MA-207 Differential Equations II
S1 - Lecture 6

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Recall : Assume $x = 0$ is a regular singular point of $x^2y'' + xB(x)y' + C(x)y = 0$ and $I(r) = 0$ has real roots $r_1 \geq r_2$. Then we get a Frobenius solution $y(x, r) = x^r \sum_{n \geq 0} a_n(r)x^n, \quad a_0(r) = 1$ at $r = r_1$.

For second solution $y_2(x)$, we have

- If $r_1 - r_2 \notin \mathbb{Z}$, then $y_2(x) = y(x, r_2)$.
- If $r_1 = r_2$, then $y_2(x) = \dfrac{\partial y(x, r)}{\partial r}\Big|_{r=r_2}$.
- If $r_1 \neq r_2$, $r_1 - r_2 \in \mathbb{Z}$, then

$$y_2(x) = \frac{\partial (r - r_2)y(x, r)}{\partial r}\Big|_{r=r_2}$$

Bessel equation $x^2y'' + xy' + (x^2 - p^2)y = 0, \;\; p \geq 0.$
Roots of $I(r) = 0$ are $p, -p$.

For $p \notin \{-1, -2, \dots, \}$

$$J_p(x) = \sum_{n\geq 0} \frac{(-1)^n}{n!\,\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}$$

For integer $m \geq 1$, define

$$J_{-m}(x) := \sum_{n\geq 0} \frac{(-1)^n}{n!\,\Gamma(n-m+1)} \left(\frac{x}{2}\right)^{2n-m}$$

$$= \sum_{n\geq m} \frac{(-1)^n}{n!\,(n-m)!} \left(\frac{x}{2}\right)^{2n-m} = \sum_{n\geq 0} \frac{(-1)^{n+m}}{(n+m)!\,(n)!} \left(\frac{x}{2}\right)^{2n+m}$$

$$= (-1)^m J_m(x)$$

So, $J_p(x)$ is defined for all real p.
Let us see some Bessel identities.

(1) $[x^p J_p(x)]' = x^p J_{p-1}(x)$.

$$J_p(x) = \sum_{n\geq 0} \frac{(-1)^n}{n!\,\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}$$

$$(x^p J_p(x))' = \left(2^p \sum_{n\geq 0} \frac{(-1)^n}{n!\,\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+2p}\right)'$$

$$= 2^p \sum_{n\geq 0} \frac{(-1)^n(2n+2p)}{n!\,\Gamma(n+p+1)} \frac{1}{2} \left(\frac{x}{2}\right)^{2n+2p-1}$$

$$= 2^p \sum_{n\geq 0} \frac{(-1)^n}{n!\,\Gamma(n+p)} \left(\frac{x}{2}\right)^{2n+2p-1}$$

$$= x^p \sum_{n\geq 0} \frac{(-1)^n}{n!\,\Gamma(n+p)} \left(\frac{x}{2}\right)^{2n+p-1} = x^p J_{p-1}(x)$$

(1) $[x^p J_p(x)]' = x^p J_{p-1}(x)$

Similarly, prove

(2) $[x^{-p} J_p(x)]' = -x^{-p} J_{p+1}(x)$

Expand LHS of (1), (2) and divide by $x^{\pm p}$ to prove

(3) $J_p'(x) + \dfrac{p}{x} J_p(x) = J_{p-1}(x)$

(4) $J_p'(x) - \dfrac{p}{x} J_p(x) = -J_{p+1}(x)$

Adding and subtracting (3) and (4), prove

(5) $J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x)$

(6) $J_{p-1}(x) + J_{p+1}(x) = \dfrac{2p}{x} J_p(x)$

Show that between any two consecutive zeros of $J_p(x)$, there exists precisely one zero of $J_{p-1}(x)$ and precisely one zero of $J_{p+1}(x)$.

Proof. Let $0 < c < d$ be two consecutive zeros of $J_p(x)$. So $x^p J_p(x)$ vanishes at c and d. By Rolle's theorem,

$$[x^p J_p(x)]'(b) = 0 \quad \text{for some} \quad b \in (c, d)$$

$$[x^p J_p(x)]' = x^p J_{p-1}(x) \implies J_{p-1}(b) = 0$$

Assume there exist another zero b' of $J_{p-1}(x)$ in (c, d). Assume $c < b < b' < d$.

Use

$$[x^{-(p-1)}J_{p-1}(x)]' = -x^{-(p-1)}J_p(x)$$

Since $x^{-(p-1)}J_{p-1}(x)$ has zeros at b, b', its derivative $-x^{-(p-1)}J_p(x)$ has a zero at $b'' \in (b, b') \subset (c, d)$.

Hence $J_p(x)$ has a zero at b''.

This is a contradiction to the assumption that c, d were consecutive zeros of $J_p(x)$.

This proves that in the interval (c, d), $J_{p-1}(x)$ has a unique zero.

Similarly, you prove that in the interval (c, d), $J_{p+1}(x)$ has a unique zero. $\square$

Example.

If $J_2(x) - J_0(x) = aJ_c''(x)$, find a and c.

Use $J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x)$ for $p = 1$, we get

$J_0(x) - J_2(x) = 2J_1'(x)$

Now use $[x^{-p}J_p(x)]' = -x^{-p}J_{p+1}$ for $p = 0$, we get

$J_0'(x) = -J_1(x)$. Therefore,

$J_2(x) - J_0(x) = -2J_1'(x) = 2J_0''(x)$.

Hence $a = 2$ and $c = 0$. $\square$

We can use $\quad J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x) \quad$ and

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

to compute $J_p(x)$ for all half integer values of p.

- $J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)$
$$= \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$$
- $J_{-3/2}(x) = -\frac{1}{x} J_{-1/2}(x) - J_{1/2}(x)$
$$= -\sqrt{\frac{2}{\pi x}} \left(\frac{\cos x}{x} + \sin x \right)$$

- $J_{\frac{5}{2}}(x) = \frac{3}{x} J_{\frac{3}{2}}(x) - J_{\frac{1}{2}}(x)$
$$= \sqrt{\frac{2}{\pi x}} \left(\frac{3 \sin x}{x^2} - \frac{3 \cos x}{x} - \sin x \right)$$

For integer m, $J_{m+\frac{1}{2}}(x)$ are elementary functions called spherical Bessel functions as they arise in solving wave equations in spherical coordinates.

Theorem (Liouville)

$J_{m+\frac{1}{2}}(x)$'s are the only Bessel functions which are elementary functions, where $m \in \mathbb{Z}$.

- $$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$$

$$\Longrightarrow \int_0^x t^p J_{p-1}(t)\,dt = x^p J_p(x)$$

- $$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$$

$$\Longrightarrow \int_0^x t^{-p} J_{p+1}(t)\,dt = -x^{-p} J_p(x)$$

For example, with $p = 1$,

$$\int_0^x t J_0(t)\,dt = x J_1(x)$$

Qualitative properties of solutions

It is rarely possible to solve 2nd order linear ODE

$$y'' + P(x)y' + Q(x)y = 0$$

in terms of familiar elementary functions.

Then how do we understand the nature and properties of solutions.

It is surprising that we can obtain quite a bit of qualitative information about the solution from the ODE itself.

Let us study some of the qualitative properties of the solution.

Theorem (Sturm separation theorem)

If $y_1(x)$ and $y_2(x)$ are linearly independent solns of

$$y'' + P(x)y' + Q(x)y = 0$$

P, Q continuous on (a, b). Then
(1) $y_1(x)$ and $y_2(x)$ have no common zero in (a, b).
(2) Between any two successive zeros of $y_1(x)$, there is exactly one zero of $y_2(x)$ and vice versa.

Proof of (1). Since y_1, y_2 are linearly independent solutions, their Wronskian

$$W(x) := W(y_1, y_2) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

is non vanishing on (a, b), hence (1) follows.

Proof of (2). Let x_1 and x_2 be successive zeros of $y_1(x)$. This means $y_1(x_1) = y_1(x_2) = 0$ and y_1 has no zeros on (x_1, x_2).
To show y_2 has a zero in (x_1, x_2).
If not, then either $y_2 > 0$ or $y_2 < 0$ on (x_1, x_2).
For $x \in \{x_1, x_2\}$, the Wronskian reduces to

$$W(x) = -y_1'(x)y_2(x) \neq 0$$

Hence $y_1'(x) \neq 0$ for $x \in \{x_1, x_2\}$. Further, $y_1'(x_1)$ and $y_1'(x_2)$ must have opposite signs. Therefore, $W(x_1)$ and $W(x_2)$ must have opposite signs.
This is a contradiction, since Wronskian is non-vanishing and continuous on (a, b).
Hence it has a constant sign. $\square$

As a consequence, if y_1 and y_2 are linearly independent solution of $y'' + P(x)y' + Q(x)y = 0$, P, Q continuous on (a, b).
Then the number of zeros of y_1 and y_2 on (a, b) differ by atmost 1.
In particular, either both have finite number of zeros or both have infinite number of zeros in (a, b).

• For further discussion, we need that any ODE in the "standard" form $y'' + P(x)y' + Q(x)y = 0$ can be written in the "normal" form $u'' + q(x)u = 0$.

Put $y(x) = u(x)v(x)$, $y' = u'v + uv'$, and $y'' = uv'' + 2u'v' + u''v$ in the ODE.

$$(uv'' + 2u'v' + u''v) + P(u'v + uv') + Quv = 0$$

$$vu'' + (2v' + Pv)u' + (v'' + Pv' + Qv)u = 0$$

Put $2v' + Pv = 0 \implies v(x) = exp\left(-\frac{1}{2}\int P(x)\,dx\right)$

Thus our ODE reduces to normal form

$$u'' + q(x)u = 0$$

where

$$q(x) = \frac{1}{v}(v''+Pv'+Qv) = Q(x)-\frac{1}{4}P(x)^2-\frac{1}{2}P'(x)$$

Theorem

Let $u(x)$: non-trivial solution of $\boxed{u'' + q(x)u = 0}$ *$q(x) < 0$ and continuous on (a, b). Then $u(x)$ has atmost one zero in (a, b).*

Proof. Assume $u(x_0) = 0$. Then $u'(x_0) \neq 0$, since Wronskian $W(x_0) \neq 0$.

Assume $\widetilde{x} \in (a, b)$ is next zero of $u(x)$ after x_0. Assume $u'(x_0) > 0$. Then $u(x) > 0$ on $(x_0, \widetilde{x})$.

Since $u''(x) = -q(x)u(x) > 0$ on $(x_0, \widetilde{x})$, $u'(x)$ is an increasing function on $(x_0, \widetilde{x})$.
Hence $u'(\widetilde{x}) > 0$. This contradicts that $u'(\widetilde{x}) < 0$.
$\square$

Theorem

Let $u(x)$: non-trivial solution of $\boxed{u'' + q(x)u = 0}$
$q(x)$: continuous and $q(x) > 0$ for all $x > x_0$.
If $\displaystyle\int_{x_0}^{\infty} q(x)\,dx = \infty$,
then $u(x)$ has infinitely many zeros in (x_0, ∞).

Proof. Assume $u(x)$ has only finitely many zeros on $(0, \infty)$. Then $\exists\, \widetilde{x} > x_0$ such that $u(x) \neq 0$ for $x \geq \widetilde{x}$. Assume $u(x) > 0$ for $x \geq \widetilde{x}$.
Then $u''(x) = -q(x)u(x) < 0$ for $x \geq \widetilde{x}$. Hence $u'(x)$ is decreasing for $x \geq \widetilde{x}$.
If we show that $u'(x) < 0$ for some $x > \widetilde{x}$, then $u(x_1) = 0$ for some $x_1 > \widetilde{x}$, a contradiction.

To show that $u'(x) < 0$ for some $x > \widetilde{x}$. Put

$$v(x) = -\frac{u'(x)}{u(x)}, \quad \text{for} \quad x \geq \widetilde{x}$$

$$v' = \frac{-u''u + u'^2}{u^2} = \frac{q(x)u^2 + u'^2}{u^2} = q(x) + v(x)^2$$

Integrate from $\widetilde{x}$ to $x > \widetilde{x}$, we get

$$v(x) - v(\widetilde{x}) = \int_{\widetilde{x}}^{x} q(x)\, dx + \int_{\widetilde{x}}^{x} v(x)^2\, dx$$

$\int_{x_0}^{\infty} q(x)\, dx = \infty \implies v(x) > 0$ for large x.
Hence for large x, $u(x)$ and $u'(x)$ have opposite signs. So claim is proved as $u(x) > 0$ for $x > \widetilde{x}$. $\square$

Theorem

In Bessel equation $x^2y'' + xy' + (x^2 - p^2)y = 0$
Substituting $u(x) = \sqrt{x}y(x)$, *we get*

$$u'' + \left[1 + \frac{1 - 4p^2}{4x^2}\right] u = 0$$

$q(x) = 1 + \dfrac{1 - 4p^2}{4x^2}$ *is continuous and* $q(x) > 0$ *for* $x > x_0 > 0$. *Further,*

$$\int_{x_0}^{\infty} \left(1 + \frac{1 - 4p^2}{4x^2}\right) dx = \infty$$

By previous theorem, $u(x)$, *hence any Bessel function has infinitely many zeros on* $(0, \infty)$.

Theorem

Let $u(x)$: non-trivial solution of $\boxed{u'' + q(x)u = 0}$ *on finite interval $[a, b]$, with $q(x)$ continuous. Then $u(x)$ has at most finite number of zeros in $[a, b]$. Hence if $u(x)$ has infinitely many zeros on $(0, \infty)$, then the set of zeros of $u(x)$ are not bounded.*

Proof. Assume $u(x)$ has infinitely many zeros in $[a, b]$. Then $\exists\, x_0 \in [a, b]$ and a sequence of zeros $x_n \neq x_0$ such that $x_n \to x_0$ as $n \to \infty$.
$u(x_0) = \lim_{x_n \to x_0} u(x_n) = 0$ (u is continuous) and

$$u'(x_0) = \lim_{x_n \to x_0} \frac{u(x_n) - u(x_0)}{x_n - x_0} = 0$$

a contradiction. $\square$

Corollary

Let $Z^{(p)}$ be the set of zeros of Bessel function $J_p(x)$ on $(0, \infty)$. Since $Z^{(p)}$ is an infinite set, it is not bounded.

We will conside the following question.

Write $Z^{(p)} = \{x_1, x_2, \ldots\}$ as increasing sequence $x_n < x_{n+1}$.

Question. What is the limit of $x_{n+1} - x_n$ as $n \to \infty$?

3 Extra slides : Not part of the course.

Assume roots of $I(r) = 0$ are complex $r_1 = t + is$ and $r_2 = t - is$, $i = \sqrt{-1}$, $s \neq 0$.
In this case, their difference $r_1 - r_2 = 2is \notin \mathbb{Z}$.
Hence we get two Frobenius solutions.
But the coefficients $a_n(r_1)$ and $a_n(r_2)$ will be complex conjugates of each other.

Further $x^{t+is} = x^t.e^{\log x^{is}} = x^t e^{is \log x}$

$x^{t+is} = x^t(\cos(s \log x) + i \sin(s \log x))$

Similarly, $x^{t-is} = x^t(\cos(s \log x) - i \sin(s \log x))$

Therefore, $y_1(x) = y(x, r_1)$ and $y_2 = y(x, r_2)$ are two L.I. solutions which are complex conjugates.

Taking real and imaginary part of y_1, we get two linearly independent <u>real</u> solutions.

Remark on solving higher order linear ODE.

$x = 0$ is a regular singular point of

$$y''' + \frac{1}{x}B(x)y'' + \frac{1}{x^2}C(x)y' + \frac{1}{x^3}D(x)y = 0$$

if $B(x), C(x), D(x)$ are analytic at 0.

In this case a Frobenius solution

$$y(x,r) = \sum_{n\geq 0} a_n(r)x^{n+r}$$

exists atleast on $(0,\rho)$, where ρ is the minimum of radius of convergence of $B(x), C(x), D(x)$ at 0.

Coefficient of x^r in

$x^3y''' + x^2B(x)y'' + xC(x)y' + D(x)y = 0$

gives the indicial equation $I(r) = 0$.

Assume $r_1 \geq r_2 \geq r_3$ be the real roots of $I(r) = 0$.

So $I(r) = (r - r_1)(r - r_2)(r - r_3)$.

- $r = r_1$: Frobenius solution $y(x, r_1)$.
- If $r_1 - r_2 \notin \mathbb{Z}$, Frobenius solution $y(x, r_2)$.
- If $r_1 - r_3, r_2 - r_3 \notin \mathbb{Z}$, Frobenius solution $y(x, r_3)$.
- If $r_1 = r_2$, Solution: $\frac{\partial}{\partial r}y(x,r)\big|_{r_1}$
- $r_1 = r_2 = r_3$, Solutions: $\dfrac{\partial}{\partial r}y(x,r)\Big|_{r_1}$ and $\dfrac{\partial^2}{\partial r^2}y(x,r)\Big|_{r_1}$ both give solutions.
- $r_1 - r_2 \in \mathbb{Z}$, Solution: $\dfrac{\partial}{\partial r}(r - r_2)y(x,r)\Big|_{r_2}$
- $r_1 - r_2, r_2 - r_3 \in \mathbb{Z}$, Solution: $\frac{\partial}{\partial r}(r - r_2)y(x,r)\big|_{r_2}$ and $\frac{\partial}{\partial r}(r - r_2)(r - r_3)y(x,r)\big|_{r_3}$