MA 205 Complex Analysis: Exponential Function

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Introduction

So what did we do in the last class? We looked at power series, saw the existence of radius of convergence for any power series, remember that this was given by

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}},$$

and let me repeat once again, it's limsup and not the limit in the above and thus it always exists. If

$$\lim_{n\to\infty} \sqrt[n]{|a_n|}$$

exists, we can replace limsup by lim in the formula since limsup equals lim in that case. If

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

exists, then this too gives R.

Consider the power series: $\frac{1}{2} + \frac{1}{3}z + \left(\frac{1}{2}\right)^2 z^2 + \left(\frac{1}{3}\right)^2 z^3 + \dots$. The radius of convergence is $\sqrt{2}$.[1] For the root test,

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

is $1/\sqrt{2}$, so $R = \sqrt{2}$. By the way note that the series does converge at $z = 1$. Can you apply the ratio test? What about $\frac{1}{2} + \left(\frac{1}{2}\right)^2 z^2 + \left(\frac{1}{2}\right)^3 z^4 + \dots$ and $\frac{1}{3}z + \left(\frac{1}{3}\right)^2 z^3 + \left(\frac{1}{3}\right)^3 z^5 + \dots$? Both the tests apply to both these series and you get R to be $\sqrt{2}$ for the first and $\sqrt{3}$ for the second. Thus R for the original series is $\sqrt{2}$!

[1] Thanks to Koustubh for the correction!

Introduction

Today, we'll use our knowledge of power series to construct a few basic functions. Before that let's first recall that the derivative of a constant function is zero. On the other hand, if the derivative is zero throughout its domain, can we say that the function has to be a constant? This is certainly true for functions of a real variable (Why? MVT), and then it's an easy check that this remains true in the complex case too.

Now consider the power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots.$$

We've seen that its radius of convergence is ∞; i.e., this function is well-defined for any $z \in \mathbb{C}$. This function will keep our company throughout this course! So let's befriend it a bit more!

Exponential Function

If

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots,$$

then $f(0) = 1$. What's $f'(z)$? Can do it term by term as we've seen in the last lecture. So,

$$f'(z) = f(z).$$

Now that this function remains invariant under differentiation, calling it by some random f is inappropriate. We'll rename the function as

$$\exp(z).$$

Note that

$$(\exp(bz))' = b\exp(bz).$$

Now consider the function

$$h(z) = \exp(z) \cdot \exp(-z).$$

This is defined throughout $\mathbb{C}$. What's $h'(z)$?

$$h'(z) = \exp(z) \cdot (-\exp(-z)) + \exp(-z) \cdot \exp(z) = -h + h = 0.$$

This happens throughout $\mathbb{C}$. Conclusion? $h(z) \equiv c$. Can we specify the constant as well? Yes, $c = 1$, since $h(0) = 1$. Thus, we have proved two things:
(i) $\exp(z)$ is non-vanishing; it's never zero.
(ii)

$$\exp(-z) = \frac{1}{\exp(z)}.$$

What else? Note that the derivative of $f(z) = a\exp(bz)$ is $f'(z) = bf(z)$. Interestingly, the converse is also true. Thus,

$$f(z) = a\exp(bz) \text{ for } a, b \in \mathbb{C} \iff f'(z) = bf(z).$$

Proof: Assume $f'(z) = bf(z)$ for $b \in \mathbb{C}$. Now consider

$$h(z) = f(z)\exp(-bz).$$

Then, $h'(z) = -bh + bh = 0$, for all z in the domain. So, $h(z) \equiv a$ for some $a \in \mathbb{C}$. Therefore,

$$f(z) = \frac{a}{\exp(-bz)} = a\exp(bz),$$

by what we already know.

Corollary: $f' = f$ and $f(0) = 1$ characterizes the exponential function.
Recall that we had earlier defined the function

$$f(z) = e^x(\cos y + \imath \sin y),$$

and observed that it's holomorphic throughout $\mathbb{C}$ and $f' = f$.
Clearly, $f(0) = 1$ as well. Thus,

$$\exp(z) = e^x(\cos y + \imath \sin y).$$

Remark: e^x here is e to the power of x, and e is the number that you know from MA 105 (base of natural logarithm). What was logarithm for you in MA 105? In any case, forget all these. We'll reconstruct everything from scratch! So as of now, we don't know e, we have no e^x, we have no logarithm! Very soon, we'll have all these!

By now we know that exp is defined throughout $\mathbb{C}$ and that 0 is not in the range of $\exp(z)$. Thus, exp is a map from $\mathbb{C} \to \mathbb{C}^\times$. Now exp has this wonderful property that it takes the "correct" operation in $\mathbb{C}$ to the "correct" operation in $\mathbb{C}^\times$.

$$\exp(w+z) = \exp(w) \cdot \exp(z).$$

In case fanciful language appeals to you, such a "correct" map in the right context, as is the case here, is called a homomorphism. Thus, exp is a homomorphism from $\mathbb{C}$ to $\mathbb{C}^\times$.

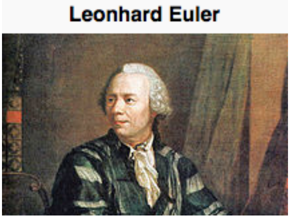
Proof: Fix $w \in \mathbb{C}$. Then the function $f(z) = \exp(w+z)$ is holomorphic in $\mathbb{C}$ and $f'(z) = f(z)$. So, $f(z) = a\exp(z)$ for some constant a. By evaluating f at 0, see that $a = \exp(w)$. Thus, $f(z) = \exp(w) \cdot \exp(z)$.

In particular:

$$\exp(\imath\theta) = \cos\theta + \imath\sin\theta$$

and

$$\exp(nz) = \exp(z)^n.$$



Euler later developed a cataract in his left eye, which was discovered in 1766. Just a few weeks after its discovery, he was rendered almost totally blind. However, his condition appeared to have little effect on his productivity, as he compensated for it with his mental calculation skills and exquisite memory. For example, Euler could repeat the Aeneid of Virgil from beginning to end without hesitation, and for every page in the edition he could indicate which line was the first and which the last. With the aid of his scribes, Euler's productivity on many areas of study actually increased. He produced on average, one mathematical paper every week in the year 1775.

"Read Euler, read Euler, he is the master of us all." (Laplace)

Exponential Function

This property of converting addition into multiplication also characterizes $\exp(z)$.

Let $0 \in \Omega$. Suppose $f : \Omega \to \mathbb{C}$ is such that f is differentiable at 0 and $f(0) \neq 0$. Suppose $f(w+z) = f(w)f(z)$ whenever $w, z, w+z \in \Omega$. Then, $f(z) = \exp(bz)$, where $b = f'(0)$.

Proof: Firstly, $f(0+0) = f(0)f(0)$ gives us $f(0) = 1$ (since it's given that $f(0) \neq 0$). Note that for small enough w, $f(w+z) = f(w)f(z)$. Thus,

$$\frac{f(z+w) - f(z)}{w} = \frac{f(w) - 1}{w} \cdot f(z) = \frac{f(w) - f(0)}{w - 0} \cdot f(z).$$

It follows that, f is differentiable in Ω (since f is differentiable at 0), and $f'(z) = f'(0)f(z)$. Thus, $f(z) = a\exp(f'(0)z)$ for some a, by an earlier result. Now $a = 1$ (evaluate at 0).

Exponential Function

Remember that we've told ourselves that we don't know e, e^x, log, etc. Now we define e as follows:

$$e = \exp(1).$$

And with this, let's reconstruct everything! First of all,

$$e^2 = e \cdot e = \exp(1) \cdot \exp(1) = \exp(2).$$

Thus, $e^n = \exp(n)$ for all $n = 0, 1, 2, \dots$. Since $e^{-n} = \frac{1}{e^n}$ and $\exp(-n) = \frac{1}{\exp(n)}$, we get $e^n = \exp(n)$ for all $n \in \mathbb{Z}$. What about $\mathbb{Q}$? Note that

$$\left[\exp\left(\frac{p}{q}\right)\right]^q = \exp\left(\frac{p}{q} + \ldots + \frac{p}{q}\right) = \exp(p) = e^p.$$

Taking q-th root on both sides, we get $e^x = \exp(x)$ for all $x \in \mathbb{Q}$.

Exponential Function

Now how are real powers defined in general? If $x \in \mathbb{R}$, take $x_n \to x$ with $x_n \in \mathbb{Q}$ and e^x is $\lim e^{x_n}$. Thus for any real x,

$$e^x = \lim e^{x_n} = \lim \exp(x_n) = \exp(\lim x_n) = \exp(x).$$

(Why is $\lim \exp(x_n) = \exp(\lim x_n)$?). In particular,

$$\frac{d}{dx} e^x = e^x,$$

since $\exp(x)$ has this property. Note that (i) $\exp(x) > 0$ for all x (why?) and (ii) it is monotonically increasing (why?). Therefore, it's one-to-one, and hence invertible. We define $\log x$ as the inverse of $\exp(x)$. i.e.,

$$\log \exp(x) = x \ \& \ \exp(\log x) = x.$$

Note that now you can prove: $\frac{d}{dx} \log x = \frac{1}{x}$.

Trigonometric Functions

Recall the Taylor expansions:

$$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots = \frac{\exp(\imath y) - \exp(-\imath y)}{2\imath}$$

$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \ldots = \frac{\exp(\imath y) + \exp(-\imath y)}{2}.$$

Motivated by this, we define complex trigonometric functions:

$$\sin z = \frac{\exp(\imath z) - \exp(-\imath z)}{2\imath} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots$$

$$\cos z = \frac{\exp(\imath z) + \exp(-\imath z)}{2} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots.$$

Exercise:

(i) Define other trigonometric functions.

(ii) Define hyperbolic trigonometric functions.

(iii) Show that $\exp : \mathbb{C} \to \mathbb{C}^\times$ is onto. Is it one-to-one?

(iv) Show that $\sin, \cos : \mathbb{C} \to \mathbb{C}$ are surjective. In particular, note the difference with real sine and cosine which were bounded by 1.

(v) Show that $\sin^2 z + \cos^2 z = 1$,

$\sin(z + w) = \sin z \cos w + \cos z \sin w$,

$\cos(z + w) = \cos z \cos w - \sin z \sin w$.