1. The number of roots of $P_{101}(x)$ lying in the open interval $(0,1)$ equals

   (A) 49 (B) 50 (C) 51 (D) 52

   SOLUTION. There are 50 roots in $(0,1)$, 50 in $(-1,0)$ and one root at 0.

2. For $x>0$, the equation $x^2y''-x(1+x)y'+y=0$ has a solution $xe^x\log x+\sum_{n=1}^{\infty} b_nH_nx^{n+1}$ with b_n equal to

   (A) $\frac{-1}{(n-1)!}$ (B) $\frac{-1}{n!}$ (C) $\frac{2^n}{n!}$ (D) $\frac{1}{(n-1)!}$

   SOLUTION. The indicial equation is $r^2-2r+1=(r-1)^2=0$ which has a repeated root. The recursion is $(r+n-1)^2a_n=(r+n-1)a_{n-1}$. Solving with $a_0=1$, we obtain

   $$a_n(r)=\frac{1}{(r+n-1)\dots(r+1)r}$$

   Note $a_n(1)=\frac{1}{n!}$ and $a_n'(1)=\frac{-H_n}{n!}$. So the first solution is xe^x, and the second solution is $xe^x\log x-\sum_{n=1}^{\infty}\frac{H_n}{n!}x^{n+1}$. Thus $b_n=\frac{-1}{n!}$.

3. The domain of analyticity of a real-valued function on $\mathbb{R}$ can be

   (A) $\{0\}$ (B) $\bigcup_{n=1}^{\infty}\{1/n\}$ (C) $[0,1]$ (D) $(-1,1)\setminus\{0\}$

   SOLUTION. The domain of analyticity is always an open set. Only $(-1,1)\setminus\{0\}$ is open. It can be realized by the function which is $1/x$ between -1 and 1 and discontinous everywhere outside.

4. A pair (a,b) of real numbers is said to be good if there exists a real number p such that $aJ_p(x)+bJ_{-p}(x)=0$ for all $x>0$. The set of all good pairs is defined by

   (A) $a^2-b^2=0$ (B) $a=b=0$ (C) $a-b=0$ (D) $a+b=0$

   SOLUTION. We can determine which pairs are good for a particular p. For p not an integer, only $a=b=0$ is possible. When p is an integer either $a=b$ or $a=-b$ is possible. So good pairs are defined by $a^2-b^2=0$.

5. If $x^{50}+x^{49}=\sum_{n=0}^{50}c_nP_n(x)$, then the sum of even coefficients $c_0+c_2+c_4+c_6+\cdots+c_{50}$ equals

   (A) 0 (B) 1 (C) 50/99 (D) 51/101

   SOLUTION. Put $x=1$ and $x=-1$ respectively, add and divide by 2 to get 1. Alternatively, we can separate odd and even parts to get $x^{50}=\sum_{n=0}^{25}c_{2n}P_{2n}(x)$, and then put $x=1$.

6. The equation $x(e^x-1)y''+(\sin x)y'+y=0$ has a

   (A) irregular singular point at $x=0$ (B) irregular singular point at $x=1$
   (C) regular singular point at $x=0$ (D) regular singular point at $x=1$

   SOLUTION. At $x=0$, $e^x-1=0$. Hence singular. All other points are ordinary. $\lim_{x\to 0}\frac{\sin x}{e^x-1}$ exists and is equal to 1. Also, $\lim_{x\to 0}\frac{x^2}{(e^x-1)x}$ exist and is equal to 1. Hence it is a regular singular point.

7. In the interval $(-1, 217)$, the equation $(1+x)y' = -y/2$ with $y(0) = 1$ has a power series solution $\sum_{n\geq 0} a_n(x-108)^n$ with the value of $a_{207}(109)^{207}$ equal to

(A) $a_0 P_{414}(0)$ (B) $a_0 P_{414}(108)$ (C) $a_0 P_{207}(0)$ (D) $a_0 J_{207}(108)$

SOLUTION. By the power series method, we get the recursion

$$a_{n+1} = \frac{-a_n(n+1/2)}{109(n+1)}.$$

for $n \geq 0$. This yields

$$a_n = \frac{(-1)^n 1/2 \cdot 3/2 \dots (1/2 + (n-1))a_0}{109^n n!} = (109)^{-n}\binom{-1/2}{n} a_0 = (109)^{-n} P_{2n}(0) a_0.$$

Now put $n = 207$. Alternatively: Observe that the solution is $(1+x)^{-1/2}$ and this is valid for $x > -1$. Using binomial theorem,

$$(1+x)^{-1/2} = 109^{-1/2}(1 + \frac{x-108}{109})^{-1/2} = 109^{-1/2}\sum_{n\geq 0}\binom{-1/2}{n}(109)^{-n}(x-108)^n.$$

The coefficient of $(x-108)^{207}$ is $109^{-1/2}\binom{-1/2}{207}(109)^{-207}$. Now use $\binom{-1/2}{207} = P_{414}(0)$ and $a_0 = \frac{1}{\sqrt{109}}$.

8. The value of ${J_0}^2(2) - {J_2}^2(2)$ equals

(A) 0 (B) $J_0(2)J_2'(2)$ (C) $J_1(2)J_1'(2)$ (D) $2J_1(2)J_1'(2)$

SOLUTION. Using identities of Bessel functions, one can find that $J_0(2) + J_2(2) = J_1(2)$ and $J_0(2) - J_2(2) = 2J_1'(2)$. So, the given expression evaluates to $2J_1(2)J_1'(2)$.

9. The radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{(2n)!}{3^{2n}(n!)^2} x^{2n}$ equals

(A) 3 (B) 9 (C) 3/2 (D) 9/4

SOLUTION. Applying ratio test, we get $\frac{a_{n+1}}{a_n} = \frac{(2n+2)(2n+1)}{9(n+1)^2}x^2$ whose limit is $4/9x^2$. This is strictly less than 1 when $|x| < 3/2$.

10. Let $g(x)$ be the quadratic polynomial with roots $\pm\sqrt{\frac{1}{3}}$ with $g(1) = 2/3$. Let $f(x)$ be the polynomial solution of the equation $((1-x^2)y')' + 6y = 0$ with $f(1) = 1$. The value of $\int_{-1}^{1} f(x)g(x)dx$ equals

(A) 0 (B) 2/3 (C) 2/5 (D) 4/15

SOLUTION. $f(x) = P_2(x)$ and $g(x) = 2/3P_2(x)$. Using $\int_{-1}^{1} P_2(x)^2 dx = 2/5$, the required value is $4/15$.

11. The recursion obtained while solving $y'' - xy' + y = 0$ by the power series method is

(A) $(n+2)(n+1)a_{n+2} = (n-1)a_n$ (B) $(n+2)(n+1)a_{n+2} = na_n$
(C) $(n+2)(n+1)a_{n+2} = (n-1)a_{n-1}$ (D) $(n+2)(n+1)a_{n+2} = (n+1)a_{n+1} - a_n$

SOLUTION. The recursion is $(n+2)(n+1)a_{n+2} = (n-1)a_n$.

12. Let a and b be the number of solutions of $J_0(x) = P_0(x)$ and $J_1(x) = P_1(x)$ respectively in the interval $[0, 1]$. Then (a, b) is

(A) $(0, 1)$ (B) $(0, 2)$ (C) $(1, 1)$ (D) $(1, 2)$

SOLUTION. (a, b) is $(1, 1)$. The function $J_0(x)$ is 1 at 0 and strictly less than 1 in $[0, 1]$. This follows from the power series of $J_0(x)$ (or also from the graph). Thus $x = 0$ is the only solution of $J_0(x) = P_0(x)$ in $[0, 1]$. Similarly, $J_1(x)$ is 0 at 0 and strictly less than $x/2$ in $[0, 1]$. Thus $x = 0$ is the only solution of $J_1(x) = P_1(x)$ in $[0, 1]$.

13. An inner product on $\mathbb{R}^2$ can be defined by setting $\langle (a_1, a_2), (b_1, b_2) \rangle$ equal to

(A) $a_1b_1 - a_2b_2$ (B) $a_1^2b_1^2 + a_2^2b_2^2$ (C) $(a_1 + a_2)(b_1 + b_2)$ (D) $2a_1b_1 - a_1b_2 - a_2b_1 + 5a_2b_2$

SOLUTION. $a_1b_1 - a_2b_2$ and $(a_1 + a_2)(b_1 + b_2)$ are not positive definite, while $a_1^2b_1^2 + a_2^2b_2^2$ is not linear. This only leaves $2a_1b_1 - a_1b_2 - a_2b_1 + 5a_2b_2$ which can be checked to be an inner product.

14. The set of all points where the Taylor series of the function $f(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1 + x^2)^n}$ around the point $x = e$ converges to $f(x)$ is

(A) $\emptyset$ (B) $(0, 2e)$ (C) $\mathbb{R} \setminus \{0\}$ (D) $\mathbb{R}$

SOLUTION. $f(x) = 0$ if $x = 0$ and $f(x) = x^2 + 1$ otherwise (by summing the geometric series). At $x = 0$, $f(x)$ is discontinuous and thus not real analytic. The Taylor series of $f(x)$ around $x = e$ will be the quadratic $((x - e) + e)^2 + 1 = (x - e)^2 + 2e(x - e) + e^2 + 1$. It will converge to $x^2 + 1$. This equals $f(x)$ on $\mathbb{R} \setminus \{0\}$.

15. The value of $\lim_{x \to 1^+} \frac{J_p(x^2 - 1)}{(x - 1)^p}$ at $p = 4$ equals

(A) 0 (B) $1/24$ (C) $1/120$ (D) ∞

SOLUTION. Write the power series of $J_p(x^2 - 1)$ and all the terms apart from the constant term go to zero. The constant term is $\frac{(x^2-1)^p}{2^p p!}$. Dividing by $(x - 1)^p$ and putting $p = 4$ yields $1/24$.

16. While solving $x^2y'' + 2x(x - 2)y' + 2(2 - 3x)y = 0$ by the Frobenius method around the point $x = 0$, the case encountered is that of

(A) roots not differing by an integer
(B) repeated roots
(C) roots differing by a positive integer with **no** log term
(D) roots differing by a positive integer with log term

SOLUTION. The indicial equation is $(r - 1)(r - 4) = 0$, so roots differ by a positive integer. The recursion is $(r + n - 1)(r + n - 4)a_n = -2(n + r - 4)a_{n-1}$. Since the problematic factor $(r + n - 4)$ from the lhs (when $r = 1$ and $n = 3$) also appears in the rhs, it can be canceled, and we will not see any log term.