1. The number of local maxima of $P_{108}(x)$ in $(-\infty, \infty)$ is

(A) 53 (B) 54 (C) 55 (D) 107

SOLUTION. $P_{108}(x)$ has 108 zeroes and between two consecutive zeroes, there is either a local minima or a local maxima. This is because $P'_{108}(x)$ has 107 zeroes and they alternate with the zeroes of $P_{108}(x)$. One can see that there are 54 local minimas and 53 local maximas.

2. Let $u = (1, 2)$ and $v = (2, 1)$ be two vectors in $\mathbb{R}^2$. Let $\langle\, , \rangle$ be an inner product on $\mathbb{R}^2$ satisfying $\langle u, u\rangle = \langle v, v\rangle = 1$ and $\langle u, v\rangle = 0$. A vector orthogonal to $(1, 0)$ with respect to this inner product is

(A) $(0, 1)$ (B) $(-2, 3)$ (C) $(3, -4)$ (D) $(4, 5)$

SOLUTION. $2v - u = (3, 0)$. A vector orthogonal to this is $v + 2u = (4, 5)$. It will also be orthogonal to $(1, 0)$.

3. Let α_n denote the n-th positive zero of $J_{2016}(x)$. Then $\int_1^{\alpha_{2016}} \big(x^{-2016} J_{2017}(x) + x^{2016} J_{2015}(x)\big) dx$ equals

(A) 0 (B) $2J_{2016}(1)$ (C) $J_{2017}(1) - J_{2015}(1)$ (D) 1

SOLUTION. We know $\frac{d}{dx}(x^p J_p(x)) = x^p J_{p-1}(x)$ and $\frac{d}{dx}(x^{-p} J_p(x)) = -x^{-p} J_{p+1}(x)$. Hence, the given expression evaluates to $[-x^{-2016} J_{2016}(x) + x^{2016} J_{2016}(x)]|_1^{\alpha_{2016}}$ which gives 0.

4. For the equation $x(x-1)(x+1)^2 y'' + y' + (x+1)^2 y = 0$, let m be the number of regular singular points and n be the number of irregular singular points. Then $m - n$ equals

(A) -3 (B) -1 (C) 1 (D) 3

SOLUTION. $x = 0$ and $x = 1$ are regular singular points and $x = -1$ is an irregular singular point.

5. A function which does *not* solve the equation $(1 - x^2)y'' - 2xy' + 2y = 0$ around $x = 0$ is

(A) $5x$ (B) $\sum_{n=0}^{\infty} \frac{x^{2n}}{1-2n}$ (C) $2x \log(\frac{1-x}{1+x}) + 4$ (D) $4 + 8x(1 - \log(\frac{1+x}{1-x}))$

SOLUTION. The given equation is the Legendre equation with $p = 1$. All options, except option D, are linear combinations of $P_1(x)$ and $1 - \frac{1}{2}x \log(\frac{1+x}{1-x})$. The latter is the Legendre function of the second kind for $p = 1$. Option B is its power series.

6. The radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^{6n+2}}{(1+\frac{1}{n})^{n^2}}$ is

(A) e (B) $e^{\frac{1}{2}}$ (C) $e^{\frac{1}{3}}$ (D) $e^{\frac{1}{6}}$

SOLUTION. The series $\sum_{n=1}^{\infty} \frac{x^{6n+2}}{(1+\frac{1}{n})^{n^2}}$ is convergent iff $\sum_{n=1}^{\infty} \frac{x^{6n}}{(1+\frac{1}{n})^{n^2}}$ is convergent. Let b_n denote the n-th term of the latter. Then $\lim_{n\to\infty} |b_n|^{1/n} = |x|^6/e$. This is less than 1 iff $|x| < e^{\frac{1}{6}}$. This is a nice example where the ratio test does not do much good.

7. The expression $(J_0(x) - J_2(x))(J_1(x) - J_3(x))(J_2(x) - J_4(x))$ for $x > 0$ equals

(A) $8\big(J_0(x) - \frac{J_1(x)}{x}\big)\big(J_1(x) - \frac{J_2(x)}{x}\big)\big(J_2(x) - \frac{J_3(x)}{x}\big)$
(B) $8\big(J_0(x) - \frac{J_1(x)}{x}\big)\big(J_1(x) - \frac{2J_2(x)}{x}\big)\big(J_2(x) - \frac{3J_3(x)}{x}\big)$
(C) $16\big(J_0(x) - \frac{J_1(x)}{x}\big)\big(J_1(x) - \frac{J_2(x)}{x}\big)\big(J_2(x) - \frac{J_3(x)}{x}\big)$
(D) $16\big(J_0(x) - \frac{J_1(x)}{x}\big)\big(J_1(x) - \frac{2J_2(x)}{x}\big)\big(J_2(x) - \frac{3J_3(x)}{x}\big)$

SOLUTION. Using the Bessel identity $J_{n+1}(x) = -J_{n-1}(x) + \frac{2n}{x}J_n(x)$, we get $J_{n-1}(x) - J_{n+1}(x) = 2J_{n-1}(x) - \frac{2n}{x}J_n(x)$. Now substitute different values of n to get the answer.

8. For any positive integer k, define $Q_k(h) = \sum_{i=1}^{\infty} \frac{(h-1)^i}{i!}\left(\left(\frac{d}{dx}\right)^i P_k(x)\right)|_{x=1}$. The value of $\int_{-1}^{1} Q_3(x)Q_7(x)dx$ equals

   (A) 0 (B) 1 (C) 2 (D) 3

   SOLUTION. Observe that $Q_k(h)$ except for a constant is just $P_k(h)$ expanded as a Taylor series around $h = 1$. In fact, $Q_k(h) = P_k(h) - P_k(1) = P_k(h) - 1$. Use this along with orthogonality of P_k to get $\int_{-1}^{1} 1\, dx = 2$.

9. Let $S = \{0, 1, 2, \dots\}$ denote the set of nonnegative integers. The number of ordered pairs (n, x) which solve the equation $P_n(J_n(x)) = 1$ for $n \in S$ and $x \in [0, 1]$ is

   (A) 0 (B) 1 (C) 2 (D) ∞

   SOLUTION. For $n \neq 0$, $P_n(y) = 1$ possibly only for $y = 1$ and $y = -1$ but $J_n(x)$ is never 1. For $n = 0$, $P_0(y)$ is always 1, so any x works.

10. Let $a = \int_0^{\frac{\pi}{4}} \frac{J_{1/2}(x)}{J_{-1/2}(x)}dx$ and $b = \int_0^{\frac{\pi}{4}} xJ_{1/2}(x)J_{-1/2}(x)dx$. Then ab equals

    (A) $\frac{\log 2}{\pi}$ (B) $\frac{\log 2}{2\pi}$ (C) $\frac{\log 2}{4\pi}$ (D) $\frac{\log 2}{8\pi}$

    SOLUTION. Use $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$ and $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$. So $a = \int_0^{\frac{\pi}{4}} \tan x\, dx = \frac{1}{2}\log 2$ and $b = \int_0^{\frac{\pi}{4}} \frac{1}{\pi}\sin 2x\, dx = \frac{1}{2\pi}$.

11. Consider the equation $x^2y'' - x^2y' + (x-6)y = 0$ for $x > 0$ with initial conditions $y(1) = 49$ and $y'(1) = -65$. The solution is of the form (where $c_1, c_2 \neq 0$)

    (A) $c_1x^3\log(x)(1 + \sum_{k=1}^{\infty} a_kx^k) + c_2x^{-2}(1 + \sum_{k=1}^{\infty} b_kx^k)$ (B) $c_2x^{-2}(1 + \sum_{k=1}^{\infty} a_kx^k)$
    (C) $c_1x^3(1 + \sum_{k=1}^{\infty} a_kx^k)$ (D) $c_1(1 + \sum_{k=1}^{\infty} a_kx^k)$

    SOLUTION. The indicial equation is $r^2 - r - 6 = 0$ which gives $r_1 = 3, r_2 = -2$. The general recursion is $(n + r + 2)(n + r - 3)a_n = (n + r - 2)a_{n-1}$. Notice that $a_5(r_2) = \frac{(r_2+3)\dots(r_2-1)}{((r_2+7)\dots(r_2+3))((r_2+2)\dots(r_2-2))}a_0$ has no singularity since $r_2 + 2$ cancels, so the log term is absent and the general solution is $y_2(x) = c_1x^{-2}(1 + \frac{3}{4}x + \frac{1}{4}x^2 + \frac{1}{24}x^3) + c_2y_1(x)$. Putting $x = 1$, we notice that the polynomial term evaluates to $\frac{49}{24}$. Inspired by this, we try the solution, $y(x) = 24x^{-2}(1 + \frac{3}{4}x + \frac{1}{4}x^2 + \frac{1}{24}x^3)$ and find that this indeed satisfies the initial conditions, and hence is the unique solution.

12. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^{1/3}$. Let $g(x) = \sum_{n=0}^{\infty} a_n(x - 3/2)^n$, where $a_n = \frac{f^{(n)}(3/2)}{n!}$ for $n \geq 0$. The largest open set contained in $\{x \mid f(x) = g(x)\}$ is

    (A) $(1/2, 5/2)$ (B) $(1, 2)$ (C) $(0, 3)$ (D) $(0, \infty)$

    SOLUTION. Write $x^{1/3} = (3/2)^{1/3}(1 + \frac{(x-3/2)}{3/2})^{1/3}$ and expand in a binomial series. This series converges iff $|\frac{x-3/2}{3/2}| < 1$ iff $x \in (0, 3)$.

13. Let r_1 and r_2 (with $r_2 \leq r_1$) be the roots of the indicial equation obtained while solving $15x^2y'' + xy' + 3y = 0$ by the Frobenius method. The number of roots of $P_3(x)$ in the interval $[r_2, r_1]$ is

    (A) 0 (B) 1 (C) 2 (D) 3

SOLUTION. The indicial equation obtained is $15r(r-1)+r+3=0$. Its solutions are $r_2=\frac{1}{3}$ and $r_1=\frac{3}{5}$. $P_3(x)=\frac{1}{2}(5x^3-3x)$ which has roots at $0, \pm\sqrt{\frac{3}{5}}$. Hence, none of the roots of $P_3(x)$ lie in $[r_2, r_1]$.

14. Let α be the smallest positive root of the equation $J_3(x)=J_5(x)$. Then

(A) $J_4(x)$ has a local maximum at α (B) $J_4(x)$ has a local minimum at α
(C) $J_4'(x)$ has a local maximum at α (D) $J_4'(x)$ has a local minimum at α

SOLUTION. $J_3(x)-J_5(x)=2J_4'(x)$ implies $J_4'(\alpha)=0$. So $J_4(x)$ has either a local minimum or a local maximum at α. Since α is the smallest extremum of $J_4(x)$, it is a local maximum. (This is true of all Bessel functions $J_n(x)$ for $n \geq 1$.)

15. Let $f(x)=\int_0^x P_{37}(y)\,dy$ for $x \in [-1,1]$. The coefficient of $P_{38}(x)$ in the Fourier-Legendre series of $f(x)$ equals

(A) $\frac{76}{37}$ (B) $\frac{1}{75}$ (C) $\frac{111}{4}$ (D) $\frac{1}{57}$

SOLUTION. From Rodrigues formula, the coefficient of x^n in $P_n(x)$ is $\frac{1}{2^n}\binom{2n}{n}$. The coefficient of x^{37} in $P_{37}(x)$ is $\frac{74!}{2^{37}(37!)^2}$ and so the coefficient of x^{38} in $f(x)$ is $\frac{74!}{2^{37}(37!)^2 38}$. Since f is a polynomial of degree 38, it will only have the first 38 Legendre polynomials in its series. Matching coefficients of x^{38} in $f(x)=\sum_{r=0}^{38} c_r P_r$ we have $\frac{76!}{2^{38}(38!)^2}c_{38}=\frac{74!}{2^{37}(37!)^2 38}$ and so $c_{38}=\frac{1}{75}$.

16. Let S be the set of values of p such that the derivative $f'(x)$ is unbounded near 0 for any solution $f(x)$ of $x^2y''+x^3y'-(p^2-\frac{1}{4})y=0$ for $x>0$. Then S contains

(A) $\{0, 0.25\}$ (B) $\{0.25, 0.5\}$ (C) $\{0, \sqrt{2}\}$ (D) $\{0.25, 0.75\}$

SOLUTION. This is regular singular about $x=0$, so we substitute $x^r\sum_{n=0}^{\infty} a_n x^n$ and get $r(r-1)-(p^2-\frac{1}{4})=0$. Hence $r=\frac{1}{2}\pm p$. The derivative will be unbounded near 0 iff the larger r is strictly less than 1 and smaller r is nonzero. This is the same as $\frac{1}{2}+|p|<1$ or $|p|<0.5$. This is also true for $p=0$ since the roots are repeated, and the second solution has a log term. Hence the required set of values of p is $(-1/2, 1/2)$. The only set among the options contained in this set is $\{0, 0.25\}$.