MA 205 Complex Analysis: Power Series

July 25, 2017

In the last lecture, you saw holomorphic functions in some detail. If $f = u + \imath v$ is holomorphic in Ω, then (i) both u and v satisfy CR equations, and (ii) $f(x, y) = (u(x, y), v(x, y))$ is real differentiable. We also saw that though neither (i) nor (ii) is sufficient to guarantee holomorphicity, both (i) and (ii) together do guarantee holomorphicity of f. We also studied harmonic functions which are closely related to holomorphic functions. The notion of a harmonic conjugate of a given harmonic function was defined, and it was stated that the existence of a harmonic conjugate for every harmonic function is guaranteed if and only if the domain is "nice". The precise mathematical condition replacing "nice" is **simply connected**. We shall come to this notion soon.

Complex differentiability implies:

- real differentiability

- real and imaginary parts satisfy CR.

What if we assume both these? Can we then say f is complex differentiable? <u>And the answer is Yes</u>.

<u>Proof</u>: Since $f = u + \imath v$ is real differentiable,

$$\lim_{(x,y)\to(a,b)} \frac{\left\| \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} - \begin{bmatrix} u(a,b) \\ v(a,b) \end{bmatrix} - \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix} \right\|}{\|(x-a, y-b)\|} = 0.$$

Note that the numerator is nothing but

$$|f(z) - f(z_0) - \alpha(x-a) - \beta(y-b)|,$$

where $\alpha = u_x + \imath v_x$, $\beta = u_y + \imath v_y$.

Cauchy-Riemann Equations

Define

$$\eta(z) = \frac{f(z) - f(z_0) - \alpha(x-a) - \beta(y-b)}{z - z_0}.$$

Observe that

$$\lim_{z \to z_0} \eta(z) = 0.$$

Thus,

$$f(z) - f(z_0) = \alpha(x-a) + \beta(y-b) + \eta(z)(z - z_0),$$

with $\eta(z) \to 0$ as $z \to z_0$. Write this as

$$f(z) - f(z_0) = \frac{\alpha - \imath\beta}{2}(z - z_0) + \frac{\alpha + \imath\beta}{2}\overline{z - z_0} + \eta(z)(z - z_0).$$

Thus,

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{\partial f}{\partial z}(z_0) + \frac{\partial f}{\partial \bar{z}}(z_0)\frac{\overline{z - z_0}}{z - z_0} + \eta(z).$$

Question is whether the lhs limit exists as $z \to z_0$. This exists if and only if the rhs limit exists. Since,

$$\lim_{z \to z_0} \frac{\overline{z - z_0}}{z - z_0}$$

does not exist (why?) and $\lim_{z \to z_0} \eta(z)$ exists, this happens if and only if

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0.$$

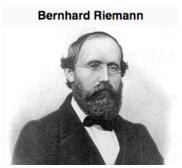i.e., CR equations are satisfied at z_0. Also, if this is the case,

$$f'(z_0) = \frac{\partial f}{\partial z}(z_0),$$

since $\lim_{z \to z_0} \eta(z) = 0$.

Augustin-Louis Cauchy

Bernhard Riemann

"More concepts and theorems have been named for Cauchy than for any other mathematician (in elasticity alone there are sixteen concepts and theorems named for Cauchy). Cauchy was a prolific writer; he wrote approximately eight hundred research articles and five complete textbooks."

Berhard Riemann(1826-1866) is one of the greatest mathematician of all times. After Newton, he is possibly the mathematician who has had the greatest impact on the development of mathematics ever since. In a relatively short mathematical career he wrote under 20 paper, most of them worth their weight in gold. His areas of contribution include - Real Analysis, Complex Analysis, Function Theory, Riemannian Geometry, Number theory - most importantly the distribution of primes, various areas of mathematical physics ... A conjecture he made in 1859, now known as the Riemann hypothesis is considered to be the most important open problem today. Proving it would also fetch one a million dollars from the Clay Mathematics Foundation !

Polynomials

Today, we'll discuss the so called analytic functions. To warm up, let's first look at the simplest of all functions. What's the most trivial function? $f(z) = a_0$, i.e, constant functions. The next easiest class of functions are polynomials:

$f(z) = a_0 + a_1 z + \ldots + a_n z^n$, $a_i \in \mathbb{C}$.

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$f(z) = a_0 + a_1 z + \ldots + a_n z^n$, $a_i \in \mathbb{C}$. These are clearly holomorphic everywhere in $\mathbb{C}$. The same polynomial $f(z)$ can be expanded along any point z_0. That is, $f(z)$ can be written as

$$b_0 + b_1(z - z_0) + \ldots + b_n(z - z_0)^n.$$

The complex numbers b_i can be easily calculated. Of course, you could expand powers of $(z - z_0)$ using binomial theorem and compare coefficients to get b_i in terms of a_i.

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The complex numbers b_i can be easily calculated. Of course, you could expand powers of $(z - z_0)$ using binomial theorem and compare coefficients to get b_i in terms of a_i. Perhaps a smarter way would be to notice that $b_i = \frac{f^{(i)}(z_0)}{i!}$, which then can be calculated from $f(z) = \sum_{i=0}^{n} a_i z^i$.

Power Series

A polynomial, by definition, is a "finite" polynomial; i.e., it comes with a finite degree. As the next class of functions, we consider functions defined by their power series. They look like:

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or more generally,

$$\sum_{i=0}^{\infty} a_i (z - z_0)^i.$$

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Of course one has to be careful; there are convergence issues. For example, $f(z) = 1 + z + z^2 + \dots$ makes sense for all z such that $|z| < 1$, but not when $|z| > 1$. (Why?) We say that this power series has radius of convergence $= 1$.

It's a beautiful fact that the radius of convergence exists for any power series; i.e., there exists R such that $\displaystyle\sum_{i=0}^{\infty} a_i(z - z_0)^i$ converges when $|z - z_0| < R$, and diverges when $|z - z_0| > R$.

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Power Series

Before we prove the existence of radius of convergence, let's recall a few definitions and observations.

We write $a = \sum\limits_{n=1}^{\infty} a_i$, $a_i \in \mathbb{C}$, if the sequence of partial sums $\{s_n\}$, where $s_n = a_1 + \ldots + a_n$, converges, and $\lim\limits_{n \to \infty} s_i = a$. The series $\sum\limits_{n=1}^{\infty} a_i$ is said be <u>absolutely convergent</u> if $\sum\limits_{n=1}^{\infty} |a_i|$ is convergent.

<u>Exercise</u>:

1. Absolute convergence $\implies$ convergence.

2. (Comparison Test) If $\sum\limits_{n=1}^{\infty} b_i$ is absolutely convergent, and if $|a_i| \leq |b_i|$ for all large enough i, then $\sum\limits_{n=1}^{\infty} a_i$ is absolutely convergent.

Easy Observation: If a power series $\sum_{i=0}^{\infty} a_i z^i$ converges for $z = z_0$, then it converges absolutely for any z with $|z| < |z_0|$.

Recall **Supremum**: Let $\{x_n\}$ be a sequence of real numbers. We say that a real number M is the supremum of this sequence if every term of the sequence is less than or equal to M and there exists terms of the sequence which are arbitrarily close to M. Equivalently it is the smallest real number having the property that it is greater than or equal to all the terms of the sequence. The supremum may or may not exist and if it exists it may or may not be equal to any of the terms of the sequence (that is the supremum may not be attained by the sequence). For example, supremum of the sequence 1.4, 1.41, 1.414, 1.4142 ,... is $\sqrt{2}$.

Upper Limit/ Limit Supremum: for a sequence of <u>real</u> numbers $x_1, x_2, \ldots$, let y_n be the supremum of the set $\{x_n, x_{n+1}, \ldots\}$. Then the sequence $y_1, y_2, \ldots$ is a monotonically decreasing sequence which diverges to ∞ or has a finite limit. This is called the upper limit (also called limit superior, denoted lim sup) of the sequence $\{x_i\}$. It can be ∞. If limit exists, then the upper limit coincides with the usual limit.

Examples:

1. the sequence $1, 2, 3, \ldots$ has upper limit ∞.
2. the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$ has upper limit 0.
3. the sequence $1, -1, 1, -1, \ldots$ has upper limit 1.

Power Series

Theorem (Cauchy's Root Test)

For a series $\sum_{n=1}^{\infty} a_i$ of complex numbers, let $C = \limsup_{i \to \infty} \sqrt[i]{|a_i|}$. Then the series converges absolutely if $C < 1$ and it diverges if $C > 1$.

The test is indecisive for $C = 1$

Theorem (Ratio Test)

For a series $\sum_{n=1}^{\infty} a_i$, let $L = \limsup_{i \to \infty} \left| \dfrac{a_{i+1}}{a_i} \right|$. Then, if $L < 1$, the series converges absolutely. The series diverges if there exists N such that $\left| \dfrac{a_{i+1}}{a_i} \right| > 1$ for $i \geq N$.

Remark $L > 1$ in the above test doesn't imply that the series diverges. (Exercise !) suppose in a series a1/a2>1 but subsequently the ratio. falls sum can exit

Proof: Let $L < 1$. Let r be such that $L < r < 1$. Then after a stage, say for $i \geq N$, $|a_{i+1}| < r|a_i|$. So $|a_{i+k}| < r^k|a_i|$.

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$$\sum_{n=1}^{\infty} |a_i| = \sum_{0}^{N} |a_i| + \sum_{N+1}^{\infty} |a_i| = \sum_{0}^{N} |a_i| + \sum_{1}^{\infty} |a_{N+i}|$$

$$< \sum_{0}^{N} |a_i| + |a_N| \sum_{1}^{\infty} r^i = \sum_{0}^{N} |a_i| + |a_N| \frac{r}{1-r} < \infty.$$

Power Series

Theorem (Existence of Radius of Convergence)

For the power series $\displaystyle\sum_{n=1}^{\infty} a_i(z - z_0)^i$, let $R = \dfrac{1}{\displaystyle\limsup_{i \to \infty} \sqrt[i]{|a_i|}}$. Then the power series converges absolutely if $|z - z_0| < R$ and diverges if $|z - z_0| > R$. If $0 < r < R$ is any real number, then the power series converges uniformly on $\{z \mid |z - z_0| < r\}$.

<u>Proof</u>: Apply root test.

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series converges uniformly on $\{z \mid |z - z_0| < r\}$.

Proof: Apply root test.

If $\lim\limits_{i \to \infty} \left| \dfrac{a_{i+1}}{a_i} \right|$ exists, then by applying the ratio test instead of the

root test, it follows that $R = \lim\limits_{i \to \infty} \left| \dfrac{a_i}{a_{i+1}} \right|$.

Power Series

Remark: If a series converges by the ratio test, then it converges by the root test as well. But not conversely. Thus the root test is better than the ratio test. But the ratio test is often easier to use whenever it succeeds.

In fact:

$$\limsup_{i \to \infty} \sqrt[i]{|a_i|} \leq \limsup_{i \to \infty} \left| \frac{a_{i+1}}{a_i} \right|$$

Exercise: Give an example of a series which can be shown to be convergent by root test but not by ratio test.

Examples:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{z^i}{i!}$. Apply ratio test. $\displaystyle\lim_{i\to\infty}\left|\frac{a_i}{a_{i+1}}\right| = \lim_{i\to\infty} i = \infty$; i.e., the series converges everywhere.

2. $z - \frac{z^3}{3} + \frac{z^5}{5} - \ldots$ Radius of convergence is 1. Both the tests apply here.

3. $\frac{1}{2} + \frac{1}{3}z + \left(\frac{1}{2}\right)^2 z^2 + \left(\frac{1}{3}\right)^2 z^3 + \ldots$. Check that the ratio test fails. Apply root test to show that the radius of convergence is $\frac{1}{\sqrt{2}}$.

ans is root 2

Power series can be added, subtracted, and multiplied in the obvious way. It can also be differentiated and integrated term by term, in its domain of convergence. Indeed,
$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots$, then,

$$\lim_{h \to 0} \frac{f(z + h) - f(z)}{h} = \sum_n a_n \left(\lim_{h \to 0} \frac{[(z - z_0 + h)^n - (z - z_0)^n]}{h} \right)$$

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Apply root test to check that a given power series, the differentiated series and the integrated series, all have the same radius of convergence.

Analytic Functions

A function $f : \Omega \to \mathbb{C}$ is said to be analytic if it is locally given by a convergent power series; i.e., every $z_0 \in \Omega$ has a neighbourhood contained in Ω such that there exists a power series centered at z_0 which converges to $f(z)$ for all z in that neighbourhood.

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> $holomorphic \implies analytic.$

This would prove our statement from Lecture 1 that once differentiable is always differentiable!

Just as in the complex case, power series and analytic functions can be defined in the real case too. But unlike in the complex case, differentiable does not mean real analytic. In fact, even infinitely differentiable does not mean real analytic. For example, $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0, \end{cases}$$

in the neighbourhood of 0 we can not define a power series that converges to 0

is infinitely differentiable but not real analytic. In this example, $f^{(i)}(0) = 0$ for all i, and thus the Taylor series of f is the zero function.

Remark: In fact such functions are in "abundance". (This statement can be made precise). Their existence is of great importance in analysis.