MA 205

End-Semester Examination (September 2013)

Time allowed : 2 hours **Maximum marks : 40**

The symbol $\mathbb{C}$ stands for the set of complex numbers, and the symbol $\mathbb{R}$ stands for the set of real numbers. Except for Question 6, you must show some work on any part of a question to claim any credit.

Figures in square brackets indicate marks.

1. (a) Let f be analytic on $\mathbb{C} \setminus \{0\}$. If $|f(z)| \leq \sqrt{|z|} + \dfrac{1}{\sqrt{|z|}}$ for all $z \in \mathbb{C} \setminus \{0\}$, then show that f is constant. [4]

   (b) Let $f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ where $n \geq 2$ and $0 \neq a_0, a_1, \ldots, a_n \in \mathbb{C}$. If $\gamma(t) = Re^{it}$, $0 \leq t \leq 2\pi$, and all the zeros of f lie in $\{z \in \mathbb{C} : |z| < R\}$, then find $\displaystyle\int_\gamma \frac{z^2 f'(z)}{f(z)} dz.$ [2]

2. (a) Prove that the equation $z^5 + 15z + 1 = 0$ has precisely four solutions in the annulus $\{z \in \mathbb{C} : \frac{3}{2} < |z| < 2\}$. [4.5]

   (b) What is the nature of the singularity of $f(z) = \dfrac{\sin z^2}{z^5 \sin z}$ at $z = 0$? Justify your answer. [1.5]

3. (a) Evaluate
$$\int_0^{2\pi} \frac{d\theta}{1 - \cos\theta + \frac{1}{4}}$$
   using the Residue Theorem. [3]

   (b) Let G be a region, let $a \in G$, and let $f : G \to \mathbb{C}$ be analytic. If, for a postive integer m, $\lim\limits_{z \to a} (z-a)^{-m} f(z)$ is a non-zero complex number, then show that f has a zero of multiplicity m at $z = a$. [3]

4. (a) Evaluate
$$\int_0^{\infty} \frac{\cos x}{(x^2+9)(x^2+4)} dx$$
   using the Residue Theorem. [3]

   (b) Let $\{f_n\}$ be a sequence of analytic functions defined on the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$ and such that $\{f_n\}$ converges uniformly to a function f on the annulus. Show that f is analytic on the annulus. [3]

5. (a) Using the Maximum Modulus Theorem, prove the following:
If f is a non-constant analytic function on a region G, then $|f|$ cannot have a positive local minimum in G. [3]

(b) Find the domain of analyticity of $f(z) = \log(\frac{1+iz}{1-iz})$ where log stands for the principal branch of the logarithm. [3]

6. State whether the following statements are True or False. A wrong answer will result in 1 negative mark.

(a) If γ is a closed piecewise smooth path in a region G such that the index $\eta(\gamma; w)$ of γ with respect to w is zero for every $w \in \mathbb{C} \setminus G$, then γ is null-homotopic. [2]

(b) If f is a function defined on an open set G such that f is differentiable at a point $a \in G$, then $f(z)$ can be represented as a power series in $(z-a)$ for z in an open disk with positive radius centered at a. [2]

(c) For a function f analytic on $S = \{x + iy : x \in \mathbb{R}, y \in \mathbb{R}; |x| < 1, |y| < 1\}$ and continuous on $\bar{S} = \{x + iy : x \in \mathbb{R}, y \in \mathbb{R}; |x| \leq 1, |y| \leq 1\}$, and satisfying that $|f|$ is bounded on the four sides γ_1, γ_2, γ_3, γ_4 of the square $\bar{S}$ respectively by 1, 2, 3, 4, it is possible to have $|f(0)| > 2$. [2]

(d) It is not possible to find a sequence of points z_n such that $0 < |z_n| < 1/2$ and $z_n \sin(1/z_n) \to i$. [2]

(e) There exists a continuous branch of $\sqrt{z}$ on $\mathbb{C} \setminus \{\theta e^{i\theta} : 0 \leq \theta < \infty\}$. [2]