MA-207 Differential Equations II
S1 - Lecture 7

M.K. Keshari

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

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Recall : Consider $u'' + q(x)u = 0$, $q(x)$ continuous in a neighbourhood of $[a, b]$. Let $u(x)$ and $\widetilde{u}(x)$ be linearly independent solutions.

- $u(x)$ and $\widetilde{u}(x)$ have no common zero in $[a, b]$.
- Between any two successive zeros of $u(x)$, there is exactly one zero of $\widetilde{u}(x)$.
- $q(x) < 0 \implies u(x)$ has atmost one zero in $[a, b]$.
- $q(x) > 0$ for all $x > x_0$ and $\displaystyle\int_{x_0}^{\infty} q(x)\,dx = \infty$,
  $\implies u(x)$ has infinitely many zeros in (x_0, ∞).
- $u(x)$ can have at most finite number of zeros in any finite interval $[a, b]$.

Corollary

$Z^{(p)}$ = set of zeros of Bessel function $J_p(x)$ on $(0, \infty)$.
Since $Z^{(p)}$ is an infinite set, it is not a bounded set.

Write $Z^{(p)} = \{\lambda_{1,p}, \lambda_{2,p}, \ldots\}$, where $\lambda_{n,p} < \lambda_{n+1,p}$.

Question. What is the limit of $\lambda_{n+1,p} - \lambda_{n,p}$ as $n \to \infty$?

Answer. this limit is π.
For proof, we will need the Sturm comparison theorem.

Theorem (Sturm Comparison theorem)

Let $y(x)$ be a non-trivial solutions of

$$y'' + q(x)y = 0$$

and $z(x)$ be a non-trivial solutions of

$$z'' + r(x)z = 0$$

where $q(x) > r(x) > 0$ are continuous in a neighbourhood of $[a, b]$.
Then $y(x)$ vanishes atleast once between any two consecutive zeros of $z(x)$ in $[a, b]$.

- Compare $y'' + 4y = 0$ and $z'' + z = 0$. Zeros of $z(x)$ are π apart and that of $y(x)$ are $\pi/2$ apart.

Proof of Sturm Comparison theorem.

Let $x_1 < x_2$ be consecutive zeros of $z(x)$ in $[a, b]$.
Assume $y(x)$ has no zero in (x_1, x_2).
We may assume $z(x) > 0$ and $y(x) > 0$ on (x_1, x_2).
Hence $z'(x_1) > 0$ and $z'(x_2) < 0$. The Wronskian

$$W(x) = W(y, z) = y(x)z'(x) - y'(x)z(x)$$

$$W'(x) = yz'' - y''z = y(-rz) - (-qy)z = (q-r)yz > 0$$

on (x_1, x_2). Integrate from x_1 to x_2, we get

$$W(x_2) - W(x_1) > 0 \implies W(x_2) > W(x_1)$$

But $W(x_1) = y(x_1)z'(x_1) > 0$ and
$W(x_2) = y(x_2)z'(x_2) < 0$, a contradiction. $\square$

Theorem. Substituting $u(x) = \sqrt{x}y(x)$ in Bessel equation, we get $(p \geq 0)$

$$u'' + q(x)u = 0, \qquad q(x) = 1 + \frac{1 - 4p^2}{4x^2}$$

- $p < 1/2 \implies q(x) > 1$
- $p = 1/2 \implies q(x) = 1$ (Well known, hence, uninteresting)
- $p > 1/2 \implies q(x) < 1$

Use $z'' + z = 0$ and Sturm comparison theorem.

Let $y_p(x)$ be a non-trivial solution of Bessel equation. Then we get ...

Theorem

- $\boxed{p < 1/2} \implies$ *Between any two roots of* $\sin(x - x_0)$, *there is a root of* $y_p(x)$.
- $\boxed{p = 1/2} \implies$ *difference of any two consecutive roots of* $y_p(x)$ *is* π
- $\boxed{p > 1/2} \implies$ *Between any two roots of* $y_p(x)$, *there is a root of* $\sin(x - x_0)$.

We can say more than the above. Suppose $\boxed{p < 1/2}$ and $a < b < c$ are consecutive roots of $u(x)$. Then $b - a < c - b$. That is, the difference between the successive roots keeps increasing.

To see a proof of this,

Consider the function $f := u(x - b + a)$ defined on the interval (b, ∞).

It is a trivial check that f satisfies the ODE

$$f'' + r(x)f = 0, \qquad r(x) := q(x - b + a)$$

Since $p < 1/2$, the function q is strictly decreasing. Thus on (b, ∞), we have $r(x) > q(x) > 0$.

Applying Sturm's comparison theorem, there is $b < x_0 < c$ such that $f(x_0) = u(x_0 - b + a) = 0$.

Clearly,

- $b < x_0 \implies a < x_0 - b + a$
- $a < b \implies x_0 - b + a < x_0$
- $\implies a < x_0 - b + a < x_0 < c$

However, $a < b < c$ are successive roots of $u(x)$. This forces that

$$x_0 - b + a = b \qquad \text{that is} \qquad x_0 = 2b - a$$

As $2b - a = x_0 < c \implies b - a < c - b$.

• Claim. The difference between any two successive roots of u is strictly less than π.
If not, then let $a < b$ be successive roots such that $b - a \geq \pi$

Since u has infinitely many roots, and their difference is strictly increasing, assume $b - a > \pi$.

But $\sin(x - x_0)$ for some x_0, has two roots in (a, b) and u has no root in (a, b), this contradicts Sturm's comparison theorem. Hence the claim is proved.

Thus, we have proved that if $\{x_n\}$ are the roots of $u(x)$ in increasing order, then the difference $x_{n+1} - x_n$ is strictly increasing and $x_{n+1} - x_n < \pi$.

• Claim. These differences converge to π.
If not, then $(x_{n+1} - x_n) \to \gamma < \pi$.
Choose $1 < \delta$, sufficiently close to 1 such that $\gamma < \frac{\pi}{\delta} < \pi$.

The function $q(x)$ is decreasing to 1. Therefore, there is a $x_0 \in \mathbb{R}$, sufficiently large, such that $q(x_0) < \delta^2 \implies q(x) < \delta^2$ for $x > x_0$.

Apply Sturm's comparison on the interval (x_0, ∞) to ODEs $u'' + q(x)u = 0$ and $z'' + \delta^2 z = 0$.

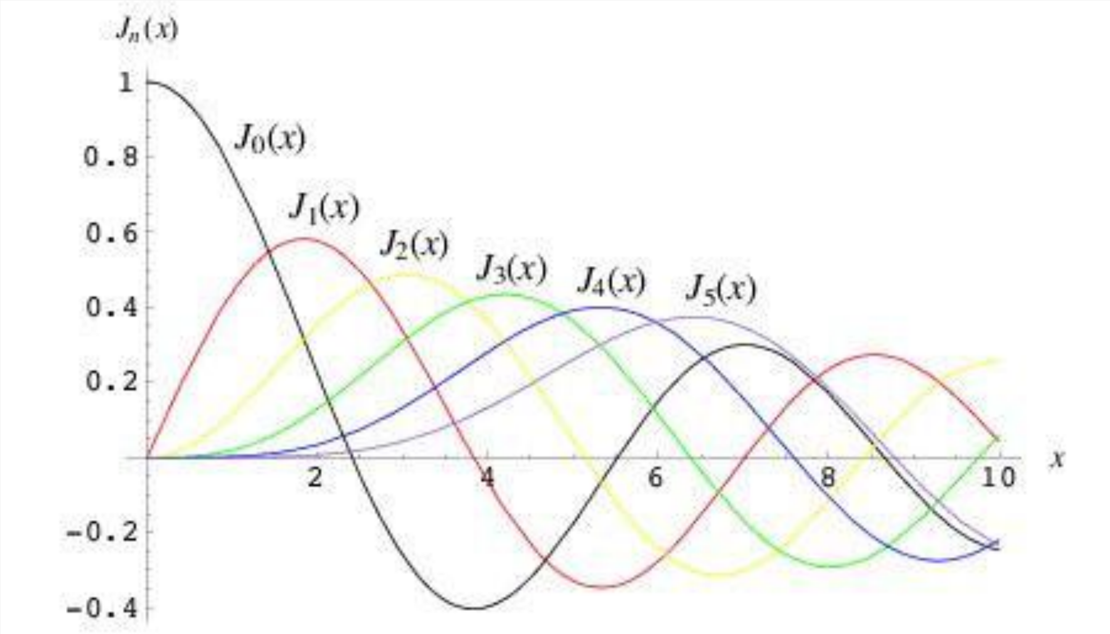
Thus, between any two roots of $u(x)$, there is a root of $z(x)$. If a and b are two consecutive roots of u, then $b - a < \gamma < \frac{\pi}{\delta} < \pi$.
We can find $\widetilde{x}$ such that $\sin(x - \widetilde{x})$ has no root in (a, b). $\square$

Assume $Z^{(p)} = \{\lambda_{1,p}, \lambda_{2,p}, \ldots\}$ be roots of Bessel function $u(x)$ with $\lambda_{n,p} < \lambda_{n+1,p}$.

Theorem

If $p < 1/2$, then $\lambda_{n+1,p} - \lambda_{n,p} < \lambda_{n+2,p} - \lambda_{n+1,p} < \pi$ and $\lambda_{n+1,p} - \lambda_{n,p} \to \pi$ as $n \to \infty$.

Similarly, if $p > 1/2$, then
$\lambda_{n+1,p} - \lambda_{n,p} > \lambda_{n+2,p} - \lambda_{n+1,p} > \pi$
and $\lambda_{n+1,p} - \lambda_{n,p} \to \pi$ as $n \to \infty$.



The first few zeroes of Bessel functions are tabulated below.

	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

Question. Why are we concerned with zeros of Bessel function $J_p(x)$?

It is often required (in mathematical physics) to expand a given function in terms of Bessel functions.

Simplest and most useful expansions are of the form

$$f(x) = a_1 J_p(\lambda_{1,p}x) + a_2 J_p(\lambda_{2,p}x) + \dots$$

where $f(x)$ is defined on $[0, 1]$ and $\lambda_{n,p}$'s are zeros of Bessel function $J_p(x)$, $p \geq 0$.

Qn. How to compute the coefficients a_n?

Remark: For a scalar $a \neq 0$, the scaled Bessel functions $J_p(ax)$ are solutions of

$$x^2y'' + xy' + (a^2x^2 - p^2)y = 0$$

known as scaled Bessel equation.

Orthogonality

Define an inner product on square-integrable functions on $[0,1]$ by

$$\langle f, g\rangle := \int_0^1 x f(x) g(x)\, dx$$

This is similar to the previous inner product except that $f(x)g(x)$ is now multiplied by x and the interval of integration is from 0 to 1.
The multiplying factor x is called a weight function.

Fix $\boxed{p \geq 0}$. Let $Z^{(p)} = \{\lambda_{1,p}, \lambda_{2,p}, \ldots\}$ denote the set of zeros of $J_p(x)$ on $(0, \infty)$.

Theorem

The set of scaled Bessel functions

$$\{J_p(\lambda_{1,p}x), J_p(\lambda_{2,p}x), \ldots\}$$

form an orthogonal family w.r.t. above inner product, i.e.

$$\langle J_p(\lambda_{k,p}x), J_p(\lambda_{l,p}x)\rangle = \int_0^1 x J_p(\lambda_{k,p}x) J_p(\lambda_{l,p}x)\, dx$$

$$= \begin{cases} \frac{1}{2}[J_{p+1}(\lambda_{k,p})]^2 & \lambda_{k,p} = \lambda_{l,p} \\ 0 & \lambda_{k,p} \neq \lambda_{l,p} \end{cases}$$

Theorem

Fix $\boxed{p \geq 0}$. *Any* $f(x) \in L^2([0,1])$ *wrt above inner product, i.e.* $\langle f, f \rangle = \int_0^1 x f^2(x)\, dx < \infty$ *can be expanded in a series of scaled Bessel functions* $J_p(\lambda_{n,p} x)$ *as*

$$\sum_{n \geq 1} c_n J_p(\lambda_{n,p} x)$$

$$c_n = \frac{2}{[J_{p+1}(\lambda_{n,p})]^2} \int_0^1 x\, f(x)\, J_p(\lambda_{n,p} x)\, dx$$

This is Fourier-Bessel series of $f(x)$ *for parameter* p.

Example. Let us compute the Fourier-Bessel series $\boxed{\text{for } p = 0}$ of $f(x) = 1$ in the interval $0 \leq x \leq 1$.

Use $\int x^p J_{p-1}(x)\, dx = x^p J_p(x) + c$ for $p = 1$.

$$\int_0^1 x\, J_0(\lambda_{n,0} x)\, dx = \frac{1}{\lambda_{n,0}} x\, J_1(\lambda_{n,0} x)\Big|_0^1 = \frac{J_1(\lambda_{n,0})}{\lambda_{n,0}}$$

$$c_n = \frac{2}{[J_1(\lambda_{n,0})]^2} \int_0^1 x\, f(x)\, J_0(\lambda_{n,0} x)\, dx = \frac{2}{\lambda_{n,0} J_1(\lambda_{n,0})}$$

Thus, the Fourier-Bessel series of $f(x)$ is

$$\sum_{n \geq 1} \frac{2}{\lambda_{n,0} J_1(\lambda_{n,0})} J_0(\lambda_{n,0} x)$$

By next theorem, this converges to 1 for $0 < x < 1$.

Convergence in norm $\boxed{p \geq 0}$

Fourier-Bessel series of $f(x) \in L^2([0,1])$, namely

$$\sum_{n\geq 1} c_n J_p(\lambda_{n,p}x), c_n = \frac{2}{J_{p+1}(\lambda_{n,p})^2} \int_0^1 x f(x) J_p(\lambda_{n,p}x)dx$$

converges to $f(x)$ in norm, i.e.

$$\|f(x) - \sum_{n=1}^{m} c_n J_p(\lambda_{n,p}x)\| \text{ converges to } 0 \text{ as } m \to \infty$$

In particular For pointwise convergence, we have

Bessel expansion theorem $\boxed{p \geq 0}$

Assume f and f' have at most a finite number of jump discontinuities in $[0,1]$, then the Bessel series

$$\sum_{n\geq 1} c_n J_p(\lambda_{n,p}x), c_n = \frac{2}{J_{p+1}(\lambda_{n,p})^2} \int_0^1 x f(x) J_p(\lambda_{n,p}x)dx$$

of $f(x)$ converges for $0 < x < 1$ to

$$\frac{f(x_-) + f(x_+)}{2}$$

At $x = 1$, the series always converges to 0 for all f as $J_p(\lambda_{n,p}) = 0$.
If $p > 0$, at $x = 0$, it converges to 0.
If $p = 0$, at $x = 0$, it converges to $f(0_+)$.

Orthogonality of scaled Bessel functions $\boxed{p \geq 0}$

If a, b are positive scalars, then $u(x) = J_p(ax)$ and $v(x) = J_p(bx)$ satisfies

$$u'' + \frac{1}{x}u' + \left(a^2 - \frac{p^2}{x^2}\right) u = 0$$

$$v'' + \frac{1}{x}v' + \left(b^2 - \frac{p^2}{x^2}\right) v = 0$$

Multiply by v and u resp. and subtract, we get

$$(vu'' - uv'') + \frac{1}{x}(vu' - uv') + (a^2 - b^2)uv = 0$$

$$(u'v - v'u)' + \frac{1}{x}(u'v - v'u) = (b^2 - a^2)uv$$

$$(x(u'v - v'u))' = (b^2 - a^2)xuv$$

$$(b^2 - a^2)\int_0^1 xuv\,dx = [x(u'v - v'u)]\Big|_0^1 = (u'v - v'u)\Big|^1$$

$$(b^2-a^2)\int_0^1 xJ_p(ax)J_p(bx)\,dx = J_p'(a)J_p(b) - J_p'(b)J_p(a)$$

So if $a = \lambda_{k,p}$ and $b = \lambda_{l,p}$ are <u>distinct</u>, then

$$\int_0^1 xJ_p(\lambda_{k,p}x)J_p(\lambda_{l,p}x)\,dx = 0$$

To compute the norm of $J_p(\lambda_{k,p}x)$, consider

$$2x^2u'\left[u'' + \frac{1}{x}u' + (a^2 - \frac{p^2}{x^2})u\right] = 0$$

$$\implies [x^2u'^2 + (a^2x^2 - p^2)u^2]' = 2a^2xu^2$$

Integrate on $[0, 1]$.

$$2a^2 \int_0^1 xu^2\,dx = [x^2u'^2 + (a^2x^2 - p^2)u^2]\Big|_0^1$$

For $p = 0$, RHS is 0 at $x = 0$.

For $p > 0$, $u(0) = J_p(0) = 0$.

Hence, when $x = 0$, RHS is zero.

Further, $u'(1) = aJ_p'(a)$, we get

$$\int_0^1 xJ_p(ax)^2\,dx = \frac{1}{2}J_p'(a)^2 + \frac{1}{2}\left(1 - \frac{p^2}{a^2}\right)J_p(a)^2$$

Put $a = \lambda_{k,p}$

$$\int_0^1 xJ_p(\lambda_{k,p}x)^2\,dx = \frac{1}{2}J_p'(\lambda_{k,p})^2 = \frac{1}{2}J_{p+1}(\lambda_{k,p})^2$$

for last equality, use $x = \lambda_{k,p}$ in

$J_p'(x) - \frac{p}{x}J_p(x) = J_{p+1}(x)$ $\square$