MA 205 Complex Analysis: Singularities

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August 27, 2015

Introduction

What did we do last time?

Quiz II on Friday, September 04. Time: 8:15-9:15 AM.

Endsem on Friday, September 11. Time: 5:30-7:30 PM.

Today we'll focus on points where a function is not defined or not holomorphic. Such a point is called a singularity of the function. We'll focus on singularities which are isolated.

Singularities

In the example $f(z) = \frac{\sin z}{z}$, though 0 is a singularity, in essence there's no singularity! The slightly modified function,

$$f(z) = \begin{cases} \frac{\sin z}{z} & \text{if } z \neq 0 \\ 1 & \text{is } z = 0, \end{cases}$$

is indeed holomorphic. We could remove the singularity. Such a singularity is called a removable singularity. In this case, existence of the limit is guaranteed.

In the example $f(z) = \frac{1}{z}$, $\lim_{z \to 0} f(z) = \infty$. Such a singularity is called

a pole.

Singularities

An isolated singularity which is not removable and not a pole is called an essential singularity. In this case the limit genuinely doesn't exist. Note: $\exp(\frac{1}{z})$ has limit ∞ as $z \to 0$ along positive x-axis; limit 0 as $z \to 0$ along negative x-axis; limit 1 as $z \to 0$ along y-axis.

Recall how we derived the power series representation of a holomorphic function on a disc centered around z_0. We used

$$f(z) = \frac{1}{2\pi\imath}\int_{|w-z_0|=r}\frac{f(w)}{w-z}dw,$$

and manipulated $\frac{1}{w-z}$ as

$$\frac{1}{w-z_0}\cdot\frac{1}{1-\frac{z-z_0}{w-z_0}}.$$

Now suppose z_0 is an isolated singularity for f. Consider an annulus with radii $R > r$ centered at z_0 such that f is holomorphic there. CIF takes the form:

$$f(z) = \frac{1}{2\pi\imath} \int_{|w-z_0|=R} \frac{f(w)}{w-z} dw - \frac{1}{2\pi\imath} \int_{|w-z_0|=r} \frac{f(w)}{w-z} dw.$$

The first integral gives rise to $\sum_{n=1}^{\infty} a_n (z-z_0)^n$ with

$$a_n = \frac{1}{2\pi\imath} \int_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^{n+1}} dz,$$

exactly as before.

In the second integral, write

$$\frac{-1}{w-z} = \frac{1}{z-z_0} \cdot \frac{1}{1-\frac{w-z_0}{z-z_0}},$$

and expand to get $\displaystyle\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ with :

$$b_n = \frac{1}{2\pi\imath} \int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{-n+1}} dz.$$

We write both together as $\displaystyle\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$. This is the Laurent series around the isolated singularity z_0. The negative part is called the principal part of the Laurent series.

Residue

If z_0 is an isolated singularity of f, then f is holomorphic in an annulus $0 < |z - z_0| < R$ for some R. The corresponding Laurent expansion is called the Laurent expansion around z_0. Consider the -1-st coefficient of this Laurent series.

$$a_{-1} = \frac{1}{2\pi\imath} \int_\gamma f(z)dz.$$

Right side has no z_0 information; it's the genuine integral of $f(z)$! If you integrate a Laurent series, only a_{-1} remains; other terms vanish. What remains is usually called a residue.

$$a_{-1} = \mathrm{Res}(f; z_0).$$

Eureka! Perhaps a_{-1} is easy to compute from $f(z)$ and if that's the case integration has become easy! This is really cool stuff! Fortunately we are not in a bathtub in Syracuse!

Cauchy Residue Theorem

Suppose f is given and γ is given. Suppose there are finitely many isolated singularities of f inside γ; say $z_1, z_2, \ldots, z_n$. What's $\int_\gamma f(z)dz$?

Theorem (Cauchy Residue Theorem)

$$\int_\gamma f(z)dz = 2\pi\imath \cdot \sum_{i=1}^{n} \mathrm{Res}(f, z_i).$$

Proof?

Thus integral of a function on a closed curve is zero not just when the function is holomorphic throughout, isolated singularities inside are okay, provided residues are zero! You'll agree that we should look for easy methods to get the residue out of the given $f(z)$. And then integrate, integrate, integrate, ...

Riemann Removable Singularity Theorem

Before we get to residue computation, let's investigate isolated singularities a bit more. Firstly, there is a very easy way to check whether a singularity is removable. z_0 is removable iff $\lim_{z\to z_0}(z-z_0)f(z)=0$. "$\implies$" is obvious. For the other way, define

$$g(z)=\begin{cases}(z-z_0)^2f(z) & \text{if } z\neq z_0\\ 0 & \text{if } z=z_0.\end{cases}$$

If f is analytic in a deleted neighbourhood of z_0, show that g is analytic throughout that neighbourhood. Write

$$g(z)=c_0+c_1(z-z_0)+c_2(z-z_0)^2+\ldots.$$

Note that $c_0=g(z_0)=0$ and $c_1=g'(z_0)=0$. Thus,

$$g(z)=c_2(z-z_0)^2+c_3(z-z_0)^3+\ldots.$$

If we define $f(z_0)=c_2$, then f is holomorphic throughout. i.e., z_0 is a removable singularity.

Using RRST, we'll understand poles better. Suppose f has a pole at z_0. By definition, this means

$$\lim_{z \to z_0} f(z) = \infty.$$

Thus, $\displaystyle\lim_{z \to z_0} (z - z_0) \cdot \frac{1}{f(z)} = 0$, and by RRST, $\frac{1}{f(z)}$ has a removable singularity at z_0. This means that the function

$$h(z) = \begin{cases} \frac{1}{f(z)} & \text{if } z \neq z_0 \\ 0 & \text{if } z = z_0 \end{cases}$$

is holomorphic in $B(z_0, R)$ for some R. Thus, z_0 is a zero of the holomorphic function $h(z)$. Of what order? Some finite m. Then $h(z) = (z - z_0)^m h_1(z)$, where h_1 is holomorphic and $h_1(z_0) \neq 0$. In other words, the given f can be written as $f(z) = \frac{g(z)}{(z-z_0)^m}$, where $g(z)$ is holomorphic throughout $B(z_0, R)$. m is the order of the pole; pole of order one is called simple.

Joke 1

There was a transatlantic flight and the pilot and copilot dropped dead. A desperate flight attendant asked if anyone knew how to fly a plane. An old polish man said: "Well, I used to fly planes in WW II, but nothing like this". When he brought him into the cockpit, his jaw dropped. There were so many buttons, levers, and fancy dials. "What's wrong?" the flight attendant asked.

"I'm just a simple pole in a complex plane", he responded.

Principal Part of the Laurent Series

If $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ is the Laurent expansion around z_0, then its principal part is

$$\sum_{n=-\infty}^{-1} a_n(z-z_0)^n.$$

Note that:

- removable iff principal part is zero.
- pole iff principal part is finite.
- essential iff principal part is infinite.

Proof (Easy exercise using previous slides).

Residue at a Pole

If the isolated singularity is removable, then the residue is trivial. If the isolated singularity is a pole, then the residue is trivial to compute. If z_0 is a pole, can write

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \ldots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots.$$

Thus,

$$g(z) = (z-z_0)^m f(z) = a_{-m} + \ldots + a_{-1}(z-z_0)^{m-1} + a_0(z-z_0)^m + \ldots.$$

Thus, g is holomorphic and

$$a_{-1} = \frac{g^{(m-1)}(z_0)}{(m-1)!}.$$

Joke 2

Why did the mathematician name her dog Cauchy?

B'coz it left a residue at every pole!

Examples

Find the isolated singularities and their residues of $f(z) = \frac{z^2}{1+z^4}$.
Singularities are clearly 4^{th} roots of -1;
$z_n = \exp\left(\imath\left[\frac{\pi}{4} + \frac{(n-1)\pi}{2}\right]\right)$, $n = 1, 2, 3, 4$. Easy to see: these are all simple poles. Can use the formula for residues.

$$\begin{aligned}
\mathrm{Res}(f; z_1) &= \lim_{z \to z_1} (z - z_1) f(z) \\
&= \frac{z_1^2}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)} \\
&= \frac{1 - \imath}{4\sqrt{2}}.
\end{aligned}$$

Similarly,

$$\mathrm{Res}(f; z_2) = \frac{-1 - \imath}{4\sqrt{2}},$$

etc.

Example 1

Compute $\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx$.
This is an MA 105 integral which we'll work out using MA 205!
The idea is to compute $\int_{-r}^{r} \frac{x^2}{1+x^4} dx$ and take limit as $r \to \infty$. Fix $r > 1$. Let γ be $[-r, r]$ together with the upper part of the circle $|z| = r$ oriented counterclockwise. Take $f(z) = \frac{z^2}{1+z^4}$. f has two poles inside γ. Now,

$$\frac{1}{2\pi\imath} \int_\gamma f(z)dz = \mathrm{Res}(f; z_1) + \mathrm{Res}(f; z_2) = \frac{-\imath}{2\sqrt{2}}.$$

This is same as

$$\frac{1}{2\pi\imath} \int_{-r}^{r} \frac{x^2}{1+x^4} dx + \frac{1}{2\pi} \int_0^\pi \frac{r^3 e^{3\imath t}}{1 + r^4 e^{4\imath t}} dt.$$

Example 1

Thus,

$$\int_{-r}^{r} \frac{x^2}{1+x^4} dx = \frac{\pi}{\sqrt{2}} \left(1 - \imath r^3 \int_0^{\pi} \frac{e^{3\imath t}}{1+r^4 e^{4\imath t}} dt \right).$$

Note that,

$$\left| \imath r^3 \int_0^{\pi} \frac{e^{3\imath t}}{1+r^4 e^{4\imath t}} dt \right| \leq \frac{\pi r^3}{r^4 - 1}.$$

Thus, in the limit, this integral is zero. Therefore,

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx = \frac{\pi}{\sqrt{2}}.$$

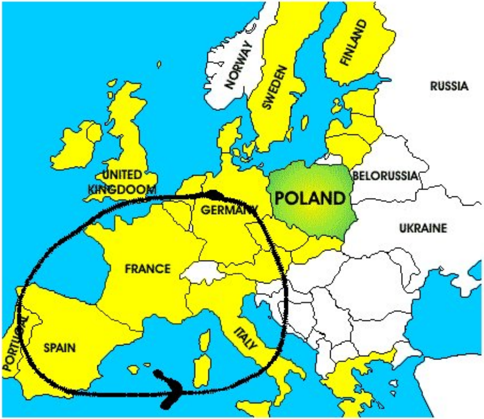
Compute $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^n}$.
Proceed as before by taking the same γ. There is just one pole inside γ which is $\imath$. Compute $\mathrm{Res}(f;\imath)$, where $f(z) = \frac{1}{(1+z^2)^n}$. This is given by $\frac{g^{(n-1)}(\imath)}{(n-1)!}$, where $g(z) = \frac{1}{(z+\imath)^n}$. Check:

$$\mathrm{Res}(f;\imath) = \frac{-\imath}{2^{2n-1}} \begin{pmatrix} 2n-2 \\ n-1 \end{pmatrix}.$$

Exactly as before, conclude that the second integral is zero in the limit. Thus, the answer is $\frac{\pi}{4^{n-1}} \begin{pmatrix} 2n-2 \\ n-1 \end{pmatrix}$.

Joke 3

What's the contour integral over Western Europe?



Zero. All the poles are in Eastern Europe.

Example 3

Let $f(z) = \frac{z}{1-z-z^2}$. Compute $\int_{\gamma_r} \frac{f(z)}{z^{n+1}} dz$ directly and using the residue theorem, where γ_r is $|z| = r$. Do this for large r.

Direct: $\left|\frac{dz}{(1-z-z^2)z^n}\right| \le \frac{1}{r}$, so the integral is zero in the limit.

Residue Theorem: poles are $0, -\alpha, -\beta$ where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$.

$$\mathrm{Res}(\cdot; -\alpha) = \frac{-1}{(-\alpha+\beta)(-\alpha)^n} = \frac{\beta^n}{\alpha-\beta};$$

$$\mathrm{Res}(\cdot; -\beta) = \frac{-1}{(\alpha-\beta)(-\beta)^n} = -\frac{\alpha^n}{\alpha-\beta};$$

$$\mathrm{Res}(\cdot; 0) = \frac{f^{(n)}(0)}{n!} = F_n.$$

Thus,

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$