MA-207 Differential Equations II

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S2 - Lecture 2

Recall the two main results from the previous class.

Theorem

- *(Ratio test) If $a_n \neq 0$ for all n and*
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$
- *(Root test)* $\limsup_{n\to\infty} |a_n|^{1/n} = L$

Then radius of convergence of the power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ *is* $\boxed{R = 1/L}$.
For $L = 0$*, we get* $R = \infty$
and for $L = \infty$*, we get* $R = 0$.

Theorem

Let $R > 0$ be the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Then the power series converges (absolutely) for all $x \in (x_0 - R, x_0 + R)$.

For $R = \infty$, we write $(x_0 - R, x_0 + R) = (-\infty, \infty) = \mathbb{R}$.

The open interval $(x_0 - R, x_0 + R)$ is called the interval of convergence of the power series.

Correction: added after the class

In the class, we stated ratio test with $\limsup$.

If we compute R' using $\limsup$ in the ratio test, then $R' \leq R$, where R is the actual radius of convergence.

For an example, take the series

$$1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + \ldots$$

Here $a_{2n+1} = 2$ and $a_{2n} = 1$ for $n \geq 0$. Now

$$\limsup \left| \frac{a_{n+1}}{a_n} \right| = 2 \implies R = 1/2.$$

The root test definition using $\limsup$ is correct.

In our example,

$$\limsup |a_n|^{1/n} = \limsup\{1, 2^{1/1}, 1, 2^{1/3}, \ldots, 1, 2^{1/2n-1}, \ldots\} = 1$$

Theorem

Let R be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$. *We assume* $\boxed{R>0}$

- *We can define a function* $f:(x_0-R, x_0+R) \to \mathbb{R}$ *by*

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$$

- f *is infinitely differentiable* $\forall\ x \in (x_0-R, x_0+R)$.
- *The successive derivatives of* f *can be computed by differentiating the power series term-by-term, that is*

$$f'(x) = \sum_{n=0}^{\infty} n a_n(x-x_0)^{n-1} \qquad \ldots$$

$$f^{(k)}(x) = \sum_{n=0}^{\infty} n(n-1)\ldots(n-k+1)\, a_n(x-x_0)^{n-k}$$

Theorem (continued ...)

- *The power series representing the derivatives* $f^{(n)}(x)$ *have same radius of convergence* R.
- *We can determine the coefficients* a_n *(in terms of derivatives of* f *at* x_0*) as*

$$f(x_0) = a_0, \quad f'(x_0) = a_1, \quad f''(x_0) = 2a_2, \ldots$$

In general,

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

- *We can also integrate the function* $f(x) = \sum_{0}^{\infty} a_n(x-x_0)^n$ *term-wise that is if* $[a,b] \subset (x_0-R, x_0+R)$, *then*

$$\int_a^b f(x)\,dx = \sum_{n=0}^{\infty} a_n \int_a^b (x-x_0)^n\,dx = \sum_{0}^{\infty} \frac{a_n}{n+1}(x-x_0)^{n+1}$$

Example (Power series representation of elementary functions)

(i) $e^x = \sum_0^\infty \frac{x^n}{n!} \qquad -\infty < x < \infty$

(ii) $\sin x = \sum_0^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad -\infty < x < \infty$

(iii) $\frac{1}{1-x} = \sum_0^\infty x^n \qquad -1 < x < 1$

(iv)
$$\frac{d}{dx}(\sin x) = \sum_0^\infty (-1)^n \frac{d}{dx}\left(\frac{x^{2n+1}}{(2n+1)!}\right)$$
$$= \sum_0^\infty (-1)^n \frac{x^{2n}}{(2n)!} = \cos x$$

Theorem

(i) Power series representation of f in an open interval I containing x_0 is unique, that is if

$$f(x) = \sum_0^\infty a_n(x-x_0)^n = \sum_0^\infty b_n(x-x_0)^n$$

for all $x \in I$, then $a_n = b_n \ \forall\ n$.
(ii) If

$$\sum_0^\infty a_n(x-x_0)^n = 0$$

for all $x \in I$, then $a_n = 0$ for all n.

Proof. (i)

$$a_n = \frac{f^{(n)}(x_0)}{n!} = b_n \qquad \text{for all} \quad n.$$

It is clear that (ii) follows from (i).

Algebraic operations on power series

Definition

If $\quad f(x) = \sum_0^\infty a_n(x - x_0)^n \qquad g(x) = \sum_0^\infty b_n(x - x_0)^n$

have radius of convergence R_1 and R_2 respectively, then

$$c_1 f(x) + c_2 g(x) := \sum_0^\infty (c_1 a_n + c_2 b_n)(x - x_0)^n$$

has radius of convergence $\boxed{R \geq \min\{R_1, R_2\}}$ for $c_1, c_2 \in \mathbb{R}$.

Further, we can multiply the series as if they were polynomials, that is

$$f(x)g(x) = \sum_0^\infty c_n(x - x_0)^n; \quad c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$$

It also has radius of convergence $R \geq \min\{R_1, R_2\}$.

Example

Find the power series expansion for $\cosh x$ in terms of powers of x^n.

$$\begin{aligned}
\cosh x &= \frac{1}{2}e^x + \frac{1}{2}e^{-x} \\
&= \frac{1}{2}\sum_{n=0}^\infty \frac{x^n}{n!} + \frac{1}{2}\sum_{n=0}^\infty (-1)^n \frac{x^n}{n!} \\
&= \sum_{n=0}^\infty \frac{1}{2}\left[1 + (-1)^n\right]\frac{x^n}{n!} \\
&= \sum_{n=0}^\infty \frac{x^{2n}}{(2n)!}
\end{aligned}$$

Since radius of convergence for Taylor series of e^x and e^{-x} are ∞, the power series expansion of $\cosh x$ is valid on $\mathbb{R}$.

Shifting the summation index

If $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \implies f'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$

Let us rewrite the series for $f'(x)$ in powers of $(x-x_0)^n$.

Put $r = n-1$, we get

$$f'(x) = \sum_{r=0}^{\infty} (r+1) a_{r+1} (x-x_0)^r$$

Similarly,

$$\begin{aligned} f^{(k)}(x) &= \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1) a_n (x-x_0)^{n-k} \\ &= \sum_{n=0}^{\infty} (n+k)(n+k-1)\dots(n+1) a_{n+k} (x-x_0)^n \end{aligned}$$

In general, $\left[\sum_{n=n_0}^{\infty} b_n (x-x_0)^{n-k} = \sum_{n=n_0-k} b_{n+k} (x-x_0)^n \right]$

Example

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Write $(x-1)f''$ as a power series around 0.

$$\begin{aligned} (x-1)f'' &= xf'' - f'' \\ &= x\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \right) - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \\ &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \\ &= \sum_{n=1}^{\infty} (n+1) n a_{n+1} x^n - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \\ &= \sum_{n=0}^{\infty} \left[(n+1) n a_{n+1} - (n+2)(n+1) a_{n+2} \right] x^n \end{aligned}$$

Example

Suppose

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

for all x in an open interval I containing $x_0 = 1$.

- Find the power series of y' and y'' in terms of $x-1$ in the interval I. Use these to express the function

  $$(1+x)y'' + 2(x-1)^2 y' + 3y$$

  as a power series in $x-1$ on I.
- Find necessary and sufficient conditions on the coefficients a_n's, so that $y(x)$ is a solution of the ODE

  $$(1+x)y'' + 2(x-1)^2 y' + 3y = 0$$

Example (Continue ...)

Solution. Write the ODE in $(x-1)$, that is

$$(1+x)y'' + 2(x-1)^2 y' + 3y = (x-1)y'' + 2y'' + 2(x-1)^2 y' + 3y$$

Express each of $(x-1)y''$, $2y''$, $2(x-1)^2 y'$ and $3y$ as a power series in powers of $(x-1)$ and add them.

$$\begin{aligned}(x-1)y'' &= (x-1)\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2}\\ &= \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1}\\ &= \sum_{n=1}^{\infty} (n+1)n a_{n+1}(x-1)^n\\ &= \sum_{n=0}^{\infty} (n+1)n a_{n+1}(x-1)^n\end{aligned}$$

Example (Continue ...)

$$2y'' = \sum_{n=2}^{\infty} 2n(n-1)a_n(x-1)^{n-2}$$
$$= \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}(x-1)^n$$
$$2(x-1)^2y' = 2(x-1)^2 \sum_{n=1}^{\infty} na_n(x-1)^{n-1}$$
$$= \sum_{n=1}^{\infty} 2na_n(x-1)^{n+1}$$
$$= \sum_{n=2}^{\infty} 2(n-1)a_{n-1}(x-1)^n$$
$$= \sum_{n=0}^{\infty} 2(n-1)a_{n-1}(x-1)^n \quad (a_{-1}=0)$$

Example (Continue ...)

We have

$$(x-1)y'' = \sum_{n=0}^{\infty} (n+1)na_{n+1}(x-1)^n$$
$$2y'' = \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}(x-1)^n$$
$$2(x-1)^2y' = \sum_{n=0}^{\infty} 2(n-1)a_{n-1}(x-1)^n \quad (a_{-1}=0)$$

Now we get

$$(x-1)y'' + 2y'' + 2(x-1)^2y' + 3y = \sum_{n=0}^{\infty} b_n(x-1)^n$$

where

$$b_n = (n+1)na_{n+1} + 2(n+2)(n+1)a_{n+2} + 2(n-1)a_{n-1} + 3a_n$$

Example (Continue ...)

For the second part,

$$y(x) = \sum_{0}^{\infty} a_n (x-1)^n$$

is the solution of the ODE

$$(x-1)y'' + 2y'' + 2(x-1)^2 y' + 3y = 0$$

on the open interval I containing 1 if and only if

$$\sum_{n=0}^{\infty} b_n (x-1)^n = 0 \text{ on } I \iff b_n = 0 \quad \text{for all} \quad n$$

that is a_n's satisfy the following recursive relation

$$(n+1)na_{n+1} + 2(n+2)(n+1)a_{n+2} + 2(n-1)a_{n-1} + 3a_n = 0$$

for all n.

Definition

If a function $f(x)$ is infinitely differentiable at x_0, then the Taylor series of f at x_0 is defined as the power series

$$TS\, f|_{x_0} := \sum_{0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

When $x_0 = 0$, the series is also called the Maclaurin series of f.

Example

The function $\quad f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

is infinitely differentiable at 0. But $f^{(n)}(0) = 0$ for all n.

Hence the Taylor series of f at 0 is the constant function taking value 0.

Therefore Taylor series of f at 0 does not converge to function $f(x)$ on any open interval around 0.

Definition

Suppose

- $f(x)$ is infinitely differentiable at x_0; and
- Taylor series of f at x_0 converges to $f(x)$ for all x in some open interval around x_0;

Then f is called analytic at x_0.

Example

The function $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

is not analytic at 0. Here 2nd condition fails.
However, f is analytic at all $x \neq 0$.

Theorem (Analytic functions)

1. *Polynomials, e^x, $\sin x$ and $\cos x$ are analytic at all $x \in \mathbb{R}$.*
2. *$f(x) = \tan x$ is analytic at all x except $x = (2n+1)\pi/2$, where $n = \pm 1, \pm 2, \ldots$.*
3. *$f(x) = x^{5/3}$ is analytic at all x except $x = 0$.*
4. *If $f(x)$ and $g(x)$ are analytic at x_0, then*
   $f(x) \pm g(x) \qquad f(x)g(x) \qquad f(x)/g(x) \quad$ *(if $g(x_0) \neq 0$)*
   are analytic at x_0.
5. *If $f(x)$ is analytic at x_0 and $g(x)$ is analytic at $f(x_0)$, then $g(f(x)) := (g \circ f)(x)$ is analytic at x_0.*
6. *If a power series $\sum_0^\infty a_n (x - x_0)^n$ has radius of convergence*

   $R > 0$, then the function $f(x) := \sum_0^\infty a_n (x - x_0)^n$

   is analytic at all points $x \in (x_0 - R, x_0 + R)$.

Example

The function $f(x) = x^2 + 1$ is analytic everywhere. Since $x^2 + 1$ is never 0, the function $h(x) := \frac{1}{x^2+1}$ is analytic everywhere. However, there is no power series around 0 which represents $h(x)$ everywhere.

If there were such a power series, then by uniqueness, it has to be the power series expansion of $h(x)$ around 0, which is

$$1 - x^2 + x^4 - x^6 + \cdots$$

However, the radius of convergence of this is $R = 1$.

In fact, for any x_0, there is no power series around x_0 which represents $h(x)$ everywhere.

Theorem

Let

$$F(x) = \frac{N(x)}{D(x)} \quad \textit{example} \quad F(x) = \frac{x^3 - 1}{x^2 + 1}$$

be a rational function, where $N(x)$ and $D(x)$ are polynomials without any common factors, that is they do not have any common (complex) zeros. Let $\alpha_1, \ldots, \alpha_r$ be distinct complex zeros of $D(x)$.

Then $F(x)$ is analytic at all x except at $x \in \{\alpha_1, \ldots, \alpha_r\}$.

If x_0 is different from $\{\alpha_1, \ldots, \alpha_r\}$, then the radius of convergence R of the Taylor series of F at x_0

$$TS\,F_{x_0} = \sum_{0}^{\infty} \frac{F^{(n)}(x_0)}{n!}(x - x_0)^n$$

is given by

$$R = \mathit{min}\,\{|x_0 - \alpha_1|, |x_0 - \alpha_2|, \ldots, |x_0 - \alpha_r|\}$$

Example

If
$$F(x) = \frac{N(x)}{D(x)} = \frac{(2+3x)}{(4+x)(9+x^2)}$$
then $D(x)$ has zeros at -4 and $\pm 3\iota$, where $\iota = \sqrt{-1}$.

Hence F is analytic at all x except at $x \in \{-4, \pm 3\iota\}$.

If $x = 2$, then radius of convergence of Taylor series of F at $x = 2$ is
$$\min\{|2+4|, |2+3\iota|, |2-3\iota|\} = \min\{6, \sqrt{13}\} = \sqrt{13}$$
If $x = -6$, then radius of convergence of Taylor series of F at $x = -6$ is
$$\min\{|-6+4|, |-6 \pm 3\iota|\} = \min\{2, \sqrt{45}\} = 2$$

Power series solution of ODE

Theorem (Existence Theorem)

If $p(x)$ and $q(x)$ are analytic functions at x_0, then every solution of
$$y'' + p(x)y' + q(x)y = 0$$
is also analytic at x_0; and therefore any solution can be expressed as
$$y(x) = \sum_0^\infty a_n(x-x_0)^n$$
If R_1 = radius of convergence of Taylor series of $p(x)$ at x_0,
R_2 = radius of convergence of Taylor series of $q(x)$ at x_0,
then radius of convergence of $y(x)$ is at least min$(R_1, R_2) > 0$.

In most applications, $p(x)$ and $q(x)$ are rational functions, that is quotient of polynomial functions.

Example

Let us solve $\boxed{y'' + y = 0 \qquad (1)}$ by power series method.

Compare with $y'' + p(x)y' + q(x)y = 0$,
$p(x) = 0$ and $q(x) = 1$ are analytic at all x.
We can find power series solution in $x - x_0$ for any x_0.
Let us assume $x_0 = 0$ for simplicity.
By existence theorem, all solution of (1) can be found in the form

$$y(x) = \sum_0^\infty a_n x^n$$

and the series will have ∞ radius of convergence.
Since

$$y'' = \sum_2^\infty n(n-1)a_n x^{n-2} = \sum_{n=0}^\infty (n+2)(n+1)a_{n+2}x^n$$

Example (Continue . . .)

$$y'' + y = \sum_0^\infty ((n+2)(n+1)a_{n+2} + a_n)x^n = 0$$

By uniqueness of power series in $x - x_0$ with positive radius of convergence, we get the recursion formula

$$(n+2)(n+1)a_{n+2} + a_n = 0$$

$$\implies a_{n+2} = \frac{-1}{(n+2)(n+1)}a_n \quad \forall n$$

Therefore,

$$a_2 = \frac{-1}{2.1}a_0, \quad a_4 = \frac{-1}{4.3}a_2 = \frac{1}{4!}a_0 \ \ldots \ a_{2n} = (-1)^n \frac{1}{(2n)!}a_0$$

$$a_3 = \frac{-1}{3.2}a_1, \quad a_5 = \frac{-1}{5.4}a_3 = \frac{1}{5!}a_1 \ \ldots \ a_{2n+1} = (-1)^n \frac{1}{(2n+1)!}a_1$$

Example (Continue ...)

Define

$$y_1(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots \qquad (a_0 = 1, a_1 = 0)$$

$$y_2(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots \qquad (a_0 = 0, a_1 = 1)$$

Then

$$y(x) = \sum_0^\infty a_n x^n = a_0 y_1(x) + a_1 y_2(x)$$

is a general solution of the ODE (1).

In this case, $y_1(x) = \cos x$ and $y_2(x) = \sin x$. Thus, $y(x)$ is an elementary function. In general, however, the solution may not be an elementary function.
We don't need to check the series for converges, since the existence theorem guarantees that the series converges for all x.

Steps for Series solution of linear ODE

1. Write ODE in standard form $y'' + p(x)y' + q(x)y = 0$.
2. Choose x_0 at which $p(x)$ and $q(x)$ are analytic. If boundary conditions at x_0 are given, choose the center of the power series as x_0.
3. Find minimum of radius of convergence of Taylor series of $p(x)$ and $q(x)$ at x_0.
4. Let $y(x) = \sum_0^\infty a_n (x - x_0)^n$, compute the power series for $y'(x)$ and $y''(x)$ at x_0 and substitute these into the ODE.
5. Set the coefficients of $(x - x_0)^n$ to zero and find recursion formula.
6. From the recursion formula, obtain (linearly independent) solutions $y_1(x)$ and $y_2(x)$. The general solution then looks like $y(x) = a_1 y_1(x) + a_2 y_2(x)$.

The following ODE's are classical:

- Bessel's equation :

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

  It occurs in problems displaying cylindrical symmetry, example diffusion of light through a circular aperture, vibration of a circular head drum, etc.

- Airy's equation :

$$y'' - xy = 0$$

  It occurs in astronomy and quantum physics.

- Legendre's equation :

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$$

  It occurs in problems displaying spherical symmetry, particularly in electromagnetism.

In this course, we will consider ODE

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$$

with $P_i(x)$ polynomials for $i = 0, 1, 2$ without any common factor.
If we write ODE in the standard form

$$y'' + \frac{P_1(x)}{P_0(x)}y' + \frac{P_2(x)}{P_0(x)}y = 0$$

we see that if x_0 is not a zero of $P_0(x)$, then $P_1(x)/P_0(x)$ and $P_2(x)/P_0(x)$ will be analytic at x_0 hence we can find the series solution of ODE in the form

$$y(x) = \sum_0^\infty a_n(x - x_0)^n$$

When x_0 is a zero of $P_0(x)$, then x_0 is called a singular point of ODE. This case will be considered later.

Example

Find the power series in x for the general solution of

$$(1+2x^2)y'' + 6xy' + 2y = 0$$

Solution. Note that 0 is not a zero of $P_0(x) = 1 + 2x^2$, hence the series solution in powers of x exists.

Put $y = \sum_0^\infty a_n x^n$ in the ODE, we get

$$\begin{aligned}
&(1+2x^2)y'' + 6xy' + 2y \\
&= y'' + 2x^2 y'' + 6xy' + 2y \\
&= \sum_0^\infty ((n+2)(n+1)a_{n+2} + 2n(n-1)a_n + 6na_n + 2a_n)x^n \\
&\implies (n+2)(n+1)a_{n+2} + [2n(n-1) + 6n + 2]a_n = 0
\end{aligned}$$

Example (Continue . . .)

$$\implies a_{n+2} = -\frac{2n^2+4n+2}{(n+2)(n+1)}\,a_n = -2\frac{n+1}{(n+2)}\,a_n \quad n \geq 0$$

Since indices on left and right differ by 2, we write separately for $n = 2m$ and $n = 2m+1$, $m \geq 0$, so

$$\begin{aligned}
a_{2m+2} &= -2\frac{2m+1}{2m+2}\,a_{2m} = -\frac{2m+1}{m+1}\,a_{2m} \\
a_{2m+3} &= -2\frac{2m+2}{2m+3}\,a_{2m+1} = -4\frac{m+1}{2m+3}\,a_{2m+1} \\
a_2 &= -\frac{1}{1}\,a_0 \\
a_4 &= -\frac{3}{2}\,a_2 = \frac{1.3}{1.2}\,a_0 \\
a_6 &= -\frac{5}{3}\,a_4 = -\frac{1.3.5}{1.2.3}\,a_0
\end{aligned}$$

Example (Continue ...)

$$a_{2m} = (-1)^m \frac{1.3.5.\ldots(2m-1)}{m!} a_0$$

$$= (-1)^m \frac{\prod_{j=1}^{m}(2j-1))}{m!} a_0$$

$$a_{2m+3} = -4\frac{m+1}{2m+3} a_{2m+1}$$

$$a_3 = -4\frac{1}{3} a_1$$

$$a_5 = -4\frac{2}{5} a_3 = 4^2 \frac{1.2}{3.5} a_1$$

$$a_7 = -4\frac{3}{7} a_5 = -4^3 \frac{1.2.3}{3.5.7} a_1$$

$$a_{2m+1} = (-1)^m 4^m \frac{m!}{\prod_{j=1}^{m}(2j+1)} a_1$$

Example (Continue ...)

We can write the solution

$$y = \sum_{0}^{\infty} a_n x^n = a_0 y_1(x) + a_1 y_2(x)$$

where a_0 and a_1 are arbitrary scalars and

$$y_1(x) = \sum_{m=0}^{\infty} (-1)^m \frac{\prod_{j=1}^{m}(2j-1)}{m!} x^{2m}$$

$$y_2(x) = \sum_{m=0}^{\infty} (-1) \frac{4^m m!}{\prod_{j=1}^{m}(2j+1)} x^{2m+1}$$

Since $P_0(x) = 1 + 2x^2$ has complex zeros $\frac{\pm\iota}{\sqrt{2}}$, the power series solution converges in the interval $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. $\square$

Example

Find the coefficients $a_0, \ldots, a_6$ in the series solution

$$y = \sum_0^\infty a_n x^n$$

of the IVP

$$(1 + x + 2x^2)y'' + (1 + 7x)y' + 2y = 0$$

with

$$y(0) = -1,\ y'(0) = -2.$$

Zeros of $P_0(x) = 1 + x + 2x^2$ are $\frac{1}{4}(-1 \pm \iota\sqrt{7})$ whose absolute values are $1/\sqrt{2}$. Hence the series solution to the IVP converges on the interval $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Example (Continue ...)

$$(1 + x + 2x^2)y'' + (1 + 7x)y' + 2y = \sum_0^\infty b_n x^n = 0$$

$$\begin{aligned} b_n &= (n+2)(n+1)a_{n+2} + (n+1)na_{n+1} + 2n(n-1)a_n \\ &\quad +(n+1)a_{n+1} + 7na_n + 2a_n = 0 \end{aligned}$$

that is

$$(n+2)(n+1)a_{n+2} + (n+1)^2 a_{n+1} + (2n^2 + 5n + 2)a_n = 0$$

Since $2n^2 + 5n + 2 = (n+2)(2n+1)$,

$$a_{n+2} = -\frac{n+1}{n+2} a_{n+1} - \frac{2n+1}{n+1} a_n \qquad n \geq 0$$

Example (Continue ...)

$$a_{n+2} = -\frac{n+1}{n+2}a_{n+1} - \frac{2n+1}{n+1}a_n \qquad n \geq 0$$

From the initial conditions $y(0) = -1, \; y'(0) = -2$ we get

$$a_0 = y(0) = -1, \quad a_1 = y'(0) = -2$$

$$a_2 = -\frac{1}{2}a_1 - a_0 = 2$$

$$a_3 = -\frac{2}{3}a_2 - \frac{3}{2}a_1 = \frac{5}{3}$$

Check that

$$y(x) = -1 - 2x + 2x^2 + \frac{5}{3}x^3 - \frac{55}{12}x^4 + \frac{3}{4}x^5 + \frac{61}{8}x^6 + \dots$$

Remark 1 - added after class

Let $f(x) = \sum_{n\geq 0} a_n x^n$. Then we can write

$$f''(x) = \sum_{n\geq 0} n(n-1)a_n x^{n-2} \qquad (*)$$

Note that the above can also be written as

$$f''(x) = \sum_{n\geq 2} n(n-1)a_n x^{n-2} \qquad (**)$$

The expression $(*)$ has the disadvantage that it gives the impression that negative powers x^{-1} and x^{-2} may occur!! This does not happen because at $n = 0$ and $n = 1$, the coefficients in the expression $(*)$ are 0.

Remark 1 - continued

So in this example, whether you use the expression $(*)$ or $(**)$ before you do the index shifting is a matter of personal taste!!

In both cases after you do the index shifting (letting $r = n - 2$), you will get

$$f''(x) = \sum_{r\geq 0}(r+2)(r+1)a_{r+2}x^r$$

Now put $n = r$ to get

$$f''(x) = \sum_{n\geq 0}(n+2)(n+1)a_{n+2}x^n$$

The main point of the discussion is to arrive at this expression.

Remark 2 - added after class

In the last example, the initial conditions are given and we need to find the solution to the ODE which satisfies this initial condition.

Plugging in the power series $y(x) = \sum_{n\geq 0} a_n x^n$ into the differential equation and computing the recursive relation we got

$$a_{n+2} = -\frac{n+1}{n+2}a_{n+1} - \frac{2n+1}{n+1}a_n \qquad n \geq 0$$

We could now proceed in two ways. Note that the coefficients a_n, for $n \geq 2$, are completely determined once we fix the values of a_0 and a_1.

(1) We know that $y(0) = a_0$ and $y'(0) = a_1$. These values are given to us as part of the initial conditions. Thus, these will determine all a_n's and hence the solution $y(x)$.

Remark 2 - continued

(2) The second method is the following. First obtain the two independent solutions $y_1(x)$ (by taking $a_0 = 1$ and $a_1 = 0$) and $y_2(x)$ (by taking $a_0 = 0$ and $a_1 = 1$) of the ODE.

We know that the general solution is $y(x) = \alpha y_1(x) + \beta y_2(x)$. Compute α and β by using the initial conditions.

$$y(0) = \alpha y_1(0) + \beta y_2(0)$$

$$y'(0) = \alpha y_1'(0) + \beta y_2'(0)$$

In class I had mentioned the second method. In the notes the first method is used.