MA-207 Differential Equations II
Lecture-13 Laplace Equation

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S1 - Lecture 13

Consider the Dirichlet problem in a disc of radius r

$$u_{xx} + u_{yy} = 0$$

with

$$u = f$$

on the boundary of the disc, which is a circle of radius r.
To solve this problem, it is better to write the Laplace operator in polar coordinates.

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

So let us solve the following problem.

Example. Solve for harmonic function $u(r,\theta)$ in unit disc i.e.

$$\Delta u(r,\theta) = 0, \quad r < 1, \ \theta \in [0, 2\pi]$$

with boundary condition

$$u(1,\theta) = f(\theta) = \begin{cases} \sin\theta, & \theta \in [0,\pi] \\ 0, & \theta \in [\pi, 2\pi] \end{cases}$$

Laplace equation in polar coordinates is

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

Assume $u(r,\theta) = R(r)\Theta(\theta)$. Then

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$

$$\frac{R''(r) + \frac{1}{r}R'(r)}{\frac{1}{r^2}R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda$$

$$\boxed{\Theta''(\theta) + \lambda\Theta(\theta) = 0}, \quad \boxed{r^2R''(r) + rR'(r) - \lambda R(r) = 0}$$

Since $u(r,\theta+2\pi) = u(r,\theta)$, the functions Θ and Θ' need to be 2π periodic.
Thus for the ODE for Θ, we need to solve

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0, \ \ \Theta(0) = \Theta(2\pi), \ \ \Theta'(0) = \Theta'(2\pi)$$

The eigenvalues and eigenfunctions for periodic eigenvalue problem in Θ are

$$\lambda_0 = 0, \quad \Theta_0 = 1$$

and for $n \geq 1$,

$$\lambda_n = n^2, \quad \Theta_n^1(\theta) = \cos(n\theta), \quad \Theta_n^2(\theta) = \sin(n\theta)$$

The problem for R-function, namely

$$r^2 R''(r) + rR'(r) - \lambda R(r) = 0$$

is Cauchy-Euler equation with solution x^m, where

$$m(m-1) + m - \lambda = m^2 - \lambda = 0$$

$$\implies m = \pm\sqrt{\lambda}$$

For $\lambda = \lambda_0 = 0$, the general solutions are

$$R_0^1(r) = 1, \quad R_0^2(r) = \ln r$$

For $\lambda = \lambda_n = n^2$, $m = \pm n$, the general solutions are

$$R_n^1(r) = r^n, \qquad R_n^2(r) = r^{-n}$$

Thus with separated variables, the solution of Laplace equation in the disc are linear combinations of

$$1, \; \ln r, \; r^n \cos(n\theta), \; r^n \sin(n\theta), \; r^{-n} \cos(n\theta), \; r^{-n} \sin(n\theta)$$

Since we are looking for solutions that are bounded in the disc (so at $r = 0$), we will discard

$$\ln r, \; r^{-n} \cos(n\theta), \; r^{-n} \sin(n\theta)$$

Thus, the series solution has the form

$$u(r,\theta) = A_0 + \sum_{n\geq 1} \left(A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta)\right)$$

$$u(1,\theta) = f(\theta) = A_0 + \sum_{n\geq 1} (A_n \cos(n\theta) + B_n \sin(n\theta))$$

Hence, A_i and B_i are Fourier coefficients of $f(\theta)$. Check that the Fourier series of $f(\theta)$ is

$$f(\theta) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n\geq 1}\frac{\cos(2n\theta)}{4n^2-1} + \frac{1}{2}\sin\theta$$

Therefore, the solution is

$$u(r,\theta) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n\geq 1}\frac{1}{4n^2-1} r^{2n}\cos(2n\theta) + \frac{1}{2} r\sin\theta$$

Example. Solve for harmonic function $u(r,\theta)$ in an annulus

$$\begin{aligned}
\Delta u(r,\theta) &= 0, \quad 1 < r < 2,\ \theta \in [0, 2\pi] \\
u(1,\theta) &= \cos\theta, \quad 0 \leq \theta \leq 2\pi \\
u_r(2,\theta) &= \sin 2\theta, \quad 0 \leq \theta \leq 2\pi
\end{aligned}$$

This BVP can be interpreted as that for the steady state temperature distribution in an annular region where on the outer boundary the heat flux is prescribed and on the inner boundary, the temperature is prescribed.

Recall that the Laplace equation in polar coordinates is

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

As the polar coordinates (r, θ) and $(r, \theta + 2\pi)$ represent the same point in the plane, any function u defined in the plane is 2π-periodic in θ. Therefore,

$$u(r, 0) = u(r, 2\pi), \quad u_\theta(r, 0) = u_\theta(r, 2\pi)$$

Assume $u(r, \theta) = R(r)\Theta(\theta)$. Then

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$

$$\frac{R''(r) + \frac{1}{r}R'(r)}{\frac{1}{r^2}R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda$$

$\boxed{\Theta''(\theta) + \lambda\Theta(\theta) = 0}$, $\boxed{r^2R''(r) + rR'(r) - \lambda R(r) = 0}$

Since $u(r, \theta + 2\pi) = u(r, \theta)$, the functions Θ and Θ' need to be 2π periodic.
Thus for the ODE for Θ, we need to solve

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0, \quad \Theta(0) = \Theta(2\pi), \quad \Theta'(0) = \Theta'(2\pi)$$

The eigenvalues and eigenfunctions for periodic eigenvalue problem in Θ are

$$\lambda_0 = 0, \quad \Theta_0 = 1$$

and for $n \geq 1$,

$$\lambda_n = n^2, \quad \Theta_n^1(\theta) = \cos(n\theta), \quad \Theta_n^2(\theta) = \sin(n\theta)$$

The problem for R-function, namely

$$r^2 R''(r) + rR'(r) - \lambda R(r) = 0$$

is Cauchy-Euler equation with solution x^m, where

$$m(m-1) + m - \lambda = m^2 - \lambda = 0$$

$$\implies m = \pm\sqrt{\lambda}$$

For $\lambda = \lambda_0 = 0$, the general solutions are

$$R_0^1(r) = 1, \quad R_0^2(r) = \ln r, \quad u_0(r,\theta) = A_0 + B_0 \ln r$$

For $\lambda = \lambda_n = n^2$, $m = \pm n$, the general solutions are

$$R_n^1(r) = r^n, \quad R_n^2(r) = r^{-n}$$

So, we get two solutions

$$u_n^1(r,\theta) = \left(A_n^1 r^n + B_n^1 r^{-n}\right)\cos(n\theta)$$

and

$$u_n^2(r,\theta) = \left(A_n^2 r^n + B_n^2 r^{-n}\right)\sin(n\theta)$$

Hence the general solution is

$$u(r,\theta) = (A_0 + B_0 \ln r) + \sum_{n\geq 1}(u_n^1(r) + u_n^2(r))$$

Since

$$u(1,\theta) = \cos\theta, \quad u_r(2,\theta) = \sin 2\theta$$

$$u(1,\theta) = A_0 + \sum_{n\geq 1} (A_n^1 + B_n^1)\cos(n\theta) + (A_n^2 + B_n^2)\sin(n\theta)$$

Compare with $u(1,\theta) = \cos\theta$, we get $A_0 = 0$,

$A_1^1 + B_1^1 = 1,\ A_n^1 + B_n^1 = 0\ (n \geq 2),\ A_n^2 + B_n^2 = 0\ (n \geq 1)$

$$u_r(r,\theta) = \frac{B_0}{r} + \sum_{n\geq 1} n(A_n^1 r^{n-1} - B_n^1 r^{-n-1})\cos n\theta$$
$$+ n(A_n^2 r^{n-1} - B_n^2 r^{-n-1})\sin n\theta$$

Compare with $u_r(2,\theta) = \sin 2\theta$, we get $B_0 = 0$,

$2(2A_2^2 - 2^{-3}B_2^2) = 1$

$A_n^1 2^{n-1} - B_n^1 2^{-n-1} = 0,\ A_n^2 2^{n-1} - B_n^2 2^{-n-1} = 0\ (n \neq 2)$

$$A_0 = 0 = B_0$$

For $n \geq 2$,

$$A_n^1 + B_n^1 = 0,\ A_n^1 2^{n-1} - B_n^1 2^{-n-1} = 0 \implies A_n^1 = 0 = B_n^1$$

$$A_1^1 + B_1^1 = 1,\ A_1^1 - B_1^1 2^{-2} = 0 \implies A_1^1 = \frac{1}{5}, B_1^1 = \frac{4}{5}$$

For $n \neq 2$,

$$A_n^2 + B_n^2 = 0,\ A_n^2 2^{n-1} - B_n^2 2^{-n-1} = 0 \implies A_n^2 = 0 = B_n^2$$

$$A_2^2 + B_2^2 = 0,\ 2A_2^2 - \frac{1}{2^3}B_2^2 = \frac{1}{2} \implies A_2^2 = \frac{4}{17}, B_2^2 = \frac{-4}{17}$$

Thus the solution is

$$u(r,\theta) = (\frac{1}{5}r + \frac{4}{5}r^{-1})\cos\theta + (\frac{4}{17}r^2 + \frac{-4}{17}r^{-2})\sin 2\theta$$