MA 205 Complex Analysis: Morera, ZAI

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Introduction

So how did the exam go? Any quick questions? In the review class, we did:

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1. The function $u(x,y) = \sqrt{\sqrt{x^2+y^2} - x}$ is the real part of
 (a) $\sqrt{z}$ (b) $-\sqrt{2z}$ (c) $\sqrt{2z}$ (d) $-\imath\sqrt{2z}$ (e) $\imath\sqrt{2z}$,
where $\sqrt{z}$ denotes ~~the principal branch~~.

A mistake was pointed out last time in this question. What was that?[1] So how about

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 (a) $\sqrt{z}$ (b) $-\sqrt{2z}$ (c) $\sqrt{2z}$ (d) $-\imath\sqrt{2z}$ (e) $\imath\sqrt{2z}$,
where $\sqrt{z}$ denotes the branch $(0, 2\pi)$.

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[1] Thanks to Reebhu for pointing this out.

Announcements

1. We'll have Quiz II on Tuesday, September 01.
(Note added after the class: Several students told me that they have the lab midsem that day. Is a slot on Friday, August 28, available which is convenient to everyone?)
2. Endsem on Friday, September 11.
3. There'll be no MA 205 class on Monday, August 24.

Morera's Theorem

In an earlier class, we had proved the path independence theorem. This said that the line integral of a continuous function on a domain is path independent if and only if it has a primitive. Suppose f is continuous such that its line integrals are path independent. Then it has a primitive; i.e., there exists a holomorphic function F such that $F' = f$. Now we know that once differentiable is infinitely differentiable. In particular $F' = f$ is differentiable. Thus we have proved: if f is continuous on a domain such that its integral along any closed path is zero, then it is holomorphic in that domain. This is called Morera's theorem.

Morera's Theorem

Morera's theorem can be thought of as a converse to Cauchy's theorem which said that integral of a function along a closed curve is zero provided it is holomorphic on and within the curve. From vanishing integrals to differentiability. Once again a story unheard of before; no analogue in the real case.

Remark: Morera's theorem can be proved from first principles and without using ODAD. In fact using ODAD is not the "right" approach!

... he was not interested in any scientific or other kind of field outside of his own realm of expertise. Morera himself, in the inaugural address as the rector of the University of Genova, after quoting a statement attributed to Peter Tait revealed the reason behind his views: "In science, the one who has a sound and solid knowledge, even in a narrow field, holds a true strength and he can use it whenever he needs: the one who has only a superficial knowledge, however wide and striking, holds nothing, and indeed he often holds a weakness pushing him towards vanity".

Zeros are Isolated

Another extremely important result which has no analogue in the real case is regarding the zeros of holomorphic functions. It goes under the name "zeros are isolated". z_0 is a zero of f if $f(z_0) = 0$. We want to prove that the zeros of a holomorphic function are isolated; i.e., if $f(z_0) = 0$, there is a small neighborhood of z_0 in which f doesn't vanish at any other point. Recall that this was not true at all for real differentiable functions. For instance,

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

is infinitely differentiable, but the whole negative real axis consists of its zeros. Not isolated at all!

Zeros are Isolated

Isolation, loneliness, etc., may have a negative connotation, but do remember it's one of the most positive results in MA 205! Perhaps "solitude" is a better word. "I live in that solitude which is painful in youth, but delicious in the years of maturity."[2] Zeros of real differentiable functions found solitude painful; in the more matured complex world, they find it delicious! And in this solitude, these tapasvins capture the true essence, the true identity, of their functions! This is what's called the identity theorem, which we'll formally state very soon.

[2] Einstein, Self-portrait, 1936.

Open, Closed, Connected

Before we start the discussion on the identity theorem, let's make some observations about openness, closedness and connectedness. Recall: open: any point has a neighborhood which is contained in the set; closed: it contains all its limit points; connected: any two points in the set can be connected via a continuous path.
Examples: $(0,1)$ is open in $\mathbb{R}$; in $\mathbb{C}$? $[0,1]$ is closed in $\mathbb{R}$; in $\mathbb{C}$? $|z|<1$ is open in $\mathbb{C}$; $|z|\leq 1$ is closed in $\mathbb{C}$; Is $\mathbb{R}$ open in $\mathbb{C}$? Is it closed in $\mathbb{C}$? Is $\mathbb{Z}$ open/closed in $\mathbb{C}$? Is $\{\frac{1}{n} \mid n \geq 1\}$ closed in $\mathbb{C}$? In $\mathbb{R}$? Is $\mathbb{Q}$ open/closed in $\mathbb{R}$?
Claim: if a set is connected, then it cannot be written as a union of two disjoint open subsets. Indeed if the set is written as $A \cup B$, with A and B disjoint, and if we take $a \in A$ and $b \in B$, then we cannot connect a to b via a continuous path. You should convince yourself that:

a set is open iff its complement is closed.

Let $A \subset X$. $a \in X$ is a limit point of A, if every neighborhood of a in X contains a point $a' \neq a$ of A. A is closed in X if A contains all its limit points. Examples: $\mathbb{Z}$ is closed in $\mathbb{R}$. What are the limit points? Nothing. So Z contains all the limit points! $[0,1]$ is closed in $\mathbb{C}$. What are the limit points? $[0,1]$. Does the set contain all its limit points? Yes. Let $A = [0,1] \cup \{2\}$. What are the limit points? $[0,1]$. Thus, A contains all its limit points, so it is closed.
Exercise: a is a limit point of $A \subset X$ iff every neighborhood of a in X contains infinitely many distinct points of A.

Thus, if a set is connected then it does not have a subset which is both open and closed. (Why?) Thus, to show that a certain non-empty subset of a domain is in fact the whole domain, it is enough to show that this subset is both open and closed. We'll use this strategy in the next slide.

Zeros are Isolated

We'll now show that zeros of a holomorphic function cannot have a limit point, unless of course $f \equiv 0$. Consider the set of zeros of f:

$$Z(f) = \{z \in \Omega \mid f(z) = 0\}.$$

Suppose this set has a limit point, say z_0. Since f is continuous, we get $f(z_0) = 0$. We first claim that $f^{(n)}(z_0) = 0$ for all $n \geq 1$. Suppose not; i.e., suppose there is an integer $n \geq 1$ such that

$$f(z_0) = f'(z_0) = \ldots = f^{(n-1)}(z_0) = 0,$$

and $f^{(n)}(z_0) \neq 0$. Take a small disc around z_0 which is contained in Ω. In this disc, expand f as a power series:

$$f(z) = \sum_{k=n}^{\infty} a_k (z - z_0)^k.$$

If $g(z) = \displaystyle\sum_{k=n}^{\infty} a_k (z - z_0)^{n-k}$, then $f(z) = (z - z_0)^n g(z)$, with $g(z_0) \neq 0$.

Zeros are Isolated

Since g is continuous, and since $g(z_0) \neq 0$, there is a small neighborhood of z_0 in which g is never zero. But remember that z_0 is a limit point of $Z(f)$, and hence this neighborhood will also contain a point, say $z' \neq z_0$, with $f(z') = 0$. Therefore, $g(z') = 0$, (why?) a contradiction. Now consider the set

$$A = \{z \in \Omega \mid f^{(n)}(z) = 0 \text{ for all } n \geq 0\}.$$

$A \neq \emptyset$ since $z_0 \in A$. We'll show that A is both open and closed, which shows that $A = \Omega$. This of course would mean $f \equiv 0$.

To show that A is closed, we need to show that A contains all its limit points. If z is a limit point of A, let $z_k \in A$ be such that $\lim z_k = z$. Since $f^{(n)}$ is continuous, it follows that $f^{(n)}(z) = 0$; i.e., $z \in A$.

To show that A is open, we need to show that every $a \in A$ has a neighborhood which is contained in A. Since Ω is open, there is a neighborhood of a which is contained in Ω. On this neighborhood, if we write $f(z) = \sum a_n(z-a)^n$, then $a_n = \frac{1}{n!}f^{(n)}(a) = 0$ for each $n \geq 0$. Thus, $f \equiv 0$ for all z in this neighborhood. Therefore, this neighborhood is in fact contained in A.

Corollary [Identity Theorem]: If f and g are holomorphic in Ω, then $f \equiv g$ iff $\{z \in \Omega \mid f(z) = g(z)\}$ has a limit point in Ω.

Zeros are Isolated

Example: You cannot have two distinct holomorphic functions which agree on all $\frac{1}{n}$.
Example: $exp(z)$ is the only holomorphic function which agrees with e^x on the real line. Similarly, for $\sin z, \cos z$ etc.
Example: Identities like $\sin^2 z + \cos^2 z = 1$ follow without any further computation since they hold true over reals.
Example: $\exp(z + w) = \exp(z) \exp(w)$ now has another proof since this is true over reals!
Corollary: If f is holomorphic, each zero of f has finite multiplicity; i.e., there exists m such that $f(z) = (z - z_0)^m g(z)$ with $g(z_0) \neq 0$.
Proof: If there is no such finite m, we get $f^{(n)}(z_0) = 0$ for all $n \geq 1$, and exactly as in the earlier proof, conclude that $f \equiv 0$.
Remark: The corollary above is a corollary to the proof of the previous result and not a corollary to the result itself.