MA 205 CA: All the Beautiful Things

U. K. Anandavardhanan
IIT Bombay

August 13, 2015

Introduction

Last time, we saw two very important theorems. Cauchy's theorem and the Cauchy integral formula. The first said that the integral along a closed curve of a function is zero if the function is holomorphic on and within the curve. The second said:

$$f(z_0) = \frac{1}{2\pi\imath} \int_\gamma \frac{f(z)}{z - z_0} dz$$

if f is holomorphic on and within the simple closed curve γ. We derived Cauchy's theorem by appealing to Green's theorem after assuming Goursat's theorem. We derived CIF from Cauchy's theorem by making use of the computation $\int_\gamma \frac{dz}{z-z_0} = 2\pi\imath$. In fact, we saw that Cauchy's theorem is equivalent to CIF.

Let f be holomorphic at z_0. This means that f is holomorphic in a neighbourhood of z_0. Let $R > 0$ be such that f is holomorphic in $|z - z_0| < R$. Let γ be a circle of radius r with $r < R$ centered at z_0. CIF gives us:

$$f(z) = \frac{1}{2\pi\imath} \int_\gamma \frac{f(w)}{w - z} dw,$$

for any z such that $|z - z_0| < r$. Here, w is over $|w - z_0| = r$. Note that we have chosen $r < R$ so that f is holomorphic on and within γ. The aim is to prove that f is analytic at z_0; i.e., to show that f can be expanded as a power series around z_0 in a neighborhood of z_0. The idea is to get a power series in $(z - z_0)$ from the rhs of CIF. The term in CIF which looks amenable to some manipulation is

$$\frac{1}{w - z}.$$

Also always keep in mind that the only series that we know well is the geometric series! Let's look at it closely.

Now,

$$\begin{aligned}\frac{1}{w-z} &= \frac{1}{w-z_0}\cdot\frac{w-z_0}{w-z}\\ &= \frac{1}{w-z_0}\cdot\frac{1}{1-\left[\frac{z-z_0}{w-z_0}\right]}\\ &= \frac{1}{w-z_0}\cdot\left[1+\left(\frac{z-z_0}{w-z_0}\right)+\left(\frac{z-z_0}{w-z_0}\right)^2+\dots,\right]\end{aligned}$$

since $\left|\frac{z-z_0}{w-z_0}\right| < 1$. We plug this in CIF.

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi\imath}\int_\gamma \frac{f(w)}{w-z}dw \\
&= \frac{1}{2\pi\imath}\int_\gamma \frac{f(w)}{w-z_0}\left[1+\left(\frac{z-z_0}{w-z_0}\right)+\left(\frac{z-z_0}{w-z_0}\right)^2+\dots\right]dw \\
&= \left[\frac{1}{2\pi\imath}\int_\gamma \frac{f(w)}{(w-z_0)}dw\right]+\left[\frac{1}{2\pi\imath}\int_\gamma \frac{f(w)}{(w-z_0)^2}dw\right](z-z_0) \\
&\quad +\left[\frac{1}{2\pi\imath}\int_\gamma \frac{f(w)}{(w-z_0)^3}dw\right](z-z_0)^2+\dots \\
&= \sum_{n=0}^{\infty} a_n(z-z_0)^n,
\end{aligned}
$$

for $|z-z_0|<r$ where

$$
a_n = \frac{1}{2\pi\imath}\int_{|w-z_0|=r}\frac{f(w)}{(w-z_0)^{n+1}}dw.
$$

Remark: Integral of sum needn't be sum of integrals in general, but in the previous slide it can be justified. The key word is "uniform convergence". We'll skip the details.

Thus, we have proved that if f is holomorphic in the disc $|z - z_0| < R$, then,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{1}{2\pi \imath} \int_{|w - z_0| = r} \frac{f(w)}{(w - z_0)^{n+1}} dw,$$

for any $r < R$. Since the power series converges to $f(z)$ for $|z - z_0| < r$, the radius of convergence is at least r. We also know that

$$a_n = \frac{f^{(n)}(z_0)}{n!},$$

whenever $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$. In particular, a_n does not depend on r. Any $r < R$ gives same a_n. Thus the radius of convergence is at least R.

Examples:
(i) $f(z) = \frac{e^z}{\sin z + \cos z}$ expanded as a power series centered at 0 has radius of convergence $= \frac{\pi}{4}$.
(ii)

$$f(z) = \begin{cases} \frac{z}{e^z - 1} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$$

expanded as a power series centered at 0 has radius of convergence $= 2\pi$.
Thus, we have proved:

$$\text{holomorphic} \implies \text{analytic.}$$

Have we proved

once differentiable is always differentiable?

Yes, analytic functions are obviously infinitely differentiable.

Cauchy's Estimate

We have also concluded that if $f : \Omega \to \mathbb{C}$ is holomorphic, and if $\{z \mid |z - z_0| \leq r\} \subset \Omega$, then,

$$f^{(n)}(z_0) = \frac{n!}{2\pi \imath} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

where γ is $|z - z_0| = r$. Now suppose f is holomorphic in $|z - z_0| < R$ and suppose f is bounded by $M > 0$ there. Can apply ML inequality in the above formula for each $r < R$ to get

$$|f^{(n)}(z_0)| \leq \left(\frac{n!}{2\pi}\right) \frac{M}{r^{n+1}} \cdot 2\pi r = \frac{n! M}{r^n}.$$

Since this is true for any $r < R$, we get,

$$|f^{(n)}(z_0)| \leq \frac{n! M}{R^n}.$$

This is called Cauchy's estimate.

Liouville's Theorem

A function defined all over $\mathbb{C}$ is called entire if it is holomorphic everywhere in $\mathbb{C}$. Examples? Polynomials, $\exp(z)$, $\sin z$, $\cos z$, etc. Clearly, sums and products of entire functions are entire. The fact that the function is defined and holomorphic everywhere puts strong restrictions on the function. For instance, we have the so called Liouville's theorem, which says:

a bounded entire function is a constant.

A non-constant entire function has to be unbounded. As we have seen $\exp(z)$ takes all values in $\mathbb{C}$ except 0, sin and cos are surjective, in particular these are all unbounded. Polynomials are also clearly unbounded. (Check this).

Proof of Liouville's theorem: Suppose $|f(z)| \leq M$ for all $z \in \mathbb{C}$. We need to show that f is a constant. We'll show this by showing that $f' \equiv 0$. It's enough to show $|f'(z)|$ can be made arbitrarily small. It's enough if we have bounds for this which can be shown to be arbitrarily small. Do we know bounds for derivatives? Yes, Cauchy's estimate.

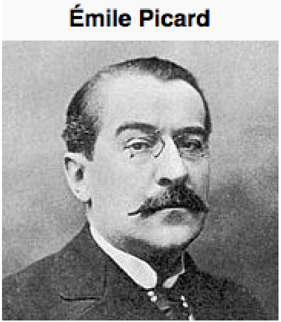
$$|f'(z)| \leq \frac{M}{R},$$

if f is holomorphic in a disc with center z and radius R. But R can be as large as we want, since f is entire. So, $f'(z) = 0$ for all $z \in \mathbb{C}$ and hence f is a constant.

Range of an Entire Function

Remark: Liouville says that the range of a non-constant entire function cannot be bounded. It has to be big enough. How big it should be? Can it be contained in the upper half plane, for example? After all, it's unbounded. Answer is No. Upper half plane can be mapped to a bounded domain via a holomorphic one-to-one function. (Check that $f(z) = \frac{z-\imath}{z+\imath}$ maps the upper half plane bijectively onto the open unit disc.) So this map composed with the given entire function is an entire function with bounded image and hence a constant. Since the second map is one-to-one, the given entire function has to be a constant. So range has to be bigger than something like the upper half plane. Should it be the other extreme; the whole of $\mathbb{C}$? Needn't be, as $\exp(z) + z_0$ misses z_0. So range can be $\mathbb{C}$ minus one point. Can it be $\mathbb{C}$ minus two points? Can you think of such an entire function? You wouldn't get one such. This is what's called the little Picard theorem: a non-constant entire function misses at most one point. It's a deep theorem, and the proof is beyond us.

Liouville (1809-1882) & Picard (1856-1941); Wiki



Besides his academic achievements, he was very talented in organisational matters. He was the first to read, and to recognize, the importance of the unpublished work of Evariste Galois which appeared in his journal in 1846.

Liouville was also involved in politics for some time, and he became a member of the Constituting Assembly in 1848. However, after his defeat in the legislative elections in 1849, he turned away from politics.

"A striking feature of Picard's scientific personality was the perfection of his teaching, one of the most marvellous, if not the most marvellous, that I have ever known." (Hadamard)

Fundamental Theorem of Algebra

Here's the power of complex analysis. We'll deduce FTA as a corollary to Liouville's theorem. Let $p(z)$ be a non-constant polynomial. We need to show that there is z_0 such that $p(z_0) = 0$. Suppose not; i.e., $p(z) \neq 0$ for any $z \in \mathbb{C}$. If something is never zero, you are itching to do what to it? Take its reciprocal! So let

$$f(z) = \frac{1}{p(z)}.$$

Is this entire? Yes, it is. The proof will be over if we can show that $f(z)$ is bounded. (Why?) So we'll show that f is bounded. Since f is a continuous function, it is bounded on any closed and bounded set of the plane. This is what you called the extreme value theorem in MA 105. In particular f is bounded in any disc of any finite radius with center at 0. Then we only need to show that there exists $R > 0$ such that f is bounded for $|z| > R$, since EVT takes care of $|z| \leq R$.

To this end, look at $\lim_{z\to\infty} f(z)$. We know this if we know $\lim_{z\to\infty} p(z)$. If $p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0$, then,

$$\lim_{z\to\infty} p(z) = \lim_{z\to\infty} z^n[1 + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n}] = \infty.$$

Thus, $\lim_{z\to\infty} f(z) = 0$. This means that $|f(z)|$ can be made as small as we want by taking $|z|$ sufficiently large. In particular, there is an $R > 0$ such that $|f(z)| < 1$ for $|z| > R$. Together with EVT, this shows that f is bounded on $\mathbb{C}$, and hence a constant by Liouville. So $p(z)$ is a constant. Contradiction, since we started with a non-constant polynomial. So there exists z_0 such that $p(z_0) = 0$.