MA205-6

Applications :

Theorem (Jordan's Lemma) :

Let f be a continuous function defined on the semicircular contour : $C_R = \{ Re^{i\theta} \mid \theta \in [0,\pi] \}$ of the form $f(z) = e^{iaz} g(z)$: with $a > 0$.

Let $M_R = \max\limits_{\theta \in [0,\pi]} |g(Re^{i\theta})|$.

Then $$\left| \int_{C_R} f(z)\, dz \right| \leq \frac{\pi}{a} M_R$$

Proof: $$\int_{C_R} f(z) dz = \int_0^{\pi} g(Re^{i\theta}) . e^{iaR(\cos\theta + i\sin\theta)} . iRe^{i\theta} d\theta$$

$$\Rightarrow \left| \int_{C_R} f(z)\,dz \right| \le R \int_0^{\pi} |g(Re^{i\theta})\, e^{aR(i\cos\theta - \sin\theta)} \cdot ie^{i\theta}|\,d\theta$$

$$= R \int_0^{\pi} |g(Re^{i\theta})|\, e^{-aR\sin\theta}\, d\theta$$

$$\le R\cdot M_R \int_0^{\pi} e^{-aR\sin\theta}\, d\theta$$

$$\le 2RM_R \int_0^{\pi/2} e^{-aR\sin\theta}\, d\theta$$

Check: $\displaystyle\int_0^{\pi/2} e^{-aR\sin\theta}\, d\theta < \frac{\pi}{2aR}$ (see the next example)

$$\Rightarrow \left| \int_{C_R} f(z)\,dz \right| \le \frac{2RM_R\,\pi}{2aR} = \frac{\pi}{a} M_R .$$

Example:

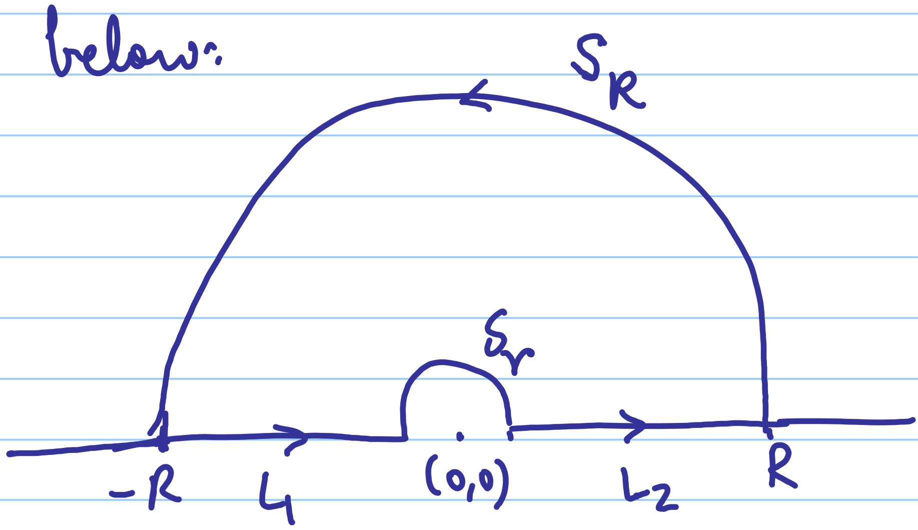
Evaluate $\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$

Take $f(z) = \frac{e^{iz}}{z}$

Note that the imaginary part of $\frac{e^{ix}}{x}$ is $\frac{\sin x}{x}$.

$f(z)$ has a simple pole at the origin.

To avoid it, we make an "indentation", as below:



$$f(z) = \frac{e^{iz}}{z}$$

By Cauchy's integral theorem,

$$\int_{L_1} f(z)dz + \int_{-S_r} f(z)dz + \int_{L_2} f(z)dz + \int_{S_R} f(z)dz = 0$$

<u>Note:</u> $\displaystyle\int_{L_1} f(z)dz + \int_{L_2} f(z)dz$

$$= \int_{-R}^{-r} \frac{e^{ix}}{x}\,dx + \int_{r}^{R} \frac{e^{ix}}{x}\,dx$$

$$= i\int_{-R}^{-r} \frac{\sin x}{x}\,dx + i\int_{r}^{R} \frac{\sin x}{x}\,dx \qquad (*)$$

As $R \to \infty$ the above $(*) \longrightarrow i\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$

We shall show $\displaystyle\int_{S_R} f(z)dz \longrightarrow 0$ as $R \to \infty$

(follows from Jordan's lemma)

Let $z = Re^{it}$ $\quad : \quad 0 \le t \le \pi$

$dz = Rie^{it}dt \quad : \quad \dfrac{dz}{z} = i\,dt$

$$\int_{S_R} f(z)\,dz = i\int_0^{\pi} e^{iRe^{it}} \cdot dt$$

$$\left|\int_{S_R} f(z)\,dz\right| \leq \int_0^{\pi} e^{-R\sin t}\,dt \qquad \text{(why?)}$$

Now use the inequality:

$$\sin t \geq \frac{2t}{\pi} \quad : \quad 0 \leq t \leq \frac{\pi}{2}$$

$$\left|\int_{S_R} f(z)\,dz\right| \leq \int_0^{\pi} e^{-R\sin t}\,dt$$

$$\leq 2\int_0^{\pi/2} e^{-R\sin t} \cdot dt$$

$$\sin t \geq \frac{2t}{\pi} \Rightarrow e^{-R\sin t} \leq e^{-\frac{2Rt}{\pi}}$$

$$\therefore \left|\int_{S_R} f(z)\,dz\right| \leq 2\int_0^{\pi/2} e^{-2Rt/\pi}\,dt$$

Show that $\int_0^{\pi/2} e^{-2Rt/\pi}\,dt \to 0$ as $R \to \infty$

Next we calculate $\lim_{r \to 0} \int_{S_r} f(z)\,dz$

$$\int_{S_r} f(z)\,dz = \int_{S_r} \frac{e^{iz}}{z}\,dz$$

$$= \int_{S_r} \frac{e^{iz}-1}{z}\,dz + \int_{S_r} \frac{dz}{z}$$

$$= \int_{S_r} \frac{e^{iz}-1}{z}\,dz - i\pi$$

Note : $\frac{e^{iz}-1}{z}$ has a removable singularity at 0. Hence $\exists$ $M > 0$ such that

$$\left| \frac{e^{iz}-1}{z} \right| \leq M \quad : \text{for } |z| \leq 1$$

(why ?)

Hence $\lim_{r \to 0} \int_{S_r} \frac{e^{iz}-1}{z}\,dz \leq \lim_{r \to 0} M \cdot (\text{length of } S_r) = 0$

$$\therefore \lim_{r\to 0} \int_{S_r} \frac{e^{iz}}{z}\,dz = -\pi i$$

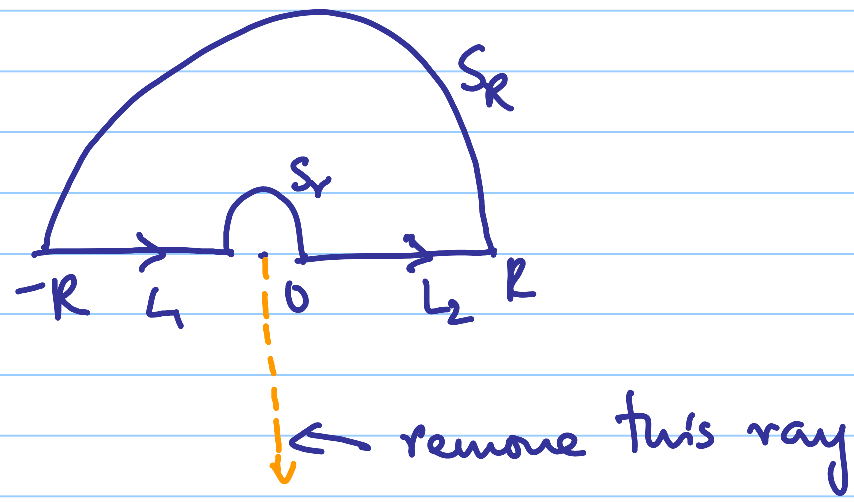
$$\therefore \quad i\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = i\pi$$

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi$$

Example: Evaluate $I = \int_0^{\infty} \frac{x^p}{(1+x^2)^2}$: $-1<p<3$

Take $f(z) = \frac{z^p}{(1+z^2)^2}$

Note: As z^p is defined using log, at $z=0$ there is a branch point. We choose a semicircular contour with indentation as below:



$$\log z = \ln|z| + i \arg z \qquad : -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$$

$$z^p = \exp(p \log z)$$

<u>Check:</u> $\displaystyle\lim_{R\to\infty} \int_{S_R} f(z)\,dz = 0$

$$\lim_{r\to 0} \int_{S_r} f(z)\,dz = 0$$

$$\int_{L_1} f(z)\,dz = \int_r^R \frac{x^p}{(1+x^2)^2} \longrightarrow I \quad \text{as } R\to\infty \;\&\; r\to 0$$

$$\int_{L_2} f(z)\,dz = \int_{-R}^{-r} \frac{|t|^p}{(1+t^2)^2} e^{i\pi p} \longrightarrow -e^{ip\pi} I \quad \text{as } R\to\infty \;\&\; r\to 0$$

If $C = S_R \cup L_1 \cup -S_r \cup L_2$ then

$$\int_C f(z)\,dz = \int_C \frac{z^p/(z+i)^2}{(z-i)^2}\,dz$$

$$= 2\pi i \frac{d}{dz} \frac{z^p}{(z+i)^2} \Bigg|_{z=-i} \quad (**)$$

$$\left(\because f^{n}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z_0)^{n+1}}\,dw \right.$$

Let $R \to \infty$ & $r \to 0$ in $(**)$ to get the value of the integral.

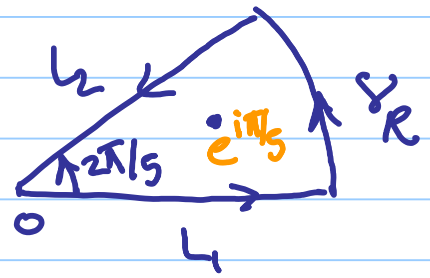
Example: Evaluate $I = \int_0^\infty \frac{dx}{1+x^5}$

Let $f(z) = \frac{1}{1+z^5}$

f has simple poles which are roots of

$1+z^5 = 0$.

$e^{i\pi/5}$ is one root. Consider the sector



$$\int_{L_1} f(z)\,dz + \int_{\gamma_R} f(z)\,dz + \int_{L_2} f(z)\,dz = 2\pi i \operatorname{Res}(f; e^{i\pi/5})$$

$\longrightarrow (*)$

check: $\lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz = 0$

$$\int_{L_1} f(z)\,dz = \int_0^R \frac{dt}{1+t^5} \longrightarrow I \quad \text{as } R\to\infty$$

On $-L_2$: $z = t e^{2\pi i/5}$: $0 \le t \le R$

$$dz = e^{2\pi i/5}\,dt$$

$$\int_{L_2} f(z)\,dz = -\int_0^R \frac{e^{2\pi i/5}\,dt}{1+t^5}$$

$$= -e^{2\pi i/5} \int_0^R \frac{dt}{1+t^5} \longrightarrow -e^{2\pi i/5}\cdot I$$

as $R\to\infty$

Letting $R\to\infty$ in $(*)$, we get:

$$I\left(1-e^{2\pi i/5}\right) = 2\pi i \operatorname{Res}\left(f; e^{\pi i/5}\right)$$

$$I = \frac{2\pi i}{\left(1-e^{2\pi i/5}\right)} \cdot \operatorname{Res}\left(f; e^{\pi i/5}\right)$$

As we have a simple pole,

$$\operatorname{Res}(f; \underset{z_0}{\underset{\parallel}{e^{\pi i/5}}}) = \lim_{z \to z_0} \frac{z - z_0}{1 + z^5}$$

$$= \frac{1}{5 z_0^4} = \frac{z_0}{5 z_0^5}$$

$$= -\frac{z_0}{5} = -\frac{1}{5} e^{i\pi/5}$$

$$\therefore I = \frac{-\pi}{5} \frac{2i \, e^{i\pi/5}}{1 - e^{2\pi i/5}}$$

$$= \frac{\pi}{5} \frac{2i}{e^{i\pi/5} - e^{-i\pi/5}} = \frac{\pi}{5} \operatorname{cosec} \frac{\pi}{5}$$

Example: Prove that if a is a rational number such that $0 < a < 1$ then

$$\int_0^\infty \frac{t^{a-1}}{1+t}\,dt = \frac{\pi}{\sin \pi a}$$

Does the result hold for all real numbers $a \in (0,1)$? (Solution in next class)

From tutorial sheet

Example: Determine the residue at each singularity of $\operatorname{cosec} z \cdot \operatorname{cosech} z$.

Let $f(z) = \operatorname{cosec} z \cdot \operatorname{cosech} z = \dfrac{1}{\sin z \cdot \sinh z}$

Singularities of f are $n\pi$ & $in\pi$: $n = 0, \pm 1, \ldots$

check these are simple poles of f, except for $n=0$, at 0 there is a double pole.

Let z_0 be a pole of f.

Then $\lim_{z \to z_0} (z - z_0) \dfrac{1}{f(z)} = 0$ $\left(\text{as } \lim_{z\to z_0} f(z) = \infty\right)$

By RRST, $\dfrac{1}{f(z)}$ has a removable singularity at z_0. Let g be holomorphic in $B_\varepsilon(z_0)$ for some $\varepsilon > 0$ & $g(z) = \dfrac{1}{f(z)}$: $z \neq z_0$

$= 0$: $z = z_0$

So $g(z) = (z - z_0) g_1(z)$: $g_1(z_0) \neq 0$

$\Rightarrow f(z) = \dfrac{1/g_1(z)}{(z - z_0)}$

$$\oint f(z) = \oint \frac{1/g_1(z)}{z - z_0}\, dz = 2\pi i \frac{1}{g_1(z_0)} \longrightarrow (*)$$

(by CIF)

Now $g(z) = \sin z \cdot \sinh z = (z - z_0) \cdot g_1(z)$

$\Rightarrow \quad g'(z) = g_1(z) + (z - z_0)\, g_1'(z_0)$

$\Rightarrow \quad g'(z_0) = g_1(z_0)$

Substitute in $(*)$ to get $\oint f(z)\,dz$

Then use CRT in $B_\varepsilon(z_0)$ to get

$\operatorname{Res}(f; z_0)$.

$\operatorname{Res}(f; 0) = ?$

Some students - especially Amiya, Anmol & Vedant - were discussing the following questions from the last tutorial sheet with me. As the explanations may be useful to you too, I'm writing them here.

1) Locate & classify the singularities of:

(iii) $f(z) = \sin(wz)\sin(1/z)\dfrac{1}{1+z^4}$: $w = \exp\left(\dfrac{i\pi}{4}\right)$

It is easy to see that solutions of $1+z^4$ form simple poles. We consider the remaining singularity at $z=0$.

We'll show that the singularity at $z = 0$ is essential by showing that

$\lim_{z \to 0} z^m \cdot f(z)$ does not exist for every positive integer $m > 0$.

(Check that implies that the principal part of the Laurent series of f about 0 is infinite, i.e., 0 is an essential singularity of f).

First note that $\sin(1/z)$ has 0 as an essential singularity (this can be seen from the Laurent series expansion of $\sin(1/z)$ about 0).

$\Rightarrow \lim\limits_{z\to 0} z^m \cdot \sin(1/z)$ does not exist for any positive integer m

(as the series expansion of $\sin(1/z)$ has infinitely many negative powers of z).

So $\lim\limits_{z\to 0} z^m \cdot f(z) = \lim\limits_{z\to 0} z^m \cdot \sin(1/z) \cdot \sin(wz) \cdot \frac{1}{1+z^4}$

$= \lim\limits_{z\to 0} z^{m+1} \cdot w \sin(1/z) \cdot \frac{\sin(wz)}{wz} \cdot \frac{1}{(1+z^4)}$

Now, $\lim\limits_{z\to 0} \frac{\sin(wz)}{wz} = 1 = \lim\limits_{z\to 0} \frac{1}{1+z^4}$.

Use: $\lim\limits_{z\to z_0} h(z) \cdot g(z)$ does not exist <u>iff</u>

$\lim\limits_{z\to z_0} g(z) \neq 0$ & $\lim\limits_{z\to z_0} f(z)$ does not exist.

$\Rightarrow \lim\limits_{z\to 0} z^m \cdot f(z)$ does not exist

i.e. 0 is an essential singularity.

Q1)(iv) $g(z) = \log z \cdot \sin(1/z) \cdot \sin(\cos z) \cdot \frac{1}{1+z^4}$

$= \log z \cdot f(z)$: f as in (iii) above.

- As in (iii) solutions of $1+z^4=0$ are simple poles.

- At $z=0$: $\log z$ has a non-isolated singularity (log is the principal branch) as its not defined on the negative real axis

$\Rightarrow$ $z=0$ is a non-isolated zero of the function g above.

Q1) (v) ~ similar to (iv).