3)\dots(r-2)}{(n+r)\dots(1+r)(n+r-5)\dots(r-4)} a_0$$

For the first solution, set $r = r_1 = 5$, we get

$$a_n(5) = \frac{(n+2)\dots(3)}{(n+5)\dots 6.n.\dots 1} = \frac{(n+2)!/2}{(n!)(n+5)!/5!}$$

$$= \frac{60}{n!(n+5)(n+4)(n+3)}$$

Thus $\quad y_1(x) = \displaystyle\sum_{n\geq 0} \frac{60}{n!(n+5)(n+4)(n+3)} x^{n+5}$

Recall $N = r_1 - r_2 = 5 - 0$ is integer, so the second solution is

$$y_2(x) = \sum_{n\geq 0} A'_n(r_2)x^{n+r_2} + \sum_{n\geq 5} A_n(r_2)x^{n+r_2}\log x$$

where $r_2 = 0$ and $A_n(r) = a_n(r)(r - r_2) = ra_n(r)$.

$$a_n(r) = \frac{(n+r-3)\dots(r-2)}{(n+r)\dots(1+r)(n+r-5)\dots(r-4)}$$

Note that $a_n(r)$ is analytic at 0 for all n.
For $n \geq 5$, r in the numerator and denominator gets cancellaed.
So $A_n(0) = 0$ for $n \geq 0$.

$$A_n'(0) = (a_n'(r).r + a_n(r))|_{r=0} = a_n(0)$$

as $a_n(r)$ is analytic at 0, so $a_n'(0)$ exists.

So we need to compute $a_n(0)$.

$$a_n(r) = \frac{(n+r-3)\dots(r-2)}{(n+r)\dots(1+r)(n+r-5)\dots(r-4)}$$

$$a_1(0) = \frac{-2}{1.(-4)} = \frac{1}{2}, \quad a_2(0) = \frac{(-1)(-2)}{2!.(-3)(-4)} = \frac{1}{12}$$

$$a_3(0) = 0, \qquad a_4(0) = 0,$$

For $n \geq 5$,

$$a_n(0) = \frac{(n-3)!.2}{n!(n-5)!4!} = \frac{1}{12.(n-5)!n(n-1)(n-2)}$$

The second solution is

$$y_2(x) = \sum_{n\geq 0} A_n'(0)x^n + \sum_{n\geq 5} A_n(0)x^n \log x$$

Recall $A_n(0) = 0$ and $A_n'(0) = a_n(0)$.

$$y_2(x) = 1+\frac{1}{2}x+\frac{1}{12}x^2+\sum_{n\geq 5}\frac{x^n}{12.(n-5)!n(n-1)(n-2)}$$

$$= 1+\frac{1}{2}x+\frac{1}{12}x^2+\sum_{n\geq 0}\frac{1}{12.n!(n+5)(n+4)(n+3)}x^{n+5}$$

$$= 1+\frac{1}{2}x+\frac{1}{12}x^2+\frac{1}{60.12}y_1(x)$$

Therefore, we can take $\quad y_2(x) = 1+\frac{1}{2}x+\frac{1}{12}x^2$

Summary

While solving an ODE around a regular singular point by the Frobenius method, the cases encountered are

- roots not differing by an integer
- repeated roots
- roots differing by a positive integer

The larger root always gives a Frobenius solution. In 2nd and 3rd cases, second solution may involve a $\log$ term.

Remark

- We also have a solution to the ODE in the interval $(-\rho, 0)$.
- Put $x = -x'$ in the ODE to get a new ODE in the variable x'.
- Using the above method, solve the new ODE in the interval $(0, \rho)$
- Let $w_1(x'), w_2(x')$ denote the solutions of the new ODE in the interval $(0, \rho)$
- Then $y_i(x) := w_i(-x)$ is a solution of the original ODE in the interval $(-\rho, 0)$.

Gamma function

Define for all $p > 0$, the Gamma function

$$\Gamma(p) := \int_0^\infty t^{p-1} e^{-t} dt$$

There is a problem if $p < 0$, as the integral is divergent.

$$\Gamma(1) = \int_0^\infty e^{-t} dt = 1$$

For $p > 0$, $\quad \Gamma(p) =$

$$\lim_{x\to\infty} \int_0^x t^{p-1} e^{-t} dt = \frac{1}{p} \left(\lim_{x\to\infty} \int_0^x t^p e^{-t} dt \right) = \frac{1}{p} \Gamma(p+1)$$

$$\Gamma(p) = \frac{\Gamma(p+1)}{p} \qquad (*)$$

Use $(*)$ to define $\Gamma(p)$ for all real $p \neq 0, -1, -2, \ldots$

For $-1 < p < 0$, $\Gamma(p+1)$ is defined; use $(*)$ to define $\Gamma(p)$.
Proceed like this for $p \in (-2, -1)$, then for $p \in (-3, -2)$, $\ldots$

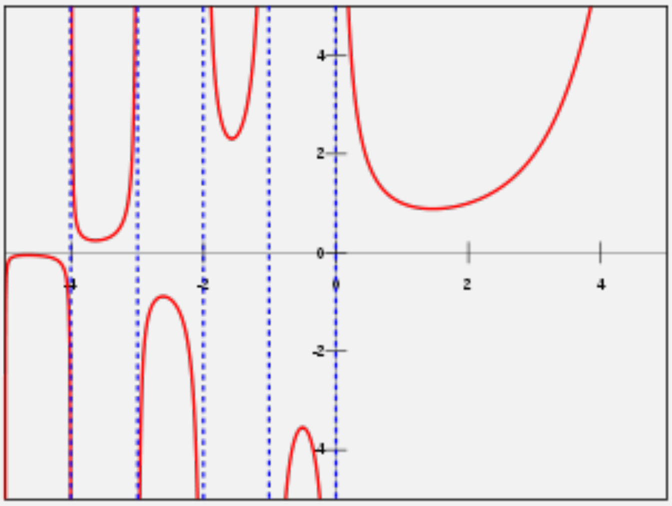
For example,

$$\Gamma(-\frac{5}{2}) = \frac{\Gamma(-\frac{3}{2})}{-\frac{5}{2}} = \frac{\Gamma(-\frac{1}{2})}{(-\frac{5}{2})(-\frac{3}{2})} = \frac{\Gamma(\frac{1}{2})(= \sqrt{\pi})}{(-\frac{5}{2})(-\frac{3}{2})(-\frac{1}{2})}$$

Further

$$\lim_{p\to 0}\Gamma(p) = \lim_{p\to 0}\frac{\Gamma(p+1)}{p} = \pm\infty$$

according as $p \to 0$ from right or left.
The graph of Gamma function is shown below.



Though the gamma function is now defined for all real numbers (except for $0, -1, -2, \ldots$), the integral representation is valid only for $p > 0$.

It is useful to rewrite

$$\frac{1}{\Gamma(p)} = \frac{p}{\Gamma(p+1)}$$

This holds for all p if we impose the condition

$$\frac{1}{\Gamma(n)} = 0, \qquad n = 0, -1, -2, \ldots$$

$$\Gamma(1/2) = \int_0^\infty t^{-1/2}e^{-t}dt = 2\int_0^\infty e^{-s^2}ds = \sqrt{\pi}$$

use the substitution $t = s^2$.

By translating,

$$
\begin{aligned}
\Gamma(1/2) & & = \sqrt{\pi} & \approx 1.772 \\
\Gamma(-1/2) & = \frac{\Gamma(1/2)}{-1/2} & = -2\sqrt{\pi} & \approx -3.545 \\
\Gamma(-3/2) & = \frac{\Gamma(-1/2)}{-3/2} & = \frac{4}{3}\sqrt{\pi} & \approx 2.363 \\
\Gamma(3/2) & = \frac{1}{2}\Gamma(1/2) & = \frac{1}{2}\sqrt{\pi} & \approx 0.886 \\
\Gamma(5/2) & = \frac{3}{2}\Gamma(3/2) & = \frac{3}{4}\sqrt{\pi} & \approx 1.329 \\
\Gamma(7/2) & = \frac{5}{2}\Gamma(5/2) & = \frac{15}{8}\sqrt{\pi} & \approx 3.323
\end{aligned}
$$

Bessel functions

Bessel equation is the ODE

$$x^2y'' + xy' + (x^2 - p^2)y = 0, \qquad p \geq 0 \qquad (*)$$

Its solutions are called Bessel functions.

Frobenius solution at $x = 0$ is called Bessel function of first kind.

The second linearly independent solution is called Bessel function of second kind.

We have already solved this equation when either $2p \notin \mathbb{Z}$ or $p = 0$. Let us recall the solutions.

$2p \notin \mathbb{Z}$

$$y_1(x) = \sum_{n\geq 0} \frac{(-1)^n}{2^{2n} n!(1+p)\ldots(n+p)} x^{2n+p}$$

$$y_2(x) = \sum_{n\geq 0} \frac{(-1)^n}{2^{2n} n!(1-p)\ldots(n-p)} x^{2n-p}$$

$$J_p(x) = \frac{y_1(x)}{2^p \Gamma(1+p)} = \sum_{n\geq 0} \frac{(-1)^n}{n!\,\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}$$

$$J_{-p}(x) = \frac{y_2(x)}{2^{-p} \Gamma(1-p)} = \sum_{n\geq 0} \frac{(-1)^n}{n!\,\Gamma(n-p+1)} \left(\frac{x}{2}\right)^{2n-p}$$

$J_p(x)$ and $J_{-p}(x)$ are Bessel functions of first and second kind of order p.

$p = 0$: Repeated root case

Frobenius solution is

$$y_1(x) = J_0(x) = \sum_{n\geq 0} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n}$$

The second solution is

$$y_2(x) = J_0(x) \ln x - \sum_{n\geq 1} \frac{(-1)^n H_n}{2^{2n}(n!)^2} x^{2n}$$

It is defined on $(0, \infty)$.

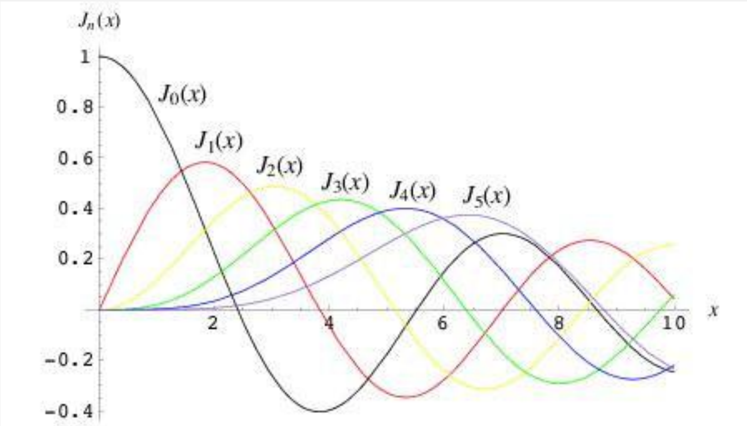
Recall the first Bessel function of order p is

$$J_p(x) := \sum_{n\geq 0} \frac{(-1)^n}{n!\,\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}, \quad x \in (0,\infty).$$

The Bessel functions of order 0 and 1 are

$$J_0(x) = \sum_{n\geq 0} \frac{(-1)^n}{n!n!} \left(\frac{x}{2}\right)^{2n}$$
$$= 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{2!2!}\left(\frac{x}{2}\right)^4 - \frac{1}{3!3!}\left(\frac{x}{2}\right)^6 + \dots$$
$$J_1(x) = \sum_{n\geq 0} \frac{(-1)^n}{n!(n+1)!} \left(\frac{x}{2}\right)^{2n+1}$$
$$= \frac{x}{2} - \frac{1}{1!2!}\left(\frac{x}{2}\right)^3 + \frac{1}{2!3!}\left(\frac{x}{2}\right)^5 + \dots$$

Both $J_0(x)$ and $J_1(x)$ have a damped oscillatory behavior having an infinite number of zeroes which occurs alternately like $\cos x$ and $\sin x$.



Further, they satisfy derivative identities similar to $\cos x$ and $\sin x$.

$$J_0'(x) = -J_1(x), \qquad [xJ_1(x)]' = xJ_0(x)$$

$2p = 2l + 1 \in \mathbb{Z}$: odd positive integer

Note $a_{2n+1} = 0$ and $I(2n + r)a_{2n} + a_{2n-2} = 0$.

Since $I(r) = (r^2 - p^2) = (r + p)(r - p)$,
for $r = -p$, $I(2n - p) = 2n(2n - 2p) \neq 0$.
So for $r = -p$ we get a second Frobenius solution

$$y_2(x) = \sum_{n \geq 0} \frac{(-1)^n}{2^{2n} n!(1 - p) \ldots (n - p)} x^{2n-p}$$

$\dfrac{y_2(x)}{2^{-p}\Gamma(1 - p)}$ gives second Bessel function

$$J_{-p}(x) := \sum_{n \geq 0} \frac{(-1)^n}{n!\,\Gamma(n - p + 1)} \left(\frac{x}{2}\right)^{2n-p}, \quad x > 0.$$

$2p = 2N$: even positive integer

The second solution is

$$y_2(x) = \sum_{n \geq 0} A'_n(-p)x^{n-p} + \sum_{n \geq 2N} A_n(-p)x^{n-p} \log x$$

$A_n(r) = (r + p)a_n(r)$, so $A_{2n+1}(r) = 0 = A'_{2n+1}(r)$.
For $\boxed{2n < 2N}$, $a_{2n}(r)$ is analytic at $-p$ and

$$a_{2n}(r) = \frac{-a_{2n-2}}{(2n + r + p)(2n + r - p)}$$
$$= \frac{(-1)^n}{\prod_{i=1}^{n}((r + 2i + p)(r + 2i - p)}$$

$$\implies A'_{2n}(-p) = a_{2n}(-p) = \frac{(-1)^n}{2^{2n} n!(1 - p) \ldots (n - p)}$$

Assume $\boxed{2n \geq 2N = 2p}$, then

$$A_{2n}(r) = (r+p)\frac{(-1)^n}{\prod_{i=1}^{n}((r+2i+p)(r+2i-p))}$$

$$= \frac{(-1)^n}{\left(\prod_{i=1}^{n}(r+2i+p)\right)\left(\prod_{i=1,i\neq p}^{n}(r+2i-p)\right)}$$

$$A_{2n}(-p) = \frac{(-1)^n}{(2^n n!)\left(2^{n-1}(1-p)\dots(-1)\cdot 1\cdots(n-p)\right)}$$

$$= \frac{2(-1)^{n-p+1}}{2^{2n}n!(p-1)!(n-p)!}$$

$$A'_{2n}(r) = -A_{2n}(r)\frac{\left[\left(\prod_{i=1}^{n}(r+2i+p)\right)\left(\prod_{i=1,i\neq p}^{n}(r+2i-p)\right)\right]'}{\left(\prod_{i=1}^{n}(r+2i+p)\right)\left(\prod_{i=1,i\neq p}^{n}(r+2i-p)\right)}$$

$$A'_{2n}(r) = -A_{2n}(r)\left[\sum_{i=1}^{n}\frac{1}{r+2i+p} + \sum_{i=1,i\neq p}^{n}\frac{1}{r+2i-p}\right]$$

$$A'_{2n}(-p) = -A_{2n}(-p)\left[\sum_{i=1}^{n}\frac{1}{2i} + \sum_{i=1,i\neq p}^{n}\frac{1}{2(i-p)}\right]$$

$$= -A_{2n}(-p)\left[\frac{H_n}{2} - \frac{H_{p-1}}{2} + \frac{H_{n-p}}{2}\right]$$

$$= \frac{2(-1)^{n-p}}{2^{2n}n!(p-1)!(n-p)!}\left[\frac{H_n}{2} - \frac{H_{p-1}}{2} + \frac{H_{n-p}}{2}\right]$$

where $H_0 = 0, \quad H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$

Thus, we get

$$y_2(x) = \sum_{n\geq 0} A_n'(-p)x^{n-p} + \sum_{n\geq 2N} A_n(-p)x^{n-p} \log x$$

$$= \sum_{n=0}^{p-1} \frac{1}{2^{2n} n!(p-n)!} x^{2n-p} +$$

$$\sum_{n\geq p} \frac{(-1)^{n-p}}{2^{2n} n!(p-1)!(n-p)!} \Big(H_n - H_{p-1} + H_{n-p}\Big) x^{2n-p}$$

$$-\sum_{n\geq p} \frac{2(-1)^{n-p}}{2^{2n} n!(p-1)!(n-p)!} x^{2n-p} \log x$$

is a second solution of Bessel equation.

Summary of $p = 1/2$

For $p = 1/2$, two independent solutions are $J_{1/2}(x)$ and $J_{-1/2}(x)$. These can be expressed in terms of the trigonometric functions (Exercise):

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x \quad \text{and} \quad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

Both exhibit singular behavior at 0.
Near 0, $J_{1/2}(x)$ is bounded but does not have a bounded derivative,
while $J_{-1/2}(x)$ is unbounded near 0.