MA 205 Complex Analysis: CR Equations

U. K. Anandavardhanan
IIT Bombay

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(class taken by Professor Ravi Raghunathan)

Introduction

In the last class, we introduced complex numbers and studied complex valued functions defined on a domain in $\mathbb{C}$. We stated the fact that every polynomial of degree n with complex coefficients has exactly n roots in $\mathbb{C}$. This is called the fundamental theorem of algebra. A beautiful proof of this beautiful fact was promised, and it will come later. We studied differentiability of a function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$. We also stated the fact that if f is once differentiable in Ω, then it is infinitely many times differentiable in Ω. Once again, a proof was promised, and this too will come later. You had promised to give me examples to show that for functions of a real variable, f may be differentiable in an interval with f' not even continuous. You were also supposed to work out the exercises given at the end of the last lecture.

Cauchy-Riemann Equations

Today, first we'll derive the so called Cauchy-Riemann equations. Notice the plural. There are two Cauchy-Riemann equations, and these are partial differential equations; i.e., equations containing partial derivatives. If f is complex differentiable at a point $z_0 = a + \imath b$, then these two equations will be satisfied at the point (a, b).

Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ be differentiable at $z_0 \in \Omega$. Thus,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

exists. In the last class, we have stressed the point that the existence of this complex limit means a lot; the limit exists as z approaches z_0 along any path. To derive the CR equations, we'll in particular look at the existence of this limit as $z \to z_0$ along the x-direction and the y-direction.

Cauchy-Riemann Equations

Let $z = x + \imath y$ and $f(z) = u(x,y) + \imath v(x,y)$. Now, as $z \to z_0$ in the x-direction:

$$\begin{aligned}
f'(z_0) &= \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \\
&= \lim_{h \to 0} \left[\frac{u(a+h,b) - u(a,b)}{h} + \imath \frac{v(a+h,b) - v(a,b)}{h} \right] \\
&= \lim_{h \to 0} \frac{u(a+h,b) - u(a,b)}{h} + \imath \lim_{h \to 0} \frac{v(a+h,b) - v(a,b)}{h} \\
&= u_x(a,b) + \imath v_x(a,b).
\end{aligned}$$

In writing the limit of a sum as the sum of the limits, we have used the fact that the individual limits exist. Why is this true in our situation?

Similarly, in the y-direction, we get

$$f'(z_0) = \lim_{k \to 0} \frac{f(z_0 + \imath k) - f(z_0)}{\imath k} = v_y(a,b) - \imath u_y(a,b).$$

Cauchy-Riemann Equations

Thus, differentiability of $f = u + \imath v$ at $z_0 = a + \imath b$ implies that u_x, u_y, v_x, v_y exist at (a, b) and they satisfy

$$u_x = v_y \;\&\; u_y = -v_x$$

at (a, b). These are the CR equations. If CR equations are not satisfied at a point, then f is not differentiable at that point.

Example: Consider $f(z) = |z|^2$. Here, $u(x, y) = x^2 + y^2$, $v(x, y) = 0$. Thus CR equations are satisfied only at the point $(0, 0)$. We conclude that f is not differentiable at any point other than $(0, 0)$. Can we conclude that f is differentiable at $(0, 0)$? Well, we need to check; CR equations give only one way. In other words, real and imaginary parts of f satisfying CR equations at a point is necessary but not sufficient for f to be differentiable at that point. In this example:

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{|z|^2}{z} = 0.$$

Example:

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$$

Here,

$$u(x,y) = \begin{cases} \frac{x^3-3xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$$

$$v(x,y) = \begin{cases} \frac{-3x^2y+y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Check that CR equations are satisfied at $(0,0)$. You'll get $u_x = v_y = 1$ and $u_y = -v_x = 0$ at $(0,0)$.

But, f is not differentiable at 0.

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{\bar{z}^2}{z^2} = \lim_{(x,y) \to (0,0)} \frac{(x - \imath y)^2}{(x + \imath y)^2}.$$

If $(x, y) \to (0, 0)$ via either of the axes, this limit is 1. If $(x, y) \to (0, 0)$ via $y = x$, this limit is -1. So limit does not exist.

If $z = x + \imath y$, then,

$$x = \frac{z + \bar{z}}{2}, \; y = \frac{z - \bar{z}}{2\imath}.$$

Suppose for a moment that z and $\bar{z}$ are independent variables! Formally applying chain rule:

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} \cdot \frac{1}{2} + \frac{\partial f}{\partial y} \cdot \frac{1}{2\imath} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \imath \frac{\partial f}{\partial y}\right).$$

Similarly,

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \imath \frac{\partial f}{\partial y}\right).$$

Motivated by this, we introduce the symbols:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \imath\frac{\partial}{\partial y}\right); \ \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \imath\frac{\partial}{\partial y}\right).$$

Note that CR equations now can be written as

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

We can of course view $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ as a function of two real variables;

$$f(x, y) = (u(x, y), v(x, y)).$$

For such functions, in MA 105, you have seen the notion of the total derivative. Recall: f is differentiable at (a, b) if there exists a 2×2 matrix $Df(a, b)$ such that

$$\lim_{(h,k)\to(0,0)} \frac{\left\| f(a+h, b+k) - f(a, b) - Df(a, b) \begin{bmatrix} h \\ k \end{bmatrix} \right\|}{\|(h, k)\|} = 0.$$

Of course, if total derivative exists, then all the partial derivatives exist, and

$$Df = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}.$$

Existence of partial derivatives does not imply the existence of total derivative, but existence of partial derivatives which are continuous throughout the domain does imply the existence of total derivative.

Exercise: Show that if f is complex differentiable, then f is real differentiable; i.e., f has a total derivative as a function of two real variables. Show that the converse is not true.

Thus, complex differentiability implies:

- real differentiability
- real and imaginary parts satisfy CR.

What if we assume both these? Can we then say f is complex differentiable? And the answer is Yes.

Proof: Since $f = u + \imath v$ is real differentiable,

$$\lim_{(x,y)\to(a,b)} \frac{\left\| \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} - \begin{bmatrix} u(a,b) \\ v(a,b) \end{bmatrix} - \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix} \right\|}{\|(x-a, y-b)\|} = 0.$$

Note that the numerator is nothing but

$$|f(z) - f(z_0) - \alpha(x-a) - \beta(y-b)|,$$

where $\alpha = u_x + \imath v_x$, $\beta = u_y + \imath v_y$.

Define

$$\eta(z) = \frac{f(z) - f(z_0) - \alpha(x - a) - \beta(y - b)}{z - z_0}.$$

Observe that

$$\lim_{z \to z_0} \eta(z) = 0.$$

Thus,

$$f(z) - f(z_0) = \alpha(x - a) + \beta(y - b) + \eta(z)(z - z_0),$$

with $\eta(z) \to 0$ as $z \to z_0$. This is a version of the increment theorem that you have seen in MA 105. Write this as

$$f(z) - f(z_0) = \frac{\alpha - \imath\beta}{2}(z - z_0) + \frac{\alpha + \imath\beta}{2}\overline{z - z_0} + \eta(z)(z - z_0).$$

Thus,

$$\frac{f(z)-f(z_0)}{z-z_0} = \frac{\partial f}{\partial z}(z_0) + \frac{\partial f}{\partial \bar{z}}(z_0)\frac{\overline{z-z_0}}{z-z_0} + \eta(z).$$

Question is whether the lhs limit exists as $z \to z_0$. This exists if and only if the rhs limit exists. Since,

$$\lim_{z \to z_0} \frac{\overline{z-z_0}}{z-z_0}$$

does not exist (why?) and $\lim_{z \to z_0} \eta(z)$ exists, this happens if and only if

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0.$$

i.e., CR equations are satisfied at z_0. Also, if this is the case,

$$f'(z_0) = \frac{\partial f}{\partial z}(z_0),$$

since $\lim_{z \to z_0} \eta(z) = 0$.

Corollary: Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ be such that it has continuous partial derivatives throughout Ω. Then if they satisfy the CR equations at a point, f is differentiable at that point. (Proof?)

Exercise: Show that $f(z) = e^x(\cos y + \imath \sin y)$ is holomorphic throughout $\mathbb{C}$. Note that $f'(z) = f(z)$. This is the complex exponential function.

The assumptions in the statement of the corollary can be weakened. In fact, the following is true:

Theorem

Let f be continuous on Ω. Suppose the partial derivatives exist and satisfy the Cauchy-Riemann equations at every point in Ω. Then f is holomorphic in Ω.

We'll not be able to prove this theorem in this course. Those of you who take a minor in mathematics will be in a position to appreciate a proof of this. Here's the name of the theorem anyway: The Looman-Menchoff Theorem.

Exercise: Show that the CR equations take the form

$$u_r = \frac{1}{r} v_\theta \ \& \ v_r = -\frac{1}{r} u_\theta$$

in polar coordinates.

Harmonic Functions

A real valued function $u : U \subset \mathbb{R}^2 \to \mathbb{R}$ is called harmonic if it is twice continuously differentiable and it satisfies the Laplace's equation everywhere on U. Recall that the Laplacian is the divergence of the gradient.

$$\nabla^2 = \nabla \cdot \nabla = \nabla \cdot (u_x, u_y) = u_{xx} + u_{yy}.$$

If $u(x, y)$ is harmonic, then $u_{xx} + u_{yy} = 0$ on U.
If $f = u + \imath v$ is holomorphic on Ω, then both u and v are harmonic on Ω. Indeed, CR equations tell us that

$$u_x = v_y \ \& \ u_y = -v_x.$$

Thus,

$$u_{xx} + u_{yy} = v_{xy} - v_{xy} = 0.$$

Similarly, $\nabla^2 v = 0$.

Suppose u and v are harmonic functions on Ω. We say that v is a harmonic conjugate of u if $f = u + \imath v$ is holomorphic in Ω.
Example: $v(x, y) = 2xy$ is a harmonic conjugate of $x^2 - y^2$ in any domain. Indeed, $f(z) = z^2$ is holomorphic everywhere.
Note that v is a harmonic conjugate of u does not mean that u is a harmonic conjugate of v! In fact:
Exercise: If u and v are harmonic conjugates of each other, show that they are constant functions.
Here's a general method to find a harmonic conjugate: given a harmonic u, find u_x. Equate $v_y = u_x$ and integrate wrt y. You'll get $v(x, y) = \ldots + \phi(x)$. Now $v_x = \ldots + \phi'(x)$. Equate this to $-u_y$ to find $\phi(x)$. That gives you v.
This might give you an impression that you can always find a harmonic conjugate, but this is not so.

Can you tell me why this method could fail in general?

But if Ω is "nice", then every harmonic u on Ω has a harmonic conjugate. Conversely, if every harmonic u on Ω has a harmonic conjugate, then Ω has to be "nice". Thus, the question in analysis: "does every harmonic function has a harmonic conjugate?" is answered by geometry: "answer depends on the shape of the domain". It's relevant at this point to recall from MA 105 that curl of grad is always zero but curl free is certainly a grad of something only when the domain has a nice property. Recall this.