MA 205 Complex Analysis: Logarithm

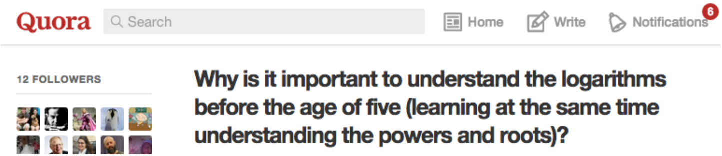
U. K. Anandavardhanan
IIT Bombay

August 03, 2015

So last lecture, almost all the time we spent discussing the exponential function. We defined it by a power series which converges everywhere. We characterized the exponential function in two distinct ways. It's the only function which is invariant under differentiation if we normalize it such that the function takes the value 1 at the point 0. It's essentially the only function with the property of converting addition in $\mathbb{C}$ to multiplication in $\mathbb{C}^\times$. We checked that, for a real variable, the exponential function matches e^x. e^x is monotonic increasing, hence one-to-one, hence invertible. This inverse is the logarithm. We checked that $\frac{d}{dx} \log x = \frac{1}{x}$. It's also an easy check that $\log(xy) = \log x + \log y$, for $x, y \in \mathbb{R}$.

Complex Logarithm & Quora

Today we'll discuss the complex logarithm. According to Quora, we should have done this long back!



Complex Logarithm

The real logarithm in MA 105 was a nice differentiable function, with derivative $\frac{1}{x}$, and in MA 205, we want a nice holomorphic function, with derivative $\frac{1}{z}$. Where will we hunt for this? Remember: $\log x$ was the inverse of $\exp(x) = e^x$; i.e.,

$$\exp(\log x) = x \ \& \ \log(\exp(x)) = x.$$

If we blindly imitate, complex log "should be" the "inverse" of the complex exponential. But $\exp(z)$ is not one-to-one:

$$\exp(z + 2n\pi\imath) = \exp(z)\exp(2n\pi\imath) = \exp(z),$$

since $\exp(2n\pi\imath) = 1$ for any $n \in \mathbb{Z}$. By the way, it's easy to see that this is the only way in which one-one-ness can be lost; i.e., $\exp(z) = \exp(w)$ implies $z - w$ is an integral multiple of $2\pi\imath$.

Wait a minute. Why do we need one-one-ness to define the inverse? If f is not one-to-one, f can take two distinct points to the same point, say $f(x) = f(y) = a$ with $x \neq y$, and then f^{-1} will take the image point a to both x and y. But a function doesn't do that; every point has a unique image under a function. In other words, a function, by the very definition, is single-valued.
Okay, what if we weaken the notion a little bit? What if we are willing to consider multi-valued functions? Then, we don't need one-one-ness to talk about inverses. The inverse will be multi-valued, that's all.

Thus, exponential not being one-to-one was just a minor head-ache. We have effective and cheap painkillers available in the market: multi-valued functions! Okay, so let's look at these multi-valued inverses of the exponential. If $z = re^{\imath\theta}$ is given, we need to get something, which when we exponentiate, we should get z back. Since there's an $e^{\imath\theta}$ in z, we immediately see that $\imath\theta$ has to be there in this something. Then there should be something else, which gives r on exponentiation. Since r is a positive real, the answer is clear: this is nothing but $\log r$. Thus, what we hunt for is something like

$$\log r + \imath\theta,$$

whose exponential is indeed z. It's multi-valued since θ can be replaced with $\theta + 2n\pi$ for any n in the above. The θ in z is called the argument of z, denoted by $\arg(z)$. We can prescribe its range to be any interval $(\alpha, \alpha + 2\pi]$ so that the argument is single-valued. When $\alpha = -\pi$, we call it $\mathrm{Arg}(z)$, the principal argument.

Have we then hunted down the logarithm? No, not yet. The candidate

$$re^{\imath\theta} \mapsto \log r + \imath\theta,$$

looks good, it is defined throughout $\mathbb{C}^{\times}$, which is fine since we don't expect 0 to have a logarithm, because exponential is never zero. But it has a very major drawback. Remember we had agreed that the complex log should be holomorphic. This map is far from being holomorphic on $\mathbb{C}^{\times}$. It's not even continuous. The real part

$$(x, y) \mapsto \log \sqrt{x^2 + y^2}$$

is fine; it's differentiable. The imaginary part

$$z \mapsto \arg(z)$$

is the problem. This is not continuous. If arg ranges over $(\alpha, \alpha + 2\pi]$, take a point z_α on the half-ray $re^{\imath\alpha}$. As $z \to z_\alpha$ from one direction, the limit of $\arg(z)$ is α, whereas from the opposite direction, the limit is $\alpha + 2\pi$. Thus, $\lim_{z \to z_\alpha} \arg(z)$ does not exist.

Complex Logarithm

The argument not being continuous on $\mathbb{C}^\times$ causes a major head-ache. It makes sure that you cannot define a holomorphic logarithm throughout $\mathbb{C}^\times$. There's no way-out. No cheap painkiller will do. The only way to proceed in order to have holomorphic logarithms defined is to do a surgery! We'll cut and throw away parts of $\mathbb{C}^\times$ in such a way that in the remaining part, maps of the kind

$$\log r + \imath\theta$$

with appropriate θ, turn out to be holomorphic. In that case, it does give a logarithm, on the new domain, since exp of this is z as we have seen already. Note the use: "a" logarithm. There are many logarithms, as we can do surgery in many different ways. Of course, a logarithm ℓ_1 on Ω_1 will agree with a logarithm ℓ_2 on Ω_2 over $\Omega_1 \cap \Omega_2$, as can be easily checked. (Agree = agree modulo the $2n\pi\imath$'s).

Let's now write down a logarithm. We first define it only on $\Omega = \{z \mid \mathrm{Re}(z) > 0\}$, the right half plane. $\ell : \Omega \to \mathbb{C}$ defined by

$$\ell(z) = \log \sqrt{x^2 + y^2} + \imath \tan^{-1}\left(\frac{y}{x}\right).$$

In the first and fourth quadrants (which constitute the right half plane), $\tan^{-1}\left(\frac{y}{x}\right)$ is nothing but $\mathrm{Arg}(z)$. Thus,

$$(\exp \circ \ell)(z) = z.$$

Both $u(x, y)$ and $v(x, y)$ are continuously differentiable, and CR equations can be checked. So $\ell(z)$ is holomorphic. Check that $\ell'(z)$ is indeed $\frac{1}{z}$. It's a logarithm!

Can we enlarge the domain? Why not $\mathrm{Re}(z) \geq 0$ (minus the origin)? i.e., include the y-axis. Only problem is the $\frac{y}{x}$ in the formula for ℓ. Easy to fix: define

$$l(z) = \begin{cases} \log\sqrt{x^2+y^2} + \imath\tan^{-1}\left(\frac{y}{x}\right) & \text{if } \mathrm{Re}(z) > 0 \\ \log y + \imath\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ \log(-y) - \imath\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0. \end{cases}$$

Once again check differentiability and $\ell'(z) = \frac{1}{z}$.
We would like to enlarge the domain further. But keep in mind that it can't be the whole of $\mathbb{C}^\times$. First note that

$$\log\sqrt{x^2+y^2} + \imath\tan^{-1}\left(\frac{y}{x}\right)$$

is not a logarithm on the left half plane. If we exponentiate this, we get $-z$ and not z. This is because

$$\mathrm{Arg}(z) = \tan^{-1}\left(\frac{y}{x}\right) + \pi$$

in the second and third quadrants.

Complex Logarithm

Define ℓ on $\mathbb{C}$ minus the negative real axis as follows:

$$l(z) = \begin{cases} \log\sqrt{x^2+y^2} + \imath\tan^{-1}\left(\frac{y}{x}\right) & \text{if } \mathrm{Re}(z) > 0 \\ \log y + \imath\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ \log(-y) - \imath\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ \log\sqrt{x^2+y^2} + \imath\tan^{-1}\left(\frac{y}{x}\right) + \imath\pi & \text{if } x < 0, y > 0 \\ \log\sqrt{x^2+y^2} + \imath\tan^{-1}\left(\frac{y}{x}\right) - \imath\pi & \text{if } x < 0, y < 0. \end{cases}$$

Check that ℓ is differentiable throughout the domain and $\ell'(z) = \frac{1}{z}$ everywhere. This is a branch of the logarithm. Since $\ell(z)$ is nothing but

$$\log|z| + \imath\mathrm{Arg}(z),$$

we call this branch, the principal branch of the logarithm. This domain cannot be further enlarged. Via a minimal surgery, we removed the negative real axis in order to get holomorphy. Thus, the negative real axis is the branch cut corresponding to the principal branch.

You should convince yourself that there is nothing sacrosanct about the negative real axis. You could define a branch of the logarithm with branch cut being any half-ray, for example. If you visualize $\mathbb{C} \cup \{\infty\}$, note that the logarithm cannot be continuously defined in any neighbourhood of 0 or ∞. Around any other point it can be continuously (and holomorphically) defined in a small enough neighbourhood. Is this clear to all of you? We say 0 and ∞ are the branch points for the logarithm.

Remark: Branch, branch cut, and branch point can be defined for any multi-valued function. Picking one branch makes the multi-valued function single-valued. Branch cut is a minimal cut you perform to make it holomorphic in the remaining part. Branch cuts are constructed by connecting branch points.

Now that we have defined log, we can define several other functions. For instance, we define

$$z^w = \exp(w \log z).$$

If log in the above formula is the principal branch, then note that

$$\imath^\imath = \exp(\imath \log \imath) = \exp(\imath \cdot \imath \frac{\pi}{2}) = \exp(-\frac{\pi}{2}) = e^{-\frac{\pi}{2}},$$

which is a real number!

$\imath$ am imaginary, but if $\imath$ apply my full power to myself, ($\imath^\imath$), then it's real!

The India opener said that defending a small total was always difficult on this pitch.

"When the opposition batsmen comes hard on you and when you are defending a small total, it is difficult."

Asked about losing their home matches, Sehwag said: "It is a good track to bat on. We have to apply ourselves, put the runs on the board and bowl well to win."

Tags Delhi Daredevils Virender Sehwag Deccan Chargers Kochi Tuskers ipl2011

Share 0 Tweet 0 Share 0 Email

If you don't apply yourself, good grades will remain imaginary!

Aside

So imaginary power imaginary can be real. Can irrational power irrational be rational?

Similarly we define inverse trigonometric functions etc. For instance, let $w = \sin^{-1} z$. Then,

$$z = \sin w = \frac{e^{\imath w} - e^{-\imath w}}{2\imath}.$$

Solve the quadratic to get:

$$e^{\imath w} = \imath z + \sqrt{1 - z^2}.$$

Thus,

$$\sin^{-1} z = \frac{1}{\imath} \log \left[\imath z + \sqrt{1 - z^2} \right].$$

Exercise: Analyse the functions

$$(i) \quad \log\left(\frac{1+z}{1-z}\right) \quad (ii) \quad \log\left(\frac{1+\imath z}{1-z}\right).$$

See the natural maximal domains on which these are holomorphic. Write down the real and imaginary parts u and v explicitly in these domains. What are the branch points?

Exercise: Describe branch points and branch cuts for

$$\sqrt{(z-z_1)(z-z_2)\ldots(z-z_n)}$$

when (i) n is even, and (ii) when n is odd.