MA-207 Differential Equation II
S1 - Lecture 3

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Orthogonality

In an inner product space $(V, \langle , \rangle)$
two vectors u and v are orthogonal if $\langle u, v\rangle = 0$.

More generally, a set $S \subset V$ is an orthogonal system if any two vectors in S are orthogonal.

An orthogonal basis is an orthogonal system which is also a basis of V.

Example. $\mathbb{R}^n$ is a vector space with coordinate-wise addition and scalar multiplication. Dot product

$$\langle (a_1, \ldots, a_n), (b_1, \ldots, b_n)\rangle := \sum_{i=1}^{n} a_i b_i$$

defines an inner product on $\mathbb{R}^n$. The standard basis $\{e_1, \ldots, e_n\}$ is an orthogonal basis of $\mathbb{R}^n$.

Example

Let V be a finite-dimensional vector space with an ordered basis $B = \{e_1, \ldots, e_n\}$.

For $u = \sum_{i=1}^{n} a_i e_i$ and $v = \sum_{i=1}^{n} b_i e_i$

$$\langle u, v\rangle := \sum_{i=1}^{n} a_i b_i = [u]_B{}^t . [v]_B$$

defines an inner product on V. Here $[u]_B = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ is the coordinate vector of u w.r.t. basis B.

Further, $\{e_1, \ldots, e_n\}$ is an orthogonal basis of V.

Theorem. Suppose V is a finite dimensional inner product space and $e_1, \ldots, e_n$ is an orthogonal basis. Then for any $v \in V$,

$$v = \sum_{i=1}^{n} \frac{\langle v, e_i\rangle}{\langle e_i, e_i\rangle} e_i$$

Proof. To see this, write $v = \sum_{i=1}^{n} a_i e_i$.

To find a_j, take inner product of v with e_j:

$$\langle v, e_j\rangle = \langle \sum_{i=1}^{n} a_i e_i, e_j\rangle = \sum_{i=1}^{n} a_i \langle e_i, e_j\rangle = a_j \langle e_j, e_j\rangle$$

Thus, $$a_j = \frac{\langle v, e_j\rangle}{\langle e_j, e_j\rangle}$$

Lemma

In a finite-dimensional inner product space, there always exists an orthogonal basis.

You can start with any basis and modify it to an orthogonal basis using Gram-Schmidt orthogonalization.

This result is not necessarily true in infinite-dimensional inner product spaces.

In general, we can only talk of a maximal orthogonal set.

Length of a vector

Let V be an inner product space. For any $v \in V$, define

$$\|v\| := \langle v, v\rangle^{1/2}$$

This is called the norm or length of the vector v. It satisfies the following properties.

- $\|0\| = 0$ and $\|v\| > 0$ if $v \neq 0$
- $\|v + w\| \leq \|v\| + \|w\|$
- $\|av\| = |a|.\|v\|$

for all $v, w \in V$ and $a \in \mathbb{R}$.

Pythagoras theorem : For orthogonal vectors v and w in any inner product space V,

$$\|v+w\|^2 = \|v\|^2 + \|w\|^2$$

Proof. $\|v+w\|^2 = \langle v+w, v+w\rangle$
$= \langle v+w, v\rangle + \langle v+w, w\rangle$
$= \langle v, v\rangle + \langle w, v\rangle + \langle v, w\rangle + \langle w, w\rangle$
$= \langle v, v\rangle + \langle w, w\rangle$, as $\langle v, w\rangle = 0$
$= \|v\|^2 + \|w\|^2$

More generally, for any orthogonal system $\{v_1, \ldots, v_n\}$

$$\|v_1 + \cdots + v_n\|^2 = \|v_1\|^2 + \cdots + \|v_n\|^2$$

Legendre equation

The following ODE is Legendre equation.

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0$$

Here p denotes a fixed real number.
The Legendre equation can also be written as

$$((1-x^2)y')' + p(p+1)y = 0$$

By Existence theorem, power series solution in x exists on the interval $(-1, 1)$.

Put $y = \sum_{0}^{\infty} a_n x^n$ in the Legendre equation.

For $n \geq 0$, the coefficient of x^n gives

$$(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + p(p+1)a_n = 0$$

$$a_{n+2} = \frac{n(n-1) + 2n - p(p+1)}{(n+2)(n+1)} a_n$$

$$= \frac{n(n+1) - p(p+1)}{(n+2)(n+1)} a_n = \frac{-(p-n)(p+n+1)}{(n+2)(n+1)} a_n$$

$$a_2 = \frac{-p(p+1)}{2.1} a_0$$

$$a_4 = \frac{-(p-2)(p+3)}{4.3} a_2 = \frac{p(p-2)(p+1)(p+3)}{4!} a_0$$

Taking $a_0 = 1$ and $a_1 = 0$, the 1st solution $y_1(x)$ is

$$\left[1 - \frac{p(p+1)}{2!} x^2 + \frac{p(p-2)(p+1)(p+3)}{4!} x^4 + \dots\right]$$

Note $y_1(x)$ is an even function.
For $p \in \{0, 2, 4, \dots, -1, -3, -5, \dots\}$, $y_1(x)$ is a polynomial function.

Recall $$a_{n+2} = \frac{-(p-n)(p+n+1)}{(n+2)(n+1)} a_n$$

$$a_3 = \frac{-(p-1)(p+2)}{3.2} a_1$$

$$a_5 = \frac{-(p-3)(p+4)}{5.4} a_3$$

$$= \frac{(p-1)(p-3)(p+2)(p+4)}{5!} a_1$$

So taking $a_0 = 0$ and $a_1 = 1$, we get the second solution

$$y_2(x) := x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-1)(p-3)(p+2)(p+4)}{5!}x^5 + \dots$$

Note $y_2(x)$ is an odd function.

For $p \in \{1, 3, 5, \dots, -2, -4, -6, \dots\}$, $y_2(x)$ is a polynomial function.

The general solution

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

is called the Legendre function.

If $p = m \geq 0$ is an integer, then precisely one of y_1 and y_2 is a polynomial, called the m-th Legendre polynomial $P_m(x)$ if even then

$P_m(x)$ is the solution of

$$(1-x^2)y'' - 2xy' + m(m+1)y = 0$$

Check that $a_m = P_m(1) \neq 0$.

Replacing $P_m(x)$ by $\dfrac{1}{a_m} P_m(x)$, we may assume that $P_m(1) = 1$ for $m \geq 0$.

Now $P_m(x)$ is uniquely defined polynomial of degree m.

Let us write down some of P_m's.

Recall $y_1(x)$ and $y_2(x)$ are defined respectively as

$$1-\frac{m(m+1)}{2!}x^2+\frac{(m(m-2)(m+1)(m+3)}{4!}x^4-\ldots,$$

$$x-\frac{(m-1)(m+2)}{3!}x^3+\frac{(m-1)(m-3)(m+2)(m+4)}{5!}x^5-\ldots$$

Let us write down few Legendre polynomials.

$P_0(x) = 1 \qquad P_1(x) = x$

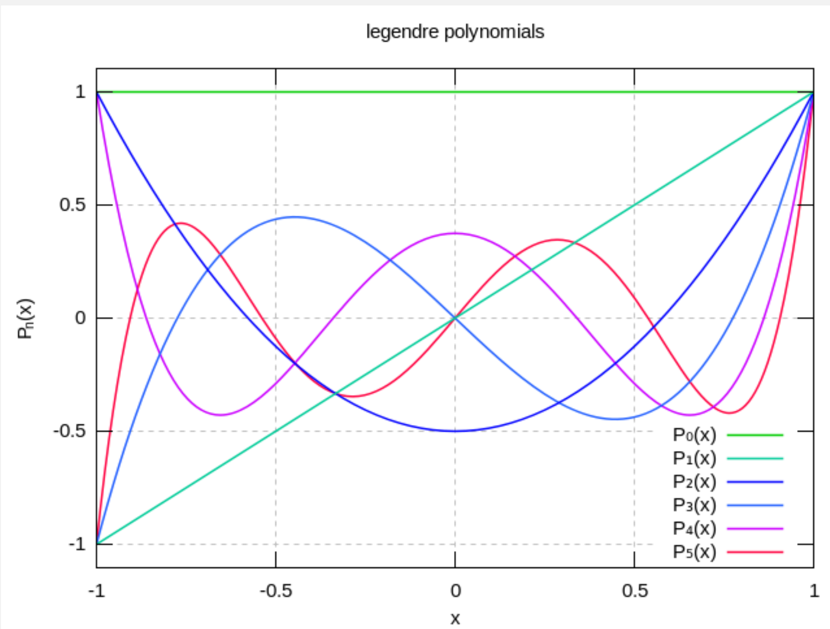
$P_2(x) = (1-3x^2)(\frac{-1}{2}) = \frac{1}{2}(3x^2-1)$

$P_3(x) = (x-\frac{5}{3}x^3)(\frac{-3}{2}) = \frac{1}{2}(5x^3-3x)$

$P_4(x) = (1-10x^2+\frac{35}{3}x^4)(\frac{3}{8}) = \frac{1}{8}(35x^4-30x^2+3)$

$P_5(x) = (x-\frac{14}{3}x^3+\frac{21}{5}x^5)(\frac{15}{8}) = \frac{1}{8}(63x^5-70x^3+15x)$

The graphs of P_m's in the interval $(-1,1)$ are given below.



Pn has n roots in -1 to 1

P_m has exactly m distinct zeros in $(-1,1)$.

2nd soln of Legendre eqn for $p = m \geq 0$ an integer

The second (non polynomial) solution is

$$\boxed{\text{m=0}} \quad x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots = \frac{1}{2}\log\left[\frac{1+x}{1-x}\right]$$

$$\boxed{\text{m=1}} \quad 1 - \frac{x^2}{1} - \frac{x^4}{3} - \frac{x^6}{5} - \ldots = 1 - \frac{x}{2}\log\left[\frac{1+x}{1-x}\right]$$

Fact. Non-polynomial solutions for any integer $m \geq 0$ always have a log factor of the above kind.

Hence they are unbounded near ± 1.

They are called the Legendre functions of the second kind.

Definition. The set $\mathcal{P}(x)$ of all polynomials in the variable x is a vector space. The set

$$\{1, x, x^2, \ldots\}$$

is an infinite basis of $\mathcal{P}(x)$.
Vector space $\mathcal{P}(x)$ carries an inner product

$$\langle f, g \rangle := \int_{-1}^{1} f(x)g(x)\,dx$$

Note that we are integrating over finite interval $[-1, 1]$ to ensures that the integral is always finite.

The norm of a polynomial f is defined by

$$\|f\| := \left(\int_{-1}^{1} f(x)f(x)dx\right)^{1/2}$$

Since $P_m(x)$ is a polynomial of degree m,

$$\{P_0(x), P_1(x), P_2(x), \dots\}$$

is a basis of $\mathcal{P}(x)$.

Orthogonality of Legendre polynomials.

$$\langle P_m, P_n \rangle = \int_{-1}^{1} P_m(x) P_n(x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m = n \end{cases}$$

i.e. Legendre polynomials form an orthogonal basis for the vector space $\mathcal{P}(x)$ and

$$\|P_n(x)\|^2 = \frac{2}{2n+1}$$

Proof. We'll use the technique of derivative-transfer which says that for differentiable functions f and g,

$$f(b)g(b) = f(a)g(a) \implies \int_a^b f g' dx = -\int_a^b f' g dx$$

Since $P_m(x)$ solves the Legendre equation

$$((1-x^2)P_m')' + m(m+1)P_m = 0$$

Multiply by P_n and integrate to get

$$\int_{-1}^{1} ((1-x^2)P_m')' P_n + m(m+1) \int_{-1}^{1} P_m P_n = 0$$

By derivative transfer, we get

$$-\int_{-1}^{1} (1-x^2) P_m' P_n' + m(m+1) \int_{-1}^{1} P_m P_n = 0$$

Interchanging the roles of m and n, we get

$$-\int_{-1}^{1}(1-x^2)P'_m P'_n + n(n+1)\int_{-1}^{1} P_m P_n = 0$$

Subtracting the two identities, we obtain

$$[m(m+1)-n(n+1)]\int_{-1}^{1} P_m P_n = 0$$

If $m \neq n$ are positive integer, then the scalar
$m(m+1)-n(n+1) = (m-n)(m+n+1)$
is non-zero. Therefore, we get

$$\int_{-1}^{1} P_m P_n = 0$$

Thus, P_m and P_n are orthogonal for $m \neq n$.

To show that $\|P_n(x)\|^2 = \dfrac{2}{2n+1}$, we use

Rodrigues formula : $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2-1)^n$

Proof. Let $w(x) = (x^2-1)^n$ and write $D = \dfrac{d}{dx}$.

Then $\qquad (x^2-1)w' - 2nxw = 0$

Use $\qquad D^n(fg) = \displaystyle\sum_{i=0}^{n}\binom{n}{i} D^i(f)D^{n-i}(g)$

$D^{n+1}\left[(x^2-1)w' - 2nxw\right] = 0$, i.e.

$$(x^2-1)D^{n+2}w + (n+1)2xD^{n+1}w + \frac{1}{2}n(n+1)2D^n w$$
$$-2nxD^{n+1}w - 2n(n+1)D^n w = 0,$$

i.e. $(x^2-1)D^{n+2}w + 2xD^{n+1}w - n(n+1)D^n w = 0$

If we write $y(x) = D^n w(x)$, then we get

$$(x^2-1)y'' + 2xy' - n(n+1)y = 0$$

So $y(x)$ is a polynomial solution of n-th Legendre equation. We need to complute $y(1)$. Since $y(x) =$

$$D^n(x^2-1)^n = \sum_{i=0}^{n} \binom{n}{i} D^i((x-1)^n)D^{n-i}((x+1)^n)$$

$$y(1) = \binom{n}{n} D^n((x-1)^n)D^0((x+1)^n)\Big|_{x=1} = n!\,2^n$$

Since $P_n(x)$ is the unique polynomial solution of n-th Legendre equation with $P_n(1) = 1$, we get

$$y(x) = n!2^n P_n(x)$$

Sketch of proof for $\|P_n(x)\|^2 = \dfrac{2}{2n+1}$

Proof. $\displaystyle\int_{-1}^{1} P_n(x)P_n(x)\,dx$

$$= \frac{1}{2^{2n}(n!)^2}\int_{-1}^{1} \frac{d^n}{dx^n}(x^2-1)^n \frac{d^n}{dx^n}(x^2-1)^n\,dx$$

$$= \frac{(-1)^n}{2^{2n}(n!)^2}\int_{-1}^{1} (x^2-1)^n \frac{d^{2n}}{dx^{2n}}(x^2-1)^n\,dx$$

$= \dfrac{(2n)!}{2^{2n}(n!)^2}\displaystyle\int_{-1}^{1}(1-x^2)^n\,dx$, by derivative transfer

put $x = \sin\theta$ and use

$$\int \cos^{2n+1}\theta\,d\theta = \frac{\cos^{2n}\theta\,\sin\theta}{2n+1} + \frac{2n}{2n+1}\int \cos^{2n-1}\theta\,d\theta$$

to complete the proof.

Expansion of polynomial in terms of P_n's

Since $P_n(x)$ is a polynomial of degree n,
$\{P_0(x), P_1(x), P_2(x), \dots\}$ is a basis of $\mathcal{P}(x)$.
If $f(x)$ is a polynomial of degree n, then

$$f(x) = \sum_{k=0}^{n} a_k P_k(x), \qquad a_k \in \mathbb{R}$$

To find a_k, we use orthogonality of P_n's.

$$\langle f(x), P_i(x)\rangle = \langle \textstyle\sum_{k=0}^{n} a_k P_k(x), P_i(x)\rangle$$

$$= \sum_{k=0}^{n} a_k \langle P_k(x), P_i(x)\rangle = a_i \|P_i\|^2$$

Thus, $$a_i = \frac{2i+1}{2}\int_{-1}^{1} f(x)P_i(x)\,dx$$

In particular, $f(x) = x^{2n} = \displaystyle\sum_{i=0}^{2n} a_i P_i(x)$

Since f is an even function, we get

$$f(x) = f(-x) = \sum_{i=0}^{2n} a_i P_i(-x) = \sum_{i=0}^{2n} a_i (-1)^i P_i(x)$$

Since a_i's are unique, as Legendre polynomials are basis for $\mathcal{P}(x)$, we get $a_i = 0$ for i odd. Hence

$$x^{2n} = \sum_{i=0}^{n} a_{2i} P_{2i}(x), \quad a_{2i} = \frac{4i+1}{2}\int_{-1}^{1} x^{2i} P_{2i}(x)\,dx$$

Similarly,

$$x^{2n+1} = \sum_{i=0}^{n} a_{2i+1} P_{2i+1}(x), \quad a_{2i+1} = \frac{4i+3}{2}\int_{-1}^{1} x^{2i+1} P_{2i+1}(x)\,dx$$

Definition. A function $f(x)$ on $[-1,1]$ is square-integrable if

$$\int_{-1}^{1} f(x)f(x)dx < \infty$$

For instance, polynomials, continuous functions, piecewise continuous functions are square-integrable.

The set of all square-integrable functions on $[-1,1]$ is a vector space and is denoted by $L^2([-1,1])$.

For square-integrable functions f and g, we define their inner product by

$$\langle f, g\rangle := \int_{-1}^{1} f(x)g(x)dx$$

Fourier-Legendre series

Theorem. Legendre polynomials form a maximal orthogonal set in $L^2([-1,1])$.

This means if $f \in L^2([-1,1])$ is such that $\langle f, P_n(x)\rangle = 0$ for all $n \geq 0$ then $f = 0$.

To any square-integrable function $f(x)$ on $[-1,1]$, we can associate a series of Legendre polynomials

$$f \sim \sum_{n=0}^{\infty} c_n P_n(x), \quad c_n = \frac{\langle f, P_n\rangle}{\langle P_n, P_n\rangle}$$

This is called the Fourier-Legendre series (or simply the Legendre series) of $f(x)$.