MA-207 Differential Equation II
S1 - Lecture 2

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21st September, 2017
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Recall

In the last lecture, we studied the following topics:

power series,

how to find its radius of convergence,

for example, the radius of convergence of $\sum_{0}^{\infty} x^{n!}$ is 1.

In today's lecture, we will discuss analytic functions and finding solution of linear ODE using power series method.

What is the crucial difference between linear and non-linear ODE?

Definition

Assume $f(x) : (x_0 - a, x_0 + a) \to \mathbb{R}$ $(a > 0)$ is infinitely differentiable at x_0.
Then the Taylor series of f at x_0 is the power series

$$TS\, f|_{x_0} := \sum_{0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

When $x_0 = 0$, the series is also called the Maclaurin series of f.

Example

The function $\quad f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$
is infinitely differentiable and $f^{(n)}(0) = 0$ for all n.

Hence the Taylor series $TS\, f|_0$ is zero function.

So, Taylor series of f at 0 does not converge to function $f(x)$ on any open interval around 0.

Definition

Suppose

- $f(x)$ is infinitely differentiable at x_0; and
- Taylor series of f at x_0 converges to $f(x)$ for all x in some open interval around x_0;

Then f is called analytic at x_0.

Example

The function $\quad f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$
is not analytic at 0, since second condition fails.

However, f is analytic at all $x \neq 0$.

Theorem (Analytic functions)

1. *Polynomials, e^x, $\sin x$ and $\cos x$ are analytic at all $x \in \mathbb{R}$.*
2. *$f(x) = x^{5/3}$ is analytic at all x except $x = 0$.* differentiable but not infinitely differentiable
3. *If f and g are analytic at x_0, then $f \pm g$, $f.g$, and f/g (if $g(x_0) \neq 0$) are analytic at x_0.*
4. *$f(x) = \tan x$ is analytic at all x except at $x = n\pi/2$, where $n = \pm 1, \pm 3, \pm 5 \ldots$.*
5. *If f is analytic at x_0 and g is analytic at $f(x_0)$, then $g \circ f$ is analytic at x_0.*
6. Text
   If a power series $\sum_0^\infty a_n(x - x_0)^n$ has radius of convergence $R > 0$, then the function $f(x) := \sum_0^\infty a_n(x - x_0)^n$ is analytic at all points $x \in (x_0 - R, x_0 + R)$.

Example

The function $f(x) = x^2 + 1$ is analytic everywhere.

Since $x^2 + 1$ is never 0, the function $h(x) := \frac{1}{x^2+1}$ is analytic everywhere.

However, there is no power series around 0 which represents $h(x)$ everywhere.

If there were such a power series, then by uniqueness, it has to be the power series expansion of $h(x)$ around 0, which is

$$1 - x^2 + x^4 - x^6 + \cdots$$

However, the radius of convergence of this is $R = 1$.

In fact, for any x_0, there is no power series around x_0 which represents $h(x)$ everywhere.

Power series solution of ODE

Theorem (Existence Theorem)

If $p(x)$ and $q(x)$ are analytic functions at x_0, then every solution of

$$y'' + p(x)y' + q(x)y = 0$$

is also analytic at x_0;
Therefore any solution can be expressed as

$$y(x) = \sum_{0}^{\infty} a_n (x - x_0)^n$$

If $R_1 =$ radius of convergence of Taylor series of $p(x)$ at x_0,
$R_2 =$ radius of convergence of Taylor series of $q(x)$ at x_0,
then radius of convergence of $y(x)$ is atleast min$(R_1, R_2) > 0$.

In most applications, $p(x)$ and $q(x)$ are rational functions, so let us see how to compute their radius of convergence.

Theorem

Let $N(x)$ and $D(x)$ are polynomials without any common factors, i.e. they do not have any common (complex) zeros. Let

$$F(x) = \frac{N(x)}{D(x)} \quad \text{e.g.} \quad F(x) = \frac{x^3 - 1}{x^2 + 1}$$

be a rational function. Let $\alpha_1, \ldots, \alpha_r$ be complex zeros of $D(x)$.
Then $F(x)$ is analytic at all x except at $x \in \{\alpha_1, \ldots, \alpha_r\}$.
If x_0 is different from $\{\alpha_1, \ldots, \alpha_r\}$, then
the radius of convergence R of the Taylor series of F at x_0

$$TS\, F_{x_0} = \sum_{0}^{\infty} \frac{F^{(n)}(x_0)}{n!}(x - x_0)^n$$

is given by

$$R = \min\{|x_0 - \alpha_1|, |x_0 - \alpha_2|, \ldots, |x_0 - \alpha_r|\}$$

Example

If

$$F(x) = \frac{N(x)}{D(x)} = \frac{1}{1-x}$$

then $D(x)$ has zeros at $x = 1$.

Hence F is analytic at all x except at $x = 1$.

If $x = 0$, then the radius of convergence of Taylor series of F at $x = 0$ is 1.

If $x = 3$, then the radius of convergence of Taylor series of F at $x = 3$ is 2.

If $x = -3$, then the radius of convergence of Taylor series of F at $x = -3$ is 4.

Example

If

$$F(x) = \frac{N(x)}{D(x)} = \frac{(2+3x)}{(4+x)(9+x^2)}$$

then $D(x)$ has zeros at -4 and $\pm 3\iota$, where $\iota = \sqrt{-1}$.

Hence F is analytic at all x except at $x \in \{-4, \pm 3\iota\}$.

If $x = 2$, then radius of convergence of Taylor series of F at $x = 2$ is

$$\text{min}\,\{|2+4|, |2+3\iota|, |2-3\iota|\} = \text{min}\,\{6, \sqrt{13}\} = \sqrt{13}$$

If $x = -6$, then radius of convergence of Taylor series of F at $x = -6$ is

$$\text{min}\,\{|-6+4|, |-6 \pm 3\iota|\} = \text{min}\,\{2, \sqrt{45}\} = 2$$

Let us solve some ODE using power series method.

Series solution of ODE

Example

Let us solve $\boxed{y'' + y = 0 \qquad (1)}$ by power series method.

Compare with $y'' + p(x)y' + q(x)y = 0$,
$p(x) = 0$ and $q(x) = 1$ are analytic at all x.
We can find power series solution in $x - x_0$ for any x_0.
Let us assume $x_0 = 0$ for simplicity.
By existence theorem, all solution of (1) can be found in the form

$$y(x) = \sum_0^\infty a_n x^n$$

and the series will have ∞ radius of convergence.
Since

$$y'' = \sum_2^\infty n(n-1)a_n x^{n-2} = \sum_{n=0}^\infty (n+2)(n+1)a_{n+2}x^n$$

Example (Continue ...)

$$y'' + y = \sum_0^\infty ((n+2)(n+1)a_{n+2} + a_n)x^n = 0$$

By uniqueness of power series with positive radius of convergence, we get the recursion formula

$$(n+2)(n+1)a_{n+2} + a_n = 0$$

$$\implies a_{n+2} = \frac{-1}{(n+2)(n+1)}a_n \quad \forall n$$

Therefore,

$$a_2 = \frac{-1}{2.1}a_0, \quad a_4 = \frac{-1}{4.3}a_2 = \frac{1}{4!}a_0 \;\; \ldots \;\; a_{2n} = (-1)^n \frac{1}{(2n)!}a_0$$

$$a_3 = \frac{-1}{3.2}a_1, \quad a_5 = \frac{-1}{5.4}a_3 = \frac{1}{5!}a_1 \;\; \ldots \;\; a_{2n+1} = (-1)^n \frac{1}{(2n+1)!}a_1$$

Example (Continue ...)

Choosing $a_0 = 1$ and $a_1 = 0$, we get a solution

$$y_1(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots$$

and choosing $a_0 = 0$ and $a_1 = 1$, we get another solution

$$y_2(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots$$

General solution of ODE $y'' + y = 0$ is

$$y(x) = a_0 y_1(x) + a_1 y_2(x) = a_1 \cos x + a_2 \sin x$$

which is an elementary function.
In general, the solution may not be an elementary function.
We don't need to check the series for converges, since the existence theorem guarantees that the series converges for all x.

Steps for Series solution of linear ODE

1. Write ODE in standard form $y'' + p(x)y' + q(x)y = 0$.
2. Choose x_0 at which $p(x)$ and $q(x)$ are analytic.
   If initial conditions are given, choose that point as x_0.
3. Find minimum of radius of convergence of Talor series of $p(x)$ and $q(x)$ at x_0.
4. Let $y(x) = \sum_0^\infty a_n (x - x_0)^n$. Compute the power series for $y'(x)$ and $y''(x)$ at x_0 and substitute these into the ODE.
5. Set the coefficients of $(x - x_0)^n$ to zero and find recursion formula.
6. From the recursion formula, obtain linearly independent solutions $y_1(x)$ and $y_2(x)$. The general solution then looks like $y(x) = a_1 y_1(x) + a_2 y_2(x)$.

The following ODE's are classical:

- Bessel's equation :

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

It occurs in problems displaying cylindrical symmetry, e.g. diffusion of light through a circular aperture, vibration of a circular head drum, etc.

- Airy's equation :

$$y'' - xy = 0$$

It occurs in astronomy and quantum physics.

- Legendre's equation :

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$$

It occurs in problems displaying spherical symmetry, particularly in electromagnetism.

In this course, we will consider ODE

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$$

with $P_i(x)$ polynomials for $i = 0, 1, 2$ without any common factor. If we write ODE in the standard form

$$y'' + \frac{P_1(x)}{P_0(x)}y' + \frac{P_2(x)}{P_0(x)}y = 0$$

we see that if x_0 is not a zero of $P_0(x)$, then $P_1(x)/P_0(x)$ and $P_2(x)/P_0(x)$ will be analytic at x_0 hence we can find the series solution of ODE in the form

$$y(x) = \sum_0^\infty a_n(x - x_0)^n$$

When x_0 is a zero of $P_0(x)$, then x_0 is called a singular point of ODE. This case will be considered later.

Example

Find the power series in x for the general solution of

$$(1+2x^2)y'' + 6xy' + 2y = 0$$

Solution. Note that 0 is not a zero of $P_0(x) = 1 + 2x^2$, hence the series solution in powers of x exists.

Put $y = \sum_0^\infty a_n x^n$ in the ODE, we get

$$\begin{aligned}
&(1+2x^2)y'' + 6xy' + 2y \\
&= y'' + 2x^2y'' + 6xy' + 2y \\
&= \sum_0^\infty ((n+2)(n+1)a_{n+2} + 2n(n-1)a_n + 6na_n + 2a_n)x^n
\end{aligned}$$

$$\implies (n+2)(n+1)a_{n+2} + [2n(n-1) + 6n + 2]a_n = 0$$

Example (Continue ...)

$$\implies a_{n+2} = -\frac{2n^2+4n+2}{(n+2)(n+1)}\, a_n = -2\frac{n+1}{(n+2)}\, a_n \quad n \geq 0$$

Since indices on left and right differ by 2, we write separately for $n = 2m$ and $n = 2m+1$, $\;m \geq 0$, so

$$\begin{aligned}
a_{2m+2} &= -2\frac{2m+1}{2m+2}\, a_{2m} = -\frac{2m+1}{m+1}\, a_{2m} \\
a_{2m+3} &= -2\frac{2m+2}{2m+3}\, a_{2m+1} = -4\frac{m+1}{2m+3}\, a_{2m+1} \\
a_2 &= -\frac{1}{1}\, a_0 \\
a_4 &= -\frac{3}{2}\, a_2 = \frac{1.3}{1.2}\, a_0 \\
a_6 &= -\frac{5}{3}\, a_4 = -\frac{1.3.5}{1.2.3}\, a_0
\end{aligned}$$

Example (Continue ...)

$$a_{2m} = (-1)^m \frac{1.3.5.\ldots.(2m-1)}{m!}\, a_0$$
$$= (-1)^m \frac{\prod_{j=1}^{m}(2j-1))}{m!}\, a_0$$
$$a_{2m+3} = -4\frac{m+1}{2m+3}\, a_{2m+1}$$
$$a_3 = -4\frac{1}{3}\, a_1$$
$$a_5 = -4\frac{2}{5}\, a_3 = 4^2\frac{1.2}{3.5}\, a_1$$
$$a_7 = -4\frac{3}{7}\, a_5 = -4^3\frac{1.2.3}{3.5.7}\, a_1$$
$$a_{2m+1} = (-1)^m 4^m \frac{m!}{\prod_{j=1}^{m}(2j+1)}\, a_1$$

Example (Continue ...)

We can write the solution

$$y = \sum_{0}^{\infty} a_n x^n = a_0 y_1(x) + a_1 y_2(x)$$

where a_0 and a_1 are arbitrary scalars and

$$y_1(x) = \sum_{m=0}^{\infty} (-1)^m \frac{\prod_{j=1}^{m}(2j-1)}{m!} x^{2m}$$

$$y_2(x) = \sum_{m=0}^{\infty} (-1) \frac{4^m m!}{\prod_{j=1}^{m}(2j+1)} x^{2m+1}$$

Since $P_0(x) = 1 + 2x^2$ has complex zeros $\frac{\pm\iota}{\sqrt{2}}$, the power series solution converges in the interval $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. $\square$

Example

Find the coefficients $a_0, \ldots, a_6$ in the series solution

$$y = \sum_0^\infty a_n x^n$$

of the IVP

$$(1 + x + 2x^2)y'' + (1 + 7x)y' + 2y = 0$$

with

$$y(0) = -1, \; y'(0) = -2.$$

Zeros of $P_0(x) = 1 + x + 2x^2$ are $\frac{1}{4}(-1 \pm \iota\sqrt{7})$ whose absolute values are $1/\sqrt{2}$. Hence the series solution to the IVP converges on the interval $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Example (Continue ...)

$$(1 + x + 2x^2)y'' + (1 + 7x)y' + 2y = \sum_0^\infty b_n x^n = 0$$

$$\begin{aligned} b_n &= (n+2)(n+1)a_{n+2} + (n+1)na_{n+1} + 2n(n-1)a_n \\ &\quad + (n+1)a_{n+1} + 7na_n + 2a_n = 0 \end{aligned}$$

i.e.

$$(n+2)(n+1)a_{n+2} + (n+1)^2 a_{n+1} + (2n^2 + 5n + 2)a_n = 0$$

Since $2n^2 + 5n + 2 = (n+2)(2n+1)$,

$$a_{n+2} = -\frac{n+1}{n+2}a_{n+1} - \frac{2n+1}{n+1}a_n \qquad n \geq 0$$

Example (Continue ...)

$$a_{n+2} = -\frac{n+1}{n+2} a_{n+1} - \frac{2n+1}{n+1} a_n \qquad n \geq 0$$

From the initial conditions $y(0) = -1, \ y'(0) = -2$ we get

$$a_0 = y(0) = -1, \quad a_1 = y'(0) = -2$$

$$a_2 = -\frac{1}{2}a_1 - a_0 = 2$$

$$a_3 = -\frac{2}{3}a_2 - \frac{3}{2}a_1 = \frac{5}{3}$$

Check that

$$y(x) = -1 - 2x + 2x^2 + \frac{5}{3}x^3 - \frac{55}{12}x^4 + \frac{3}{4}x^5 + \frac{61}{8}x^6 + \dots$$

Vector spaces

We will recall the notion of Inner product space from Linear Algebra.

Recall the notion of a vector space V over $\mathbb{R}$.
Elements of V are called vectors and elements of $\mathbb{R}$ are called scalars.
There are two operations in a vector space, namely,

- addition

$$v + w, \quad v, w \in V$$

- scalar multiplication

$$cv, \quad c \in \mathbb{R}, \ \ v \in V$$

Any vector space V has a dimension, which may not be finite.

Inner product spaces

Let V be a vector space over $\mathbb{R}$ (not necessarily finite-dimensional).
A bilinear form on V is a map

$$\langle , \rangle : V \times V \to \mathbb{R}$$

which is linear in both coordinates, that is,

$$\langle au + v, w \rangle = a\langle u, w \rangle + \langle v, w \rangle$$

$$\langle u, av + w \rangle = a\langle u, v \rangle + \langle u, w \rangle$$

for $a \in \mathbb{R}$ and $u, v \in V$.
An inner product on V is a bilinear form on V which is

- symmetric: $\langle v, w \rangle = \langle w, v \rangle$
- positive definite: $\langle v, v \rangle \geq 0$ for all v and $\langle v, v \rangle = 0$ iff $v = 0$

A vector space with an inner product is called an inner product space.