MA-207 Differential Equations II
S1 - Lecture 4

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Recall

• For integer $n \geq 0$, $P_n(x)$ is a polynomial of degree n and is a solution of Legendre equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

• Set $\{P_0(x), P_1(x), \ldots, P_n(x), \ldots\}$ is an orthogonal basis of $\mathcal{P}(x)$, the vector space of polynomial functions, with respect to inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\, dx$$

• The length or norm of $P_n(x)$ is

$$\|P_n(x)\|^2 = \frac{2}{2n+1}$$

• If f is a polynomial of degree n, then we can write

$$f(x) = \sum_{i=0}^{n} a_i P_i(x), \qquad a_i = \frac{2i+1}{2} \int_{-1}^{1} f(x) P_i(x)\, dx$$

• Let $L^2([-1,1])$ be the vector space consisting of all functions f on $[-1,1]$ which is square-integrable i.e.

$$\int_{-1}^{1} f(x) f(x) dx < \infty$$

• For square-integrable functions f and g, we define their "inner product"

$$\langle f, g \rangle := \int_{-1}^{1} f(x) g(x) dx$$

Strictly speaking, above definition of bilinear form on $L^2([-1,1])$ $\langle f, g \rangle := \int_{-1}^{1} f(x) g(x) dx$
is not an inner product.

Take a function f which is defined as 0 for $x \neq 0$ on $[-1,1]$ and $f(0) = 1$.

Then f is a non-zero function. But $\langle f, f \rangle = 0$. So our bilinear form is not positive definite. or any other odd funtion

To rectify the problem, we replace a function f in $L^2([-1,1])$ by equivalence class of f,
where we define two functions f and g to be equivalent if norm of $f - g$ is zero.

But we will not bother about this technicality.

Theorem. Set $\{P_0, P_1, \ldots, P_n, \ldots\}$ of Legendre polynomials is a maximal orthogonal set in $L^2([-1,1])$, i.e. for $f \in L^2([-1,1])$

$$\langle f, P_n(x)\rangle = 0 \ \ \forall n \geq 0 \implies f = 0$$

• To any square-integrable function $f(x)$ on $[-1,1]$, we associate a series of Legendre polynomials

$$f \sim \sum_{n=0}^{\infty} c_n P_n(x), \qquad c_n = \frac{\langle f, P_n\rangle}{\langle P_n, P_n\rangle}$$

The series is the Fourier-Legendre series (or simply the Legendre series) of $f(x)$.

Convergence of Legendre series in L^2 norm.

The Fourier-Legendre series of $f \in L^2([-1,1])$ converges in norm to $f(x)$, that is

$$\|f(x) - \sum_{n=0}^{m} c_n P_n(x)\| \to 0 \quad \text{as} \quad m \to \infty$$

• Pointwise convergence of Legendre series to $f(x)$ is more delicate. There are two issues here:

- Does the Legendre series converge at x?
- If yes, then does it converge to $f(x)$?

Theorem (Legendre expansion theorem)

If both $f(x)$ and $f'(x)$ have at most a finite number of jump discontinuities in the interval $[-1,1]$, then the Legendre series converges to

$$\frac{1}{2}(f(x_-)+f(x_+)) \quad \text{for } -1<x<1$$

$$f(-1_+) \quad \text{for } x=-1$$

$$f(1_-) \quad \text{for } x=1$$

In particular, the series converges to $f(x)$ at every point of continuity x.

Example

Consider the function

$$f(x)=sgn(x)=\begin{cases} 1 & \text{if } 0<x<1 \\ -1 & \text{if } -1<x<0 \end{cases}$$

The Legendre series of $f(x)$ is

$$\sum_{n=0}^{\infty} c_n P_n(x), \quad c_n=\frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx$$

Since $P_{2n}(x)$ is an even function and f is an odd function, we get $\boxed{c_{2n}=0, \quad n\geq 0}$

The odd coefficients are computed as follows.

$$c_{2k+1} = \frac{4k+3}{2}\int_{-1}^{1} \mathsf{sgn}(x) P_{2k+1}(x)\,dx =$$ n=2k+1

$$(4k+3)\int_0^1 P_{2k+1}(x)\,dx = (4k+3)\frac{D^{2k}(x^2-1)^{2k+1}}{2^{2k+1}(2k+1)!}\Big|_0^1$$

$$= \frac{4k+3}{2^{2k+1}(2k+1)!}\left[-D^{2k}(x^2-1)^{2k+1}\big|_{x=0}\right]$$

$$= \frac{4k+3}{2^{2k+1}(2k+1)!}\left[-(2k)!\binom{2k+1}{k}(-1)^{k+1}\right]$$

$$= \frac{(-1)^k(4k+3)}{2^{2k+1}(2k+1)}\binom{2k+1}{k}$$

$$c_{2k+1} = \frac{(-1)^k(4k+3)}{2^{2k+1}(2k+1)}\binom{2k+1}{k}$$

The Legendre series of f is

$$\frac{3}{2}P_1(x) - \frac{7}{8}P_3(x) + \frac{11}{16}P_5(x) - \cdots$$ How

Since f and f' have jump discontinuity at 0, by the Legendre expansion theorem, this series converges to $f(x)$ for $x \neq 0$ and to 0 for $x = 0$.

Theorem

Suppose we want to approximate $f \in L^2([-1,1])$ in the sense of least square by polynomials $p(x)$ of degree $\leq n$; i.e. we want to minimize

$$I = \|f(x) - p(x)\|^2 = \int_{-1}^{1} [f(x) - p(x)]^2 \, dx$$

Then the minimizing polynomial is precisely the first $n+1$ terms of the Legendre series of $f(x)$, i.e.

$$\sum_{i=0}^{n} c_i P_i(x), \quad c_k = \frac{2k+1}{2} \int_{-1}^{1} f(x) P_k(x) dx$$

Proof. Write degree $\leq n$ minimizing polynomial $p(x) = \sum_{k=0}^{n} b_k P_k(x)$. We claim that $b_k = c_k$.

$$I = \int_{-1}^{1} \left[f(x) - \sum_{k=0}^{n} b_k P_k(x) \right]^2 dx =$$

$$\int_{-1}^{1} f(x)^2 \, dx + \sum_{k=0}^{n} \frac{2b_k^2}{2k+1} - 2 \sum_{k=0}^{n} b_k \int_{-1}^{1} f(x) P_k(x) \, dx$$

$$= \int_{-1}^{1} f(x)^2 \, dx + \sum_{k=0}^{n} \frac{2}{2k+1} b_k^2 - 2 \sum_{k=0}^{n} b_k \frac{2c_k}{2k+1}$$

$$= \int_{-1}^{1} f(x)^2 \, dx + \sum_{k=0}^{n} \frac{2}{2k+1} (b_k - c_k)^2 - \sum_{k=0}^{n} \frac{2}{2k+1} c_k^2$$

Clearly, I is minimum when $b_k = c_k$ for $k = 0, \ldots, n$.

This completes the proof.

Caution. If f has a power series expansion on $[-1, 1]$, then best "least square polynomial approximation" to $f(x)$ is not the partial sums of the power series, in general.

This completes our study of Legendre polynomials which were obtained by solving Legendre equation using power series method.

Now we will study another class of ODE for which power series method does not work.

Consider $$y'' + q(x)y' + r(x)y = 0 \quad (*)$$

- We say x_0 is an ordinary point of $(*)$ if $q(x), r(x)$ analytic at x_0 In this case, a power series solution in $x - x_0$ exists.
- x_0 is a regular singular point of $(*)$ if it is not ordinary but $(x - x_0)q(x)$ and $(x - x_0)^2 r(x)$ are analytic at x_0, i.e. there exists functions $q_1(x)$ and $r_1(x)$ analytic at x_0 such that

$$q(x) = \frac{q_1(x)}{x - x_0}, \qquad r(x) = \frac{r_1(x)}{(x - x_0)^2}$$

- A non-regular singular point is irregular singular.

Let $a(x), b(x), c(x)$: analytic at x_0 with $a(x_0) \neq 0$.

$$(x - x_0)^2 a(x)y'' + (x - x_0)b(x)y' + c(x)y = 0,$$

Then x_0 is a regular singular point.

- $x^2y'' + xy' + (x + 1)y = 0$
  All $x \neq 0$ are ordinary points.
  $x = 0$ is a regular singular point.
- $x^3y'' + xy' + (x + 1)y = 0$
  All $x \neq 0$ are ordinary points.
  $x = 0$ is an irregular singular point.
- $x^2(1 - x^2)y'' + xy' + (x + 1)y = 0$
  All $x \neq 0, 1, -1$ are ordinary points.
  $x = 0, 1, -1$ are regular singular points.

Example. Consider Cauchy-Euler equation

$$x^2y'' + b_0xy' + c_0y = 0 \qquad b_0, c_0 \in \mathbb{R}$$

Here $x = 0$ is a regular singular point.

All $x \neq 0$ are ordinary points.

Assume $\boxed{x > 0}$. Note that $y = x^r$ solves the equation if

$$r(r - 1) + b_0r + c_0 = 0$$

$$\implies r^2 + (b_0 - 1)r + c_0 = 0$$

Let r_1 and r_2 denote the roots of this quadratic equation.

- If the roots $r_1 \neq r_2$ are real, then

$$x^{r_1} \quad \text{and} \quad x^{r_2}$$

  are two independent solutions.
- If the roots $r_1 = r_2$ are real, then

$$x^{r_1} \quad \text{and} \quad (\log x)x^{r_1}$$

  are two independent solutions.
- If the roots are complex (written as $a \pm ib$), then

$$x^a \cos(b \log x) \quad \text{and} \quad x^a \sin(b \log x)$$

  are two independent solutions.

Fuchs-Frobenius Theorem.

Consider the ODE

$$x^2 a(x) y'' + x b(x)\, y' + c(x)\, y = 0 \qquad (*)$$

where $a(x), b(x), c(x)$ are analytic at 0 with $a(0) \neq 0$.

So $x = 0$ is a regular singular point of ODE.

Then $(*)$ always has atleast one solution of the form

$$y(x, r) = x^r \sum_{n \geq 0} a_n x^n, \quad a_0 = 1, \quad r \in \mathbb{R} \qquad (**)$$

The solution $(**)$ is called Frobenius solution or fractional power series solution.

The power series $\sum_{n\geq 0} a_n x^n$ converges on $(-\rho, \rho)$, where $\rho > 0$ is the minimum of radius of convergence of $a(x), b(x)$ and $c(x)$.

Thus, we will consider the solution $y(x)$ only in the open interval $(0, \rho)$.

First solution

Consider $\quad x^2y'' + xB(x)y' + C(x)y = 0$
with

$$B(x) = \sum_{i\geq 0} b_i x^i \quad \text{and} \quad C(x) = \sum_{i\geq 0} c_i x^i$$

analytic functions at 0. Then $x = 0$ is a regular singular point. By Fuchs - Frobenius theorem, there exists a solution of the type

$$y(x, r) = \sum_{n\geq 0} a_n(r) x^{n+r}$$

with $a_0 = 1$. Substituting it into the ODE, we get

$$0 = \left(\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n(r)x^{n+r}\right) + \left(\sum_{i=0}^{\infty} b_i x^i\right)$$

$$\left(\sum_{n=0}^{\infty}(n+r)a_n(r)x^{n+r}\right) + \left(\sum_{i=0}^{\infty} c_i x^i\right)\left(\sum_{n=0}^{\infty} a_n(r)x^{n+r}\right)$$

Note $a_0(r) = 1$. Coefficient of x^r gives

$I(r) := r(r-1)+b_0 r+c_0 = 0,\quad b_0 = B(0),\ c_0 = C(0)$

For $n \geq 1$, the coefficient of x^{n+r} gives

$$a_n(r)I(n+r)+\sum_{i=0}^{n-1} b_{n-i}\,(i+r)a_i(r)+\sum_{i=0}^{n-1} c_{n-i}a_i(r) = 0$$

Let r_1 and r_2 be the roots of indicial equation $I(r) = 0$.

We will assume that r_1 and r_2 are real and $r_1 \geq r_2$.

Note that $I(r_1) = 0$ and $I(n+r_1) \neq 0$ for any $n \geq 1$.
Hence we can define $a_n(r_1)$ for $n \geq 1$, inductively by the equation

$$a_n(r_1) = -\frac{\sum_{i=0}^{n-1} b_{n-i}(i+r_1)a_i(r_1) + \sum_{i=0}^{n-1} c_{n-i}a_i(r_1)}{I(n+r_1)}$$

We get that

$$y_1(x) = y(x,r_1) = \sum_{n\geq 0} a_n(r_1)x^{n+r_1}$$

is a solution to ODE.

Note $r_1 \geq r_2$.

• If $r_1 - r_2$ is not an integer, then $I(n + r_2) \neq 0$ for any $n \geq 1$.

Hence we can define $a_n(r_2)$ for $n \geq 1$, inductively by the equation

$$a_n(r_2) = -\frac{\sum_{i=0}^{n-1} b_{n-i}(i + r_2)a_i(r_2) + \sum_{i=0}^{n-1} c_{n-i}a_i(r_2)}{I(n + r_2)}$$

We get that

$$y_2(x) = y(x, r_2) = \sum_{n \geq 0} a_n(r_2)x^{n+r_2}$$

is a 2nd linearly independent solution.

• To find second linearly independent solution when either $r_1 = r_2$ or $r_1 - r_2$ is an integer will be considered next.

But first an example.

Example

Consider the ODE $\quad 2x^2y'' - xy' + (1 + x)y = 0$

Observe that $x = 0$ is a regular singular point.

Further, Frobenius solution exist on $(0, \infty)$.

Substitute

$$y(x, r) = \sum_{0}^{\infty} a_n(r)x^{n+r}, \qquad a_0 = 1$$

we get the indicial equation, i.e. coefficient of x^r is

$$I(r) = 2r(r-1)-r+1 = 2r^2-3r+1 = (r-1)(2r-1)$$

Roots of indicial equation are $\boxed{r_1 = 1}$ and $\boxed{r_2 = 1/2}$

Their difference $r_1 - r_2 = 1/2$ is not an integer.

So we will get two linearly independent Frobenius solutions on $(0, \infty)$ one each for r_1 and r_2 respectively.

$2x^2y'' - xy' + (1+x)y = 0, \ I(r) = (r-1)(2r-1)$

For $n \geq 1$, the coefficient of x^{n+r} is

$$I(n+r)a_n(r) + a_{n-1}(r) = 0$$

$$\implies a_n(r) = -\frac{a_{n-1}(r)}{(n+r-1)(2n+2r-1)}$$

Thus, $\quad a_n(r_1) = a_n(1) = -\dfrac{a_{n-1}}{n(2n+1)}$

$$= (-1)^n \frac{1}{n!((2n+1)(2n-1)\dots 3} = (-1)^n \frac{2^n}{(2n+1)!}$$

Thus the first solution is

$$y_1(x) = y(x,1) = x + \sum_{n\geq 1} \frac{(-2)^n x^{n+1}}{(2n+1)!}$$

Second solution for $r_2 = 1/2$,

$$a_n(1/2) = -\frac{a_{n-1}}{(n+\frac{1}{2}-1)2n} = -\frac{a_{n-1}}{n(2n-1)}$$

$$= (-1)^n \frac{1}{n!(2n-1)(2n-3)\dots 1} = \frac{(-2)^n}{(2n)!}$$

So the second solution is

$$y_2(x) = y(x, 1/2) = x^{1/2}\left(1 + \sum_{n\geq 1} \frac{(-2)^n x^n}{(2n)!}\right)$$

Both the solutions converge on $(0, \infty)$.

Theorem (Repeated roots case $r_1 = r_2$.)

Consider $\quad Ly = x^2y'' + xB(x)y' + C(x)y = 0,$
where $B(x) = \sum_0^\infty b_i x^i$ *and* $C(x) = \sum_0^\infty c_i x^i$ *are analytic at* 0. *Assume indicial equation* $I(r) = 0$ *has repeated roots* r_1, r_1.

Define $$y(x, r) = \sum_0^\infty a_n(r)x^{n+r}, \qquad a_0 = 1$$

where $a_n(r)$ *for* $n \geq 1$ *are rational functions in the variable* r *defined inductively by*

$$I(n+r)a_n(r) + \sum_{i=0}^{n-1} b_{n-i}\,(i+r)a_i(r) + \sum_{i=0}^{n-1} c_{n-i}a_i(r) = 0$$

Then we get $\quad Ly(x,r) = I(r)x^r$

Let us differentiate both sides with respect to r.

$$\frac{\partial}{\partial r}(Ly(x,r)) = \frac{\partial}{\partial r}(I(r)x^r)$$

$$\implies L\left(\frac{\partial}{\partial r}y(x,r)\right) = I'(r)x^r + I(r)x^r \ln x$$

Since $I(r) = 0$ has repeated roots r_1, r_1, we get

$$L\left(\frac{\partial}{\partial r}y(x,r)\big|_{r=r_1}\right) = 0$$

Hence, the second solution is

$$y_2(x) = \frac{\partial}{\partial r}y(x,r)\big|_{r=r_1}$$

Recall, $\quad y(x,r) = x^r \sum_0^\infty a_n(r)x^n, \qquad a_0 = 1.$

Hence

$$y_2(x) = \frac{\partial}{\partial r}\left(x^r \sum_0^\infty a_n(r)x^n\right)\big|_{r=r_1}$$

$$= y_1(x)\ln x + x^{r_1}\sum_{n\geq 1} a_n'(r_1)x^n$$

Example

Consider the ODE

$$x^2y'' + 3xy' + (1-2x)y = 0$$

This has a regular singularity at $x = 0$.

The indicial equation

$$\begin{aligned} I(r) &= r(r-1) + 3r + 1 \\ &= r^2 + 2r + 1 \end{aligned}$$

has a repeated roots $-1, -1$.

For Frobenius solution, substitute

$$y = x^r \sum_{n \geq 0} a_n(r) x^n, \qquad a_0 = 1$$

in the ODE.

The coefficient of x^{n+r} for $n \geq 1$ is

$I(n+r)a_n(r) - 2a_{n-1}(r) = 0$

$$\implies a_n(r) = \frac{2a_{n-1}(r)}{(n+r+1)^2}$$

Setting $r = -1$, we get

$$a_n(-1) = \frac{2a_{n-1}(-1)}{n^2} = \frac{2^n}{(n!)^2}$$

Thus, the fractional power series solution is

$$y_1(x) = y(x, -1) = \frac{1}{x} \sum_{n \geq 0} \frac{2^n}{(n!)^2} x^n$$

The second solution is

$$y_2(x) = y_1(x) \ln x + x^{-1} \sum_{n \geq 1} a_n'(-1) x^n$$

$$a_n(r) = \frac{2a_{n-1}(r)}{(n+r+1)^2} = \frac{2^n}{[(n+r+1)(n+r)\dots(r+2)]^2}$$

$$a_n'(r) = \frac{-2.2^n \, [(n+r+1)(n+r)\dots(r+2)]'}{[(n+r+1)(n+r)\dots(r+2)]^3}$$

$$= -2a_n(r)\left(\frac{1}{n+r+1} + \frac{1}{n+r} + \cdots + \frac{1}{r+2}\right)$$

$$a_n'(-1) = -\frac{2^{n+1}}{(n!)^2} H_n,$$

where $\qquad H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$

(These are the partial sums of the harmonic series.)

So the second solution is

$$y_2(x) = y_1(x)\ln x - \frac{1}{x}\sum_{n\geq 1} \frac{2^{n+1} H_n}{(n!)^2} x^n$$

Both solutions are defined on $(0, \infty)$.