MA 205 Complex Analysis: Review

U. K. Anandavardhanan
IIT Bombay

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Announcement

Those who have not yet collected their Quiz I papers, please collect it from me at the end of the class.

Quiz II

1. If $u(x,y)$ is harmonic then both u_x and u_y are harmonic.
(a) True (b) False.

Justification: Laplacian commutes with partial derivatives:

$$\frac{\partial}{\partial x}\nabla u = \nabla \frac{\partial u}{\partial x}.$$

(♣) True (b) False.

Quiz II

2. The largest domain Ω among the following to which the function $u(x, y) = \sqrt{\sqrt{x^2 + y^2} + x}$ can be harmonically extended throughout (i.e., there is $h : \Omega \to \mathbb{R}$ such that h is harmonic in Ω and $h(x, y) = u(x, y)$ wherever $u(x, y)$ is harmonic) is

(a) $\mathbb{C}$ (b) $\mathbb{C}\setminus\{0\}$ (c) $\mathbb{C}\setminus\{x \mid x < 0\}$ (d) $\mathbb{C}\setminus\{x \mid x \leq 0\}$ (e) $\mathbb{C}\setminus\mathbb{R}$.

~~**Justification:** Take $x = r\cos\theta$ and $y = r\sin\theta$ with $r \geq 0$ and $-\pi < \theta \leq \pi$. Then $u(x, y) = \sqrt{2r}\,|\cos\frac{\theta}{2}|$ (which is harmonic in $\mathbb{C}\setminus\{x \mid x \leq 0\}$). The function $f(r, \theta) = \sqrt{2r}\cos\frac{\theta}{2}$ extends $u(x, y)$ and is harmonic when $r \neq 0$. Note that f cannot be further extended harmonically to $\mathbb{C}$ since $\sqrt{2z}$ cannot be holomorphically defined throughout $\mathbb{C}$.~~

(a) $\mathbb{C}$ ~~(♣) $\mathbb{C}\setminus\{0\}$~~ (c) $\mathbb{C}\setminus\{x \mid x < 0\}$ (d) $\mathbb{C}\setminus\{x \mid x \leq 0\}$ (e) $\mathbb{C}\setminus\mathbb{R}$.

Note added after the class: The correct answer would be[1]
(a) $\mathbb{C}$ (b) $\mathbb{C}\backslash\{0\}$ (c) $\mathbb{C}\backslash\{x \mid x < 0\}$ (♣) $\mathbb{C}\backslash\{x \mid x \leq 0\}$ (e) $\mathbb{C}\backslash\mathbb{R}$.

Justification: $u(x,y) = \sqrt{2r}\ |\cos\frac{\theta}{2}|$ and the largest domain in which this is harmonic in $\mathbb{C}\backslash\{x \mid x \leq 0\}$. If $h(x,y)$ is any continuous function defined in a larger domain in $\mathbb{C}$ which matches $u(x,y)$ in $\mathbb{C}\backslash\{x \mid x \leq 0\}$ then $h(x,y)$ has to agree with $u(x,y)$ everywhere since $u(x,y)$ is continuous everywhere and $\mathbb{C}\backslash\{x \mid x \leq 0\}$ is dense in $\mathbb{C}$.

[1] Thanks to several students, especially Sandesh and Karan, for clarifying this point.

3. Let $f(z) = \sum_{n=0}^{\infty} F_n \left(z - \frac{1}{2}\right)^n$, where F_n is the n^{th} Fibonacci number. Expand $f(z)$ as a power series $\sum_{n=0}^{\infty} a_n z^n$ around $z = 0$.

Then its radius of convergence of is

(a) $\frac{\sqrt{5}-1}{2}$ (b) $\frac{\sqrt{5}+1}{2}$ (c) $\frac{\sqrt{5}-2}{2}$ (d) $\frac{\sqrt{5}+2}{2}$ (e) $\frac{3-\sqrt{5}}{2}$.

Justification: The original series centered at $z = 1/2$ has radius of convergence $\frac{\sqrt{5}-1}{2}$. A point on this circle of convergence is given by $\frac{1}{2} + re^{\imath\theta}$. The minimum of $|\frac{1}{2} + re^{\imath\theta}|$ is when $\theta = \pi$.

(a) $\frac{\sqrt{5}-1}{2}$ (b) $\frac{\sqrt{5}+1}{2}$ (♣) $\frac{\sqrt{5}-2}{2}$ (d) $\frac{\sqrt{5}+2}{2}$ (e) $\frac{3-\sqrt{5}}{2}$.

4. The function $f(z) = (1 - \sin z)^2$ has a zero at $z = \frac{\pi}{2}$ of multiplicity (i.e., order) equal to

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5.

Justification: Since

$$\sin z = \cos(z - \frac{\pi}{2}) = 1 + (z - \frac{\pi}{2}) + \frac{(z - \frac{\pi}{2})^2}{2!} + \dots,$$

we get $(1 - \sin z)^2 = \left(z - \frac{\pi}{2}\right)^2 g(z)$, where $g(\pi/2) \neq 0$.

OR

Show that $f(\pi/2) = f'(\pi/2) = f^{(2)}(\pi/2) = f^{(3)}(\pi/2) = 0$ and $f^{(4)}(\pi/2) \neq 0$.

OR

For $g(z) = 1 - \sin z$, $g(\pi/2) = g'(\pi/2) = 0$ and $g''(\pi/2) \neq 0$.

(a) 1 (b) 2 (c) 3 (♣) 4 (e) 5.

Quiz II

5. The singularity at $z = 0$ of the function $f(z) = \sin z \cdot \sin \frac{1}{z}$ is
(a) bp (b) removable (c) pole (d) essential (e) nota.

Justification: The Laurent expansion of $f(z)$ around the isolated singularity $z = 0$ has infinite principal part.

(a) bp (b) removable (c) pole (♣) essential (e) nota.

Quiz II

6. The singularity at $z = 0$ of $f(z) = \sin^{-1} z \cdot \sin^{-1} \frac{1}{z}$ is
   (a) bp (b) removable (c) pole (d) essential (e) nota.

Justification: $\sin^{-1} z$ is holomorphic in a small enough neighborhood of $z = 0$ and $z = 0$ is a branch point for $\sin^{-1} \frac{1}{z}$.

(♣) bp (b) removable (c) pole (d) essential (e) nota.

7. The gamma function is never vanishing.

(a) True (b) False.

Justification:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

(♣) True (b) False.

8. The residue at $z = 0$ of $f(z) = \frac{z}{(1-\cos z)^2}$ is

(a) 0 (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) 1 (e) nota.

Justification:

$$1 - \cos z = \frac{z^2}{2}\left[1 - \frac{z^2}{12} + \frac{z^4}{360} - \dots\right]$$

$$f(z) = \frac{4}{z^3}\left[1 + \left(\frac{z^2}{12} - \frac{z^4}{360} + \dots\right) + \left(\frac{z^2}{12} - \frac{z^4}{360} + \dots\right)^2 + \dots\right]^2$$

(a) 0 (b) $\frac{1}{3}$ (♣) $\frac{2}{3}$ (d) 1 (e) nota.

9. The value of $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1} dx$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{e}$ (c) πe (d) $\frac{\pi}{2e}$ (e) $\frac{\pi e}{2}$.

Justification: Can take $f(z) = \frac{e^{\imath z}}{z^2+1}$ and the semicircular contour: L from $-R$ to R and $C = Re^{\imath\theta}$ where θ is from 0 to π. By CRT, the total integral is $2\pi\imath$ Res $(f, \imath) = 2\pi\imath \cdot \frac{e^{-1}}{2\imath} = \frac{\pi}{e}$. Now,

$$\left| \int_C \frac{e^{\imath z}}{z^2+1} dz \right| \leq \frac{\pi R}{R^2 - 1}$$

and this goes to 0 as $R \to \infty$. The required integral is the real part of the integral over L in the limit.

(a) $\frac{\pi}{2}$ (♣) $\frac{\pi}{e}$ (c) πe (d) $\frac{\pi}{2e}$ (e) $\frac{\pi e}{2}$.

10. Among the following theorems my most favourite is
(a) Morera (b) ZAI (c) CRT (d) RRST (e) MMT.

Justification: Any reasonable justification.

Any one answer.

1. Let $D = \{z \mid |z| < 1\}$. Let $f : D \to D$ be holomorphic such that $f(0) = 0$ and $|f(z)| < 1$ for all $z \in D$. Consider

$$g(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0. \end{cases}$$

Then,

(i) g is holomorphic in D.
(a) True (b) False.

(ii) On $\{z \mid |z| \leq r < 1\}$, $|g|$ takes its maximum at some z_0.
(a) True (b) False.

(iii) In (ii), $|z_0|$ is ____.

(iv) $|f(z)| \leq |z|$ on D.
(a) True (b) False.

2. Let f be entire and it is given that $|f(z)| \leq K|z|^n$ for all $|z| \geq M$. Then,

(i) Can write $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$ for all $z \in \mathbb{C}$.
(a) True (b) False.

(ii) $|f^{(m)}(0)| \leq \frac{m! K |z|^n}{M^m}$ follows from
(a) ODAD (b) ZAI (c) CRT (d) MMT (e) Cauchy's Est.

(iii) $f^{(m)}(0) = 0$ for $m \geq n$.
(a) True (b) False.

(iv) $f^{(m)}(0) = 0$ for $m \geq n + 1$.
(a) True (b) False.

(v) f is a polynomial of degree at most n.
(a) True (b) False.

3. Let f be on a region Ω bounded by a simple closed curve γ. Then $\frac{1}{2\pi\imath}\int_\gamma \frac{f'(z)}{f(z)}dz$ is

(i) when f has one zero and no pole in Ω
(a) 1 (b) 2 (c) 3 (d) nota (e) can't say.

(ii) when f has one zero of multiplicity one and no pole in Ω
(a) 1 (b) 2 (c) 3 (d) nota (e) can't say.

(iii) when f has one zero of multiplicity one and another zero of multiplicity two and one pole of order one in Ω
(a) 1 (b) 2 (c) 3 (d) nota (e) can't say.

(iv) when f has one pole of order one and another of order two and no zeros
(a) 1 (b) 2 (c) 3 (d) nota (e) can't say.