MA-207 Differential Equations II

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Now we will start the study of Partial differential equations.

A partial differential equation (PDE) is an equation for an unknown function u that involves independent variables $x, y, \ldots$, the function u and the partial derivatives of u.
The order of the PDE is the order of the highest partial derivative of u in the equation.
Examples of some famous PDEs.

1. $u_t - k(u_{xx} + u_{yy}) = 0$ two dimensional Heat equation, order 2.
2. $u_{tt} - c^2(u_{xx} + u_{yy}) = 0$ two dimensional wave equation, order 2.
3. $u_{xx} + u_{yy} = 0$ two dimensional Laplace equation, order 2.
4. $u_{tt} + u_{xxxx}$ Beam equation, order 4.

Examples of non-famous PDE's (I made it up).

1. $u_x + \sin(u_y) = 0$, order 1.
2. $3x^2 \sin(xy) e^{-xy^2} u_{xx} + \log(x^2 + y^2) u_y = 0$, order 2.

A PDE is said to be "linear" if it is linear in u and its partial derivatives i.e. it is a degree 1 polynomial in u and its partial derivatives.

Heat equation, Wave equation, Laplace equation and Beam equation are linear PDEs.

In the above two non-famous examples, the first is non-linear while the second is linear.

The general form of first order linear PDE in two variables x, y is

$$A(x, y)u_x + B(x, y)u_y + C(x, y)u = f(x, y)$$

The general form of first order linear PDE in three variables x, y, z is

$$Au_x + Bu_y + Cu_z + Du = f$$

where coefficients A, B, C, D and f are functions of x, y and z.
The general form of second order linear PDE in two variables x, y is

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = f$$

where coefficients A, B, C, D, E, F and f are functions of x and y.
When $A \ldots, F$ are all constants, then its a linear PDE with constant coefficients.

Linear Partial Differential Operator

Second order linear PDE in two variable can be written as $Lu = f$, where

$$L = A\frac{\partial^2}{\partial x^2} + 2B\frac{\partial^2}{\partial x \partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F$$

is the linear differential operator. It is called linear since the map $u \mapsto Lu$ is a linear map.

Examples. Laplace operator in $\mathbb{R}^2$ is

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Heat and Wave operator in one space variable are

$$H = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}, \qquad \Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$$

Classification of second order linear PDE

Consider the linear differential operator L in $\mathbb{R}^2$.

$$L = A\frac{\partial^2}{\partial x^2} + 2B\frac{\partial^2}{\partial x \partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F$$

where $A, \ldots, F$ are functions of x and y.

To the operator L, we associate the discriminant $\mathbb{D}(x, y)$ given by

$$\mathbb{D}(x, y) = A(x, y)C(x, y) - B^2(x, y)$$

The operator L or the PDE $Lu = f$ is said to be

- elliptic at (x_0, y_0), if $\mathbb{D}(x_0, y_0) > 0$,
- hyperbolic at (x_0, y_0), if $\mathbb{D}(x_0, y_0) < 0$,
- parabolic at (x_0, y_0), if $\mathbb{D}(x_0, y_0) = 0$.

If L is elliptic at each point (x, y) in a domain $\Omega \subset \mathbb{R}^2$, then L is called elliptic in Ω.

Similarly for hyperbolic and parabolic. Recall

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},\ H = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2},\ \Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$$

- Two dimensional Laplace operator Δ is elliptic in $\mathbb{R}^2$, since $\mathbb{D} = 1$.
- One dimensional Heat operator H is parabolic in $\mathbb{R}^2$, since $\mathbb{D} = 0$.
- One dimensional Wave operator $\Box$ is hyperbolic in $\mathbb{R}^2$, since $\mathbb{D} = -1$.

When the coefficients of an operator L are not constant, the type may vary from point to point.

Example. Consider the Tricomi operator (well known)

$$T = \frac{\partial^2}{\partial x^2} + x\frac{\partial^2}{\partial y^2}$$

The discriminant $\mathbb{D} = x$.
Hence T is elliptic in the half-plane $x > 0$,
hyperbolic in the half-plane $x < 0$ and
parabolic on the y-axis.

Remark about terminology

Consider

$$L = A\frac{\partial^2}{\partial x^2} + 2B\frac{\partial^2}{\partial x \partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F$$

at the point (x_0, y_0). If we replace $\partial/\partial x$ by ξ and $\partial/\partial y$ by η and evaluate $A, \dots, F$ at (x_0, y_0), then L becomes a polynomial in 2 variables

$$P(\xi, \eta) = A\xi^2 + 2B\xi\eta + C\eta^2 + D\xi + E\eta + F$$

Consider the curves in (ξ, η)-plane given by

$$P(\xi, \eta) = \text{constant}$$

then these curves are elliptic if $\mathbb{D}(x_0, y_0) > 0$, hyperbolic if $\mathbb{D}(x_0, y_0) < 0$ and parabolic if $\mathbb{D}(x_0, y_0) = 0$.

Second order linear operators in $\mathbb{R}^3$

The classification is done analogously by associating a polynomial of degree 2 in three variables to L and considering the surfaces defined by level sets of the polynomial.
These surfaces are either ellipsoids, hyperboloids, or paraboloids.
The operator L is accordingly labeled as elliptic, hyperbolic or parabolic.
We can also proceed as follows; Consider

$$L = a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x \partial y} + 2c\frac{\partial^2}{\partial x \partial z} + d\frac{\partial^2}{\partial y^2} + 2e\frac{\partial^2}{\partial y \partial z} + f\frac{\partial^2}{\partial z^2}$$
$$+ \text{ lower order terms}$$

where $a, b, \ldots$ are functions of (x, y, z).

To L, we associate the symmetric matrix

$$M(x,y,z) = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

Here the (i,j)-th entry is the coefficient of $\dfrac{\partial^2}{\partial x_i \partial x_j}$.
Since M is symmetric, it has 3 real eigenvalues.

- L is elliptic at (x_0, y_0, z_0) if all three eigen values of $M(x_0, y_0, z_0)$ are of same sign.
- L is hyperbolic at (x_0, y_0, z_0) if two eigen values are of same sign and one of different sign.
- L is parabolic at (x_0, y_0, z_0) if one of the eigenvalue is zero.

Principle of superposition

Let L be a linear differential operator.
The PDE $Lu = 0$ is called homogeneous and
the PDE $Lu = f$, $(f \neq 0)$ is non-homogeneous.

Principle 1. If $u_1, \ldots, u_N$ are solutions of $Lu = 0$ and $c_1, \ldots, c_N$ are constants, then $\sum_{i=1}^{N} c_i u_i$ is also a solution of $Lu = 0$.

In general, space of solutions of $Lu = 0$ contains infinitely many independent solutions and we may need to use infinite linear combinations of them.

Principle 2.

Assume

- $u_1, u_2, \ldots$ are infinitely many solutions of $Lu = 0$.
- the series $w = \sum_{i \geq 1} c_i u_i$ with $c_1, c_2, \ldots$ constants, converges to a twice differentiable function;
- term by term partial differentiation is valid for the series, i.e. $Dw = \sum_{i \geq 1} c_i D u_i$, D is any partial differentiation of order 1 or 2.

Then w is again a solution of $Lu = 0$.

Principle 3 for non-homogeneous PDE.

If u_i is a solution of $Lu = f_i$, then

$$w = \sum_{i=1}^{N} c_i u_i$$

with constants c_i, is a solution of $Lu = \sum_{i=1}^{N} c_i f_i$.

One-dimensional heat equation

The temperature evolution of a thin rod of length L is decribed by the PDE

$$u_t = k^2 u_{xx}, \quad 0 < x < L,\ t > 0,$$

called one-dimensional heat equation.
Here k is a positive constant.

x is the space variable and t is the time variable.

$u(x,t)$ is the temperature at point x and time t.

At time $t = 0$, we must specify temperature at every point. That is, specify $u(x,0)$.

We must also specify boundary conditions that u must satisfy at the two endpoints of the rod for all $t > 0$.

We call this problem an initial-boundary value problem IBVP.

We consider different kinds of boundary conditions.

In each case, we use method of separation of variables.
Suppose

$$v(x,t) = X(x)\,T(t)$$

Substituting this in the Heat equation $\boxed{u_t = k^2 u_{xx}}$

$$T'(t)X(x) = k^2 X''(x)T(t).$$

We can now separate the variables:

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{k^2 T(t)}$$

The equality is between a function of x and a function of t,
so both must be constant, say $-\lambda$.
We need to solve

$$X''(x) + \lambda X(x) = 0 \quad \text{and} \quad T'(t) = -k^2 \lambda T(t).$$

Dirichlet boundary conditions $u(0,t) = u(L,t) = 0$

Initial-boundary value problem is

$$
\begin{aligned}
&u_t = k^2 u_{xx} && 0 < x < L, \quad t > 0 \\
&u(0,t) = 0 && t > 0 \\
&u(L,t) = 0, && t > 0 \\
&u(x,0) = f(x), && 0 \leq x \leq L
\end{aligned}
$$

The endpoints of the rod are maintained at temperature 0 at all time t.

(The rod is isolated from the surroundings except at the endpoints from where heat will be lost to the surrounding.)

Assuming the solution in the form $\boxed{v(x,t) = X(x)T(t)}$

$$v(0,t) = X(0)T(t) = 0 \quad \text{and} \quad v(L,t) = X(L)T(t) = 0$$

we don't want T to be identically zero, we get

$$X(0) = 0 \quad \text{and} \quad X(L) = 0.$$

We need to solve eigenvalue problem

$$X''(x) + \lambda X(x) = 0, \quad X(0) = 0, \quad X(L) = 0, \qquad (*)$$

and $$T'(t) = -k^2 \lambda T(t) \implies T(t) = exp(-k^2 \lambda t)$$

The eigenvalues of $(*)$ are

$$\lambda_n = \frac{n^2 \pi^2}{L^2}$$

with associated eigenfunctions

$$X_n = \sin \frac{n\pi x}{L}, \ \ n \geq 1.$$

We get infinitely many solutions for IBVP, one for each $n \geq 1$

$$\begin{aligned} v_n(x,t) &= T_n(t) X_n(x) \\ &= exp\left(\frac{-n^2 \pi^2 k^2}{L^2} t \right) \sin \frac{n\pi x}{L} \end{aligned}$$

Note $$v_n(x,0) = \sin \frac{n\pi x}{L}$$

Therefore

$$v_n(x,t) = exp\left(\frac{-n^2\pi^2k^2}{L^2}\,t\right)\sin\frac{n\pi x}{L}$$

satisfies the IBVP

$$\begin{aligned}
&u_t = k^2u_{xx} && 0 < x < L, \quad t > 0\\
&u(0,t) = 0 && t > 0\\
&u(L,t) = 0 && t > 0\\
&u(x,0) = \sin\frac{n\pi x}{L} && 0 \le x \le L
\end{aligned}$$

More generally, if $\alpha_1, \ldots, \alpha_m$ are constants and

$$u_m(x,t) = \sum_{n=1}^{m}\alpha_n\, exp\left(\frac{-n^2\pi^2k^2}{L^2}\,t\right)\sin\frac{n\pi x}{L}$$

then $u_m(x,t)$ satisfies the IBVP with initial condition

$$u_m(x,0) = \sum_{n=1}^{m}\alpha_n \sin\frac{n\pi x}{L}.$$

Let us consider the formal series

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2}\, t\right) \sin\frac{n\pi x}{L}$$

Setting $t = 0$ we get

$$u(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin\frac{n\pi x}{L}$$

To solve our IBVP we would like to have

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \sin\frac{n\pi x}{L} \qquad 0 \le x \le L$$

Is it possible that f has such an expansion?

Given f on $[0, L]$, it has a Fourier sine series

$$f(x) = \sum_{n\ge 1} b_n \sin\frac{n\pi x}{L}$$

Definition

The formal solution of IBVP

$$\begin{aligned} u_t &= k^2 u_{xx} & 0 < x < L, \quad t > 0 \\ u(0,t) &= 0 & t > 0 \\ u(L,t) &= 0 & t > 0 \\ u(x,0) &= f(x) & 0 \le x \le L \end{aligned}$$

is

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2}\, t\right) \sin\frac{n\pi x}{L}$$

where

$$S(x) = \sum_{n=1}^{\infty} \alpha_n \sin\frac{n\pi x}{L}$$

is the Fourier sine series of f on $[0, L]$ i.e.

$$\alpha_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\, dx.$$

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2}\, t\right) \sin\frac{n\pi x}{L}$$

We say $u(x,t)$ is a formal solution, since the series for $u(x,t)$ may NOT satisfy all the requirements of IBVP.

When it does, we say it is an actual solution of IBVP.

Because of negative exponential in $u(x,t)$, the series in $u(x,t)$ converges for all $t > 0$.

Each term in $u(x,t)$ satisfies the heat equation and boundary condition.

If u_t and u_{xx} can be obtained by differentiating the series term by term, once w.r.t. t and twice w.r.t. x for $t > 0$, then u also satisfies these properties.

If $f(x)$ is continuous and piecewise smooth on $[0,L]$, then we can do it. Hence we get next result.

Theorem

$f(x)$ *: continuous and piecewise smooth on* $[0, L]$
$f(0) = f(L) = 0$

$$S(x) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi x}{L} \quad \text{with} \quad \alpha_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

is Fourier sine series of f *on* $[0, L]$*. Then the IBVP*

$$\begin{aligned}
& u_t = k^2 u_{xx} && 0 < x < L, \quad t > 0 \\
& u(0,t) = 0 && t > 0 \\
& u(L,t) = 0 && t > 0 \\
& u(x,0) = f(x) && 0 \le x \le L
\end{aligned}$$

has a solution

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n exp\left(\frac{-n^2\pi^2 k^2}{L^2}\, t \right) \sin \frac{n\pi x}{L}$$

Here u_t *and* u_{xx} *can be obtained by term-wise differentiation for* $t > 0$*.*

Example

Let $f(x) = x(x^2 - 3Lx + 2L^2)$. Solve IBVP

$$\begin{aligned} &u_t = k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ &u(0,t) = 0 && t > 0 \\ &u(L,t) = 0 && t > 0 \\ &u(x,0) = f(x) && 0 \le x \le L \end{aligned}$$

The Fourier sine expansion of $f(x)$ is

$$S(x) = \frac{12L^3}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{L}.$$

Therefore, the solution of IBVP is

$$u(x,t) = \frac{12L^3}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} exp\left(\frac{-n^2\pi^2k^2}{L^2}\, t \right) \sin \frac{n\pi x}{L}.$$

□

Neumann boundary conditions

Initial-boundary value problem is

$$\begin{aligned} u_t &= k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ u_x(0,t) &= 0 && t > 0 \\ u_x(L,t) &= 0, && t > 0 \\ u(x,0) &= f(x), && 0 \le x \le L \end{aligned}$$

Assuming the solution in the form $\boxed{v(x,t) = X(x)T(t)}$

$$v_x(0,t) = X'(0)T(t) = 0 \quad \text{and} \quad v_x(L,t) = X'(L)T(t) = 0$$

we don't want T to be identically zero, we get

$$X'(0) = 0 \quad \text{and} \quad X'(L) = 0.$$

We need to solve eigenvalue problem

$$X''(x) + \lambda X(x) = 0, \quad X'(0) = 0, \quad X'(L) = 0, \qquad (*)$$

and $\quad T'(t) = -k^2 \lambda T(t) \implies T(t) = exp(-k^2 \lambda t)$

The eigenvalues of $(*)$ are

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

with associated eigenfunctions

$$X_n = \cos\frac{n\pi x}{L}, \;\; n \geq 0.$$

We get infinitely many solutions for IBVP, one for each $n \geq 0$

$$\begin{aligned} v_n(x,t) &= T_n(t)X_n(x) \\ &= exp\left(\frac{-n^2\pi^2k^2}{L^2}\,t\right)\cos\frac{n\pi x}{L} \end{aligned}$$

Note $\qquad v_n(x,0) = \cos\frac{n\pi x}{L}$

Therefore

$$v_n(x,t) = exp\left(\frac{-n^2\pi^2k^2}{L^2}\,t\right)\cos\frac{n\pi x}{L}$$

satisfies the IBVP

$$\begin{aligned} &u_t = k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ &u_x(0,t) = 0 && t > 0 \\ &u_x(L,t) = 0 && t > 0 \\ &u(x,0) = \cos\frac{n\pi x}{L} && 0 \le x \le L \end{aligned}$$

More generally, if $\alpha_0, \ldots, \alpha_m$ are constants and

$$u_m(x,t) = \sum_{n=0}^{m} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2}\, t\right) \cos\frac{n\pi x}{L}$$

then $u_m(x,t)$ satisfies the IBVP with initial condition

$$u_m(x,0) = \sum_{n=0}^{m} \alpha_n \cos\frac{n\pi x}{L}.$$

Let us consider the formal series

$$u(x,t) = \sum_{n=0}^{\infty} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2}\, t\right) \cos\frac{n\pi x}{L}$$

Setting $t = 0$ we get

$$u(x,0) = \sum_{n=0}^{\infty} \alpha_n \cos\frac{n\pi x}{L}$$

To solve our IBVP we would like to have

$$f(x) = \sum_{n=0}^{\infty} \alpha_n \cos\frac{n\pi x}{L} \qquad 0 \leq x \leq L$$

Is it possible that f has such an expansion?

Given f on $[0, L]$, it has a Fourier cosine series

$$f(x) = \sum_{n\geq 0} a_n \cos\frac{n\pi x}{L}$$

Definition

The formal solution of IBVP

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is

$$u(x,t) = \sum_{n=0}^{\infty} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2}\, t\right) \cos \frac{n\pi x}{L}$$

where

$$S(x) = \sum_{n=0}^{\infty} \alpha_n \cos \frac{n\pi x}{L}$$

is the Fourier sine series of f on $[0, L]$ i.e.

$$\alpha_0 = \frac{1}{L} \int_0^L f(x)\, dx \qquad\qquad \alpha_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx.$$

$$u(x,t) = \sum_{n=0}^{\infty} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2}\,t\right)\cos\frac{n\pi x}{L}$$

We say $u(x,t)$ is a formal solution, since the series for $u(x,t)$ may NOT satisfy all the requirements of IBVP.

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If u_t and u_{xx} can be obtained by differentiating the series term by term, once w.r.t. t and twice w.r.t. x for $t > 0$, then u also satisfies these properties.

If $f(x)$ is continuous and piecewise smooth on $[0, L]$, then we can do it. Hence we get next result.

Theorem

$f(x)$ is continuous, piecewise smooth on $[0, L]$; $f'(0) = f'(L) = 0$.

$$S(x) = \sum_{n=1}^{\infty} \alpha_n \cos \frac{n\pi x}{L} \quad \textit{with}$$

$$\alpha_0 = \frac{1}{L} \int_0^L f(x)\, dx \qquad \alpha_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx$$

is Fourier sine series of f on $[0, L]$. Then the IBVP

$$\begin{aligned} u_t &= k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ u_x(0,t) &= 0 && t > 0 \\ u_x(L,t) &= 0 && t > 0 \\ u(x,0) &= f(x) && 0 \le x \le L \end{aligned}$$

has a solution

$$u(x,t) = \sum_{n=0}^{\infty} \alpha_n exp\left(\frac{-n^2\pi^2k^2}{L^2}\, t\right) \cos \frac{n\pi x}{L}$$

Here u_t and u_{xx} can be obtained by term-wise differentiation for $t > 0$.

Example

Let $f(x) = x$ on $[0, L]$. Solve IBVP

$$\begin{aligned} & u_t = k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ & u_x(0,t) = 0 && t > 0 \\ & u_x(L,t) = 0 && t > 0 \\ & u(x,0) = f(x) && 0 \le x \le L \end{aligned}$$

The Fourier cosine expansion of $f(x)$ is

$$C(x) = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{L}.$$

Therefore, the solution of IBVP is

$$u(x,t) =$$

$$\frac{L}{2} - \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} exp\left(\frac{-(2n-1)^2 \pi^2 k^2}{L^2} t \right) \cos \frac{(2n-1)n\pi x}{L}.$$