MA 205 Complex Analysis: Some More Theorems

September 2, 2017

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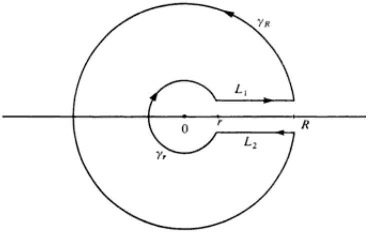

Real Integrals

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The integral is the sum of four integrals; one on L_1, one on γ_R, one on L_2, one on γ_r. Note that

$$\int_r^R \frac{t^{-c}}{1+t} dt = \lim_{\delta \to 0} \int_{L_1} \frac{z^{-c}}{1+z} dz.$$

Similarly,

$$\lim_{\delta \to 0} \int_{L_2} \frac{z^{-c}}{1+z} dz = -e^{-2\pi \imath c} \int_r^R \frac{t^{-c}}{1+t} dt.$$

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This is zero in the limit as $\rho \to 0$ or $\rho \to \infty$. Thus we get:

$$2\pi\imath e^{-\imath\pi c} = (1-e^{-2\imath\pi c})\int_0^\infty \frac{t^{-c}}{1+t}dt.$$

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Thus,

$$\int_0^\infty \frac{t^{-c}}{1+t} dt = \frac{2\pi\imath e^{-\imath\pi c}}{1 - e^{-2\imath\pi c}} = \frac{\pi}{\sin \pi c}.$$

Integrate $I = \int_{-\infty}^{\infty} \frac{e^{x/2}dx}{coshx}$

In this case coshx has infinitely many poles along the imaginary axis, namely at $z = i(\pi/2 + n\pi), n \in \mathbb{Z}$ and so we do not choose the previous kind of contours. Instead we choose a rectangular contour γ consisting of vertices $L, -L, L + i\pi$ and $-L + i\pi$.

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By residue theorem, $\int_\gamma \frac{e^{z/2}dz}{coshz} = 2\pi i Res(f, i\frac{\pi}{2}) = 2\pi e^{i\frac{\pi}{4}}$.

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From this it follows from the ML-inequality that as L tends to ∞, the integral along the right vertical side tends to zero. Similarly one checks that the integral along the left vertical side also tend to zero.

Example cont ..

Now since $cosh(x + i\pi) = -coshx$, the integrals along the horizontal sides are related by

$$\int_{L}^{-L} \frac{e^{(x+i\pi)/2}dx}{cosh(x + i\pi)} = e^{i\pi/2} \int_{-L}^{L} \frac{e^{x/2}dx}{coshx}$$

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Taking L tending to ∞ , we see that
$I = \frac{2\pi e^{i\pi/4}}{(1+e^{i\pi/2})} = \frac{\pi}{cos(\pi/4)} = \pi\sqrt{2}$.

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Suppose the improper integral is of the form $\int_{-\infty}^{\infty} f(x)dx$. The general idea of course is to find a contour which contains the real line as part of the contour in the "limit". The choice should be made so that by residue theory one knows the integral over the full contour and such that the integal over the extra added part goes to zero in the limit.

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I. For instance suppose there exists a constant C such that $|f(z)| \leq \frac{C}{|z^r|}$ for sufficiently large $|z|$ and for some $r > 1$ (here $f(z)$ is an extension of $f(x)$ to a function of the complex variable). Note that this happens for instance in the case when $f(x) = P(x)/Q(x)$ where $\deg Q(x) \geq \deg\, P(x) + 2$. Then close up the interval with a semicircle into the upper half plane and integrate along the contour and take limit as the radius of semicircle goes to infinity. Use ML inequality to show that the integral along the semicircle goes to zero as radius goes to ∞.

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In case the integral is from 0 to ∞ , try and relate it to some integral from $-\infty$ to ∞. For instance the function may have a natural continuation to the negative reals. In case this is not possible, often because $f(x)$ has a singularity at origin; usually a pole, then try using a half annular region $A(0; r, R)$ like we have done in earlier examples.

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Otherwise, try a rectangular contour and show that the integral over the extra sides goes to zero in the limit.

Choice of contour for integration

II. If the integrand is of the form $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} sin(x)dx$, where $P(x)$ and $Q(x)$ are polynomials with $deg(Q(x)$ atleast one more than that of $P(x)$, close up the interval by the semicircular region in the upper half plane and use Jordan's lemma to show that the integral over the semicircle goes to zero using Jordan's lemma. (We have seen this when we integrated $sin(x)/x$).

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III. If the integrand is of the type $\int_0^{2\pi} P(cos(t), sin(t))dt$, set $z = e^{it}$ and use $cos(t) = \frac{z+z^{-1}}{2}$ and $sin(t) = \frac{z-z^{-1}}{2i}$. dt becomes $\frac{dz}{iz}$ and then the integral assumes the form $\int_{|z|=1} P(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i})\frac{dz}{iz}$ which can then be computed by using residue theorem.

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IV. If the integrand has infinitely many poles going to infinity, you are usually better off using a rectangular contour which emcompasses only finitely many poles.

Choice of contour for integration

As before one tries to show that in the limit, the integral over the extra added vertical sides goes to zero in the limit and the intergals over the two horizontal sides are related ; usually proportional to each other. Thus taking limit as the length of the rectangular sides goes to infinity, one gets the desired answer.

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Fractional Residue Theorem : Suppose z_0 is a simple pole of $f(z)$ and C_δ is an arc of the circle $|z - z_0| = \delta$ of angle α, then

$$\lim_{\delta \to 0} \int_{C_\delta} f(z)dz = \alpha i \text{Res}(f(z), z_0)$$

Argument principle

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Argument Principle : Let γ be a closed contour contained in $\mathbb{C}$ and let $f(z)$ be a meromorphic function on an open set containing γ and its interior such that γ does not pass through any of the zeros and poles of $f(z)$. Then

$$\int_{\gamma} \frac{f'(z)}{f(z)} dz = N - P$$

where N and P denote the number of zero's and poles enclosed by γ with each zero and pole counted as many times as its order.

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(Proof sketched on the board)

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The proof of Rouche's theorem follows easily from the argument principle applied to the function f/g and noting that after cancelling common factors, the zeros of f (resp. g) are the zero's (resp. poles) of f/g. I'll leave the details of the proof as an exercise.

Example

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Take $g(z) = 11z^4$. Then $|g(z) - f(z)| < |g(z)|$ on the unit circle. Hence $g(z)$ has the same number of roots as $f(z)$ inside the unit circle. But the number of roots of $g(z)$ inside unit circle is 4 (counting mutiplicity) which therefore equals number of roots of $f(z)$.

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Let us consider $g(z) = -2z$. Then

$$|g(z) - f(z)| = |e^z - 1| = |\sum_1^{\infty} \frac{z^n}{n!}| \leq \sum_1^{\infty} \frac{|z^n|}{n!} = e - 1 < |g(z)|$$

on the unit circle. Hence by Rouche's theorem $f(z)$ and $g(z)$ have equal number of roots in the unit circle, namely 1.

FTA

Here's another quick and pretty proof of FTA using Rouche's theorem.
Let $f(z) = a_0 + a_1 z + \cdots + z^n$ be a non-constant polynomial. Take $g(z) = z^n$. Then on a sufficiently large circle around 0 of radius R, $|f(z) - g(z)| < |f(z)|$. Hence $f(z)$ and $g(z)$ have same number of zero's in the disc of radius R. Since $g(z)$ has n zero's, so does $f(z)$!

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A consequence of this theorem is the **Little Picard theorem** which states that any non-constant entire function can miss atmost one point.

The little Picard Theorem can be seen to be a corollary of the Big Picard Theorem as follows :

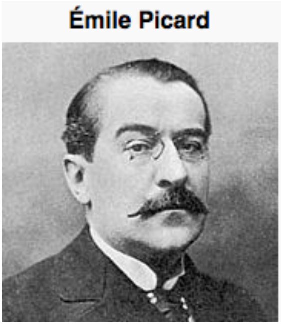
Picard's Theorem

Recall the following fact mentioned earlier: An entire function has a pole at infinity if and only if it is a non-constant polynomial.

Let $f(z)$ be a non-constant entire function. We wish to show it misses atmost one point. If $f(z)$ is a polynomial, then it is surjective by FTA. If $f(z)$ is not a polynomial, then it has an essential singularity at infinity (WHY ?). That is $f(\frac{1}{z})$ has an essential singularity at 0. Thus by Big Picard theorem, in any punctured neighborhood of 0, say of radius r, $f(\frac{1}{z})$ misses atmost one point. But this implies that in the complement of the circle of radius $1/r$, $f(z)$ misses atmost one point. This is what we wanted.

In combination with an earlier theorem discussed in the course, we now have the following theorem: A non-constant entire function is either surjective or misses one point in which case it is of the form $e^{f(z)} + c$ for some holomorphic function $g(z)$ and some constant c.

Liouville (1809-1882) & Picard (1856-1941); Wiki



Joseph Liouville

Émile Picard

Picard was a top rate mathematician who did fundamental work in many disciples; analysis, function theory, differential equations, and analytic geometry to name a few. In physics he worked on elasticity, heat and electricity. Hadamard wrote about his teacher Picard:- A striking feature of Picard's scientific personality was the perfection of his teaching, one of the most marvellous, if not the most marvellous, that I have ever known.

It is a remarkable fact that between 1894 and 1937 he trained over 10000 engineers who were studying at the cole

Centrale des Arts et Manufactures.