Conformal Mappings

September 9, 2017

Motivation

Today we look at conformal mappings. Roughly speaking a conformal map between two subsets U and V of $\mathbb{R}^n$ is a differentiable mapping that preserves magnitude and orientation of angles between directed curves. A more general class of mappings which only preserve magnitude of angles between directed curves but not neccessarily their orientation are called isogonal mappings. We will focus our attention on studying conformal mappings between open subsets of $\mathbb{C}$. Conformal mappings also happen to be of great importance in Physics and Engineering; an aspect I am not well familar with.

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and hence $\arg\, w'(t_0) = \arg\, f'[z(t_0)] + \arg\, z'(t_0)$

Preservation of Angles

Thus if the directed tangent to C at z_0 makes an angle θ with the x-axis, then γ makes an angle $\theta + \arg f'(z_0)$ with the x-axis. Consequently if C_1 and C_2 are two smooth, parametrized (hence also directed) curves passing through z_0 and intersecting at an angle ϕ at z_0 (meaning their tangents at z_0 make an angle ϕ), then their images also make an angle ϕ at $f(z_0)$.

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Because of this angle preserving preoperty, a holomorphic function $w = f(z)$ is said to be conformal at z_0 if $f'(z_0)$ is non-zero.

Examples

The mapping $w = e^z$ is conformal at all points since the derivative is everwhere non-vanishing. Consider two lines $C_1 = \{x = c_1\}$ and $C_2 = \{y = c_2\}$ in the domain; the first one directed upwards and the second one directed to the right.

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Non-Examples

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The mapping $z \to z^2$ is not conformal at 0 and does not preserve angles; the images of the real and imaginary axis are the real axis and the real axis with opposite orientation resp. Thus the angle between curves through 0 gets doubled. This is true for any two smooth curves passing through 0. This is a special case of the following more general fact:

If z_0 is a point at which first $m-1$ derivatives vanish, then then angle between two smooth curves passing through z_0 gets multiplied by m.

Inverse Function Theorem (Special Case)

Let us understand holomorphic mappings in a neighborhood of a conformal point. Recall the basic fact from calculus: If $f \in C^1(\mathbb{R})$ and $f'(x_0) \neq 0$, then in a neighborhood of x_0, $f(x)$ is either strictly increasing or strictly decreasing. In particular it is injective in a neighborhood of x_0.

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In this setting even the converse is true: If z_0 is a point such that $f'(z_0) = 0$ then in no neighborhood of z_0 is $f(z)$ injective.
For example note that while $x \to x^3$ is injective in a neighborhood of 0, $z \to z^3$ is not injective in any neighborhood.

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If such a mapping exists U and V are said to be biholomorphic.
Note in particular by the earlier remark that such an $f(z)$ is conformal at all points in U. An easy exercise show that the inverse mapping is automatically holomorphic.
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Thus a biholomorphism is a bijective map, holomorphic both ways. Clearly composite of biholomorphisms is a biholomorphism and the inverse of a biholomorphism is a biholomorphism. In view of the special case of the inverse function theorem stated earlier, a holomorphic map which is conformal at a point z_0 is a biholomorphism in a neighborhood of z_0 (what's called a local biholomorphism).

Motivation for this notion

The motivation for this definition is simple: If two open subsets are biholomorphic, then (loosely speaking) studying complex analysis on one of them is equivalent to studying it on the other. For example if $f : U \to V$ is a biholomorphism, then one of them is path-connected (resp. simply connected) if and only if the other too is path-connected (resp. simply-connected). More generally if one of them is a simply connected subset minus n points, so is the other.

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Examples

1. The identity map $f : U \to U$ is clearly a biholomorphism. More generally, multiplication by a non-zero scalar defines a biholomorphism from $\mathbb{C}$ to $\mathbb{C}$. Similarly open discs of any two radii are biholomorphic.

2. The mapping from the open unit disc to the upper half plane $\mathbb{D} \to \mathbb{H}$ given by $z \to i\frac{1-z}{1+z}$ is a biholomorphism as one can check.

If $U \subseteq \mathbb{C}$ is open, then a biholomorphism from U to U is called an automorphism.
3. A basic fact is that the only automorphisms of $\mathbb{C}$ are of the form $az + b$ with $a \neq 0$. This is because biholomorphisms can be easily seen to be proper maps and hence polynomial. But the only injective polynomial functions are linear polynomials with non-zero linear coefficient !

Examples

4. As a consequence of Schwartz lemma, one can show that the automorphisms of the unit disc are

$$z \to \lambda \frac{z - a}{\bar{a}z - 1}$$

where $|\lambda| = 1$ and $|a| < 1$.

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Given the rigid nature of holomorphic functions, the theorem is hugely surprising and beautiful. This theorem was conjecture by Riemann in 1851 in his thesis. He gave an incomplete proof based on Dirichlet principle stated roughly as : Minimizer of a certain energy functional is a solution to Poisson's equation.
Weierstrass found an error in the proof. The first complete proof was due to Constantin Carathodory in 1922 and simplified by Paul Koebe 2 years later. Here is a link to the history of the Riemann mapping theorem.
https://www.math.stonybrook.edu/ bishop/classes/math401.F09/GrayRMT.pdf

Uniformization Theorem

The Riemann mapping theorem is a special case of a even deeper and more powerful theorem called the Uniformization theorem. The precise statement involves notions that we don't have time to go into in detail, but here is the idea.
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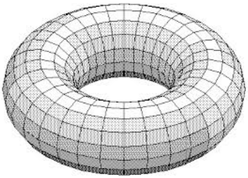
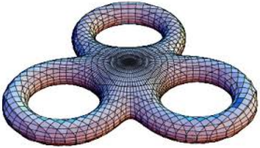
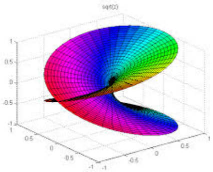
Examples of Riemann Surfaces







Surface for root z.png

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One can deduce the Riemann Mapping Theorem from the Uniformization Theorem. The first rigorous proofs of the uniformization theorem were given by Poincare and Paul Koebe in 1907. The theorem has many important applications. For example, one of the proofs of the Little Picard's Theorem follows from the uniformization theorem.

LET'S CELEBRATE. CHEERS !