MA-207 Differential Equations II

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S2 - Lecture 5

Recall

Definition

Consider the second-order linear ODE

$$y'' + p(x)y' + q(x)y = 0 \qquad (*)$$

1. $x_0 \in \mathbb{R}$ is called an ordinary point of $(*)$ if $p(x)$ and $q(x)$ are analytic at x_0.
2. $x_0 \in \mathbb{R}$ is called regular singular point if $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are analytic at x_0.
3. If $x_0 \in \mathbb{R}$ is not ordinary or regular singular, then we call it irregular singular.

In the regular singular case we look for solution of the type

$$y(x) = \sum_{n \geq 0} a_n x^{n+r}$$

Recall

Consider the differential operator

$$L := x^2 \frac{d^2}{dx^2} + xb(x)\frac{d}{dx} + c(x)$$

Consider the function of two variables

$$\psi(r, x) := \sum_{n \geq 0} a_n(r) x^{n+r}$$

Then one checks easily that

$$L\psi(r, x) = \sum_{n \geq 0} E(n) x^{n+r}$$

where

$E(0) := I(r)a_0,$ and for $n \geq 1$

$$E(n) := I(n+r)a_n(r) + \sum_{i=0}^{n-1} (i+r) b_{n-i} a_i(r) + \sum_{i=0}^{n-1} c_{n-i} a_i(r)$$

Recall

$I(r)$ is the indicial equation, given by $r(r-1)+b_0r+c_0$. The roots are $r_1 \geq r_2$.

Setting $a_0(r)=1$ and $E(n)=0$ allows us to inductively define functions $a_n(r)$.

Note that each $a_n(r)$ is a rational function in r, in fact, the denominator of $a_n(r)$ is $\prod_{i=1}^{n} I(i+r)$.

The functions $a_n(r)$ are analytic at r_1. They are analytic at r_2 if $r_1 - r_2 \notin \mathbb{Z}$.

In particular, if we put $r = r_1$, then it gives a solution since

$$L\psi(r_1, x) = I(r_1)x^{r_1} = 0$$

Explicitly this solution is

$$y_1(x) = x^{r_1} \sum_{n \geq 0} a_n(r_1)x^n$$

If $r_1 - r_2 \notin \mathbb{Z}$ then the second solution is given by

$$y_2(x) = x^{r_2} \sum_{n \geq 0} a_n(r_2) x^n$$

Now let us consider the case when I has repeated roots

Since I has repeated roots $r_1 = r_2$, it follows that, for every $n \geq 1$, the polynomial $\prod_{i=1}^{n} I(i + r)$ does not vanish at $r = r_1$

Consequently, it is clear that the $a_n(r)$ are analytic in a small neighborhood around $r = r_1 = r_2$.

Second solution: $r_1 = r_2$

Now let us apply the differential operator $\frac{d}{dr}$ on both sides of the equation $L\psi(r,x) = I(r)x^r$. Clearly the operators L and $\frac{d}{dr}$ commute with each other, and so we get

$$\begin{aligned}
\frac{d}{dr}L\psi(r,x) &= L\frac{d}{dr}\psi(r,x) \\
&= L\sum_{n\geq 0}\left(a_n'(r)x^{n+r} + a_n(r)x^{n+r}\log x\right) = \frac{d}{dr}I(r)x^r \\
&= I'(r)x^r + I(r)x^r\log x
\end{aligned}$$

Thus, if we plug in $r = r_1 = r_2$ in the above, then we get

$$L\Big(\sum_{n\geq 0} a_n'(r_2)x^{n+r_2} + a_n(r_2)x^{n+r_2}\log x\Big) = 0$$

Second solution: $r_1 = r_2$

Theorem (Second solution: $r_1 = r_2$)

A second solution to the differential equation is given by

$$\sum_{n\geq 0} a_n'(r_2)x^{n+r_2} + \sum_{n\geq 0} a_n(r_2)x^{n+r_2}\log x$$

Second solution: $r_1 = r_2$

Example

Consider the ODE

$$x^2 y'' + 3xy' + (1-2x)y = 0$$

This has a regular singularity at $x = 0$.

$$\begin{aligned} I(r) &= r(r-1) + 3r + 1 \\ &= r^2 + 2r + 1 \end{aligned}$$

has a repeated roots $-1, -1$.

Let us find the Frobenius solution directly by putting

$$y = x^r \sum_{n\geq 0} a_n(r) x^n \qquad a_0 = 1$$

$$y' = \sum_{n\geq 0} (n+r) a_n(r) x^{n+r-1}$$

$$y'' = \sum_{n\geq 0}^{\infty} (n+r)(n+r-1) a_n(r) x^{n+r-2}$$

Second solution: $r_1 = r_2$

Example (continues ...)

$$
\begin{aligned}
&x^2 y(x,r)'' + 3xy(x,r)' + (1-2x)y(x,r) \\
&= \sum_{n=0}^{\infty} \left[(n+r)(n+r-1) + 3(n+r) + 1\right] a_n(r) x^{n+r} \\
&\quad - \sum_{n=0}^{\infty} 2a_n(r) x^{n+r+1}
\end{aligned}
$$

Recursion relations for $n \geq 1$ are

$$
\begin{aligned}
a_n(r) &= \frac{2a_{n-1}(r)}{(n+r)(n+r-1) + 3(n+r) + 1} \\
&= \frac{2a_{n-1}(r)}{(n+r+1)^2} \\
&= \frac{2^n}{[(n+r+1)(n+r)\ldots(r+2)]^2}\, a_0
\end{aligned}
$$

Example (continues ...)

Setting $r = -1$ (and $a_0 = 1$) yields the fractional power series solution

$$y_1(x) = \frac{1}{x} \sum_{n \geq 0} \frac{2^n}{(n!)^2} \, x^n$$

The power series converges on $(0, \infty)$.

The second solution is

$$y_2(x) = y_1(x) \log x + x^{-1} \sum_{n \geq 1} a_n'(-1) x^n$$

where

$$a_n(r) = \frac{2^n}{[(n+r+1)(n+r)\dots(r+2)]^2}$$

$$a_n'(r) = \frac{-2.2^n \, [(n+r+1)(n+r)\dots(r+2)]'}{[(n+r+1)(n+r)\dots(r+2)]^3}$$

Second solution: $r_1 = r_2$

Example (continued)

$$= -2a_n(r)\left(\frac{1}{n+r+1} + \frac{1}{n+r} + \cdots + \frac{1}{r+2}\right)$$

Putting $r = -1$, we get

$$a_n'(-1) = -\frac{2^{n+1}H_n}{(n!)^2}$$

where

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

(These are the partial sums of the harmonic series.)

So the second solution is

$$y_2(x) = y_1(x)\log x - \frac{1}{x}\sum_{n\geq 1} \frac{2^{n+1}H_n}{(n!)^2}\, x^n$$

It is clear that this series converges on $(0, \infty)$.

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Define

$$N := r_1 - r_2$$

Note that each $a_n(r)$ is a rational function in r, in fact, the denominator is exactly $\prod_{i=1}^{n} I(i+r)$.

The polynomial $\prod_{i=1}^{n} I(i+r)$ evaluated at r_2 vanishes iff $n \geq N$. For $n \geq N$ it vanishes to order exactly 1.

Thus, if we define

$$A_n(r) := a_n(r)(r - r_2)$$

then it is clear that for every $n \geq 0$, the function $A_n(r)$ is analytic in a neighborhood of r_2.

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

In particular, $A_n(r_2)$ and $A_n'(r_2)$ are well defined real numbers.

Multiplying the equation $L\psi(r,x) = I(r)x^r$ with $r - r_2$ we get

$$(r - r_2)L\psi(r,x) = L(r - r_2)\psi(r,x) = (r - r_2)I(r)x^r$$

Note that

$$(r - r_2)\psi(r,x) = \sum_{n\geq 0} A_n(r)x^{n+r}$$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

Now let us apply the differential operator $\frac{d}{dr}$ on both sides of the equation $L(r - r_2)\psi(r, x) = (r - r_2)I(r)x^r$ to get

$$\begin{aligned}\frac{d}{dr}L(r - r_2)\psi(r, x) &= L\frac{d}{dr}(r - r_2)\psi(r, x)\\ &= \frac{d}{dr}(r - r_2)I(r)x^r\\ &= I(r)x^r + (r - r_2)I'(r)x^r + (r - r_2)I(r)x^r \log x\end{aligned}$$

Thus we get

$$\begin{aligned}L\frac{d}{dr}\Big(\sum_{n\geq 0} A_n(r)x^{n+r}\Big) &= L\frac{d}{dr}\Big(\sum_{n\geq 0} A_n(r)x^{n+r}\Big)\\ &= L\Big(\sum_{n\geq 0} A_n'(r)x^{n+r} + A_n(r)x^{n+r}\log x\Big)\\ &= I(r)x^r + (r - r_2)I'(r)x^r +\\ &\qquad (r - r_2)I(r)x^r \log x\end{aligned}$$

Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$

If we set $r = r_2$ into the equation

$$L\Big(\sum_{n\geq 0} A_n'(r)x^{n+r} + A_n(r)x^{n+r}\log x\Big) = I(r)x^r + (r - r_2)I'(r)x^r + (r - r_2)I(r)x^r \log x$$

we get the second solution

$$L\Big(\sum_{n\geq 0} A_n'(r_2)x^{n+r_2} + A_n(r_2)x^{n+r_2}\log x\Big) = 0$$

Theorem (Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$)

A second solution to the differential equation is given by

$$\sum_{n\geq 0} A_n'(r_2)x^{n+r_2} + \sum_{n\geq 0} A_n(r_2)x^{n+r_2}\log x$$

Example

Consider the ODE $$xy'' - (4+x)y' + 2y = 0 \qquad (*)$$

Multiplying $(*)$ with x, we get $x = 0$ is a regular singular point.

$$I(r) = r(r-1) - 4r + 0 = r(r-5) = 0$$

with the roots differing by a positive integer.

Put $y(x,r) = x^r \sum_{n=0}^{\infty} a_n(r) x^n$, $a_0(r) = 1$, into the ODE to get

$$x \sum_{n\geq 0} (n+r)(n+r-1)a_n(r)x^{n+r-2}$$
$$-(4+x)\sum_{n\geq 0}(n+r)a_n(r)x^{n+r-1} + 2\sum_{n\geq 0} a_n(r)x^{n+r} = 0$$

the coefficient of x^{n+r-1} for $n \geq 1$ gives

Example (continues ...)

$$(n+r)(n+r-1)a_n(r) - 4(n+r)a_n(r) - (n+r-1)a_{n-1}(r) \\ +2a_{n-1}(r) = 0$$

For $n \geq 1$,

$$(n+r)(n+r-5)a_n = (n+r-3)a_{n-1}$$

$$a_n(r) = \frac{(n+r-3)}{(n+r)(n+r-5)}\, a_{n-1}$$

$$= \frac{(n+r-3)\dots(r-2)}{(n+r)\dots(1+r)(n+r-5)\dots(r-4)}\, a_0$$

For the first solution, set $r = r_1 = 5$ (and $a_0 = 1$), we get

$$a_n(5) = \frac{(n+2)\dots(3)}{(n+5)\dots 6(n)\dots 1}$$

$$= \frac{(n+2)!/2}{(n!)(n+5)!/5!}$$

Example (continues ...)

$$= \frac{60}{n!(n+5)(n+4)(n+3)}$$

Thus

$$y_1(x) = \sum_{n\geq 0} \frac{60}{n!(n+5)(n+4)(n+3)}\, x^{n+5}$$

Recall $N = r_1 - r_2 = 5 - 0$ is integer, so the second solution is

$$y_2(x) = \sum_{n\geq 0} A'_n(r_2)x^{n+r_2} + \sum_{n\geq 0} A_n(r_2)x^{n+r_2}\log x$$

where, for $n \geq 0$

$$A_n(r) = (r - r_2)a_n(r)$$

Since $r_2 = 0$, the above becomes

$$A_n(r) = ra_n(r)$$

Example

In this example, we can easily check that none of the $a_n(r)$ have a singularity at $r = 0$.

Thus, $A_n(0) = 0$ for all $n \geq 0$; and $A'_n(0) = a_n(0)$ for all $n \geq 0$.

$a_1(0) = \frac{1}{2}$; $a_2(0) = \frac{1}{12}$;
It is easily checked that for $n \geq 3$

$$a_n(r) = \frac{(n+r-3)(n+r-4)}{n!12}$$

Thus, $a_3(0) = a_4(0) = 0$.

Example

Therefore a second solution is

$$y_2(x) = 1 + \frac{x}{2} + \frac{x^2}{12} + \sum_{n\geq 5} \frac{(n-3)(n-4)}{n!12} x^n$$

$$= 1 + \frac{x}{2} + \frac{x^2}{12} + \sum_{k\geq 0} \frac{1}{k!(k+5)(k+4)(k+3)12} x^{k+5}$$

Since

$$\sum_{k\geq 0} \frac{1}{k!(k+5)(k+4)(k+3)12} x^{k+5}$$

is a multiple of $y_1(x)$,
we get that a second solution is

$$y_2(x) = 1 + \frac{x}{2} + \frac{x^2}{12}.$$

Summary

While solving an ODE around a regular singular point by the Frobenius method, the cases encountered are

- roots not differing by an integer
- repeated roots
- roots differing by a positive integer

The larger root always yields a fractional power series solution.

In the first case, the smaller root also yields a fractional power series solution.

In the second and third cases, the second solution may involve a log term.

Let us write down some classical ODE's.

- (Euler equation) $\alpha x^2 y'' + \beta x y' + \gamma y = 0$
- (Bessel equation) $x^2 y'' + x y' + (x^2 - \nu^2) y = 0$. We will next look at this case more closely.
- (Laguerre equation) $x y'' + (1 - x) y' + \lambda y = 0$

Gamma function

Define for all $p \geq 1$, the Gamma function

$$\Gamma(p) := \int_0^\infty t^{p-1} e^{-t} dt$$

There is a problem if $p < 1$, since t^{p-1} is unbounded near 0.
For $p > 1$, there is no problem because e^{-t} is rapidly decreasing.

$$\Gamma(1) = \int_0^\infty e^{-t} dt = 1$$

For any real number $p \geq 1$,

$$\Gamma(p+1) = \lim_{x\to\infty} \int_0^x t^p e^{-t} dt = p \left(\lim_{x\to\infty} \int_0^x t^{p-1} e^{-t} dt \right) = p\Gamma(p)$$

$$\Gamma(p+1) = p\,\Gamma(p) \implies \Gamma(p) = \frac{\Gamma(p+1)}{p} \qquad (*)$$

We use $(*)$ to extend the gamma function to all real numbers except non-positive integers $0, -1, -2, \ldots$

Note $0 < p < 1 \implies 1 < p+1 < 2$, hence $\Gamma(p+1)$ is defined. We use $(*)$ to define $\Gamma(p)$.

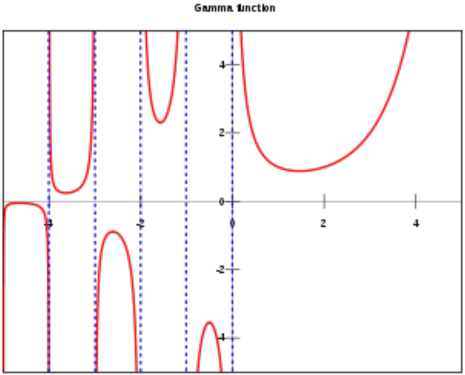
Next, $-1 < p < 0 \implies 0 < p+1 < 1$. Since $\Gamma(p+1)$ is defined above; use $(*)$ to define $\Gamma(p)$. Proceed like this

For example, $\Gamma(-\dfrac{5}{2}) = \dfrac{\Gamma(-\frac{3}{2})}{-\frac{5}{2}} = \dfrac{\Gamma(-\frac{1}{2})}{(-\frac{5}{2})(-\frac{3}{2})} = \dfrac{\Gamma(\frac{1}{2})(=\sqrt{\pi})}{(-\frac{5}{2})(-\frac{3}{2})(-\frac{1}{2})}$

Further

$$\lim_{p\to 0} \Gamma(p) = \lim_{p\to 0} \frac{\Gamma(p+1)}{p} = \pm\infty$$

according as $p \to 0$ from right or left.
The graph of Gamma function is shown below.



Though the gamma function is now defined for all real numbers (except the non positive integers),
the integral representation is valid only for $p > 0$.

It is useful to rewrite

$$\frac{1}{\Gamma(p)} = \frac{p}{\Gamma(p+1)}$$

This holds for all p if we impose the natural condition that the reciprocal of Γ evaluated at a non positive integer is 0.

$$\begin{aligned}\Gamma(1/2) &= \int_0^\infty t^{-1/2} e^{-t} dt \\ &= 2\int_0^\infty e^{-s^2} ds \qquad \text{(use the substitution } t = s^2\text{)} \\ &= \sqrt{\pi}\end{aligned}$$

By translating,

$$\begin{aligned}
\Gamma(1/2) & & &= \sqrt{\pi} &&\approx 1.772\\
\Gamma(-1/2) &= \frac{\Gamma(1/2)}{-1/2} & &= -2\sqrt{\pi} &&\approx -3.545\\
\Gamma(-3/2) &= \frac{\Gamma(-1/2)}{-3/2} & &= \frac{4}{3}\sqrt{\pi} &&\approx 2.363\\
\Gamma(3/2) &= \frac{1}{2}\Gamma(1/2) & &= \frac{1}{2}\sqrt{\pi} &&\approx 0.886\\
\Gamma(5/2) &= \frac{3}{2}\Gamma(3/2) & &= \frac{3}{4}\sqrt{\pi} &&\approx 1.329\\
\Gamma(7/2) &= \frac{5}{2}\Gamma(5/2) & &= \frac{15}{8}\sqrt{\pi} &&\approx 3.323
\end{aligned}$$

Bessel equation is the second-order linear ODE

$$x^2y'' + xy' + (x^2 - p^2)y = 0 \qquad p \geq 0 \qquad\qquad (*)$$

Its solutions are called Bessel functions.

Bessel functions have applications in physics and engineering:

Since $x = 0$ is a regular singular point of $(*)$, we get a Frobenius solution, called Bessel function of first kind.

The second linearly independent solution of $(*)$ is called Bessel function of second kind.

For Frobenius solution, put $y = x^r \sum_{n=0}^{\infty} a_n(r)x^n \quad a_0 = 1.$

The indicial equation, i.e. coefficient of x^r, for Bessel equation $x^2y'' + xy' + (x^2 - p^2)y = 0$ is

$$I(r) = r(r-1) + r - p^2 = r^2 - p^2 = 0$$

The roots are $r_1 = p$ and $r_2 = -p$.
For recurrence relations, equating coefficient of x^{n+r} to 0 (for $n \geq 1$) we get

$$[(r+n)^2 - p^2]a_n(r) + a_{n-2}(r) = 0 \qquad n \geq 2$$
$$((r+1)^2 - p^2)a_1(r) = 0 \implies a_1(r) = 0$$

So all odd terms $a_{2n+1}(r) = 0$.

$$a_{2n}(r) = \frac{-1}{(r+2n)^2 - p^2}\, a_{2n-2}$$
$$= \frac{(-1)^n}{((r+2)^2 - p^2)((r+4)^2 - p^2)\ldots((r+2n)^2 - p^2)}$$

For Frobenius solution, set $r = p$ the larger root.

$$a_{2n}(p) = \frac{(-1)^n}{((p+2)^2 - p^2)((p+4)^2 - p^2)\dots((p+2n)^2 - p^2)}$$

$$= \frac{(-1)^n}{(2(2p+2))(4(2p+4))\dots(2n(2p+2n))}$$

$$= \frac{(-1)^n}{2^{2n} n!(1+p)\dots(n+p)}$$

The solution $\qquad y_1(x) = x^p \sum_{n\geq 0} \frac{(-1)^n}{2^{2n} n!(1+p)\dots(n+p)} x^{2n}$

converges on $(0, \infty)$.

Multiply $y_1(x)$ by $\dfrac{1}{2^p \Gamma(1+p)}$

$$J_p(x) := \left(\frac{x}{2}\right)^p \sum_{n\geq 0} \frac{(-1)^n}{n!\,\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n} \qquad x > 0.$$

This is called the Bessel function of first kind of order p.

$$J_p(x) := \sum_{n\geq 0} \frac{(-1)^n}{n!\,\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p} \qquad x > 0.$$

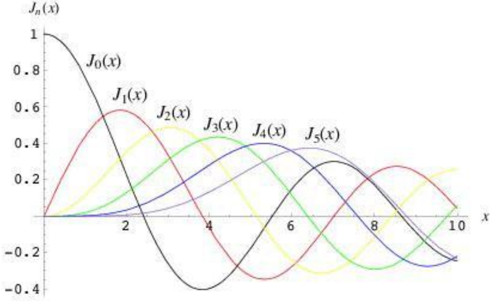
The Bessel function of order 0 is

$$\begin{aligned} J_0(x) &= \sum_{n\geq 0} \frac{(-1)^n}{n!n!} \left(\frac{x}{2}\right)^{2n} \\ &= 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{2!2!}\left(\frac{x}{2}\right)^4 - \frac{1}{3!3!}\left(\frac{x}{2}\right)^6 + \dots \end{aligned}$$

The Bessel function of order 1 is

$$\begin{aligned} J_1(x) &= \sum_{n\geq 0} \frac{(-1)^n}{n!(n+1)!} \left(\frac{x}{2}\right)^{2n+1} \\ &= \frac{x}{2} - \frac{1}{1!2!}\left(\frac{x}{2}\right)^3 + \frac{1}{2!3!}\left(\frac{x}{2}\right)^5 + \dots \end{aligned}$$

Both $J_0(x)$ and $J_1(x)$ have a damped oscillatory behavior having an infinite number of zeros, these zeros occur alternately like functions $\cos x$ and $\sin x$.



Further, they satisfy derivative identities similar to $\cos x$ and $\sin x$.

$$J_0'(x) = -J_1(x) \qquad\qquad [xJ_1(x)]' = xJ_0(x)$$

Second independent solution of Bessel equation

Recall $r_1 = p$ and $r_2 = -p$ are roots of indicial equation.
So that $r_1 - r_2 = 2p$.
The analysis to get a second independent solution of the Bessel equation splits into the following cases

- $2p$ is not an integer
- $2p$ is an odd positive integer
- $2p$ is an even positive integer
- $p = 0$

Second independent solution of Bessel equation

Case 1: $2p$ is not an integer.

Solving the recursion

$$[(r+n)^2 - p^2]a_n(r) + a_{n-2}(r) = 0 \quad n \geq 2 \qquad a_1(r) = 0.$$

for $r = -p$, we obtain

$$y_2(x) = x^{-p} \sum_{n \geq 0} \frac{(-1)^n}{2^{2n} n! (1-p) \dots (n-p)} x^{2n}$$

Multiplying by $\dfrac{1}{2^{-p}\Gamma(1-p)}$

$$J_{-p}(x) := \left(\frac{x}{2}\right)^{-p} \sum_{n \geq 0} \frac{(-1)^n}{n!\, \Gamma(n-p+1)} \left(\frac{x}{2}\right)^{2n} \qquad x > 0.$$

This is a second solution of the Bessel equation linearly independent of $J_p(x)$.
It is unbounded near $x = 0$.

Case 2: $2p$ is a positive integer.

Recall that the second solution is given by

$$y_2(x) = \sum_{n \geq 0} A_n'(-p)x^{n-p} + \sum_{n \geq 0} A_n(-p)x^{n-p} \log x$$

where

$$A_n(r) := (r+p)a_n(r)$$

Case 2(a): $2p$ is an odd positive integer, that is, $p = \frac{2l+1}{2}$ for some $l > 0$

We have seen that $A_{2n+1}(r) = (r+p)a_{2n+1}(r) = 0$

$$a_{2n}(r) = \frac{(-1)^n}{\prod_{i=1}^{n}((r+2i)^2 - p^2)}$$

Second independent solution of Bessel equation

Since the polynomial $\prod_{i=1}^{n}((r+2i)^2-p^2)$ evaluated at $r=-p$, is $\prod_{i=1}^{n} 4i(i-p)\neq 0$,

the function $a_{2n}(r)$ is analytic in a neighborhood of $-p$.

Thus, $A_{2n}(-p)=0$ and $A'_{2n}(-p)=a_{2n}(-p)$.
Thus, in this case we obtain that the second solution is

$$y_2(x)=\sum_{n\geq 0}\frac{(-1)^n}{2^{2n}n!(1-p)\dots(n-p)}\,x^{2n-p}$$

Multiplying by $\dfrac{1}{2^{-p}\Gamma(1-p)}$

$$J_{-p}(x):=\left(\frac{x}{2}\right)^{-p}\sum_{n\geq 0}\frac{(-1)^n}{n!\,\Gamma(n-p+1)}\left(\frac{x}{2}\right)^{2n}\qquad x>0.$$

Case 2(b): $2p$ is an even positive integer, that is, p is a positive integer.

As before, $A_{2n+1}(r) = 0$.
The polynomial $\prod_{i=1}^{n}((r+2i)^2 - p^2)$ evaluated at $r = -p$, is $\prod_{i=1}^{n} 4i(i-p)$,
Thus, if $n < p$, then $a_{2n}(r)$ is analytic in a neighborhood of $-p$.
Thus, if $n < p$, then $A_{2n}(-p) = 0$ and

$$A'_{2n}(-p) = a_{2n}(-p) = \frac{(-1)^n}{2^{2n} n! (1-p) \dots (n-p)} = \frac{1}{2^{2n} n! (p-n)!}$$

If $n \geq p$, then

$$\begin{aligned} A_{2n}(-p) &= \frac{2(-1)^n}{2^{2n} n! (1-p) \dots (-1) \cdot 1 \cdot 2 \cdots (n-p)} \\ &= \frac{-2(-1)^{n-p}}{2^{2n} n! (p-1)! (n-p)!} \end{aligned}$$

Define

$$f(r) := \Big(\prod_{i=1}^{p-1} ((r+2i)^2 - p^2) \Big) (r+3p) \Big(\prod_{i=p+1}^{n} ((r+2i)^2 - p^2) \Big) \quad (*)$$

Then

$$A_{2n}(r)f(r) = (-1)^n$$

Differentiating the above and setting $r = -p$ we get

$$A'_{2n}(-p)f(-p) + A_{2n}(-p)f'(-p) = 0$$

Taking log and differentiating $(*)$ we get

$$f'(-p) = f(-p)\Big(\frac{1}{2p} + \sum_{i\in\{1,2,\ldots,n\}\setminus p} \frac{1}{2i} + \frac{1}{2(i-p)}\Big)$$

$$= f(-p)\Big(\frac{H_n}{2} - \frac{H_{p-1}}{2} + \frac{H_{n-p}}{2}\Big),$$

where

$$H_0 = 0, \qquad H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

Thus,

$$A'_{2n}(-p) = -A_{2n}(-p)\Big(\frac{H_n}{2} - \frac{H_{p-1}}{2} + \frac{H_{n-p}}{2}\Big)$$
$$= \frac{2(-1)^{n-p}}{2^{2n}n!(p-1)!(n-p)!}\Big(\frac{H_n}{2} - \frac{H_{p-1}}{2} + \frac{H_{n-p}}{2}\Big)$$

Thus, we get

$$y_2(x) = \sum_{n=0}^{p-1} \frac{1}{2^{2n}n!(p-n)!}x^{2n-p} +$$
$$\sum_{n\geq p} \frac{(-1)^{n-p}}{2^{2n}n!(p-1)!(n-p)!}\Big(H_n - H_{p-1} + H_{n-p}\Big)x^{2n-p} +$$
$$-\sum_{n\geq p} \frac{2(-1)^{n-p}}{2^{2n}n!(p-1)!(n-p)!}x^{2n-p}\log x$$

is a second solution.

Case 3: $p = 0$ (Repeated root case)
The indicial equation has a repeated root $r_1 = r_2 = 0$,

$$a_{2n}(r) = \frac{(-1)^n}{(r+2)^2(r+4)^2\dots(r+2n)^2} \qquad a_{2n+1}(r) = 0$$

Differentiating $a_{2n}(r)$ with respect to r, we get

$$a_{2n}(r)' = -2a_{2n}(r)\left(\frac{1}{r+2} + \frac{1}{r+4} + \cdots + \frac{1}{r+2n}\right)$$

$$a'_{2n}(0) = \frac{(-1)^{n-1}H_n}{2^{2n}(n!)^2}, \quad H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

By theorem stated earlier, the second solution is

$$y_2(x) = J_0(x)\ln x - \sum_{n\geq 1}\frac{(-1)^n H_n}{2^{2n}(n!)^2}x^{2n} \qquad x > 0$$

where $y_1(x) = J_0(x) = \sum_{n\geq 0}\frac{(-1)^n}{2^{2n}(n!)^2}x^{2n}$ is Frobenius solution.