Midterm Solutions

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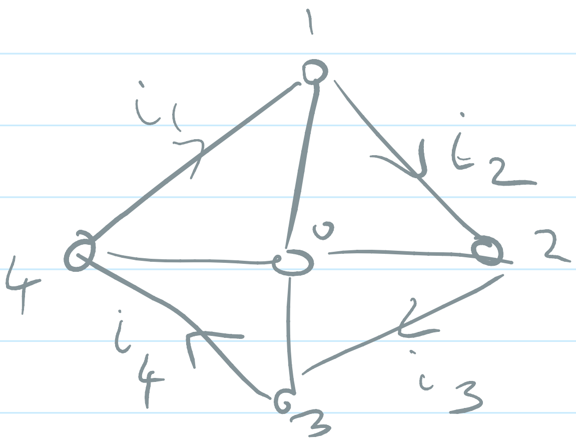
1.)



Loop 4104: $\frac{1}{C_1}\int_{-\infty}^{t} i_1\,dt + V = R_1(i_4 - i_1)$

Loop 1201: $V = i_2 R_2 + \frac{1}{C_2}\int_{-\infty}^{t}(i_2 - i_3)\,dt$

Loop 2302: $i_3 R_2 + (i_3 - i_4) R_L + \frac{1}{C_2}\int_{-\infty}^{t}(i_3 - i_2)\,dt = 0$

Loop 3403: $\frac{1}{C_3}\int_{-\infty}^{t} i_4\,dt + (i_4 - i_1) R_1$

$+ (i_4 - i_3) R_L = 0$

2) (a) Let $S = \{v_1, \ldots, v_k\}$.

$\mathrm{Span}(S) := \left\{ \sum_{i=1}^{k} c_i v_i \mid c_i \in F \right\}$

(b) If $\sum c_i v_i \in Span(S)$ and $\sum d_i v_i \in Span\, S$

then $a \sum c_i v_i + b \sum d_i v_i$

$$= \sum (a c_i + b d_i) v_i \in Span(S)$$

Hence $Span(S)$ is a vector space.

(d) $S \subseteq W_S$.

$Span(S)$ is a vector space containing S. By definition, W_S is the intersection of all vector spaces containing S.

$$\therefore W_S \subseteq Span(S). \quad -(1)$$

If V is a vector space containing S then $Span(S) \subseteq V$. Hence

$$Span(S) \subseteq \bigcap_{\substack{S \subseteq V \\ V \text{ vector space}}} V = W_S. \quad -(2)$$

From (1), (2), $Span(S) = W_S$.

(e) $Span(S)$ is a vector space, and $W_S = Span(S)$. Hence W_S is a vector space.

3) $$v - L_1 \frac{di_1}{dt} = v_1 \qquad (1)$$

$$L_1 \frac{d}{dt}(i - i_1) = v_1 \qquad (2)$$

$$\Rightarrow L_1 \frac{di}{dt} = v$$ makes the circuits

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4) (a) Let (N, E), (N', E') be graphs

A homomorphism is a function
$f : N \to N'$
s.t. $\forall (n_1, n_2) \in E$, we have

$f(n_1), f(n_2) \in E'$.

(b) A tree T is a subgraph (N_T, E_T) of $\mathcal{G} = (N, E)$ s.t.
$\forall \; n_1, n_2 \in N(T)$, $\exists$ unique

path in T from n_1 to n_2.

(c) $(G, 0, +)$ is an Abelian group iff

$+ : G \times G \to G$

(1) $a + 0 = 0 + a = 0$

Inverse : $G \to G$

(2) $g + (-g) = 0$

(3) Associativity $g + (h+k) = (g+h) + k$

(4) Abelian: $g + h = h + g$

(d) $V^* := \{ f : V \to F \mid f \text{ linear map} \}$

(e) $L : U \to V$ is a linear map iff

$$L(au_1 + bu_2) = aL(u_1) + bL(u_2).$$

(f) Transpose $L^T : V^* \to U^*$ is the map
$f \mapsto f \circ L$

(g) $S = \{v_1, \ldots, v_k\}$ is linearly independent iff

$$\sum c_i v_i = 0 \Rightarrow c_i = 0 \ \forall i$$

5) $g = 1_A \circ g = (h \circ f) \circ g = h \circ (f \circ g) = h \circ 1_B = h$

$\therefore$ $g = h$. Further $f \circ g = 1_B$ and $g \circ f = 1_A$, so $g = f^{-1}$, and f is an isomorphism.

$\square$

6) (a) A tree T in G with $N(T) = N(G)$ is a spanning tree.

(b) Let $H \hookrightarrow G$ be a minimal subgraph of G which is connected i.e. $\forall n_1, n_2 \in N_G$, there exists

a path $n_1 \to m_1 \to m_2 \to \cdots m_{k-1} \to n_2$ in H.

and further every edge of H is a cut. (If not, H is not minimal, since we can drop an edge).

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<u>Claim:</u> $\forall\, n_1, n_2 \in N_G$, there exists a unique path $n_1 \rightsquigarrow n_2$ in H

<u>Proof:</u> Fix n_1, n_2. We know there is a path $n_1 \rightsquigarrow n_2$ in H, by assumption. Suppose there is more than one path, for contradiction. Let the two paths be

$$\pi_1 = n_1 \to m_1 \to \cdots \to m_{k-1} \to n_2 \quad \text{and}$$

$$\pi_2 = n_1 \to l_1 \to \cdots \to l_{q-1} \to n_2 .$$

Without loss of generality, suppose the edge $m_j \to m_{j+1}$ does not belong to π_2 (there is such an edge either in π_1 or in π_2 because $\pi_1 \neq \pi_2$.)

We claim that edge $m_j \to m_{j+1}$ is not a cut. This will contradict the assumption and complete the proof.

To see this, note that there is a path

$$\pi = m_j \to m_{j-1} \to \cdots m_1 \to n_1 \to l_1 \cdots \to l_{q-1} \to n_2 \to m_{k-1} \to \cdots \to m_{j+1}$$

Hence any path that used the edge $m_j \to m_{j+1}$ can instead use this path π.

So $m_j \to m_{j+1}$ is not a cut. Hence there is a unique path $n_1 \rightsquigarrow n_2$ in H.

Hence H is a spanning tree. $\square$