MA-207 Differential Equations II

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23rd October, 2017
S2 - Lecture 9

Recall

Theorem

Let $f \in L^2[-L, L]$. Then f can be written as a series

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

which is called the Fourier series of f on $[-L, L]$. Here

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx \qquad \text{and for } n > 0$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx$$

The above series converges to f in norm, that is,

$$\lim_{N \longrightarrow \infty} \left\| f - a_0 - \sum_{n=1}^{N} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \right\| = 0$$

Recall

- In the previous lecture there was a typo in the formula for a_0. This has now been corrected.
- Suppose we have a maximal orthogonal set, say $\{\phi_1, \phi_2, \ldots\}$. Assume that every function can be written as a series in these functions. Then the coefficient of ϕ_n in the expansion of f is given by

$$a_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}$$

$f = \sum_{n \geq 1} a_n \phi_n$.

- EVP 1. $y'' + \lambda y = 0, \;\; y(0) = 0, \, y(L) = 0$

has infinitely many positive eigenvalues $\lambda_n = \dfrac{n^2\pi^2}{L^2}$ for $n \geq 1$ with associated eigenfunctions

$$y_n = \sin \frac{n\pi x}{L}.$$

- EVP 2. $y'' + \lambda y = 0, \;\; y'(0) = 0, \, y'(L) = 0$

has eigenvalue $\lambda_0 = 0$ with eigenfunction $y_0 = 1$.
has infinitely many positive eigenvalues $\lambda_n = \dfrac{n^2\pi^2}{L^2}$ for $n \geq 1$ with associated eigenfunctions

$$y_n = \cos \frac{n\pi x}{L}.$$

• EVP 3. $y'' + \lambda y = 0, \;\; y(0) = 0, \; y'(L) = 0$
has infinitely many positive eigenvalues
$\lambda_n = \dfrac{(2n-1)^2\pi^2}{4L^2}, \quad n = 1, 2, \ldots$
with associated eigenfunctions

$$y_n = \sin \frac{(2n-1)\pi x}{2L}.$$

• EVP 4. $y'' + \lambda y = 0, \;\; y'(0) = 0, \; y(L) = 0$
has infinitely many positive eigenvalues
$\lambda_n = \dfrac{(2n-1)^2\pi^2}{4L^2}, \quad n = 1, 2, \ldots$
with associated eigenfunctions

$$y_n = \cos \frac{(2n-1)\pi x}{2L}.$$

• EVP 5. $y'' + \lambda y = 0$, $y(-L) = y(L)$, $y'(-L) = y'(L)$
has an eigenvalue $\lambda_0 = 0$ with eigenfunction $y_0 = 1$
and infinitely many positive eigenvalues $\lambda_n = \dfrac{n^2\pi^2}{L^2}$, $n = 1, 2, \ldots$
with associated eigenfunctions

$$y_{1n} = \cos\frac{n\pi x}{L} \quad \text{and} \quad y_{2n} = \sin\frac{n\pi x}{L}.$$

• Eigenfunctions of EVP (1-4) are orthogonal on $[0, L]$ w.r.t. inner product $\langle f, g\rangle = \int_0^L f(x)g(x)dx$

• Eigenfunctions of EVP 5 is orthogonal on $[-L, L]$ w.r.t. inner product $\langle f, g\rangle = \int_{-L}^L f(x)g(x)dx$.

Fourier Series.

Let $f \in L^2([-L, L])$ be piecewise smooth. Extend f to $\mathbb{R}$ as a periodic function of period $2L$.

The Fourier series of f is

$$F(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx, \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx, \quad n > 0$$

- $F(x) = \frac{1}{2}[f(x^+) + f(x^-)]$ for all $x \in \mathbb{R}$.

Fourier sine series

Let f be a function on $[0, L]$. Then we claim that f can be written as a series

$$f(x) = \sum_{n \geq 1} b_n \sin \frac{n\pi x}{L}$$

To see this, let us first extend f to $[-L, L]$ by defining $f(x) = -f(-x)$ for $x \in [-L, 0)$. Denote the extension by $\tilde{f}$.

Then we know that $\tilde{f}$ has a Fourier expansion

$$\tilde{f}(x) = a_0 + \sum_{n \geq 1} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} \tilde{f}(x) dx \qquad a_n = \frac{1}{L} \int_{-L}^{L} \tilde{f}(x) \cos \frac{n\pi x}{L} dx \qquad n > 0$$

$$b_n = \frac{1}{L} \int_{-L}^{L} \tilde{f}(x) \sin \frac{n\pi x}{L} dx$$

Now note that by the way $\tilde{f}$ has been defined, it is an odd function. Thus, $a_0 = 0$.

Since $\cos \frac{n\pi x}{L}$ is an even function and $\tilde{f}$ is odd, it follows $\tilde{f}(x) \cos \frac{n\pi x}{L}$ is an odd function. Thus, $a_n = 0$.

This proves that

$$\tilde{f}(x) = \sum_{n \geq 1} a_n \sin \frac{n\pi x}{L}$$

Restricting this expansion to $[0, L]$ we get the required expansion of f.

Fourier cosine series

Let f be a function on $[0, L]$. Then we claim that f can be written as a series

$$f(x) = a_0 + \sum_{n \geq 1} a_n \cos \frac{n\pi x}{L}$$

To see this, let us first extend f to $[-L, L]$ by defining $f(x) = f(-x)$ for $x \in [-L, 0)$. Denote the extension by $\tilde{f}$.

Then we know that $\tilde{f}$ has a Fourier expansion

$$\tilde{f}(x) = a_0 + \sum_{n \geq 1} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} \tilde{f}(x) dx \qquad a_n = \frac{1}{L} \int_{-L}^{L} \tilde{f}(x) \cos \frac{n\pi x}{L} dx \qquad n > 0$$

$$b_n = \frac{1}{L} \int_{-L}^{L} \tilde{f}(x) \sin \frac{n\pi x}{L} dx$$

Now note that by the way $\tilde{f}$ has been defined, it is an even function.

Since $\sin \frac{n\pi x}{L}$ is an odd function and $\tilde{f}$ is even, it follows $\tilde{f}(x) \sin \frac{n\pi x}{L}$ is an odd function. Thus, $b_n = 0$.

This proves that

$$\tilde{f}(x) = a_0 + \sum_{n \geq 1} a_n \cos \frac{n\pi x}{L}$$

Restricting this expansion to $[0, L]$ we get the required expansion of f.

Expansion in terms of eigenfunctions of EVP3

Let f be a function on $[0, L]$. Then we claim that f can be written as a series

$$f(x) = \sum_{n \geq 1} a_n \sin \frac{(2n-1)\pi x}{2L}$$

Let $f \in L^2([0, L])$. Extend f to f_1 on $[0, 2L]$ as $f_1(x) = f(2L - x)$ for $x \in (L, 2L)$.

Fourier sine series of f_1 on $[0, 2L]$ is

$$F(x) = \sum_{n \geq 1} b_n \sin \frac{n\pi x}{2L}$$

$$b_n = \frac{2}{2L} \int_0^{2L} f_1(x) \sin \frac{n\pi x}{2L}\, dx$$

$$= \frac{1}{L} \int_0^{L} f(x) \sin \frac{n\pi x}{2L} dx + \frac{1}{L} \int_L^{2L} f(2L - x) \sin \frac{n\pi x}{2L} dx$$

Expansion in terms of eigenfunctions of EVP3

$$\int_L^{2L} f(2L-x)\sin\frac{n\pi x}{2L}dx$$

$$(x' = 2L - x), \qquad = \int_L^0 f(x')\sin(n\pi - \frac{n\pi x'}{2L})(-dx')$$

$$\int_0^L (-1)^{n+1} f(x)\sin\frac{n\pi x}{2L}\,dx$$

$$b_n = \frac{1}{L}\int_0^L f(x)\sin\frac{n\pi x}{2L}dx + \frac{1}{L}\int_0^L (-1)^{n+1} f(x)\sin\frac{n\pi x}{2L}dx$$

So $b_{2n} = 0$, $b_{2n-1} = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{(2n-1)\pi x}{2L}\,dx$.

Thus $F(x) = \displaystyle\sum_{n\geq 1} b_{2n-1}\sin\frac{(2n-1)\pi x}{2L}$.

Expansion in terms of eigenfunctions of EVP3

The Mixed Fourier sine series of $f \in L^2([0,L])$ is the restriction of Fourier sine series of f_1 to $[0,L]$, i.e.

$$F(x) = \sum_{n \geq 1} c_n \sin \frac{(2n-1)\pi x}{2L}$$

$$c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{(2n-1)\pi x}{2L} \, dx$$

This is the Fourier series of f on $[0,L]$ w.r.t. orthogonal system of eigenfunctions

$$B = \{\sin \frac{\pi x}{2L}, \sin \frac{3\pi x}{2L}, \ldots, \sin \frac{(2n-1)\pi x}{2L}, \ldots\}$$

of EVP 3 : $\boxed{y'' + \lambda y = 0, \quad y(0) = 0 = y'(L)}$.

Mixed Fourier cosine series

Let $f \in L^2([0,L])$. Extend f to f_1 on $[0,2L]$ as
$f_1(x) = -f(2L-x)$ for $x \in (L, 2L)$.

Fourier cosine series of f_1 on $[0,2L]$ is

$$F(x) = \sum_{n=1}^{\infty} d_n \cos\frac{(2n-1)\pi x}{2L}, d_n = \frac{2}{L}\int_0^L f(x)\cos\frac{(2n-1)\pi x}{2L}\,dx$$

This is the Fourier series of f on $[0,L]$ w.r.t. orthogonal system of eigenfunctions

$$B = \{\cos\frac{\pi x}{2L}, \cos\frac{3\pi x}{2L}, \ldots, \cos\frac{(2n-1)\pi x}{2L}, \ldots\}$$

of EVP 4 : $\boxed{y'' + \lambda y = 0,\ y'(0) = 0 = y(L)}$.

A useful observation

Often we need to find Fourier expansion of polynomial functions in terms of the eigenfunctions of Problems 1-4 satisfying the boundary conditions.

We can use "derivative transfer principle" to find Fourier coefficients.

In EVP 1 with $f(0) = 0 = f(L)$, we get Fourier sine series on $[0, L]$.

$$F(x) = \sum_{n\geq 1} b_n \sin \frac{n\pi x}{L}\, dx$$

$$b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

$$= \frac{2}{n\pi}\int_0^L f'(x) \cos \frac{n\pi x}{L}\, dx$$

$$= \frac{-2}{L}\left(\frac{L}{n\pi}\right)^2 \int_0^L f''(x) \sin \frac{n\pi x}{L}\, dx$$

In EVP (2) with $f'(0) = 0 = f'(L)$, we get Fourier cosine series on $[0, L]$, where for $n \geq 1$,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad 0 \leq x \leq L$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx = \frac{-2}{n\pi} \int_0^L f'(x) \sin \frac{n\pi x}{L} \, dx$$

$$= \frac{-2L}{n^2\pi^2} \int_0^L f''(x) \cos \frac{n\pi x}{L}$$

$$a_n = \frac{2}{L} \left(\frac{L}{n\pi} \right)^3 \int_0^L f'''(x) \sin \frac{n\pi x}{L}$$

$$a_0 = \frac{1}{L} \int_0^L f(x) \, dx$$

In EVP 3 with $f(0) = 0 = f'(L)$, we get Mixed Fourier sine series on $[0, L]$.

$$F(x) = \sum_{n\geq 1} c_n \sin \frac{(2n-1)\pi x}{2L}\, dx$$

$$c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{(2n-1)\pi x}{2L}\, dx$$

$$= \frac{4}{(2n-1)\pi} \int_0^L f'(x) \cos \frac{(2n-1)\pi x}{2L}\, dx$$

$$= \frac{-2}{L} \left(\frac{2L}{(2n-1)\pi} \right)^2 \int_0^L f''(x) \sin \frac{(2n-1)\pi x}{2L}\, dx$$

In EVP 4 with $f'(0) = 0 = f(L)$, we get Mixed Fourier cosine series on $[0, L]$.

$$F(x) = \sum_{n \geq 1} d_n \cos \frac{(2n-1)\pi x}{2L} \, dx$$

$$d_n = \frac{2}{L} \int_0^L f(x) \cos \frac{(2n-1)\pi x}{2L} \, dx$$

$$= \frac{-4}{(2n-1)\pi} \int_0^L f'(x) \sin \frac{(2n-1)\pi x}{2L} \, dx$$

$$= \frac{-2}{L} \left(\frac{2L}{(2n-1)\pi} \right)^2 \int_0^L f''(x) \cos \frac{(2n-1)\pi x}{2L} \, dx$$

Example. Find the Fourier sine expansion of

$$f(x) = x(x^2 - 3Lx + 2L^2) \quad \text{on} \quad [0, L]$$

Note $f(0) = 0 = f(L)$, $f''(x) = 6(x - L)$, Fourier sine coefficient

$$b_n = \frac{-2}{L}\left(\frac{L}{n\pi}\right)^2 \int_0^L f''(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{-12L}{n^2\pi^2}\int_0^L (x - L)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{12L^2}{n^3\pi^3}\left[(x - L)\cos\frac{n\pi x}{L}\Big|_0^L - \int_0^L \cos\frac{n\pi x}{L}\,dx\right]$$

$$= \frac{12L^2}{n^3\pi^3}\left[L - \frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_0^L\right] = \frac{12L^3}{n^3\pi^3}$$

Therefore, the Fourier sine expansion of $f(x)$ on $[0, L]$ is

$$\frac{12L^3}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{L} \qquad \square$$

Example. Find the Fourier cosine expansion of

$$f(x) = x^2(3L - 2x) \quad \text{on} \quad [0, L]$$

$$\begin{aligned} a_0 &= \frac{1}{L}\int_0^L (3Lx^2 - 2x^3)\,dx \\ &= \frac{1}{L}\left(Lx^3 - \frac{x^4}{2}\right)_0^L \\ &= \frac{L^3}{2} \end{aligned}$$

$$f'(x) = 6Lx - 6x^2 \implies f'(0) = f'(L) = 0$$

Note $f'''(x) = -12$. We get

$$a_n = \frac{2}{L}\left(\frac{L}{n\pi}\right)^3 \int_0^L f'''(x) \sin\frac{n\pi x}{L}\,dx$$

$$= \frac{-24}{L}\left(\frac{L}{n\pi}\right)^3 \int_0^L \sin\frac{n\pi x}{L}\,dx$$

$$= \frac{24}{L}\left(\frac{L}{n\pi}\right)^4 \cos\frac{n\pi x}{L}\Big|_0^L = \frac{24L^3}{n^4\pi^4}\,\left[(-1)^n - 1\right]$$

Thus $a_{2n} = 0$ and $a_{2n-1} = \dfrac{-48L^3}{(2n-1)^4\pi^4}$.

Thus Fourier cosine expansion of $f(x)$ on $[0, L]$ is

$$\frac{L^3}{2} - \frac{48L^3}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\cos\frac{(2n-1)\pi x}{L}$$

Example Find the mixed Fourier sine expansion of

$$f(x) = x(2x^2 - 9Lx + 12L^2) \quad \text{on} \quad [0, L]$$

Since $f(0) = 0 = f'(L)$ and $f''(x) = 6(2x - 3L)$, we get

$$c_n = \frac{-2}{L}\left(\frac{2L}{(2n-1)\pi}\right)^2 \int_0^L f''(x) \sin\frac{(2n-1)\pi x}{2L}\, dx$$

$$= \frac{-48L}{(2n-1)^2\pi^2} \int_0^L (2x - 3L) \sin\frac{(2n-1)\pi x}{2L}\, dx$$

$$= \frac{96L^2}{(2n-1)^3\pi^3}\Big[(2x - 3L)\cos\frac{(2n-1)\pi x}{2L}\Big|_0^L$$

$$-2\int_0^L \cos\frac{(2n-1)\pi x}{2L}\, dx\Big]$$

$$= \frac{96L^2}{(2n-1)^3\pi^3}\left[3L - \frac{4L}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{2L}\Big|_0^L\right]$$
$$= \frac{96L^3}{(2n-1)^3\pi^3}\left[3 + (-1)^n\frac{4}{(2n-1)\pi}\right]$$

Therefore, the mixed Fourier sine expansion of $f(x)$ on $[0, L]$ is

$$c\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[3 + (-1)^n\frac{4}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi x}{2L}$$

with $c = \dfrac{96L^3}{\pi^3}$.

Example. Find the mixed Fourier cosine expansion of $f(x) = 3x^3 - 4Lx^2 + L^3$ on $[0, L]$

Soln. $f'(0) = 0 = f(L)$ $f''(x) = 2(9x - 4L)$, we get

$$d_n = \frac{-2}{L}\left(\frac{2L}{(2n-1)\pi}\right)^2 \int_0^L f''(x)\cos\frac{(2n-1)\pi x}{2L}\,dx$$

$$= \frac{-16L}{(2n-1)^2\pi^2}\int_0^L (9x-4L)\cos\frac{(2n-1)\pi x}{2L}\,dx$$

$$= \frac{-32L^2}{(2n-1)^3\pi^3}\Big[(9x-4L)\sin\frac{(2n-1)\pi x}{2L}\Big|_0^L$$

$$-9\int_0^L \sin\frac{(2n-1)\pi x}{2L}\,dx\Big]$$

$$= \frac{-32L^2}{(2n-1)^3\pi^3}\left[(-1)^{n+1}5L + \frac{18L}{(2n-1)\pi}\cos\frac{(2n-1)\pi x}{2L}\Big|_0^L\right]$$

$$= \frac{32L^3}{(2n-1)^3\pi^3}\left[(-1)^n 5 + \frac{18}{(2n-1)\pi}\right]$$

Therefore, the Mixed Fourier cosine expansion of $f(x)$ on $[0, L]$ is

$$\frac{32L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[(-1)^n 5 + \frac{18}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi x}{2L}$$

$\square$