MA 205 Complex Analysis: Cauchy & CIF

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Introduction

We'll continue with complex integration. Last time, we first defined the integral of a complex valued function on an interval in $\mathbb{R}$. This was easy; integrate real part and imaginary part separately. We saw that this complex integral is a complex linear functional on the space of continuous functions. It has the property:

$$\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt.$$

We also saw the complex version of the fundamental theorem of calculus. Then we defined the complex line integral.

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

(What was γ here?) We stated the path independence theorem: a holomorphic f on a domain has a primitive if and only if its line integral depends only on the end points of the path. We ended the lecture by observing that $\int_{|z-z_0|=r} \frac{dz}{z-z_0} = 2\pi\imath$.

ML Inequality

Before we proceed further, let's recall from MA 105, the formula for the length of a parametrized curve. If γ is a smooth parametrized curve, $\gamma = z(t) = (x(t), y(t))$, $t \in [a, b]$, then,

$$\ell(\gamma) = \int_a^b |z'(t)|dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

Now if we know that our holomorphic function is bounded in its domain, i.e., there is an $M > 0$ such that $|f(z)| \leq M$ for all $z \in \Omega$, then,

$$\begin{aligned} \left| \int_\gamma f(z)dz \right| &= \left| \int_a^b f(z(t))z'(t)dt \right| \\ &\leq \int_a^b |f(z(t))||z'(t)|dt \\ &\leq M \int_a^b |z'(t)|dt \\ &= M \cdot \ell(\gamma). \end{aligned}$$

Proof of Path Independence

Let's see a quick proof of the path independence theorem. We need to show that f has a primitive iff $\int f(z)dz$ is path independent. Suppose f has a primitive; i.e., there is F such that $F' = f$. Then,

$$\begin{aligned}\int f(z)dz = \int F'(z)dz &= \int_a^b F'(z(t))z'(t)dt \\ &= \int_a^b \left[\frac{d}{dt}F(z(t))\right] dt = F(z(b)) - F(z(a)).\end{aligned}$$

Thus, the integral depends only on the end points.

Proof of Path Independence

On the other hand, suppose the integral depends only on the end points. This means that the integral is independent of the path on which you integrate. We need to find an F, show that it is differentiable, and $F'(z) = f(z)$ for all $z \in \Omega$. How do we go about getting such an F? Something whose derivative is the given function should be an integral of that function! To get a function of z, we'll integrate up to z. From where? From any fixed z_0. How? Along <u>any</u> path joining z_0 to z. Why should there be a path joining z_0 and z? Because Ω is connected. Thus, our candidate for the primitive is

$$F(z) = \int_{\gamma(z_0,z)} f(z)dz.$$

The first question to ask as soon as you see the definition of a function is: is it well-defined? What if I had taken another path from z_0 to z? Will it give another value of $F(z)$? No, because the integral is given to be path independent.

Proof of Path Independence

Okay, so we have a good candidate for the primitive. We only have to check that it is indeed a primitive. To this end, consider a small neighborhood of z which is completely contained in Ω. (Why should there be a such a neighborhood?) Let $h \in \mathbb{C}$ be such that $|h|$ is very small. Join z to $z + h$ via a straight line; $z + ht$, $t \in [0, 1]$. Now,

$$\begin{aligned}
\lim_{h\to 0} \frac{F(z+h) - F(z)}{h} &= \lim_{h\to 0} \frac{1}{h}\left[\int_{\gamma(z_0,z+h)} f(w)dw - \int_{\gamma(z_0,z)} f(w)dw\right] \\
&= \lim_{h\to 0} \frac{1}{h}\int_{z}^{z+h} f(w)dw \\
&= \lim_{h\to 0} \frac{1}{h}\int_{0}^{1} f(z+ht)hdt \\
&= f(z).
\end{aligned}$$

This finishes the proof.

Theorem (Cauchy's Theorem)

If f is holomorphic on and within a closed curve γ, then $\int_\gamma f(z)dz = 0$.

If $f = u + \imath v$, consider the corresponding vector field

$$f(x,y) = (u(x,y), v(x,y)).$$

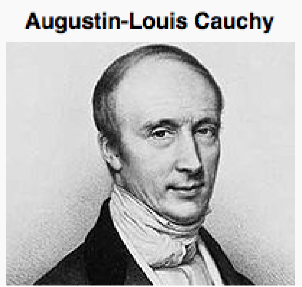
Holomorphy of f implies that the vector field is differentiable. Now,

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_a^b f(z(t))z'(t)dt \\
&= \int_a^b [u(x(t),y(t)) + \imath v(x(t),y(t))][x'(t) + \imath y'(t)]dt \\
&= \int_\gamma udx - vdy + \imath \int_\gamma vdx + udy.
\end{aligned}$$

First is the line integral of the vector field $(u, -v)$ and the second that of (v, u).

Note that both the vector fields $(u, -v)$ and (v, u) have curl zero because of CR equations. What's that theorem from MA 105 that relates a line integral with a double integral? What's the exact statement? Can we apply that here to conclude that both the real line integrals above, and hence the complex line integral, are zero? Everything is okay, except that we do not know for sure that the partial derivatives are continuous throughout our region. But this is indeed true! If f is holomorphic, then f' is automatically continuous. This is called Goursat's theorem, not difficult to prove, but we do not do it right now. However, we'll assume this fact so that we can proceed and finish the proof of Cauchy's theorem by appealing to Green's theorem from MA 105.

Cauchy (1789-1857), Green (1793-1841), Goursat (1858-1936)



Note that without Goursat we would have had to assume that f is continuously differentiable. This improvement came much much later!

"Goursat was born the year after Cauchy died, so it was a long time coming!"[1]

[1] http://mathoverflow.net/questions/3819

Example: Let γ be any closed curve which goes around the point z_0 once (in the counterclockwise direction). Calculate $\int_\gamma \frac{dz}{z-z_0}$. Choose r small enough so that the circle $C : |z - z_0| = r$ lies inside γ. Consider the curve $\gamma \cup (-C)$. Note that $f(z) = \frac{1}{z-z_0}$ is holomorphic in the region contained on and inside this curve. Therefore, by Cauchy's theorem,

$$\int_{\gamma \cup (-C)} \frac{dz}{z - z_0} = 0.$$

Thus,

$$\int_\gamma \frac{dz}{z - z_0} = \int_C \frac{dz}{z - z_0} = 2\pi \imath.$$

Cauchy Integral Formula

We'll now use Cauchy's theorem to prove the Cauchy Integral Formula. In the next lecture, we'll deduce both the fundamental theorem of algebra and the fact that once differentiable is always differentiable as applications of this beautiful result.

Theorem (Cauchy Integral Formula)

Let f be holomorphic everywhere within and on a simple closed curve γ (oriented positively). If z_0 is interior to γ, then,

$$f(z_0) = \frac{1}{2\pi\imath} \int_\gamma \frac{f(z)dz}{z - z_0}.$$

Proof: We need to show that

$$\int_\gamma \frac{f(z)dz}{z - z_0} = \int_\gamma \frac{f(z_0)dz}{z - z_0},$$

since the latter integral is $2\pi\imath \cdot f(z_0)$.

Thus, we need to show that the difference

$$\int_\gamma \frac{f(z) - f(z_0)}{z - z_0} dz$$

is zero. Since f is continuous at z_0, given $\epsilon > 0$, there is $\delta > 0$ such that

$$|z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon.$$

Choose $r < \delta$ and consider the circle $C_r : |z - z_0| = r$. By Cauchy's theorem applied to $\gamma \cup (-C_r)$, we get

$$\int_\gamma \frac{f(z) - f(z_0)}{z - z_0} dz = \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz.$$

Now,

$$\left| \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \leq \int_{C_r} \frac{|f(z) - f(z_0)|}{|z - z_0|} dz = \frac{\epsilon}{r} 2\pi r = 2\pi\epsilon.$$

Thus, $\left| \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0} dz \right|$ can be made arbitrarily small; i.e., it is zero.

Example:

(i) $\int_{|z|=2} \frac{e^z dz}{(z+1)(z-3)^2} = \int_{|z|=2} \frac{f(z)dz}{z+1}$, where $f(z) = \frac{e^z}{(z-3)^2}$. So by CIF, answer is $2\pi\imath f(-1) = \frac{\pi\imath}{8e}$.

(ii)
$\int_{|z|=6} \frac{dz}{z^3-1} = 2\pi\imath\left[\frac{1}{(1-\omega)(1-\omega^2)} + \frac{1}{(\omega-1)(\omega-\omega^2)} + \frac{1}{(\omega^2-\omega)(\omega^2-1)}\right] = 0.$

(iii) $\int_{|z|=3} \frac{\cos \pi z}{z^2-1} dz = \frac{1}{2}\int_{|z|=3}\left[\frac{\cos \pi z}{z-1} - \frac{\cos \pi z}{z+1}\right] dz = 0$

OR

$$\begin{aligned}\int_{|z|=3} \frac{\cos \pi z}{z^2-1} dz &= \int_{|z-1|=\varepsilon} \frac{\frac{\cos \pi z}{z+1}}{z-1} dz + \int_{|z+1|=\varepsilon} \frac{\frac{\cos \pi z}{z-1}}{z+1} dz \\ &= 0.\end{aligned}$$

Summing Up

Today we saw two very important theorems. Cauchy's theorem and the Cauchy integral formula. The first said that the integral along a closed curve of a function is zero if the function is holomorphic on and within the curve. The second said:

$$f(z_0) = \frac{1}{2\pi\imath} \int_\gamma \frac{f(z)}{z - z_0} dz$$

if f is holomorphic on and within the simple closed curve γ. We derived Cauchy's theorem by appealing to Green's theorem after assuming Goursat's theorem. We derived CIF from Cauchy's theorem by making use of the computation $\int_\gamma \frac{dz}{z-z_0} = 2\pi\imath$. In fact, Cauchy's theorem is equivalent to CIF. Do you see this? CIF is indeed a remarkable thing; here's a scenario where external appearance tells us so much about internal reality! Value at any interior point is gotten by averaging just on the border! Don't we sniff a whiff of great things to come?

Next time ...

Indeed, both the promised surprises of the first lecture will come out as consequences of CIF. But before we proceed further, we should stop and think for a moment as to what makes the whole thing click. You did a full semester of MA 105; functions of two real variables and their differentiability properties. There were neat results like the Green's theorem which were extremely useful. But anything like the CIF that enables us to see through the interior? No, nothing of that sort. So what's going on here? Cauchy's theorem gave us CIF. And Green's theorem gave us Cauchy. It gave us Cauchy because two vector fields - $(u, -v)$ and (v, u) - were automatically of curl zero. And why were these of curl zero? Thanks to CR equations.

Next time ...

And why were CR equations there in the first place? Existence of

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

contains a lot more info than the existence of a similar one real variable limit. And in the two real variable case, $\frac{f(x,y) - f(a,b)}{(x-a,y-b)}$ doesn't make sense. In short, a limit in the plane has much more meat in it than a limit in the line, and in the setting of complex numbers we are able to take limit of quotients in the plane. The only way in which quotienting in the plane can be done is by introducing multiplication of complex numbers. It's a very general theme in mathematics, physics, and hence in nature, that functions which obey nontrivial partial differential equations exhibit wonderful properties. Complex differentiable functions satisfy a nontrivial pde (or nontrivial pde's in terms of real and imaginary parts); their real counterparts don't. This is the key!