Partial Differential Equations

Contents

CHAPTER 1

Tutorial Problems

1.1. Power series and series solutions

Problems.

(1) Find the radius of convergence of the following power series:

SOLUTION. Method used: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{R}$

(a) $\sum x^n$ SOLUTION. $R = 1$

(b) $\sum \frac{x^m}{m!}$ SOLUTION. $R = \infty$

(c) $\sum m!x^m$ SOLUTION. $R = 0$

(d) $\sum_{m=k}^{\infty} m(m-1)\cdots(m-k+1)x^m$ SOLUTION. $R = 1$

(e) $\sum \frac{(2n)!}{2^{2n}(n!)^2}x^n$ SOLUTION. $R = 1$

(f) $\sum_1^{\infty} \frac{x^m}{m(m+1)\cdots(m+k+1)}$ SOLUTION. $R = 1$

(g) $\sum_1^{\infty} \frac{n^n}{n!}x^n$ SOLUTION. $R = e^{-1}$

(h) $\sum_1^{\infty} \frac{(2n)!}{n^n}x^n$ SOLUTION. $R = 0$

(i) $\sum_1^{\infty} \frac{(3n)!}{2^n(n!)^3}x^n$ SOLUTION. $R = 2/27$

(2) Determine the radius of convergence of

$$\sum n!x^{n^2}, \quad \sum x^{n!} \quad \text{and} \quad \sum_{p \text{ prime}} x^p.$$

SOLUTION. (i) Let $a_n = n!x^{n^2}$. Then $\left|\frac{a_{n+1}}{a_n}\right| = (n+1)|x|^{2n+1}$. Hence

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} 0 & \text{if } |x| < 1, \\ \infty & \text{if } |x| \geq 1. \end{cases}$$

Therefore, convergence if $|x| < 1$ and divergence otherwise. Hence $R = 1$.
(ii) Let $b_n = x^{n!}$. Then

$$\lim_{n\to\infty} |\frac{b_{n+1}}{b_n}| = \lim_{n\to\infty} |x|^{n!n} = \begin{cases} 0 & \text{if } |x| < 1 \\ 1 & \text{if } |x| = 1 \\ \infty & \text{if } |x| > 1. \end{cases}$$

(iii) Let p_n denote the n^{th} prime number. Then the series is $\sum x^{p_n}$. Let $c_n = x^{p_n}$. Then

$$|\frac{c_{n+1}}{c_n}| = |x|^{p_{n+1}-p_n}.$$

Since $p_{n+1} - p_n \geq 2$, $|x|^{p_{n+1}-p_n}$ is greater, less or equal to $|x|$, according as $|x|$ is greater, less or equal to 1. Therefore, $\limsup |x|^{p_{n+1}-p_n}$ is strictly smaller than 1 if $|x| < 1$, $\liminf |x|^{p_{n+1}-p_n}$ is strictly greater than 1 if $|x| > 1$. Hence, by the refined ratio test, the series converges if $|x| < 1$ and diverges if $|x| > 1$, so $R = 1$. Note the series diverges at $|x| = 1$ due to terms not converging to 0.
Aliter: Since $p_n > n$, $|x|^{p_n}$ is greater, equal or less than $|x|^n$ according as $|x|$ is greater, equal or less than 1. By comparison test against $\sum |x|^n$, $\sum x^{p_n}$ converges if $|x| < 1$ and diverges if $|x| \geq 1$. Similar *Aliter* works for (ii).

(3) Show that if $\sum_{n=1}^{\infty} a_n x^n$ has radius of convergence R, then $\sum_{n=1}^{\infty} a_n x^{2n}$ has radius of convergence $\sqrt{R}$ and $\sum_{n=1}^{\infty} a_n^2 x^n$ has radius of convergence R^2.

Solution. (i) Let $x^2 = z$. Then $\sum a_n x^{2n} = \sum a_n z^n$ converges for $|z| < R$ and diverges for $|z| > R$. Equivalently, $\sum a_n x^{2n}$ converges for $|x| < \sqrt{R}$ and diverges for $|x| > \sqrt{R}$. Hence the radius of convergence is $\sqrt{R}$.
(ii) We know that $\limsup |a_n|^{1/n} = 1/R$. So $\limsup |a_n^2|^{1/n} = 1/R^2$. Hence the radius of convergence is R^2.

(4) Apply the power series method around $x = 0$ to solve the following differential equations. What step recursion do you get in each case?
(a) $(1 - x^2)y' = y$

Solution. Let $y = \sum a_n x^n$. Substitution yields $a_0 = a_1$ and

$$(n + 1)a_{n+1} = (n - 1)a_{n-1} + a_n, \ n \geq 1.$$

(This is a 3-step recursion.) By induction on k, one can show that

$$a_{2k} = a_{2k+1} \quad \text{and} \quad 2ka_{2k} = (2k - 1)a_{2k-2}.$$

Now

$$a_{2k} = \frac{2k - 1}{2k} a_{2k-2} = \cdots = \frac{(2k)!}{(2^k k!)^2} a_0.$$

Combining with $a_{2k+1} = a_{2k}$,

$$y = a_0 \sum_{k=0}^{\infty} \frac{(2k)!}{(2^k k!)^2} (x^{2k} + x^{2k+1}) = a_0(x + 1) \sum \frac{(2k)! x^{2k}}{(2^k k!)^2}.$$

This can be written in closed form as follows.

$$y = a_0\sqrt{\frac{1+x}{1-x}} = a_0(1+x)(1-x^2)^{-1/2}.$$

(b) $y' = xy,\ y(0) = 1$

SOLUTION. Let $y = \sum a_n x^n$. Then

$$(n+1)a_{n+1} = a_{n-1}, \quad n \geq 0,\ a_{-1} = 0,$$

which is a 3-step recursion. The initial condition $y(0) = 1$ implies $a_0 = 1$. Since $a_{-1} = 0$, we have $a_{od} = 0$, and for the even coefficients

$$a_{2n} = \frac{a_0}{2^n n!} = \frac{1}{2^n n!}.$$

Therefore,

$$y = \sum \frac{x^{2n}}{2^n n!} = e^{x^2/2}.$$

(c) $(1-x^2)y' = 2xy$

SOLUTION. Let $y = \sum a_n x^n$. Then $a_1 = 0$ and the recursion is $a_{n+1} = a_{n-1}$. (This is a 3-step recursion.) Hence $a_{od} = 0$ and $a_{2n} = a_0$ and

$$y = a_0 \sum x^{2n} = \frac{a_0}{1-x^2}.$$

(d) $y' - 2xy = 1,\ y(0) = 0$. Use the solution to deduce the Taylor series for $e^{x^2}\int_0^x e^{-t^2}\,dt$.

SOLUTION. Let $y = \sum a_n x^n$. The initial condition $y(0) = 0$ implies $a_0 = 0$. Further $a_1 - 2a_0 = 1$ which implies $a_1 = 1$. The general recursion is

$$(n+1)a_{n+1} = 2a_{n-1}, \quad n \geq 1.$$

(This is a 3-step recursion.) Hence $a_{2n} = 0$ and

$$a_{2n+1} = \frac{2a_{2n-1}}{2n+1} = \cdots = \frac{2^n a_1}{(2n+1)(2n-1)\dots 3} = \frac{2^{2n} n!}{(2n+1)!}$$

since $a_1 = 1$. Hence

$$y = \sum \frac{2^{2n} n! x^{2n+1}}{(2n+1)!}.$$

Using integrating factor e^{-x^2}, the differential equation can be written in an exact form to yield the solution

$$y = e^{x^2}\int_0^x e^{-t^2}\,dt.$$

By uniqueness of solutions, we conclude that the above power series is the Taylor series of this function.

(5) Find the power series solutions for the following differential equations around $x = 1$, that is in powers of $(x-1)$.

(a) $y'' + y = 0$

SOLUTION. Let $y = \sum a_n x^n$. Then

$$n(n-1)a_n + a_{n-2} = 0, \quad n \geq 2$$

with a_0, a_1 arbitrary. This gives

$$y = a_0 \sum \frac{(-1)^k (x-1)^{2k}}{(2k)!} + a_1 \sum \frac{(-1)^k (x-1)^{2k+1}}{(2k+1)!}$$
$$= a_0 \cos(x-1) + a_1 \sin(x-1).$$

(b) $y'' - y = 0$

SOLUTION. Let $y = \sum a_n x^n$. Then

$$n(n-1)a_n - a_{n-2} = 0, \quad n \geq 2$$

with a_0, a_1 arbitrary. This gives

$$y = a_0 \sum \frac{(x-1)^{2k}}{(2k)!} + a_1 \sum \frac{(x-1)^{2k+1}}{(2k+1)!} = a_0 \cosh(x-1) + a_1 \sinh(x-1).$$

(6) Find the power series solutions for the following differential equations around $x = 0$. What step recursion do you get in each case?

(a) Legendre equation:

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0.$$

When do we have polynomial solutions?

SOLUTION. Let $y = \sum a_n x^n$. Then

$$a_{n+2} = \frac{(n-p)(n+p+1)}{(n+2)(n+1)} a_n, \quad n \geq 0,$$

which is a 3-step recursion. This implies a_0 and a_1 are arbitrary. Further,

$$a_2 = -\frac{p(p+1)}{2!} a_0, \; a_4 = +\frac{[(p(p-2)][(p+1)(p+3)]}{4!} a_0,$$
$$a_6 = -\frac{[p(p-2)(p-4)][(p+1)(p+3)(p+5)]}{6!} a_0, \ldots$$

and

$$a_3 = -\frac{(p-1)(p+2)}{3!} a_1, \; a_5 = +\frac{[(p-1)(p-3)][(p+2)(p+4)]}{5!} a_1,$$
$$a_7 = -\frac{[(p-1)(p-3)(p-5)][(p+2)(p+4)(p+6)]}{7!} a_1, \ldots$$

Write $y(x) = a_0 y_0(x) + a_1 y_1(x)$ where the notation is self-explanatory, y_0 is an even function while y_1 is an odd function.

We have polynomial solutions if and only if p is an integer: The coefficients of the power series are zero only for integer values of p, so this is a necessary condition to have a polynomial solution. It is also sufficient since in this case either the series y_0 or the series y_1 terminates,

(b) Tchebychev equation:

$$(1-x^2)y'' - xy' + p^2y = 0.$$

When do we have polynomial solutions?

SOLUTION. Let $y = \sum a_n x^n$. Then

$$a_{n+2} = \frac{n^2 - p^2}{(n+2)(n+1)} a_n, \quad n \geq 0,$$

which is a 3-step recursion. This implies a_0 and a_1 are arbitrary. Explicitly,

$$a_2 = -\frac{p^2}{2!}a_0,\ a_4 = +\frac{p^2(p^2-2^2)}{4!}a_0,\ a_6 = -\frac{p^2(p^2-2^2)(p^2-4^2)}{6!}a_0, \dots$$

and

$$a_3 = -\frac{p^2-1^2}{3!}a_1,\ a_5 = \frac{(p^2-1^2)(p^2-3^2)}{5!}a_1,\ a_7 = -\frac{(p^2-1^2)(p^2-3^2)}{7!}a_1, \dots$$

Write $y = a_0y_0 + a_1y_1$, with

$$y_0(x) = 1 - \frac{p^2}{2!}x^2 + \frac{p^2(p^2-2^2)}{4!}x^4 - \dots$$

and

$$y_1(x) = x - \frac{p^2-1^2}{3!}x^3 + \frac{(p^2-1^2)(p^2-3^2)}{5!}x^5 - \dots.$$

We have polynomial solutions if and only if p is an integer. (Suppose p is an integer. Then either the series y_0 or the series y_1 terminates, according as p is even or odd. Accordingly, on setting either $a_1 = 0$ or $a_0 = 0$, we get a polynomial solution of degree p.)

(c) Airy equation:

$$y'' - xy = 0.$$

SOLUTION. Let $y = \sum a_n x^n$. Then

$$a_{n+2} = \frac{a_{n-1}}{(n+1)(n+2)}, \quad n \geq 0,$$

which is a 4-step recursion. This implies a_0 and a_1 are arbitrary. Further, since $a_{-1} = 0$, $a_2 = a_5 = \dots = a_{3n-1} = \dots = 0$. The remaining coefficients are

$$a_{3n} = \frac{1.4.7\dots(3n-2)}{(3n)!}a_0 \quad \text{and} \quad a_{3n+1} = \frac{2.5.8\dots(3n-1)}{(3n+1)!}a_1.$$

Hence

$$y(x) = a_0\left[1 + \frac{1}{3!}x^3 + \frac{1.4}{6!}x^6 + \frac{1.4.7}{9!}x^9 + \dots\right] + a_1\left[x + \frac{2}{4!}x^4 + \frac{2.5}{7!}x^7 + \frac{2.5.8}{10!}x^{10} + \dots\right].$$

Note:

$$a_n = \frac{(n-2)(n-5)\dots(0 \textit{ or } 1 \textit{ or } 2)a_{-1 \textit{ or } 0 \textit{ or } 1}}{n!}$$

according as $n \equiv (-1 \textit{ or } 0 \textit{ or } 1) \bmod 3$.

(d) Hermite equation :

$$y'' - x^2 y = 0.$$

SOLUTION. Let $y = \sum a_n x^n$. Then

$$a_{n+2} = \frac{a_{n-2}}{(n+1)(n+2)}, \quad n \geq 0$$

which is a 5-step recursion. This implies a_0 and a_1 are arbitrary. Further, since $a_{-1} = 0 = a_{-2}$, $a_n = 0$ for $n \equiv 2, 3 \bmod 4$. The remaining coefficients are

$$a_{4n} = \frac{[1.5.9 \ldots (4n-3)][2.6.10 \ldots (4n-2)]}{(4n)!} a_0$$

and

$$a_{4n+1} = \frac{[2.6.10 \ldots (4n-2)][3.7.11 \ldots (4n-1)]}{(4n+1)!} a_1.$$

(7) Show that the function $(\sin^{-1} x)^2$ satisfies the initial value problem (IVP):

$$(1-x^2)y'' - xy' = 2, \quad y(0) = 0,\ y'(0) = 0.$$

Hence find the Taylor series for $(\sin^{-1} x)^2$ around 0. What is its radius of convergence ?

SOLUTION. Direct substitution gives the first part. To fing the Taylor series, let us apply the power series method. Accordingly let $y = \sum a_n x^n$ be a solution of the IVP. Then $a_0 = a_1 = 0$ due to the initial conditions, and $a_2 = 1$ and

$$a_{n+2} = \frac{n^2 a_n}{(n+1)(n+2)}, \quad n \geq 1.$$

This implies $a_{od} = 0$ and

$$a_{2n} = \frac{2^2.4^2 \ldots (2n-2)^2}{(2n)!} a_2 = \frac{2^{2n-1}((n-1)!)^2}{(2n)!}$$

on substituting $a_2 = 2$. For the radius of convergence, let $a_{2n} = b_n$ and $x^2 = z$. The radius of convergence of $\sum b_n z^n$ is

$$\lim \frac{b_n}{b_{n+1}} = \lim \frac{(2n+2)(2n+1)}{4n^2} = 1.$$

Hence radius of convergence is unity for both the series since $|z| < 1$ is equivalent to $|x| < 1$.

(8) Show that the even and odd parts of the binomial series of $(1-x)^{-m}$ are two linearly independent power series solutions of

$$(1-x^2)y'' - 2(m+1)xy' - m(m+1)y = 0$$

around $x = 0$. Hence deduce that $\{(1-x)^{-m}, (1+x)^{-m}\}$ is another linearly independent set of solutions.

SOLUTION. Let $y = \sum_{n\geq 0} a_n x^n$ be a power series solution. Substitution in the equation gives the recursion

$$a_{n+2} = \frac{(m+n+1)(m+n)}{(n+2)(n+1)} a_n, \quad n \geq 0,$$

with a_0, a_1 arbitrary. For $n \geq 2$,

$$\begin{aligned}
a_n &= \frac{(m+n-1)(m+n-2)}{n(n-1)} a_{n-2} \\
&= \frac{(m+n-1)(m+n-2)(m+n-3)(m+n-4)}{n(n-1)(n-2)(n-3)} a_{n-4} = \dots \\
&= \begin{cases} \dfrac{(m+n-1)(m+n-2)\dots(m+1)m}{n!} a_0 & \text{if } n \text{ is even,} \\ \dfrac{(m+n-1)(m+n-2)\dots(m+2)(m+1)}{n!} a_1 & \text{if } n \text{ is odd.} \end{cases} \\
&= \begin{cases} \dbinom{m+n-1}{n} a_0 & \text{if } n \text{ is even} \\ \dbinom{m+n-1}{n} \dfrac{a_1}{m} & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

Replace a_1/m by a new constant a_1 to conclude that

$$a_n = \begin{cases} \dbinom{m+n-1}{n} a_0 & \text{if } n \text{ is even,} \\ \dbinom{m+n-1}{n} a_1 & \text{if } n \text{ is odd.} \end{cases}$$

The general solution therefore, is

$$y(x) = a_0 \sum_{n \text{ even}} \binom{m+n-1}{n} x^n + a_1 \sum_{n \text{ odd}} \binom{m+n-1}{n} x^n.$$

The n-th coefficient of $(1-x)^{-m}$ equals

$$(-1)^n \frac{-m(-m-1)\dots(-m-n+1)}{n!} = \frac{m(m+1)\dots(m+n-1)}{n!} = \binom{m+n-1}{n}.$$

This proves the first part. Setting $a_0 = 1 = a_1$, we get $(1-x)^{-m}$ as a solution, while on letting $a_0 = 1 = -a_1$, we get $(1+x)^{-m}$ as another independent solution. This proves the last statement.

1.2. Legendre equation and Legendre polynomials

Problems.

(1) Express x^2, x^3, and x^4 as a linear combination of the Legendre polynomials. (This is possible since the Legendre polynomials form a basis for the vector space of polynomials.)

SOLUTION. We first express x^2 and x^4 using the Legendre polynomials of even degree. Since $P_0 = 1$ and $P_2 = \frac{3}{2}x^2 - \frac{1}{2}$,

$$x^2 = \frac{2}{3}P_2 + \frac{1}{3}P_0.$$

Substituting this,

$$P_4 = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8} = \frac{35}{8}x^4 - \frac{15}{4}(\frac{2}{3}P_2 + \frac{1}{3}P_0) + \frac{3}{8}P_0 = \frac{35}{8}x^4 - \frac{5}{2}P_2 - \frac{7}{8}P_0.$$

Therefore

$$x^4 = \frac{8}{35}P_4 + \frac{4}{7}P_2 + \frac{1}{5}P_0.$$

Similarly, x^3 can be expressed in terms of the Legendre polynomials of odd degree. Since $P_1 = x$ and $P_3 = \frac{1}{2}(5x^3 - 3x)$,

$$x^3 = \frac{2}{5}P_3 + \frac{3}{5}P_1.$$

(2) Show that

$$\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n = \sum_{m=0}^{[n/2]}(-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}\, x^{n-2m}$$

where $[n/2]$ denotes the greatest integer less than or equal to $n/2$.

Both expressions equal $P_n(x)$, the n-th Legendre polynomial. The expression in the lhs is known as the Rodrigues formula.

SOLUTION. Start with the lhs. The binomial expansion gives

$$(x^2-1)^n = \sum_{m=0}^{n}\binom{n}{m}(-1)^m x^{2n-2m}.$$

Differentiating n times,

$$\begin{aligned}\frac{d^n}{dx^n}(x^2-1)^n &= \sum_{m=0}^{[n/2]}\binom{n}{m}(-1)^m(2n-2m)(2n-2m-1)\dots(n-2m+1)\, x^{n-2m}\\ &= \sum_{m=0}^{[n/2]}\frac{n!}{m!(n-m)!}(-1)^m\frac{(2n-2m)!}{(n-2m)!}\, x^{n-2m}.\end{aligned}$$

Dividing both sides by $2^n n!$ yields the required identity.

(3) Show that if $f(x)$ is a polynomial with double roots at a and b then $f''(x)$ vanishes at least twice in (a, b). (This is also true if $f(x)$ is a smooth function.)

Generalize this and show (using Rodrigues' formula) that $P_n(x)$ has n distinct roots in $(-1, 1)$.

SOLUTION. Let $f(a) = f'(a) = 0 = f(b) = f'(b)$. By Rolle's theorem, there is a $c \in (a, b)$ such that $f'(c) = 0$. Applying Rolle's theorem to $f'|_{[a,c]}$ and $f'|_{[c,b]}$, we get $c_1 \in (a, c)$ and $c_2 \in (c, b)$ where f'' vanishes. More generally: If $f(x)$ is a smooth function with roots of multiplicity n at both a and b, then $f^{(n)}$ vanishes at least n times in (a, b). (The hypothesis says $f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0 = f(b) = f'(b) = \cdots = f^{(n-1)}(b)$.) We prove this result by induction. Assuming the result for $n-1$, there are $n-1$ points $a < t_1 < \cdots < t_{n-1} < b$ where $f^{(n-1)}(t_i) = 0$. Applying Rolle's theorem to $f^{(n-1)}|_{[t_{i-1},t_i]}$, we get n distinct zeroes of $f^{(n)}$ in the intervals (t_{i-1}, t_i). (Here $t_0 = a$ and $t_n = b$ is implicit.) This completes the induction step.
Now consider

$$f(x) = \frac{(x^2-1)^n}{2^n n!}.$$

This polynomial has roots of multiplicity n at $x = \pm 1$, Therefore, by the above result $P_n(x) = f^{(n)}(x)$ has at least n distinct zeroes in $(-1, 1)$. Being a polynomial of degree n, these can be the only zeroes and each of them must be simple.

(4) Take the Rodrigues formula as the definition for $P_n(x)$, and show the following relations.

(a) $P_n(-x) = (-1)^n P_n(x)$

SOLUTION. Note that $P_n(x)$ is an even or an odd function according as n is even or odd. Hence $P_n(-x) = (-1)^n P_n(x)$.

(b) $P_n'(-x) = (-1)^{n+1} P_n'(x)$

SOLUTION. Note that $P_n'(x)$ is an even or an odd function according as n is odd or even. Hence $P_n'(-x) = (-1)^{n+1} P_n'(x)$.

(c) $P_n(1) = 1$ and $P_n(-1) = (-1)^n$

SOLUTION.

$$P_n(x) = \frac{1}{2^n n!} D^n (x^2-1)^n = \frac{1}{2^n n!} \sum_{r=0}^{n} \binom{n}{r} D^r (x-1)^n D^{n-r} (x+1)^n.$$

Now,

$$D^r (x-1)^n \big|_{x=1} = \begin{cases} 0 & \text{if } r < n, \\ n! & \text{if } r = n. \end{cases}$$

Hence evaluating at $x = 1$,

$$P_n(1) = \frac{1}{2^n n!} n! (1+1)^n = 1.$$

Similarly, or by part (a), $P_n(-1) = (-1)^n$.

(d) $P_{2n+1}(0) = 0$ and $P_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}$

SOLUTION. $P_{2n+1}(0) = 0$ since it is an odd function, while

$$P_{2n}(0) = \frac{1}{4^n (2n)!} D^{2n} (x^2-1)^{2n} \Big|_{x=0} = \frac{1}{4^n (2n)!} \times \text{ the constant term in } D^{2n}(x^2-1)^{2n}.$$

The constant term in $D^{2n}(x^2-1)^{2n}$ is

$$(2n)! \times \text{ the coefficient of } x^{2n} \text{ in } (x^2-1)^{2n} = (2n)!\binom{2n}{n}(-1)^n.$$

Hence

$$P_{2n}(0) = \frac{1}{4^n(2n)!}(2n)!\frac{(2n)!}{n!n!}(-1)^n = (-1)^n\frac{(2n)!}{2^{2n}(n!)^2}.$$

(e) $P_n'(1) = \frac{1}{2}n(n+1)$ and $P_n'(-1) = (-1)^{n-1}\frac{1}{2}n(n+1)$

SOLUTION.

$$\begin{aligned}
P_n'(1) &= \frac{1}{2^n n!}D^{n+1}(x^2-1)^2\big|_{x=1} \\
&= \frac{1}{2^n n!}\left[\sum_{r=0}^{n+1}\binom{n+1}{r}D^r(x-1)^n \cdot D^{n+1-r}(x+1)^n\right]_{x=1} \\
&= \frac{1}{2^n n!}\binom{n+1}{n}n! \cdot D(x+1)^n\big|_{x=1} \\
&= \frac{n+1}{2^n}\cdot n(1+1)^n = \frac{n(n+1)}{2}
\end{aligned}$$

The main point to note is that only the n-th term in the summation survives when we substitute $x = 1$.

Similarly, or by part (b),

$$P_n'(-1) = (-1)^{n+1}P_n'(1) = (-1)^{n+1}\frac{n(n+1)}{2}.$$

(f) $P_{2n}'(0) = 0$ and $P_{2n+1}'(0) = (-1)^n\dfrac{(2n+1)!}{2^{2n}(n!)^2}$.

SOLUTION. Since P_{2n}' is an odd function, $P_{2n}'(0) = 0$.

$$\begin{aligned}
P_{2n+1}'(0) &= \frac{1}{2^{2n+1}(2n+1)!}D^{2n+2}(x^2-1)^{2n+1}\big|_{x=0} \\
&= \frac{1}{2^{2n+1}(2n+1)!}\binom{2n+1}{n+1}(2n+2)!(-1)^n \\
&= (-1)^n\frac{(2n+1)!}{2^{2n}(n!)^2}
\end{aligned}$$

(5) Show that

$$\int_{-1}^{1}(1-x^2)P_m'(x)P_n'(x)\,dx = \begin{cases} \dfrac{2n(n+1)}{2n+1} & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

SOLUTION. Recall the self-adjoint form of the Legendre equation

$$[(1-x^2)P_n']' + n(n+1)P_n = 0.$$

Multiplying by P_m and integrating over $[-1,1]$,

$$\int_{-1}^{1} P_m[(1-x^2)P_n']' + n(n+1)P_mP_n = 0.$$

Integrating the first term by parts,

$$-\int_{-1}^{1} P_m'[(1-x^2)P_n']dx + n(n+1)\int_{-1}^{1} P_m P_n dx = 0.$$

Now use

$$\int_{-1}^{1} P_m P_n dx = \begin{cases} \frac{2}{2n+1} & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

(6) Show the following relations when $n-m$ is even and nonnegative.

(a) $\displaystyle\int_{-1}^{1} P_m' P_n' dx = m(m+1)$

SOLUTION.

$$\int_{-1}^{1} P_m' P_n' dx = P_m' P_n \Big|_{-1}^{1} - \int_{-1}^{1} P_m'' P_n dx$$
$$= 2P_m'(1) - \int_{-1}^{1} P_m'' P_n dx$$

since $n-m$ being even makes $P_m' P_n$ odd and $P_n(1) = 1$

$$= m(m+1) - \int_{-1}^{1} P_m'' P_n dx.$$

To evaluate the integral, we repeatedly integrate by parts

$$\int_{-1}^{1} P_m'' D^n (x^2-1)^n dx$$

to get

$$(-1)^n \int_{-1}^{1} (D^{n+2} P_m)(x^2-1)^n dx.$$

Now $n-m \geq 0$ implies $n+2 > m$ so that $D^{n+2} P_m \equiv 0$. This makes the integral vanish.

(b) $\displaystyle\int_{-1}^{1} x^m P_n'(x) dx = 0$. What is the value of the integral if $n-m$ is odd (instead of even)?

SOLUTION. (i) Since $n-m$ is even, $x^m P_n'$ is an odd function. So the integral is zero. (ii) If $n-m$ is odd, then $n > m$ and $x^m P_n'$ is an even function. Now

$$\int_{-1}^{1} x^m P_n'(x) dx = 2\int_{0}^{1} x^m P_n' dx = 2\Big[x^m P_n \Big|_{0}^{1} - m\int_{0}^{1} x^{m-1} P_n dx\Big]$$
$$= 2 - 2m\int_{0}^{1} x^{m-1} P_n dx = 2.$$

In the last step, we used that x^{m-1} belongs to the span of $P_0, \dots, P_{m-1}$, and hence is orthogonal to P_n. Alternatively,

$$\begin{aligned}2^n n! \int_{-1}^{1} x^{m-1} P_n dx &= \int_{-1}^{1} x^{m-1} D^n (x^2-1)^n dx \\ &= (-1)^{m-1}(m-1)! \int_{-1}^{1} D^{n-m+1}(x^2-1)^n dx \\ &= (-1)^{m-1}(m-1)! D^{n-m}(x^2-1)^n \Big|_{-1}^{1} = 0.\end{aligned}$$

(7) If $x^n = \sum_{r=0}^{n} a_r P_r(x)$, then show that $a_n = \dfrac{2^n (n!)^2}{(2n)!}$.

SOLUTION.

$$\begin{aligned}a_n &= \frac{2n+1}{2} \frac{1}{2^n n!} \int_{-1}^{1} x^n D^n (x^2-1)^n dx \\ &= (-1)^n \frac{2n+1}{2} \cdot \frac{1}{2^n n!} \cdot n! \int_{-1}^{1} (x^2-1)^n dx \\ &= \frac{2n+1}{2^{n+1}} \int_{-1}^{1} (1-x)^n (1+x)^n dx \\ &= \frac{2n+1}{2^{n+1}} \frac{n(n-1)\dots 1}{(n+1)(n+2)\dots(2n)} \int_{-1}^{1} (1+x)^{2n} dx \\ &= \frac{2n+1}{2^{n+1}} \frac{n(n-1)\dots 1}{(n+1)(n+2)\dots(2n)} \frac{2^{2n+1}}{2n+1} \\ &= \frac{2^n (n!)^2}{(2n)!}\end{aligned}$$

(8) Expand the following functions $f(x)$ in a series of Legendre polynomials:

$$f(x) \approx \sum_{n \geq 0} c_n P_n \quad \text{with} \quad c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) dx.$$

The Rodrigues formula is useful to evaluate these integrals. The Legendre expansion theorem (stated in the lecture notes) applies in each case.

(a)

$$f(x) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1. \end{cases}$$

SOLUTION. Since $f(x) =: \operatorname{sgn}(x)$ is an odd function, $c_{even} = 0$. The odd coefficients are computed below.

$$\begin{aligned}c_{2k+1} &= \frac{4k+3}{2} \int_{-1}^{1} \operatorname{sgn}(x) P_{2k+1}(x) dx = (4k+3) \int_{0}^{1} P_{2k+1}(x) dx \\ &= (4k+3) \frac{D^{2k}(x^2-1)^{2k+1}}{2^{2k+1}(2k+1)!} \Big|_0^1 \\ &= \frac{2k+1}{2^{2k+1}(2k+1)!} [-D^{2k}(x^2-1)^{2k+1}\big|_{x=0}]\end{aligned}$$

$$= \frac{2k+1}{2^{2k+1}(2k+1)!}[-(2k)!\binom{2k+1}{k}(-1)^{k+1}]$$
$$= \frac{(-1)^k(4k+3)}{2^{2k+1}(k+1)}\binom{2k}{k}.$$

(b)

$$f(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1. \end{cases}$$

SOLUTION. Note that

$$f(x) = \frac{1}{2}(\operatorname{sgn}(x) + 1) = \frac{1}{2}(\operatorname{sgn}(x) + P_0).$$

Hence, using part (a),

$$f(x) = \frac{1}{2}P_0 + \frac{1}{2}\sum_{k\geq 0} c_{2k+1}P_{2k+1}, \quad \text{where} \quad c_{2k+1} = \frac{(-1)^k(4k+3)}{2^{2k+1}(k+1)}\binom{2k}{k}.$$

(c)

$$f(x) = \begin{cases} -x & \text{if } -1 < x < 0 \\ x & \text{if } 0 < x < 1. \end{cases}$$

SOLUTION. Since $f(x) = |x|$ is an even function, $c_{od} = 0$. Now

$$c_{2k} = (4k+1)\int_0^1 xP_{2k}(x)dx.$$

Clearly, $c_0 = 1/2$. For $k \geq 1$,

$$c_{2k} = -\frac{4k+1}{2^{2k}(2k)!}[-D^{2k-2}(x^2-1)^{2k}]\big|_{x=0} = (-1)^{k-1}\frac{4k+1}{4^k k(k+1)}\binom{2k-2}{k-1}.$$

Explicitly, for $k = 1$,

$$c_2 = 5\int_0^1 \frac{1}{2}x(3x^2-1)dx = 5/8.$$

(d)

$$f(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ x & \text{if } 0 < x < 1 \end{cases}$$

SOLUTION. Using

$$f(x) = x/2 + |x|/2 = \frac{1}{2}P_1 + \frac{1}{2}|x|,$$

we obtain from part (c),

$$f(x) = \frac{1}{2}P_0 + \frac{1}{2}P_1 + \frac{1}{2}\sum_{k\geq 1}\frac{(-1)^{k-1}(4k+1)}{4^k k(k+1)}\binom{2k-2}{k-1}P_{2k}.$$

(9) Consider the *associated Legendre equation*

$$(1-x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0 \tag{1}$$

which occurs in quantum physics. Substituting

$$y(x) = (1-x^2)^{m/2}v(x),$$

show that v satisfies

$$(1-x^2)v'' - 2(m+1)xv' + [n(n+1) - m(m+1)]v = 0 \tag{2_m}$$

Show that $v = D^m P_n$ satisfies (2_m). Thus

$$y(x) = (1-x^2)^{m/2}D^m P_n(x)$$

is the bounded solution of (1) and is called an *associated Legendre function*.

Solution. Given $y(x) = (1-x^2)^{m/2}v(x)$. Then

$$y'(x) = -mx(1-x^2)^{m/2-1}v(x) + (1-x^2)^{m/2}v'(x),$$

$$\begin{aligned} y''(x) = -m(1-x^2)^{m/2-1}v(x) &+ m(m-2)x^2(1-x^2)^{m/2-2}v(x) \\ &- mx(1-x^2)^{m/2-1}v'(x) - mx(1-x^2)^{m/2-1}v'(x) \\ &+ (1-x^2)^{m/2}v''(x). \end{aligned}$$

Therefore,

$$\begin{aligned} &(1-x^2)y''(x) - 2xy'(x) \\ &\quad = (1-x^2)^{m/2+1}v''(x) - 2(m+1)x(1-x^2)^{m/2}v'(x) \\ &\qquad + (1-x^2)^{m/2-1}\Big[m(m-2)x^2 - m(1-x^2) + 2mx^2\Big]v(x). \end{aligned}$$

The lhs is

$$\left[\frac{m^2}{1-x^2} - n(n+1)\right](1-x^2)^{m/2}v(x).$$

Now simplify to obtain (2_m). Let n be a fixed natural number. That (2_m) is satisfied by $D^m P_n$ is obviously true for $m = 0$. Assume for m and check for $m+1$ by substituting $D^m P_n$ in (2_m) and differentiating once to check the validity of (2_{m+1}).

Remark: By applying the solution to the last problem of Section 1.1, one can show that

$$\left(\frac{1-x}{1+x}\right)^{\pm m/2}$$

form a basis of the solution of equation (1) in the special case when $n = 0$. Clearly, the only bounded solution $D^m P_0$ is identically zero, if $m > 0$.

Optional problems.

(1) Show that the substitution $y = u/\sqrt{\sin\varphi}$, with $0 \le \varphi \le \pi$ transforms the spherical form of the Legendre equation

$$\frac{d^2y}{d\varphi^2} + \cot\varphi \frac{dy}{d\varphi} + n(n+1)y = 0$$

into

$$\frac{d^2u}{d\varphi^2} + \left[\left(n + \frac{1}{2}\right)^2 + \frac{1}{4\sin^2\varphi}\right] u = 0.$$

SOLUTION. $y(\varphi) = \dfrac{u(\varphi)}{\sqrt{\sin\varphi}}$ implies

$$y' = \frac{u'}{\sqrt{\sin\varphi}} - \frac{u\cot\varphi}{2\sqrt{\sin\varphi}} = \frac{u'}{\sqrt{\sin\varphi}} - \frac{1}{2}y\cot\varphi$$

and

$$y''(\varphi) = \frac{u''}{\sqrt{\sin\varphi}} - \frac{u'\cot\varphi}{2\sqrt{\sin\varphi}} - \frac{1}{2}y'\cot\varphi + \frac{1}{2}y\csc^2\varphi.$$

Now substitute and simplify.

(2) (a) Show that for large integral values of n and φ close to $\pi/2$ (the center of the interval),

$$P_n(\cos\varphi) \approx A_n \cos\left[(n + \frac{1}{2})\varphi + \alpha_n\right].$$

[*Hint:* Use previous exercise.]

SOLUTION. For $n >> 0$ and $\varphi \approx \pi/2$,

$$(n + 1/2)^2 + 1/(4\sin^2\varphi) \approx (n + 1/2)^2.$$

Therefore,

$$u(\varphi) \approx A_n \cos[(n + 1/2)\varphi + \alpha_n] \quad \text{(phase-amplitude form)}$$

Finally, since $\sin\varphi \approx 1$, for φ close to $\pi/2$, hence $y \approx u$. But $y = P_n(\cos\varphi)$ is a solution of the Legendre equation,

$$P_n(\cos\varphi) \approx u \approx A_n \cos[(n + 1/2)\varphi + \alpha_n]$$

for φ close to $\pi/2$.

(b) By considering parity under the reflection $\varphi \mapsto \pi - \varphi$ show that the phase α_n is $-\dfrac{\pi}{4}$.
[*Hint:* $P_n(-\cos\varphi) = (-1)^n P_n(\cos\varphi)$.]

SOLUTION.

$$P_n(\cos(\pi - \varphi) = (-1)^n P_n(\cos\varphi)$$

allows us to choose the phase α_n so that the approximate solution also has this property, that is,

$$\cos\left[\left(n + \frac{1}{2}\right)\left(\pi - \varphi\right) + \alpha_n\right] = (-1)^n \cos\left[\left(n + \frac{1}{2}\right)\varphi + \alpha_n\right].$$

This is equivalent to

$$\left[\left(n+\frac{1}{2}\right)\left(\pi-\varphi\right)+\alpha_n\right]+\left[\left(n+\frac{1}{2}\right)\varphi+\alpha_n\right]=\frac{(2n+1)\pi}{2}+2\alpha_n$$

being an odd multiple of π for n odd and an even multiple of π for n even. Choosing $\alpha_n=-\dfrac{\pi}{4}$ makes this dream come true.

(c) Show that A_n approaches $\sqrt{\dfrac{2}{n\pi}}$ as n approaches infinity,
[*Hint:* Consider $y(\pi/2)=P_n(0)$ for n even, and $y'(\pi/2)=-P_n'(0)$ for n odd. Invoke the estimate (without proof)

$$\frac{(2m)!}{(2^m m!)^2}\approx\frac{1}{\sqrt{m\pi}},\qquad m>>0.]$$

Solution. For n even,

$$P_n(0)=(-1)^{n/2}\frac{n!}{2^n((n/2)!)^2}\quad\text{and}\quad y(\pi/2)=(-1)^{n/2}A_n.$$

Therefore

$$A_n=\frac{n!}{2^n((n/2)!)^2}\approx\sqrt{\frac{2}{n\pi}}\ \text{for } n>>0.$$

On the other hand for n odd, first we note that

$$\frac{dP_n(\cos\varphi)}{d\varphi}=-\sin\varphi P_n'(\cos\varphi).$$

At $\varphi=\pi/2$, this reads

$$y'(\pi/2)=-P_n'(0)=-\frac{(-1)^{(n-1)/2}n!}{\left[2^{(n-1)/2}\big((n-1)/2\big)!\right]^2}.$$

Also

$$y'(\pi/2)=-\left(n+\frac{1}{2}\right)A_n(-1)^{(n-1)/2}.$$

Therefore,

$$\left(n+\frac{1}{2}\right)A_n=\frac{n!}{\left[2^{(n-1)/2}\big((n-1)/2\big)!\right]^2}$$

or

$$A_n=\frac{n}{n+1/2}\frac{(n-1)!}{\left[2^{(n-1)/2}\big((n-1)/2\big)!\right]^2}\approx\sqrt{\frac{2}{(n-1)\pi}}\approx\sqrt{\frac{2}{n\pi}}$$

when n is very large.
Remark: In the computations above, notice that $u(\pi/2)=y(\pi/2)$ and $u'(\pi/2)=y'(\pi/2)$. $A_n\cos[(n+1/2)\varphi+\alpha_n]$ agree with the actual solution $y(\varphi)=P_n(\cos\varphi)$ to a similar extent.

1.3. Frobenius method for regular singular equations

Problems.

(1) Attempt a power series solution around $x = 0$ for

$$x^2 y'' - (1+x)y = 0.$$

Explain why the procedure does not give any nontrivial solutions.

SOLUTION. Write $y = \sum_{n\geq 0} a_n x^n$. Hence

$$n(n-1)a_n = a_n + a_{n-1},$$

or

$$a_n = \frac{1}{n^2 - n - 1} a_{n-1}.$$

This holds for all $n \geq 0$ with the convention $a_{-1} = 0$. This implies

$$a_0 = 0, a_1 = 0, \ldots, a_n = 0, \ldots.$$

Reason: The differential equation can be written as $y'' - \dfrac{1+x}{x^2} y = 0$ and the coefficient $-\dfrac{1+x}{x^2}$ does not have a power series around $x = 0$. In fact 0 is a regular singular point.

(2) Attempt a Frobenius series solution for the differential equation

$$x^2 y'' + (3x-1)y' + y = 0.$$

Why does the method fail?

SOLUTION. Write

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n, \quad a_0 \neq 0.$$

This implies $ra_0 = 0$ and hence $r = 0$ since $a_0 \neq 0$. Further with $r = 0$, we get

$$a_{n+1} = (n+1)a_n.$$

The radius of convergence of the resulting power series is 0. The method fails because $x = 0$ is a not a regular singular point.

(3) Locate and classify the singular points for the following differential equations. (All letters other than x and y such as p, λ, etc are constants.)

(a) Bessel equation:

$$x^2 y'' + xy' + (x^2 - p^2)y = 0.$$

SOLUTION. $x = 0$ is the only singular point and it is regular singular. We can write

$$y'' + \frac{1}{x} y' + \frac{x^2 - p^2}{x^2} y = 0$$

and both 1 and $(x^2 - p^2)$ are real analytic everywhere, in fact polynomials.

(b) Laguerre equation:

$$xy'' + (1-x)y' + \lambda y = 0.$$

SOLUTION. $x = 0$ is the only singular point and it is regular singular.

(c) Jacobi equation:

$$x(1-x)y'' + (\gamma - (\alpha+1)x)y' + n(n+\alpha)y = 0.$$

SOLUTION. $x = 0$ and $x = 1$ are the only singular points and both are regular singular.

(d) Hypergeometric equation:

$$x(1-x)y'' + [c - (a+b+1)x)]y' - aby = 0.$$

SOLUTION. $x = 0$ and $x = 1$ are the only singular points and both are regular singular.

(e) Associated Legendre equation:

$$(1-x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right] y = 0$$

SOLUTION. $x = \pm 1$ are the singular points and both are regular singular.

(f)

$$xy'' + (\cot x)y' + xy = 0.$$

SOLUTION. $x = 0$ is the only singular point and it is not regular singular. We can write

$$y'' + \frac{\cot x}{x} y' + \frac{x^2}{x^2} y = 0.$$

Though the second coefficient x^2 is a polynomial, the first coefficient $\cot x$ cannot be expanded as a power series about $x = 0$.

(4) In Problem (3) above find the indicial equations corresponding to all the regular singular points.

SOLUTION. The basic method is as follows: If x_0 is a regular singular point of a second order linear ODE, first write it in the form

$$y'' + \frac{b(x)}{(x-x_0)} y' + \frac{c(x)}{(x-x_0)^2} y = 0.$$

Now the indicial equation for the purpose of expanding in fractional powers of $(x - x_0)$ is

$$r(r-1) + b(x_0)r + c(x_0) = 0.$$

(a) $x_0 = 0$ is the only singular point which is regular. $b(x) = 1,\ c(x) = x^2 - p^2$. The indicial equation is $r^2 - p^2 = 0$.

(b) $x_0 = 0$ is the only singular point which is regular. $b(x) = 1-x,\ c(x) = \lambda x$. The indicial equation is $r^2 = 0$.

(c) $x_0 = 0$ and $x_0 = 1$ are both regular singular points. For $x_0 = 0$,
$$b(x) = \frac{\gamma - (\alpha+1)x}{1-x} \quad \text{and} \quad c(x) = n(n+\alpha)x^2.$$
The indicial equation is $r(r-1) + \gamma r = 0$.
For $x_0 = 1$,
$$b(x) = \frac{\gamma - (\alpha+1)x}{-x} \quad \text{and} \quad c(x) = n(n+\alpha)(x-1)^2.$$
The indicial equation is $r(r-1) + (\alpha + 1 - \gamma)r = 0$.

(d) Again $x_0 = 0$ and $x_0 = 1$ are both regular singular points. For $x_0 = 0$,
$$b(x) = \frac{c - (a+b+1)x}{1-x} \quad \text{and} \quad c(x) = -abx^2.$$
The indicial equation is $r(r-1) + cr = 0$.
For $x_0 = 1$,
$$b(x) = \frac{c - (a+b+1)x}{-x} \quad \text{and} \quad c(x) = -ab(x-1)^2.$$
The indicial equation is $r(r-1) + (a + b + 1 - c)r = 0$.

(e) $x_0 = \pm 1$ are regular singular. For $x_0 = 1$,
$$b(x) = \frac{2x}{x+1} \quad \text{and} \quad c(x) = \frac{[n(n+1)(1-x^2) - m^2}{(1+x)^2}.$$
The indicial equation is $r(r-1) + r - m^2/4 = 0$, that is, $r^2 = m^2/4$.
By symmetry, the same is true for $x_0 = -1$.

(f) $x_0 = 0$ is the only singular point. It is not regular, so no indicial equation.

(5) Find two linearly independent solutions of the following differential equations.

(a) $x(x-1)y'' + (4x-2)y' + 2y = 0$.

Solution. Observe that
$$x(x-1)y'' + (4x-2)y' + 2y = D^2[x(x-1)y].$$
Hence
$$x(x-1)y = Ax + B$$
is the general solution with A and B arbitrary constants, and
$$y = \frac{1}{x-1} \quad \text{and} \quad y = \frac{1}{x(x-1)}$$
are two linearly independent solutions. The first one has a singularity at 1, while the second has a singularity at both 1 and -1.
(One may also attempt a Frobenius series solution.)

(b) $(1-x^2)y'' - 2xy' + 2y = 0$.

SOLUTION. This is the Legendre equation for $p = 1$ which was solved in class.
Aliter: Guess that $y_1(x) = x$ is a solution. Employing the method of variation of parameters, let the second solution be $y_2(x) = xu(x)$. Substituting in the ODE and simplifying, we get

$$x(1-x^2)u'' = (4x^2-2)u'.$$

So

$$\frac{u''}{u'} = \frac{1}{1-x} - \frac{2}{x} - \frac{1}{1+x}.$$

So

$$u' = \frac{1}{x^2(1+x)(1-x)} = \frac{1}{x^2} + \frac{1}{2(1-x)} + \frac{1}{2(1+x)}.$$

So

$$u(x) = -\frac{1}{x} + \frac{1}{2}\log\frac{1+x}{1-x}.$$

So

$$y_2(x) = -1 + \frac{1}{2}x\log\frac{1+x}{1-x}.$$

(c) $x^2y'' + x^3y' + (x^2-2)y = 0$.

SOLUTION. Applying the Frobenius method, gives the indicial equation $r(r-1) - 2 = 0$, which implies $r_2 = -1$ and $r_1 = 2$. The recursion is

$$[(n+r)(n+r-1)-2]a_n = -(n+r-1)a_{n-2}.$$

For $r = -1$,

$$a_n = -\frac{(n-2)}{n(n-3)}a_{n-2} \quad \text{for } n \geq 0,\ n \neq 0, 3$$

and with $a_{-1} = a_{-2} = 0$. Now $n = 1, 2$ yield $a_1 = 0 = a_2$. Thus we see that along with a_2, $a_4 = a_6 = \cdots = a_{2k} = \cdots = 0$. Further,

$$a_{2k+1} = (-1)^{k-1}\frac{3a_3}{(2k+1)2^{k-1}(k-1)!} \quad \text{for } k \geq 1.$$

Thus

$$y_1(x) = a_0/x + a_3x^2 \times \text{ an even power series}$$

is the form of the solution with a_0, a_3 arbitrary. Thus we already get the general solution. If we try $r = 2$ now we will get

$$x^2\sum_{k\geq 0} A_{2k}x^{2k}, \quad A_{2k} = (-1)^k\frac{3A_0}{(2k+3)2^k k!}$$

which is aready present in y_1 if we take $a_0 = 0,\ a_3 = A_0$.
Aliter: Guess that $y_1(x) = 1/x$ is a solution. By the method of variation of parameters, let the second solution be $y_2(x) = u(x)/x$.

On substitution, we find that u satisfies $xu'' + (x^2 - 2)u' = 0$ whence $u' = x^2 e^{-x^2/2}$. Therefore,

$$y_2(x) = \frac{1}{x}\int x^2 e^{-x^2/2} dx.$$

On expanding we see that it matches the power series part of the solution obtained above.

(d) $xy'' + 2y' + xy = 0$.

SOLUTION. $0 = xy'' + 2y' + xy = (xy)'' + (xy)$. Hence two linearly independent solutions are

$$\frac{\cos x}{x} \quad \text{and} \quad \frac{\sin x}{x}.$$

Optional problems.

(1) Show that the hypergeometric equation has a regular singular point at infinity[1], but that the point of infinity is an irregular singular point for the Airy equation.

SOLUTION.

(a) The hypergeometric equation is (essentially)

$$x(1-x)y'' + (c - ax)y' + by = 0.$$

Let $t = 1/x$, $u(t) = y(x)$. $y'(x) = \dot{u}(t)t'(x) = -t^2\dot{u}$ and $y''(x) = -t^2(-t^2)\ddot{u} - 2t(-t^2)\dot{u} = t^4\ddot{u} + 2t^3\dot{u}$.

$$\begin{aligned} 0 &= x(1-x)y'' + (c-ax)y' + by \\ &= \frac{1}{t}\left(1 - \frac{1}{t}\right)\left(t^4\ddot{u} + 2t^3\dot{u}\right) + \left(c - \frac{a}{t}\right)(-t^2 u) + bu \end{aligned}$$

This gives $\ddot{u} + \dfrac{[(a-2) + (2-c)t]}{t-1}\dfrac{\dot{u}}{t} + \dfrac{b}{t-1}\dfrac{u}{t^2} = 0$. Clearly, $t = 0$ is a regular singular point of this equation.

(b) The Airy equation is $y'' - xy = 0$. Letting $t = 1/x$, $y(x) = u(t)$ as before, we come across

$$\ddot{u} + 2\frac{\dot{u}}{t} + (-\frac{1}{t^3})\frac{y}{t^2} = 0.$$

Clearly the third coefficient $= 1/t^3$ cannot be a power series in t. Hence irregular singularity at $x = \infty$.

[1] The differential equation $y'' + p(x)y' + q(x)y = 0$ has a regular singular point at infinity, if after substitution of $x = 1/t$ in the ODE, the resulting ODE has a regular singular point at the origin.

1.4. Bessel equation and Bessel functions

Problems.

(1) Using the indicated substitutions, reduce the following differential equations to the Bessel equation and find the general solution in term of the Bessel functions.

(a) $x^2y'' + xy' + (\lambda^2x^2 - p^2)y = 0, \quad (\lambda x = z)$

SOLUTION. Let $\lambda x = z$ and $u(z) = y(x)$. Then

$$\frac{du}{dz} = \frac{1}{\lambda} \cdot \frac{dy}{dx} \quad \text{and} \quad \frac{d^2u}{dz^2} = \frac{1}{\lambda^2}\frac{d^2y}{dx^2}.$$

Hence

$$z\frac{du}{dz} = \frac{x}{d}ydx \quad \text{and} \quad z^2\frac{d^2u}{dz^2} = x^2\frac{d^y}{dz^2}.$$

The given equation transforms to

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} + (z^2 - p^2)u = 0.$$

The general solution is $u(z) = c_1J_p(z) + c_2Y_p(z)$ which gives

$$y(x) = c_1J_p(\lambda x) + c_2Y_p(\lambda x).$$

(b) $xy'' - 5y' + xy = 0, \quad (y = x^3u)$.

SOLUTION. $y = x^3u(x)$. Therefore,

$$y' = 3x^2u + x^3u' \quad \text{and} \quad y'' = x^3u'' + 6x^2u' + 6xu.$$

So

$$0 = xy'' - 5y' + xy = (x^4u'' + 6x^3u' + 6x^2u) - 5(3x^2u + x^3u') + x^4u$$
$$= x^4u'' + x^3u' + (x^4 - 9x^2)u.$$

This implies $x^2u'' + xu' + (x^2 - 3^2)u = 0$. The general solution therefore, is $u(x) = c_1J_3(x) + c_2Y_3(x)$, or equivalently,

$$y(x) = x^3[c_1J_3(x) + c_2Y_3(x)].$$

(c) $y'' + k^2xy = 0, \quad (y = u\sqrt{x}, \ \frac{2}{3}kx^{3/2} = z)$.

SOLUTION. $y(x) = x^{1/2}u(z), \ z = (2k/3)x^{2/3}$. Let us write $D \equiv \dfrac{d}{dx}$ and $' \equiv \dfrac{d}{dz}$. Then

$$D^2y = k^2x^{3/2}u'' + \frac{3k}{2}u' - \frac{u}{4x^{3/2}}.$$

So

$$0 = D^2y + k^2xy = k^2x^{3/2}u'' + \frac{3k}{2}u' - \frac{u}{4x^{3/2}} + k^2x^{3/2}u.$$

On simplification,

$$z^2u'' + zu' + (z^2 - \frac{1}{3^2})u = 0.$$

Therefore, $u(z) = AJ_{1/3}(z) + BJ_{-1/3}(z)$, or equivalently,

$$y(x) = AJ_{1/3}(\frac{2k}{3}x^{2/3}) + BJ_{-1/3}(\frac{2k}{3}x^{2/3}).$$

(d) $x^2y'' + (1-2p)xy' + p^2(x^{2p} + 1 - p^2)y = 0, \ \ (y = x^p u, \ \ x^p = z)$.

SOLUTION. Let us write $D \equiv \frac{d}{dx}$ and $' \equiv \frac{d}{dz}$ as before. Then

$$Dy = px^{p-1}u + x^p u'(px^{p-1}).$$

This implies $xDy = pzu + pz^2u'$. Similarly,

$$\begin{aligned}
x^2D^2y &= p(p-1)zu + 2p^2z^2u' + x^{2+p}D^2u \\
&= p(p-1)zu + 2p^2z^2u' + x^2z(u'px^{p-1})'px^{p-1} \quad (\text{since } D = px^{p-1}\frac{d}{dz}) \\
&= p(p-1)zu + 2p^2z^2u' + p^2xz^2(u''x^{p-1} + u'(p-1)x^{p-2}x' \\
&= p(p-1)zu + 2p^2z^2u' + p^2z^2(u''z + (p-1)u'/p) \\
&= p^2z^3u'' + p(3p-1)z^2u' + p(p-1)zu.
\end{aligned}$$

Now the given equation is

$$\begin{aligned}
0 = x^2D^2y &+ (1-2p)xDy + p^2(x^{2p} + 1 - p^2)y \\
&= (p(p-1)zu + 2p^2z^2u' + p^2z^2(u''z + (p-1)u'/p) \\
&\qquad + (1-2p)(pzu + pz^2u') + p^2(z^2 + 1 - p^2)zu.
\end{aligned}$$

On dividing by p^2z, this simplifies to $z^2u'' + zu' + (z^2 - p^2)u = 0$. The general solution is $u(z) = AJ_p(z) + BY_p(z)$ which gives

$$y(x) = x^p[AJ_p(x^p) + BY_p(x^p)].$$

(2) Show that

(a) $J_{1/2} = \sqrt{\frac{2}{\pi x}}\sin x$

SOLUTION. From the expression for J_p,

$$J_{1/2}(x) = \sqrt{\frac{x}{2}}\sum_{m\geq 0}\frac{(ix/2)^{2m}}{m!(m+\frac{1}{2})!}.$$

Also,

$$m!(m+\frac{1}{2})! = m!(m+\frac{1}{2})(m-\frac{1}{2})\ldots\frac{1}{2}\Gamma(1/2) = \frac{(2m+1)!}{2^{2m+1}}\sqrt{\pi}.$$

This implies,

$$J_{1/2}(x) = \sqrt{\frac{x}{2}}\sum_{m\geq 0}\frac{(ix/2)^{2m}2^{2m+1}}{(2m+1)!\sqrt{\pi}} = \sqrt{\frac{2}{\pi x}}\sin x.$$

(b) $J_{-1/2} = \sqrt{\frac{2}{\pi x}} \cos x$

SOLUTION. Similarly, using

$$m!(m - \frac{1}{2})! = \frac{(2m-1)!\sqrt{\pi}}{2^{2m}},$$

one can see that

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

(c) $J_{3/2} = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$

SOLUTION. From 2(a),

$$[x^{-1/2} J_{1/2}(x)]' = -x^{-1/2} J_{3/2}(x).$$

Therefore

$$\sqrt{\frac{2}{\pi}} \left[\frac{\sin x}{x} \right]' = -x^{-1/2} J_{3/2}(x).$$

This implies

$$J_{3/2}(x) = -\sqrt{\frac{2x}{\pi}} \left(\frac{\cos x}{x} - \frac{\sin x}{x^2} \right) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right).$$

(d) $J_{-3/2} = -\sqrt{\frac{2}{\pi x}} \left(\frac{\cos x}{x} + \sin x \right)$

SOLUTION. Similarly from 2(a) again,

$$[x^{-1/2} J_{-1/2}(x)]' = x^{-1/2} J_{-3/2}(x).$$

Therefore

$$J_{-3/2}(x) = \sqrt{\frac{2x}{\pi}} \left[\frac{\cos x}{x} \right]' = \sqrt{\frac{2x}{\pi}} \left[-\frac{\sin x}{x} - \frac{\cos x}{x^2} \right] = -\sqrt{\frac{2}{\pi x}} \left(\frac{\cos x}{x} + \sin x \right).$$

(3) For an integer n show that $J_n(x)$ is an even (resp. odd) function if n is even (resp. odd).

SOLUTION. It is clear for nonnegative integers directly from

$$J_n(x) = \left(\frac{x}{2} \right)^n \sum_{m \geq 0} \frac{(ix/2)^{2m}}{m!(m+n)!}.$$

We only have to observe that for $n < 0$, the terms corresponding to $0 \leq m \leq |n| - 1$ will vanish due to the presence of the factorials of negative integers in the denominators.

(4) Show that between any two consecutive positive zeros of $J_n(x)$ there is precisely one zero of $J_{n+1}(x)$ and one zero of $J_{n-1}(x)$.

SOLUTION. Let $J_n(a) = J_n(b) = 0$, where $0 < a < b$ are consecutive zeroes of J_n. Then $x^{\pm n} J_n(x) = 0$ for $x = a, b$. By Rolles' theorem there exist $c_\pm \in (a, b)$ such that $[x^{\pm n} J_n(x)]'(c_\pm) = 0$. This implies $\pm x^{\pm n} J_{n \mp 1}(c_\pm) = 0$. (We take corresponding signs only.) In other words, $J_{n-1}(c_+) = 0$ and $J_{n+1}(c_-) = 0$. If possible let there be $c < d$ in (a, b) such that $J_{n+1}(c) = 0 = J_{n+1}(d)$. Then there is another $k \in (c, d)$ where

$J_{[(n+1)-1]}(k) = 0$. This contradicts that a and b are consecutive zeroes of J_n. Therefore, J_{n+1} vanishes exactly once in (a, b). Similarly, J_{n-1} vanishes exactly once in (a, b).
Remark: n need not be an integer in this problem.

(5) Show the following.

(a) $J_3 + 3J_0' + 4J_0''' = 0$.

SOLUTION. The relation $J_0' = \dfrac{1}{2}(J_{-1} - J_1) = -J_1$ implies that

$$\begin{aligned} J_3 + 3J_0' + 4J_0''' &= J_3 - 3J_1 - 4J_1'' = J_3 - 3J_1 - 2(J_0 - J_2)' \\ &= J_3 - 3J_1 + 2J_1 + (J_1 - J_3) = 0. \end{aligned}$$

(b) $J_2 - J_0 = aJ_c''$ find a and c.

SOLUTION. $J_2 - J_0 = -2J_1' = +2J_0''$. Thus $a = 2$ and $c = 0$.

(c) $\int J_{p+1} dx = \int J_{p-1} dx - 2J_p$.

SOLUTION. $2J_p' = J_{p-1} - J_{p+1}$. This implies $2J_p = \int J_{p-1} - \int J_{p+1}$ (indefinite integrals) and the result follows.

(6) If y_1 and y_2 are any two solutions of the Bessel equation of order p, then show that $y_1 y_2' - y_1' y_2 = c/x$ for a suitable constant c.

SOLUTION. Let $W(y_1, y_2) = y_1 y_2' - y_1' y_2$. This is called the Wronskian. Then

$$\begin{aligned} W' &= y_1 y_2'' - y_1'' y_2 \\ &= -y_1(y_2'/x + (x^2 - p^2)y_2^2/x^2) + (y_1'/x + (x^2 - p^2)y_1^2/x^2)y_2 = -\frac{W}{x}. \end{aligned}$$

Integrating, $\log W = -\log x + \log c$ or $W = c/x$.

(7) Show that

$$\int x^\mu J_p(x) dx = x^\mu J_{p+1}(x) - (\mu - p - 1) \int x^{\mu-1} J_{p+1}(x) dx.$$

SOLUTION.

$$\begin{aligned} \int x^\mu J_p(x) dx &= \int x^{\mu-p-1}(x^{p+1} J_p(x)) dx = \int x^{\mu-p-1}(x^{p+1} J_{p+1}(x))' dx \\ &= x^{\mu-p-1} x^{p+1} J_{p+1} - \int (\mu - p - 1) x^{\mu-p-2} x^{p+1} J_{p+1} dx \\ &= x^\mu J_{p+1} - (\mu - p - 1) \int x^{\mu-1} J_{p+1} dx. \end{aligned}$$

In one of the steps, we used integration by parts.

(8) Expand the indicated function in Fourier-Bessel series over the given interval and in terms of the Bessel function of given order. (The Bessel expansion theorem applies in each case.)

(a) $f(x) = 1$ over $[0, 3]$, $p = 0$.

SOLUTION.

$$f(x) = \sum_{z \in Z^{(0)}} c_z J_0(zx/3), \quad 0 \le x \le 3,$$

where

$$\begin{aligned} c_z &= \frac{2}{9J_1(z)^2} \int_0^3 f(x) J_0(zx/3) x dx \\ &= \frac{2}{z^2 J_1(z)^2} \int_0^z t J_0(t) dt \quad (\text{on setting } x = 3t/z,\ f(x) = 1) \\ &= \frac{2}{z^2 J_1(z)^2} z J_1(z) = \frac{2}{z J_1(z)} \quad (\text{since } xJ_0(x) = [xJ_1(x)]'). \end{aligned}$$

Sample values from the tables

$z \in Z^{(0)}$	2.405	5.52	8.65	11.79	14.93
$J_1(z)$	0.52	-0.34	0.27	-0.23	0.21
c_z	1.60	-1.07	0.86	-0.74	0.64

Explicitly, substituting the first few values, the Bessel series is

$$1 \approx 1.60 J_0(0.80x) - 1.07 J_0(1.84x) + 0.86 J_0(2.88x) - \ldots$$

for $0 \le x \le 3$.

(b) $f(x) = x$ over $[0, 1]$, $p = 1$.

SOLUTION.

$$f(x) = \sum_{z \in Z^{(1)}} c_z J_1(zx), \quad 0 \le x \le 1,$$

where

$$c_z = \frac{2}{J_0(z)^2} \int_0^1 f(x) J_1(zx) x dx$$

On setting $x = t/z = f(x)$

$$= \frac{2}{z^3 J_0(z)^2} \int_0^z t^2 J_1(t) dt$$

Integrating by parts using $J_1 = -J_0'$

$$\begin{aligned} &= \frac{2}{z^3 J_0(z)^2} [-z^2 J_0(z) + \int_0^z 2t J_0(t) dt] \\ &= \frac{2}{z^3 J_0(z)^2} [-z^2 J_0(z) + 2z J_1(z)] \quad (\text{since } xJ_0(x) = [xJ_1(x)]') \\ &= \frac{-2}{z J_0(z)}, \quad z \in Z^{(1)}. \end{aligned}$$

Sample values from the tables

$z \in Z^{(1)}$	3.83	7.02	10.17	13.32
$J_0(z)$	-0.40	0.30	-0.25	0.22
c_z	1.31	-0.95	0.79	-0.68

Explicitly, substituting the first few values, the Bessel series is

$$x \approx 1.31J_1(3.83x) - 0.95J_1(7.02x) + 0.79J_1(10.17x) - 0.68J_1(13.32x) + \dots$$

for $0 \leq x \leq 1$.

(c) $f(x) = x^3$ over $[0, 3]$, $p = 1$.

SOLUTION.

$$f(x) = \sum_{z \in Z^{(1)}} c_z J_1(zx/3), \quad 0 \leq x \leq 3,$$

where

$$c_z = \frac{2}{9J_0(z)^2} \int_0^3 f(x) J_1(zx/3) x dx$$

On setting $x = 3t/z,\ f(x) = x^3 = 27t^3/z^3$

$$= \frac{18}{z^4 J_0(z)^2} \int_0^z t^4 J_1(t) dt$$

Integrating by parts using $x^2 J_1 = [x^2 J_2]'$

$$= \frac{18}{z^4 J_0(z)^2} [-z^2(z^2 J_2(z)) - \int_0^z 2t.t^2 J_2(t) dt]$$
$$= \frac{18}{z^4 J_0(z)^2} [-z^4 J_0(z) - 2z^3 J_3(z)]$$

since $x^3 J_2(x) = [x^3 J_3(x)]'$ and $J_2(z) = -J_0(z);\ z \in Z^{(1)}$

$$= \frac{-18}{J_0(z)} - \frac{36 J_3(z)}{z J_0(z)^2}, \quad z \in Z^{(1)}.$$

Further,

$$J_1(z) + J_3(z) = \frac{4}{z} J_2(z) = -\frac{4}{z} J_0(z).$$

Hence $J_3(z) = -4J_0(z)/z$ and

$$c_z = \frac{18}{J_0(z)} \left[\frac{8}{z^2} - 1 \right];\ z \in Z^{(1)}.$$

Sample values from the tables

$z \in Z^{(1)}$	3.83	7.02	10.17	13.32
$J_0(z)$	-0.40	0.30	-0.25	0.22
c_z	??	??	??	??

(d) $f(x) = x^2$ over $[0, 2]$, $p = 2$.

SOLUTION.

$$f(x) = \sum_{z \in Z^{(2)}} c_z J_1(zx/2), \quad 0 \leq x \leq 2,$$

where

$$c_z = \frac{2}{4J_1(z)^2}\int_0^2 f(x)J_2(zx/2)xdx$$

On setting $x = 2t/z,\ f(x) = x^2 = 4t^2/z^2$

$$\begin{aligned}
&= \frac{8}{z^4 J_1(z)^2}\int_0^z t^3 J_2(t)dt \\
&= \frac{8}{z^4 J_1(z)^2} z^3 J_3(z) \quad (\text{Since } x^3 J_2 = [x^3 J_3]') \\
&= \frac{8J_3(z)}{zJ_1(z)^2} = \frac{-8}{zJ_1(z)}, \qquad z \in Z^{(2)}.
\end{aligned}$$

The last equality is due to

$$J_3(z) + J_1(z) = \frac{4}{z}J_2(z) = 0.$$

Sample values from the tables

$z \in Z^{(2)}$	5.1356	8.4172	11.6198	14.7960
$J_1(z)$	-0.3397	0.2713	-0.2324	0.2065
c_z	4.5857	-3.5033	2.9625	-2.6183

Explicitly, substituting the first few values, the Bessel series is

$$x^2 \approx 4.5857J_1(2.5678x) - 3.5033J_1(4.2086x) + 2.9625J_1(5.8099x) - 2.6183J_1(7.3980x) + \ldots$$

for $0 \le x \le 2$.

(e) $f(x) = \sqrt{x}$ over $[0, \pi]$, $p = \frac{1}{2}$.

SOLUTION.

$$f(x) = \sum_{z \in Z^{(1/2)}} c_z J_{1/2}(zx/\pi), \quad 0 \le x \le \pi,$$

where

$$c_z = \frac{2}{\pi^2 J_{-1/2}(z)^2}\int_0^\pi f(x)J_{1/2}(zx/\pi)xdx.$$

Now

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x \quad \text{and} \quad Z^{(1/2)} = \{\pi, 2\pi, 3\pi, \ldots\}.$$

Hence writing c_n for $z = n\pi$, we have

$$c_n = \frac{2}{\pi^2 J_{-1/2}(n\pi)^2}\int_0^\pi \sqrt{x}\sqrt{\frac{2}{n\pi x}}\sin nx.xdx = \frac{2n\pi^2}{\pi^2.2}\int_0^\pi \sqrt{\frac{2}{n\pi}}x\sin nxdx.$$

In the last equality we have evaluated $[J_{-1/2}(x)]^2 = \dfrac{2}{\pi x}\cos^2 x$ at $x = n\pi$.

Thus

$$c_n = \sqrt{\frac{2n}{\pi}}\int_0^\pi x\sin xdx = \sqrt{\frac{2n}{\pi}}\left(\left|x\frac{-\cos x}{n}\right|_0^\pi + \int_0^\pi \frac{\cos nx}{n}dx\right) = \frac{2}{n\pi}(-1)^{n+1}.$$

Hence

$$\sqrt{x} = \sqrt{2\pi} \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} J_{1/2}(nx).$$

Remark: Putting $J_{1/2}(nx) = \sqrt{\frac{2}{n\pi x}} \sin nx$ and simplifying, we get

$$x = 2 \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \sin nx,$$

the Fourier sine series of x over $[0, \pi]$.

(9) Show Schlömilch's formula

$$\exp\left(\frac{tx}{2} - \frac{x}{2t}\right) = \sum_{-\infty}^{\infty} J_n(x) t^n.$$

Use this formula to show that

$$J_0^2 + 2\sum_{n=1}^{\infty} J_n^2 = 1.$$

Deduce that $|J_0| \leq 1$ and $|J_n| \leq \frac{1}{\sqrt{2}}$.

Solution.

$$e^{tx/2 - x/2t} = e^{tx/2} e^{-x/2t} = \Big[\sum_{k\geq 0} \frac{(tx)^k}{2^k k!}\Big]\Big[\sum_{j\geq 0} \frac{(-1)^j x^j}{2^j t^j j!}\Big].$$

For $n \in \mathbb{Z}$, the coefficient of t^n in the above is

$$\sum_{k-j=n} \frac{(-1)^j x^{j+k}}{2^{j+k} j! k!} \;=\; \sum_{j\geq 0} \frac{(-1)^j x^{2j+n}}{2^{2j+n}(j+n)! j!} \;=\; \left(\frac{x}{2}\right)^n \sum_{j\geq 0} \frac{(ix/2)^j}{j!(j+n)!} \;=\; J_n(x).$$

This proves Schlömilch's formula. Now replace t by $-t$ and take product to get

$$1 = \left[\sum_{-\infty}^{\infty} J_n(x) t^n\right]\left[\sum_{-\infty}^{\infty} (-1)^m J_m(x) t^m\right] = \left[\sum_{-\infty}^{\infty} J_n(x) t^n\right]\left[\sum_{-\infty}^{\infty} J_{-m}(x) t^m\right].$$

This shows that $J_0^2 + 2\sum_{n=1}^{\infty} J_n^2 = 1$, along with a sequence of identities:

$$\sum_{j\in\mathbb{Z}} J_{m+j} J_m = 0 \text{ for } m \in \mathbb{Z} \setminus \{0\}.$$

(Just look at the coefficients of various powers of t.) The bounds on $|J_n|$ are now obvious.

(10) Show that

$$
\begin{aligned}
\int J_0(x)dx &= J_1(x) + \int \frac{J_1(x)dx}{x} \\
&= J_1(x) + \frac{J_2(x)}{x} + 1.3\int \frac{J_2(x)dx}{x^2} \\
&= J_1(x) + \frac{J_2(x)}{x} + \frac{1.3J_3(x)}{x^2} + 1.3.5\int \frac{J_3(x)dx}{x^3} \\
&. \\
&. \\
&. \\
&= J_1(x) + \frac{J_2(x)}{x} + \frac{1.3J_3(x)}{x^2} + \cdots + \frac{1.3.5\ldots(2n-3)J_n(x)}{x^{n-1}} \\
&\qquad + 1.3.5\ldots(2n-1)\int \frac{J_n(x)dx}{x^n}
\end{aligned}
$$

SOLUTION. We use induction on n.

$$
\begin{aligned}
\int \frac{J_n(x)dx}{x^n} &= \int \frac{x^{n+1}J_n(x)dx}{x^{2n+1}} = \int \frac{[x^{n+1}J_{n+1}(x)]'dx}{x^{2n+1}} \\
&= x^{-2n-1}[x^{n+1}J_{n+1}(x)] - \int (-2n-1)x^{-2n-2}[x^{n+1}J_{n+1}(x)]dx \\
&= \frac{J_{n+1}(x)}{x^n} + (2n+1)\int \frac{J_{n+1}dx}{x^{n+1}}.
\end{aligned}
$$

Substituting in the n-th step, assumed to be valid by induction hypothesis, we get the validity of the $(n+1)$-th step.

Optional problems.

(1) Show that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dx}[J_n^2 + J_{n+1}^2] &= \frac{n}{x}J_n^2 - \frac{n+1}{x}J_{n+1}^2, \\
\frac{d}{dx}[xJ_nJ_{n+1}] &= x(J_n^2 - J_{n+1}^2),
\end{aligned}
$$

and deduce that

$$
J_0^2 + 2\sum_{n=1}^{\infty} J_n^2 = 1 \quad \text{and} \quad \sum_{0}^{\infty}(2n+1)J_nJ_{n+1} = \frac{x}{2}.
$$

Hint for the second identity: Look at

$$
\frac{d}{dx}\Big[x\sum_{n=0}^{\infty}(2n+1)J_nJ_{n+1}\Big]
$$

SOLUTION. Recall that $J_{n-1} \pm J_{n+1} = \dfrac{2nJ_n}{x}, 2J_n'$ respectively, and $[x^{\pm n}J_n]' = \pm x^{\pm n}J_{n\mp 1}$.

(a)

$$\begin{aligned}
\frac{1}{2}\frac{d}{dx}\Big[J_n^2+J_{n-1}^2\Big] &= J_nJ_n'+J_{n+1}J_{n+1}' \\
&= J_n\Big[\frac{J_{n-1}-J_{n+1}}{2}\Big]-J_{n+1}\Big[\frac{J_n-J_{n+2}}{2}\Big] \\
&= \frac{J_nJ_{n-1}-J_{n+1}J_{n+2}}{2} \\
&= \frac{1}{2}\Big[J_n\Big(\frac{2n}{x}J_n-J_{n+1}\Big)-J_{n+1}\Big(\frac{2n+2}{x}J_{n+1}-J_n\Big)\Big] \\
&= \frac{n}{x}J_n^2-\frac{n+1}{x}J_{n+1}^2.
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{1}{2}\frac{d}{dx}\Big[xJ_nJ_{n+1}\Big] &= \frac{1}{2}\frac{d}{dx}\Big[x^{n+1}J_n.x^{-n}J_{n+1}\Big] \\
&= [x^{n+1}J_{n+1}]'x^{-n}J_n+x^{n+1}J_{n+1}[x^{-n}J_n]' \\
&= x^{n+1}J_n.x^{-n}J_n-x^{n+1}J_{n+1}.x^{-n}J_{n+1} \\
&= x[J_n^2-J_{n+1}^2].
\end{aligned}$$

(c)

$$J_0^2+2\sum_{n\ge1}J_n^2=(J_0^2+J_1^2)(J_1^2+J_2^2)(J_2^2+J_3^2)+\cdots=\sum_{n\ge1}(J_{n-1}^2+J_n^2).$$

Therefore,

$$\begin{aligned}
\frac{d}{dx}(J_0^2+2\sum_{n=1}^{\infty}J_n^2) &= \sum_{n\ge1}\frac{d}{dx}\Big[(J_{n-1}^2+J_n^2)\Big] \\
&= \sum_{n\ge1}\Big[\frac{2n-2}{x}J_{n-1}^2-\frac{2n}{x}J_n^2\Big] \qquad \text{(a telescopic sum)} \\
&= 0.
\end{aligned}$$

Therefore, $J_0^2+2\sum_{n\ge1}J_n^2$ is a constant which is 1, at $x=0$.

(d)

$$\begin{aligned}
\frac{d}{dx}\Big(x\sum_{n=0}^{\infty}(2n+1)J_nJ_{n+1}\Big) &= \sum_{n=0}^{\infty}(2n+1)[x(J_nJ_{n+1})]' \\
&= \sum_{n=0}^{\infty}(2n+1)x[J_n^2-J_{n+1}^2] \qquad \text{(from (b))} \\
&= x\Big[\sum_{n=0}^{\infty}(2n+1)J_n^2-\sum_{n=0}^{\infty}(2n+1)J_{n+1}^2\Big] \\
&= x\Big[\sum_{n=0}^{\infty}(2n+1)J_n^2-\sum_{n=1}^{\infty}(2n-1)J_n^2\Big] \\
&= x[J_0^2+2\sum_{n\ge1}J_n^2]=x \qquad \text{(form (c))}.
\end{aligned}$$

(2) (i) Use the method of variation of parameters to find a solution to the Bessel equation of integral order n that is linearly independent of J_n.
(ii) Use the theory of regular singular equations to show that this solution will be of the form

$$K(\log x)J_n(x) + x^{-n}h(x),\ K \neq 0$$

and $h(x)$ being entire and $h(0) \neq 0$.

SOLUTION. (i) For convenience we will write J for J_n. Let $Y(x) = v(x)J(x)$ be a second solution. Then

$$Y' = vJ' + v'J \quad \text{and} \quad Y'' = vJ'' + 2v'J' + v''J.$$

Substitution in $x^2y'' + xy' + (x^2 - n^2)y = 0$, gives

$$2v'x^2J' + (x^2v'' + xv')J = 0.$$

On simplifying we get,

$$\frac{J'}{J} = -\frac{xv'' + v'}{2v'x} = -\frac{[xv']'}{2xv'}.$$

Now integration gives $xv' = J^{-2}$ so that $v = \displaystyle\int \frac{dx}{xJ^2}$. Thus $Y = J\displaystyle\int \frac{dx}{xJ^2}$.
Remark: From the theory of infinite products applied to the entire function J, we can conclude the claims of part (ii) now.
(ii) From the theory of differential equations, which are regular singular at 0, and whose indicial equation has roots differing by an integer, we have the following: Let the first solution corresponding to the 'larger' root r_1 be

$$y(x) = x^{r_1}\sum_{n\geq 0} a_n x^n.$$

Then the second solution is of the form

$$Ky(x)\log x + x^{r_2}\sum_{m\geq 0} A_m x^m.$$

Further, (i) $K \neq 0$ if $r_1 - r_2 = 0$ and (ii) $A_0 \neq 0$ if $r_1 - r_2$ is a positive integer.
In the case of the Bessel equation of order n, the indicial equation has roots $\pm n$ and therefore $r_1 - r_2 = 2n$. For $n = 0$, the logarithmic term has to be there. Any solution which is linearly independent of J_0 is of the form

$$KJ_0(x)\log x + \sum_{m\geq 0} A_m x^m = KJ_0(x)\log x + h(x).$$

The radius of convergence of $h(x)$ is infinite and so it is an entire function. $h(0) = A_0$ may be zero. If so, to make it nonzero, just add $J_0(x)$. When $n > 0$, suppose $K = 0$. Then the second solution will be

$$Y(x) = x^{-n}\sum_{m\geq 0} A_m x^m, \quad A_0 \neq 0.$$

Substitution in the Bessel equation of order n, gives the recursions $A_m = (m+2)(m+2-2n)A_{m+2}$. However, $A_{2n-2} = 2n[(2n-2)+2-2n]A_{2n} = 0$. Clearly, then $A_{2n-4} = A_{2n-6} = \cdots = A_0 = 0$ which is a contradiction. Finally, $h(x) = \sum A_m x^m$ also has infinite radius of convergence and so is an entire function. $h(0) = A_0 \neq 0$.

Remark: For m odd, the recursions $A_m = \dfrac{A_{m-2}}{m(m-2n)}$ go right down to $A_{-1} = 0$, since the denominator never becomes 0. Therefore, only A_{2m} can be nonzero. In fact if $K = 0$, then Y is just a constant multiple of J_n.

1.5. Fourier series

Problems.

(1) Show that

$$\sum_{n=1}^{\infty} \frac{1}{n} \sin nx \sin^2 n\alpha = \begin{cases} \text{constant} & (0 < x < 2\alpha) \\ 0 & (2\alpha < x < \pi) \end{cases}$$

SOLUTION. Let

$$f(x) = \begin{cases} c \text{ if } 0 < x < 2\alpha, \\ 0 \text{ if } 2\alpha < x < \pi. \end{cases}$$

The Fourier sine series is

$$f(x) = \sum_{n\geq 1} b_n \sin nx, \quad \text{with } b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx.$$

Thus

$$b_n = \frac{2}{\pi} \int_0^{2\alpha} c \sin nx dx = \frac{4c \sin^2 n\alpha}{n\pi}.$$

Therefore,

$$\frac{4c}{\pi} \sum \frac{1}{n} \sin^2 n\alpha \sin nx = f(x).$$

or equivalently,

$$\sum \frac{1}{n} \sin^2 n\alpha \sin nx = \frac{\pi f}{4c} = \begin{cases} \pi/4 & \text{if } 0 < x < 2\alpha, \\ 0 & \text{if } 2\alpha < x < \pi. \end{cases}$$

Thus in fact the constant is $\pi/4$.

(2) Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos nx}{n^2} = \frac{\pi^2}{12} - \frac{x^2}{4}, \quad (-\pi \leq x \leq \pi).$$

SOLUTION. Let

$$\frac{\pi^2}{2} - \frac{x^2}{4} = a_0 + \sum_{n\geq 1} [a_n \cos nx + b_n \sin nx], \quad -\pi < x < \pi.$$

Then $b_n = 0$ for all $n \geq 1$.

$$a_0 = \frac{1}{\pi} \int_0^{\pi} \left(\frac{\pi^2}{12} - \frac{x^2}{4} \right) dx = 0.$$

For $n \geq 1$,

$$a_n = \frac{2}{\pi} \int_0^{\pi} \left(\frac{\pi^2}{12} - \frac{x^2}{4} \right) \cos nx dx = -\frac{2}{\pi} \int_0^{\pi} (-x/2) \frac{\sin nx}{n} dx$$
$$= \frac{1}{\pi} \left[\left| x \frac{-\cos nx}{n^2} \right|_0^{\pi} + \int_0^{\pi} \frac{\cos nx}{n^2} dx \right] = \frac{(-1)^{n-1}}{n^2}.$$

Hence

$$\frac{\pi^2}{12} - \frac{x^2}{4} = \sum_{n\geq 1} \frac{(-1)^{n-1} \cos nx}{n^2}.$$

(3) Show that

$$\sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{(2n+1)^3} = \frac{1}{8}\pi x(\pi - x), \quad (0 \le x \le \pi).$$

SOLUTION. We have to find the Fourier sine series of the given function. For $n \ge 1$,

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi \frac{\pi x(\pi-x)}{8} \sin nx dx \\ &= \frac{1}{4}\left[\left| x(\pi - x)\frac{\cos nx}{n}\right|_0^\pi \right] + \int_0^\pi (\pi - 2x)\frac{\cos nx}{n} \\ &= \frac{1-(-1)^n}{2n^3} \\ &= \begin{cases} 0 \text{ if } n \text{ is even,} \\ \frac{1}{n^3} \text{ if } n \text{ is odd.} \end{cases} \end{aligned}$$

(4) Use the Fourier expansions given in problems (1), (2) and (3) along with Fourier's Theorem to deduce the following results.

(a) $1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \ldots = \frac{2\pi}{3\sqrt{3}}$

SOLUTION. In Problem (1) take $\alpha = \pi/3$. Then $x = \pi/3$ is a point of continuity of f. Hence evaluating at $x = \pi/3$,

$$\sum \frac{1}{n}\sin^3\frac{n\pi}{3} = \pi/4.$$

Further

$$\sin\frac{n\pi}{3} = \begin{cases} \sqrt{3}/2 & \text{if } n \equiv 1, 2 (\text{mod} 6) \\ -\sqrt{3}/2 & \text{if } n \equiv 4, 5 (\text{mod} 6) \\ 0 & \text{if } n \equiv 3, 6 (\text{mod} 6). \end{cases}$$

Hence

$$\frac{\pi}{4} = \frac{3\sqrt{3}}{8}\left[1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \ldots\right]$$

which is equivalent to the given identity.

(b) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \frac{1}{10} - \frac{1}{11} + \ldots = \frac{\pi}{3\sqrt{3}}$

SOLUTION. In Problem (1) again, take $\alpha = \pi/3$ but this time evaluate at $x = 2\pi/3$ which is a simple jump discontinuity.

$$\frac{f_+(2\pi/3) + f_-(2\pi/3)}{2} = \frac{0 + \pi/4}{2} = \frac{\pi}{8}.$$

Hence

$$\frac{\pi}{8} = \sum \frac{1}{n}\sin\frac{2n\pi}{3}\sin^2\frac{n\pi}{3}.$$

Since

$$\sin^2\frac{n\pi}{3} = \begin{cases} 3/4 & \text{if } n \equiv 1, 2, 4, 5 (\text{mod} 6) \\ 0 & \text{if } n \equiv 3, 6 (\text{mod} 6) \end{cases},$$

and

$$\sin\frac{2n\pi}{3} = \begin{cases} \sqrt{3}/2 & \text{if } n \equiv 1,4(\text{mod}6) \\ -\sqrt{3}/2 & \text{if } n \equiv 2,5(\text{mod}6) \\ 0 & \text{if } n \equiv 3,6(\text{mod}6) \end{cases},$$

we have

$$\sin^2\frac{n\pi}{3}\sin\frac{2n\pi}{3} = \begin{cases} 3\sqrt{3}/8 & \text{if } n \equiv 1,4(\text{mod}6) \\ -3\sqrt{3}/8 & \text{if } n \equiv 2,5(\text{mod}6) \\ 0 & \text{if } n \equiv 3,6(\text{mod}6). \end{cases}$$

Hence

$$\frac{3\sqrt{3}}{8}\left[1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \frac{1}{10} - \frac{1}{11} + \dots\right] = \frac{\pi}{8}.$$

(c) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$

SOLUTION. In Problem (2) take $x = 0$. Then

$$\sum_{n\geq 1}\frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots.$$

(d) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6}$ (Euler's formula)

SOLUTION. In Problem (2) again, take $x = \pi$ which is a point of continuity on periodic extension of f. Then

$$\sum\frac{(-1)^{n-1}(-1)^n}{n^2} = \frac{\pi^2}{12} - \frac{\pi^2}{4} = -\frac{\pi^2}{6}$$

which implies

$$\sum\frac{1}{n^2} = \frac{\pi^2}{6}.$$

(e) $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - + \dots = \frac{\pi^3}{32}$

SOLUTION. In Problem (3), set $x = \pi/2$ which is a point of continuity. Then

$$\sum_{n\geq 0}\frac{\sin(n\pi + \frac{\pi}{2})}{(2n+1)^3} = \frac{\pi}{8}\cdot\frac{\pi}{2}\cdot\frac{\pi}{2}.$$

Since $\sin(n\pi + \frac{\pi}{2}) = (-1)^n$, we obtain the necessary identity.

(f) $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2}\dots = \frac{\pi^2}{8}$

SOLUTION. From Problems (4(c)) and (4(d)) (that is essentially using Problem (2)),

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$$

and

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6}$$

Taking their average gives

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots=\frac{\pi^2}{8}.$$

(g) $\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\frac{1}{7.9}+\ldots=\frac{\pi}{4}-\frac{1}{2}$

SOLUTION. In Problem (1), take $\alpha=\pi/2$ and evaluate at $x=\pi/2$, to obtain

$$\sum_{n\geq 0}\frac{(-1)^n}{2n+1}=\frac{\pi}{4}.$$

Now in the given equation, the lhs is

$$\begin{aligned}\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\frac{1}{7.9}+\ldots &= \frac{1}{2}(1-\frac{1}{3})-\frac{1}{2}(\frac{1}{3}-\frac{1}{5})+\frac{1}{2}(\frac{1}{5}-\frac{1}{7})-\frac{1}{2}(\frac{1}{7}-\frac{1}{9})+\ldots\\ &=\frac{1}{2}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots\\ &=-\frac{1}{2}+(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots)=-\frac{1}{2}+\frac{\pi}{4}.\end{aligned}$$

(5) Using the Parseval identity, show that

$$1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\ldots=\frac{\pi^4}{96}.$$

Hint: Use

$$f(x)=\begin{cases}x & \text{if } -\pi/2<x<\pi/2,\\ \pi-x & \text{if } \pi/2<x<3\pi/2.\end{cases}$$

or problem (2).

SOLUTION. The given periodic function is equivalent to an odd 2π-periodic extension of

$$f(x)=\frac{\pi}{2}-\left|\frac{\pi}{2}-x\right|,\ 0\leq x\leq\pi.$$

Hence in the Fourier series all $a_n=0$ and

$$b_n=\frac{2}{\pi}\int_0^\pi\left(\frac{\pi}{2}-\left|\frac{\pi}{2}-x\right|\right)\sin nx dx=\frac{4}{\pi n^2}\sin\frac{n\pi}{2}.$$

This implies

$$b_n^2=\begin{cases}0 & \text{if } n \text{ is even},\\ \dfrac{16}{\pi^2 n^4} & \text{if } n \text{ is odd}.\end{cases}$$

Further

$$\frac{1}{\pi}\int_{-\pi}^{\pi}f^2dx=\frac{2}{\pi}\int_0^\pi f^2dx=\frac{\pi^2}{2}.$$

Hence etc.

Aliter: In Problem (2), $f(x)=\frac{\pi^2}{12}-\frac{x^2}{4}$. The Fourier cefficients are

$a_n = \dfrac{(-1)^{n-1}}{n^2}$, $n \geq 1$ and the remaining Fourier coefficients are zero. By the Parseval identity,

$$\sum_{n\geq 1} \frac{1}{n^4} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx = \frac{1}{72\pi}\int_0^{\pi} (\pi^2 - 3x^2)^2 dx.$$

Hence $\displaystyle\sum_{n\geq 1} \frac{1}{n^4} = \frac{\pi^4}{90}$. Now the sum of the even terms

$$\sum \frac{1}{(2n)^4} = \frac{1}{2^4}\sum \frac{1}{n^4} = \frac{\pi^4}{2^4.90}.$$

Hence

$$\sum_{n\ odd} \frac{1}{n^4} = \frac{\pi^4}{90} - \frac{\pi^4}{2^4.90} = \frac{\pi^4}{96}.$$

(6) Find the Fourier series of the function $f(x)$ which is assumed to have the period 2π, where

(a) $f(x) = x, \quad 0 < x < 2\pi$.

Solution.

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} x dx = \pi \quad \text{and} \quad a_n = \frac{1}{\pi}\int_0^{2\pi} x\cos nx dx = 0,\ n \geq 1.$$

Next

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x \sin nx dx = -\frac{2}{n}.$$

Thus

$$f(x) = x = \pi - 2\sum \frac{\sin nx}{n}, \quad 0 < x < \pi.$$

(b) $f(x) = \begin{cases} -x & -\pi \leq x < 0 \\ x & 0 \leq x < \pi \end{cases}$

Solution. Since $f(x) = |x|$ is an even function, all $b_n = 0$. Also

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x dx = \frac{\pi}{2}.$$

For $n \geq 1$,

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx dx = \begin{cases} 0 & \text{if } n \text{ is even,} \\ -4/\pi n^2 & \text{if } n \text{ is odd.} \end{cases}$$

Hence

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n\geq 0} \frac{\cos(2n+1)x}{(2n+1)^2}.$$

(c) $f(x) = x + |x|, \quad -\pi < x < \pi.$

SOLUTION. x has a sine series and $|x|$ has a cosine series. Hence combining,

$$x + |x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n\ odd} \frac{\cos nx}{n^2} + \frac{2}{\pi} \sum_{n \geq 1} \frac{\sin nx}{n}.$$

(7) Find the Fourier series of the periodic function $f(x)$ of period $p = 2$ where

$$f(x) = \begin{cases} 0 & -1 < x < 0, \\ x & 0 < x < 1. \end{cases}$$

SOLUTION.

$$a_0 = \int_{-1}^{1} f(x) dx/2 = \int_0^1 x dx = 1/4.$$

For $n \geq 1$,

$$a_n = \int_0^1 x \cos n\pi x dx = \begin{cases} 0 \text{ if } n \text{ is even,} \\ \frac{-2}{n^2\pi^2} \text{ if } n \text{ is odd.} \end{cases}$$

Further,

$$b_n = \int_0^1 x \sin n\pi x dx = \frac{(-1)^{n+1}}{n\pi}.$$

Therefore,

$$f(x) = \frac{1}{4} - \frac{2}{\pi^2} \sum_{n \geq 0} \frac{\cos(2n+1)\pi x}{(2n+1)^2} + \frac{1}{\pi} \sum_{n \geq 1} \frac{((-1)^{n+1} \sin n\pi x}{n}.$$

(8) State whether the given function is even or odd. Find its Fourier series.

(a)

$$f(x) = \begin{cases} k & -\pi/2 < x < \pi/2, \\ 0 & \pi/2 < x < 3\pi/2. \end{cases}$$

SOLUTION. By 2π-periodicity, the function on $(-\pi, \pi)$ reads

$$f(x) = \begin{cases} k & \text{if } 0 \leq |x| \leq \frac{\pi}{2}, \\ 0 & \text{if } \frac{\pi}{2} \leq |x| \leq \pi. \end{cases}$$

Thus f is an *even* function. Hence $b_n = 0$ for all n. Also $a_0 = (1/\pi) \int_0^\pi f(x) dx = k/2$. For $n \geq 1$,

$$a_n = (2/\pi) \int_0^{\pi/2} k \cos nx dx = \frac{2k}{\pi} \sin \frac{n\pi}{2} = \begin{cases} \pm \frac{2k}{n\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

As for the sign, if $n = 2m+1$, then $\pm = (-1)^m$. The Fourier series is

$$f(x) = \frac{k}{2} + \frac{2k}{\pi} \sum_{m \geq 0} \frac{(-1)^m \cos(2m+1)x}{2m+1}.$$

(b)

$$f(x) = 3x(\pi^2 - x^2), \quad -\pi < x < \pi.$$

SOLUTION. Clearly, $f(x)$ is an odd function. Hence $a_n = 0$ for all n while $b_n = \dfrac{2}{\pi}\displaystyle\int_0^\pi f(x)\sin nx dx$. (The calculation is a *third degree* torture.)

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^\pi 3x(\pi^2 - x^2)\sin nx dx \\
&= \frac{6}{\pi}\left[\left| x(\pi^2 - x^2)\frac{-\cos nx}{n}\right|_0^\pi + \int_0^\pi (\pi^2 - 3x^2)\frac{\cos nx}{n}dx\right] \\
&= \frac{6}{\pi}\left[0 + \left|(\pi^2 - 3x^2)\frac{\sin nx}{n^2}\right|_0^\pi + \int_0^\pi 6x\frac{\sin nx}{n^2}dx\right] \\
&= \frac{36}{\pi n^2}\left[\left| x\frac{-\cos nx}{n}\right|_0^\pi + \int_0^\pi \frac{\cos nx}{n}dx\right] = \frac{(-1)^{n+1}36}{n^3}.
\end{aligned}$$

Hence

$$3x(\pi^2 - x^2) = 36\sum_{n\geq 1}\frac{(-1)^{n+1}\sin nx}{n^3}.$$

(9) Find the Fourier series for the given functions f on the prescribed interval.

(a)

$$f(x) = \begin{cases} -1 & \text{if } -1 \leq x < 0 \\ 1 & \text{if } 0 \leq x \leq 1 \end{cases}$$

for $|x| \leq 1$.

SOLUTION. f is an odd function. Hence $a_n = 0$ for all n while

$$b_n = 2\int_0^1 1.\sin n\pi x dx = \begin{cases} 0 & \text{if } n \text{ even,} \\ 4/n\pi & \text{if } n \text{ odd.} \end{cases}$$

Hence

$$f(x) = \operatorname{sgn}(x) = \frac{4}{\pi}\sum_{n\geq 0}\frac{\sin(2n+1)\pi x}{2n+1}, -1 < x < 1.$$

(b)

$$f(x) = \begin{cases} -x, & -1 \leq x < 0 \\ x, & 0 \leq x \leq 1 \end{cases}$$

for $|x| \leq 1$.

SOLUTION. In short $f(x) = |x|$. So it is an even function. Hence $b_n = 0$ for all $n \geq 1$. Also

$$a_0 = \int_0^1 x dx = 1/2.$$

Further,

$$a_n = 2\int_0^1 x\cos n\pi x dx = \begin{cases} 0 & \text{if } n \text{ even,} \\ -4/n^2\pi^2 & \text{if } n \text{ odd.} \end{cases}$$

Hence

$$|x| = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k\geq 0}\frac{\cos(2k+1)\pi x}{(2k+1)^2}, \quad 1 \leq x \leq 1.$$

(c)

$$f(x) = \begin{cases} 0, & -2 \leq x < 1 \\ 3, & 1 \leq x \leq 2 \end{cases}$$

for $|x| \leq 2$.

Solution. Observe that $\frac{2}{3}f(x)-1$ is just the signum function $\operatorname{sgn}(x)$. The Fourier cefficients of the latter are

$$b_n = \frac{2}{2}\int_0^2 \sin\frac{n\pi x}{2} = \begin{cases} 0 & \text{if } n \text{ even,} \\ 4/n\pi & \text{if } n \text{ odd,} \end{cases}$$

and $a_n = 0$. Hence

$$f(x) = \frac{3}{2} + \frac{6}{\pi}\sum_{n\geq 0}\frac{1}{2n+1}\sin\frac{(2n+1)\pi x}{2}, \quad -2 < x < 2.$$

(d)

$$f(x) = e^{x/a}, \quad |x| \leq l.$$

Solution. We have

$$a_0 = \frac{1}{2l}\int_{-l}^{l} e^{x/a}dx = \frac{a\sinh(l/a)}{l}.$$

For $n \geq 1$,

$$\begin{aligned} a_n &= \frac{1}{l}\int_{-l}^{l}\cos\frac{n\pi x}{l}\cdot e^{x/a}dx \\ &= \frac{1}{l}\left[\left|\cos\frac{n\pi x}{l}\cdot ae^{x/a}\right|_{-l}^{l} + \frac{n\pi a}{l}\int_{-l}^{l}\sin\frac{n\pi x}{l}\cdot e^{x/a}dx\right] \\ &= \frac{2a(-1)^n}{l}\cdot\sinh(l/a) + \frac{n\pi a}{l}b_n. \end{aligned}$$

Similarly, $b_n = -\dfrac{n\pi a}{l}a_n$. These imply

$$a_n = (-1)^n\frac{2al\sinh(l/a)}{l^2+n^2\pi^2} \quad \text{and} \quad b_n = (-1)^{n+1}\frac{2n\pi a^2\sinh(l/a)}{l^2+n^2\pi^2}, \ n \geq 1.$$

(e)

$$f(x) = \sin^2 x, \quad |x| \leq \pi.$$

Solution. $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$ is already the required Fourier series.

(10) Expand each of the following functions in a Fourier cosine series on the prescribed interval.

(a)

$$f(x) = e^{-x}, \quad 0 \le x \le 1.$$

SOLUTION.

$$e^{-x} = (1 - 1/e) + \sum_{n\ge 1} \frac{2\left(1 - \frac{(-1)^n}{e}\right)}{n^2\pi^2 + 1} \cos n\pi x, \quad 0 \le x \le 1.$$

(b)

$$f(x) = \begin{cases} 0, & 0 \le x \le 1 \\ 1, & 1 \le x \le 2 \end{cases}$$

for $0 \le x \le 2$.

SOLUTION. We have

$$a_0 = \frac{1}{2}\int_0^2 f(x)dx = \frac{1}{2}\int_1^2 dx = \frac{1}{2}.$$

Next for $n \ge 1$,

$$a_n = \int_1^2 \cos\frac{n\pi x}{2}dx - \frac{2}{n\pi}\sin\frac{n\pi}{2} = \begin{cases} 0 & \text{if } n \text{ even,} \\ \dfrac{2(-1)^{\frac{n+1}{2}}}{n\pi} & \text{if } n \text{ odd.} \end{cases}$$

Hence

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\sum_{k\ge 0} \frac{(-1)^k}{2k+1}\cos\frac{(2k+1)\pi x}{2}, \quad 0 \le x \le 2.$$

(c)

$$f(x) = 2\sin x \cos x, \quad 0 \le x \le \pi.$$

SOLUTION. $f(x) = \sin 2x$. For cosine series,

$$a_0 = \frac{1}{\pi}\int_0^\pi \sin 2x dx = 0.$$

For $n \ge 1$,

$$a_n = \frac{2}{\pi}\int_0^\pi \sin 2x \cos nx dx = \frac{1}{\pi}\int_0^\pi [\sin(n+2)x - \sin(n-2)x]dx = 0.$$

This is 0 if $n = 2$. For $n \ne 2$, the calculation continues as follows.

$$\begin{aligned} a_n &= \frac{1}{\pi}\left[\frac{\cos(n-2)x}{n-2} - \frac{\cos(n+2)x}{n+2}\right]_0^\pi \\ &= \frac{1}{\pi}\left[\frac{(-1)^n - 1}{n-2} - \frac{(-1)^n - 1}{n+2}\right] \\ &= \begin{cases} 0 & \text{if } n \text{ even (including } n = 0, 2), \\ \dfrac{-8}{\pi(n^2-4)} & \text{if } n \text{ odd.} \end{cases} \end{aligned}$$

Hence

$$\sin 2x = \frac{-8}{\pi}\sum_{k\ge 0} \frac{\cos(2k+1)x}{(2k+1)^2 - 4}, \quad 0 \le x \le \pi.$$

Remark: We can similarly show that

$$\sin x = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k\geq 1}\frac{\cos 2kx}{4k^2-1}, \quad 0 \leq x \leq \pi.$$

(11) Expand each of the following functions in a Fourier sine series on the prescribed interval.

(a)

$$f(x) = e^{-x}, \quad 0 < x < 1.$$

SOLUTION.

$$b_n = 2\int_0^1 e^{-x}\sin n\pi x dx = \frac{2n\pi}{n^2\pi^2+1}\left(1 - \frac{(-1)^n}{e}\right).$$

(b)

$$f(x) = \begin{cases} x, & 0 < x < a \\ a, & a \leq x \leq 2a \end{cases}$$

for $0 < x < 2a$.

SOLUTION.

$$b_n = \frac{2}{2a}\int_0^{2a} f(x)\sin\frac{n\pi x}{2a}dx = \frac{1}{a}\left[\int_0^a x\sin\frac{n\pi x}{2a}dx + \int_a^{2a} a\sin\frac{n\pi x}{2a}dx\right]$$

$$= \frac{4a}{n^2\pi^2}\sin\frac{n\pi}{2} - (-1)^n\frac{2a}{n\pi} = \begin{cases} \dfrac{-2a}{n\pi} & \text{if } n \text{ even,} \\ \dfrac{2a}{n\pi} + \dfrac{(-1)^{\frac{n-1}{2}}4a}{n^2\pi^2} & \text{if } n \text{ odd.} \end{cases}.$$

(c)

$$f(x) = 2\sin x\cos x, \quad 0 < x < \pi.$$

SOLUTION. $f(x) = \sin 2x$ is already a Fourier sine series over $(0, \pi)$. Here,

$$b_n = \delta_{2,n} = \begin{cases} 0 & \text{if } \ n \neq 2 \\ 1 & \text{if } \ n = 2. \end{cases}$$

(d)

$$f(x) = \cos x, \quad 0 < x < \pi.$$

SOLUTION.

$$b_n = \frac{2}{\pi}\int_0^{\pi}\cos x\sin nx dx$$

$$= \frac{1}{\pi}\int_0^{\pi}[\sin(n+1)x + \sin(n-1)x]dx = \begin{cases} 0 & \text{if } n \text{ odd,} \\ \dfrac{4n}{(n^2-1)\pi} & \text{if } n \text{ even.} \end{cases}$$

Hence

$$\cos x = \frac{4}{\pi}\sum_{n\ even}\frac{n\sin nx}{n^2-1} = \frac{8}{\pi}\sum_{k\geq 1}\frac{k\sin 2kx}{4k^2-1}.$$

1.6. Heat equation by separation of variables

For the two-dimensional heat equation, the following are relevant.

(a) The Laplacian in polar coordinates in the plane is

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

(b) The Laplacian in spherical polar coordinates for the sphere of radius b is

$$\Delta = \frac{1}{b^2}\Big(\frac{\partial^2}{\partial \varphi^2} + \cot\varphi\frac{\partial}{\partial \varphi} + \frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial \theta^2}\Big).$$

Problems.

(1) Which of the following PDEs can be reduced to two or more ODEs by the method of separation of variables?

(a) $au_{xy} + bu = 0$

Solution. Yes. Let $u(x,y) = X(x)Y(y)$. Then,

$$a\frac{X'(x)}{X(x)} = -b\frac{Y(y)}{Y'(y)}.$$

(b) $au_{xx} + 2bu_{xy} + cu_{yy} = 0$

Solution. No, if $abc \neq 0$. Let $u(x,y) = X(x)Y(y)$. Then,

$$aX''(x)Y(y) + 2bX'(x)Y'(y) + cX(x)Y''(y) = 0$$

does not separate.

(c) $au_{xx} + 2bu_{xy} + cu_y = 0$

Solution. Yes. Let $u(x,y) = X(x)Y(y)$. Then,

$$\frac{aX''(x)}{2bX'(x) + c} = -\frac{Y'(y)}{Y(y)}.$$

(d) $z_{xx} + xyz_y = 0$

Solution. Yes. Let $z(x,y) = X(x)Y(y)$. Then,

$$\frac{X''(x)}{xX(x)} = -\frac{yY'(y)}{Y(y)}.$$

(e) $f(x)\theta_{tt} = a^2[f(x)\theta_x]_x$

Solution. Yes. Let $\theta(x,t) = X(x)Y(t)$. Then,

$$\frac{Y''(t)}{Y(t)} = a^2\left[\frac{f'(x)X''(x)}{f(x)X(x)} + \frac{X''(x)}{X(x)}\right].$$

(2) The curved surface of a thin rod of length ℓ is insulated. The temperature throughout the rod is 100. If at each end of the rod the temperature is suddenly reduced to 0 at time $t = 0$, find the temperature subsequently. What is the explicit temperature at the mid-point of the rod and how does it behave with respect to the time variable t?

Solution. The homogeneous heat equation is

$$u_t = ku_{xx}, \quad 0 < x < \ell.$$

We are given Dirichlet boundary conditions and initial condition $u_0(x) = u(x,0) = 100$. So

$$u(x,t) = \sum_{n\geq 1} Y_n(t) \sin \frac{n\pi x}{\ell},$$

where $Y_n(t)$ solves the ODE

$$\dot{Y}_n(t) + n^2(\pi/\ell)^2 k Y_n(t) = 0, \quad Y_n(0) = b_n,$$

with

$$b_n = \frac{2}{\ell}\int_0^\ell 100 \sin\frac{n\pi x}{\ell} dx = \begin{cases} 0 & \text{if } n \text{ even,} \\ \dfrac{400}{n\pi} & \text{if } n \text{ odd.} \end{cases}$$

Explicitly,

$$Y_n(t) = b_n e^{-n^2(\pi/\ell)^2 kt}.$$

Substituting,

$$u(x,t) = \frac{400}{\pi}\sum_{n \text{ odd}} \frac{1}{n}\sin\frac{n\pi x}{\ell} e^{-n^2(\pi/\ell)^2 kt}.$$

At the *midpoint* $x = \ell/2$ and

$$u(\ell/2,t) = \frac{400}{\pi}\sum_{n \text{ odd}} (-1)^{\frac{n-1}{2}} \frac{e^{-n^2(\pi/\ell)^2 kt}}{n}.$$

$u(\ell/2,t) \to 0$ exponentially fast as $t \to \infty$. For a rigorous proof, note that

$$\begin{aligned} |u(x,t)| &< \sum_{n \text{ odd}} (e^{-(\pi/\ell)^2 kt})^n \text{ (termwise)} \\ &= \frac{e^{-(\pi/\ell)^2 kt}}{1 - e^{-2(\pi/\ell)^2 kt}} \\ &< 2e^{-(\pi/\ell)^2 kt} \qquad \text{for } t > \frac{\ell^2 \log 2}{2\pi^2 k}. \end{aligned}$$

The last expression is of the form Ce^{-kt} showing exponential decay.

(3) Solve the following nonhomogeneous differential equation

$$u_t - u_{xx} = 8e^{-t}\sin 3x$$

with boundary and initial conditions:

$$u(0,t) = 0 = u(\pi,t) \quad \text{and} \quad u(x,0) = 2\sin 2x.$$

Solution. Expanding in the Fourier sine series over $(0,\pi)$,

$$f(x,t) = 8e^{-t}\sin 3x = \sum B_n(t)\sin nx.$$

Hence

$$B_n(t) = 0,\ n \neq 3 \quad \text{and} \quad B_3(t) = 8e^{-t}.$$

Similarly,

$$u_0(x) := u(x,0) = 2\sin 2x = \sum_{n\geq 1} b_n \sin nx$$

implies

$$b_n = 0,\ n \neq 2 \quad \text{and} \quad b_2 = 2.$$

Writing

$$u(x,t) = \sum_{n\geq 1} Y_n(t) \sin nx$$

yields the ODE

$$\dot{Y}_n(t) + n^2 Y_n(t) = B_n(t),\ Y_n(0) = b_n.$$

Substituting, we obtain $Y_n(t) = 0,\ n \neq 2, 3$, while

$$\dot{Y}_2(t) + 4Y_2(t) = 0,\ Y_2(0) = 2 \quad \text{and} \quad \dot{Y}_3(t) + 9Y_3(t) = 8e^{-t},\ Y_3(0) = 0.$$

Thus $Y_2(t) = 2e^{-4t}$. By the method of undetermined coefficients, $Y_3(t) = Ce^{-9t} + De^{-t}$. This implies $C + D = 0$ and $D = 1$ on substituting in the equations for $Y_3(t)$. Thus $Y_3(t) = e^{-t} - e^{-9t}$ and

$$u(x,t) = 2e^{-4t} \sin 2x + (e^{-t} - e^{-9t}) \sin 3x.$$

(4) Solve

$$u_t - u_{xx} = e^{-t} \cos 2x$$

with boundary and initial conditions:

$$u_x(0,t) = e^{-t},\ u_x(\pi,t) = -e^{-t} \quad \text{and} \quad u(x,0) = \sin x.$$

Hint: Start with $z(x,t) = e^{-t} \sin x$ to homogenize the boundary conditions.

Solution. Observe that $z(x,t) = e^{-t} \sin x$ solves the homogeneous heat equation with the given boundary and initial conditions. Put

$$v(x,t) := u(x,t) - z(x,t).$$

Then $v(x,t)$ solves the equation

$$v_t - v_{xx} = [u_t - u_{xx}] - [z_t - z_{xx}] = e^{-t} \cos 2x,$$

with boundary and initial conditions

$$v_x(0,t) = 0 = v_x(\pi,t) \quad \text{and} \quad v(x,0) = 0.$$

Due to the Neumann boundary conditions, let

$$v(x,t) = Y_0(t) + \sum_{n\geq 1} Y_n(t) \cos nx.$$

Since $f(x,t) = e^{-t} \cos 2x$ has only the second Fourier cosine coefficient nonzero, and $v(x,0) = 0$ has all Fourier cosine coefficients zero, we obtain $Y_n(t) = 0$, for $n \neq 2$, and

$$\dot{Y}_2(t) + 4Y_2(t) = e^{-t}, \quad Y_2(0) = 0.$$

Solving, $Y_2(t) = 1/3(e^{-t} - e^{-4t})$. Hence

$$u(x,t) = v(x,t) + z(x,t) = \frac{e^{-t} - e^{-4t}}{3} \cos 2x + e^{-t} \sin x.$$

(5) For the heat equation:

$$u_t - ku_{xx} = 0, \quad 0 < x < \ell,\ t > 0$$

with initial condition $u(x, 0) = u_0(x)$, and Neumann boundary conditions $u_x(0, t) = u_x(\ell, t) = 0$, show that

$$\int_0^\ell u(x, t)\, dx = C,$$

where C is a constant. In other words, the average temperature stays constant. Further, show that

$$\lim_{t \mapsto \infty} u(x, t) = \frac{1}{\ell}\int_0^\ell u_0(x)\, dx.$$

Compute the solution, when u_0 is:

$$(i)\, u_0(x) = x \quad \text{and} \quad (ii)\, u_0(x) = \sin^2(\frac{\pi x}{\ell}).$$

Solution.

$$\frac{d}{dt}\int_0^\ell u(x, t)\, dx = \int_0^\ell u_t(x, t)\, dx = \int_0^\ell ku_{xx}(x, t)\, dx = [ku_x(x, t)]_0^\ell = 0$$

The solution to the homogeneous heat equation with Neumann boundary conditions is

$$u(x, t) = a_0 + \sum_{n \geq 1} a_n e^{-n^2(\pi/\ell)^2 kt} \cos\frac{n\pi x}{\ell}$$

where a_n are computed by expanding the initial condition

$$u(x, 0) = u_0(x) = a_0 + \sum_{n \geq 1} a_n \cos\frac{n\pi x}{\ell}.$$

In particular, $a_0 = \dfrac{1}{\ell}\displaystyle\int_0^\ell u_0(x)dx$. Now evidently,

$$\lim_{t \mapsto \infty} u(x, t) = a_0 = \frac{1}{\ell}\int_0^\ell u_0(x)\, dx.$$

assuming that the limit can be taken, term by term, inside the summation.

(i) For $u_0(x) = x$, we have $a_0 = \ell/2$ and

$$a_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 4\ell/n^2\pi^2 & \text{if } n \text{ is odd.} \end{cases}$$

$$u(x, t) = \frac{\ell}{2} + \frac{4\ell}{\pi^2}\sum_{n \text{ odd}} \frac{e^{-n^2(\pi/\ell)^2 kt}}{n^2} \cos\frac{n\pi x}{\ell}.$$

(ii) For

$$u_0(x) = \sin^2\frac{\pi x}{\ell} = \frac{1}{2}\left(1 - \cos\frac{2\pi x}{\ell}\right),$$

we have $a_0 = 1/2,\ a_2 = -1/2$ and $a_n = 0$, for $n \neq 0, 2$. Hence

$$u(x, t) = \frac{1}{2} - \frac{1}{2}e^{-4(\pi/\ell)^2 kt} \cos\frac{2\pi x}{\ell}.$$

(6) Show that in a thin rod with insulated ends, the average temperature

$$\bar{u} = \frac{1}{\ell}\int_0^\ell u(x,t)dx$$

remains constant. Show that the temperature distribution coverges to this constant at all points uniformly and exponentially fast as $t \to \infty$. For convenience, assume that $u(x,0) = u_0(x)$ is a Riemann integrable function on $[0,\ell]$.

Solution. Since there is no external source, $u_t = ku_{xx}$. Insulated ends means that no heat can flow across the end points, so that the temperature gradient vanishes at both ends for all time that is $u_x(0,t) = 0 = u_x(\ell,t)$ and we have (homogeneous) Neumann boundary conditions. Hence the total heat content $Q = \displaystyle\int_0^\ell u(x,t)dx$ as also the average temperature $\bar{u} = Q/\ell$ remains constant. See previous problem for a proof. If

$$u_0(x) = a_0 + \sum_{n\geq 1} a_n \cos\frac{n\pi x}{\ell}$$

is the Fourier cosine series of the initial temperature distribution, then

$$u(x,t) = a_0 + \sum_{n\geq 1} a_n e^{-n^2(\pi/\ell)^2kt}\cos\frac{n\pi x}{\ell}.$$

Here $a_0 = \dfrac{1}{\ell}\displaystyle\int_0^\ell u_0(x)dx = \bar{u}$. Hence

$$u - \bar{u} = \sum_{n\geq 1} a_n e^{-n^2(\pi/\ell)^2kt}\cos\frac{n\pi x}{\ell}.$$

Since u_0 is Riemann integrable, it is bounded, say $|u_0| \leq M$. Employing this bound on

$$a_n = \frac{2}{\ell}\int_0^\ell u_0(x)\cos\frac{n\pi x}{\ell}dx$$

implies $|a_n| \leq 2M$. Therefore

$$|u-\bar{u}| \leq 2M\sum_{n\geq 1} e^{-n^2(\pi/\ell)^2kt} \leq 2M\sum_{n\geq 1}(e^{-(\pi/\ell)^2kt})^n = \frac{2Me^{-(\pi/\ell)^2kt}}{1-e^{-(\pi/\ell)^2kt}}.$$

The last expression is independent of x and tends to zero as $t \to \infty$. Hence $u(x,t) \to \bar{u}$ uniformly as $t \to \infty$.

(7) Compute the solution of

$$u_t - ku_{xx} + a^2u = 0, \quad 0 < x < \ell,\ t > 0$$

with initial condition $u(x,0) = u_0(x)$, and Dirichlet boundary conditions $u(0,t) = u(\ell,t) = 0$. Find $\lim_{t\mapsto\infty} u(x,t)$.

Solution. Put $u(x,t) = e^{-a^2t}v(x,t)$. Then v satisfies the homogeneous heat equation with Dirichlet boundary conditions. Hence

$$u(x,t) = e^{-a^2t}\sum_{n\geq 1} b_n e^{-n^2(\pi/\ell)^2kt}\sin\frac{n\pi x}{\ell},$$

where b_n are the coefficients of the Fourier sine series of $u_0(x)$ on $(0, \ell)$, that is
$$b_n = \frac{2}{\ell}\int_0^\ell u_0(x)\sin\frac{n\pi x}{\ell}.$$
This gives as before that
$$|u(x,t)| \le \frac{2M}{\ell}\frac{e^{-a^2 t}.e^{-(\pi/\ell)^2 kt}}{1-e^{-(\pi/\ell)^2 kt}},$$
and so $u(x,t) \to 0$ as $t \to \infty$, uniformly in x.

Aliter: Let $u(x,t) = X(x)T(t)$. Substituting in the given equation, we get
$$X(x)T'(t) - kX''(x)T(t) + a^2X(x)T(t) = 0, \quad \text{or} \quad \frac{T'(t)}{T(t)} = -a^2 + \frac{X''(x)}{X(x)}.$$
So the variables separate. Therefore,
$$\frac{T'(t)}{T(t)} + a^2 = k\frac{X''(x)}{X(x)} = \text{const.}$$
If we try a nonnegative constant, then due to Dirichlet boundary conditions, we get only the trivial solution. So let the constant be $-\mu^2; \mu > 0$. The Dirichlet boundary conditions imply $\frac{\mu}{c} = \frac{n\pi}{\ell}$ and
$$u_n(x,t) = b_n e^{-(n^2(\pi/\ell)^2 k + a^2)t}\sin\frac{n\pi x}{\ell}, \quad n = 1, 2, 3, \ldots.$$
Summing over all n yields the general solution given earlier.

(8) Solve the following heat equation:
$$u_t - ku_{xx} = 0, \quad 0 < x < \ell,\ t > 0$$

(a) with zero initial condition $u(x,0) = 0$ and Dirichlet boundary conditions: $u(0,t) = 0$, and $u(\ell,t) = e^{-t}$. Assume $\ell/\pi c$ is not an integer.

Solution. Note that the boundary conditions are nonhomogeneous. Consider the function
$$z(x,t) = \frac{xe^{-t}}{\ell}$$
which satisfies these boundary conditions. Set $v := u - z$. Then $v(x,t)$ is the solution to
$$v_t - kv_{xx} = \frac{xe^{-t}}{\ell}$$
with Dirchlet boundary conditions $v(0,t) = 0 = v(\ell,t)$ and the initial condition $v(x,0) = -x/\ell$. Expanding everything in Fourier sine series on $(0,\ell)$,
$$f(x,t) := \frac{xe^{-t}}{\ell} = \sum_{n\ge 1} B_n(t)\sin\frac{n\pi x}{\ell}, \quad \text{with } B_n(t) = (-1)^{n+1}\frac{2e^{-t}}{n\pi},$$
$$v_0(x) := -x/\ell = \sum_{n\ge 1} b_n \sin\frac{n\pi x}{\ell}, \quad \text{with } b_n = (-1)^n\frac{2}{n\pi},$$
and
$$v(x,t) = \sum_{n\ge 1} Y_n(t)\sin\frac{n\pi x}{\ell}.$$

The functions $Y_n(t)$ solve the IVP

$$\dot{Y}_n(t) + n^2(\pi/\ell)^2 k Y_n(t) = (-1)^{n+1}\frac{2e^{-t}}{n\pi}, \quad Y_n(0) = (-1)^n \frac{2}{n\pi}.$$

A particular integral is of the form $K_n e^{-t}$. Substitution gives

$$K_n = (-1)^{n+1}\frac{2}{n\pi}\cdot\frac{\ell^2}{kn^2\pi^2 - \ell^2}.$$

(Here we use the assumption that $\ell/\pi c$ is not an integer. Else for one particular n, a different formula will be needed.) The general solution of the ODE now is

$$Y_n(t) = C_n e^{-n^2(\pi/\ell)^2 kt} + K_n.$$

The initial condition $Y_n(0) = (-1)^n\frac{2}{n\pi}$ implies

$$C_n = (-1)^n \frac{2kn\pi}{kn^2\pi^2 - \ell^2}.$$

Thus $Y_n(t)$ and hence $v(x,t)$ are now explicit. Finally

$$u(x,t) = v(x,t) + \frac{xe^{-t}}{\ell}$$

is the required solution.

(b) with zero initial condition $u(x,0) = 0$ and Neumann boundary conditions: $u_x(0,t) = 0$ and $u_x(\ell,t) = e^{-t}$. Assume $\ell/\pi\sqrt{k}$ is not an integer.

Solution. To homogenize the boundary conditions, let

$$z(x,t) = \frac{x^2 e^{-t}}{2\ell}.$$

Let $v := u - z$. Then v satisfies the nonhomogeneous heat equation

$$v_t - kv_{xx} = f(x,t),$$

with

$$f(x,t) = -\left(\frac{x^2}{2} + k\right)\frac{e^{-t}}{\ell},$$

the initial condition $v(x,0) = -\frac{x^2}{2\ell}$, and homogeneous Neumann boundary conditions

$$v_x(0,t) = 0 = v_x(\ell,t).$$

We expand everything in a Fourier cosine series. Let

$$v(x,0) = -\frac{x^2}{2\ell} = a_0 + \sum_{n\geq 1} a_n \cos\frac{n\pi x}{\ell},$$

where

$$a_0 = \frac{1}{\ell}\int_0^\ell v(x,0)dx = -\frac{1}{\ell}\int_0^\ell \frac{x^2}{2\ell}dx = -\ell/6,$$

and for $n \geq 1$,

$$a_n = \frac{2}{\ell}\int_0^\ell \frac{-x^2}{2\ell}\cos\frac{n\pi x}{\ell}dx = (-1)^{n+1}\frac{2\ell}{n^2\pi^2}.$$

Since

$$f(x,t) = e^{-t}\left(v(x,0) - \frac{k}{\ell}\right),$$

we have

$$f(x,t) = A_0(t) + \sum_{n\geq 1} A_n(t)\cos\frac{n\pi x}{\ell},$$

with

$$A_0(t) = (a_0 - k/\ell)e^{-t} \quad \text{and} \quad A_n(t) = a_n e^{-t} \text{ for } n \geq 1.$$

Write

$$v(x,t) = Y_0(t) + \sum_{n\geq 1} Y_n(t)\cos\frac{n\pi x}{\ell}.$$

So $Y_0(t)$ satisfies

$$\dot{Y}_0(t) = (a_0 - k/\ell)e^{-t}, \quad Y_0(0) = a_0,$$

yielding

$$Y_0(t) = (k/\ell - a_0)e^{-t} + (2a_0 - k/\ell)$$

while for $n \geq 1$, $Y_n(t)$ satisfies

$$\dot{Y}_n(t) + n^2(\pi/\ell)^2 k Y_n(t) = a_n e^{-t}, \quad Y_n(0) = a_n.$$

yielding

$$Y_n(t) = \frac{n^2\pi^2 k a_n}{n^2\pi^2 k - \ell^2} e^{-n^2(\pi/\ell)^2 kt} - \frac{\ell^2 a_n e^{-t}}{n^2\pi^2 k - \ell^2},$$

This makes $v(x,t)$ and hence $u(x,t)$ explicit. Also note that

$$\lim_{t\to\infty} u(x,t) = 2a_0 - k/\ell = -\left(\frac{\ell}{3} + \frac{k}{\ell}\right),$$

and hence

$$u(x,0) = 0 > -\left(\frac{\ell}{3} + \frac{k}{\ell}\right) = u(x,\infty).$$

This reflects the fact that the right end of the rod is uninsulated, and heat is escaping through it since $u_x(\ell,t) = e^{-t} > 0$.

(c) with initial condition $u(x,0) = u_0(x)$ and Dirichlet boundary conditions: $u(0,t) = 0$ and $u(\ell,t) = t$. Discuss the behaviour of the solution for large t.

Solution. Once again to homogenize the boundary conditions, let

$$z(x,t) = \frac{xt}{\ell}$$

and $v = u - z$. Then v solves the heat equation

$$v_t - kv_{xx} = -x/\ell = \frac{2}{\pi}\sum_{n\geq 1}\frac{(-1)^n}{n}\sin\frac{n\pi x}{\ell},$$

with

$$v(0,t) = 0 = v(\ell,t),$$

and initial condition

$$v(x,0) = u_0(x) = \sum_{n\geq 1} b_n \sin\frac{n\pi x}{\ell}.$$

Writing

$$v(x,t) = \sum_{n\geq 1} Y_n(t)\sin\frac{n\pi x}{\ell}.$$

yields the ODE

$$\dot{Y}_n(t) + n^2(\pi/\ell)^2 k Y_n(t) = \frac{2(-1)^n}{n\pi}, \quad Y_n(0) = b_n.$$

This gives

$$Y_n(t) = \left[\left(b_n - (-1)^n\frac{2\ell^2}{n^3\pi^3 k}\right)e^{-n^2(\pi/\ell)^2 kt} + (-1)^n\frac{2\ell^2}{n^3\pi^3 k}\right]$$

As $t\to\infty$, $Y_n(t)$ approaches $(-1)^n\frac{2\ell^2}{n^3\pi^3 k}$. This implies that

$$v(x,\infty) = \sum_{n\geq 1}(-1)^n\frac{2\ell^2}{n^3\pi^3 k}\sin\frac{n\pi x}{\ell}$$

is a continuous and hence a bounded function on $[0,\ell]$. Hence due to the xt factor, $u(x,\infty)=\infty$ for all $x>0$.

(9) A thin circular disc of radius R whose upper and lower faces are insulated is initially at the temperature $u(r,\theta) = f(r)$.

(a) If the temperature along the circumference of the disc is suddenly reduced to 0 and maintained at that value, find the temperature in the disc as a function of (r,t).

SOLUTION. The heat equation in a region is $u_t = k\Delta u$ when there are no sources or sinks inside the region. For a circular disc (centered at origin) we use polar coordintes. In these coordinates,

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

Since the initial temperature only depends on r, the solution u will depend on r and t only. In this situation, the heat equation, the (Dirichlet) boundary condition and initial condition read:

$$u_t = k(u_{rr} + r^{-1}u_r), \quad u(R,t) = 0,\ u(r,0) = f(r).$$

We employ the method of separation of variables. Accordingly, let $u = X(r)T(t)$ be a separated solution of the above system without the initial condition. Substituting and separating variables, we obtain

$$\frac{X''(r)}{X(r)} + \frac{1}{r}\frac{X'(r)}{X(r)} = \frac{T'(t)}{kT(t)}.$$

Since lhs is a function of r and rhs that of t only, both must have a common constant value, say λ. For $\lambda > 0$, write $\lambda = \mu^2,\ \mu > 0$. The equation for X reads

$$r^2X''(r) + rX'(r) - \mu^2X(r) = 0, \quad X(R) = 0.$$

It is a scaled Bessel equation of order $n = 0$ scaled by the *imaginary* number $i\mu$. The general solution is $AJ_0(i\mu r) + BY_0(i\mu r)$. Since u must be bounded as $r \to 0$, B must vanish. Further, since

$$J_0(i\mu r) = \sum_{m\geq 0} \frac{(\mu r)^{2m}}{4^m (m!)^2}$$

is a series of positive terms only, it cannot satisfy $X(R) = 0$. Hence $A = 0$ too. Hence we look for nontrivial solutions when $\lambda \leq 0$. So let now $\lambda = 0$. Then $r^2 X'' + rX' = 0$. The general solution is $A \log r + B$ and again we get $A = B = 0$. So finally let $\lambda = -\mu^2,\ \mu > 0$. The general solution is $X(r) = AJ_0(\mu r) + BY_0(\mu r)$ and due to boundedness at $r = 0$, $B = 0$. Moreover, the condition $X(R) = 0$ implies $\mu = \dfrac{z}{R}$ for $z \in Z^{(0)}$. For these values of μ, we have

$$T(t) = De^{-\mu^2 k}t.$$

Therefore, the general solution of the system *sans* the initial condition is

$$u(r,t) = \sum_{z\in Z^{(0)}} c_z e^{-(z/R)^2 kt} J_0\left(\frac{zr}{R}\right).$$

We now invoke the initial condition to find that

$$f(r) = \sum_{z\in Z^{(0)}} c_z J_0\left(\frac{zr}{R}\right)$$

is the Fourier-Bessel expansion of $f(r)$ over the interval $[0, R]$ in terms of J_0. This allows us to compute the coefficients

$$c_z = \frac{2}{R^2 J_1(z)^2} \int_0^R f(r) J_0\left(\frac{zr}{R}\right) r dr = \frac{2}{z^2 J_1(z)^2} \int_0^z f\left(\frac{Rt}{z}\right) J_0(t).tdt.$$

(b) For $f(r) = 100(1-r^2/R^2)$, if the temperature along the circumference of the disc is suddenly raised to 100 and maintained at that value, then find the temperature in the disc subsequently.

Solution. Let the temperature distribution be

$$U(r,t) = 100u(r,t) + 100.$$

Then u must solve the equations

$$u_t = k(u_{rr} + r^{-1}u_r), \qquad \text{(Heat equation)},$$

$$u(R,t) = 0 \qquad \text{(Homogeneous boundary condition)},$$

$$u(r,0) = -r^2/R^2 \qquad \text{(Initial condition)}.$$

From part (a), it only remains to find $c_z,\ z \in Z^{(0)}$. We know that

$$\int_0^z t^2 J_0(t).tdt = z^2.zJ_1(z) - \int_0^z 2t.tJ_1(t)dt = z^3 J_1(z) - 2z^2 J_2(z)$$
$$= z^3 J_1(z) - 2z^2 \frac{2}{z} J_1(z) = (z^3 - 4z) J_1(z).$$

On substitution,

$$c_z = \frac{2}{z^2 J_1(z)^2} \int_0^z (-t^2/z^2) J_0(t).tdt = \frac{8-2z^2}{z^3 J_1(z)}, \quad z \in Z^{(0)}.$$

Thus

$$U(r,t) = 100 + 100 \sum_{z \in Z^{(0)}} \frac{8-2z^2}{z^3 J_1(z)} e^{-z^2 k R^{-2} t} J_0\left(\frac{zr}{R}\right).$$

(10) A thin upper hemisphere of radius R whose outer and inner surfaces are insulated, is initially at temperature $u(\theta, \varphi) = f(\varphi)$, with φ being the polar angle. If the temperature around the boundary of the shell (the equator) is suddenly reduced to 0 and maintained at that value, find the subsequent temperature in the hemisphere as a function of (φ, t).

Solution. The Laplacian on the sphere of radius R in terms of *spherical polar coordinates* is

$$\Delta = \frac{1}{R^2}\left[\frac{\partial^2}{\partial \varphi^2} + \cot\varphi \frac{\partial}{\partial \varphi} + \frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial \theta^2}\right]$$

Let the given hemisphere be described by $x^2+y^2+z^2=R^2,\ z \geq 0$ Then in the spherical polar coordinates, it is the 'rectangle' $0 \leq \varphi \leq \pi/2,\ 0 \leq \theta \leq 2\pi$. The heat equation on the sphere, or on any subregion thereof is, $u_t = k\Delta u$. Since the region is hemispherical, and the boundary and the initial conditions are independent of the azimuthal angle θ, we expect the solution too to be independent of θ. Accordingly, let $u = u(\varphi, t)$ be the temperature distribution at time t. Then u will solve

$$u_t = \frac{1}{R^2}\left[u_{\varphi\varphi} + \cot\varphi u_\varphi\right] \qquad \text{(Heat equation)},$$

$$u(\varphi,t)|_{\varphi=\frac{\pi}{2}} = 0 \qquad \text{(Homogeneous boundary condition)},$$

$$u(\varphi,0) = f(\varphi) \qquad \text{(Initial condition)}.$$

Ignoring the initial condition we seek solutions of the form $u = X(\varphi)T(t)$. On substitution in the heat equation, we get

$$\frac{X''(\varphi)}{X(\varphi)} + \cot\varphi\frac{X'(\varphi)}{X(\varphi)} = \frac{R^2 T'(t)}{kT(t)} \qquad \text{(Variables separated)}.$$

Both the sides must now be equal to a common constant and it can be seen that it must be nonpositive, else for example, $T(t)$ will blow up exponentially as time t increases. (Mathematical reasoning is more tedious. The Legendre's equation with $\lambda = n(n+1) < 0$, has all nontrivial solutions *unbounded* at ± 1.) Let the common nonpositive constant be written as $-n(n+1),\ n \geq 0$. Then

$$X''(\varphi) + \cot\varphi X'(\varphi) + n(n+1)X(\varphi) = 0, \quad X(\pi/2) = 0,$$

and

$$T(t) = C_n e^{-n(n+1)kR^{-2}t}.$$

Now the differential equation for $X(\varphi)$ is the Legendre equation if the change of variables $\varphi = \cos^{-1} z$ is used. The general solution then is

$$X(\varphi) = Ay_0(z) + By_1(z), \quad \varphi \in [0, \pi/2] \equiv z \in [0,1],$$

where $y_0 = a_0 + a_2 z^2 + \dots$ is an even power series while $y_1 = a_1 z + a_3 z^3 + \dots$ is odd. Since it is known that $a_0 \neq 0$, and $X(\pi/2) = 0$, we deduce $A = 0$. Therefore, $X(\varphi) = By_1(z)$ is an odd Legendre function on $(-1, 1)$ restricted to $[0, 1)$. Now it is implicit that $X(\varphi)$ remains bounded as $\varphi \to 0$, that is the temperature at the north pole is finite at all time $t > 0$. This means that y_1 remains bounded as $z \to \pm 1$. This forces n to be an odd integer and $y_1 = P_n(z)$ is the Legendre polynomial. Hence the separated solutions are of the form

$$u(\varphi, t) = C_n P_n(\cos\varphi) e^{-n(n+1)kR^{-2}t}, \quad n = 1, 3, 5, \dots.$$

The general solution then is

$$u(\varphi, t) = \sum_{n \text{ odd}} C_n e^{-n(n+1)kR^{-2}t} P_n(\cos\varphi).$$

Finally, we use the initial condition to see that

$$f(\varphi) = \sum_{n \text{ odd}} C_n P_n(\cos\varphi)$$

is the Fourier-Legendre series of f thought of as an odd function on $(0, \pi)$. $(f(\varphi) = -f(\pi - \varphi).)$ Thus

$$C_n = 2 \cdot \frac{2n+1}{2} \int_0^1 f(\cos^{-1} z) P_n(z) dz$$

for odd integers n.

Find the explicit solution if
(a) $f(\varphi) = \cos 3\varphi$

Solution. Since $P_1(z) = z,\ P_3(z) = \dfrac{5z^3 - 3z}{2}$,

$$f(\varphi) = \cos 3\varphi = 4\cos^3\varphi - 3\cos\varphi = -0.2P_1(\cos\varphi) + 1.6P_3(\cos\varphi).$$

Hence

$$u(\varphi, t) = -0.2e^{-2kR^{-2}t} P_1(\cos\varphi) + 1.6e^{-12kR^{-2}t} P_3(\cos\varphi).$$

At the topmost point the polar angle $\varphi = 0$, so

$$u(0, t) = -\frac{1}{5} e^{-2kR^{-2}t} + \frac{8}{5} e^{-12kR^{-2}t}.$$

(b) $f(\varphi) = \cos 2\varphi$.
What is the temperature at its topmost point as a function of t?

Solution. This case is harder since $\cos 2\varphi$ is an even function on $[0, \pi]$. We must consider the odd extension of $\cos\varphi,\ 0 \le \varphi \le \pi/2$. Thus we get an infinite series unlike in (a). Since $\cos 2\varphi = 2\cos^2\varphi - 1$, we need to expand $2z^2 - 1$ in terms of $P_1, P_3, P_5, \dots$ over $[0, 1]$. For any odd integer n,

$$C_n = (2n+1) \int_0^1 (2z^2 - 1) P_n(z) dz = \frac{(2n+1)}{2^n n!} \int_0^1 (2z^2 - 1) D^n (z^2 - 1)^n dz.$$

It can be seen on integration by parts that

$$\int_0^1 (2z^2-1)D^n(z^2-1)^n dz = [D^{n-1} - 4D^{n-3}](z^2-1)^n|_{z=0}$$

for $n = 3, 5, 7, \ldots$. For $n = 1$ we have

$$\int_0^1 (2z^2-1)D(z^2-1)dz = 0$$

which implies $C_1 = 0$. Further,

$$C_n = \frac{2n+1}{2^n n!}\left[(n-1)!(-1)^{\frac{n+1}{2}}\binom{n}{\frac{n-1}{2}} - 4(n-3)!(-1)^{\frac{n+3}{2}}\binom{n}{\frac{n-3}{2}}\right]$$
$$= (-1)^{\frac{n+1}{2}}\frac{2n+1}{2^n n!}\left[(n-1)!\binom{n}{\frac{n-1}{2}} + 4(n-3)!\binom{n}{\frac{n-3}{2}}\right]$$

for $n = 3, 5, 7, \ldots$. At the topmost point

$$u(0,t) = \sum_{n \text{ odd}} C_n e^{-n(n+1)kR^{-2}t},$$

since $P_n(\cos 0) = P_n(1) = 1$.

1.7. Wave equation by separation of variables

Problems.

(1) Consider the wave equation

$$u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < \ell,\ t > 0$$

with initial position $f(x)$, initial velocity $g(x)$ and Neumann boundary conditions $u_x(0,t) = u_x(\ell,t) = 0$. Compute the solution for:

(a) $f(x) = x^2(x-\ell), \quad g(x) = 0$

SOLUTION. The solution is of the form

$$u(x,t) = Y_0(t) + \sum_{n\ge1} Y_n(t)\cos\frac{n\pi x}{\ell},$$

and $Y_n(t)$ are the unique solutions to the IVP

$$\ddot{Y}_n + n^2(\pi/\ell)^2 c^2 Y_n = 0, \quad Y_n(0) = a_n,\ \dot{Y}_n(0) = 0.$$

The a_n are the coefficients of the Fourier cosine series of $f(x)$:

$$x^2(x-\ell) = a_0 + \sum_{n\ge1} a_n \cos\frac{n\pi x}{\ell}, \quad 0 < x < \ell.$$

Thus

$$Y_n(t) = a_n \cos\frac{n\pi ct}{\ell} \quad \text{and} \quad u(x,t) = a_0 + \sum_{n\ge1} a_n \cos\frac{n\pi ct}{\ell}\cos\frac{n\pi x}{\ell}.$$

Now routine calculations show that

$$a_0 = \ell^{-1}\int_0^\ell f(x)dx = -l^3/12,$$

and

$$a_n = 2\ell^{-1}\int_0^\ell f(x)\cos\frac{n\pi x}{\ell}dx = (-1)^n\frac{2\ell^3}{n^4\pi^4}[n^2\pi^2 + 6(1+(-1)^n)].$$

(b) $f(x) = \sin^2(\frac{\pi x}{\ell}), \quad g(x) = 0$

SOLUTION. As above

$$u(x,t) = a_0 + \sum_{n\ge1} a_n \cos\frac{n\pi ct}{\ell}\cos\frac{n\pi x}{\ell},$$

where

$$\sin^2\frac{\pi x}{\ell} = a_0 + \sum_{n\ge1} a_n\cos\frac{n\pi x}{\ell} = \frac{1}{2} - \frac{1}{2}\cos\frac{2\pi x}{\ell}.$$

Thus

$$a_0 = 1/2,\ a_2 = -1/2 \text{ and } a_n = 0,\ n \ne 0, 2.$$

Hence

$$u(x,t) = \frac{1}{2}\left[1 - \cos\frac{2\pi ct}{\ell}\cos\frac{2\pi x}{\ell}\right].$$

(c) $f(x) = 0, \quad g(x) = 1.$

SOLUTION. Here all $a_n = 0$ and $a_{10} = 1,\ a_{1n} = 0,\ n \geq 1$. Consequently, $Y_n(t) = 0,\ n \geq 1$ and $Y_0(t) = t$. Therefore, $u(x,t) = t$.

(2) Solve the wave equation

$$u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < \ell,\ t > 0$$

with zero initial conditions and inhomogeneous Neumann boundary conditions: $u_x(0,t) = t$ and $u_x(\ell,t) = 0$.

SOLUTION. We have to homogenize the Neumann boundary conditions at the cost of making the PDE and initial conditions inhomogeneous. So set $z(x,t) = tx(1 - x/2\ell)$, and put

$$v(x,t) = u(x,t) - z(x,t).$$

Then $v(x,t)$ solves the equation

$$v_{tt} - c^2 v_{xx} = -\frac{c^2 t}{\ell}$$

with boundary conditions

$$v_x(0,t) = 0 = v_x(\ell,t)$$

and initial conditions

$$v(x,0) = 0, \quad v_t(x,0) = -x\left(1 - \frac{x}{2\ell}\right).$$

Assuming

$$v(x,t) = X_0(t) + \sum_{n\geq 1} X_n(t) \cos\frac{n\pi x}{\ell},$$

the $X_n(t)$ solve the IVP

$$\ddot{X}_0(t) = -\frac{c^2 t}{\ell}, \quad X_0(0) = 0,\ \dot{X}_0(0) = a_{10}$$

and for $n \geq 1$,

$$\ddot{X}_n(t) + n^2(\pi/\ell)^2 c^2 X_n(t) = 0, \quad X_n(0) = 0,\ \dot{X}_n(0) = a_{1n}.$$

This is because $f(x,t) = -\frac{c^2 t}{\ell}$ is a one-term cosine series of the rhs of the PDE for v.

$$a_{10} + \sum_{n\geq 1} a_{1n} \cos\frac{n\pi x}{\ell} = -x\left(1 - \frac{x}{2\ell}\right)$$

is the cosine series which gives the constants $a_{1n},\ n \geq 0$. Now for $n \geq 1$,

$$X_n(t) = \frac{a_{1n}\ell}{n\pi c} \sin\frac{n\pi ct}{\ell}$$

while

$$X_0(t) = a_{10}t - \frac{c^2 t^3}{6\ell}.$$

Except for explicitly determining a_{1n}, we get

$$u(x,t) = tx\left(1 - \frac{x}{2\ell}\right) + \left(a_{10}t - \frac{c^2 t^3}{6\ell}\right) + \sum_{n\geq 1} \frac{a_{1n}\ell}{n\pi c} \sin\frac{n\pi ct}{\ell} \cos\frac{n\pi x}{\ell}.$$

Finally, you can check that

$$a_{10} = -\frac{1}{\ell}\int_0^\ell x\left(1-\frac{x}{2\ell}\right)dx = -\frac{\ell}{6}$$

and for $n \geq 1$,

$$a_{1n} = -\frac{2}{\ell}\int_0^\ell x\left(1-\frac{x}{2\ell}\right)\cos\frac{n\pi x}{\ell}dx = \frac{2\ell}{n^2\pi^2}.$$

(3) Solve the wave equation

$$u_{tt} - c^2 u_{xx} = -xe^{-t}/\ell, \quad 0 < x < \ell,\ t > 0$$

with initial conditions: $u(x,0) = u_t(x,0) = 0$ and boundary conditions: $u(0,t) = e^{-t}, u(\ell,t) = 1$.

SOLUTION. To homogenize the Dirichlet boundary conditions, let

$$z(x,t) = \frac{x + (\ell - x)e^{-t}}{\ell},$$

and put

$$v(x,t) = u(x,t) - z(x,t).$$

Then $v(x,t)$ solves the equation

$$v_{xx} - c^2 v_{tt} = -e^{-t},$$

with initial conditions

$$v(x,0) = -1, \quad v_t(x,0) = 1 - \frac{x}{\ell}$$

and homogeneous Dirichlet boundary conditions

$$v(0,t) = 0 = v(\ell,t).$$

So the function $v(x,t)$ will have an expansion

$$v(x,t) = \sum_{n\geq 1} Y_n(t)\sin\frac{n\pi x}{\ell}.$$

The other Fourier sine expansions are

$$f(x,t) := -e^{-t} = \sum_{n\geq 1} B_n(t)\sin\frac{n\pi x}{\ell} = -\frac{4e^{-t}}{\pi}\sum_{n\text{ odd}}\frac{1}{n}\sin\frac{n\pi x}{\ell}.$$

$$v(x,0) = -1 = \sum_{n\geq 1} b_n \sin\frac{n\pi x}{\ell} = -\frac{4}{\pi}\sum_{n\text{ odd}}\frac{1}{n}\sin\frac{n\pi x}{\ell}.$$

$$v_t(x,0) = 1 - \frac{x}{\ell} = \sum_{n\geq 1} b_{1n}\sin\frac{n\pi x}{\ell} = \sum_{n\geq 1}\frac{2}{n\pi}\sin\frac{n\pi x}{\ell}.$$

Therefore, for even n,

$$\ddot{Y}_n(t) + n^2(\pi/\ell)^2 c^2 Y_n(t) = 0, \quad Y_n(0) = 0,\ \dot{Y}_n(0) = 2/n\pi$$

which implies

$$Y_n(t) = \frac{2\ell}{n^2\pi^2 c}\sin\frac{n\pi ct}{\ell}.$$

The case of odd n is complicated.

$$\ddot{Y}_n(t) + n^2(\pi/\ell)^2 c^2 Y_n(t) = -\frac{4e^{-t}}{n\pi}, \quad Y_n(0) = -\frac{4}{n\pi},\ \dot{Y}_n(0) = 2/n\pi.$$

Let $K_n e^{-t}$ be a particular solution of this ODE.Direct substitution gives $K_n = \dfrac{-4\ell^2}{n\pi(\ell^2 + n^2\pi^2c^2)}$. The general solution, therefore, is

$$Y_n(t) = C_n \cos\frac{n\pi ct}{\ell} + D_n \sin\frac{n\pi ct}{\ell} - \frac{4\ell^2 e^{-t}}{n\pi(\ell^2 + n^2\pi^2c^2)}.$$

Using the initial values, we get

$$C_n = \frac{2\ell(n^2\pi^2c^2 - 2\ell^2)}{n^2\pi^2c^2 + \ell^2} \quad \text{and} \quad D_n = \frac{2\ell(n^2\pi^2c^2 - \ell^2)}{n^2\pi^2c(n^2\pi^2c^2 + \ell^2)}.$$

This determines the Fourier sine series of $v(x,t)$ completely. Finally,

$$u(x,t) = v(x,t) + \frac{x + (\ell - x)e^{-t}}{\ell}.$$

(4) Use separation of variables to solve the telegrapher equation

$$u_{tt} - \gamma^2 u_{xx} + 2\alpha^2 u_t = 0, \quad 0 < x < \ell, \ t > 0$$

with initial condition $u(x,0) = f(x)$, $u_t(x,0) = g(x)$ and Dirichlet boundary conditions $u(0,t) = u(\ell,t) = 0$. Show that the solution $u(x,t)$ tends to zero as $t \mapsto \infty$.

Solution. Let $u(x,t) = X(x)T(t)$ be an elementary solution of the linear PDE. Substitution and simplification separates the variables:

$$\frac{1}{\gamma^2}\frac{T''(t)}{T(t)} + 2\frac{\alpha^2}{\gamma^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}.$$

Using the, by now standard arguments, both the sides equal a common constant which due to the homogeneous Dirichlet boundary conditions must of the form $-\dfrac{n^2\pi^2}{\ell^2}$, $n = 1, 2, 3, \ldots$. Then

$$X(x) = constt. \sin\frac{n\pi x}{\ell}$$

while $T(t)$ satisfies

$$T''(t) + 2\alpha^2 T'(t) + \frac{n^2\pi^2\gamma^2}{\ell^2}T(t) = 0.$$

This is a constant coefficient linear second-order ODE. The auxillary equation has the roots

$$m_\pm = -\alpha^2 \pm \sqrt{\alpha^4 - \frac{n^2\pi^2\gamma^2}{\ell^2}}, \quad n = 1, 2, 3, \ldots.$$

The general solution then is

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty}\left(C_n e^{\left(-\alpha^2 + \sqrt{\alpha^4 - \frac{n^2\pi^2\gamma^2}{\ell^2}}\right)t} + D_n e^{\left(-\alpha^2 - \sqrt{\alpha^4 - \frac{n^2\pi^2\gamma^2}{\ell^2}}\right)t}\right)\sin\frac{n\pi x}{\ell} \\ &= \sum_{n=1}^{\infty}\left(C_n e^{\sqrt{\alpha^4 - \frac{n^2\pi^2\gamma^2}{\ell^2}}\,t} + D_n e^{-\sqrt{\alpha^4 - \frac{n^2\pi^2\gamma^2}{\ell^2}}\,t}\right)e^{-\alpha^2 t}\sin\frac{n\pi x}{\ell}. \end{aligned}$$

The coefficients C_n and D_n can be determined using the Fourier sine series of $f(x)$ and $g(x)$.

We observe that if $\alpha^4 \geq \dfrac{n^2\pi^2\gamma^2}{\ell^2}$, then both the roots will be negative, and if $\alpha^4 < \dfrac{n^2\pi^2\gamma^2}{\ell^2}$, then the real part of both the roots will be negative. Hence it follows that each elementary solution and thus u also decays to 0 as $t \to \infty$. Note that for each n,

$$\Re(m_\pm) \leq -\alpha^2 + \Re\left(\sqrt{\alpha^4 - \frac{\pi^2\gamma^2}{\ell^2}}\right)$$

which is a fixed negative constant. So u will decay to 0 exponentially fast.

(5) An elastic string whose linear density is $\rho(x) = \rho_0(1+\alpha x)$, where x is the distance from one end of the string, is stretched under tension T between two points at a distance ℓ apart. What are the natural frequencies of the string? More precisely, write down the equation whose solutions are the natural frequencies of the string. (This equation is known as the *characteristic equation* of the vibration problem.)

SOLUTION. The equation governing the vibrations is

$$\rho(x)u_{tt} = Tu_{xx}, \quad u(0,t) = 0 = u(\ell,t),$$

where $\rho(x)$ is the, possibly nonconstant, linear density of the string at x, and T is the tension. In the given problem, $\rho(x) = \rho_0(1+\alpha x)$. In the fundamental modes of vibrations, $u(x,t) = X(x)Y(t)$. On substitution in the given wave equation, the variables separate:

$$\frac{Y''(t)}{Y(t)} = \frac{T}{\rho_0(1+\alpha x)} \cdot \frac{X''(x)}{X(x)} = -\mu^2 < 0$$

(If the common constant is ≥ 0, no nontrivial solution will be possible due to the homogeneous Dirichlet boundary conditions.) Clearly,

$$Y(t) = C_\mu \cos\mu t + D_\mu \sin\mu t.$$

The differential equation for $X(x)$ is

$$X''(x) + 2.25c^2\mu^2(\alpha^{-1}+x)X(x) = 0,$$

where we have, for convenience set $c = \sqrt{\dfrac{4\rho_0\alpha}{9T}}$. Now make a change of variables

$$z = c(\alpha^{-1}+x)^{3/2} \quad \text{and} \quad (\alpha^{-1}+x)^{1/2}v(z) = X(x).$$

(Compare with Problem 1(c) in Section 1.4.) This yields

$$z^2v'' + zv' + (z^2 - \frac{1}{3^2}) = 0.$$

The general solution is

$$v(z) = A_\mu J_{1/3}(z) + B_\mu J_{-1/3}(z).$$

It is interesting to note that for $x = 0$, $z = c\mu\alpha^{-1} > 0$, so that the $J_{-1/3}$ term is also allowed. Now the Dirichlet boundary conditions imply that $v(z(0)) = 0 = v(z(\ell))$. This gives two linear equations for A_μ, B_μ namely

$$J_{1/3}(z(0))A_\mu + J_{-1/3}(z(0))B_\mu = 0,$$

and

$$J_{1/3}(z(\ell))A_\mu + J_{-1/3}(z(\ell))B_\mu = 0.$$

For a nontrivial mode of vibrations to exist, at least one of the A_μ, B_μ must be nonzero. This can happen if and only if the determinant

$$\begin{vmatrix} J_{1/3}(z(0)) & J_{-1/3}(z(0)) \\ J_{1/3}(z(\ell)) & J_{-1/3}(z(\ell)) \end{vmatrix} = 0.$$

Going back to the original variables, the characteristic equation is

$$\begin{vmatrix} J_{1/3}(\frac{2k\mu}{3\alpha^{3/2}}) & J_{-1/3}(\frac{2k\mu}{3\alpha^{3/2}}) \\ J_{1/3}(\frac{2k\mu}{3\alpha^{3/2}}[1+\alpha\ell]^{3/2}) & J_{-1/3}(\frac{2k\mu}{3\alpha^{3/2}}[1+\alpha\ell]^{3/2}) \end{vmatrix} = 0.$$

Here $k^2 = \dfrac{\rho_0\alpha}{T}$. The above equation must be solved for μ. The solution set is precisely the set of fundamental frequencies and the corresponding $X(x)$ are the associated amplitudes. The abstract Sturm-Liouville theory shows that there is a sequence of values of μ increasing to infinity and which tend to be regularly spaced, that is, $\mu_{n+1} - \mu_n \to const. > 0$ as $n \to \infty$.

(6) Work the previous exercise if the linear density is $\rho(x) = \rho_0 e^{\alpha x}$.

SOLUTION. Repeating the method of separation of variables as in the previous problem, this time we get for $u(x,t) = X(x)Y(t)$,

$$X''(x) + \frac{\rho_0\alpha\mu^2}{T}e^{\alpha x}X(x) = 0.$$

For convenience again, we let $k^2 = \dfrac{\rho_0\alpha}{T}$. Then

$$X''(x) + k^2\mu^2 e^{\alpha x}X(x) = 0.$$

Letting $z = e^{\alpha x/2},\ X(x) = v(z)$, we obtain after rearranging

$$z^2v'' + zv' + \left[\frac{4k^2\mu^2}{\alpha^2}z^2\right]v = 0.$$

The general solution is

$$v(z) = A_\mu J_0\left(\frac{2k\mu z}{\alpha}\right) + B_\mu Y_0\left(\frac{2k\mu z}{\alpha}\right).$$

Again Y_0 term will remain bounded as $z = 1$ when $x = 0$. The characteristic equation this time will be

$$\begin{vmatrix} J_0(\frac{2k\mu}{\alpha}) & Y_0(\frac{2k\mu}{\alpha}) \\ J_0(\frac{2k\mu}{\alpha}e^{\ell\alpha/2}) & Y_0(\frac{2k\mu}{\alpha}e^{\ell\alpha/2}) \end{vmatrix} = 0.$$

(7) Find the characteristic equation to determine the frequencies of the pure harmonics of an annulus $\{0 < a \le r \le b\}$ which do not depend on the angle θ in polar coordinates.

SOLUTION. This time we have $u = u(r,t) = X(r)T(t)$. The wave equation $\Delta u = u_{rr} + r^{-1}u_r = u_{tt}$ yields on separation of variables,

$$\frac{X''(r) + r^{-1}X'(r)}{X(r)} = \frac{T''(t)}{T(t)}.$$

Again we assume the that the common constant is negative say, $-\mu^2$. $T(t) = A\cos(\mu t + \alpha)$ and so μ are the frequencies of the fundamental modes. Now the differential equation for $X(r)$ is $r^2X'' + rX' + \mu^2r^2X = 0$ which is a scaled Bessel equation of order 0. The general solution is $X(r) = AJ_0(\mu r) + BY_0(\mu r)$. The boundary conditions are $X(a) = 0 = X(b)$. Note that we cannot eliminate Y_0 since it is smooth on $[a, b]$. The boundary conditions mean that the equations

$$AJ_0(\mu a) + BY_0(\mu a) = 0 \quad \text{and} \quad AJ_0(\mu b) + BY_0(\mu b) = 0$$

must have $(A, B) \neq (0, 0)$ as a solution, the condition for which is

$$\begin{vmatrix} J_0(\mu a) & Y_0(\mu a) \\ J_0(\mu b) & Y_0(\mu b) \end{vmatrix} = 0.$$

The above equation determines the frequencies $\mu_k,\ k = 1, 2, 3, \ldots$ increasing to infinity by the theory of *regular* Sturm-Liouville equations.

(8) An elastic membrane in the shape of a plane circular sector of radius R and angle β is clamped along its boundary. Find the fundamental modes of vibrations.

Solution. Let the vertex of the sector (that is, the center of the circle) as origin, and the plane of the membrane as xy-plane. We assume that one arm of the sector is along the +ve x-axis, and so the other arm is along the ray $\theta = \beta$. In polar coordinates in the plane, the Laplacian is

$$\Delta_{\mathbb{R}^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

The wave equation is

$$u_{tt} = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}$$

in the domain $(r, \theta, t) \in [0, R] \times [0, \beta] \times \mathbb{R}$. The boundary conditions are

$$(i)\ u(r, 0, t) = 0 = u(r, \beta, t)$$
$$(ii)\ u(R, \theta, t) = 0, \quad u(0, \theta, t) \text{ is finite.}$$

We solve for $u(r, \theta, t) = X(r)Y(\theta)Z(t)$. Substituting in the wave equation and separating variables, we obtain

$$\frac{Y''(\theta)}{Y(\theta)} = -\frac{r^2X''(r) + rX'(r)}{X(r)} + \frac{r^2Z''(t)}{Z(t)}.$$

Due to the boundary conditions (i), the common constant has to be $-\dfrac{n^2\pi^2}{\beta^2}$ for $n = 1, 2, 3, \ldots$ and correspondingly,

$$Y(\theta) = B\sin\frac{n\pi\theta}{\beta}.$$

Now the rhs gives

$$\frac{Z''(t)}{Z(t)} = \frac{X''(r) + r^{-1}X'(r)}{X(r)} - \frac{n^2\pi^2}{r^2\beta^2},$$

with variables again separated. Let the common constant be $-\mu^2$. Then $Z(t) = A\cos(\mu t + \varphi)$. Thus μ is a fundamental frequency which is to be precisely determined. The ODE for $X(r)$ is

$$r^2X''(r) + rX'(r) + \left(\mu^2r^2 - \frac{n^2\pi^2}{\beta^2}\right)X(r) = 0.$$

The only solution which is bounded at $r = 0$ is the scaled Bessel function $J_{n\pi/\beta}(\mu r)$. The final condition $u(R, \theta, t) = 0$ forces $\mu R \in Z^{(n\pi/\beta)}$, the set of positive zeroes of $J_{n\pi/\beta}$. Thus the fundamental frequencies form the set

$$R^{-1}\bigcup_{n=1}^{\infty} Z^{(n\pi/\beta)}.$$

The associated pure harmonics (waves) are

$$J_{n\pi/\beta}\left(\frac{zr}{R}\right)\sin\frac{n\pi\theta}{\beta}\cos\left(\frac{zt}{R} + \varphi\right) \qquad \text{for } z \in Z^{(n\pi/\beta)},\ n = 1, 2, 3, \ldots.$$

(9) The portion of the cone $x^2 + y^2 = z^2\tan^2\alpha$ between its vertex O and the rim $z = R\cos\alpha$ is clamped along its rim. Find the fundamental harmonics. What are the orders of the Bessel equations that you get? What is the connection with the previous question?

Solution. The cone in question is a right circular cone of semi-vertical angle α and slant height R. If we take a plane circular sector of radius R and angle $\beta = 2\pi\sin\alpha$ and join together its boundary arms, we get this cone. So to find its fundamental modes of vibrations, all we have to do is to replace the Dirichlet boundarry conditions (i) in the previous problem by the periodic boundary conditions

$$(i')u(r, 0, t) = u(r, \beta, t) \quad u_\theta(r, 0, t) = u_\theta(r, \beta, t).$$

This finally gives the frequency set as

$$R^{-1}\bigcup_{k=0}^{\infty} Z^{(2k\pi/\beta)}.$$

Thus the frequency set is 'halved' since only even n occur. However, for each frequency, except from $R^{-1}Z^{(0)}$, there are now *two* linearly independent amplitudes, namely

$$J_{n\pi/\beta}\left(\frac{zr}{R}\right)\sin\frac{n\pi\theta}{\beta} \quad \text{and} \quad J_{n\pi/\beta}\left(\frac{zr}{R}\right)\cos\frac{n\pi\theta}{\beta},\ n = 0, 2, 4, 6, \ldots.$$

(10) Find all the pure harmonics and their associated frequencies of the unit sphere.

Solution. We have to find the solutions of the wave equation

$$u_{tt} = \Delta_{\mathbb{S}^2}u$$

which are of the form

$$u(\theta, \varphi; t) = X(\theta)Y(\varphi)Z(t).$$

As usual on substituting in

$$\Delta_{\mathbb{S}^2} u = u_{\varphi\varphi} + \cot\varphi u_\varphi + \csc^2\varphi u_{\theta\theta} = u_{tt},$$

we get

$$\frac{X''(\theta)}{X(\theta)} = \sin^2\varphi \frac{Z''(t)}{Z(t)} - \sin^2\varphi \left(\frac{Y''(\varphi)}{Y(\varphi)} + \cot\varphi \frac{Y'(\varphi)}{Y(\varphi)} \right).$$

Since lhs is depending only on θ while rhs only on (φ, t), the two sides must be equal to a common constant. Further due to the implicit periodic boundary conditions $X(0) = X(2\pi)$ and $X'(0) = X'(2\pi)$, the common constant must be $-m^2$, $m = 0, 1, 2, \ldots$ and

$$X(\theta) = A\cos m\theta + B\sin m\theta.$$

Next from rhs we also have a further separation of variables

$$\frac{Z''(t)}{Z(t)} = \frac{Y''(\varphi)}{Y(\varphi)} + \cot\varphi \frac{Y'(\varphi)}{Y(\varphi)} - \frac{m^2}{1-\cos^2\varphi}.$$

Again both the sides equal a common constant μ, say. This gives an associated Legendre equation for $Y(\varphi)$ in *polar form* and nontrivial bounded solutions can exist if and only if $\mu = -n(n+1)$ with $n = m, m+1, m+2, \ldots$. Recall that these solutions are $(1-x^2)^{m/2} D^m P_n(x)$ in the standard form. In polar form they give

$$Y(\varphi) = \sin^m \varphi P_n^{(m)}(\cos\varphi).$$

Now

$$Z(t) = C\cos(\sqrt{n(n+1)}t + \alpha),$$

which shows that u will have period $\dfrac{2\pi}{\sqrt{n(n+1)}}$ in time. Thus frequencies which can occur are $\omega_n = \sqrt{n(n+1)}$ for $n = 1, 2, 3, \ldots$. ($n = 0$ does not give oscillations in time.) For each fundamental frequency ω_n, the possible amplitudes are

$$(A\cos m\theta + B\sin m\theta)\sin^m\varphi P_n^{(m)}(\cos\varphi), m = 0, 1, 2, \ldots n.$$

These are $2n+1$ linearly independent amplitudes with a basis

$$\cos m\theta \sin^m \varphi P_n^{(m)}(\cos\varphi);\ m = 0, 1, 2, \ldots n$$

$$\sin m\theta \sin^m \varphi P_n^{(m)}(\cos\varphi);\ m = 1, 2, \ldots n.$$

Another *more useful* (but complex) basis is

$$\{e^{im\theta}\sin^m\varphi P_n^{(m)}(\cos\varphi) :\ m = 0, \pm 1, \pm 2, \cdots \pm n\}.$$

The pure harmonics (waves) are

$$u = A\cos(\sqrt{n(n+1)}t + \alpha)e^{im\theta}\sin^m\varphi P_n^{(m)}(\cos\varphi).$$

Remark: The integer n corresponding to the frequency ω_n is known as the *principal quantum number.* The numbers $m \in \{-n, -n+1, \ldots, 0, \ldots, n\}$ which describe the (complex) amplitudes are known as the *magnetic quantum numbers* underlying the corresponding principal quantum number n.

(11) Find the azimuthal angle independent pure harmonics and their associated frequencies of a thin hemisphere of unit radius whose equator is clamped.

SOLUTION. Since the vibrations are independent of azimuthal angle, we write $u = u(\varphi, t)$. It satisfies

$$u_{tt} = u_{\varphi\varphi} + \cot\varphi\, u_\varphi$$

over the region $[0, \pi/2] \times \mathbb{R}$ with boundary conditions $u(0, t)$ finite and $u(\pi/2, t) = 0$. For pure harmonics, we need to find solutions of the form $u = Y(\varphi)Z(t)$. Substitution gives

$$\frac{Z''(t)}{Z(t)} = \frac{Y''(\varphi)}{Y(\varphi)} + \cot\varphi \frac{Y'(\varphi)}{Y(\varphi)}.$$

Let both the sides be equal to $-\mu$. We first look at $Y(\varphi)$. It satisfies the polar form of the Legendre equation

$$Y''(\varphi) + \cot\varphi\, Y'(\varphi) + \mu Y(\varphi) = 0.$$

Also $Y(0)$ is finite and $Y(\pi/2) = 0$. The corresponding system in the standard form will be

$$(1 - z^2)y'' - 2zy' + \mu y = 0, \quad |y(1)| < \infty,\ y(0) = 0.$$

The general solution is $y = a_0 y_0 + a_1 y_1$, where y_0 and y_1 are respectively even and odd power series in z with $y_0(0) \neq 0$. Therefore, $y(0) = 0$ implies $a_0 = 0$. $y(z)$ is thus an odd power series so that the condition $|y(1)|$ finite forces $|y(-1)|$ finite too. Consequently, y must be a Legendre polynomial of odd degree n and $\mu = n(n+1)$, $n = 1, 3, 5, \ldots$. Then

$$Y(\varphi) = P_n(\cos\varphi) \quad \text{and} \quad Z(t) = A\cos(\sqrt{n(n+1)}t + \alpha).$$

The frequencies of the pure harmonics are therefore, of the form $\sqrt{n(n+1)}$ for $n = 1, 3, 5, \ldots$, and each frequency has a unique θ-independent amplitude $P_n(\cos\varphi)$.

Remark: If we allow θ dependency, then we have all fundamental harmonics of the full sphere which vanish along the equator $z = 0$ or equivalently $\varphi = \pi/2$. Thus the frequencies are $\{\sqrt{n(n+1)} : n = 1, 2, 3, \ldots\}$, and the amplitudes are

$$\{e^{im\theta}\sin^m\varphi P_n^{(m)}(\cos\varphi) \mid |m| \leq n \text{ and } n - m \text{ odd.}\}.$$

(12) A polar cap of the standard sphere $\mathbb{S}^2$ between the polar angles $\varphi = 0$ and $\varphi = \varphi_0$ is clamped along its rim. Find the characteristic equation to determine the frequencies of its fundamental modes (pure harmonics) which do not depend on the azimuthal angle θ.

SOLUTION. The fundamental modes of vibrations are of the form $u(\theta, \varphi, t) = Y(\theta)X(\varphi)T(t)$ which satisfy the wave equation $u_{tt} = \Delta_{\mathbb{S}^2} u$. In this problem we look for u independent of θ. Hence $u = X(\varphi)T(t)$ and

$$\Delta_{\mathbb{S}^2} u = u_{\varphi\varphi} + (\cot\varphi)u_\varphi = XT'' + \cot\varphi X'T = u_{tt} = XT''.$$

The variables separate to give

$$\frac{T''(t)}{T(t)} = \frac{X''(\varphi)}{X(\varphi)} + \cot\varphi \frac{X'(\varphi)}{X(\varphi)}.$$

As usual we loosely argue that the common constant must be negative $-\mu^2$, say. So $T(\mu, t) = A\cos(\mu t + \alpha)$. Now $X(\varphi)$, $0 \le \varphi \le \varphi_0$ must satisfy $X(\varphi_0) = 0$ as the polar cap is clamped along its rim. Further, we must have $X(0)$ to be finite. $X(\varphi)$ satisfies the polar form of the Legendre equation $X''(\varphi) + \cot\varphi X'(\varphi) + \mu^2 X(\varphi) = 0$ over the interval $[0, \varphi_0]$. $\varphi = 0$ corresponds to $x = 1$ in the standard Legendre equation, and it is a regular singular point with indicial equation being $r^2 = 0$. Hence there is a unique nonzero solution $X_0(\mu, \varphi)$ which is finite at $\varphi = 0$. All others will have a $\log\varphi$ term. Now the characteristic equation determining the frequencies μ is

$$X_0(\mu, \varphi_0) = 0.$$

Remark: In the special case of a hemisphere ($\varphi_0 = \pi/2$) by a clever argument one can show that $X_0 = P_{od}$- odd degree Legendre polynomials, so that $\mu \in \{\sqrt{n(n+1)}; n = 1, 3, 5, \dots\}$.

1.8. Laplace equation by separation of variables

Problems.

(1) Assuming that term-wise differentiation is permissible, show that a solution of Laplace equation: $\Delta u = 0$ in the disc of radius 1 with the boundary condition : $u(1,\theta) = f(\theta)$ is given by

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

where a_n, b_n are the Fourier coefficients of f.

Solution. In the plane, the Laplacian in polar coordinates is given by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

For each $n \geq 0$,

$$\begin{aligned}
&\Delta[r^n(a_n \cos n\theta + b_n \sin n\theta)] \\
&\quad = n(n-1)r^{n-2}(a_n \cos n\theta + b_n \sin n\theta) + nr^{n-2}(a_n \cos n\theta + b_n \sin n\theta) \\
&\qquad - n^2 r^{n-2}(a_n \cos n\theta + b_n \sin n\theta) = 0.
\end{aligned}$$

This implies that $\Delta u = 0$ and

$$u(1,\theta) = a_0 + \sum_{n\geq 1}(a_n \cos n\theta + b_n \sin n\theta) = f(\theta).$$

(2) Using separation of variables, solve the Neumann problem for the Laplace equation: $\Delta u = 0$ in the disc of radius 1 with the boundary condition : $\frac{\partial u}{\partial r}(1,\theta) = \sin^3\theta$.

Solution. Neumann boundary conditions mean that $u_r(1,\theta) = g(\theta)$. If $u(r,\theta) = X(r)Y(\theta)$ in polar coordinates, then $\Delta u = 0$ gives

$$X''Y + (1/r)X'Y + (1/r^2)XY'' = 0 \qquad \text{or} \qquad r^2X''/X + rX'/X = -Y''/Y$$

with variables separated. Now $u(r,\theta) = u(r,\theta + 2\pi)$ (implicit Periodic boundary conditions). Therefore, the common constant can only be of the type

$$n^2,\ n = 0, 1, 2, \ldots.$$

Then

$$Y(\theta) = a_n \cos n\theta + b_n \sin n\theta,$$

and at the same time

$$r^2X'' + rX' - n^2X = 0$$

whose general solution is

$$X(r) = \begin{cases} A_0 + B_0 \log r & \text{if } n = 0, \\ A_n r^n + B_n / r^{n+1} & \text{if } n \geq 1. \end{cases}$$

Now another unstated boundary condition is $u(r,0)$ remains bounded as $r \to 0$. So $B_n = 0$, for all n. The general solution now is

$$u(r,\theta) = A_0 + \sum r^n (a_n \cos n\theta + b_n \sin n\theta).$$

The given Neumann boundary conditions means that

$$g(\theta) = \sum_{n\geq 1} n(a_n \cos n\theta + b_n \sin n\theta).$$

In particular for the solution to exist, $\int_0^{2\pi} g(\theta)d\theta = 0$ and the coefficients $a_n, b_n,\ n \geq$ are determined from the Fourier coefficients of g. a_0 remains arbitrary. Thus for example, in the given problem

$$g(\theta) = \sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4},$$

so $a_n = 0$ for all $n \geq 0$, $b_1 = 3/4$, $b_3 = -1/12$, while the rest of the b_n's vanish. Hence

$$u(r,\theta) = \frac{3r}{4}\sin\theta - \frac{r^3}{12}\sin 3\theta + C.$$

(3) A thin sheet of metal bounded by x-axis and the lines $x = 0$ and $x = 1$ and extending to infinity in the y direction has its vertical edges maintained at the constant temperature $u = 0$. Over its lower edge the temperature distribution $u(x, 0) = 100$ is maintained. Find the steady-state temperature distribution. Solve the problem when the vertical edges are insulated and the lower edge is maintained at $u(x, 0) = \sin\pi x$.

Solution. (i) We need to solve the Laplace equation $u_{xx} + u_{yy} = 0$. The boundary conditions $u(0, y) = u(1, y) = 0$ imply that

$$u(x,y) = \sum Y_n(y)\sin n\pi x,$$

where

$$Y_n''(y) = n^2\pi^2 Y_n(y).$$

Now

$$u(x,0) = 100 = \sum Y_n(0)\sin n\pi x$$

implies

$$Y_n(0) = \begin{cases} \dfrac{400}{n\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Therefore, $Y_n = 0$ for n even. For odd n, we have

$$Y_n(y) = C_n e^{n\pi y} + D_n e^{-n\pi y}$$

subject to $C_n + D_n = 400/n\pi$. Physical reality forces $C_n = 0$ else the temperature will grow exponentially as $y \to \infty$. So the steady state is

$$u(x,t) = \frac{400}{\pi}\sum_{n \text{ odd}} \frac{1}{n} e^{-n\pi y}\sin n\pi x.$$

(ii) In this case, the vertical edges have Neumann boundary conditions $u_x(0, y) = 0 = u_x(1, y)$. Hence

$$u(x,y) = Y_0(y) + \sum Y_n(y)\cos n\pi x,$$

where

$$Y_n'' = n^2\pi^2 Y_n$$

$$Y_n(y) = \begin{cases} C_0 y + D_0 & \text{if } n = 0, \\ C_n e^{n\pi y} + D_n e^{-n\pi y} & \text{if } n \geq 1. \end{cases}$$
$$= \begin{cases} D_0 & \text{if } n = 0, \\ D_n e^{-n\pi y} & \text{if } n \geq 1. \end{cases} \qquad \text{(Physical reality)}$$
$$= D_n e^{-n\pi y}$$

for all $n \geq 0$. Hence

$$u(x,y) = \sum_{n \geq 0} D_n e^{-n\pi y} \cos n\pi x.$$

Finally

$$u(x,0) = \sin \pi x = \sum_{n \geq 0} D_n \cos n\pi x, \quad 0 < x < 1$$

allows us to see that $D_0 = 2/\pi$, while

$$D_n = \begin{cases} 0 & \text{for odd } n, \\ -4/\pi(n^2-1) & \text{for even } n, \end{cases}$$

for $n \geq 1$. Finally,

$$u(x,y) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n \text{ even}} \frac{e^{-n\pi y} \cos n\pi x}{n^2 - 1}.$$

(4) Use separation of variables to solve the Laplace equation $u_{xx} + u_{yy} = 0$, subject to the homogeneous boundary conditions $u(x,0) = u_x(\pi, y) = u_x(0,y) = 0$ and the nonhomogeneous boundary condition $u(x,2) = 4 + 3\cos x - 2\cos 2x$.

SOLUTION. $u_{xx} + u_{yy} = 0$ and $u_x(0,y) = 0 = u_x(\pi, y)$ implies that

$$u(x,y) = (C_0 y + D_0) + \sum_{n \geq 1} (C_n e^{ny} + D_n e^{-ny}) \cos nx.$$

Next $u(x,0) = 0$ implies

$$D_0 = 0, \; C_n + D_n = 0, \; n \geq 1.$$

Hence

$$u(x,y) = C_0 y + \sum 2C_n \sinh ny \cos nx.$$

Finally,

$$u(x,2) = 4 + 3\cos x - 2\cos 2x = 2C_0 + \sum 2C_n \sinh 2n \cos nx$$

implies

$$C_0 = 2, \; 2C_1 = \frac{3}{\sinh 2}, \; 2C_2 = \frac{-2}{\sinh 4},$$

while the remaining C_n's are zero. Hence

$$u(x,y) = 2y + \frac{3}{\sinh 2} \cos x \sinh y - \frac{2}{\sinh 4} \cos 2x \sinh 2y$$
$$\approx 2y + 0.82716 \cos x \sinh y - 0.073287 \cos 2x \sinh 2y.$$

(5) A right circular solid cylinder of radius b and height h has its lower base maintained at the constant temperature $u = 100$ and its upper base at $u = 0$. If the curved surface is insulated, then find the steady state temperature distribution in the cylinder. What if the curved surface is maintained at $u = 50$ instead of being insulated?

SOLUTION. Let the cylinder be

$$\{x^2 + y^2 \le b^2,\ 0 \le z \le h\}.$$

Then in *cylidrical polar coordinates* (ρ, θ, z), the cylinder is

$$\{0 \le \rho \le b,\ 0 \le z \le h\}.$$

The steady state equation is $\Delta T = 0$. In cylindrical polar coordinates,

$$\Delta = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

We can expect the temperature distribution $T = T(\rho, z)$ to be independent of the azimuthal angle θ. The boundary conditions are

$$T_\rho(b, z) = 0 \text{ (insulated lateral surface)} \quad \text{and} \quad T(\rho, 0) = 100,\ T(\rho, h) = 0.$$

By separation of variables method, let $T(\rho, z) = R(\rho)Z(z)$. Then

$$\Delta T = R''Z + \rho^{-1}R'Z + RZ'' = 0 \qquad \text{or} \qquad \frac{\rho R''}{R} + \frac{R'}{\rho R} = -\frac{Z''}{Z}.$$

The lateral Neumann boundary conditions implies that $R'(b) = 0$. First we try the common constant to be 0. Then $R(\rho) = A_0$ and $Z(z) = C_0 z + D_0$ or equivalently $T = C_0 z + D_0$. If we take $C_0 = -\dfrac{100}{h}$, $D_0 = 100$, we note that $T(\rho, z) = 100 - 100z/h$ already satisfies all the boundary conditions and hence must the solution to the problem. *This is an example of mixed boundary conditions: Neumann along the curved surface and Dirichlet along the flat ends.*

$$T = 100\left[1 - \frac{z}{h}\right].$$

Second part: In this case let us homogenize the Dirichlet boundary conditions in the variable z by letting $T_0 = 100 - 100z/h$ and $S = T - T_0$. Then S satisfies

$$S_{\rho\rho} + \frac{1}{\rho}S_\rho + S_{zz} = 0,$$

with boundary conditions

$$S(\rho, 0) = 0 = S(\rho, h) \quad \text{and} \quad S(b, z) = 100z/h - 50.$$

By separation of variables method we see in this case that the elementary solutions are of the type

$$S_n = R_n(\rho) \sin\frac{n\pi z}{h},\ n = 1, 2, 3, \ldots,$$

and $R_n(\rho)$ is a 'bounded near 0' solution of the scaled Bessel equation of order 0:

$$\rho^2 R''(\rho) + \rho R'(\rho) - n^2\pi^2 h^{-2}\rho^2 R(\rho) = 0.$$

However, the scaling is by the imaginary number $\dfrac{in\pi}{h}$. Thus

$$R_n = J_0(in\pi\rho/h) =: I_0(n\pi\rho/h).$$

Note that I_0 is an even power series with positive coefficients;

$$I_0(x) = \sum_{m\geq 0} \frac{x^{2m}}{4^m (m!)^2}$$

and hence can never vanish along the real line. I_0 is called the *modified Bessel function of the first kind*. (J_0 viewed as a function of a complex variable is entire. On the real part, it behaves like cosine, and on the imaginary part, it behaves like exponential. The latter is what we are calling I_0.) So now

$$S(\rho, z) = \sum C_n I_0(\frac{n\pi\rho}{h}) \sin \frac{n\pi z}{h}.$$

Using the curved boundary condition,

$$100z/h - 50 = \sum C_n I_0(\frac{n\pi b}{h}) \sin \frac{n\pi z}{h}.$$

This implies

$$C_n I_0(\frac{n\pi b}{h}) = \frac{2}{h}\int_0^h \left[100\frac{z}{h} - 50\right] \sin \frac{n\pi z}{h} dz = \begin{cases} 0 & \text{for odd } n, \\ -\dfrac{200}{n\pi} & \text{for even } n. \end{cases}$$

Thus

$$S(\rho, z) = -\frac{200}{\pi} \sum_{n \text{ even}} \frac{I_0(n\pi h^{-1}\rho)}{nI_0(n\pi h^{-1} b)} \sin \frac{n\pi z}{h},$$

and finally

$$T = S(\rho, z) + 100 - \frac{100z}{h}.$$

(6) The upper half of the sphere of radius b is maintained at a temperature $u = 100$, and the lower half is maintained at $u = 0$. Find the steady-state temperature distribution in the solid enclosed by the sphere.

SOLUTION. Let in the spherical polar coordinates (r, θ, φ) the upper half of the sphere be $r = b$, $0 \leq \varphi \leq \pi/2$ and the lower half be $r = b$, $\pi/2 \leq \varphi \leq \pi$. The enclosed solid is $r \leq b$. The 3-dimensional Laplacian in the solid ball is

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left[\frac{\partial^2}{\partial \varphi^2} + \cot\varphi \frac{\partial}{\partial \varphi} + \frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial \theta^2}\right].$$

In steady state $\Delta u(r, \theta, \varphi) = 0$. From the symmetry of the data, we expect that the temperature distribution is independent of θ. Thus $u = u(r, \varphi)$ satisfies the PDE

$$u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}(u_{\varphi\varphi} + \cot\varphi u_\varphi) = 0,$$

along with the Dirichlet boundary conditions

$$u(b, \varphi) = \begin{cases} 100 & \text{if } 0 < \varphi < \pi/2, \\ 0 & \text{if } \pi/2 < \varphi < \pi, \end{cases}$$

and the implicit boundary conditions $u(r,0),\ u(r,\pi),\ u(0,\varphi)$ are bounded. By separation of variables method, let us find the elementary solutions $u(r,\varphi) = Y(r)X(\varphi)$. Substitution in the PDE and separating variables

$$\frac{r^2Y''(r) + 2rY'(r)}{Y(r)} = -\frac{X''(\varphi) + \cot\varphi X'(\varphi)}{X(\varphi)}$$

If the common constant is λ, then $X(\varphi)$ satisfies the Legendre equation in polar form and hence for $X(0),\ X(\pi)$ to be bounded, $\lambda = n(n+1),\ n = 0, 1, 2, \dots,$ and

$$X(\varphi) = P_n(\cos\varphi).$$

Further, for this constant, (solving a Cauchy-Euler equation),

$$Y(r) = Ar^n + B/r^{n+1}$$

and for $Y(0)$ to be bounded, $B = 0$. Thus the elementary solutions are $C_n r^n P_n(\cos\varphi)$. So

$$u(r,\varphi) = \sum_{n\geq 0} C_n r^n P_n(\cos\varphi).$$

Finally the stated boundary conditions implies

$$\sum_{n\geq 0} C_n P_n(z) = 100\chi_{[0,1]}(z), \quad -1 \leq z \leq 1,$$

where $z = \cos\varphi$. The above is the Fourier-Legendre series and so

$$C_n = 50(2n+1)\int_0^1 P_n(z)dz = \begin{cases} 0 & \text{for even } n, \\ \frac{(-1)^{(n-1)/2}50}{2^n n!}\binom{n}{(n-1)/2} & \text{for odd } n. \end{cases}$$

(7) Previous problem if the upper half is maintained at $u = 50\cos\varphi$ and the lower half at $u = -50\cos\varphi$.

SOLUTION. Identical to above. Except in the end

$$\sum_{n\geq 0} C_n P_n(z) = 50|z|, \quad -1 \leq z \leq 1.$$

$$C_n = 25(2n+1)\int_{-1}^1 |z|P_n(z)dz = \begin{cases} 0 & \text{for odd } n, \\ \frac{50(2n+1)}{2^n n!}\int_0^1 zD^n(z^2-1)^n dz & \text{for even } n. \end{cases}$$

On computing for even n,

$$C_n = \begin{cases} \frac{1}{2} & \text{for } n = 0, \\ \frac{(-1)^{n/2-1}50(2n+1)}{2^n n(n-1)}\binom{n}{n/2-1} & \text{for even } n \geq 2. \end{cases}$$

(8) What is the gravitational potential of a thin circular disc of radius a and mass M if the potential on the perpendicular axis of the disc at a distance r from the centre of the disc is $\dfrac{2M}{a^2}\left(\sqrt{r^2+a^2} - r\right)$?

SOLUTION. Assume the disc to to be in the centre of the xy-plane, so that the z-axis is the perpendicular axis of the disc. From symmetry we expect the gravitational potential to be independent of the azimuthal angle θ. So $u = u(r, \varphi)$. In spherical polar coordinates in $\mathbb{R}^3$, the Laplacian is

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left[\frac{\partial^2}{\partial \varphi^2} + \cot\varphi \frac{\partial}{\partial \varphi} + \frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial \theta^2}\right].$$

Therefore, for $u = u(r, \varphi)$,

$$\Delta u = u_{rr} + 2r^{-1}u_r + r^{-2}[u_{\varphi\varphi} + \cot\varphi u_\varphi].$$

To solve $\Delta u = 0$ by separation of variables, $u = X(r)Y(\varphi)$ gives,

$$\frac{r^2 X''(r) + 2rX'(r)}{X(r)} = -\left[\frac{Y''(\varphi)}{Y(\varphi)} + \cot\varphi \frac{Y'(\varphi)}{Y(\varphi)}\right].$$

The common constant must be nonnegative and of the form $n(n+1)$, $n = 0, 1, 2, 3, \ldots$ for $Y(\varphi)$ to be bounded near $0, \pi$. ($Y(\varphi)$ must satisfy the Legendre equation in polar form.) Thus

$$Y(\varphi) = P_n(\cos\varphi)$$

and correspondingly,

$$X(r) = Ar^n + \frac{B}{r^{n+1}}.$$

Hence the general solution is of the form

$$u(r, \varphi) = \sum_{n\geq 0}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)P_n(\cos\varphi).$$

To determine A_n, B_n, first we consider the case $r > a$. In this case, $u \to 0$ as $r \to \infty$ along the z-axis, forces $A_n = 0$. For B_n, we have

$$u(r, \varphi)\Big|_{\varphi=0,\pi} = \frac{2M}{a^2}\left(\sqrt{r^2 + a^2} - r\right),$$

so that $B_{\text{odd}} = 0$. Finally, at $\varphi = 0$,

$$u(r, 0) = \sum_{m\geq 0}\frac{B_{2m}}{r^{2m+1}} = \frac{2Mr}{a^2}\left[\left(1 + \frac{a^2}{r^2}\right)^{1/2} - 1\right].$$

Now by binomial expansion in the powers of $\frac{a}{r}$, we get

$$B_{2m} = (-1)^m M\frac{(2m)!a^{2m}}{4^m m!(m+1)!},$$

and

$$u(r, \varphi) = \frac{M}{a}\sum_{m\geq 0}(-1)^m\frac{(2m)!P_{2m}(\cos\varphi)}{4^m m!(m+1)!}\left(\frac{a}{r}\right)^{2m+1}.$$

Next we look at the case $r < a$. This case is tricky due to a discontinuity when we pass from $z > 0$ to $z < 0$, wherein we must cross the disc. This time by boundedness of u at $r = 0$, all the $B_n's$ must vanish. We first consider the upper half-ball region $\{r < a\} \cap \{z > 0\}$.

$$u(r, \varphi) = \sum_{n\geq 0}A_n r^n P_n(\cos\varphi).$$

Along the positive z-axis,

$$u(r,0) = \sum_{n\geq 0} A_n r^n = \frac{2M}{a}\left[\left(1+\frac{r^2}{a^2}\right)^{1/2} - \frac{r}{a}\right].$$

Again expanding by binomial series and on comparing the coefficients, $A_0 = 2M/a,\ A_1 = -2M/a^2$, the remaining $A_{\text{odd}} = 0$. Also

$$A_{2m} = (-1)^{m-1}\frac{M(2m-2)!}{4^{m-1}m!(m-1)!a^{2m+1}}$$

for $m \geq 1$. Putting together,

$$u(r,\varphi) = \frac{2M}{a}\left[1 - P_1(\cos\varphi)\frac{r}{a}\right] + \frac{M}{a}\left[\sum_{m\geq 1}(-1)^{m-1}\frac{(2m-2)!P_{2m}(\cos\varphi)}{4^{m-1}m!(m-1)!}\left(\frac{r}{a}\right)^{2m}\right].$$

Finally, in the lower half-ball, due to symmetry wrt $\varphi \mapsto \pi - \varphi$, the P_1 term should be relaced by its negative. In short we can write,

$$u = \frac{2M}{a}\left[1 - |\cos\varphi|\frac{r}{a}\right] + \frac{M}{a}\left[\sum_{m\geq 1}(-1)^{m-1}\frac{(2m-2)!P_{2m}(\cos\varphi)}{4^{m-1}m!(m-1)!}\left(\frac{r}{a}\right)^{2m}\right].$$

Remark: The function $r|\cos\varphi|$ is actually $|z|$, which is not harmonic, but it is so, separately in the open half regions, $z > 0$ and $z < 0$.

(9) Find the steady-state temperature distribution in a thin unit spherical frustum between $z = \pm\frac{1}{2}$, whose upper boundary is maintained at the constant temperature T and the lower boundary as per (i) T (ii) $-T$.

(The frustum is the portion of the unit sphere whose z coordinate is between $-1/2$ and $1/2$.)

Solution. In spherical polar coordinates,

$$\Delta_{\mathbb{S}^2} = \frac{\partial^2}{\partial\varphi^2} + \cot\varphi\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}.$$

In both cases, we expect u to be independent of θ. So let $u = u(\varphi)$. In the steady state, $\Delta_{\mathbb{S}^2}u = 0$. So we find $u(\varphi)$ satisfying the ODE:

$$u'' + \cot\varphi u' = 0,$$

which is the Legendre equation with parameter $n = 0$. The general solution is

$$u = AP_0(\cos\varphi) + BQ_0(\cos\varphi) = A + B\log\left[\frac{1+z}{1-z}\right].$$

The second equality holds since $z = \cos\varphi$ on a *unit* sphere.
In (i), the boundary conditions give $A = T$, and $B = 0$. Hence $u \equiv T$.
In (ii), the boundary conditions give $A = 0$, and $B = \dfrac{T}{\log 3}$. Hence

$$u = \frac{T}{\log 3}\log\left[\frac{1+z}{1-z}\right].$$

Generalize to $T = f(\theta)$, instead of being a constant.

SOLUTION. We will only generalize (i). (ii) is similar. We have to solve for $u(\theta, \varphi)$ satisfying the PDE

$$u_{\varphi\varphi} + \cot\varphi u_\varphi + \csc^2\varphi u_{\theta\theta} = 0,$$

in the 'rectangle'

$$0 \le \theta \le 2\pi, \quad \pi/3 \le \varphi \le 2\pi/3$$

subject to the boundary conditions

$$u(\theta, \pi/3) = f(\theta) = u(\theta, 2\pi/3)$$

and implicitly the periodic boundary conditions

$$u(0, \varphi) = u(2\pi, \varphi), \quad u_\theta(0, \varphi) = u_\theta(2\pi, \varphi).$$

As usual, since periodic boundary conditions is a homogeneous boundary conditions, the separation of variables will give an expansion,

$$u(\theta, \varphi) = X_0(\varphi) + \sum_{n\ge 1}[X_n(\varphi)\cos n\theta + Y_n(\varphi)\sin n\theta.$$

Substitution in the PDE leads to the ODEs

$$(X_n'' + \cot\varphi X_n')\sin^2\varphi = n^2 X_n(\varphi), \ n \ge 0$$

and

$$(Y_n'' + \cot\varphi Y_n')\sin^2\varphi = n^2 Y_n(\varphi), \ n \ge 1.$$

The equation for X_0 is the Legendre equation in polar form with parameter $n = 0$. Hence the general solution is

$$X_0(\varphi) = A_0 + B_0 \log\left[\frac{1+\cos\varphi}{1-\cos\varphi}\right] = A_0 + B_0 \log\cot(\varphi/2).$$

For $n \ge 1$, the X_n and Y_n both satisfy the equation

$$(*) \qquad y'' + \cot\varphi y' + \left[-\frac{n^2}{1-\cos^2\varphi}\right] y = 0.$$

The above is the associated Legendre equation of parameter value 0 in *polar form*. By Problem 8 in Section 1.1 combined with Problem 11 in Section 1.2 and putting $x = \cos\varphi$, we find that the general solution of (*) is

$$y(\varphi) = C_n \tan^n(\varphi/2) + D_n \cot^n(\varphi/2).$$

Thus we can let

$$X_n = A_n \tan^n(\varphi/2) + B_n \cot^n(\varphi/2) \quad \text{and} \quad Y_n = C_n \tan^n(\varphi/2) + D_n \cot^n(\varphi/2).$$

Thus

$$u(\theta, \varphi) = \left(A_0 + B_0 \log\cot\frac{\varphi}{2}\right) + \sum_{n\ge 1}\left(A_n \tan^n\frac{\varphi}{2} + B_n\cot^n\frac{\varphi}{2}\right)\cos n\theta$$
$$+ \sum_{n\ge 1}\left(C_n \tan^n\frac{\varphi}{2} + D_n\cot^n\frac{\varphi}{2}\right)\sin n\theta$$

Finally, let

$$f(\theta) = a_0 + \sum_{n\ge 1}[a_n\cos n\theta + b_n \sin n\theta].$$

The values $\tan(\pi/6) = 1/\sqrt{3}$ and $\tan(\pi/3) = \sqrt{3}$ allow us to find all the unknown constants in terms of 'known' constants a_n, b_n. The final answers are
Generalized (i):

$$\begin{aligned} A_0 &= a_0, B_0 = 0, \\ A_n &= B_n = \frac{a_n}{3^{n/2} + 3^{-n/2}}, \\ C_n &= D_n = \frac{b_n}{3^{n/2} + 3^{-n/2}}, \end{aligned}$$

and

$$u(\theta, \varphi) = a_0 + \sum_{n \geq 1} \frac{(\tan^n \frac{\varphi}{2} + \cot^n \frac{\varphi}{2})}{3^{n/2} + 3^{-n/2}} [a_n \cos n\theta + b_n \sin n\theta].$$

Generalized (ii):

$$\begin{aligned} A_0 &= 0, B_0 = \frac{2a_0}{\log 3}, \\ -A_n &= B_n = \frac{a_n}{3^{n/2} - 3^{-n/2}}, \\ -C_n &= D_n = \frac{b_n}{3^{n/2} - 3^{-n/2}}, \end{aligned}$$

and

$$u(\theta, \varphi) = \frac{2a_0}{\log 3} \log \cot \frac{\varphi}{2} + \sum_{n \geq 1} \frac{(\cot^n \frac{\varphi}{2} - \tan^n \frac{\varphi}{2})}{3^{n/2} - 3^{-n/2}} [a_n \cos n\theta + b_n \sin n\theta].$$

(10) Show that in solving for the steady-state temperature distribution in space using the separation of spherical coordinates (r, θ, φ), we are naturally led to solving for the amplitudes of the pure harmonics of the unit sphere.

Solution. In polar coordinates,

$$\Delta_{\mathbb{R}^3} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left[\frac{\partial^2}{\partial \varphi^2} + \cot\varphi \frac{\partial}{\partial \varphi} + \frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial \theta^2}\right].$$

The steady state equation is $\Delta u(r, \theta, \varphi) = 0$, which we attempt to solve by separation of variables. So let $u = X(r)Y(\theta)Z(\varphi)$. The domain of the equation is $[0, \infty) \times [0, 2\pi] \times [0, \pi]$ and the implicit homogeneous boundary conditions are (i) $u(r, \theta, \varphi)$ remains bounded as $\varphi \to 0, \pi$ and (ii) $u(r, 0, \varphi) = u(r, 2\pi, \varphi)$, $u_\theta(r, 0, \varphi) = u_\theta(r, 2\pi, \varphi)$. On substituting $u = XYZ$ in the steady state equation and simplifying, we get

$$\frac{r^2 X''(r)}{X(r)} + \frac{2rX'(r)}{X(r)} = -\left[\frac{Z''(\varphi)}{Z'(\varphi)} + \cot\varphi \frac{Z'(\varphi)}{Z(\varphi)} + \frac{1}{\sin^2\varphi}\frac{Y''(\theta)}{Y(\theta)}\right].$$

The lhs is a function of r alone while rhs is that of (θ, φ), hence both sides must equal to a common constant. At this stage solving for $Y(\theta)$ and $Z(\varphi)$ under the boundary conditions stated above is solving for the amplitudes of the fundamental harmonics of the unit sphere $\mathbb{S}^2$.

(11) Find the steady-state temperature function in the shell enclosed between two concentric spheres of radii b_1 and b_2 respectively, if the temperature distributions $u(b_1, \varphi) = f_1(\varphi)$ and $u(b_2, \varphi) = f_2(\varphi)$ are maintained over the inner and outer surfaces, respectively. Solve explicitly when $b_1 = 1,\ b_2 = 2$ and $f_1(\varphi) = \cos\varphi$ and $f_2(\varphi) = 3\cos 2\varphi$.

Solution. In spherical polar coordinates, we expect the temperature distribution to be independent of θ due to the boundary temperatures being independent of the same. Hence in effect we have to solve the PDE

$$u_{rr} + 2r^{-1}u_r + r^{-2}(v_{\varphi\varphi} + \cot\varphi u_\varphi) = 0,$$

in the 'rectangle' $[b_1, b_2] \times [0, \pi]$ subject to the boundary conditions (i) $u(b_1, \varphi) = f_1(\varphi),\ u(b_2, \varphi) = f_2(\varphi)$ and (ii) $u(r, \varphi)$ is bounded as $\varphi \to 0, \pi$. Let $u(r, \varphi) = X(r)Y(\varphi)$. Then we get

$$\frac{r^2X''(r) + 2rX'(r)}{X(r)} = -\frac{Y''(\varphi) + \cot\varphi Y'(\varphi)}{Y(\varphi)}.$$

Due to the boundary conditions (ii), the common constant has to be $n(n+1),\ n \in \{0, 1, 2, \dots\}$ and

$$Y(\varphi) = P_n(\cos\varphi).$$

Correspondingly,

$$X(r) = Ar^n + B/r^{n+1}$$

and the general solution is

$$u(r, \varphi) = \sum_{n\geq 0} \left(A_n r^n + B_n/r^{n+1}\right) P_n(\cos\varphi).$$

Finally, expand the boundary data in Fourier-Legendre series:

$$f_1(\varphi) = \sum c_n P_n(\cos\varphi) = \sum \left(A_n b_1^n + B_n/b_1^{n+1}\right) P_n(\cos\varphi)$$

and

$$f_2(\varphi) = \sum d_n P_n(\cos\varphi) = \sum \left(A_n b_2^n + B_n/b_2^{n+1}\right) P_n(\cos\varphi).$$

Hence,

$$b_1^n A_n + b_1^{-n-1} B_n = c_n \quad \text{and} \quad b_2^n A_n + b_2^{-n-1} B_n = d_n.$$

This allows us to solve for unknown constants A_n, B_n in terms of 'known' b_1, b_2, c_n, d_n for $n \geq 0$. In the numerical example,

$$f_1(\varphi) = P_1(\cos\varphi) \quad \text{and} \quad f_2(\varphi) = 6\cos^2\varphi - 3 = -P_0 + 4P_2(\cos\varphi)$$

which implies $c_1 = 1, c_n = 0, n \neq 1$ and $d_0 = -1, d_2 = 4, d_n = 0, n \neq 0, 2$. This trivially implies $A_n = 0 = B_n,\ n \geq 3$. The equations

$$A_0 + B_0 = 0 \quad \text{and} \quad A_0 + \frac{B_0}{2} = -1$$

imply $A_0 = -2, B_0 = 2$. The equations

$$A_1 + B_1 = 1 \quad \text{and} \quad 2A_1 + \frac{B_1}{4} = 0$$

imply $A_1 = \dfrac{-1}{7}, B_1 = \dfrac{8}{7}$, and the equations

$$A_2 + B_2 = 0 \quad \text{and} \quad 4A_2 + \frac{B_2}{8} = 4$$

imply $A_2 = \dfrac{32}{31} = -B_2$.