MA 205 Complex Analysis: Cauchy Integral Formula and its Beautiful Consequences

August 8, 2017

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Theorem (Cauchy Integral Formula)

Let $\Omega \subseteq \mathbb{C}$ be a domain. Let f be holomorphic everywhere within and on a simple closed contour γ (oriented positively). If z_0 is interior to γ, then,

$$f(z_0) = \frac{1}{2\pi\imath} \int_\gamma \frac{f(z)dz}{z - z_0}.$$

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We derived the CIF from Cauchy's theorem and I remarked that it is easy to go the other way round.

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Consider $\int_{C_1} |z|^2 dz$
$dz = (1+i)dt.$
Hence $\int_{C_1} |z|^2 dz = \int_0^1 ((-1+t)^2 + t^2)(1+i)dt$
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Now consider:
$\int_{C_2} |z|^2 dz$.
Here $C_2 : z_2(t) = e^{it}$. Hence $dz = ie^{it} dt$

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Example 2: $\int_{|z|=3} \frac{\cos \pi z}{z^2-1} dz = \frac{1}{2} \int_{|z|=3} \left[\frac{\cos \pi z}{z-1} - \frac{\cos \pi z}{z+1} \right] dz = 0$

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$$\int_{|z|=3} \frac{\cos \pi z}{z^2 - 1} dz = \int_{|z-1|=\varepsilon} \frac{\frac{\cos \pi z}{z+1}}{z-1} dz + \int_{|z+1|=\varepsilon} \frac{\frac{\cos \pi z}{z-1}}{z+1} dz$$

(**Discussed on the board**)

Examples

A easy modification of the previous argument shows that if Ω is a domain in $\mathbb{C}$, γ is a simple closed curve, f a holomorphic function on an open set containing γ and its interior and $z_1, \ldots, z_n$ are distinct points in the interior of γ and $a_i \in \mathbb{Z}$, then:

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$$\int_\gamma \frac{f(z)}{(z-z_1)^{a_1}(z-z_2)^{a_2}\cdots(z-z_n)^{a_n}}$$

$$= \sum_{i=1}^{n} \int_{C_i} \frac{f(z)}{(z-z_1)^{a_1}(z-z_2)^{a_2}\cdots(z-z_n)^{a_n}}$$

where C_i is a small circle around z_i not containing any of the other z_j's.
We will soon see how to compute this integral.

As a consequence of the Cauchy Integral formula, we showed that a holomorphic function is analytic, i.e, can be expanded around any point in its domain of holomorphicity. Since analytic functions are infinitely differentiable, we have shown that once differentiable implies infinitely differentiable... a long awaited claim !

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(ii)

$$f(z) = \begin{cases} \frac{z}{e^z - 1} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$$

expanded as a power series centered at 0 has radius of convergence $= 2\pi$.

Morera's theorem is a converse to Cauchy's theorem. It states that if Ω is a domain in $\mathbb{C}$ and if $f : \Omega \to \mathbb{C}$ is a continuous, complex valued function on Ω such that $\int_\gamma f(z)dz = 0$ for every closed curve γ in $\mathbb{C}$, then f is holomorphic on Ω.

Proof ??

Cauchy's estimate

We have also concluded that if $f : \Omega \to \mathbb{C}$ is holomorphic, and if $\{z \mid |z - z_0| \leq r\} \subset \Omega$, then,

$$f^{(n)}(z_0) = \frac{n!}{2\pi\imath} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

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where γ is $|z - z_0| = r$. Now suppose f is holomorphic in $|z - z_0| < R$ and suppose f is bounded by $M > 0$ there. Can apply ML inequality in the above formula. Since this is true for any $r < R$, we get,

$$|f^{(n)}(z_0)| \leq \frac{n! M}{R^n}.$$

This is called Cauchy's estimate.

Liouville's Theorem

A function defined all over $\mathbb{C}$ is called **entire** if it is holomorphic everywhere in $\mathbb{C}$. Examples? Polynomials, $\exp(z)$, $\sin z$, $\cos z$, etc. Clearly, sums and products of entire functions are entire. The fact that the function is defined and holomorphic everywhere puts strong restrictions on the function. For instance, we have the so called Liouville's theorem, which says:

A bounded entire function is a constant.

A non-constant entire function has to be unbounded. As we have seen $\exp(z)$ takes all values in $\mathbb{C}$ except 0, sin and cos are surjective, in particular these are all unbounded. Polynomials are also clearly unbounded.

polynomial are only function that are entire as well as proper

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$$|f'(z)| \leq \frac{M}{R},$$

if f is holomorphic in a disc with center z and radius R. But R can be as large as we want, since f is entire. So, $f'(z) = 0$ for all $z \in \mathbb{C}$ and hence f is a constant.

Fundamental Theorem of Algebra

Recall that the fundamental theorem of algebra asserts that every non-constant polynomial with complex coefficients has a complex root.

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Suppose $f(z)$ does not have any zero, then $\frac{1}{f(z)}$ defines a holomorphic function which, by the above computation, is bounded and hence by Lioville's theorem is constant. A contradiction.

Proper Maps

Definition

A continuous function $f : \mathbb{C} \to \mathbb{C}$ is said to be proper if $|f(z)| \to \infty$ as $|z| \to \infty$.

It can be checked easily that the above definition is equivalent to saying that a continuous function f is proper if and only if $f^{-1}(K)$ is compact whenever $K \subset \mathbb{C}$ is compact. This equivalent criterion allows one to talk about proper maps between much more general spaces (for instance all the spaces we have encountered in geometry since school and much more). Proper functions are of tremendous importance in various areas of mathematics such as analysis (including the area of several complex variables), topology, algebraic geometry etc.

Now it might appear from the above proof of FTA that we have proved the following seemingly more general statement : Any proper, holomorphic function from $\mathbb{C}$ to $\mathbb{C}$ is surjective. However that is not the case as the following theorem shows.

Theorem: Any proper, holomorphic function from $\mathbb{C}$ to $\mathbb{C}$ is neccessarily a non-constant polynomial.

We just showed that polynomial functions over the complex numbers are surjective. Similarly sin, cos and various other elementary functions are surjective. On the other hand e^z is not surjective; misses zero. A striking result here is that any entire function whose image misses the origin is of the form $e^{g(z)}$ for some entire function $g(z)$. Here is a brief sketch of the proof (The details are beyond us):

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Let $f(z)$ be any entire function which misses the origin. Then $f(z)$ defines a map $\mathbb{C} \to \mathbb{C}^*$. One also has the natural map $p : \mathbb{C} \to \mathbb{C}^*$ given by e^z. One shows that there exists a holomorphic map (called a lift), say $g : \mathbb{C} \to \mathbb{C}$ such that $p(g(z)) = f(z)$. Hence $f(z) = e^{g(z)}$.
Thus also, any non-surjective holomorphic function from $\mathbb{C}$ to $\mathbb{C}$ is of the form $e^{g(z)} + c$ for some analytic function $g(z)$ and some constant c.

Logarithm Revisited

Theorem

Let Ω be a simply connected domain in $\mathbb{C}$ with $1 \in \Omega$ and $0 \notin \Omega$. Then there exists a unique holomorphic function $F(z)$ on Ω, (denoted $\log(z)$) such that:

1. $F(1) = 0$ *and* $F'(z) = 1/z$

2. $e^{F(z)} = z \;\; \forall z \in \Omega$

3. $F(r) = \ln(r)$ *when r is a positive real number close to 1. (With the usual definition of $\ln$ for real numbers)*

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3. $F(r) = \ln(r)$ when r is a positive real number close to 1. (With the usual definition of $\ln$ for real numbers)

Proof:
Since $1 \in \Omega$ and $0 \notin \Omega$, define the function $F(z) = \int_1^z \frac{1}{w} dw$. Since Ω is simply-connected, it follows by Cauchy's theorem, that this function is well defined. We have seen before that this defines a holomorphic function on Ω. Clearly $F(1) = 0$ and $F'(z) = \frac{1}{z}$ proving 1.

Proof cont ..

One checks that the function $ze^{-F(z)}$ has its derivative identically vanishing and hence is a constant. Substituting $z = 1$, this constant is seen to be 1. This proves 2. For proving 3, take a straight path joining 1 and r, for a small real number r. Then $F(r) = \int_1^r \frac{1}{t} dt = ln(r)$.

Note that such a function is unique. (Why ?)

Exercise: Let f be an entire function such that there exists a real constant C such that for all $z \in \mathbb{C}$, $|f(z)| \leq C|z|^n$, then $f(z)$ is a polynomial of degree less than or equal to n.