MA 205 – 3

Complex integration :

Definite integrals:

Let $f : [a,b] \to \mathbb{C}$ be a piecewise continuous function.

$$f(t) = u(t) + i\,v(t)$$

Define : $$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Basic properties:

1. $$\mathrm{Re}\int_a^b f(t)\,dt = \int_a^b \mathrm{Re}\, f(t)\,dt = \int_a^b u(t)\,dt$$

2. $\operatorname{Im}\int_a^b f(t)\,dt = \int_a^b \operatorname{Im} f(t)\,dt = \int_a^b v(t)\,dt.$

3. $\int_a^b \left(c_1 f_1(t) + c_2 f_2(t)\right) dt = c_1\int_a^b f_1(t)\,dt$

$+ c_2\int_a^b f_2(t)\,dt.$

for $c_1, c_2 \in \mathbb{C}$.

4. $\left|\int_a^b f(t)\,dt\right| \leq \int_a^b |f(t)|\,dt$

Proof of 4: This inequality is clear if $\int_a^b f(t)\,dt = 0$. Suppose $\int_a^b f(t)\,dt = w \neq 0 \in \mathbb{C}$.

Let $\theta = \operatorname{Arg}(w) = \operatorname{Arg}\left(\int_a^b f(t)\,dt\right).$

$$\left| \int_a^b f(t)\,dt \right| = e^{-i\theta} \int_a^b f(t)\,dt = \int_a^b e^{-i\theta} f(t)\,dt.$$

On the other hand,

$$\left| \int_a^b f(t)\,dt \right| = \operatorname{Re} \int_a^b e^{-i\theta} f(t)\,dt$$

$$= \int_a^b \operatorname{Re}\left(e^{-i\theta} f(t)\right) dt$$

$$\leq \int_a^b \left| e^{-i\theta} f(t) \right| dt$$

$$= \int_a^b |f(t)| \; dt .$$

<u>Example:</u> Suppose $\theta \in \mathbb{R}$ then show that

$$\left| e^{2\pi i\theta} - 1 \right| \leq 2\pi |\theta| .$$

Solution: Let $f(t) = e^{i\theta t}$: $\theta, t \in \mathbb{R}$.

Then $\left| \int_0^{2\pi} e^{i\theta t} dt \right| \leq \int_0^{2\pi} |e^{i\theta t}| \, dt = 2\pi.$

On the other hand,

$$\left| \int_0^{2\pi} e^{i\theta t} \, dt \right| = \left| \frac{e^{i\theta t}}{i\theta} \right|_0^{2\pi} = \frac{|e^{2\pi i\theta} - 1|}{|\theta|}$$

$$\Rightarrow |e^{2\pi i\theta} - 1| \leq |\theta| \cdot 2\pi \quad : \theta \in \mathbb{R}$$

Length of curve

Recall from calculus the formula for the length of a parameterised curve.

If γ is a smooth parameterised curve

$\gamma = z(t) = (x(t), y(t))$, $t \in [a,b]$,

then,

$$l(\gamma) = \int_a^b |z'(t)|\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, .\, dt$$

Contour integrals

We now consider the complex analogue of the line integrals from calculus.

Definition of a contour integral:

Consider a curve i.e., a continuous map $z : [a,b] \to \mathbb{C}$.

Let $z(t) = x(t) + i\, y(t)$.

So $x(t)$ & $y(t)$ are continuous functions.

We say the curve is smooth if $z(t)$

has continuous derivative $z'(t) \neq 0$

for all points along the curve.

A <u>contour</u> is a curve consisting of

a finite number of smooth curves

joined end to end. It is said to

be <u>simple</u> if $z(t)$ is one-one

except possibly for the end points

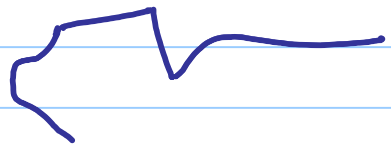
a & b (i.e, the curve does not cross

itself).

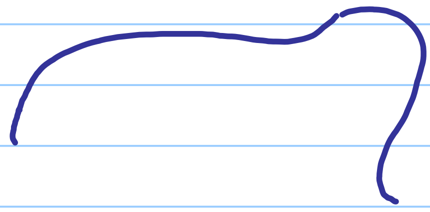
Further, it is said to be closed if the initial & final values of $z(t)$ at a & b respectively, are the same.

Examples:

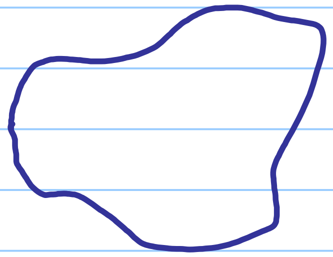
1. Contour :



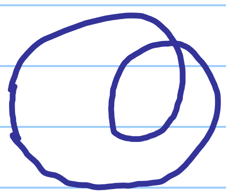
2. Simple contour, not closed :



3. Simple & closed contour :



4. Closed but not simple :



Let $f : \Omega \to \mathbb{C}$ be a complex function on a domain Ω and $C \subseteq \Omega$ be any contour with initial point z_0 & terminal point z. We say f is integrable along the contour C & write:

$$\int_C f(z)\, dz := \int_a^b f(z(t)) \frac{d}{dt} z(t) \cdot dt$$

$$= \int_a^b f(z(t)) \cdot z'(t) \cdot dt$$

For $f(z) = u(x,y) + iv(x,y)$ &

$dz = dx + idy$, we have

$$\int_C f(z)\,dz = \int_C u\,dx - v\,dy + i\int_C u\,dy + v\,dx$$

$$= \int_a^b \left[u(x(t),y(t))\frac{dx(t)}{dt} - v(x(t),y(t))\cdot\frac{dy(t)}{dt}\right]dt$$

$$+ i\int_a^b \left[u(x(t),y(t))\frac{dy(t)}{dt} + v(x(t),y(t))\cdot\frac{dx(t)}{dt}\right]dt$$

Claim: The usual properties of real line integrals are carried through in their complex analogues.

1. $\int_C f(z)\,dz$ is independent of the parameterization of C.

2. $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$, where $-C$ is the opposite curve of C.

3. $$\int_{C_1 \cup \cdots \cup C_r} f(z)\,dz = \int_{C_1} f\,dz + \cdots + \int_{C_r} f\,dz$$

where the C_i's are <u>suitably</u> defined.

<u>Example:</u> Evaluate the integral $\int_C \frac{1}{z-z_0}\,dz$

C = a circle of any radius centered at z_0, in the anti-clockwise direction.

<u>Solution:</u> C can be parameterized by:

$$z(t) = z_0 + re^{it} \quad : 0 \le t \le 2\pi$$

where $r > 0$ is the radius.

The contour integral becomes:

$$\oint_C \frac{1}{z-z_0}\,dz = \int_0^{2\pi} \frac{1}{z(t)-z_0} \cdot z'(t)\,dt$$

$$= \int_0^{2\pi} \frac{1}{re^{it}} \cdot ire^{it}\,dt$$

$$= 2\pi i$$

Note: The value of the integral is independent of the radius r.

Path independence:

Example: Let $C_1 =$ the line segment with initial point -1 & final point i &

let $C_2 =$ the arc of the unit circle $\operatorname{Im} z \geq 0$ with initial point -1 & final point i.

Parameterise them as

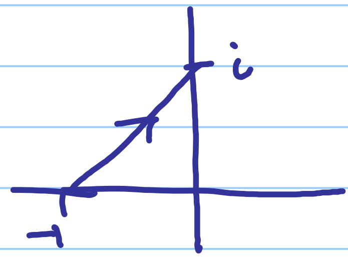
C_1 : $z(t) = -1 + (1+i)t$; $0 \leq t \leq 1$

$= (-1+t) + it$

C_2 : $|z| = 1$ & $z = e^{i\theta}$

Evaluate: $\int_{C_j} |z|^2\, dz$ & $\int_{C_j} \frac{1}{z^2}\, dz$: $j=1,2$

1. → $\int_{C_1} |z|^2\, dz$



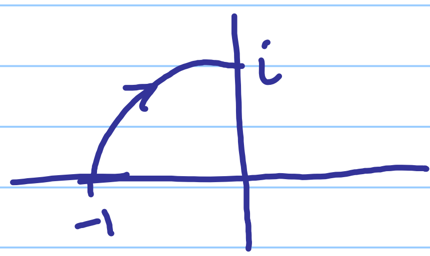
C_1 : $z(t) = -1 + (1+i)\,t$: $0 \le t \le 1$

So $|z|^2 = (-1+t)^2 + t^2$ & $dz = (1+i)\, dt$

$$\int_{C_1} |z|^2 dz = \int_0^1 (2t^2 - 2t + 1)(1+i)\, dt$$

$$= \frac{2}{3}(1+i)$$

→ $\int_{C_2} |z|^2\, dz$



C_2 : $|z| = 1$, $z = e^{i\theta}$ & $dz = i e^{i\theta} d\theta$

The initial & final points of the path correspond to $\theta = \pi$ & $\theta = \frac{\pi}{2}$ respectively

The contour integral is:

$$\int_{C_2} |z|^2 dz = \int_{\pi}^{\pi/2} i e^{i\theta} d\theta = e^{i\theta}\Big|_{\pi}^{\pi/2} = 1+i$$

So the results do not agree! That is, the value of this contour integral depends on the path of integration.

Note here that $|z|^2$ is not holomorphic for $z \neq (0,0)$. We will see later that this explains the path dependence of the integral.

C.W

$\rightarrow$ $$\int_{C_1} \frac{1}{z^2}\,dz$$

$$= \int_0^1 \frac{1+i}{[-1+(1+i)t]^2}\,dt = \frac{1}{-1+(1+i)t}\Big|_0^1 = \underline{\underline{-1+i}}$$

$$\int_{C_2} \frac{1}{z^2}\,dz = \int_{\pi}^{\pi/2} \frac{1}{e^{2i\theta}}\, i e^{i\theta}\,d\theta = \int_{\pi}^{\pi/2} i e^{-i\theta}\,d\theta$$

$$= -e^{-i\theta}\Big|_{\pi}^{\pi/2} = \underline{\underline{-1+i}}$$

Check: Holomorphy of $\frac{1}{z^2}$!

Absolute value of a complex integral

Let f & C be as earlier.

Then $\left| \int_C f(z)\,dz \right| \leq M \cdot L$

where $M =$ the upper bound of $|f(z)|$ along C

$L =$ length of the contour C.

Proof: $\left| \int_C f(z)\,dz \right| = \left| \int_a^b f(z(t)) \cdot z'(t)\,dt \right|$

$$\leq \int_a^b |f(z(t))| \cdot |z'(t)|\,dt$$

$$\leq \int_a^b M \cdot |z'(t)|\,dt$$

$$= M \cdot L$$

Path independence:

Under what conditions do we get

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

where C_1 & C_2 are contours in a domain Ω with same initial & final points & $f(z)$ is piecewise continuous inside Ω. We have seen that this fails for $f(z) = |z|^2$ & will show that it holds for $f(z) = \frac{1}{z^2}$. This is the same as asking when does $\oint_C f(z)\,dz = 0$

any closed contour lying inside Ω.

(We can treat $C_1 \cup -C_2$ as a closed contour). This is answered by the following:

<u>Cauchy's integral theorem:</u>

Let $f(z) = u(x,y) + i\,v(x,y)$ be holomorphic on & inside a simple closed contour C & let $f'(z)$ be continuous on & inside C then $\oint_C f(z)\,dz = 0$

<u>Proof:</u> We use Green's theorem: if two real functions $P(x,y)$ & $Q(x,y)$ have

continuous 1^{st} order partial derivatives on & inside C then

$$\oint_C P\,dx + Q\,dy = \iint_\Omega (Q_x - P_y)\,dx\,dy$$

where Ω is a simply connected domain bounded by C.

Suppose $f(z) = u(x,y) + iv(x,y)$, $z = x+iy$.

Then $\oint_C f(z)\,dz = \oint_C u\,dx - v\,dy + i\oint_C v\,dx + u\,dy$.

Now $f'(z)$ is continuous $\Rightarrow$ u & v have continuous derivatives on & inside C.

Green's theorem $\Rightarrow$ the 2 line integrals can be transformed into double integrals:

$$\oint_C f(z)\,dz = \iint_\Omega (-v_x - u_y)\,dx\,dy + i\iint_\Omega (u_x - v_y)\,dx\,dy$$

C.R. equations $\Rightarrow$ $u_x = v_y$ & $u_y = -v_x$.

Hence the integrands on the R.H.S are both zero, proving the theorem.

Note Goursat in 1903 obtained the same result <u>without</u> assuming continuity of $f'(z)$.

Goursat's theorem:

If a function $f(z)$ is analytic throughout a simply connected domain Ω then for any simple closed contour C lying completely in Ω, we have:

$$\oint_C f(z)\,dz = 0$$

Corollary 1: Let $f(z)$ be analytic on a simply connected domain Ω. Suppose $z_1, z_2 \in \Omega$ & C_1, C_2 are contours inside Ω joining z_1 to z_2. Then,

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

i.e., the integral is independent of the path chosen & depends only on the end points.

Corollary 2: Fundamental Theorem of Calculus:

Let $f(z)$ be analytic on a simply connected domain Ω. Consider a fixed point $z_0 \in \Omega$. Then the function

$$F(z) := \int_{z_0}^{z} f(w)\,dw \quad \text{is well defined}$$

for any $z \in \Omega$. Further, $F'(z) = f(z)$ for $z \in \Omega$.

Proof: F is well defined by Corollary 1.

To show $F' = f$, consider

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} [f(w) - f(z)]\, dw$$

By Cauchy's theorem, the last integral is independent of the path joining z & $z+\Delta z$ so long as the path is completely inside Ω. We choose the path as the straight line segment joining z & $z+\Delta z$ & choose $|\Delta z|$ small enough so that it lies completely in Ω (recall Ω is connected & open!)

Hence

$$\left|\frac{F(z+\Delta z)-F(z)}{\Delta z}-f(z)\right| = \frac{1}{|\Delta z|}\left|\int_{z}^{z+\Delta z}[f(w)-f(z)]\,dw\right|$$

$$\leq \frac{1}{|\Delta z|}\int_{z}^{z+\Delta z}|f(w)-f(z)|\,dw$$

$$< \frac{\Delta z}{|\Delta z|}\cdot\varepsilon$$

(by continuity of $f(z)$, given $\varepsilon > 0$ $\exists$ $\delta > 0$ such that $|f(w)-f(z)| < \varepsilon$ if $|w-z| < \delta$ for w in the straight line path).

That is,

$$\lim_{\Delta z \to 0}\frac{F(z+\Delta z)-F(z)}{\Delta z} = f(z)$$

That is, $F'(z) = f(z)$ for $z \in \Omega$

Hence F' is analytic on Ω. Further, for any contour joining z_1, z_2 inside Ω,

$$\int_{z_1}^{z_2} f(z)\,dz = \int_{z_0}^{z_2} f(w)\,dw - \int_{z_0}^{z_1} f(w)\,dw$$

$$= F(z_2) - F(z_1) \quad : z_1, z_2 \in \Omega$$

Note 1: Existence of such an antiderivative or primitive for $f(t) = u(t) + iv(t) : a \leq t \leq b$

If such an f is continuous then

$$t \longmapsto \int_a^t f(x)\,dx$$

is an antiderivative of f. If F is

any antiderivative of f on $[a,b]$, then

$$\int_r^s f(x)\,dx = F(s) - F(r), \text{ for } r, s \in [a,b].$$

The proof of the above follows by applying the Fundamental theorem of Calculus for the reals to u & v.

Note 2: Jordan curve theorem :

Let C be a simple closed curve in $\mathbb{R}^2$. Then its complement $\mathbb{R}^2 \setminus C$ consists of exactly 2 connected components. One of these components is bounded (the interior) & the other is unbounded (the exterior) & the curve C is the boundary of each component.

<u>Example:</u> Let γ be a closed curve which goes round the point z_0 once in the counter clockwise direction. Evaluate $\int_\gamma \frac{dz}{z-z_0}$.

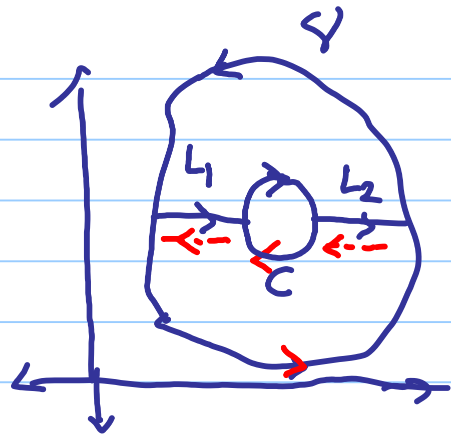
<u>Sol^n:</u> Let $C_r = \{z \in \mathbb{C} \mid |z-z_0| = r\}$.

Choose r small enough so that $C_r \subseteq \Omega$.

Consider the curve $\gamma \cup (-C)$.

Note that $f(z)$ is holomorphic in the region contained on & inside the curve-

Hence by Cauchy's theorem,



$$\int_{\gamma \cup -C} \frac{dz}{z - z_0} = 0 \text{ (why ?)}$$

i.e., $$\int_{\gamma} \frac{dz}{z - z_0} = \int_{C} \frac{dz}{z - z_0} = 2\pi i$$

Theorem : Cauchy's integral formula (CIF) :

Let f be holomorphic on & inside a simple closed curve γ (oriented positively). If z_0 is in the interior of γ then

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0}\, dz.$$

Proof: We need to show that:

$$\int_\gamma \frac{f(z)}{z-z_0}\,dz = \int_\gamma \frac{f(z_0)}{z-z_0}\,dz$$

$$\left(\because \int_\gamma \frac{f(z_0)}{z-z_0} = 2\pi i f(z_0) \right)$$

i.e., to show that

$$\int_\gamma \frac{f(z)-f(z_0)}{z-z_0} = 0$$

As f is continuous at z_0, given $\varepsilon > 0$ $\exists$

$\delta > 0$ such that

$$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$$

Choose $r < \delta$ & such that

$C_r = \{ z \mid |z - z_0| = r \} \subseteq \text{Int}(\gamma)$.

Apply Cauchy's theorem to $\gamma \cup (-C_r)$

to get:

$$\int_{\gamma} \frac{f(z)-f(z_0)}{z-z_0}\, dz = \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\, dz$$

Now $\left| \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\, dz \right| \leq \int_{C_r} \frac{|f(z)-f(z_0)|}{|z-z_0|} \cdot dz$

$$< \frac{\varepsilon}{r} 2\pi r$$

$$= 2\pi\varepsilon$$

That is, $\left| \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0} \cdot dz \right| = 0$

as it can be made arbitrarily small

Note: The above theorem says the value of f at any interior point is gotten by averaging on the boundary.

Examples: Evaluate :

1. $$\int_{|z|=2} \frac{e^{z}\, dz}{(z+1)(z-3)^2}$$

2. $$\int_{|z|=6} \frac{dz}{z^3-1}$$

3. $$\int_{|z|=3} \frac{\cos \pi z}{z^2-1}\, dz$$

4. Let k be a real constant. Show that

$$\int_0^{2\pi} e^{k\cos\theta} \cdot \sin(k\sin\theta)\, d\theta = 0 \quad \&$$

$$\int_0^{2\pi} e^{k\cos\theta} \cos(k\sin\theta)\, d\theta = 2\pi$$

Solutions:

1. Let $f(z) = \dfrac{e^z}{(z-3)^2}$. So by CIF,

$$\int_{|z|=2} \frac{f(z)}{z+1}\, dz = 2\pi i\, f(-1) = \frac{\pi i}{8e}$$

2. $$\int_{|z|=6} \frac{dz}{z^3-1} = 2\pi i \left[\frac{1}{(1-\omega)(1-\omega^2)} + \frac{1}{(\omega-1)(\omega-\omega^2)} + \frac{1}{(\omega^2-\omega)(\omega^2-1)} \right]$$

$$= 0 \qquad \because\ \omega^3 = 1,\ \omega \neq 1$$

3. $$\int_{|z|=3} \frac{\cos \pi z}{z^2-1}\, dz = \frac{1}{2} \int_{|z|=3} \left[\frac{\cos \pi z}{z-1} - \frac{\cos \pi z}{z+1} \right] dz$$

$$= 0$$

alternatively,

$$\int_{|z|=3} \frac{\cos \pi z}{z^2-1}\, dz = \int_{|z-1|=\varepsilon} \frac{\frac{\cos \pi z}{z+1}}{z-1} \cdot dz$$

$$= \int_{|z+1|=\varepsilon} \frac{\frac{\cos \pi z}{z-1}}{z-1} \cdot dz$$

$$= 0$$

4. Apply CIF to $\oint_{|z|=1} \frac{e^{kz}}{z} \cdot dz \quad ; k \in \mathbb{R}$

$$= (2\pi i)\, e^{k \cdot 0}$$

$$= 2\pi i$$

On the other hand,

$$2\pi i = \oint_{|z|=1} \frac{e^{kz}}{z} \cdot dz$$

$$= \int_0^{2\pi} \frac{e^{k(\cos\theta + i\sin\theta)}}{e^{i\theta}} \; i \cdot e^{i\theta} \, d\theta$$

$$= i \int_0^{2\pi} e^{k\cos\theta} \left[\cos(k\sin\theta) + i\sin(k\sin\theta)\right] d\theta.$$

Equating the real & imaginary parts

gives the answer.

We next take derivatives in the CIF & show that

holomorphic $\Rightarrow$ analytic.

Let f be holomorphic at z_0, i.e. f is holomorphic in a neighbourhood of z_0, say f is holomorphic in $B_R(z_0)$ for some $R > 0$. Let γ be a circle of radius $r < R$ centered at z_0.

By CIF, we have

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z} \, dw$$

for any z such that $|z - z_0| < r$.

Note w lies on γ, so $|w-z_0|=r$.

To write $f(z)$ as a power series in $z-z_0$, consider

$$\frac{1}{w-z} = \frac{1}{w-z_0} \cdot \frac{w-z_0}{w-z}$$

$$= \frac{1}{w-z_0} \cdot \frac{1}{1-\left(\frac{z-z_0}{w-z_0}\right)}$$

$$= \frac{1}{w-z_0}\left[1+\frac{z-z_0}{w-z_0}+\left(\frac{z-z_0}{w-z_0}\right)^2+\cdots\right]$$

$$\because \text{ as } \left|\frac{z-z_0}{w-z_0}\right|<1$$

Substituting this in the CIF, we get

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z}\, dw$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z_0} \left[1 + \left(\frac{z-z_0}{w-z_0}\right) + \left(\frac{z-z_0}{w-z_0}\right)^2 + \cdots \right] dw$$

$$= \left[\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z_0}\, dw\right] + \left[\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-z_0)^2}\, dw\right](z-z_0)$$

$$+ \left[\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-z_0)^3}\, dw\right](z-z_0)^2 + \cdots$$

{ <u>**Note:**</u> the interchange of integration & sum is justified because the series converges <u>uniformly</u>. $\longrightarrow$ check }

$$= \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad : \quad \text{for } |z-z_0| < r$$

where,

$$a_n = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{(w-z_0)^{n+1}}\, dw$$

Hence if f is holomorphic in $B_R(z_0)$

then $$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

where, $$a_n = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{(w-z_0)^{n+1}}\, dw$$

for any $r < R$.

Since the power series converges to $f(z)$ in $B_r(z_0)$, the radius of convergence is atleast r.

We also know that $a_n = \frac{f^n(z_0)}{n!}$

whenever $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$.

In particular, a_n does not depend on r.

Any $r < R$ gives the same a_n.

Thus radius of convergence is atleast R.

Thus

holomorphic $\Rightarrow$ analytic $\Rightarrow$ infinitely differentiable.

Cauchy's estimate:

Suppose $f: \Omega \to \mathbb{C}$ is holomorphic &

$\{z \mid |z - z_0| \leq r\} \subseteq \Omega$. Then we have,

$$f^n(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{n+1}} \cdot dz$$

where $\gamma = \{z \mid |z - z_0| = r\}$.

Now suppose f is holomorphic in $B_R(z_0)$

& suppose f is bounded by $M > 0$ there.

For each $r < R$ we get

$$|f^n(z_0)| \leq \left(\frac{n!}{2\pi}\right) \cdot \frac{M}{r^{n+1}} \cdot 2\pi r = \frac{n! \cdot M}{r^n}$$

(use the arc length bound proved earlier).

As this inequality is true for every $r < R$, we get

$$|f^{n}(z_0)| \leq \frac{n!\, M}{R^n}$$

This is called Cauchy's estimate.

Liouville's theorem – revisited

A bounded entire function is a constant.

Proof: Suppose $|f(z)| \leq M \;\; \forall\; z \in \mathbb{C}$.

To show f is a constant function, we will show that $f' = 0$.

By Cauchy's estimate,

$$|f'(z)| \le \frac{M}{R}$$

if f is holomorphic in a disc with center z & radius R. As f is entire R may be chosen to be as large as we require. This implies $|f'(z)|$ can be made arbitrarily small. Hence $f'(z) = 0$ for every $z \in \mathbb{C}$. That is, f is a constant.