Problem Set 3
Data Analysis and Interpretation (EE 223)

Instructor: Prof. Prasanna Chaporkar
EE Department, IIT Bombay

1. The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} ce^{-x}e^{-2y} & 0 < x < \infty,\ 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

   Compute (a) c, (b) $P\{X > 1, Y < 1\}$, (c) $P\{X < Y\}$, and (d) $P\{X < a\}$.

2. The joint density of X and Y is given by

$$f(x,y) = \begin{cases} e^{-(x+y)} & 0 < x < \infty,\ 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

   Find the density function of the random variable X/Y.

3. Sonia and Narendra decide to meet at a certain location. If each of them independently arrives at a time uniformly distributed between 5 PM and 6 PM, find the probability that the first to arrive has to wait longer than 10 minutes.

4. If the joint density function of X and Y is

$$f(x,y) = 6e^{-2x}e^{-3y} \quad 0 < x < \infty,\ 0 < y < \infty$$

   and is equal to 0 outside this region, are the random variables independent? What if the joint density function is

$$f(x,y) = 24xy \quad 0 < x < 1,\ 0 < y < 1,\ 0 < x + y < 1$$

   and is equal to 0 otherwise?

5. Nitish and Lalu shoot at a target. The distance of each shot from the center of the target is uniformly distributed on $(0,1)$, independently of the other shot. What is the PDF of the distance of the losing shot from the center?

6. Let X, Y, Z be independent and uniformly distributed over $(0,1)$. Compute $P\{X \geq YZ\}$.

7. Sum of two independent random variables

   (a) If X and Y are independent random variables, both uniformly distributed on $(0,1)$, calculate the probability density of $X + Y$.

   (b) If X and Y are independent Gamma random variables with respective parameters $(\alpha_1,\ \beta)$ and $(\alpha_2,\ \beta)$, then prove that $X + Y$ is also a Gamma random variable with parameters $(\alpha_1 + \alpha_2,\ \beta)$. Recall that a Gamma random variable has a density of the form

$$f(x;\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \qquad x, \alpha, \beta > 0.$$

   (c) If X and Y are independent Gaussian random variables with respective parameters $(\mu_1,\ \sigma_1^2)$ and $(\mu_2,\ \sigma_2^2)$, then prove that $X + Y$ is also a Gaussian random variable with parameters $(\mu_1 + \mu_2,\ \sigma_1^2 + \sigma_2^2)$. Recall that a Gaussian random variable has a density of the form

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2}}.$$

8. Conditional distribution of random variable

(a) Suppose that the joint density of X and Y is given by

$$f(x,y) = \begin{cases} \dfrac{e^{-x/y}e^{-y}}{y} & 0 < x < \infty,\ 0 < y < \infty \\ 0 \quad \text{otherwise} \end{cases}$$

Find $P\{X > 1|Y = y\}$.

(b) If X and Y are independent Poisson random variables with respective parameters λ_1 and λ_2. Prove that the conditional distribution of X given that $X + Y = n$ is a binomial distribution. A Poisson random variable has a pmf as

$$f(k;\lambda) = P\{X = k\} = \frac{\lambda^k e^{-\lambda}}{k!}.$$

(c) Rajan goes to the bank to make a deposit, and is equally likely to find 0 or 1 customer ahead of him. The times of service of these customers are independent and exponentially distributed with parameter λ. What is the CDF of Rajan's waiting time? Recall that a Exponential random variable has a density of the form

$$f(x;\lambda) = \lambda e^{-\lambda x} \quad x \geq 0.$$

9. Joint probability distribution of functions of random variables

(a) X and Y have joint density function

$$f(x,y) = \frac{1}{x^2y^2} \quad x \geq 1\ y \geq 1$$

Compute the joint density function of $U = XY,\ V = X/Y$. What are the marginal densities?

(b) Let X be exponentially distributed with mean 1. Once we observe the experimental value x of X, we generate a normal random variable Y with zero mean and variance $x + 1$. What is the joint PDF of X and Y?