MA108 – Lecture 16

Improper integrals of the first kind

Definition: A function $f:[a,b]\to\mathbb{R}$ is said to be piecewise continuous if there is a partition:

$$a=t_0<t_1<\cdots<t_{n-1}<t_n=b$$

such that
(i) f is continuous on (t_{i-1}, t_i) for $i=1,2,\dots,n$

(ii) $\lim\limits_{t\to t_i^+} f(t)$ & $\lim\limits_{t\to t_i^-} f(t)$ both exist for $i=1,2,\dots,n-1$ and $\lim\limits_{t\to t_0^+} f(t)$ and $\lim\limits_{t\to t_n^-} f(t)$ both exist.

Let $f:[a,\infty)\to\mathbb{R}$ be a function. If f is such that, for any $b\geq a$, $f:[a,b]\to\mathbb{R}$ is piecewise continuous, then we say f is so on $[a,\infty)$.

Note 1) Such an f is bounded on $[a,b]$ for every $b\geq a$.

Note 2) For any $b\geq a$, the usual Riemann integral

$$I(b)=\int_a^b f(x)\,dx \quad \text{exists.}$$

Definition: An improper integral of first kind is defined to be

$$\int_a^\infty f(x)\,dx := \lim_{b\to\infty}\int_a^b f(x)\,dx,$$

if this limit exists.

If the above limit exists, we say that $\int_a^\infty f(x)\,dx$ <u>converges</u>, otherwise it is said to <u>diverge</u>.

<u>Example 1)</u>: Consider the improper integral $\int_1^\infty \frac{dx}{x^s}$: $s \in \mathbb{R}$.

$$I(b) = \int_1^b \frac{dx}{x^s} = \left. \frac{x^{1-s}}{1-s} \right|_1^b \quad : \text{if } s \neq 1$$

$$= \frac{b^{1-s} - 1}{1-s}$$

$$\Rightarrow \quad I(b) = \begin{cases} \dfrac{b^{1-s} - 1}{1-s} & : s \neq 1 \\[2ex] \ln b & : s = 1 \end{cases}$$

$$\Rightarrow \lim_{b \to \infty} I(b) = \begin{cases} \dfrac{1}{s-1} & : \text{if } s > 1 \\[2ex] \infty & : \text{if } s \leq 1 \end{cases}$$

Example 2) The integral $\int_0^\infty \sin x \, dx$ diverges because

$$I(b) = \int_0^b \sin x \, dx = 1 - \cos b$$

and $\lim_{b \to \infty} I(b)$ does not exist.

Note: We can define similarly,

$$\int_{-\infty}^b f(x) \, dx = \lim_{a \to -\infty} \int_a^b f(x) \, dx$$

We say that the integral $\int_{-\infty}^\infty f(x) \, dx$ is <u>convergent</u> if there is a $c \in \mathbb{R}$ such that $\int_{-\infty}^c f(x) \, dx$ is convergent & $\int_c^\infty f(x) \, dx$ is convergent.

We then define:

$$\int_{-\infty}^{\infty} f(x)\,dx := \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx.$$

Exercise: Show that the definition is independent of the choice of c.

Convergence tests for improper integral

I Theorem 1: Suppose there is a real number $M > 0$ such that $\int_a^b |f(x)|\,dx \leq M$; for every $b \geq a$.

Then $\int_a^{\infty} f(x)\,dx$ and $\int_a^{\infty} |f(x)|\,dx$ are convergent.

II Theorem 2: Comparison test:

Suppose $0 \leq f(x) \leq g(x)$; for every $x \geq a$.

If $\int_a^\infty g(x)\,dx$ converges, then

$\int_a^\infty f(x)\,dx$ also converges &

$$\int_a^\infty f(x)\,dx \leq \int_a^\infty g(x)\,dx$$

<u>Proof</u>: <u>Exercise</u>:

<u>Example</u>: As $0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$

on $[1, \infty)$ and $\int_1^\infty \frac{1}{x^2}\,dx$

converges, we have $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$

also converges.

<u>III</u> <u>Theorem 3 (Limit Comparison test)</u>:

Suppose $f(x) \geq 0$ and $g(x) > 0$ on $[a, \infty)$ & $\int_a^b f(x)\,dx$ & $\int_a^b g(x)\,dx$

exist, for every $b \geq a$.

If $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = c \;:\; c \neq 0,$

then both $\int_a^\infty f(x)\,dx$ & $\int_a^\infty g(x)\,dx$

converge or both diverge.

If $c = 0$ and $\int_a^\infty g(x)\,dx$ converges

then $\int_a^\infty f(x)\,dx$ converges.

<u>Proof</u>: Exercise.

<u>Example</u>: Consider $\int_1^\infty e^{-x} x^s\,dx \;; s \in \mathbb{R}$

We have $\lim\limits_{x\to\infty} \dfrac{e^{-x} x^s}{x^{-2}} = 0$

and $\int_1^\infty \dfrac{1}{x^2}\,dx$ converges.

Hence by the above theorem,

$\int_1^\infty e^{-x} x^s\,dx$ converges for every $s \in \mathbb{R}$

Example: Define the Gamma function:

$\Gamma : (0,\infty) \to \mathbb{R}$ as:

$$\Gamma(y) = \int_0^\infty e^{-x} x^{y-1}\, dx$$

As $\displaystyle\int_0^\infty e^{-x} x^{y-1}\, dx = \int_0^1 e^{-x} x^{y-1}\, dx$

$$+ \int_1^\infty e^{-x} x^{y-1}\, dx,$$

$\Gamma(y)$ is well-defined for $y>0$.

Functional equation:

$$\Gamma(y+1) = y \cdot \Gamma(y)$$

Let $0<a<b$. Use integration by parts to evaluate:

$$\int_a^b e^{-x} x^{y}\, dx = -x^y e^{-x}\Big|_a^b + y\int_a^b e^{-x} x^{y-1}\, dx$$

$$= a^y e^{-a} - b^y e^{-b} + y \cdot \int_a^b e^{-x} x^{y-1}\, dx$$

Taking limit as $b \to \infty$ & $a \to 0^+$

on both sides, we get:

$$\int_0^\infty e^{-x} x^y \, dx = y \, \Gamma(y)$$

i.e. $\Gamma(y+1) = y \, \Gamma(y)$

Note: $\Gamma(n+1) = n!$: $n = 0, 1, 2, \ldots$

To see this, apply induction on n. (check!)

Improper integrals of the second kind

Let $f : (a, b] \to \mathbb{R}$ be such that

$\int_x^b f(t) \, dt$ exists for every $x \in (a, b]$.

Set $I(x) = \int_x^b f(t) \, dt$: $x \in (a, b]$.

Definition: If $\lim_{x \to a^+} I(x)$ exists,

then we say that $\int_a^b f(t) \, dt$

is convergent and call it an improper integral of the second kind.

<u>Note</u>: In a similar fashion, we can define improper integrals of the second kind if $f:[a,b) \longrightarrow \mathbb{R}$.

<u>Example</u>: $f(t) = \dfrac{1}{t^s}$: $t > 0$

Let $b, x > 0$. Then

$$I(x) = \int_x^b \frac{dt}{t^s} = \begin{cases} \dfrac{b^{1-s} - x^{1-s}}{1-s} & : s \neq 1 \\ \ln b - \ln x & : s = 1 \end{cases}$$

Thus $\lim\limits_{x \to 0^+} \displaystyle\int_x^b \frac{dt}{t^s}$ exists if and only if $s < 1$.

<u>Definition</u>: Let $a = t_0 < t_1 < \cdots < t_{n+1} = b$ be a partition of $[a,b]$ and

$f : [a,b] \setminus \{t_1, t_2, \ldots, t_n\} \longrightarrow \mathbb{R}$.

We say $\int_a^b f(t)\,dt$ converges if

each of $\int_{t_{i-1}}^{t_i} f(t)\,dt \quad : i = 1, \dots, n$

converges and define:

$$\int_a^b f(t)\,dt = \int_a^{t_1} f(t)\,dt + \int_{t_1}^{t_2} f(t)\,dt + \dots + \int_{t_n}^{b} f(t)\,dt$$

<u>Example:</u> Consider $\int_0^3 \frac{dx}{(x-1)^{2/3}}$.

The function $(x-1)^{-2/3}$ is not defined at $x=1$.

We have $\int_0^1 \frac{dx}{(x-1)^{2/3}} = 3$

& $\int_1^3 \frac{dx}{(x-1)^{2/3}} = 3\sqrt[3]{2}$

Hence $\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$

$$= 3 + 3\sqrt[3]{2}$$

Example: Prove that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

As e^{-x^2} is continuous for every x

and $\int_0^1 e^{-x^2} dx$ is a proper integral.

We will show that the improper

integral $\int_1^\infty e^{-x^2} dx$ converges.

Note that $\int_1^\infty e^{-x^2} dx \leq \int_1^\infty e^{-x} dx = \frac{1}{e}$.

Hence $\int_1^\infty e^{-x^2} dx$ converges. To

find its value, note that

$$I = \int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-y^2} dy$$

$$\Rightarrow I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy$$

Put $x = r\cos\theta$, $y = r\sin\theta$,

$dx\, dy = r\, dr\, d\theta$

Hence $I^2 = \int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} e^{-r^2} r\,dr\,d\theta$

$$= \frac{\pi}{4} \int_0^{\infty} 2r\, e^{-r^2} dr$$

$$= \frac{\pi}{4}$$

Hence, $I = \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

<u>Example</u>: Find $\Gamma(1/2)$, $\Gamma(3/2)$, $\Gamma(5/2)$, ...

By definition,

$$\Gamma(1/2) = \int_0^{\infty} e^{-x} x^{\frac{1}{2}-1} dx = \int_0^{\infty} \frac{dx}{e^x \sqrt{x}}$$

Put $x = t^2$ to get

$$\Gamma(1/2) = \int_0^{\infty} e^{-t^2} dt = \frac{2\sqrt{\pi}}{2} = \sqrt{\pi}$$

i.e., $\Gamma(1/2) = \sqrt{\pi}$

Now, $\Gamma(3/2) = \Gamma(1/2 + 1) = \frac{1}{2} \cdot \Gamma(1/2)$

$$= \frac{\sqrt{\pi}}{2}$$

Similarly, $\Gamma(5/2) = \frac{3}{2} \cdot \Gamma(3/2)$

$$= \frac{3}{4}\sqrt{\pi}.$$