MA-207 Differential Equation II

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18th September, 2017
S1 - Lecture 1

Some Class Policies

EVALUATION: 50 marks are waiting to be earned:

Two In-Tutorial Quizzes	2×2.5 marks
Main Quiz	13 marks
Final	32 marks
Total	**50** marks

ATTENDANCE:

- Attendance in the first week of classes is mandatory.
- Attendance $< 80\% \implies$ you *may* be awarded a DX grade.
- You may give biometric attendance from 5:23 to 5:45.

ACADEMIC HONESTY: Be honest. Do not violate the academic integrity of the Institute. Any form of academic dishonesty will invite severe penalties.

Books

Elementary differential equations with boundary value problems by William F. Trench (available online)

Differential Equations with Applications and Historical Notes by George F. Simmons

Welcome to MA 207, a sequel to MA 108.

Since most of the functions encountered in MA 108 were elementary, i.e. algebraic or transcendental functions, let us begin by recalling their definitions.

An algebraic function is a polynomial function, e.g.

$$x^3 + 3x + 2$$

rational function or equivalently quotient of polynomial functions, e.g.

$$\frac{x^3 + 3x + 2}{x^5 + 2x^3 + 5}$$

or more generally, any function $y = f(x)$ that satisfies an equation of the form

$$P_n(x)y^n + P_{n-1}(x)y^{n-1} + \ldots + P_1(x)y + P_0(x) = 0$$

for some n, where each $P_i(x)$ is a polynomial.

The elementary functions are algebraic functions defined on the last frame, trigonometric functions, e.g.

$$\sin x, \quad \cos x, \quad \tan x$$

inverse trigonometric functions, e.g.

$$\sin^{-1} x, \quad \cos^{-1} x, \quad \tan^{-1} x$$

exponential and logarithmic functions, e.g.

$$e^{x^2}, \quad \log(x^2 + x + 1)$$

and all other functions that can be constructed from these functions by adding, subtracting, multiplying, dividing or forming a composition of such functions. Thus

$$y = tan \left[\frac{xe^{1/x^2} + \tan^{-1}(1 + x^2) + \sqrt{x^2 + 3}}{\sin x \cos 2x - \sqrt{\log x} + x^{3/2}} \right]^{1/3}$$

is an elementary function.

Beyond elementary functions lie the special functions, e.g. Gamma function, Beta function, Riemann zeta function etc.

Definition

The Riemann zeta function is defined on the set $\{s \in \mathbb{C} \,|\, \mathrm{Re(s)} > 1\}$ by

$$\zeta(s) := \sum_{n \geq 1} \frac{1}{n^s}$$

It is a non trivial theorem that the zeta function extends to the whole plane as a meromorphic function.

$s \in \{-2, -4, \ldots\}$ are zeros of $\zeta(s)$ called trivial zeros.

The Riemann hypothesis states that all the non-trivial zeros of the zeta function lie on the line $\mathrm{Re}(s) = \frac{1}{2}$.

This is one of the millennium problems and has a prize of 1 million US dollars.

Large number of special functions arise as solutions of 2nd order linear ODE.

Suppose we want to solve

$$y'' + y = 0$$

Then elementary functions $y = \sin x$ and $y = \cos x$ are solutions.

Suppose we want to solve

$$xy'' + y' + xy = 0$$

This equation can not be solved in terms of elementary functions.

In fact 2nd order linear ODE with constant coefficients can be solved in terms of elementary functions.

There is no other known class of 2nd order linear ODE which can be solved in terms of elementary functions.

If we know one solution $y_1(x)$ of ODE

$$y'' + p(x)y' + q(x)y = 0$$

with $p(x), q(x)$ continuous, then we can try to use the method of variation of parameters to find another linearly independent solution, i.e. put

$$y_2 = u(x)y_1(x)$$

in ODE and solve for $u(x)$.

Question. How to find the 1st solution?

For this, we will solve our ODE in terms of power series.

Let us review power series, which is used throughout in this course.

Definition

For real numbers $x_0, a_0, a_1, a_2, \ldots$, an infinite series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n := a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots.$$

is called a power series in $x - x_0$ with center x_0.
The power series converges at a point $x = x_1$ if the limit

$$\lim_{N \to \infty} \sum_{n=0}^{N} a_n (x_1 - x_0)^n$$

exists and is finite.
In this case, the sum of the series is the value of the limit.

If the series does not converge at x_1, i.e. either limit does not exist or it is $\pm\infty$, then we say the power series diverges at x_1.

A power series always converges at its center $x = x_0$.

Theorem

For any power series,

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

exactly one of these statements is true.

1. *The power series converges only for $x = x_0$.*
2. *The power series converges for all values of x.*
3. *There is a positive number $0 < R < \infty$ such that the power series converges if $|x - x_0| < R$ and diverges if $|x - x_0| > R$.*

R is called the radius of convergence of the power series.

We define $R = 0$ in case (i)
and $R = \infty$ in case (ii).

Question. How to compute the radius of convergence?

Theorem

- *(Ratio test) Assume $a_n \neq 0$ for all n and*

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

- *(Root test)* $\limsup_{n\to\infty} |a_n|^{1/n} = L$

Then radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is $\boxed{R = 1/L}$.

For $L = 0$, we get $R = \infty$
and for $L = \infty$, we get $R = 0$.

Theorem

Let $R > 0$ be the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Then the power series converges (absolutely) for all $x \in (x_0 - R, x_0 + R)$.

For $R = \infty$, we write $(x_0 - R, x_0 + R) = (-\infty, \infty) = \mathbb{R}$.

The open interval $(x_0 - R, x_0 + R)$ is called the interval of convergence of the power series.

Example

Find the radius of convergence and interval of convergence (if $R > 0$) of the following series.

$$(i)\ \sum_{0}^{\infty} n! x^n \quad (ii)\ \sum_{10}^{\infty} (-1)^n \frac{x^n}{n^n} \quad (iii)\ \sum_{0}^{\infty} 2^n n^3 (x-1)^n$$

$$(i)\ \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{n!}\right| = \lim_{n\to\infty} (n+1) = \infty$$

So $R = 0$ in case (i).

In case (ii), $R = \infty$ so interval of convergence is $(-\infty, \infty)$

In case (iii) $R = 1/2$ so interval of convergence is $(1/2, 3/2)$.

Theorem

Let R be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$. *We assume* $\boxed{R > 0}$

- *We can define a function $f : (x_0 - R, x_0 + R) \to \mathbb{R}$ by*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

- *f is infinitely differentiable $\forall\ x \in (x_0 - R, x_0 + R)$.*
- *The successive derivatives of f can be computed by differentiating the power series term-by-term, i.e.*

$$f'(x) = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1} \qquad \ldots$$

$$f^{(k)}(x) = \sum_{n=0}^{\infty} n(n-1)\ldots(n-k+1)\, a_n (x - x_0)^{n-k}$$

Theorem (continued ...)

- *The power series representing the derivatives* $f^{(n)}(x)$ *have same radius of convergence* R.
- *We can determine the coefficients* a_n *as*

$$f(x_0) = a_0, \quad f'(x_0) = a_1, \quad f''(x_0) = 2a_2, \ldots$$

In general,

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

- *We can also integrate the function* $f(x) = \sum_0^\infty a_n (x - x_0)^n$

term-wise i.e. if $[a, b] \subset (x_0 - R, x_0 + R)$, *then*

$$\int_a^b f(x)\, dx = \sum_{n=0}^\infty a_n \int_a^b (x - x_0)^n\, dx = \sum_0^\infty \frac{a_n}{n+1}(x - x_0)^{n+1}$$

Example (Power series representation of elementary functions)

(i) $e^x = \sum_0^\infty \frac{x^n}{n!} \qquad -\infty < x < \infty$

(ii) $\sin x = \sum_0^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad -\infty < x < \infty$

(iii) $\frac{1}{1-x} = \sum_0^\infty x^n \qquad -1 < x < 1$

(iv) $\frac{d}{dx}(\sin x) = \sum_0^\infty (-1)^n \frac{d}{dx}\left(\frac{x^{2n+1}}{(2n+1)!}\right)$

$$= \sum_0^\infty (-1)^n \frac{x^{2n}}{(2n)!} = \cos x$$

Theorem

(i) Power series representation of f in an open interval I containing x_0 is unique, i.e. if

$$f(x) = \sum_0^\infty a_n (x - x_0)^n = \sum_0^\infty b_n (x - x_0)^n$$

for all $x \in I$, then $a_n = b_n \; \forall \; n$.
(ii) If

$$\sum_0^\infty a_n (x - x_0)^n = 0$$

for all $x \in I$, then $a_n = 0$ for all n.

Proof. (i)

$$a_n = \frac{f^{(n)}(x_0)}{n!} = b_n \qquad \text{for all} \quad n.$$

It is clear that (ii) follows from (i).

algebraic operations on power series

Definition

If $\quad f(x) = \sum_0^\infty a_n (x - x_0)^n \qquad g(x) = \sum_0^\infty b_n (x - x_0)^n$

have radius of convergence R_1 and R_2 respectively, then

$$c_1 f(x) + c_2 g(x) := \sum_0^\infty (c_1 a_n + c_2 b_n)(x - x_0)^n$$

has radius of convergence *atleast* $\boxed{R = \min \{R_1, R_2\}}$ for $c_1, c_2 \in \mathbb{R}$.

Further, we can multiply the series as if they were polynomials, i.e.

$$f(x)g(x) = \sum_0^\infty c_n (x - x_0)^n; \quad c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$$

It also has radius of convergence *atleast* R.

Example

Find the power series expansion for $\cosh x$ in terms of powers of x^n.

$$\begin{aligned}\cosh x &= \frac{1}{2}e^x + \frac{1}{2}e^{-x}\\ &= \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{n!} + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}\\ &= \sum_{n=0}^{\infty}\frac{1}{2}\left[1+(-1)^n\right]\frac{x^n}{n!}\\ &= \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}\end{aligned}$$

Since radius of convergence for Taylor series of e^x and e^{-x} are ∞, the power series expansion of $\cosh x$ is valid on $\mathbb{R}$.

Shifting the summation index

If $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n \implies f'(x) = \sum_{n=1}^{\infty} na_n(x-x_0)^{n-1}$

Let us rewrite the series for $f'(x)$ in powers of $(x-x_0)^n$.

Put $r = n-1$, we get

$$f'(x) = \sum_{r=0}^{\infty}(r+1)a_{r+1}(x-x_0)^r$$

Similarly,

$$\begin{aligned}f^{(k)}(x) &= \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)a_n(x-x_0)^{n-k}\\ &= \sum_{n=0}^{\infty}(n+k)(n+k-1)\dots(n+1)a_{n+k}(x-x_0)^n\end{aligned}$$

In general, $\left[\sum_{n=n_0}^{\infty} b_n(x-x_0)^{n-k} = \sum_{n=n_0-k} b_{n+k}(x-x_0)^n\right]$

Example

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Write $(x-1)f''$ as a power series in which a general term is constant multiple of x^n.

$$
\begin{aligned}
(x-1)f'' &= xf'' - f'' \\
&= x\left(\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}\right) - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \\
&= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \\
&= \sum_{n=1}^{\infty} (n+1)n a_{n+1} x^n - \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n \\
&= \sum_{n=0}^{\infty} \left[(n+1)n a_{n+1} - (n+2)(n+1)a_{n+2}\right] x^n
\end{aligned}
$$

Example

Suppose

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

for all x in an open interval I containing $x_0 = 1$.

- Find the power series of y' and y'' in terms of $x-1$ in the interval I. Use these to express the function

  $$(1+x)y'' + 2(x-1)^2 y' + 3y$$

  as a power series in $x-1$ on I.
- Find necessary and sufficient conditions on the coefficients a_n's, so that $y(x)$ is a solution of the ODE

  $$(1+x)y'' + 2(x-1)^2 y' + 3y = 0$$

Example (Continue ...)

Solution. Write the ODE in $(x-1)$, i.e.

$$(1+x)y'' + 2(x-1)^2 y' + 3y = (x-1)y'' + 2y'' + 2(x-1)^2 y' + 3y$$

Express each of $(x-1)y''$, $2y''$, $2(x-1)^2 y'$ and $3y$ as a power series in powers of $(x-1)$ and add them.

$$\begin{aligned}
(x-1)y'' = (x-1)\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} &= \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1} \\
&= \sum_{n=1}^{\infty}(n+1)na_{n+1}(x-1)^n \\
&= \sum_{n=0}^{\infty}(n+1)na_{n+1}(x-1)^n
\end{aligned}$$

Example (Continue ...)

$$\begin{aligned}
2y'' &= \sum_{n=2}^{\infty} 2n(n-1)a_n(x-1)^{n-2} \\
&= \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}(x-1)^n
\end{aligned}$$

$$\begin{aligned}
2(x-1)^2 y' = 2(x-1)^2\sum_{n=1}^{\infty} na_n(x-1)^{n-1} &= \sum_{n=1}^{\infty} 2na_n(x-1)^{n+1} \\
&= \sum_{n=2}^{\infty} 2(n-1)a_{n-1}(x-1)^n \\
&= \sum_{n=0}^{\infty} 2(n-1)a_{n-1}(x-1)^n \quad (a_{-1}=0)
\end{aligned}$$

Example (Continue ...)

We have

$$(x-1)y'' = \sum_{n=0}^{\infty}(n+1)na_{n+1}(x-1)^n$$

$$2y'' = \sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}(x-1)^n$$

$$2(x-1)^2y' = \sum_{n=0}^{\infty}2(n-1)a_{n-1}(x-1)^n \quad (a_{-1}=0)$$

Now we get

$$(x-1)y''+2y''+2(x-1)^2y'+3y = \sum_{n=0}^{\infty}b_n(x-1)^n$$

where

$$b_n = (n+1)na_{n+1}+2(n+2)(n+1)a_{n+2}+2(n-1)a_{n-1}+3a_n$$

Example (Continue ...)

For the second part,

$$y(x) = \sum_{0}^{\infty}a_n(x-1)^n$$

is the solution of the ODE

$$(x-1)y''+2y''+2(x-1)^2y'+3y=0$$

on the open interval I containing 1 if and only if

$$\sum_{n=0}^{\infty}b_n(x-1)^n = 0 \text{ on } I \Longleftrightarrow b_n = 0 \quad \text{for all} \quad n$$

i.e. a_n's satisfy the following recursive relation

$$(n+1)na_{n+1}+2(n+2)(n+1)a_{n+2}+2(n-1)a_{n-1}+3a_n = 0$$

for all n.

Correction: added after the class

In the class, we stated ratio test with $\limsup$ instead of using $\lim$ which we have corrected now.

The $\limsup$ definition in the ratio test does not give radius of convergence, though it gives convergence of the series for $x \in (x_0 - R, x_0 + R)$, where $R = 1/L$ and $L = \limsup |a_{n+1}|/|a_n|$.

For an example, take the series

$$1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + \dots.$$

Here the coefficient of x^n is 1 if n is even and 2 is n is odd. Now

$$\limsup \frac{a_{n+1}}{a_n} = \lim b_n, \qquad b_n = \sup\{\frac{a_{n+1}}{a_n}, \frac{a_{n+2}}{a_{n+1}}, \dots, \}$$

Since all $b_n = 2$ for all n, we get

$$\limsup \frac{a_{n+1}}{a_n} = 2 \implies R = 1/2.$$

Correction : added after the class

Note that the series

$$1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + \dots = \sum_0^\infty x^n + \sum_0^\infty x^{2n+1}$$

Since both series have radius of convergence 1, then sum has radius of convergence atleast 1, whereas we found $R = 1/2$.

This example shows that the ratio test using $\limsup$ does not give radius of convergence, in general.

The root test definition using $\limsup$ is correct.

In our example,

$$\limsup a_n^{1/n} = \limsup\{1, 2^{1/1}, 1, 2^{1/3}, \dots, 1, 2^{1/2n-1}, \dots\} = 1$$

Hence the radius of convergence of the series is exactly 1.

Corrections: added after the class

If two power series

$$f(x) = \sum_0^\infty a_n x^n \quad \text{and} \quad g(x) = \sum_0^\infty b_n x^n$$

have radius of convergence R_1 and R_2 respectively, then their sum

$$cf(x) + dg(x) = \sum_0^\infty (ca_n + db_n)x^n$$

has radius of convergence **atleast** minimum of $\{R_1, R_2\}$. Similarly, their product

$$f(x)g(x) = \sum_0^\infty c_n x^n, \qquad c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$$

has radius of convergence **atleast** minimum of $\{R_1, R_2\}$.
In the class slide, the word "atleast" was missing.

Corrections: added after the class

We can see a simple example as follows:

(1) If $f = \sum_0^\infty a_n x^n$ has radius of convergence $R > 0$, then $-f = \sum_0^\infty -a_n x^n$ has same radius of convergence $R > 0$.
But $f + (-f) = 0$ has radius of convergence ∞.

(2) We will see that the Taylor series expansion in x for functions

$$f = \frac{x-2}{x-1}, \quad \text{and} \quad g = \frac{x-1}{x-2}$$

have radius of convergence 1 and 2 respectively. But the product $fg = 1$ has radius of convergence ∞.

added after the class: Not important for exam purpose

Consider a sequence of functions f_n defined from $(-1, 1)$ to $\mathbb{R}$. Assume that for all $x \in (-1, 1)$, $f(x)$ defined by $\lim_{n\to\infty} f_n(x)$ exists.
We say that $f(x)$ is the pointwise limit of $f_n(x)$.

Defn. We say the $f_n(x)$ converges to $f(x)$ uniformly on $(-1, 1)$ if given $\epsilon > 0$, there exists N such that for all $n \geq N$ and for all $x \in (-1, 1)$, we have that $f_n(x)$ belongs to the interval $(f(x) - \epsilon, f(x) + \epsilon)$.

Geometrically, the graph of all f_n for $n \geq N$ lies in the ϵ neighbourhood of the graph of $f(x)$.

If we want to define pointwise convergence or uniform convergence of series $\sum_0^\infty f_n(x)$, we can define it using the sequence of partial sums of the series.

added after the class: Not important for exam purpose

Defn. We say that the series $\sum_0^\infty f_n(x)$ converges to the function $g(x)$ pointwise or uniformly if the sequence of functions $s_n(x)$ converges to $g(x)$ pointwise or uniformly, where $s_n(x) = f_0(x) + \ldots + f_n(x)$.

We have seen that the sequence of functions $f_n(x) = \cos^n(x)$ on $(-\pi/2, \pi/2)$ converges pointwise, but not uniformly.

If we want an example of a series converging pointwise, but not uniformly, then we can take

$$g_0 = f_0, \quad g_1 = f_1 - f_0, \quad \ldots, \quad g_n = f_n - f_{n-1}, \ldots,$$

Then partial sums

$$s_n = \sum_0^n g_i(x) = f_n$$

Hence above sequence gives an example of a series converging pointwise, but not uniformly.

added after the class: Not important for exam purpose

We will state some results for uniform convergence.

Theorem

Assume f_n are continuous functions from closed interval $[a,b]$ to $\mathbb{R}$. If $\sum_0^\infty f_n(x)$ converges to $f(x)$ uniformly, then limit function $f(x)$ is also continuous. Further,

$$\int_a^b f(x)\,dx = \int_a^b \sum_0^\infty f_n(x)\,dx = \sum_0^\infty \int_a^b f_n(x)\,dx$$

added after the class : Not important for exam purpose

Theorem

Assume f_n are differentiable functions from closed interval $[a,b]$ to $\mathbb{R}$. Assume $\sum_0^\infty f_n(x)$ converges to $f(x)$ pointwise in $[a,b]$ and further $\sum_0^\infty f_n'(x)$ converges uniformly on $[a,b]$. Then

(1) $\sum_0^\infty f_n(x)$ converges uniformly on $[a,b]$ to $f(x)$ and

(2) $\sum_0^\infty f_n'(x) = f'(x)$.

i.e. limit function $f(x)$ is differentiable and its derivative can be obtained by term-wise differentiation.