HW (4):

3 b) Consider any two spanning trees of the given graph G. Let the trees be T_1 and T_2.

Let the set of all edges of $T_1 = E_1$, the set of edges for $T_2 = E_2$.

Consider the set $E_1 - E_2$ and $E_2 - E_1$.

~~These are two sets such that the edges in one set are~~

If $k \in E_1 - E_2$ and we remove k from the tree T_1, we get two seperate trees T_{11} and T_{12}.

Now, we claim that $\exists$ at least one edge $l \in E_2 - E_1$, which, when put back into the ~~graph~~ tree (instead of k) gives us another tree.

Proof for claim: Suppose there is no such l, then there are two possibilities:

(1) $l \notin E_2$: This is impossible since removal of k isolates at least one node, and, by def^n, each node must be connected to atleast one other node in a tree.

(2) $l \in E_2$, $l \notin E_2 - E_1$: If this is true, then $l \in E_2 \cap E_1 \Rightarrow l \in E_1 \Rightarrow k = l$

But this is impossible since $k \in E_1 - E_2$ by hypothesis.

$\therefore \exists \; l \in E_2 - E_1$ s.t $T_1 + l - k$ is a tree.

Let us call ~~this~~ $T_1 + l - k = T'$. Now, the argument that we put for T_1, T_2 can be applied to T', T_2 but we will have one less element (edge) in $E' - E_2$ and $E_2 - E'$.

Inductively, we can therefore write

$T_2 = (T_1 + l_1 - k_1) + (l_2 - k_2) + \ldots + (l_m - k_m)$

where adding each $l_i - k_i$ results in a spanning tree. $\therefore$ Any node of graph G is connected to another node, i.e G is connected.

(Note that $|E_1 - E_2| = |E_2 - E_1| = m$ as the number of edges in a spanning tree is always $|N| - 1$)