Exercises :

1. Show that is $f(z)$ if a proper function from $\mathbb{C}$ to $\mathbb{C}$, then f is a closed map, i.e, f maps closed subsets to closed subsets.
(This fact will be useful for us later)

2. Show that the only holomorphic function $f : \mathbb{C} \to \mathbb{C}$ with the property that $f(x+iy) = u(x) + iv(y)$ is given by $f(z) = \lambda z + a$ for some $\lambda \in \mathbb{R}$ and $a \in C$.

3. Find the radius of convergence of :

a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n z^n}{n}$. z<1/2

b) $\displaystyle\sum_{n=1}^{\infty} n! z^n$ 0

4. Draw the following paths :

(i) $\gamma(t) = 1 + i + 2e^{it}; \quad 0 \leq t \leq 2\pi$
(ii) $\gamma(t) = t + i\ \mathrm{cos}ht, \quad 1 \leq t \leq 1$

5. Find the Taylor expansion of the following functions around 0 and find the radius of convergence:

(i) $(2z+1)^{-1}$
(ii) $f(z) = \log(1+z)$

6. (Sometimes one can use Cauchy Integral formula even in the case when f is not holomorphic.) Let $f(z) = |z+1|^2$. Let $\gamma(t) = e^{it},\ 0 \leq t \leq 2\pi$.

(i) Show that f is not holomorphic on any domain that contains γ.
(ii) Find a function g that is holomorphic on some domain that contains γ and such that $f(z) = g(z)$ at all points on the unit circle γ.
(iii) Using the above and Cauchy Integral formula, show that:

$$\int_{\gamma} |z+1|^2 = 2\pi i$$

7. Prove the uniqueness part in the theorem discussed in lecture 7 under the heading titled "Logarithm Revisited".
(I had discussed this in class, but it seems it was not clear to many of you and so prove it as an exercise).