MA-207 Differential Equations II

Ronnie Sebastian

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

12th October, 2017
S2 - Lecture 7

Bessel functions

Bessel equation is the second-order linear ODE

$$x^2y'' + xy' + (x^2 - p^2)y = 0 \qquad p \geq 0 \qquad\qquad (*)$$

For real p, define

$$J_p(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(p+n+1)} \left(\frac{x}{2}\right)^{2n+p}$$

1. The above is a well defined power series once we know that the Gamma function never vanishes.
2. If $p \notin \{0, 1, 2, \ldots\}$ $J_p(x)$ and $J_{-p}(x)$ are the two independent solutions of the Bessel equation.
3. If $p \in \{0, 1, 2, \ldots\}$ then $J_{-p}(x) = (-1)^p J_p(x)$. Thus, in this case the second solution is not $J_{-p}(x)$.

Bessel identities

1. $\dfrac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$
2. $\dfrac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$

   The above two can be obtained by formally differentiating the power series.

3. $J_p'(x) + \dfrac{p}{x} J_p(x) = J_{p-1}(x)$
4. $J_p'(x) - \dfrac{p}{x} J_p(x) = -J_{p+1}(x)$

   These follow from (1) and (2). Expand LHS and divide by $x^{\pm p}$;

5. $J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x)$
6. $J_{p-1}(x) + J_{p+1}(x) = \dfrac{2p}{x} J_p(x)$

Add and subtract (3) and (4) to get (5) and (6).

Consequences of Bessel identities

Problem: Show that between any two consecutive zeros of $J_p(x)$, there exists precisely one zero of $J_{p-1}(x)$ and precisely one zero of $J_{p+1}(x)$

Problem: Find a and c so that $J_2(x) - J_0(x) = aJ_c''(x)$.

Theorem (Sturm separation theorem)

If $y_1(x)$ and $y_2(x)$ are linearly independent solns of

$$y'' + P(x)y' + Q(x)y = 0$$

P, Q continuous on (a, b). Then
(1) $y_1(x)$ and $y_2(x)$ have no common zero in (a, b).
(2) Between any two successive zeros of $y_1(x)$, there is exactly one zero of $y_2(x)$ and vice versa.

Given any ODE in the "standard" form $y'' + P(x)y' + Q(x)y = 0$ can be written in the "normal" form $u'' + q(x)u = 0$.

Define $v(x) := \exp\Big(\int_{a_0}^{x} -\frac{1}{2}P(t)dt\Big)$ and set $u(x) = \frac{y(x)}{v(x)}$.

One easily checks that $u(x)$ satisfies the differential equation

$$u'' + q(x)u = 0 \qquad\qquad q(x) := Q(x) - \frac{1}{4}P(x)^2 - \frac{1}{2}P'(x)$$

It is clear that the zeros of u are the same as those of y.

Theorem

Let $u(x)$ be a non-trivial solution of $u'' + q(x)u = 0$ on finite interval (a, b), with $q(x)$ continuous. Then $u(x)$ has at most finite number of zeros in (a, b).
Hence if $u(x)$ has infinitely many zeros on $(0, \infty)$, then the set of zeros of $u(x)$ are not bounded.

Theorem

Let $u(x)$ be a non-trivial solution of $u'' + q(x)u = 0$. If $q(x) < 0$ in (a, b) and continuous then $u(x)$ has atmost one zero in (a, b).

Remark

In the previous class, we had stated application 1 of the Bessel identity only for $p > 0$. This condition is not required, as the Bessel identities hold for all p for the functions $J_p(x)$.

Correction.

The function $y(x) = x \sin \frac{1}{x}$, satisfies the differential equation $y'' + \frac{1}{x^2} y = 0$ on the interval $(0, \infty)$. In the interval $(0, 1)$ this function has infinitely many zeros, contradicting the theorem stated in the previous lecture.

The problem in this example is that the zeros$=\{x_n = \frac{1}{n\pi}\}_{n \geq 1}$ tend to 0, which is not a point of $(0, 1)$. The proof that we gave in the previous class breaks down as $x_0 = 0$ is not in the domain of definition of the function $y(x)$.

Remarks and Corrections

Therefore, the correct statement of the theorem is the following

Theorem (Corrected)

Let $u(x)$ be a non-trivial solution of $u'' + q(x)u = 0$ on the interval (α, β), with $q(x)$ continuous. Let $[a, b] \subset (\alpha, \beta)$ be a finite interval. Then $u(x)$ has <u>at most</u> finite number of zeros in $[a, b]$.

With this statement, the proof given in the previous class works.

Theorem

Let $u(x)$ be a non-trivial solution of $u'' + q(x)u = 0$ Let $q(x)$ be continuous and $q(x) > 0$ for all $x > x_0 > 0$.
If $\displaystyle\int_{x_0}^{\infty} q(x)\,dx = \infty$,
then $u(x)$ has infinitely many zeros on $(0, \infty)$.

Proof. Assume $u(x)$ has only finitely many zeros on $(0, \infty)$.
Then there is $x_1 > x_0$ such that $u(x) \neq 0$ for $x \geq x_1$. Assume $u(x) > 0$ for $x \geq x_1$.
Then $u''(x) = -q(x)u(x) < 0$ for $x \geq x_1$. Hence $u'(x)$ is decreasing for $x \geq x_1$.
If we show that $u'(x_2) < 0$ for some $x_2 > x_1$, then we get for $x > x_2$

$$u(x) = \int_{x_2}^{x} u'(t)dt + u(x_2) \leq \int_{x_2}^{x} u'(x_2)dt + u(x_2)$$
$$\leq u'(x_2)(x - x_2) + u(x_2)$$

Thus if x is sufficiently large, then $u(x) < 0$, a contradiction.

To show that $u'(x) < 0$ for some $x > x_1$. Put

$$v(x) = -\frac{u'(x)}{u(x)}, \quad \text{for} \;\; x \geq x_1$$

$$v' = \frac{-u''u + u'^2}{u^2} = \frac{q(x)u^2 + u'^2}{u^2} = q(x) + v(x)^2$$

Integrating we get

$$v(x) - v(x_1) = \int_{x_1}^{x} q(x)\,dx + \int_{x_1}^{x} v(x)^2\,dx$$

$\int_{x_0}^{\infty} q(x)\,dx = \infty \implies v(x) > 0$ for large x.

Thus, $u'(x) = -u(x)v(x)$ and this shows that $u'(x) < 0$ for x large.

Theorem

In Bessel equation $x^2y'' + xy' + (x^2 - p^2)y = 0$ *Substituting* $u(x) = \sqrt{x}y(x)$, *we get*

$$u'' + \left[1 + \frac{1-4p^2}{4x^2}\right]u = 0$$

$q(x) = 1 + \dfrac{1-4p^2}{4x^2}$ *is continuous and* $q(x) > 0$ *for* $x > x_0 > 0$. *Further,*

$$\int_{x_0}^{\infty} \left(1 + \frac{1-4p^2}{4x^2}\right) dx = \infty$$

By previous theorem, $u(x)$, *hence any Bessel function has infinitely many zeros on* $(0, \infty)$.

Corollary

Let $Z^{(p)}$ be the set of zeros of Bessel function $J_p(x)$ on $(0, \infty)$. Since $Z^{(p)}$ is an infinite set, it is not bounded.

We will conside the following question.

Write $Z^{(p)} = \{x_1, x_2, \ldots\}$ as increasing sequence $x_n < x_{n+1}$.

Question. What is the limit of $x_{n+1} - x_n$ as $n \to \infty$?

We will need the Sturm comparison theorem.

Theorem (Sturm Comparison theorem)

Let $y(x)$ be a non-trivial solutions of

$$y'' + q(x)y = 0$$

and $z(x)$ be a non-trivial solutions of

$$z'' + r(x)z = 0$$

where $q(x) > r(x) > 0$ are continuous.
Then $y(x)$ vanishes at least once between any two consecutive zeros of $z(x)$.

Compare $y'' + 4y = 0$ and $z'' + z = 0$.

Here $(q(x) =)\, 4 > (r(x) =)\, 1 > 0$

Zeros of $y(x)$ are $\pi/2$ apart and that of $z(x)$ are π apart.

Proof of Sturm Comparison theorem.

Let $x_1 < x_2$ be consecutive zeros of $z(x)$.

Assume $y(x)$ has no zero in (x_1, x_2).

We may assume $z(x) > 0$ and $y(x) > 0$ on (x_1, x_2). Hence $z'(x_1) > 0$ and $z'(x_2) < 0$.

Consider the function $W(x) = y(x)z'(x) - y'(x)z(x)$

$$W'(x) = yz'' - y''z = y(-rz) - (-qy)z = (q - r)yz > 0$$

on (x_1, x_2).

Integrating from x_1 to x_2, we get

$$W(x_2) - W(x_1) > 0 \implies W(x_2) > W(x_1)$$

But $W(x_1) = y(x_1)z'(x_1) > 0$ and $W(x_2) = y(x_2)z'(x_2) < 0$, a contradiction. □

Theorem

Substituting $u(x) = \sqrt{x}y(x)$ *in Bessel equation, we get Bessel equation in normal form* $(p \geq 0)$

$$u'' + q(x)u = 0, \qquad q(x) = 1 + \frac{1 - 4p^2}{4x^2}$$

- $p < 1/2 \implies q(x) > 1$
- $p = 1/2 \implies q(x) = 1$ *(Well known, hence, uninteresting)*
- $p > 1/2 \implies q(x) < 1$

Use $z'' + z = 0$ *and Sturm comparison theorem.*

Let $y_p(x)$ *be a non-trivial solution of Bessel equation. Then we get ...*

Theorem

- $\boxed{p < 1/2} \implies$ *Between any two roots of* $\alpha\cos x + \beta\sin x$ *there is a root of* $y_p(x)$*.*
- $\boxed{p = 1/2} \implies x_2 - x_1 = \pi$
- $\boxed{p > 1/2} \implies$ *Between any two roots of* $y_p(x)$ *there is a root of* $\alpha\cos x + \beta\sin x$*.*

We can say more than the above. Suppose $p < 1/2$ and $a < b < c$ are consecutive roots of $u(x)$. Then $b - a < c - b$. That is, the difference between the successive roots keeps increasing.

To see this, consider the function $f := u(x - b + a)$ defined on the interval (b, ∞).

It is a trivial check that f satisfies the differential equation

$$f'' + r(x)f = 0 \qquad\qquad r(x) := q(x - b + a)$$

Since $p < 1/2$ the function q is strictly decreasing. Thus, on (b, ∞) we have $r(x) > q(x) > 0$.

Applying Sturm's comparison theorem we get that there is a $b < x_0 < c$ such that $f(x_0) = u(x_0 - b + a) = 0$.

Clearly,

- $b < x_0 \implies a < x_0 - b + a$
- $a < b \implies x_0 - b + a < x_0$

Thus,

$$a < x_0 - b + a < x_0 < c$$

However, $a < b < c$ are successive roots of $u(x)$. This forces that

$$x_0 - b + a = b \qquad \text{that is} \qquad x_0 = 2b - a$$

As $x_0 < c$ we get that $2b - a < c$, that is, $b - a < c - b$.

Next we claim that the difference between any two successive roots of u is strictly less than π.

If not, then let $a < b$ be successive roots such that $b - a \geq \pi$

Since u has infinitely many roots, and their difference is strictly increasing, we may assume that $b - a > \pi$.

But now we can choose $\alpha, \beta \in \mathbb{R}$ such that $\alpha \cos x + \beta \sin x$ has two roots in (a, b), which contradicts Sturm's comparison theorem.

Thus, we have proved that if $\{x_n\}$ are the roots of u in increasing order, then the difference $x_{n+1} - x_n$ is strictly increasing and bounded above by π.

Next let us show that these differences converge to π. If not, then $(x_{n+1} - x_n) \to \gamma < \pi$. Choose $1 < \delta$, sufficiently close to 1 such that $\gamma < \frac{\pi}{\delta} < \pi$.

The function $q(x)$ is decreasing to 1. Therefore, there is a $x_0 \in \mathbb{R}$, sufficiently large, such that $q(x_0) < \delta^2$. Apply Sturm's comparison on the interval (x_0, ∞) to the differential equations $u'' + q(x)u = 0$ and $z'' + \delta^2 z = 0$.

Thus, between any two roots of u there is a root of z. Let a and b be two consecutive roots of u such that $x_0 < a < b$. Since $b - a < \gamma < \frac{\pi}{\delta}$, find a' and b' such that $x_0 < a' < a < b < b'$ and $b' - a' = \frac{\pi}{\delta}$.

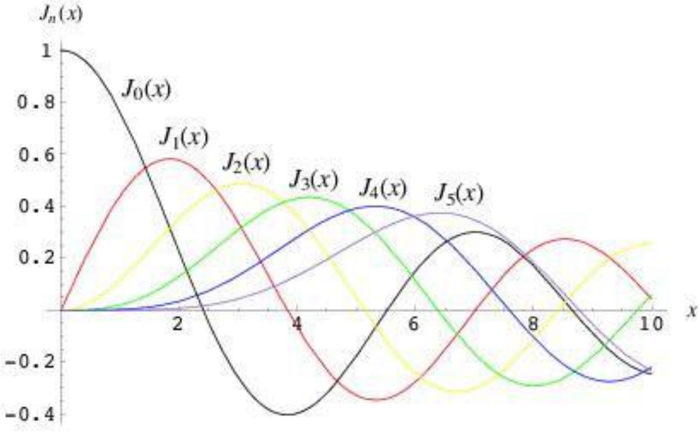
Find α and β such that the function $\alpha \cos \delta\, x + \beta \sin \delta\, x$ vanishes at a'. This function is a solution to the ODE $z'' + \delta^2 z = 0$. The next root of this function is at $a' + \frac{\pi}{\delta} = b'$. Thus, we get a contradiction to Sturm's theorem which says that there is a root of this function in the interval (a, b).

Thus, we have proved

Theorem

If $p < 1/2$ then the sequence of differences of roots of u, $x_{n+1} - x_n$ is increasing and tends to π.

Similarly, we can prove that if $p > 1/2$ then the sequence of difference of roots of u is decreasing and tends to π.


$J_n(x)$
1
0.8
0.6
0.4
0.2
-0.2
-0.4
2
4
6
8
10
x
$J_0(x)$
$J_1(x)$
$J_2(x)$
$J_3(x)$
$J_4(x)$
$J_5(x)$

The first few zeroes of Bessel functions are tabulated below.

	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

Question. Why are we concerned with zeros of Bessel function $J_p(x)$?

It is often required in mathematical physics to expand a given function in terms of Bessel functions.

Simplest and most useful expansions are of the form

$$f(x) = \sum_{n=1}^{\infty} a_n J_p(\lambda_{p,n} x) = a_1 J_p(\lambda_{p,1} x) + a_2 J_p(\lambda_{p,2} x) + \dots$$

where $f(x)$ is defined on, (say) $[0,1]$, and $\lambda_{p,n}$'s are zeros of Bessel function $J_p(x)$, $p \geq 0$.

Qn. How to compute the coefficients a_n?

Remark: For a scalar a, the scaled Bessel functions $J_p(ax)$ are solutions of

$$x^2 y'' + xy' + (a^2 x^2 - p^2) y = 0$$

known as scaled Bessel equation.

Orthogonality

Define an inner product on functions on $[0, 1]$ by

$$\langle f, g \rangle := \int_0^1 x f(x) g(x)\, dx$$

This is similar to the previous inner product except that $f(x)g(x)$ is now multiplied by x and the interval of integration is from 0 to 1.

We call a function on $[0, 1]$ square integrable with respect to this inner product if

$$\int_0^1 x f(x)^2 dx < \infty$$

The multiplying factor x is called a weight function.

Fix $p \geq 0$. Let $Z^{(p)} = \{\lambda_{p,1}, \lambda_{p,2}, \ldots\}$ denote the set of zeros of $J_p(x)$ on $(0, \infty)$.

Theorem

The set of scaled Bessel functions

$$\{J_p(\lambda_{p,1}x), J_p(\lambda_{p,2}x), \ldots\}$$

form an orthogonal family w.r.t. above inner product, i.e. $\langle J_p(\lambda_{p,k}x), J_p(\lambda_{p,l}x)\rangle :=$

$$\int_0^1 xJ_p(\lambda_{p,k}x)J_p(\lambda_{p,l}x)\,dx = \begin{cases} \frac{1}{2}[J_{p+1}(\lambda_{p,k})]^2 & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}$$

Theorem

Fix $p \geq 0$ *and* $Z^{(p)} = \{\lambda_{p,1}, \lambda_{p,2}, \ldots\}$ *: zeros of* $J_p(x)$ *on* $(0, \infty)$*. Any square-integrable function* $f(x)$ *on* $[0, 1]$ *can be expanded in a series of scaled Bessel functions* $J_p(\lambda_{p,n} x)$ *as*

$$f(x) = \sum_{n \geq 1} c_n J_p(\lambda_{p,n} x)$$

where

$$c_n = \frac{2}{[J_{p+1}(\lambda_{p,n})]^2} \int_0^1 x\, f(x)\, J_p(\lambda_{p,n} x)\, dx$$

This is Fourier-Bessel series of $f(x)$ *for parameter* p*.*

Example. Let us compute the Fourier-Bessel series (for $p = 0$) of $f(x) = 1$ in the interval $0 \leq x \leq 1$.

Use $\frac{d}{dx}\big(x^p J_p(x)\big) = x^p J_{p-1}(x)$ for $p = 1$.

$$\int_0^1 x\, J_0(\lambda_{0,n} x)\, dx = \frac{1}{\lambda_{0,n}} x\, J_1(\lambda_{0,n} x)\Big|_0^1 = \frac{J_1(\lambda_{0,n})}{\lambda_{0,n}}$$

$$c_n = \frac{2}{[J_1(\lambda_{0,n})]^2} \int_0^1 x\, f(x)\, J_0(\lambda_{0,n} x)\, dx = \frac{2}{\lambda_{0,n} J_1(\lambda_{0,n})}$$

Thus, the Fourier-Bessel series of $f(x)$ is

$$\sum_{n \geq 1} \frac{2}{\lambda_{0,n} J_1(\lambda_{0,n})} J_0(\lambda_{0,n} x)$$

By next theorem, this converges to 1 for $0 < x < 1$.

Convergence in norm

Fourier-Bessel series converges to $f(x)$ in norm, i.e.

$$\|f(x) - \sum_{n=1}^{m} c_n J_p(\lambda_{p,n} x)\| \text{ converges to } 0 \text{ as } m \to \infty$$

For pointwise convergence, we have

Bessel expansion theorem

Assume f and f' have at most a finite number of jump discontinuities in $[0, 1]$, then the Bessel series converges for $0 < x < 1$ to

$$\frac{f(x_-) + f(x_+)}{2}$$

At $x = 1$, the series always converges to 0 for all f,
at $x = 0$, if $p = 0$ then it converges to $f(0_+)$.
at $x = 0$, if $p > 0$ then it converges to 0.