I DIFFERENTIABILITY FOR FUNCTIONS OF MANY VARIABLES

In Section 14.3 of the textbook we gave an informal introduction to the concept of differentiability. We called a function $f(x, y)$ *differentiable* at a point (a, b) if it is well-approximated by a linear function near (a, b). This section focuses on the precise meaning of the phrase "well-approximated." By looking at examples, we shall see that local linearity requires the existence of partial derivatives, but they do not tell the whole story. In particular, existence of partial derivatives at a point is not sufficient to guarantee local linearity at that point.

We begin by discussing the relation between continuity and differentiability. As an illustration, take a sheet of paper, crumple it into a ball and smooth it out again. Wherever there is a crease it would be difficult to approximate the surface by a plane—these are points of nondifferentiability of the function giving the height of the paper above the floor. Yet the sheet of paper models a graph which is continuous—there are no breaks. As in the case of one-variable calculus, continuity does not imply differentiability. But differentiability does *require* continuity: there cannot be linear approximations to a surface at points where there are abrupt changes in height.

Differentiability For Functions Of Two Variables

For a function of two variables, as for a function of one variable, we define differentiability at a point in terms of the error in a linear approximation as we move from the point to a nearby point. If the point is (a, b) and a nearby point is $(a+h, b+k)$, the distance between them is $\sqrt{h^2+k^2}$. (See Figure I.34.)

Definition of Differentiability

A function $f(x, y)$ is **differentiable at the point** (a, b) if there is a linear function $L(x, y) = f(a, b) + m(x-a) + n(y-b)$ such that if the *error* $E(x, y)$ is defined by

$$f(x, y) = L(x, y) + E(x, y),$$

and if $h = x-a$, $k = y-b$, then the *relative error* $E(a+h, b+k)/\sqrt{h^2+k^2}$ satisfies

$$\lim_{(h,k)\to(0,0)} \frac{E(a+h, b+k)}{\sqrt{h^2+k^2}} = 0.$$

The function f is **differentiable** if it is differentiable at each point of its domain. The function $L(x, y)$ is called the *local linearization* of $f(x, y)$ near (a, b).

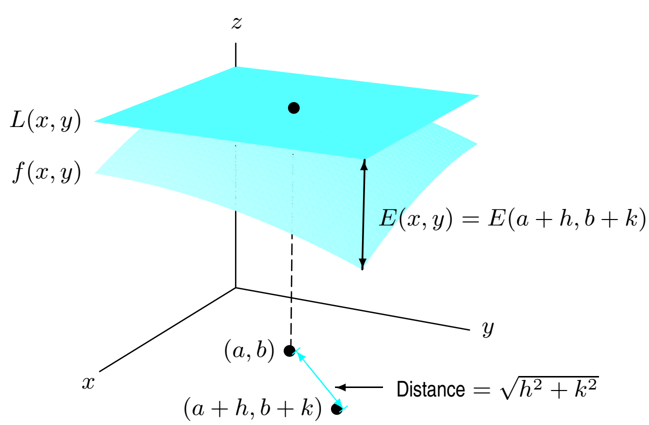

Figure I.34: Graph of function $z = f(x, y)$ and its local linearization $z = L(x, y)$ near the point (a, b)

Partial Derivatives and Differentiability

In the next example, we show that this definition of differentiability is consistent with our previous notion — that is, that $m = f_x$ and $n = f_y$ and that the graph of $L(x, y)$ is the tangent plane.

Example 1 Show that if f is a differentiable function with local linearization $L(x, y) = f(a, b) + m(x - a) + n(y - b)$, then $m = f_x(a, b)$ and $n = f_y(a, b)$.

Solution Since f is differentiable, we know that the relative error in $L(x, y)$ tends to 0 as we get close to (a, b). Suppose $h > 0$ and $k = 0$. Then we know that

$$\begin{aligned} 0 = \lim_{h\to 0} \frac{E(a+h,b)}{\sqrt{h^2+k^2}} = \lim_{h\to 0} \frac{E(a+h,b)}{h} &= \lim_{h\to 0} \frac{f(a+h,b) - L(a+h,b)}{h} \\ &= \lim_{h\to 0} \frac{f(a+h,b) - f(a,b) - mh}{h} \\ &= \lim_{h\to 0} \left(\frac{f(a+h,b) - f(a,b)}{h} \right) - m = f_x(a,b) - m. \end{aligned}$$

A similar result holds if $h < 0$, so we have $m = f_x(a, b)$. The result $n = f_y(a, b)$ is found in a similar manner.

The previous example shows that if a function is differentiable at a point, it has partial derivatives there. Therefore, if any of the partial derivatives fail to exist, then the function cannot be differentiable. This is what happens in the following example of a cone.

Example 2 Consider the function $f(x, y) = \sqrt{x^2 + y^2}$. Is f differentiable at the origin?

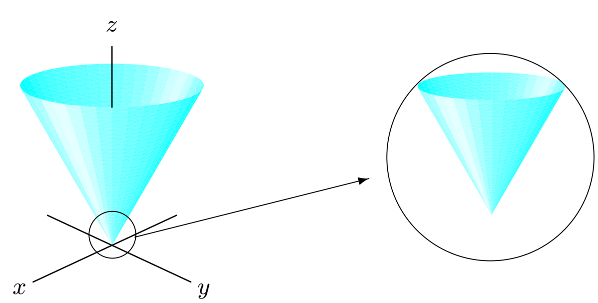

Figure I.35: The function $f(x, y) = \sqrt{x^2 + y^2}$ is not locally linear at $(0, 0)$: Zooming in around $(0, 0)$ does not make the graph look like a plane

Solution If we zoom in on the graph of the function $f(x, y) = \sqrt{x^2 + y^2}$ at the origin, as shown in Figure I.35, the sharp point remains; the graph never flattens out to look like a plane. Near its vertex, the graph does not look like it is well approximated (in any reasonable sense) by any plane.

Judging from the graph of f, we would not expect f to be differentiable at $(0, 0)$. Let us check this by trying to compute the partial derivatives of f at $(0, 0)$:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\sqrt{h^2+0} - 0}{h} = \lim_{h\to 0} \frac{|h|}{h}.$$

Since $|h|/h = \pm 1$, depending on whether h approaches 0 from the left or right, this limit does not exist and so neither does the partial derivative $f_x(0, 0)$. Thus, f cannot be differentiable at the origin. If it were, both of the partial derivatives, $f_x(0, 0)$ and $f_y(0, 0)$, would exist.

Alternatively, we could show directly that there is no linear approximation near $(0,0)$ that satisfies the small relative error criterion for differentiability. Any plane passing through the point $(0,0,0)$ has the form $L(x,y) = mx + ny$ for some constants m and n. If $E(x,y) = f(x,y) - L(x,y)$, then

$$E(x,y) = \sqrt{x^2+y^2} - mx - ny.$$

Then for f to be differentiable at the origin, we would need to show that

$$\lim_{\substack{h\to 0\\ k\to 0}} \frac{\sqrt{h^2+k^2} - mh - nk}{\sqrt{h^2+k^2}} = 0.$$

Taking $k = 0$ gives

$$\lim_{h\to 0} \frac{|h| - mh}{|h|} = 1 - m \lim_{h\to 0} \frac{h}{|h|}.$$

This limit exists only if $m = 0$ for the same reason as before. But then the value of the limit is 1 and not 0 as required. Thus, we again conclude f is not differentiable.

In Example 2 the partial derivatives f_x and f_y did not exist at the origin and this was sufficient to establish nondifferentiability there. We might expect that if both partial derivatives do exist, then f *is* differentiable. But the next example shows that this not necessarily true: the existence of both partial derivatives at a point is *not* sufficient to guarantee differentiability.

Example 3 Consider the function $f(x,y) = x^{1/3}y^{1/3}$. Show that the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ exist, but that f is not differentiable at $(0,0)$.

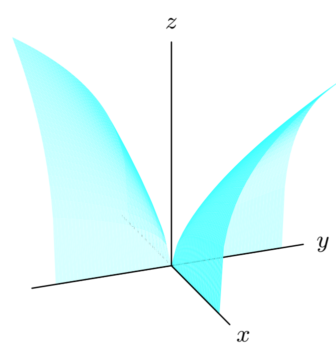

Figure I.36: Graph of $z = x^{1/3}y^{1/3}$ for $z \geq 0$

Solution See Figure I.36 for the part of the graph of $z = x^{1/3}y^{1/3}$ when $z \geq 0$. We have $f(0,0) = 0$ and we compute the partial derivatives using the definition:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0,$$

and similarly

$$f_y(0,0) = 0.$$

So, if there did exist a linear approximation near the origin, it would have to be $L(x,y) = 0$. But we can show that this choice of $L(x,y)$ does not result in the small relative error that is required for differentiability. In fact, since $E(x,y) = f(x,y) - L(x,y) = f(x,y)$, we need to look at the limit

$$\lim_{\substack{h\to 0\\ k\to 0}} \frac{h^{1/3}k^{1/3}}{\sqrt{h^2+k^2}}.$$

If this limit exists, we get the same value no matter how h and k approach 0. Suppose we take $k = h > 0$. Then the limit becomes

$$\lim_{h\to 0} \frac{h^{1/3}h^{1/3}}{\sqrt{h^2+h^2}} = \lim_{h\to 0} \frac{h^{2/3}}{h\sqrt{2}} = \lim_{h\to 0} \frac{1}{h^{1/3}\sqrt{2}}.$$

But this limit does not exist, since small values for h will make the fraction arbitrarily large. So the only possible candidate for a linear approximation at the origin does not have a sufficiently small relative error. Thus, this function is *not* differentiable at the origin, even though the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ exist. Figure I.36 confirms that near the origin the graph of $z = f(x,y)$ is not well approximated by any plane.

In summary,

- If a function is differentiable at a point, then both partial derivatives exist there.
- Having both partial derivatives at a point does not guarantee that a function is differentiable there.

Continuity and Differentiability

We know that differentiable functions of one variable are continuous. Similarly, it can be shown that if a function of two variables is differentiable at a point, then the function is continuous there.

In Example 3 the function f was continuous at the point where it was not differentiable. Example 4 shows that even if the partial derivatives of a function exist at a point, the function is not necessarily continuous at that point if it is not differentiable there.

Example 4 Suppose that f is the function of two variables defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

Problem 4 on page 46 showed that $f(x,y)$ is not continuous at the origin. Show that the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ exist. Could f be differentiable at $(0,0)$?

Solution From the definition of the partial derivative we see that

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \left(\frac{1}{h} \cdot \frac{0}{h^2+0^2}\right) = \lim_{h\to 0} \frac{0}{h} = 0,$$

and similarly

$$f_y(0,0) = 0.$$

So, the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ exist. However, f cannot be differentiable at the origin since it is not continuous there.

In summary,

- If a function is differentiable at a point, then it is continuous there.
- Having both partial derivatives at a point does not guarantee that a function is continuous there.

How Do We Know If a Function Is Differentiable?

Can we use partial derivatives to tell us if a function is differentiable? As we see from Examples 3 and 4, it is not enough that the partial derivatives exist. However, the following condition *does* guarantee differentiability:

> **Condition for Differentiability**
>
> If the partial derivatives, f_x and f_y, of a function f exist and are continuous on a small disk centered at the point (a, b), then f is differentiable at (a, b).

We will not prove this fact, although it provides a criterion for differentiability which is often simpler to use than the definition. It turns out that the requirement of continuous partial derivatives is more stringent than that of differentiability, so there exist differentiable functions which do not have continuous partial derivatives. However, most functions we encounter will have continuous partial derivatives. The class of functions with continuous partial derivatives is given the name C^1.

Example 5 Show that the function $f(x, y) = \ln(x^2 + y^2)$ is differentiable everywhere in its domain.

Solution The domain of f is all of 2-space except for the origin. We shall show that f has continuous partial derivatives everywhere in its domain (that is, the function f is in C^1). The partial derivatives are

$$f_x = \frac{2x}{x^2 + y^2} \quad \text{and} \quad f_y = \frac{2y}{x^2 + y^2}.$$

Since each of f_x and f_y is the quotient of continuous functions, the partial derivatives are continuous everywhere except the origin (where the denominators are zero). Thus, f is differentiable everywhere in its domain.

Most functions built up from elementary functions have continuous partial derivatives, except perhaps at a few obvious points. Thus, in practice, we can often identify functions as being C^1 without explicitly computing the partial derivatives.

The Error in Linear and Quadratic Taylor Approximations

On page 729 of the textbook, we saw how to approximate a function $f(x, y)$ by Taylor polynomials. (The Taylor polynomial of degree 1 is the local linearization.) We now compare the magnitudes of the errors in the linear and quadratic approximations.

Let's return to the function $f(x, y) = \cos(2x + y) + 3\sin(x + y)$. The contour plots in Example 4 on page 730 of the textbook suggest that the quadratic approximation, $Q(x, y)$, is a better approximation to f than the linear approximation, $L(x, y)$. Consider approximations about the point $(0, 0)$. The errors in the linear and the quadratic approximations are defined as

$$E_L(x, y) = f(x, y) - L(x, y) \qquad E_Q(x, y) = f(x, y) - Q(x, y).$$

Table I.3 shows how the magnitudes of these errors, $|E_L|$ and $|E_Q|$, depend on the distance, $d(x, y) = \sqrt{x^2 + y^2}$, of the point (x, y) from $(0, 0)$. The values in Table I.3 suggest that, in this example,

$$E_L(x, y) \text{ is approximately proportional to } d^2 \quad \text{and} \quad E_Q(x, y) \text{ is approximately proportional to } d^3.$$

Table I.3 *Magnitude of the error in the linear and quadratic approximations to* $f(x, y) = \cos(2x + y) + 3\sin(x + y)$

Point, (x, y)	Distance, d	Error, $\lvert E_L(x, y)\rvert$	Error, $\lvert E_Q(x, y)\rvert$
$x = y = 0$	0	0	0
$x = y = 10^{-1}$	$1.4 \cdot 10^{-1}$	$5 \cdot 10^{-2}$	$4 \cdot 10^{-3}$
$x = y = 10^{-2}$	$1.4 \cdot 10^{-2}$	$5 \cdot 10^{-4}$	$4 \cdot 10^{-6}$
$x = y = 10^{-3}$	$1.4 \cdot 10^{-3}$	$5 \cdot 10^{-6}$	$4 \cdot 10^{-9}$
$x = y = 10^{-4}$	$1.4 \cdot 10^{-4}$	$5 \cdot 10^{-8}$	$4 \cdot 10^{-12}$

To use these approximations in practice, we need bounds on the magnitudes of the errors. If the distance between (x, y) and (a, b) is represented by $d(x, y) = \sqrt{(x-a)^2 + (y-b)^2}$, it can be shown that the following results hold:

Error Bound for Linear Approximation

Suppose $f(x, y)$ is a function with continuous second-order partial derivatives such that for $d(x, y) \le d_0$,

$$|f_{xx}|,\ |f_{xy}|,\ |f_{yy}| \le M_L.$$

Suppose

$$\begin{aligned} f(x, y) &= L(x, y) + E_L(x, y) \\ &= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + E_L(x, y). \end{aligned}$$

Then we have

$$|E_L(x, y)| \le 2M_L d(x, y)^2 \quad \text{for} \quad d(x, y) \le d_0.$$

Note that the upper bound for the error term $E_L(x, y)$ has a form reminiscent of the second-order term in the Taylor formula for $f(x, y)$.

Error Bound for Quadratic Approximation

Suppose $f(x, y)$ is a function with continuous third-order partial derivatives such that for $d(x, y) \le d_0$,

$$|f_{xxx}|,\ |f_{xxy}|,\ |f_{xyy}|,\ |f_{yyy}| \le M_Q.$$

Suppose

$$\begin{aligned} f(x, y) &= Q(x, y) + E_Q(x, y) \\ &= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ &\quad + \frac{f_{xx}(a, b)}{2}(x - a)^2 + f_{xy}(a, b)(x - a)(y - b) + \frac{f_{yy}(a, b)}{2}(y - b)^2 + E_Q(x, y). \end{aligned}$$

Then we have

$$|E_Q(x, y)| \le \frac{4}{3} M_Q d(x, y)^3 \quad \text{for} \quad d(x, y) \le d_0.$$

Problem 15 shows how these error estimates and the coefficients (2 and $4/3$) are obtained. The important thing to notice is the fact that, for small d, the magnitude of E_L is much smaller than d and the magnitude of E_Q is much smaller than d^2. In other words we have the following result:

As $d(x,y) \to 0$:

$$\frac{E_L(x,y)}{d(x,y)} \to 0 \quad \text{and} \quad \frac{E_Q(x,y)}{(d(x,y))^2} \to 0.$$

This means that near the point (a,b), we can view the original function and the approximation as indistinguishable and behaving the same way.

Example 6 Suppose that the Taylor polynomial of degree 2 for f at $(0,0)$ is $Q(x,y) = 5x^2 + 3y^2$. Suppose we are also told that

$$|f_{xxx}|, |f_{xxy}|, |f_{xyy}|, |f_{yyy}| \leq 9.$$

Notice that $Q(x,y) > 0$ for all (x,y) except $(0,0)$. Show that, except at $(0,0)$, we have

$$f(x,y) > 0 \quad \text{for all } (x,y) \text{ such that } \sqrt{x^2+y^2} = d < 0.25.$$

Solution By the error bound for the Taylor polynomial of degree 2, we have

$$|E_Q(x,y)| = |f(x,y) - Q(x,y)| \leq \frac{4}{3}(9)d^3 = 12d^3$$

which can be written as

$$-12d^3 \leq f(x,y) - Q(x,y) \leq 12d^3.$$

Therefore we know that

$$Q(x,y) - 12d^3 \leq f(x,y).$$

Since $Q(x,y) = 5x^2 + 3y^2$, we have

$$5x^2 + 3y^2 - 12d^3 \leq f(x,y).$$

Since $5x^2 + 3y^2 \geq 3x^2 + 3y^2 = 3d^2$, we have

$$3d^2 - 12d^3 \leq f(x,y).$$

Now d^3 approaches 0 faster than d^2, so when d is small, we have

$$0 \leq 3d^2 - 12d^3 \leq f(x,y).$$

In fact, writing $3d^2 - 12d^3 = 3d^2(1-4d)$ shows that $d < 1/4$ ensures that $f(x,y) > 0$, except at $(0,0)$ where $f = 0$. Thus, f has the same sign as Q for points near $(0,0)$.

Problems for Section I

For the functions f in Problems 1–4 answer the following questions. Justify your answers.

(a) Use a computer to draw a contour diagram for f.

(b) Is f differentiable at all points $(x,y) \neq (0,0)$?

(c) Do the partial derivatives f_x and f_y exist and are they continuous at all points $(x,y) \neq (0,0)$?

(d) Is f differentiable at $(0,0)$?

(e) Do the partial derivatives f_x and f_y exist and are they continuous at $(0,0)$?

1. $f(x,y)=\begin{cases}\dfrac{x}{y}+\dfrac{y}{x}, & x\neq 0 \text{ and } y\neq 0,\\ 0, & x=0 \text{ or } y=0.\end{cases}$

2. $f(x,y)=\begin{cases}\dfrac{2xy}{(x^2+y^2)^2}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$

3. $f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$

4. $f(x,y)=\begin{cases}\dfrac{x^2y}{x^4+y^2}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$

5. Consider the function

$$f(x,y)=\begin{cases}\dfrac{xy^2}{x^2+y^2}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$$

(a) Use a computer to draw the contour diagram for f.
(b) Is f differentiable for $(x,y)\neq(0,0)$?
(c) Show that $f_x(0,0)$ and $f_y(0,0)$ exist.
(d) Is f differentiable at $(0,0)$?
(e) Suppose $x(t)=at$ and $y(t)=bt$, where a and b are constants, not both zero. If $g(t)=f(x(t),y(t))$, show that

$$g'(0)=\frac{ab^2}{a^2+b^2}.$$

(f) Show that

$$f_x(0,0)x'(0)+f_y(0,0)y'(0)=0.$$

Does the chain rule hold for the composite function $g(t)$ at $t=0$? Explain.
(g) Show that the directional derivative $f_{\vec u}(0,0)$ exists for each unit vector $\vec u$. Does this imply that f is differentiable at $(0,0)$?

6. Consider the function

$$f(x,y)=\begin{cases}\dfrac{xy^2}{x^2+y^4}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$$

(a) Use a computer to draw the contour diagram for f.
(b) Show that the directional derivative $f_{\vec u}(0,0)$ exists for each unit vector $\vec u$.
(c) Is f continuous at $(0,0)$? Is f differentiable at $(0,0)$? Explain.

7. Consider the function $f(x,y)=\sqrt{|xy|}$.

(a) Use a computer to draw the contour diagram for f. Does the contour diagram look like that of a plane when we zoom in on the origin?
(b) Use a computer to draw the graph of f. Does the graph look like a plane when we zoom in on the origin?
(c) Is f differentiable for $(x,y)\neq(0,0)$?
(d) Show that $f_x(0,0)$ and $f_y(0,0)$ exist.
(e) Is f differentiable at $(0,0)$? [Hint: Consider the directional derivative $f_{\vec u}(0,0)$ for $\vec u=(\vec i+\vec j)/\sqrt{2}$.]

8. Suppose a function f is differentiable at the point (a,b). Show that f is continuous at (a,b).

9. Suppose $f(x,y)$ is a function such that $f_x(0,0)=0$ and $f_y(0,0)=0$, and $f_{\vec u}(0,0)=3$ for $\vec u=(\vec i+\vec j)/\sqrt{2}$.

(a) Is f differentiable at $(0,0)$? Explain.
(b) Give an example of a function f defined on 2-space which satisfies these conditions. [Hint: The function f does not have to be defined by a single formula valid over all of 2-space.]

10. Consider the following function:

$$f(x,y)=\begin{cases}\dfrac{xy(x^2-y^2)}{x^2+y^2}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0).\end{cases}$$

The graph of f is shown in Figure I.37, and the contour diagram of f is shown in Figure I.38.

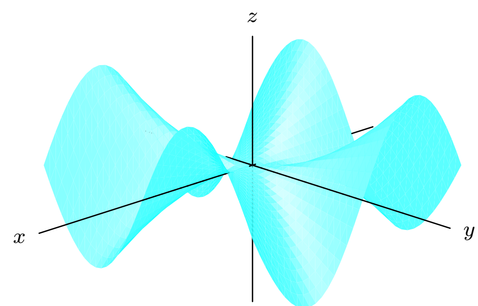

Figure I.37: Graph of $\dfrac{xy(x^2-y^2)}{x^2+y^2}$

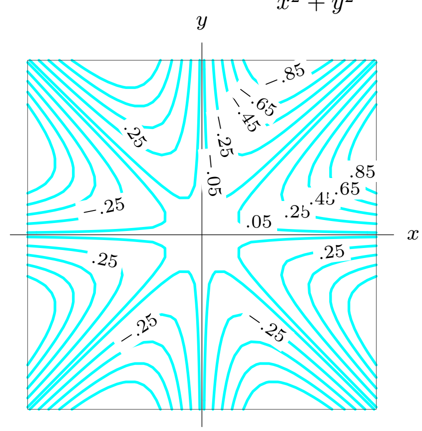

Figure I.38: Contour diagram of $\dfrac{xy(x^2-y^2)}{x^2+y^2}$

(a) Find $f_x(x,y)$ and $f_y(x,y)$ for $(x,y)\neq(0,0)$.
(b) Show that $f_x(0,0)=0$ and $f_y(0,0)=0$.
(c) Are the functions f_x and f_y continuous at $(0,0)$?
(d) Is f differentiable at $(0,0)$?

For Problems 11–14:

(a) Find the local linearization, $L(x, y)$, to the function $f(x, y)$ at the origin. Estimate the error $E_L(x, y) = f(x, y) - L(x, y)$ if $|x| \leq 0.1$ and $|y| \leq 0.1$.

(b) Find the degree 2 Taylor polynomial, $Q(x, y)$, for the function $f(x, y)$ at the origin. Estimate the error $E_Q(x, y) = f(x, y) - Q(x, y)$ if $|x| \leq 0.1$ and $|y| \leq 0.1$.

(c) Use a calculator to compute exactly $f(0.1, 0.1)$ and the errors $E_L(0.1, 0.1)$ and $E_Q(0.1, 0.1)$. How do these values compare with the errors predicted in parts (a) and (b)?

11. $f(x, y) = (\cos x)(\cos y)$

12. $f(x, y) = (e^x - x)\cos y$

13. $f(x, y) = e^{x+y}$

14. $f(x, y) = (x^2 + y^2)e^{x+y}$

15. It is known that if the derivatives of a one-variable function, $g(t)$, satisfy

$$|g^{(n+1)}(t)| \leq K \quad \text{for } |t| \leq d_0,$$

then the error, $E_n(t)$, in the n^{th} Taylor approximation, $P_n(x)$, is bounded as follows:

$$|E_n(t)| = |g(t) - P_n(t)| \leq \frac{K}{(n+1)!}|t|^{n+1} \quad \text{for } |t| \leq d_0.$$

In this problem, we use this result for $g(t)$ to get the error bounds for the linear and quadratic Taylor approximations to $f(x, y)$. For a particular function $f(x, y)$, let $x = ht$ and $y = kt$ for fixed h and k, and define $g(t)$ as follows:

$$g(t) = f(ht, kt) \quad \text{for } 0 \leq t \leq 1.$$

(a) Calculate $g'(t)$, $g''(t)$, and $g'''(t)$ using the chain rule.

(b) Show that $L(ht, kt) = P_1(t)$ and that $Q(ht, kt) = P_2(t)$, where L is the linear approximation to f at $(0, 0)$ and Q is the Taylor polynomial of degree 2 for f at $(0, 0)$.

(c) What is the relation between $E_L(x, y) = f(x, y) - L(x, y)$ and $E_1(t)$? What is the relation between $E_Q(x, y) = f(x, y) - Q(x, y)$ and $E_2(t)$?

(d) Assuming that the second and third-order partial derivatives of f are bounded for $d(x, y) \leq d_0$, show that $|E_L|$ and $|E_Q|$ are bounded as on page 52.