MA 205 Complex Analysis: Integration

U. K. Anandavardhanan
IIT Bombay

August 06, 2015

Recall

In the last lecture, we saw the notion of a multi-valued function. The most basic example was the complex logarithm. Using log, we defined other multi-valued functions such as power function, inverse trigonometric functions, etc. We saw that we can make these functions single valued by taking s branches which we get by throwing away branch cuts. A branch point was the end point of a branch cut. In other words, a branch point of a multi-valued function has the property that in any neighbourhood of that point, the function takes multiple values. You have to throw away this point in order to achieve single valued-ness. In the basic example of log, you saw that the only branch points are 0 and ∞. You also saw that the various half-rays are branch cuts. We also discussed that any path connecting 0 and ∞ would be a branch cut for log. This is because Ω is simply connected iff every harmonic function on Ω has a harmonic conjugate, and therefore the harmonic function $\frac{1}{2}\log(x^2 + y^2)$ will have a harmonic conjugate in any simply connected domain. Did we prove this fact yet?

Recall

In particular, we discussed $\sqrt{z}$. For a positive real number, this function gives the positive square root in certain branches and the negative square root in certain other branches. When do you get the positive square root and when do you get the negative square root?[1]

[1] Thanks to Ayush for correcting a mistake in the last class.

Fix the exam date

We'll have a quiz for 10 marks in the next tutorial (Tuesday, August 11). Tutorial will be from 5:00 to 6:00 and the quiz during 6:00-6:30.

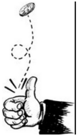
Syllabus: Whatever I cover till Monday, August 10.

Fix the exam date

It'll not be subjective type because unknown questions are usually answered like this:

> *Question: What is an array? Student: ARRAY is the word used to call a friend. eg. "ARRAY, idhar aa re ...".*

It'll not be True/False type because unknown questions are usually answered like this:



It'll be like:

MA 205 Complex Analysis Fall 2015 Quiz I 11-08-2015

Name: Division: **S1**
Roll No: Tutorial Batch: **LT** 0 0 _

Circle the correct answer and briefly justify your choice in the space given.

Now that we've differentiated enough, it's time to integrate. Recall integration from MA 105. You first integrated real valued functions on an interval. Remember Riemann sum, Riemann integration, area under a curve, etc? Integrals had nice properties: well behaved under addition and scalar multiplication. In the language of MA 106, integral is a linear functional from the vector space of integrable functions to $\mathbb{R}$. Integral also respected monotonicity. But in spite of all the nice properties, one struggled hard to integrate. To effortlessly integrate, we needed the fundamental theorem. Integral is nothing but the anti-derivative. If you remember, this was a high point of the initial days of MA 105!

The next step in MA 105 was to view an interval as a curve; a parametrized curve. $[a, b]$ is nothing but $a(1-t)+bt$, $0 \leq t \leq 1$. Then you asked: if one could integrate along these simple curves, why not over other parametrized curves in the plane? Indeed, you did this. This was called the line integral. What did you integrate over curves in the plane? You integrated vector fields. Given a vector field $f(x, y) = (P(x, y), Q(x, y))$, given a parametrized curve $\gamma = \gamma(t)$, $a \leq t \leq b$, the line integral

$$\int_\gamma f \cdot ds$$

meant

$$\int_a^b f(\gamma(t)) \cdot \gamma'(t) dt.$$

Of course, γ was assumed to be smooth or piecewise smooth, so that γ' makes sense at least after breaking $[a, b]$ into finitely many subintervals, it is continuous, and $\gamma'(t) \neq 0$.

The same line integral was also written as

$$\int Pdx + Qdy.$$

Sometimes, it was easier to integrate this way. What else? What was the physical meaning of the line integral? Does the value of the line integral depend on the particular parametrization that we have or it depends only on the underlying path? As you would expect, work done depends only on the path. You also might remember a certain theorem which went by the name path independence: line integral is path independent iff it's a gradient.

Theorem (Path Independence)

A vector field f on a connected open domain is the gradient of some function F is and only if for any two paths C_1 and C_2 with the same endpoints, $\int_{C_1} f \cdot ds = \int_{C_2} f \cdot ds$.

Before we get into complex integration, let's recall perhaps the most important line integral calculation that you ever did in MA 105. Line integral of

$$f(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

along the unit circle.

$$\begin{aligned} \int_C f \cdot ds &= \int_a^b f(c(t)) \cdot c'(t) dt \\ &= \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t) dt \\ &= 2\pi. \end{aligned}$$

So that was a quick summary of real integration from MA 105. Now to complex integration in MA 205. First we integrated $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$. Now let's start with $f : I = [a, b] \subset \mathbb{R} \to \mathbb{C}$. f is of the form $u + \imath v$; i.e., $f(t) = u(t) + \imath v(t)$. We define

$$\int_a^b f(t)dt$$

as

$$\int_a^b u(t)dt + \imath \int_a^b v(t)dt.$$

The real integral was a real linear functional from the vector space of integrable functions on I to $\mathbb{R}$. Check that our definition makes the complex integral a complex linear functional from the vector space of integrable functions on I to $\mathbb{C}$. What does this mean?

Next we observed monotonicity for the real integral. If $f(t) \leq g(t)$ for all $t \in I$, then $\int_a^b f(t)dt \leq \int_a^b g(t)dt$. Since "$\leq$" is a unique real concept, we need to think for a second about the correct complex analogue. Note that for a real valued function f, $-|f(t)| \leq f(t) \leq |f(t)|$. Thus, monotonicity of integrals would say: $-\int_a^b |f(t)|dt \leq \int_a^b f(t)dt \leq \int_a^b |f(t)|dt$, or

$$\left| \int_a^b f(t)dt \right| \leq \int_a^b |f(t)|dt.$$

It makes perfect sense to ask whether such an inequality is true for complex integrals? And the answer is Yes. This is complex monotonicity.

Proof: f is complex valued and so $\int_a^b f(t)dt \in \mathbb{C}$, say w_0. Let $c = \frac{|w_0|}{w_0}$. Thus, $|c| = 1$ and $c\int_a^b f(t)dt \in \mathbb{R}$. Therefore, $c\int_a^b f(t)dt = \int_a^b \mathrm{Re}(cf(t))dt$. But, $\mathrm{Re}(cf(t)) \leq |cf(t)| = |f(t)|$. Thus,

$$\begin{aligned}
\left|\int_a^b f(t)dt\right| &= \left|c\int_a^b f(t)dt\right| \\
&= \left|\int_a^b \mathrm{Re}(cf(t))dt\right| \\
&\leq \int_a^b |\mathrm{Re}(cf(t))|dt \\
&\leq \int_a^b |f(t)|dt.
\end{aligned}$$

Then, we had recollected the fundamental theorem, which we noticed made integration easier. Do we have a complex version? Of course we do.

Theorem (Fundamental Theorem of Calculus)

Let $f : I = [a, b] \to \mathbb{C}$ be a continuous function. Then,

$$x \mapsto \int_a^x f(t)dt$$

is an anti-derivative (or a primitive) of f. If F is any anti-derivative of f on I, then,

$$\int_r^s f(t)dt = F(s) - F(r),$$

for any $r, s \in I$.

What about the proof? It's easy. Just apply the fundamental theorem from MA 105 to real and imaginary parts of f.

Next we did line integrals. A function $f : \Omega \to \mathbb{C}$ indeed gives a vector field $f(x, y) = (u(x, y), v(x, y))$. If $\gamma = z(t)$ is a smooth parametrized curve in Ω, then we define the complex line integral

$$\int_\gamma f(z)dz$$

as

$$\int_a^b f(z(t))z'(t)dt.$$

If f is holomorphic, the vector field is differentiable. In fact, it is continuously differentiable, but this we haven't proved yet. Once again, the line integral does not depend on the parametrization; it depends only on the path.

And yes, we have a path independence theorem as well.

Theorem (Path Independence)

Let $f : \Omega \to \mathbb{C}$ be continuous. Then, there exists a holomorphic F on Ω with $F' = f$ if and only if for every $w, z \in \Omega$ and every path γ from w to z,

$$\int_\gamma f(z)dz = F(z) - F(w).$$

The proof of the path independence theorem from MA 105 more or less works here too. In any case, I'll sketch a proof in the next lecture.

Finally, we had recalled the most important example from MA 105. Let's see the most important example in MA 205 as well. Recall that the line integral of

$$f(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

along the unit circle produced 2π and not zero. This did not contradict Green's theorem since the vector field is not defined at $(0, 0)$ which is interior to the unit circle. The analogous example here is: the line integral along the unit circle of $f(z) = \frac{1}{z}$.

$$\int_{|z|=1} \frac{dz}{z} = \int_0^{2\pi} \frac{e^{\imath\theta} \imath d\theta}{e^{\imath\theta}} = 2\pi\imath.$$

Similarly,

$$\int_{|z-z_0|=r} \frac{dz}{z - z_0} = 2\pi\imath.$$