MA 205

Complex Analysis

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(Course ends with mid-sem break)

Introduction

$\mathbb{R}$ = set of real numbers

(Your old friend from MA105 - Calculus 🙂)

Recall : $\mathbb{R}$ $(+, \cdot)$

addition multiplication

<u>Question:</u> Can we extend these operations to

$\mathbb{R} \times \mathbb{R}$, $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$, ... ?

<u>Answer:</u> $\underline{+}$: Can easily be extended,

in fact, these are vector spaces over $\mathbb{R}$

(remember MA106).

$\underline{\cdot}$: One way is to use the dot products

$$(x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = \sum_{i=1}^{n} x_i y_i$$

$$\bar{x} \cdot \bar{y}$$

Here $\bar{x} \cdot \bar{y}$ could be 0 even if $\bar{x} \neq 0$ & $\bar{y} \neq 0$ (why?)

We would like a multiplication such that $\bar{x}\cdot\bar{y} \neq 0$ if $\bar{x} \neq 0$ & $\bar{y} \neq 0$ as in $\mathbb{R}$.

$\boxed{\text{Fact}}$ Such an "abelian multiplication" can only be defined on $\mathbb{R}^2$ & that is as follows:

$$(*) \quad \underbrace{(x_1, y_1)\cdot(x_2, y_2)}_{\bar{z}_1 \cdot \bar{z}_2} = (x_1x_2 - y_1y_2,\ x_1y_2 + y_1x_2)$$

: for $x_1, x_2, y_1, y_2 \in \mathbb{R}$

Note: ① $\bar{z}_1 \cdot \bar{z}_2 = \bar{z}_2 \cdot \bar{z}_1$

② $\bar{z}_1 \cdot \bar{z}_2 = 0 \Rightarrow \bar{z}_1 = 0$ or $\bar{z}_2 = 0$

③ $(0,1)\cdot(0,1) = (-1, 0)$

i.e., -1 is a square with this multiplication!

The above formula is derived as follows:

Let i = an "imaginary number" such that $i^2 = -1$.

Then $i \notin \mathbb{R}$ (why?)

Let $\mathbb{C} = \mathbb{R} + i\mathbb{R} = \{x+iy \mid x, y \in \mathbb{R}\}$

Check: $\mathbb{C}$ is a vector space over $\mathbb{R}$ with addition and scalar multiplication defined as:

$$(x_1 + iy_1) + (x_2 + iy_2) = x_1 + x_2 + i(y_1 + y_2)$$

$$c \cdot (x_1 + iy_1) = cx_1 + i\,cy_1$$

for $x_1, y_1, x_2, y_2, c \in \mathbb{R}$.

Clearly, $\mathbb{C} \cong \mathbb{R} \times \mathbb{R}$ as $\mathbb{R}$-vector spaces.

Dimension of $\mathbb{C}$ over $\mathbb{R}$ $= 2$

(Recall MA106 ☺)

- Define multiplication on $\mathbb{C}$ as the usual term by term multiplication along with $i^2 = -1$

i.e., $(x_1 + i y_1) \cdot (x_2 + i y_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$

Note: Clearly, the multiplication on $\mathbb{R}^2$ defined in $(*)$ above is the same as the one in $\mathbb{C}$ as defined above. (Check)

Note what we have done above is to "adjoin" a root of the polynomial x^2+1 which is irreducible over $\mathbb{R}$ (why is it irreducible?) to $\mathbb{R}$.

Exercises:

1. Show that if $z = x+iy \in \mathbb{C}$, $x, y \in \mathbb{R}$ then
   (a) $z_1 \cdot z_2 = z_2 \cdot z_1$

   (b) $z_1(z_2 z_3) = (z_1 z_2) z_3$

2. Define the complex conjugation map:

$$\overline{} : \mathbb{C} \longrightarrow \mathbb{C} \quad \text{defined as}$$

$$\overline{x+iy} = x - iy \quad : \quad x, y \in \mathbb{R}.$$

Show that $\bar{\ }$ as defined above is $\mathbb{R}$-linear map on $\mathbb{C}$ which is a bijection.

<u>Notation:</u> For $z = x+iy \in \mathbb{C}$ with $x, y \in \mathbb{R}$

we set: $x = \mathrm{Re}(z) =$ real part of z

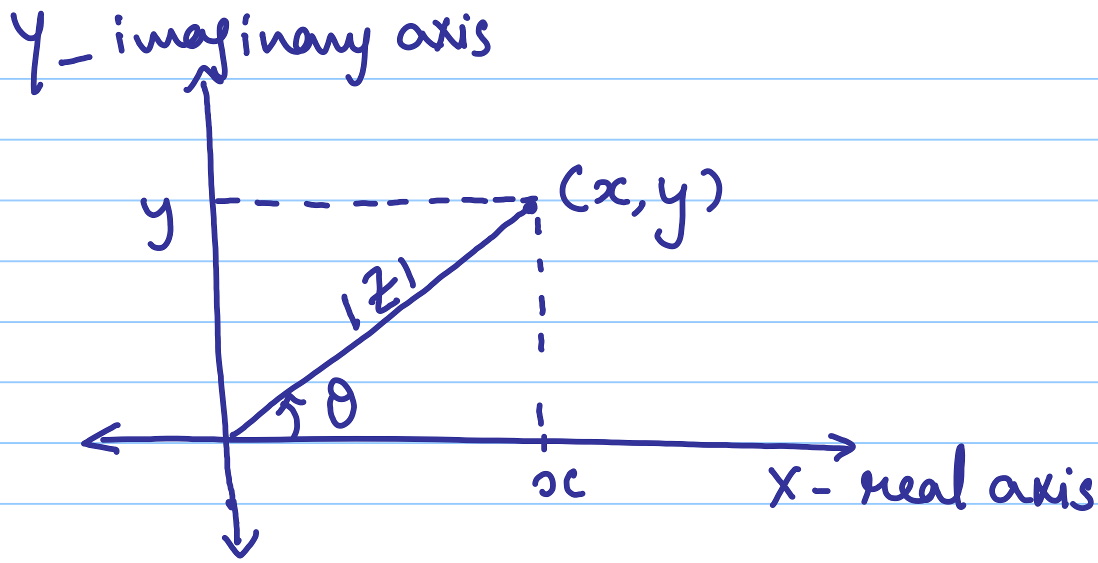
$y = \mathrm{Im}(z) =$ imaginary part of z.

<u>Modulus or absolute value of a complex no:</u>

Let $z = x+iy \in \mathbb{C}$, $x, y \in \mathbb{R}$

<u>Define:</u> Modulus or absolute value of z as:

$$|z| = \sqrt{x^2+y^2} \in \mathbb{R}$$



This gives us the distance of z from the origin in the complex plane.

Clearly, $|Re(z)| \le |z|$

$|Im(z)| \le |z|$

Suppose $\boxed{z \neq 0}$

Let $\underline{\theta}$ be the angle in radians that z makes with the positive real axis.

Then $x = r\cos\theta$: where $r = |z|$

(polar coordinates!)

$y = r\sin\theta$

Note : θ can have infinitely many values that differ by $2\pi n$, where n is an integer.

Definition: Each such value of θ is called an argument (or general argument) of z and this set is denoted by $\arg(z)$.

- In the interval $-\pi < \theta \leq \pi$ there is a unique such value

θ which is called the <u>principal value</u> of $\arg(z)$ and is denoted by $\operatorname{Arg}(z)$.

i.e., $\arg(z) = \operatorname{Arg}(z) + 2\pi n \quad : n = 0, \pm 1, \pm 2 \cdots$

<u>Note</u> 1 when z is a negative real number,

$\operatorname{Arg}(z) = \pi$ and not $-\pi$!

<u>Note</u> 2 $\arg(z)$ is a multivalued function of z.

$\operatorname{Arg}(z)$ is a single valued function of z.

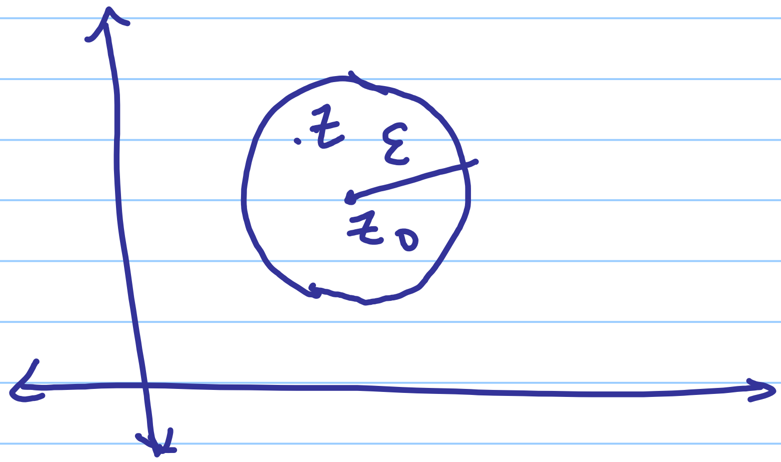
Point to ponder: After we define the notion of a continuous function, come back here and check the continuity of Arg(z).

Regions/Domains in the Complex plane:

Here we discuss analogues in $\mathbb{C}$ of the open/closed intervals and sets in $\mathbb{R}$.

- For $\varepsilon > 0$ & $z_0 \in \mathbb{C}$, an ε-neighborhood of z_0 is:

$$B_\varepsilon(z_0) = \{ z \in \mathbb{C} \mid |z - z_0| < \varepsilon \}$$



Let $S \subseteq \mathbb{C}$ be any subset.

Then $z_0 \in \mathbb{C}$ is called an:

① Interior point of S : if $B_\varepsilon(z_0) \subseteq S$ for some $\varepsilon > 0$.

② Exterior point of S : if there exists some $\varepsilon > 0$ such that $B_\varepsilon(z_0) \cap S = \phi$

③ A boundary point of S : if z_0 is neither an interior nor an exterior point of S.

ie $B_\varepsilon(z_0) \cap S \neq \phi$ & $B_\varepsilon(z_0) \cap S^c \neq \phi$ for every $\varepsilon > 0$

We denote by

$\mathrm{Int}(S)$ = set of interior points of S

$\mathrm{Ext}(S)$ = set of exterior points of S

$\partial(S)$ = set of boundary points of S

clearly, $\mathbb{C} = \mathrm{Int}(S) \cup \mathrm{Ext}(S) \cup \partial(S)$

(check!)

Further, the intersection of any two of the above 3 sets is empty. (check)

<u>Examples</u>: $S_1 = \{ z \in \mathbb{C} \mid |z| < 1 \}$
(C.W.)

$S_2 = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$

Find their interior, exterior & boundary.

$$\mathrm{Int}(S_1) = \mathrm{Int}(S_2) = S_1$$

$$\mathrm{Ext}(S_1) = \mathrm{Ext}(S_2) = \{ z \in \mathbb{C} \mid |z| > 1 \}$$

$$\partial(S_1) = \partial(S_2) = \{ z \in \mathbb{C} \mid |z| = 1 \}$$

<u>Definitions:</u> Open set : if $S = \mathrm{Int}(S)$

Closed set : complement of an open set

closure of S : smallest closed set containing S.

<u>Exercises:</u> ① Show that S is open if and only if $S \cap \partial S = \phi$.

② closure of $S = \mathrm{Int}(S) \cup \partial(S)$.

Examples: ① $B_\varepsilon(z_0)$ is open for every $\varepsilon > 0$.

② $\{z \in \mathbb{C} \mid |z| \geq \varepsilon\}$ is a closed set for every $\varepsilon > 0$.

③ $\{z \in \mathbb{C} \mid |z| = 1\}$ is a closed set.

④ Punctured disc $= \{z \in \mathbb{C} \mid 0 < |z| \leq \varepsilon\}$ is neither open nor closed.

⑤ $\Omega = \left\{ x+iy \in \mathbb{C} \,/\, x, y \in \mathbb{R} \text{ and } \frac{x^2}{4} + \frac{y^2}{9} < 1 \right\}$ is open.

Definition: <u>Connected set</u> : Every $z_1, z_2 \in S$ can be joined by a continuous map $\gamma : [0,1] \to \mathbb{C}$ such that $\gamma([0,1]) \subseteq S$ & $\gamma(0) = z_1$, $\gamma(1) = z_2$.

Example: ① $B_\varepsilon(z_0)$ is connected

② $\{ z \in \mathbb{C} \mid 1 < |z| < 2 \}$ is connected

③ Is the unit square in the complex plane connected?

Definition: <u>Domain</u> : An open & connected set is called a domain.

Example: $B_\varepsilon(z_0)$ is a domain.

Limits

Let f be a function defined on

$B_\varepsilon(z_0) \setminus \{z_0\}$. We say limit of

$f(z)$ as z approaches z_0 is a

complex number w_0 & write

$$\lim_{z \to z_0} f(z) = w_0$$

if for every $\varepsilon > 0$ there exists a $\delta > 0$

such that

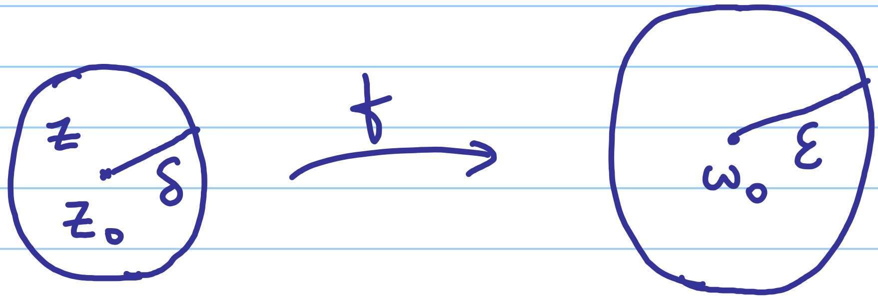
$$|f(z) - w_0| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta.$$

- Compare this with the definition of

$\lim\limits_{x \to x_0} f(x)$ in the set of real numbers $\mathbb{R}$.

Example: Limits exist for constant functions.

C.W. Find z & $f(z)$ which satisfy the two inequalities in the definition of limits for constant functions.



Note: Let f be a function defined on a set S. We can extend the definition of limits to points in the boundary $\partial(S)$ as follows. (This is the analogue of one-sided limits in $\mathbb{R}$).

Let $z_0 \in \partial(S)$. Then given $\varepsilon > 0$

there exists $\delta > 0$ such that

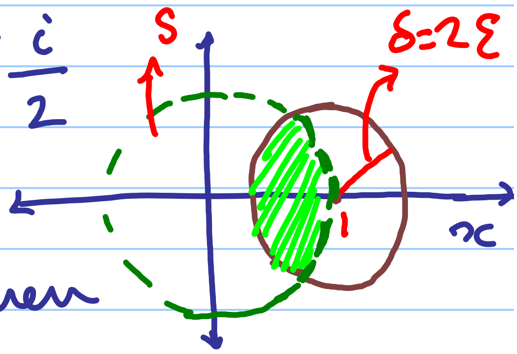
$|f(z) - w_0| < \varepsilon$ whenever $\underline{0 < |z - z_0| < \delta}$

and $\underline{z \in S \cup \partial(S)}$.

<u>Examples:</u> ① Let $S =$ open disc $= \{ z \in \mathbb{C} \mid |z| < 1 \}$

& choose $1 \in \partial(S)$. Let $f(z) = iz/2$.

Show that $\lim\limits_{z \to 1} f(z) = \dfrac{i}{2}$



note that if $|z| < 1$ then

$$\left| f(z) - \frac{i}{2} \right| = \left| \frac{iz}{2} - \frac{i}{2} \right| = \left| \frac{z-1}{2} \right|$$

i.e, $\left| f(z) - \frac{i}{2} \right| < \varepsilon$ whenever $0 < |z - 1| < 2\varepsilon$

Again, if $z \in S \cup \partial(S)$ then $|z| \leq 1$.

(2) Let $f(z) = \dfrac{z}{\bar{z}}$

Show that $\lim_{z \to 0} f(z)$ does not exist.

If this limit exists, we could approach it from any direction, say the x-axis:

$z = (x, 0) \longrightarrow (0, 0)$
$\quad\quad \ne 0$

$f(z) = f(x + i0) = \dfrac{x}{\bar{x}} = 1$.

If we approach from the y-axis:

$z = (0, y) \longmapsto (0, 0)$
$\quad\quad \ne 0$

Then $f(z) = f(0+iy) = f(iy) = \dfrac{iy}{\overline{iy}}$

$= \dfrac{iy}{-iy}$: check $\overline{iy} = -iy$

$= -1$

Limit laws:

It is easy to show that the usual limit laws hold:

1. $\lim_{z \to z_0} f(z)$ if it exists is unique.

Suppose $a \in \mathbb{C}$ & $\lim_{z \to z_0} f(z)$ & $\lim_{z \to z_0} g(z)$

exist. Then

② $$\lim_{z \to z_0} \left(f(z) + g(z)\right) = \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z)$$

③ $$\lim_{z \to z_0} a \cdot f(z) = a \cdot \lim_{z \to z_0} f(z)$$

④ $$\lim_{z \to z_0} f(z) \cdot g(z) = \lim_{z \to z_0} f(z) \cdot \lim_{z \to z_0} g(z)$$

⑤ $$\lim_{z \to z_0} \frac{1}{f(z)} = \frac{1}{\lim_{z \to z_0} f(z)} \quad : \text{if } \lim_{z \to z_0} f(z) \neq 0.$$

H.W. Check the proofs of the above limit laws.

Continuity

Let $\Omega \subseteq \mathbb{C}$ be a subset.

<u>Definition</u>: A function $f : \Omega \to \mathbb{C}$ is continuous at $z_0 \in \Omega$ if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

The function f is continuous on a domain if it is continuous at every point in the domain.

<u>Reading Assignment</u>: Check the sum, product & composition rules hold for continuous functions.

Differentiation and the Cauchy-Riemann equations

Recall from calculus on $\mathbb{R}$ that to define derivatives we consider open intervals. Similarly, in complex analysis we define derivatives on domains – i.e., open & connected subsets of $\mathbb{C}$.

Definition: Let $\Omega \subseteq \mathbb{C}$ be a domain.

A function $f : \Omega \to \mathbb{C}$ is complex differentiable at $z_0 \in \Omega$ if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

Then f is said to be differentiable at z_0 and $f'(z_0)$ is called the derivative of f at z_0.

Check: The sum, product and chain rule hold for differentiable functions at a point z_0.

Connections with real valued functions:

Let $\Omega \subseteq \mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be a function. We write

$$f(x+iy) = u(x,y) + i\,v(x,y) \quad : x,y \in \mathbb{R}.$$

Thus a complex valued function

$$f : \Omega \longrightarrow \mathbb{C}$$

gives rise to two real valued functions

$$u : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$v : \mathbb{R}^2 \longrightarrow \mathbb{R}.$$

Example: $f(z) = z^2$

If $z = x + iy$: $x, y \in \mathbb{R}$, then

$$f(x+iy) = x^2 - y^2 + i\,2xy$$

Hence $u(x,y) = x^2 - y^2$

$v(x,y) = 2xy$

Theorem : Let $z = x + iy$, $x, y \in \mathbb{R}$.

Fix $z_0 = x_0 + iy_0$ & $w_0 = u_0 + iv_0 \in \mathbb{C}$.

Suppose $f(z) = u(x,y) + i\,v(x,y)$. Then,

$\lim_{z \to z_0} f(z) = w_0$ if and only if

$\lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0$ & $\lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0$.

Note The limit laws follow directly from the above theorem and the limit laws for real-valued functions of two real variables.

- Similarly, a function $f(x+iy) = u(x,y) + iv(x,y)$ is continuous at $z_0 = x_0 + iy_0$ if and only if u & v are continuous at (x_0, y_0).

The above interpretation leads to useful applications. For instance:

Supp f is continuous in a region $R \subseteq \mathbb{C}$, that is both closed and bounded - (bounded simply means that $R \subseteq B_\varepsilon(z_0)$ for some $z_0 \in \mathbb{C}$ & $\varepsilon > 0$).

Let $f(z) = u(x,y) + i\,v(x,y)$: $x, y \in \mathbb{R}$,

$z = x + iy$

Then u & v are also continuous functions on R by our above discussion.

$\Rightarrow \sqrt{u(x,y)^2 + v(x,y)^2}$ is continuous on R and reaches its maximum in that region (by Heine-Borel theorem).

$\Rightarrow f$ is bounded on R and

$|f(z)|$ attains a maximum value in R.

i.e., $\exists$ $M \geq 0$, $M \in \mathbb{R}$ such that

$$|f(z)| \leq M \quad \forall z \in R$$

& equality holds for atleast one such z.

Definition: We say f is holomorphic in a domain $\Omega \subseteq \mathbb{C}$ if f is differentiable at each point of Ω.

Exercises: Check for differentiability & holomorphicity:

1. $f(z) = a$: i.e., f is a constant function.

2. $f(z) = z$

3. $f(z) = z^n$; $n \in \mathbb{Z}$

4. $f(z) = \mathrm{Re}(z)$

5. $f(z) = |z|$

⑥ $f(z) = |z|^2$

⑦ $f(z) = \bar{z}$

⑧ $f(z) = \frac{z}{\bar{z}}$: if $z \neq 0$

$= 0$: if $z = 0$

9] Show that if f is differentiable at z_0 then f is continuous at z_0.

Cauchy-Riemann equations:

Let $z = x+iy$, $x, y \in \mathbb{R}$.

Let $f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$

be differentiable at a point $z_0 = x_0 + iy_0$, $x_0, y_0 \in \mathbb{R}$. Then the 1st order partial derivatives of u & v satisfy a pair of equations called the CR equations as seen below.

As f is differentiable at z_0, we have

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists.}$$

Now, as $z \to z_0$ in the x-direction,

we get :

$$f'(z_0) = \lim_{h \to 0} \frac{u(x_0+h, y_0) - u(x_0, y_0)}{h}$$

$$+ i \lim_{h \to 0} \frac{v(x_0+h, y_0) - v(x_0, y_0)}{h}$$

$$= \lim_{h \to 0} \frac{u(x_0+h, y_0) - u(x_0, y_0)}{h} + i \lim_{h \to 0} \frac{v(x_0+h, y_0) - v(x_0, y_0)}{h}$$

(why ?)

$$= u_x(x_0, y_0) + i\, v_x(x_0, y_0)$$

Similarly in the y-direction we get:

$$f'(z_0) = v_y(x_0, y_0) - i\, u_y(x_0, y_0).$$

Thus differentiability of $f = u + iv$ at $z_0 = x_0 + iy_0$ implies that u_x, u_y, v_x, v_y exist at (x_0, y_0) & they satisfy :

$$\boxed{u_x = v_y \quad \& \quad u_y = -v_x \text{ at } (x_0, y_0).}$$

Further, $f'(z_0) = u_x + i v_x$ at (x_0, y_0).

These are the CR equations.

__Note:__ If the CR equations are __not__ satisfied at a point then f is __not__ differentiable at that point.

Example ① $f(z) = |z|^2 = x^2 + y^2$

Here $u(x,y) = x^2 + y^2$, $v(x,y) = 0$

Thus CR equations are satisfied only at $(0,0)$.

$\Rightarrow f$ is <u>not</u> differentiable at any point other than possibly $(0,0)$

<u>at $(0,0)$:</u> $\displaystyle \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$

$$= \lim_{z \to 0} \frac{|z|^2}{z}$$

$$= 0.$$

Example : $f(z) = \dfrac{\bar{z}^2}{z}$

- Conversely, we can prove

if $u : \Omega \to \mathbb{R}$ & $v : \Omega \to \mathbb{R}$ are a pair of real valued functions on the domain Ω such that:

u_x, u_y, v_x, v_y exist in a neighborhood of z_0, are continuous at z_0 and satisfy the CR equations

then $f(z) = u(x,y) + i\, v(x,y)$ is complex differentiable at z_0.

<u>Note:</u> We shall not prove this converse in class.

Exercise: Show that the CR-equations take the form:

$$u_r = \frac{1}{r} v_\theta \quad \& \quad v_r = \frac{-1}{r} u_\theta \quad : r > 0$$

in polar coordinates.

Example: Let $u(x,y) = x^2+y^2$ & $v(x,y)=0$.

Then $f(z) = u(x,y) + i v(x,y) = |z|^2$.

By the above discussion, f is differentiable at $z=0$. In fact, $f'(0) = 0 + i0 = 0$.

But as seen earlier this function cannot have a derivative at any non-zero point as the CR equations are not satisfied there.

Theorem: If $f'(z) = 0$ everywhere in a domain Ω then $f(z)$ must be a constant function in Ω.

Proof: Let $f(z) = u(x,y) + i v(x,y)$

As $f'(z) = 0$ on Ω, we have

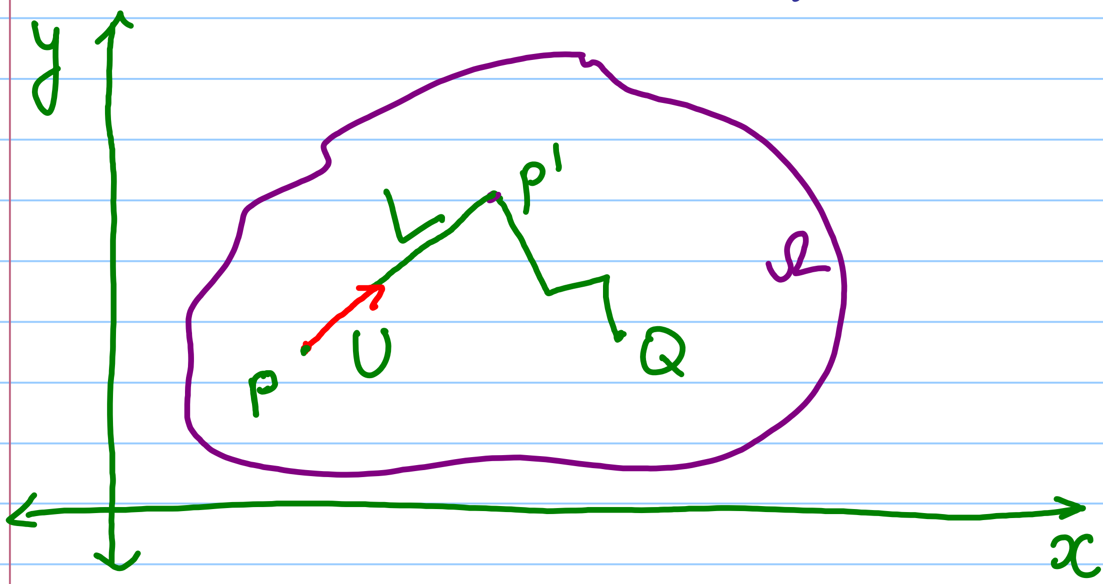
$$u_x + i v_x = 0 = v_y - i u_y \quad \text{(why?)}$$

$$\Rightarrow \quad u_x = u_y = v_x = v_y = 0 \quad \text{on } \Omega$$

Let $P, P' \in \Omega$ be points connected by a line segment L lying entirely in Ω.

Let s = distance along L

U = unit vector along L in the direction of increasing s.



The directional derivative $\frac{du}{ds}$ can be written as the dot product:

(★★) $$\frac{du}{ds} = (\text{grad}\, u) \cdot U$$ (recall from MA105 Calculus)

where $\text{grad}\, u = u_x i + u_y j$.

As u_x & u_y are 0 on Ω, grad u is the zero vector at all points on L.

$\Rightarrow$ By (**) $\dfrac{du}{ds} = 0$ along L

$\Rightarrow$ u is constant on L.

Since Ω is connected, any other point $Q \in \Omega$ can be joined to P by a polygonal path made up of finitely many such line segments lying in Ω & joined end to end.

Hence $u(P) = u(Q)$, for every $Q \in \Omega$.

$\Rightarrow \quad u(x,y) = a$, for a fixed $a \in \mathbb{C}$
$\forall \; x+iy \in \Omega$.

Similarly, $v(x,y) = b$, for a fixed $b \in \mathbb{C}$
$\forall \; x+iy \in \Omega$.

$\Rightarrow \; f(x+iy) = a+ib \quad$ in Ω.

Example: Suppose $\quad f(z) = u(x,y) + i\,v(x,y)$

& its conjugate $\overline{f(z)} = u(x,y) - i\,v(x,y)$

are both holomorphic in a domain Ω.

Show that : f must be constant

in Ω.

Solution:

CR equations for $f(z) \Rightarrow u_x = v_y,\ u_y = -v_x$

CR equations for $\overline{f(z)} \Rightarrow u_x = -v_y,\ u_y = v_x$

$\Rightarrow u_x = 0$ on Ω & $v_x = 0$ on Ω.

Then $f'(z) = u_x + i v_x = 0 + i0 = 0$.

Hence by the earlier theorem,

$f(z)$ is constant in Ω.

Harmonic functions:

Let $\Omega \subseteq \mathbb{R}^2$ be a domain.

Definition: A real valued function

$u: \Omega \longrightarrow \mathbb{R}$ is called harmonic

if it has continuous partial derivatives

of 1st & 2nd order & it satisfies

Laplace's differential equation:

$$u_{xx} + u_{yy} = 0 \quad \text{on } \Omega.$$

Recall that the Laplacian is the

divergence of the gradient:

$$\nabla^2 = \nabla \cdot \nabla = \nabla \cdot (u_x, u_y) = u_{xx} + u_{yy}.$$

This is zero if u is harmonic.

Theorem : If $f = u + iv$ is holomorphic on Ω then both u & v are harmonic on Ω.

Proof : CR equations $\Rightarrow u_x = v_y$ & $u_y = -v_x$

Hence $u_{xx} + u_{yy} = v_{yx} - v_{xy}$.

Fact The continuity of the partial derivatives of u & v ensures that $v_{yx} = v_{xy}$ & $u_{yx} = u_{xy}$.

$$\Rightarrow u_{xx} + u_{yy} = 0$$

Similarly, $v_{xx} + v_{yy} = 0$

That is, u & v are harmonic on Ω.

<u>Example:</u> $f(z) = \dfrac{i}{z^2}$

$$= \frac{i}{z^2} \cdot \frac{\bar{z}^2}{\bar{z}^2}$$

$$= \frac{2xy + i(x^2 - y^2)}{(x^2+y^2)^2}$$

Both $u = \dfrac{2xy}{(x^2+y^2)^2}$ & $v = \dfrac{x^2-y^2}{(x^2+y^2)^2}$

are harmonic throughout any domain in the xy-plane that does not contain $(0,0)$.

<u>(check)</u>

Definition: Suppose u & v are harmonic functions in a domain Ω. We say v is a harmonic conjugate of u if $f = u + iv$ is holomorphic in Ω.

Theorem: A function $f = u + iv$ is holomorphic in a domain Ω if and only if v is a harmonic conjugate of u.

Remarks:

1. v is a harmonic conjugate of u implies u is a harmonic conjugate of $-v$.

2. Show that any two harmonic conjugates of u differ by a constant function.

3. Show that if v is a harmonic conjugate of u & u is a harmonic conjugate of v then u & v are constant functions.

Question: Does every harmonic function u on Ω conjugate?

Answer: If the domain Ω is "good" then Yes. (e.g. $\Omega = \mathbb{R}^2$ or an open disc).