MA 2017, Tutorial Sheet-6
Heat equation by separation of variables

1. Which of the following PDEs can be reduced to two or more ODEs by the method of separation of variables?

   (a) $au_{xy} + bu = 0$ (b) $au_{xx} + 2bu_{xy} + cu_{yy} = 0$

   (c) $au_{xx} + 2bu_{xy} + cu_y = 0$ (d) $z_{xx} + xyz_y = 0$

   (e) $f(x)\theta_{tt} = a^2[f(x)\theta_x]_x$

2. Solve the following heat equations.

   (a) $L = 1,\quad u_t = u_{xx}$, with $u(0,t) = 0 = u(1,t),\ u(x,0) = x(1-x)$,

   (b) $L = \pi,\ u_t = 3u_{xx}$, with $u(0,t) = 0 = u(\pi,t),\ u(x,0) = x\sin x$,

   (c) $L = 2,\ u_t = 4u_{xx}$, with $u_x(0,t) = 0 = u_x(2,t),\ u(x,0) = \cos(\frac{\pi x}{2})$,

   (d) $L = 1,\ u_t = u_{xx}$, with $u_x(0,t) = 0 = u_x(1,t),\ u(x,0) = x^2(3x^2 - 8x + 6)$,

   (e) $L = 1,\ u_t = u_{xx}$, with $u_x(0,t) = 0 = u_x(1,t),\ u(x,0) = \cos\pi x$,

3. Solve the following non-homogeneous IBVP.

   (a) $L = 4,\ u_t = 9u_{xx} - 54x$, with $u(0,t) = 1,\ u(4,t) = 61,\ u(x,0) = 1 - x + x^3$.

   (b) $L = 1,\ u_t = u_{xx} - 2$, with $u(0,t) = 1,\ u(1,t) = 3,\ u(x,0) = 2x^2 + 1$,

   (c) $L = 1,\ u_t = 3u_{xx} - 18x$, with $u_x(0,t) = -1,\ u_x(1,t) = -1,\ u(x,0) = -x$,

   (d) $L = 1,\ u_t = 3u_{xx} + \pi^2\sin\pi x$, with $u_x(0,t) = 0,\ u_x(1,t) = -\pi,\ u(x,0) = 2\cos\pi x$.

   (e) $L = \pi,\quad u_t - u_{xx} = 8e^{-t}\sin 3x$, with $u(0,t) = 0 = u(\pi,t),\ u(x,0) = 2\sin 2x$.

4. The curved surface of a thin rod of length ℓ is insulated. The temperature throughout the rod is 100. If at each end of the rod the temperature is suddenly reduced to 0 at time $t = 0$, find the temperature subsequently. What is the explicit temperature at the mid-point of the rod and how does it behave with respect to the time variable t?

5. (a) Solve $u_t - u_{xx} = e^{-t}\cos 2x$, with $u_x(0,t) = e^{-t},\ u_x(\pi,t) = -e^{-t},\ u(x,0) = \sin x$.

   Remaining problems are not for the exam but only for your intellectual curiosity.

6. For the heat equation: $u_t - ku_{xx} = 0,\ 0 < x < \ell,\ t > 0$ with $u(x,0) = u_0(x)$ and $u_x(0,t) = u_x(\ell,t) = 0$, show that $\displaystyle\int_0^\ell u(x,t)\,dx = C$, where C is a constant. In other words, the average temperature stays constant.

Further, show that $\lim\limits_{t \mapsto \infty} u(x,t) = \dfrac{1}{\ell} \displaystyle\int_0^\ell u_0(x)\,dx$.

Compute the solution, when u_0 is: $(i)\, u_0(x) = x$, $(ii)\, u_0(x) = \sin^2(\frac{\pi x}{\ell})$.

7. Compute the solution of $u_t - k u_{xx} + a^2 u = 0, \quad 0 < x < \ell,\ t > 0$ with $u(x,0) = u_0(x)$ and $u(0,t) = u(\ell,t) = 0$. Find $\lim_{t \mapsto \infty} u(x,t)$.