Integration

August 3, 2017

Definite Integrals

Let $f : [a, b] \to \mathbb{C}$ be a piecewise continuous function. Let $f(t) = u(t) + iv(t)$. We define

$$\int_a^b f(t)dt$$

to be

$$\int_a^b u(t)dt + i\int_a^b v(t)dt$$

where both these integrals are defined to be the usual Riemann Integrals.

Some basic properties:

$$1. Re \int_a^b f(t)dt = \int_a^b Ref(t)dt = \int_a^b u(t)dt$$

$$2. Im \int_a^b f(t)dt = \int_a^b Imf(t)dt = \int_a^b v(t)dt$$

$$3. \int_a^b (c_1 f_1(t) + c_2 f_2(t))dt = c_1 \int_a^b f_1(t) + c_2 \int_a^b f_2(t)dt$$

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Length of a parametrized curve

Recall that if $\gamma(t) = x(t) + iy(t)$ is a parametrized curve, then we define the lenth of γ to be

$$\int_a^b |\gamma'(t)|dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt$$

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Contour Integration

We now discuss the complex analogue of line integrals form calculus.

We say a curve $\gamma(t) = x(t) + iy(t)$ is **smooth** if $\gamma'(t) \neq 0$ for all t. Such a curve is also called regular parametrized curve.
A **contour** is a curve consisting of a finite number of smooth curves joined end to end. It is said to be **simple** if the parametrization map is one to one except possibly at the end-points. (Intuitively it means that the curve does not cross itself). It is said to be **closed** if the initial and end-point are the same. i.e, $\gamma(a) = \gamma(b)$.

Jordan Curve Theorem

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theorem was first discovered by Camille Jordan in 1887 although his proof was not rigorous. The first rigorous proof was due to Oswald Veblenin 1905. Although intuitively very believable the proof of this theorem is non-trivial. We shall not prove this here. Generalizations of this theorem to higher dimensions also exist. Note that in the above theorem, one does not make any assumptions of differentiability of the curve. This is what makes the proof all the more non-trivial.

Complex Integration

Let $f : \Omega \to \mathbb{C}$ be a complex function defined on a domain Ω and let C be a contour with initial point z_0 and terminal point z. We define the integral of f along C to be

$$\int_C f(z) \stackrel{def}{=} \int_a^b f(z(t))z'(t)dt$$

For $f(z) = u(x, y) + v(x, y)$ and $dz = dx + idy$, we have

$$\int_C f(z)dz = \int_C udx - vdy + i \int_C udy + vdx$$

$$= \int_a^b [(u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t)]dt$$

$$+i \int_a^b [(u(x(t), y(t))y'(t) + v(x(t), y(t))x'(t)]dt$$

Properties

The usual properties of real line integrals get carried over to the complex analogues:

1. This integral is independent of parametrization.
2. $\int_{-C} f(z)dz = -\int_C f(z)dz$ where $-C$ is the opposite curve, i,e curve with the opposite parametrization.
3. $\int_{C_1 \cup C_2 \cup \ldots C_n} f(z)dz = \int_{C_1} f(z)dz + \ldots + \int_{C_n} f(z)dz$
4. $|\int_C f(z)dz| \leq \int_C |f(z)|dz$

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Basic Example

Consider $f(z) = \int_C \frac{1}{z-z_0}dz$ where C is any circle around z_0.

We can parametrize C as $z(t) = z_0 + re^{it}$ with $0 \leq t \leq 2\pi$. Then $\int_C f(z)dz = \int_C \frac{1}{z-z_0}dz = \int_0^{2\pi} \frac{1}{re^{it}} ire^{it}dt = 2\pi i$

Note that the integral is independent of the circle chosen around z_0. This is an instance of a more general situation

Path independence

We will show that f has a primitive iff $\int f(z)dz$ is path independent. Suppose f has a primitive; i.e., there is F such that $F' = f$. Then,

$$\begin{aligned}\int_C f(z)dz = \int_C F'(z)dz &= \int_a^b F'(z(t))z'(t)dt \\ &= \int_a^b \left[\frac{d}{dt}F(z(t))\right] dt = F(z(b)) - F(z(a)).\end{aligned}$$

Thus, the integral depends only on the end points.

Proof of Path Independence

On the other hand, suppose the integral depends only on the end points of the path and not the path itself. This means that the integral is independent of the path on which you integrate. We need to find an F, show that it is differentiable, and $F'(z) = f(z)$ for all $z \in \Omega$.

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$$F(z) = \int_{\gamma(z_0,z)} f(z)dz.$$

This function is well defined because of the hypothesis of independence of integral on the path.

Proof of Path Independence

We have a good candidate for the primitive. We only have to check that it is indeed a primitive. To this end, consider a small neighborhood of z which is completely contained in Ω. Let $h \in \mathbb{C}$ be such that $|h|$ is very small. Join z to $z + h$ via a straight line; $z + ht$, $t \in [0, 1]$. Now,

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$$\begin{aligned}
\lim_{h\to 0} \frac{F(z+h)-F(z)}{h} &= \lim_{h\to 0} \frac{1}{h}\left[\int_{\gamma(z_0,z+h)} f(w)dw - \int_{\gamma(z_0,z)} f(w)dw\right] \\
&= \lim_{h\to 0} \frac{1}{h}\int_{z}^{z+h} f(w)dw \\
&= \lim_{h\to 0} \frac{1}{h}\int_{0}^{1} f(z+ht)hdt \\
&= f(z).
\end{aligned}$$

This finishes the proof.

Cauchy's theorem

We now come to the most important, central theorem in this subject on which most of complex analysis depends, namely Cauchy's theorem.

Theorem

Let C be a simple closed contour and let f be a holomorphic theorem defined on an open set containing C as well as its interior. Then $\int_C f(z)dz = 0$.

<u>**Remark**</u> Note that by Jordan curve theorem, interior of C makes sense.

Cauchy's theorem

Proof.

Let $f(z) = u(x, y) + v(x, y)$. The proof uses Green's theorem. Recall that by Greens theorem if P and Q are two real valued functions with continuous first partial derivatives, then

$$\int_C (Pdx + Qdy) = \int\int_\Omega (Q_x - P_y)dxdy$$

Note that a priori we do not have the hypothesis to guarantee continuity of the first partial derivatives of u and v since we do not know if $f'(z)$ is continuous. However it is a fact that if $f(z)$ is holomorphic, then $f'(z)$ is continuous. This is called Goursat's theorem. We will assume this theorem without proof.

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Thus by Green's theorem,
$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (vdx + udy) =$
$\int\int_\Omega (-v_x - u_y)dxdy + i\int\int_\Omega (u_x - v_y)dxdy = 0$(By CR equations) □

Simply-Connectedness

Definition

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Examples: $\mathbb{C}$, any open disc in $\mathbb{C}$, $\mathbb{C}$ minus negative reals etc. Open annulus i.e, area between two concentric circle is **NOT** simply connected. Similarly any open set minus a non-empty set of finitely many points is **NOT** simply connected.

Remark The notion of simply connectedness makes sense for much more general spaces but the definition is slightly complicated. This notion is the beginning of a very deep and beautiful area of mathematics called Algebraic Topology. This area also has strong connections with complex analysis and its higher dimensional analogue, namely the area of several complex variables.

Theorem

(More general form of Cauchy's theorem) *Let Ω be a simply connected domain in $\mathbb{C}$. Let $f(z)$ be a holomorphic function defined on Ω. Let C be a simple closed contour in Ω. Then* $\int_C f(z)dz = 0$

Integral when the curve is non-smooth

In case the path of integration is not differentiable, i.e, $x(t)$ and $y(t)$ are not differentiable functions of the parameter t, one can still make sense of the integral much the same way as we define the Riemann Integral. C be a continuous rectifiable curve lying in a domain Ω joining points p and q. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function on Ω. Divide the curve into n parts in any manner $p = z_0, z_1, ..., z_n = q$. On each of the paths z_{i-1} to z_i on the curve, choose points ζ_i on the curve between z_i and z_{i+1}.

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Still more general form of Cauchy's theorem

Theorem

(Even stronger form of Cauchy's theorem) *Let Ω be a simply connected domain in $\mathbb{C}$. Let $f(z)$ be a holomorphic function defined on Ω. Let C be a simple closed rectifiable curve in Ω. Then $\int_C f(z)dz = 0$.*

curve of finite length