Tutorial-5

1. Locate & classify the singularities of :

(i) $\left(\sin \frac{1}{z}\right)\left(\frac{1}{1+z^4}\right)$ (ii) $z^5\left(\sin \frac{1}{z}\right)\left(\frac{1}{1+z^4}\right)$

(iii) $\sin(wz) \cdot \sin\left(\frac{1}{z}\right)\left(\frac{1}{1+z^4}\right)$: where $w = \exp\left(\frac{i\pi}{4}\right)$

(iv) $(\log z)\left(\sin(wz)\ \sin\left(\frac{1}{z}\right)\right)\left(\frac{1}{1+z^4}\right)$

(v) $z^5 \log z \left(\sin(wz)\ \sin\left(\frac{1}{z}\right)\right)\left(\frac{1}{1+z^4}\right)$

2. Determine the residue at each singularity of the foll:

(i) $\operatorname{cosec} z \cdot \operatorname{cosech} z$ (ii) $\dfrac{\operatorname{cosec} z \cdot \operatorname{cosech} z}{z^3}$

3. Is $\dfrac{\cos(\pi\sqrt{z})-1}{z^3-z}$ analytic? Determine the residue at each of its poles otherwise.

4. Show that if $a_1, \dots, a_n$ are the distinct roots of a monic polynomial $P(z)$ of degree n. for each k we have the formula:

$$\prod_{j \neq k} (a_k - a_j) = P'(a_k)$$

5. Evaluate residues for the following.

(i) Show that if θ is a root of $z^4 + a^4 = 0$,

$$\text{Res}\left(\frac{1}{z^4+a^4}; \theta\right) = -\frac{\theta}{4a^4}$$

(ii) Generalise (i) to the case $\dfrac{1}{z^n+a^n}$: $a>0$.

(iii) Show that the residues of $\dfrac{z^{n-1}}{z^n+a^n}$ at all the poles are equal to $\dfrac{1}{n}$.

(iv) Show that $\text{Res}\left(\frac{1}{(1+z^2)^n}; i\right) = \frac{-i}{2^{2n-1}} \binom{2n-2}{n-1}$

(v) Show that $\operatorname{Res}\left(\dfrac{z^{2n}}{(1+z)^n} ; -1\right) = \dfrac{(-1)^{n-1}(2n)!}{(n-1)!\,(n+1)!}$

(vi) Show that the singularities of $\dfrac{\sqrt{z}}{\sin\sqrt{z}}$

are all poles at $z = n^2 z^2$, where n is a

positive integer & the residue there is $(-1)^n 2\pi^2 n^2$.

(vii) Show that $\operatorname{Res}\left(\dfrac{1}{(1+z)^2\cosh(\pi z/2)} ; i\right) = \dfrac{1}{2\pi i}$

(viii) $\operatorname{Res}\left(\dfrac{\exp(a\log z)}{(1+z^2)^2} ; i\right) = \dfrac{1-a}{4i}\exp\left(\dfrac{1}{2}a\pi i\right)$

; $a \in \mathbb{R}$

(ix) Supp f is analytic in a neighbourhood of

the real axis & $z_k = \left(k+\frac{1}{2}\right)\pi$ then

show that $\operatorname{Res}\left(\dfrac{f(z)}{\cos^2 z} ; z_k\right) = f'(z_k)$

6. Discuss the singularities of the function

$z^{-1} \log(1-az)$.

7. By transforming into an integral over the unit circle;

(i) Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$: if $a>b>0$.

(ii) Show that $\int_0^{2\pi} \frac{d\theta}{a^2+1-2a\cos\theta} = \frac{2\pi}{1-a^2}$, for $a>1$.

What is the value of the integral for $a<1$?

8. Prove that $\int_0^{2\pi} \frac{d\theta}{1-ae^{i\theta}} = 0$ or 2π

according as $|a|<1$ or $|a|>1$.

9. Show that, for $n \in \mathbb{N}$,

$$\int_0^{\pi} \frac{\sin(nx)}{\sin x}\, dx = \frac{\pi}{2}\left(1+(-1)^n\right)$$

10. By transforming into an integral over the unit circle, show that

$$\int_0^{2\pi} \frac{d\theta}{a^2\cos^2\theta + b^2\sin^2\theta} = \frac{2\pi}{ab}.$$

11. Do the exercise 10. above by integrating $\oint \frac{dz}{z}$ over a suitable ellipse.