TUTORIAL 5

1. Locate and classify the type of singularities of :

a) $\frac{sin(1/z)}{(1+z^4)}$
b) $\frac{z^5 sin(1/z)}{(1+z^4)}$
c) $\frac{1}{sin(1/z)}$
d) $tan(1/z)$

2. Find Laurent expansions for the function $f(z) = \frac{2(z-1)}{z^2-2z-3}$ valid on the annuli (i) $0 \leq |z| < 1, (ii) 1 < |z| < 3, (iii) |z| > 3$.

3. Let Ω be a domain in $\mathbb{C}$ and let $z_0 \in D$. Suppose that z_0 is an isolated singularity of $f(z)$ and $f(z)$ is bounded in some punctured neighborhood of z_0 (that is, there exists $M > 0$ such that $|f(z)| \leq M$ for all $z \in D - z_0$). Show that $f(z)$ has a removable singularity at z_0.

4. By integrating e^{-z^2} around a sector of radius R one arm of which is along the real axis and the other making an angle $\pi/4$ with the real axis, show that:

$$\int_0^\infty sin(x^2)dx = \frac{\sqrt{\pi}}{2\sqrt{2}} = \int_0^\infty cos(x^2)dx$$

(Here use the well known integral $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$).

5. Compute using residue theory :

$$\int_{-\infty}^\infty \frac{cos(x)dx}{(1+x^2)^2}$$

6. Show by transforming into an integral over the unit circle, that $\int_0^{2\pi} \frac{d\theta}{a^2+1-2acos\theta} = \frac{2\pi}{1-a^2}$, where $a > 1$. Also compute the value when $a < 1$.

7. Show that if $a_1, a_2, ..., a_n$ are the distinct roots of a monic polynomial $P(z)$ of degree n, for each $1 \leq k \leq n$ we have the formula:

$$\prod_{j \neq k} (a_j - a_k) = P'(a_k)$$