MA-207 Differential Equations II

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S2 - Lecture 3

Announcements

1. As tutorials are being held on different days for different batches, it has been decided to cancel the in-tutorial quizzes.
2. Now the one hour quiz will have 15 marks and the end semester exam will have 35 marks. Recall the total score will be out of 50.

Last week

The two important things we did last week were

1. How to compute the radius of convergence of a power series
2. Power series defines a nice function in its interval of convergence
3. Suppose we are given an ODE: $y'' + p(x)y' + q(x)y = 0$, and $p(x)$ and $q(x)$ are analytic (given by power series) in an interval I around x_0, then the solution y is also analytic on I.
4. We can compute the two independent solutions, to an ODE as above, by plugging in a power series into the ODE and getting recursive relation for coefficients.

Legendre equation

The following ODE is known as the Legendre equation.

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0$$

Here p denotes a fixed real number.

By Existence theorem, power series solution in x exists on the interval $(-1,1)$.

Put $y(x) = \sum_{n=0}^{\infty} a_n x^n$ in the Legendre equation.

Equating the coefficient of x^n in the resulting equation, we get the recursive relation

$$(n+2)(n+1)a_{n+2} - n(n+1)a_n + p(p+1)a_n = 0, \quad n \geq 0$$

Legendre equation: Two independent solutions

$$\Longrightarrow \qquad a_{n+2} = \frac{(n-p)(p+n+1)}{(n+2)(n+1)}\, a_n$$

Let us set $\boxed{a_0 = 1 \text{ and } a_1 = 0}$ in the recursion formula to find a first solution.

The solution is given by (note it is an even function)

$$y_1(x) := a_0 \left[1 - \frac{p(p+1)}{2!}x^2 + \frac{p(p+1)(p-2)(p+3)}{4!}x^4 + \dots\right]$$

Let us find a second solution by setting $\boxed{a_0 = 0 \text{ and } a_1 = 1}$ in the recursion formula.

The second solution is given by (note it is an odd function)

$$y_2(x) := a_1 \left[x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-1)(p+2)(p-3)(p+4)}{5!}x^5 + \dots\right]$$

Legendre polynomials

Thus, the two independent solutions are

$$y_1(x) := a_0 \left[1 - \frac{p(p+1)}{2!} x^2 + \frac{p(p+1)(p-2)(p+3)}{4!} x^4 + \ldots \right]$$

$$y_2(x) := a_1 \left[x - \frac{(p-1)(p+2)}{3!} x^3 + \frac{(p-1)(p+2)(p-3)(p+4)}{5!} x^5 + \ldots \right]$$

Remark

If $p \in \{0, 2, 4, \ldots\} \cup \{-1, -3, -5, \ldots\}$ then $y_1(x)$ is a polynomial function. It is an even function.

If $p \in \{1, 3, 5, \ldots\} \cup \{-2, -4, -6, \ldots\}$ then $y_2(x)$ is a polynomial function. It is an odd function.

Thus, if p is an integer then exactly one solution is a polynomial and the other is an infinite power series.

Legendre polynomials

The general solution

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

is called a Legendre function.

If $p = m$ is an integer, then precisely one of y_1 or y_2 is a polynomial, and it is called the m-th Legendre polynomial $P_m(x)$.

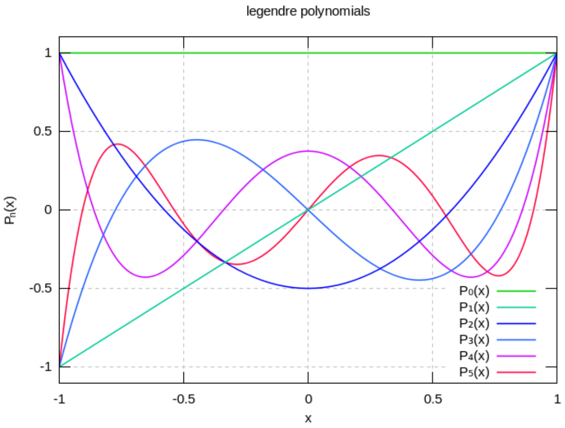
For $m \geq 0$ note that $P_m(x)$ is a polynomial of degree m. It is an even function if m is even and an odd function if m is odd.

Let us write down few Legendre polynomials.

$$\begin{array}{rclcl}
P_0(x) & = & 1 & & \\
P_1(x) & = & x & & \\
P_2(x) & = & (1-3x^2)(\frac{-1}{2}) & = & \frac{1}{2}(3x^2-1) \\
P_3(x) & = & (x-\frac{5}{3}x^3)(\frac{-3}{2}) & = & \frac{1}{2}(5x^3-3x) \\
P_4(x) & = & (1-10x^2+\frac{35}{3}x^4)(\frac{3}{8}) & = & \frac{1}{8}(35x^4-30x^2+3) \\
P_5(x) & = & (x-\frac{14}{3}x^3+\frac{21}{5}x^5)(\frac{15}{8}) & = & \frac{1}{8}(63x^5-70x^3+15x)
\end{array}$$

Legendre polynomials

The graphs of P_m's in the interval $(-1, 1)$ are given below.



What is so interesting about the collection of Legendre polynomials?

To answer this question we need some linear algebra.

We will recall the notion of Inner product space from Linear Algebra.

First recall the notion of a vector space V over $\mathbb{R}$.

A vector space is a set equipped with two operations

- addition

$$v + w, \qquad v, w \in V$$

- scalar multiplication

$$cv, \qquad c \in \mathbb{R}, \quad v \in V$$

A vector space V has a dimension, which may <u>not</u> be finite.

Inner product spaces

Let V be a vector space over $\mathbb{R}$ (not necessarily finite-dimensional).

A bilinear form on V is a map

$$\langle , \rangle : V \times V \to \mathbb{R}$$

which is linear in both coordinates, that is,

$$\langle au + v, w \rangle = a\langle u, w \rangle + \langle v, w \rangle$$

$$\langle u, av + w \rangle = a\langle u, v \rangle + \langle u, w \rangle$$

for $a \in \mathbb{R}$ and $u, v \in V$.

An inner product on V is a bilinear form on V which is

- symmetric:$\langle v, w \rangle = \langle w, v \rangle$
- positive definite: $\langle v, v \rangle \geq 0$ for all v and $\langle v, v \rangle = 0$ iff $v = 0$

A vector space with an inner product is called an inner product space.

Orthogonality

In an inner product space V, two vectors u and v are orthogonal if $\langle u, v \rangle = 0$.

More generally, a set of vectors forms an orthogonal system if they are mutually orthogonal.

An orthogonal basis is an orthogonal system which is also a basis.

Example

Consider the vector space $\mathbb{R}^n$ with coordinate-wise addition and scalar multiplication. The rule

$$\langle (a_1, \ldots, a_n), (b_1, \ldots, b_n) \rangle := \sum_{i=1}^{n} a_i b_i$$

defines an inner product on $\mathbb{R}^n$.

The standard basis $\{e_1, \ldots, e_n\}$ is an orthogonal basis of $\mathbb{R}^n$.

The previous example can be formulated more abstractly as follows.

Example

Let V be a finite-dimensional vector space with ordered basis $B = \{e_1, \ldots, e_n\}$.

For $u = \sum_{i=1}^{n} a_i e_i$ and $v = \sum_{i=1}^{n} b_i e_i$ define

$$\langle u, v \rangle := \sum_{i=1}^{n} a_i b_i$$

This defines an inner product on V

With this definition, $\{e_1, \ldots, e_n\}$ is an orthogonal basis of V.

Lemma

Suppose V is a **finite** *dimensional inner product space, and $e_1, \ldots, e_n$ is an orthogonal basis.*

Then for any $v \in V$

$$v = \sum_{i=1}^{n} \frac{\langle v, e_i \rangle}{\langle e_i, e_i \rangle} e_i$$

Proof.

Write $v = \sum_{i=1}^{n} a_i e_i$.

We want to find the coefficients a_j.

Take inner product of v with e_j:

$$\langle v, e_j \rangle = \langle \sum_{i=1}^{n} a_i e_i, e_j \rangle = \sum_{i=1}^{n} a_i \langle e_i, e_j \rangle = a_j \langle e_j, e_j \rangle$$

Thus, $$a_j = \frac{\langle v, e_j \rangle}{\langle e_j, e_j \rangle}$$ $\square$

Lemma

In a finite-dimensional inner product space, there always exists an orthogonal basis.

Start with any basis and modify it to an orthogonal basis by Gram-Schmidt orthogonalization.

This result is not necessarily true in infinite-dimensional inner product spaces.

For infinite dimensional vector spaces, we can only talk of a maximal orthogonal set.

A subset $\{e_1, e_2, \ldots\}$ is called a maximal orthogonal set for V if

- $\langle e_i, e_j \rangle = \delta_{ij}$
- $\langle v, e_i \rangle = 0$ for all i iff $v = 0$.

Length of a vector

For a vector v in an inner product space, define

$$\|v\| := \langle v, v\rangle^{1/2}$$

This is called the norm or length of the vector v.

It satisfies the following three properties.

- $\|0\| = 0$ and $\|v\| > 0$ if $v \neq 0$
- $\|v + w\| \leq \|v\| + \|w\|$
- $\|av\| = |a|\|v\|$

for all $v, w \in V$ and $a \in \mathbb{R}$.

Pythagoras theorem

Theorem

For **orthogonal** *vectors v and w in any inner product space V,*

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2$$

Proof.

$$\begin{aligned}\|v + w\|^2 &= \langle v + w, v + w\rangle \\ &= \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle \\ &= \langle v, v\rangle + \langle w, w\rangle \\ &= \|v\|^2 + \|w\|^2\end{aligned}$$

□

More generally, for any orthogonal system $\{v_1, \dots, v_n\}$

$$\|v_1 + \cdots + v_n\|^2 = \|v_1\|^2 + \cdots + \|v_n\|^2$$

The vector space of polynomials

The set of all polynomials in the variable x is a vector space denoted by $\mathcal{P}(x)$.

The set

$$\{1, x, x^2, \dots\}$$

is an infinite basis of the vector space $\mathcal{P}(x)$.

$\mathcal{P}(x)$ carries an inner product defined by

$$\langle f, g \rangle := \int_{-1}^{1} f(x)g(x)\, dx$$

We are integrating over finite interval $[-1, 1]$ which ensures that the integral is finite.

The norm of a polynomial is by definition $\langle f, f \rangle$

$$\|f\| := \left(\int_{-1}^{1} f(x)f(x)dx \right)^{1/2}$$

Derivative transfer

Note that

$$\frac{d}{dx}(fg) = g\frac{df}{dx} + f\frac{dg}{dx}$$

Integrating both sides we get

$$\int_{-1}^{1} \frac{d}{dx}(fg) = \int_{-1}^{1} g\frac{df}{dx} + \int_{-1}^{1} f\frac{dg}{dx}$$

$$\implies \qquad f(1)g(1) - f(-1)g(-1) = \int_{-1}^{1} g\frac{df}{dx} + \int_{-1}^{1} f\frac{dg}{dx}$$

Thus if

$$f(1)g(1) - f(-1)g(-1) = 0$$

then we get

$$\int_{-1}^{1} g\frac{df}{dx} = -\int_{-1}^{1} f\frac{dg}{dx}$$

This will be referred to as derivative-transfer

Orthogonality of Legendre polynomials

Since $P_m(x)$ is a polynomial of degree m, it follows that

$$\{P_0(x), P_1(x), P_2(x), \dots\}$$

is a basis of the vector space of polynomials $\mathcal{P}(x)$.

Theorem

We have

$$\langle P_m, P_n \rangle = \int_{-1}^{1} P_m(x) P_n(x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m = n \end{cases}$$

i.e. Legendre polynomials form an orthogonal basis for the vector space $\mathcal{P}(x)$ and

$$\|P_n(x)\|^2 = \frac{2}{2n+1}$$

Orthogonality of Legendre polynomials

The Legendre equation may be written as

$$((1-x^2)y')' + p(p+1)y = 0$$

In particular, $P_m(x)$ satisfies

$$((1-x^2)P_m'(x))' + m(m+1)P_m(x) = 0 \qquad (*)$$

Proof of Orthogonality.

Multiply $(*)$ by P_n and integrate to get

$$\int_{-1}^{1} ((1-x^2)P_m')'P_n + m(m+1)\int_{-1}^{1} P_mP_n = 0$$

By derivative transfer ($f = (1-x^2)P_m'$ and $g = P_n$), we get

$$-\int_{-1}^{1} (1-x^2)P_m'P_n' + m(m+1)\int_{-1}^{1} P_mP_n = 0$$

continued ...

Interchanging the roles of m and n, we get

$$-\int_{-1}^{1}(1-x^2)P_m'P_n' + n(n+1)\int_{-1}^{1} P_mP_n = 0$$

Subtracting the two identities, we obtain

$$[m(m+1)-n(n+1)]\int_{-1}^{1} P_mP_n = 0$$

If $m \neq n$ we get

$$\int_{-1}^{1} P_mP_n = 0$$

Thus, P_m and P_n are orthogonal. $\square$

Rodrigues formula

It only remains to show that $\|P_n(x)\|^2 = \dfrac{2}{2n+1}$.

We need some intermediate steps before we can show this.

Denote by D the differential operator $\frac{d}{dx}$.

Let us first note that for $0 \leq i < n$

$$\big(D^i(x^2-1)^n\big)(1) = 0$$

This is clear once we observe

$$\begin{aligned} D^i(x^2-1)^n &= D^i\big((x-1)^n(2+x-1)^n\big) \\ &= D^i\big(2^n(x-1)^n + (*)(x-1)^{n+1} + \cdots\big) \end{aligned}$$

By the same reasoning we get for $0 \leq i < n$

$$\big(D^i(x^2-1)^n\big)(-1) = 0$$

Consider the polynomial of degree n given by

$$y(x) = D^n(x^2-1)^n$$

Rodrigues formula

For $k < n$ consider the integral

$$\int_{-1}^{1} P_k(x)y(x) = \int_{-1}^{1} P_k(x)D(D^{n-1}(x^2-1)^n)$$

applying derivative transfer with $f = D^{n-1}(x^2-1)^n$ and $g = P_k(x)$ we get

$$\begin{aligned}\int_{-1}^{1} P_k(x)y(x) &= -\int_{-1}^{1} DP_k(x)D^{n-1}(x^2-1)^n \\ &= \int_{-1}^{1} D^2P_k(x)D^{n-2}(x^2-1)^n \\ &= \int_{-1}^{1} D^nP_k(x)(x^2-1)^n = 0\end{aligned}$$

We have repeatedly applied derivative transfer with $f = D^{n-i}(x^2-1)^n$ and $g = D^{i-1}P_k(x)$.

Since $P_k(x)$ is a polynomial of degree k we get that $D^nP_k(x) = 0$.

Rodrigues formula

This forces that $y(x) = cP_n(x)$ for some nonzero constant c as we know that $P_k(x)$'s are orthogonal to each other.

$$\begin{aligned} D^n(x^2-1)^n &= D^n\big((x-1)^n(2+x-1)^n\big) \\ &= D^n\big(2^n(x-1)^n + (*)(x-1)^{n+1} + \cdots\big) \end{aligned}$$

From the above it is clear that

$$y(1) = n!2^n$$

Thus, we can normalize our Legendre polynomials so that $P_m(1) = 1$. That is, take

$$P_m(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

This is called Rodrigues formula.

Computing $\|P_n(x)\|$

Proof.

$$\int_{-1}^{1} P_n(x)P_n(x)\,dx = \frac{1}{2^{2n}(n!)^2}\int_{-1}^{1}\frac{d^n}{dx^n}(x^2-1)^n\frac{d^n}{dx^n}(x^2-1)^n\,dx$$

$$= \frac{(-1)^n}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n\frac{d^{2n}}{dx^{2n}}(x^2-1)^n\,dx$$

by derivative transfer

$$= \frac{(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}(1-x^2)^n\,dx$$

$$I_n = \int_{-1}^{1}(1-x^2)^n\,dx = \int_{-1}^{1}(1-x^2)^n\frac{dx}{dx}dx$$

$$\overset{dt}{=} 2n\int_{-1}^{1}(1-x^2)^{n-1}x^2dx = -2nI_n + 2nI_{n-1}$$

Proof.

We get the recursive relation

$$(2n+1)I_n = 2nI_{n-1}$$

We conclude that

$$I_n = \frac{2n}{2n+1}\frac{2(n-1)}{2n-1}\cdots\frac{2}{3}I_0$$

We conclude that

$$\|P_n(x)\| = \frac{(2n)!}{2^{2n}(n!)^2}\frac{2n}{2n+1}\frac{2(n-1)}{2n-1}\cdots\frac{2}{3}I_0$$
$$= \frac{I_0}{2n+1} = \frac{2}{2n+1}$$

$\square$

This completes the proof of the theorem.

Exercises added after class

These exercises are related to some facts from linear algebra that we used in the lecture today.

1. Recall the proof of the Gram Schmidt orthogonalization lemma.
2. Let $f_i(x)$ (for $i \geq 0$) be a collection of nonzero polynomials. Assume that $f_i(x)$ has degree i. Show that $\{f_0(x), f_1(x), \ldots, f_n(x)\}$ is a basis for the vector space consisting of polynomials of degree $\leq n$.