MA-207 Differential Equations II
Lecture-11 Wave Equation

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S1 - Lecture 11

Recall : The (formal) solution $u(x,t)$ of Heat equation with Dirichlet boundary conditions

$$\begin{aligned} u_t &= k^2 u_{xx}, && 0 < x < L, \quad t > 0 \\ u(0,t) &= 0 = u(L,t), && t > 0 \\ u(x,0) &= f(x), && 0 \leq x \leq L \end{aligned}$$

is $$u(x,t) = \sum_{n=1}^{\infty} \alpha_n \, e^{-\nu_n^2 k^2 t} \sin(\nu_n x)$$

where $\nu_n = \dfrac{n\pi}{L}$, and

$$\alpha_n = \frac{2}{L} \int_0^L f(x) \sin(\nu_n x) \, dx$$

are the Fourier sine coefficients of f on $[0, L]$,

Theorem. The (formal) solution $u(x,t)$ of Heat equation with Neumann boundary condition

$$\begin{aligned} &u_t = k^2 u_{xx}, && 0 < x < L, \quad t > 0 \\ &u_x(0,t) = 0 = u_x(L,t), && t > 0 \\ &u(x,0) = f(x), && 0 \le x \le L \end{aligned}$$

is $$u(x,t) = \sum_{n=0}^{\infty} \alpha_n e^{-\nu_n^2 k^2 t} \cos(\nu_n x)$$

where $\nu_n = \dfrac{n\pi}{L}$, and

$$\alpha_n = \frac{2}{L} \int_0^L f(x) \cos(\nu_n x)\, dx$$

are Fourier cosine coefficients of f on $[0, L]$.

Non homogeneous Heat Equation: Dirichlet boundary condition

Let us now consider the following PDE

$$\begin{aligned} &u_t - k^2 u_{xx} = F(x,t) && 0 < x < L, \quad t > 0 \\ &u(0,t) = f_1(t), \ u(L,t) = f_2(t) && t > 0 \\ &u(x,0) = f(x) && 0 \le x \le L \end{aligned}$$

Substitute

$$z(x,t) = u(x,t) - \left(1 - \frac{x}{L}\right) f_1(t) - \frac{x}{L} f_2(t)$$

Then we get

$$\begin{aligned} &z_t - k^2 z_{xx} = G(x,t) \\ &z(0,t) = 0, \quad z(L,t) = 0 \\ &z(x,0) = g(x) \end{aligned}$$

To find $u(x,t)$, it is enough to solve for $z(x,t)$.

Let us take the special case when

$$G(x,t) = \theta(t)\sin(\nu_n x), \quad \nu_n = \frac{n\pi}{L}$$

Then take $z(x,t) = Z(t)\sin(\nu_n x)$.

Then $g(x) = z(x,0)$ should be $Z(0)\sin(\nu_n x)$.

$$z_t - z_{xx} = Z'(t)\sin(\nu_n x) + \nu_n^2 \sin(\nu_n x) Z(t) = G(x,t)$$

gives

$$Z'(t) + \nu_n^2 Z(t) = \theta(t)$$

We can find unique solution $Z(t)$ and hence $z(x,t)$. Let us consider the general case when $G(x,t)$ is not in the above form.

Assume $\quad z(x,t) = \sum_{n\geq 1} Z_n(t)\sin(\nu_n x)$

$$z_t - k^2 z_{xx} = \sum_{n\geq 1} \left(Z_n'(t) + k^2\nu_n^2 Z_n(t)\right)\sin(\nu_n x)$$

Fourier sine series, $\quad G(x,t) = \sum_{n\geq 1} G_n(t)\sin(\nu_n x)$

where $\quad G_n(t) = \frac{2}{L}\int_0^L G(x,t)\sin(\nu_n x)\, dx$

$$\boxed{Z_n'(t) + k^2\nu_n^2 Z_n(t) = G_n(t)}$$

$$z(x,0) = g(x) = \sum_{n\geq 1} b_n \sin(\nu_n x) \implies \boxed{Z_n(0) = b_n}.$$

Find the unique solution $Z_n(t)$ and hence $z(x,t)$.

Example. Let us consider the following PDE

$$\begin{aligned} &u_t - u_{xx} = e^t && 0 < x < 1, \quad t > 0 \\ &u(0,t) = 0 = u(1,t) && t > 0 \\ &u(x,0) = x(x-1) && 0 \le x \le 1 \end{aligned}$$

Since boundary conditions $u(0,t) = u(1,t) = 0$ are Dirichlet, we find solution in Fourier sine series.

Fourier sine series : $e^t = \sum_{n \ge 1} \frac{4 \sin(\nu_{2n-1} x)}{\nu_{2n-1}} e^t$.

$$x(x-1) = \sum_{n \ge 1} \frac{-8}{(\nu_{2n-1})^3} \sin(\nu_{2n-1} x)$$

Put $\boxed{u(x,t) = \sum_{n \ge 1} u_n(t) \sin(\nu_n x)}$ into

$u_t - u_{xx} = e^t$,

$$\sum_{n \ge 1} \left[u_n'(t) + \nu_n^2 u_n(t) \right] \sin(\nu_n x) = \sum_{n \ge 1} \frac{4 \sin(\nu_{2n-1} x)}{\nu_{2n-1}} e^t$$

$$u_{2n}'(t) + (\nu_{2n})^2 u_{2n}(t) = 0 \implies u_{2n}(t) = C e^{-(\nu_{2n})^2 t}$$

$$u_{2n-1}'(t) + (\nu_{2n-1})^2 u_{2n-1}(t) = \frac{4}{\nu_{2n-1}} e^t \implies$$

$$u_{2n-1}(t) = e^{-(\nu_{2n-1})^2 t} \left[\int_0^t \frac{4}{\nu_{2n-1}} e^s e^{(\nu_{2n-1})^2 s} ds + C' \right]$$

Use the initial condition $u(x,0) = x(x-1) \implies$

$$\sum_{n \ge 1} u_n(0) \sin(\nu_n x) = \sum_{n \ge 1} \frac{-8 \sin(\nu_{2n-1} x)}{(\nu_{2n-1})^3}$$

we get $u_{2n}(0) = 0, \quad u_{2n-1}(0) = \frac{-8}{(\nu_{2n-1})^3}$

$$u_{2n}(0) = 0 \implies C = 0 \implies u_{2n}(t) = 0$$

$$u_{2n-1}(t) = e^{-(\nu_{2n-1})^2 t}\left[\int_0^t \frac{4}{\nu_{2n-1}} e^s e^{(\nu_{2n-1})^2 s} ds + C'\right]$$

$$u_{2n-1}(0) = \frac{-8}{(\nu_{2n-1})^3} \implies C' = \frac{-8}{(\nu_{2n-1})^3}$$

Thus, the solution is

$$u(x,t) = \sum_{n\geq 1} u_{2n-1}(t)\sin(\nu_{2n-1}x)$$

$$u_{2n-1}(t) = \frac{4(e^t - e^{-(\nu_{2n-1})^2 t})}{\nu_{2n-1}((\nu_{2n-1})^2 + 1)} + \frac{-8e^{-(\nu_{2n-1})^2 t}}{(\nu_{2n-1})^3}$$

Non homogeneous Heat Equation: Neumann boundary condition

Let us now consider the following PDE

$$u_t - k^2 u_{xx} = F(x,t), \qquad 0 < x < L, \quad t > 0$$
$$u_x(0,t) = f_1(t), \; u_x(L,t) = f_2(t), \qquad t > 0$$
$$u(x,0) = f(x), \qquad 0 \leq x \leq L$$

We reduce to zero boundary conditions by making the substitution

$$z(x,t) = u(x,t) - \left(x - \frac{x^2}{2L}\right) f_1(t) - \frac{x^2}{2L} f_2(t)$$

$$z_t - k^2 z_{xx} = G(x,t)$$
$$z_x(0,t) = 0, \; z_x(L,t) = 0$$
$$z(x,0) = g(x)$$

It is clear that we would have solved for $u(x,t)$ if and only if we have solved for $z(x,t)$.
In view of this observation, let us try and solve the problem for $z(x,t)$.

Since boundary conditions are Neumann, we should try and look for a solution of the type

$$z(x,t) = \sum_{n\geq 0} Z_n(t)\cos(\nu_n x), \quad \nu_n = \frac{n\pi}{L}$$

Differentiating the above term by term,

$$z_t - k^2 z_{xx} = \sum_{n\geq 0} \left(Z_n'(t) + k^2\nu_n^2 Z_n(t)\right)\cos(\nu_n x)$$

Let us write $G(x,t) = \sum_{n\geq 0} G_n(t)\cos(\nu_n x)$

We need $z_t - k^2 z_{xx} = G(x,t)$, so

$$Z_n'(t) + k^2\nu_n^2\, Z_n(t) = G_n(t) \qquad (*)$$

$$z(x,0) = g(x) = \sum_{n\geq 0} b_n \cos\nu_n x \implies \boxed{Z_n(0) = b_n}$$

Find the unique solution $Z_n(t)$ and the solution of IBVP is

$$z(x,t) = \sum_{n\geq 0} Z_n(t)\cos(\nu_n x)$$

Example. Consider the following PDE

$$
\begin{aligned}
&u_t - u_{xx} = e^t, && 0 < x < 1, \quad t > 0\\
&u_x(0,t) = 0,\ u_x(1,t) = 0, && t > 0\\
&u(x,0) = x(x-1), && 0 \le x \le 1
\end{aligned}
$$

The boundary conditions $u_x(0,t) = u_x(1,t) = 0$ is Neumann, so we should find solution in Fourier cosine series.

Fourier cosine series of e^t is e^t (1 is an eigen function). Fourier cosine series of $x(x-1)$ is

$$x(x-1) = -\frac{1}{6} + \sum_{n\ge1} \frac{4\cos(\nu_{2n}x)}{(\nu_{2n})^2}$$

Put $\boxed{u(x,t) = \sum_{n\ge0} u_n(t)\cos(\nu_n x)}$ into

$$\sum_{n\ge0} \left(u_n'(t) + (\nu_n)^2 u_n(t)\right)\cos(\nu_n x) = e^t$$

For $n = 0$, $\quad u_0'(t) = e^t \implies u_0(t) = e^t + C_0$

$u_n'(t) + (\nu_n)^2 u_n(t) = 0 \implies u_n(t) = C_n e^{-(\nu_n)^2 t}$, for $n \ge 1$.

$$u(x,0) = x(x-1) = -\frac{1}{6} + \sum_{n\ge1} \frac{4\cos(\nu_{2n}x)}{(\nu_{2n})^2}$$

gives $u_0(0) = \dfrac{-1}{6}$, $u_{2n-1}(0) = 0$, $u_{2n}(0) = \dfrac{4}{(\nu_{2n})^2}$.

Thus, $C_0 = -\dfrac{7}{6}$, $\ C_{2n-1} = 0$, $\ C_{2n} = \dfrac{4}{(\nu_{2n})^2}$.

$$u(x,t) = e^t - \frac{7}{6} + \sum_{n\ge1} \left(\frac{2}{(\nu_{2n})^2} e^{-(\nu_{2n})^2 t}\right)\cos(\nu_{2n}x)$$

One-dimensional wave equation

Consider a flexible string of length L which is tightly stretched along the x-axis so that its end points are at $x = 0$ and $x = L$.
Assume that the tension force on the string is the dominant force and all other forces acting on the string are negligible. For example, damping force and mass of the string is negligible.
Assume that each point on the string moves only vertically.
Let $u(x,t)$ denote the vertical displacement at time t of the point x on the string.
Then the mathematical model describing the motion of string is

$$u_{tt} = a^2 u_{xx}, \qquad 0 < x < L,\ t > 0,$$

called the one-dimensional wave equation.

Here a is a positive constant called the wave's speed. We can interpret a as the speed at which the crest of the wave travels horizontly.
Some solutions of wave equation are

$u_1(x,t) = \sin(a\omega_1 t)\sin(\omega_1 x)$ and

$u_2(x,t) = \cos(a\omega_2 t)\sin(\omega_2 x)$

where ω_1, ω_2 are arbitrary real constants.

We can interchange any $\sin$ with $\cos$ and obtain again a solution. We can also take finite linear combination of solutions and obtain a solution.

Initial boundary value problem for Wave equation

Example 1. A flexible string of length L is stretched horizonty so that one end is at $x = 0$ and the other at $x = L$.

Both ends are fixed.
The string is moved vertically from equilibrium so that the point x on the string is displaced by $f(x)$.

Hence the shape of the string is the graph of $f(x)$.

The string is then released from rest,
i.e. each point of the string has 0 velocity at time $t = 0$ (when the string is released).

The displacement function $u(x,t)$ on the string satisfies the following IBVP

Wave equation
$u_{tt}(x,t) = a^2 u_{xx}(x,t), \quad 0 < x < L, \ t > 0;$

Initial condition
$u(x,0) = f(x), \ \ u_t(x,0) = 0, \quad 0 < x < L;$

Boundary condition
$u(0,t) = 0, \ \ u(L,t) = 0, \quad t > 0.$

Example 2. Start with the string of Example 1 at equilibrium (sitting horizontly and end points fixed).

Suppose that at time $t = 0$, each point of the string is given a velocity $g(x)$.

Then displacement $u(x,t)$ satisfies the following IBVP:

$$u_{tt}(x,t) = a^2 u_{xx}(x,t), \quad 0 < x < L, \ t > 0;$$

$$u(x,0) = 0, \ \ u_t(x,0) = g(x), \quad 0 < x < L;$$

$$u(0,t) = 0, \ \ u(L,t) = 0, \quad t > 0.$$

Example 3. The combination of Example 1 and Example 2.

i.e. the string is initially moved from the equilibrium position by $f(x)$ and each point of the string is given an initial velocity $g(x)$.

Then displacement $u(x,t)$ satisfies the following IBVP:

$$u_{tt}(x,t) = a^2 u_{xx}(x,t), \quad 0 < x < L, \ t > 0;$$

$$u(x,0) = f(x), \ \ u_t(x,0) = g(x), \quad 0 < x < L;$$

$$u(0,t) = 0, \ \ u(L,t) = 0, \quad t > 0.$$

Example 4. Suppose that ends of the string are not fixed but is allowed to move vertically in such a way that the shape of the string always remains horizontal at both end points.

This can be realised by attaching the ends of the string to a mechanical apparatus containing bearing or springs.

Now $u(0, t)$ and $u(L, t)$ are not zero.

If the string is moved initially from the equilibrium position by $f(x)$ and each point of the string is given an initial velocity $g(x)$.

Then displacement $u(x, t)$ satisfies the following IBVP:

Wave equation
$u_{tt}(x, t) = a^2 u_{xx}(x, t), \quad 0 < x < L, \ t > 0;$

Initial condition
$u(x, 0) = f(x), \ \ u_t(x, 0) = g(x), \quad 0 < x < L;$

Boundary condition
$u_x(0, t) = 0, \ \ u_x(L, t) = 0, \quad t > 0.$

Solving Wave equation $u_{tt} = a^2 u_{xx}$

We will use the method of separation of variables.

Assume $\boxed{u(x,t) = X(x)\,T(t)}$. Substituting this in wave equation $u_{tt} = a^2 u_{xx}$, we get

$$X(x)T''(x) = a^2 X''(x)T(t).$$

Separate the variables: $\dfrac{X''(x)}{X(x)} = \dfrac{T''(t)}{a^2 T(t)}$

The equality is between a function of x and a function of t, so both must be constant, say $-\lambda$.

We need to solve

$\boxed{X''(x) + \lambda X(x) = 0}$ and $\boxed{T''(t) + a^2 \lambda T(t) = 0}$.

Dirichlet boundary conditions $u(0,t) = u(L,t) = 0$

Initial boundary value problem is

$$\begin{aligned} &u_{tt}(x,t) = a^2 u_{xx}(x,t), \quad 0 < x < L,\ t > 0; \\ &u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0; \\ &u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L; \end{aligned}$$

Assuming the solution is $u(x,t) = X(x)T(t)$,

$$u(0,t) = X(0)T(t) = 0 = u(L,t) = X(L)T(t)$$

Since we don't want T to be identically zero, we get

$$X(0) = 0 \quad \text{and} \quad X(L) = 0.$$

We need to solve the eigenvalue problem

$$\boxed{X''(x) + \lambda X(x) = 0, \quad X(0) = X(L) = 0}$$

The eigenvalues are $\lambda_n = \nu_n^2$ with $\nu_n = \dfrac{n\pi}{L}$, $n \geq 1$ with associated eigenfunctions $X_n(x) = \sin \nu_n x$. For each λ_n, we consider the equation

$$T''(t) + a^2 \lambda T(t) = 0$$

For each λ_n, we get a solution for T given by

$$T_n(t) = A_n \cos(a\nu_n t) + \frac{B_n}{a\nu_n} \sin(a\nu_n t)$$

where A_n and B_n are real numbers.

Thus, we get a solution for each $n \geq 1$

$$u_n(x,t) = \left[A_n \cos(a\nu_n t) + \frac{B_n}{a\nu_n} \sin(a\nu_n t) \right] \sin \nu_n x$$

Therefore, the function

$$u_n(x,t) = \left[A_n \cos(a\nu_n t) + \frac{B_n}{a\nu_n} \sin(a\nu_n t) \right] \sin \nu_n x$$

satisfies the wave equation

$$u_{tt}(x,t) = a^2 u_{xx}(x,t), \qquad 0 < x < L, \ t > 0;$$
$$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0;$$

with initial conditions

$$u(x,0) = A_n \sin \nu_n x, \quad u_t(x,0) = B_n \sin \nu_n x$$

By Principle of superposition, any linear combination of these solutions is again is a solution. Therefore, we get the following result.

Theorem. Let $\nu_n = \frac{n\pi}{L}$. Then $u(x,t) =$

$$\sum_{n\geq 1}\left(A_n\cos(a\nu_n t) + \frac{B_n}{a\nu_n}\sin(a\nu_n t)\right)\sin\nu_n x$$

is a formal solution of Dirichlet IBVP

$$u_{tt}(x,t) = a^2 u_{xx}(x,t), \quad 0 < x < L,\ t > 0;$$
$$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0; \qquad \text{with}$$

$$u(x,0) = f(x) = \sum_{n\geq 1} A_n \sin\nu_n x$$

$$u_t(x,0) = g(x) = \sum_{n\geq 1} B_n \sin\nu_n x$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\nu_n x\,dx,\ B_n = \frac{2}{L}\int_0^L g(x)\sin\nu_n x dx$$

Dirichlet boundary conditions: Actual solution

Let f and g be continuous and piecewise smooth functions on $[0,L]$ such that $f(0) = f(L) = 0$.

Then the series defining $u(x,t)$ is uniformly convergent and $u(x,t)$ is a continuous function for $t \geq 0$ and $0 \leq x \leq L$.

Moreover, if f, f', f'', g, g' are continuous functions on $[0,L]$, then the function $u(x,t)$ defined by the infinite series is twice differential in (x,t) and term by term differentiation in the series are valid.

This gives $u(x,t)$ as the unique actual solution of IBVP.

Example. Consider the wave equation with IBVP

$$\begin{array}{ll} u_{tt} = 5u_{xx} & 0 < x < 1, \quad t > 0 \\ u(0,t) = u(L,t) = 0 & t > 0 \\ u(x,0) = \sin \pi x + 3 \sin 5\pi x & 0 \le x \le 1 \\ u_t(x,0) = \sin 5\pi x - 26 \sin 9\pi x & 0 \le x \le 1 \end{array}$$

Both f and g are given by their Fourier sine series,

$A_1 = 1, \; A_5 = 3, \; B_5 = 1, \; B_9 = -26$

Thus with $\boxed{\nu_n = n\pi}$, the solution is

$$u(x,t) = \cos(\sqrt{5}\nu_1 t)\sin(\nu_1 x)$$

$$+ \left(3\cos(\sqrt{5}\nu_5 t) + \frac{1}{\sqrt{5}\nu_5}\sin(\sqrt{5}\nu_5 t)\right)\sin(\nu_5 x)$$

$$+ \frac{-26}{\sqrt{5}\nu_9}\sin(\sqrt{5}\nu_9 t)\sin(\nu_9 x)$$

Neumann boundary conditions

$u_x(0,t) = u_x(L,t) = 0$

Initial boundary value problem is

$$\begin{array}{l} u_{tt}(x,t) = a^2 u_{xx}(x,t), \qquad 0 < x < L, \; t > 0; \\ u_x(0,t) = 0, \quad u_x(L,t) = 0, \quad t > 0; \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L; \end{array}$$

Assuming the solution is $u(x,t) = X(x)T(t)$,

$$u_x(0,t) = X'(0)T(t) = 0 = u_x(L,t) = X'(L)T(t)$$

Since we don't want T to be identically zero, we get

$$X'(0) = 0 \quad \text{and} \quad X'(L) = 0.$$

We need to solve the eigenvalue problem

$$\boxed{X''(x) + \lambda X(x) = 0, \quad X'(0) = X'(L) = 0}$$

The eigen values are $\lambda_n = \nu_n^2$, $\nu_n = \frac{n\pi}{L}$, $n \geq 0$ with associated eigenfunctions $X_n(x) = \cos \nu_n x$. For each λ_n, we consider the equation

$$T''(t) + a^2 \lambda T(t) = 0$$

So $T_0(t) = (A_0 + B_0 t)$. For each $n \geq 1$,

$$T_n(t) = A_n \cos(a\nu_n t) + \frac{B_n}{a\nu_n} \sin(a\nu_n t)$$

where A_n and B_n are real numbers.

Thus, $u_0(x,t) = (A_0 + B_0 t)$ and for each $n \geq 1$

$$u_n(x,t) = \left[A_n \cos(a\nu_n t) + \frac{B_n}{a\nu_n} \sin(a\nu_n t) \right] \cos \nu_n x$$

Therefore, the function $u_0(x,t) = (A_0 + B_0 t)$ and

$$u_n(x,t) = \left[A_n \cos(a\nu_n t) + \frac{B_n}{a\nu_n} \sin(a\nu_n t) \right] \cos \nu_n x$$

satisfy the wave equation

$u_{tt}(x,t) = a^2 u_{xx}(x,t), \quad 0 < x < L, \; t > 0;$
$u_x(0,t) = 0, \quad u_x(L,t) = 0, \quad t > 0;$

with initial conditions $u(x,0) = A_0, \; u_x(x,0) = B_0$ when $n = 0$ and for $n \geq 1$

$$u(x,0) = A_n \cos \nu_n x, \quad u_t(x,0) = B_n \cos \nu_n x$$

By Principle of superposition, any linear combination of these solutions is again is a solution. Therefore, we get the following result.

Theorem. Let $\nu_n = \dfrac{n\pi}{L}$. Then $u(x,t) = (A_0 + B_0 t)$

$$+\sum_{n\geq 1}\left(A_n\cos(a\nu_n t) + \frac{B_n}{a\nu_n}\sin(a\nu_n t)\right)\cos\nu_n x$$

is a formal solution of Neumann IBVP

$$u_{tt}(x,t) = a^2 u_{xx}(x,t), \qquad 0 < x < L,\ t > 0;$$
$$u_x(0,t) = 0, \quad u_x(L,t) = 0, \quad t > 0; \qquad \text{with}$$

$$u(x,0) = f(x) = A_0 + \sum_{n\geq 1} A_n \cos\nu_n x$$

$$u_t(x,0) = g(x) = B_0 + \sum_{n\geq 1} B_n\ \cos\nu_n x$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\nu_n\,dx,\ B_n = \frac{2}{L}\int_0^L g(x)\cos\nu_n x\,dx$$

Neumann boundary conditions: Actual solution

Let f' and g be continuous and piecewise smooth functions on $[0,L]$ such that $f'(0) = f'(L) = 0$.

Then the series defining $u(x,t)$ is uniformly convergent and $u(x,t)$ is a continuous function for $t \geq 0$ and $0 \leq x \leq L$.

Moreover, if f', f'', g' are continuous functions on $[0,L]$, then the function $u(x,t)$ defined by the infinite series is twice differential in (x,t) and term by term differentiation in the series are valid.

This gives $u(x,t)$ as the unique actual solution of IBVP.

Example. Consider the wave equation with IBVP

$$\begin{aligned}
&u_{tt} = 5u_{xx}, && 0 < x < 1, \quad t > 0 \\
&u_x(0,t) = u_x(L,t) = 0, && t > 0 \\
&u(x,0) = 34 + \cos \pi x + 3 \cos 5\pi x, && 0 \le x \le 1 \\
&u_t(x,0) = 23 + \cos 5\pi x - 26 \cos 9\pi x, && 0 \le x \le 1
\end{aligned}$$

Both f and g are given by their Fourier cosine series

$$A_0 = 34,\ A_1 = 1,\ A_5 = 3,\ B_0 = 23,\ B_5 = 1,\ B_9 = -26$$

Thus, the solution is given by (here $\nu_n = n\pi$)

$$\begin{aligned}
u(x,t) &= 34 + 23t + \cos(\sqrt{5}\nu_1 t)\cos(\nu_1 x) \\
&\quad + \left(3\cos(\sqrt{5}\nu_5 t) + \frac{1}{\sqrt{5}\nu_5}\sin(\sqrt{5}\nu_5 t)\right)\cos(\nu_5 x) \\
&\quad \frac{-26}{\sqrt{5}\nu_9}\sin(\sqrt{5}\nu_9 t)\cos(\nu_9 x)
\end{aligned}$$