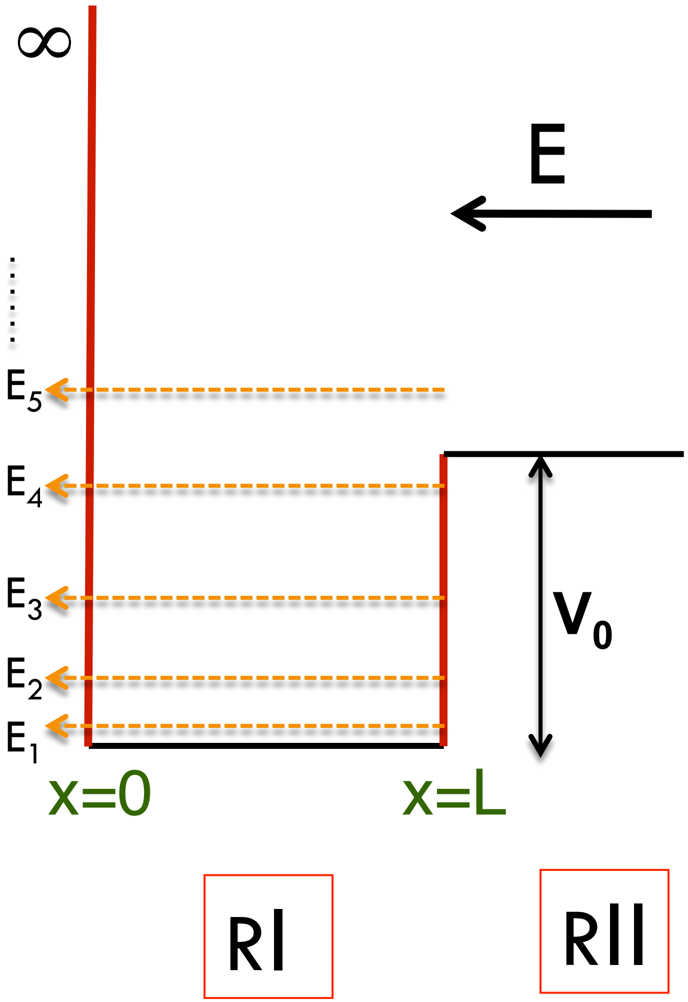
Semi-infinite potential well and Potential Barrier:

Quantum Tunneling

Semi-infinite rectangular potential well (contd.)



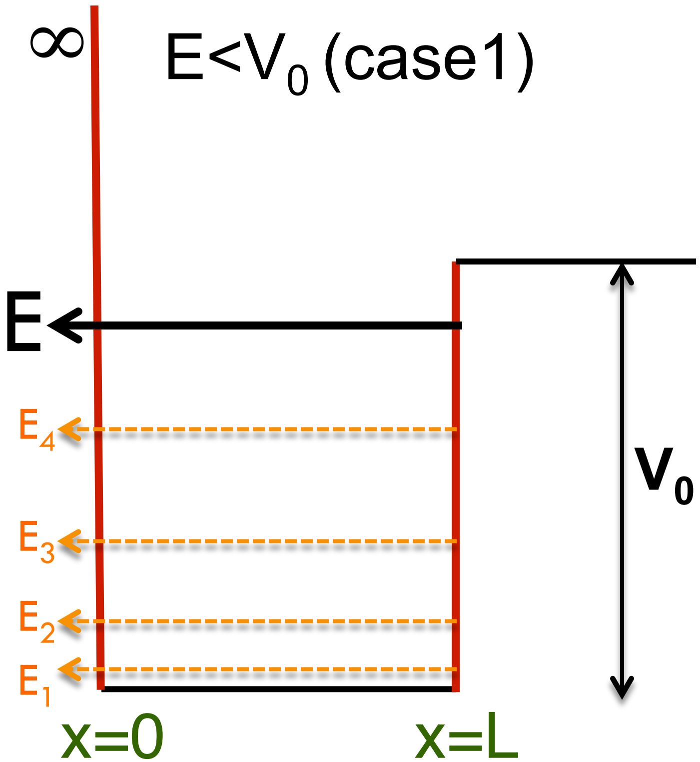
$E > V_0$ (free)

$$\psi_I(x) = A\sin k_1 x + B\cos k_1 x \qquad k_1 = \sqrt{\frac{2mE)}{\hbar^2}}$$

$$\psi_{II}(x) = Ce^{ik_2 x} + De^{-ik_2 x} \qquad k_2 = \sqrt{\frac{2m\left(E - V_0\right)}{\hbar^2}}$$

Bound States

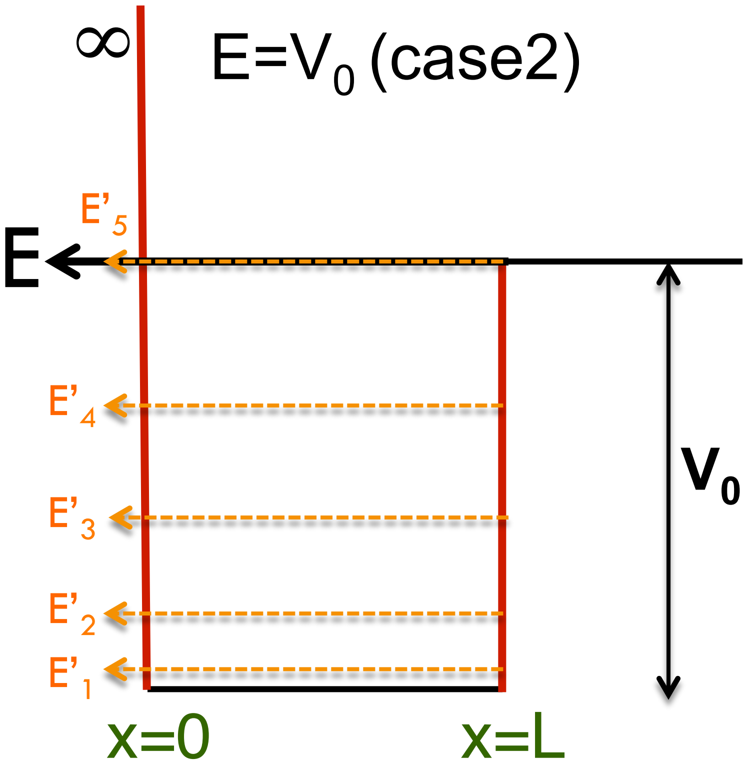
$E<V_0$ (case1)



$$\cot kL = -\frac{\alpha}{L}$$

$$k = \sqrt{\frac{2mE_n}{\hbar^2}}, \quad \alpha = \sqrt{\frac{2m\left(V_0 - E_n\right)}{\hbar^2}}$$

$E=V_0$ (case2)

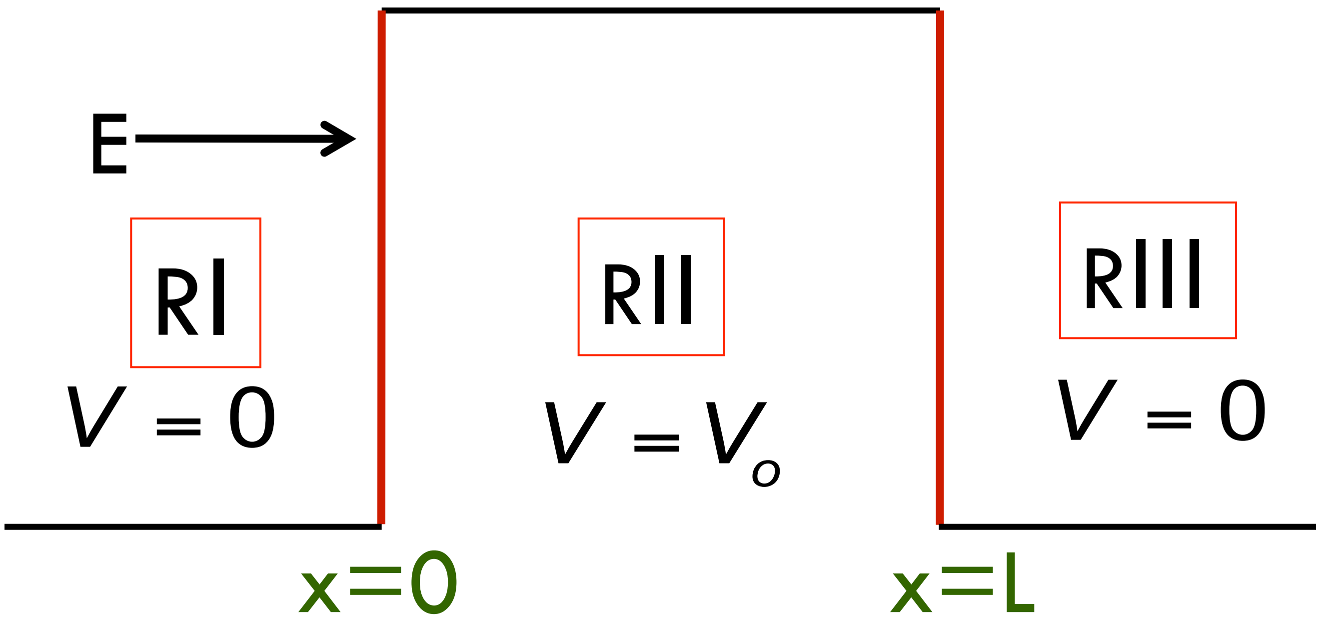


$$\cos kL = 0, \quad E_n^{'} = \frac{\hbar^2\left[(2n+1)\frac{\pi}{2}\right]^2}{2mL^2}$$

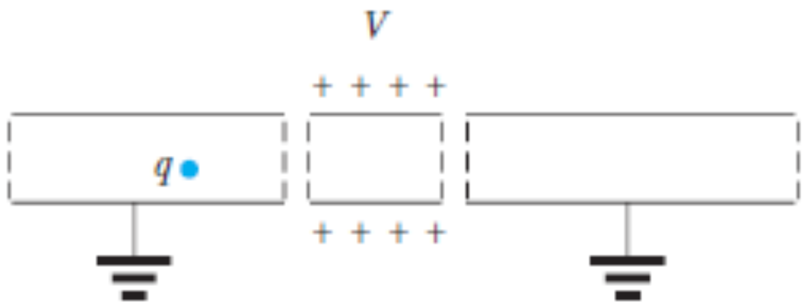
Quantum mechanical tunneling through a potential barrier

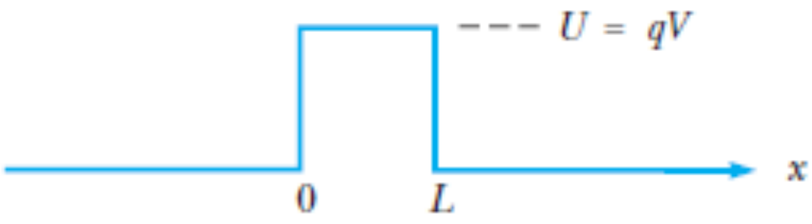
$$V(x) = V_o \quad \text{for } 0 < x < L$$

$$= 0 \quad \text{for } x < 0 \text{ or } x > L$$



Electron encountering a barrier





$$\frac{d^2\psi_I(x)}{dx^2} + \frac{2m}{\hbar^2} E\psi_I(x) = 0 \qquad \text{Region 1}$$

$$\frac{d^2\psi_{II}(x)}{dx^2} + \frac{2m}{\hbar^2} \left(E - V_0\right)\psi_{II}(x) = 0 \qquad \text{Region 2}$$

$$\frac{d^2\psi_{III}(x)}{dx^2} + \frac{2m}{\hbar^2} E\psi_{III}(x) = 0 \qquad \text{Region 3}$$

Note that E is positive and ($E-V_0$) is negative

RI $$\psi_I(x) = Ae^{ikx} + Be^{-ikx} \qquad k = \sqrt{\frac{2mE}{\hbar^2}}$$

RII $$\psi_{II}(x) = Ce^{\alpha x} + De^{-\alpha x} \qquad \alpha = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

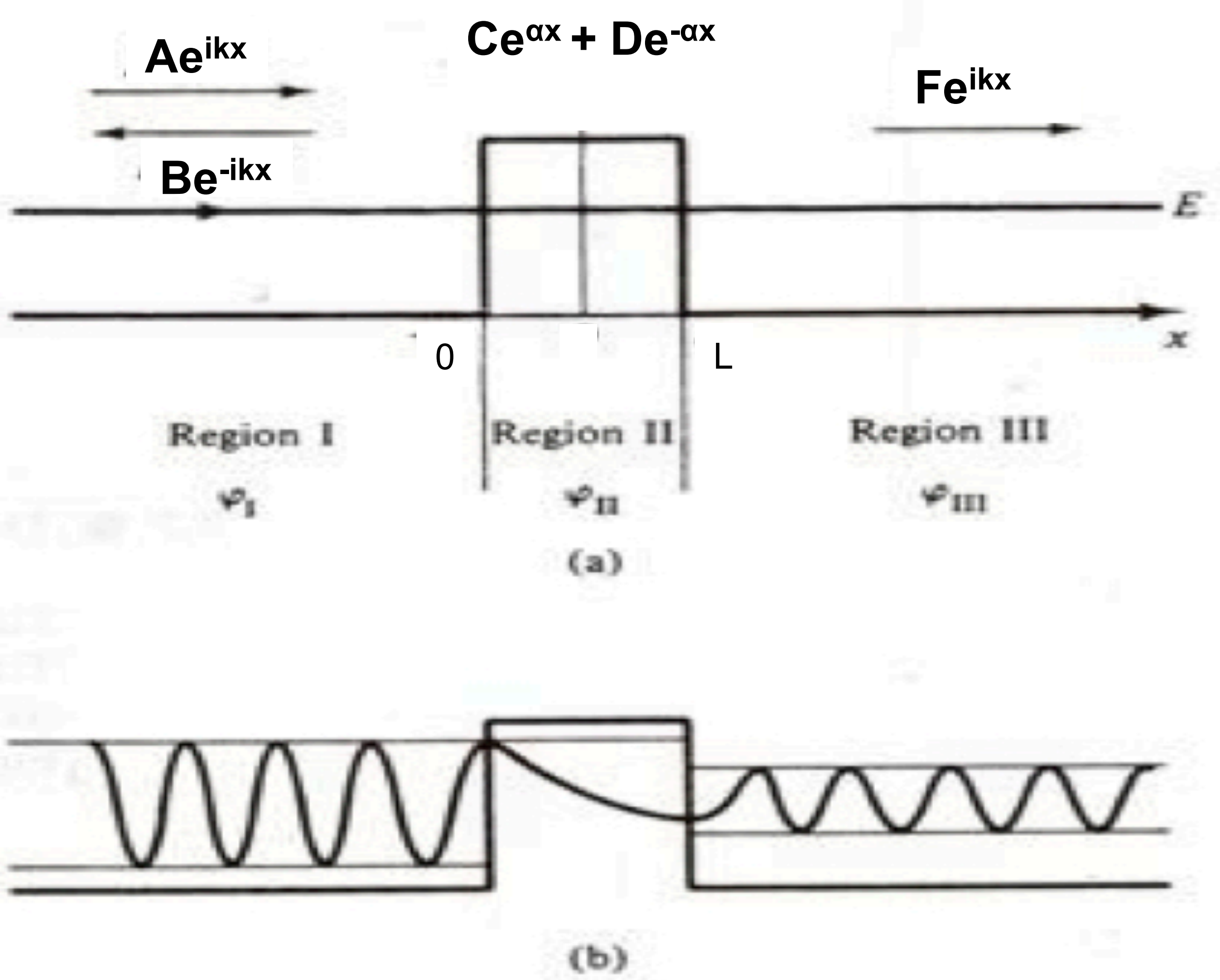
RIII $$\psi_{III}(x) = Fe^{ikx} + Ge^{-ikx} \qquad k = \sqrt{\frac{2mE}{\hbar^2}}$$

In region 1, there is the original wave and the reflected wave. Hence both A and B are non zero.

In region two, the ends do not go to infinity and hence both C and D survive.

In region 3, since no reflection at $x=+\infty$, $G=0$

Pictorial Representation



Boundary Conditions yields

$$G = 0$$

$$A + B = C + D$$

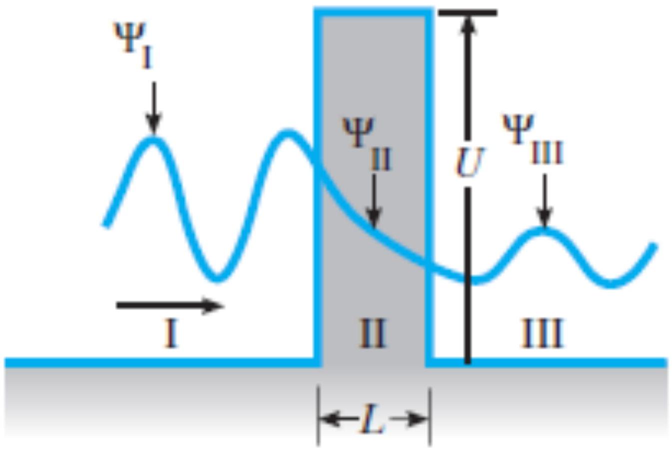
$$ik(A - B) = \alpha(C - D)$$

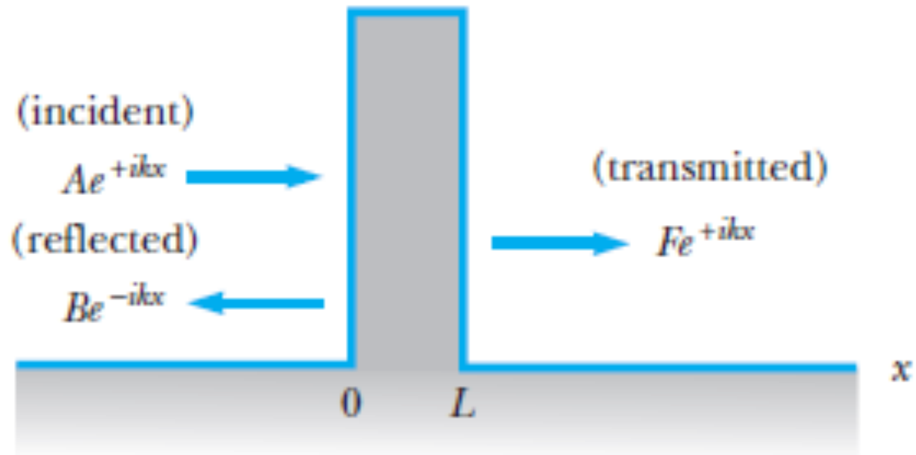
$$Ce^{\alpha L} + De^{-\alpha L} = Fe^{ikL}$$

$$\alpha(Ce^{\alpha L} - De^{-\alpha L}) = Fike^{ikL}$$

$$\frac{1}{T} = \left|\frac{A}{F}\right|^2 = 1 + \frac{1}{4}\left(\frac{\alpha}{k} + \frac{k}{\alpha}\right)^2 \sinh^2 \alpha L$$

Quantum Mechanical Tunneling





Dominant term determining the tunneling is $e^{-\alpha x}$

Example:

Consider

- An electron of E=1 eV.
- Potential barrier of V_0=2eV and L=1 nm
- The probability that the electron will tunnel through the barrier is about 80%!!!

Consequences

- If $E<V_0$, the beam gets reflected back. But the wave gets attenuated in the region II and then enters the region III. In region III, it travels with the same wave length (energy), but with less amplitude.

Limiting Case

$$T = \left[1 + \frac{1}{4}\left(\frac{\alpha}{k} + \frac{k}{\alpha}\right)^2 \sinh^2 \alpha L\right]^{-1} = \left[1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2 \alpha L\right]^{-1}$$

If $\alpha L >>> 1$ ($E <<< V_0$), then $\sinh(\alpha L) \approx e^{\alpha L}/2$

$$T = \left[\frac{16E}{V_0}\left(1 - \frac{E}{V_0}\right) e^{-2\alpha L}\right]$$

Transmission probability (T) decreases with

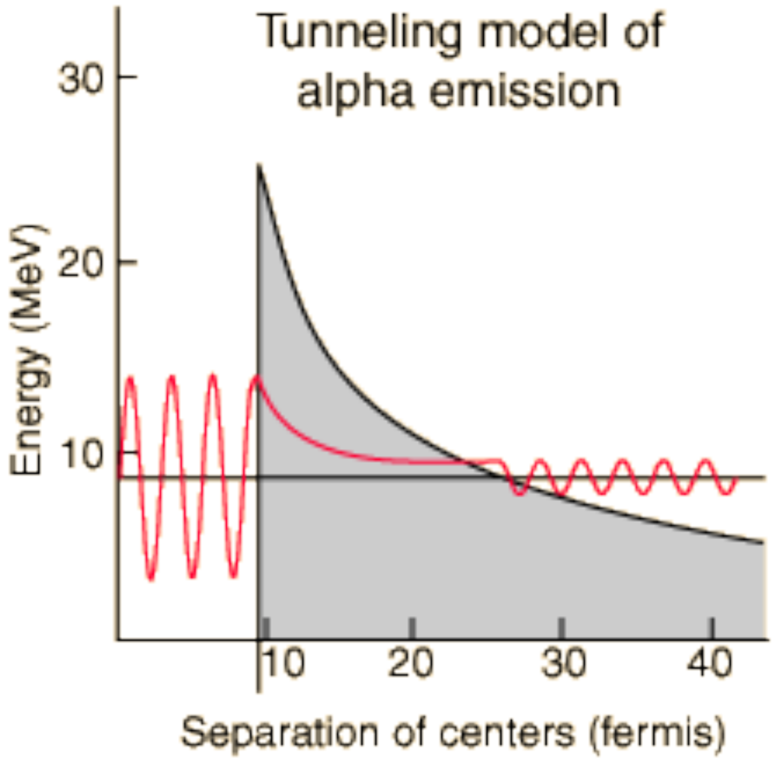
(i) increasing the width of the barrier (L)

(ii) decreasing the energy E w.r.to V_0

Experimental evidences for tunneling

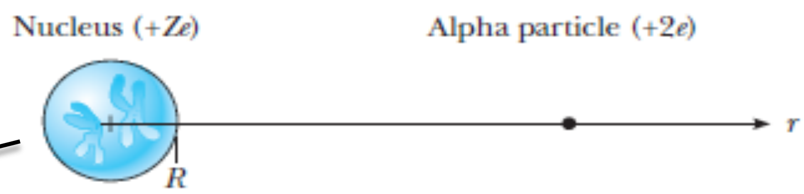
1. Gamow's theory of Alpha decay

Though the energy (E) of nucleons (α-particle) is smaller than the potential barrier produced by the nuclear forces (V_0), the alpha particle escapes the nucleus because of tunneling effect



$\Delta E . \Delta t \geq h/4\pi$

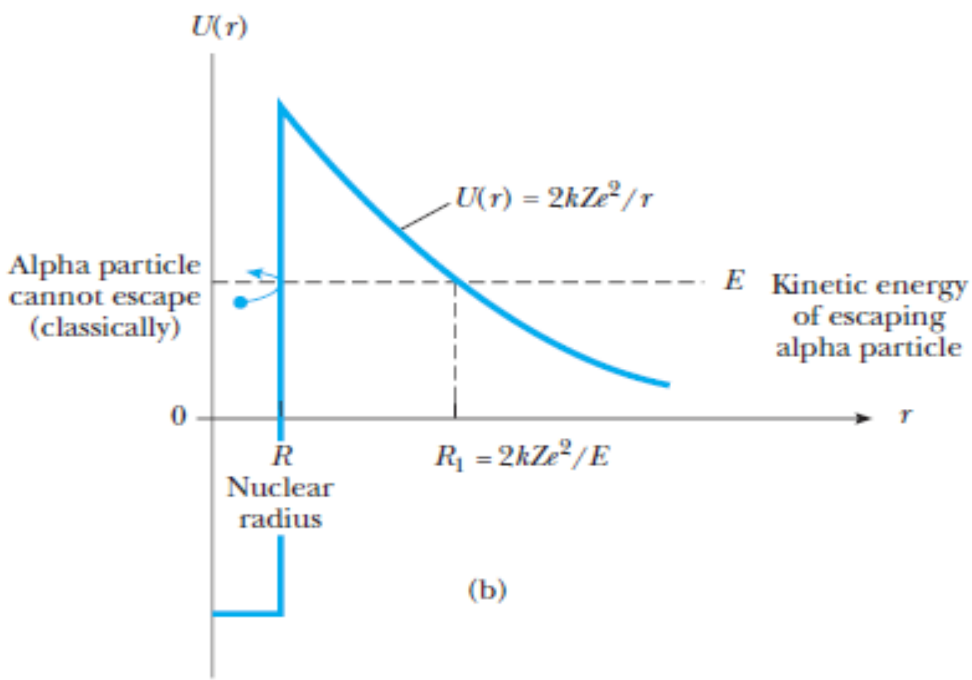
Alpha decay (contd.)



(a)

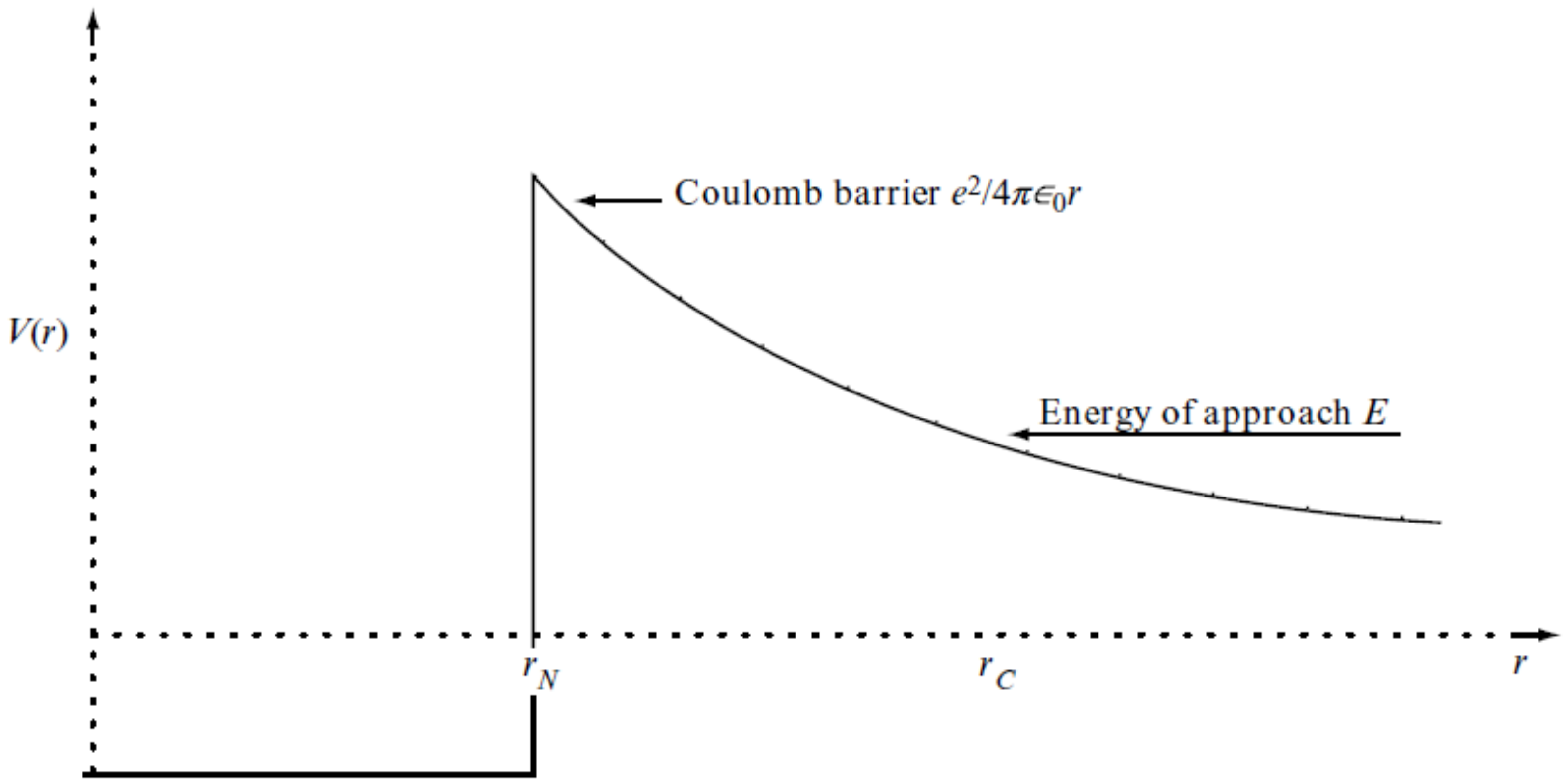
Unstable heavy nucleus

$$^{238}_{92}U \rightarrow ^{234}_{90}Th + ^{4}_{2}He$$



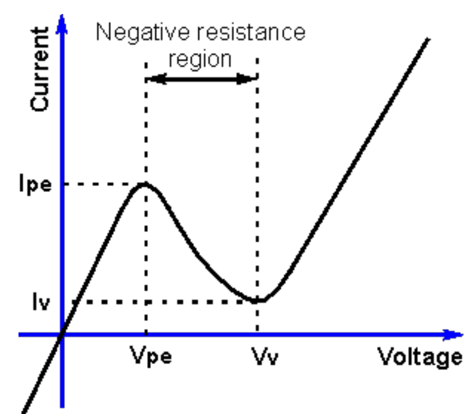
(b)

2. Tunneling of protons: energy of sun



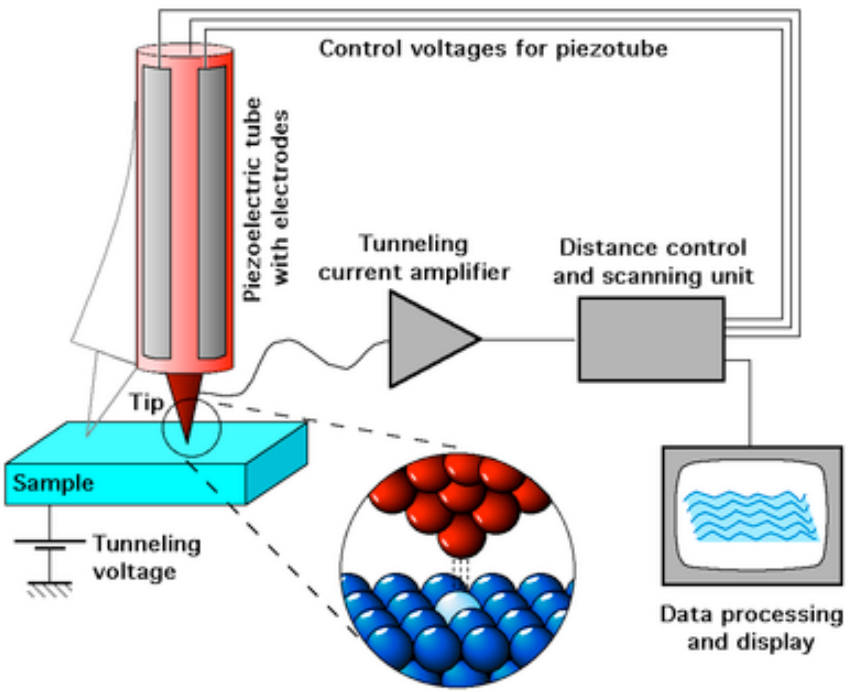
3. Tunnel diode

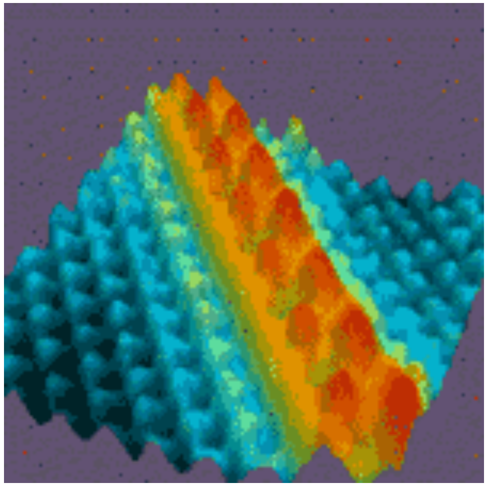
- Ordinary *pn* diodes depend on diffusion current and hence the switching speed is very low.
- A heavily doped *pn* junction has a very thin depletion layer and hence has tunneling current.
- The tunneling current can be controlled by changing the potential barrier (the step) by applied voltage.
- Very high switching frequencies (~ GHz).

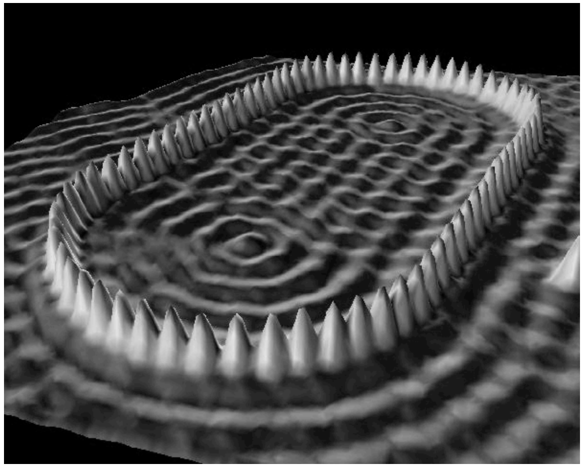


4. Scanning tunneling microscope

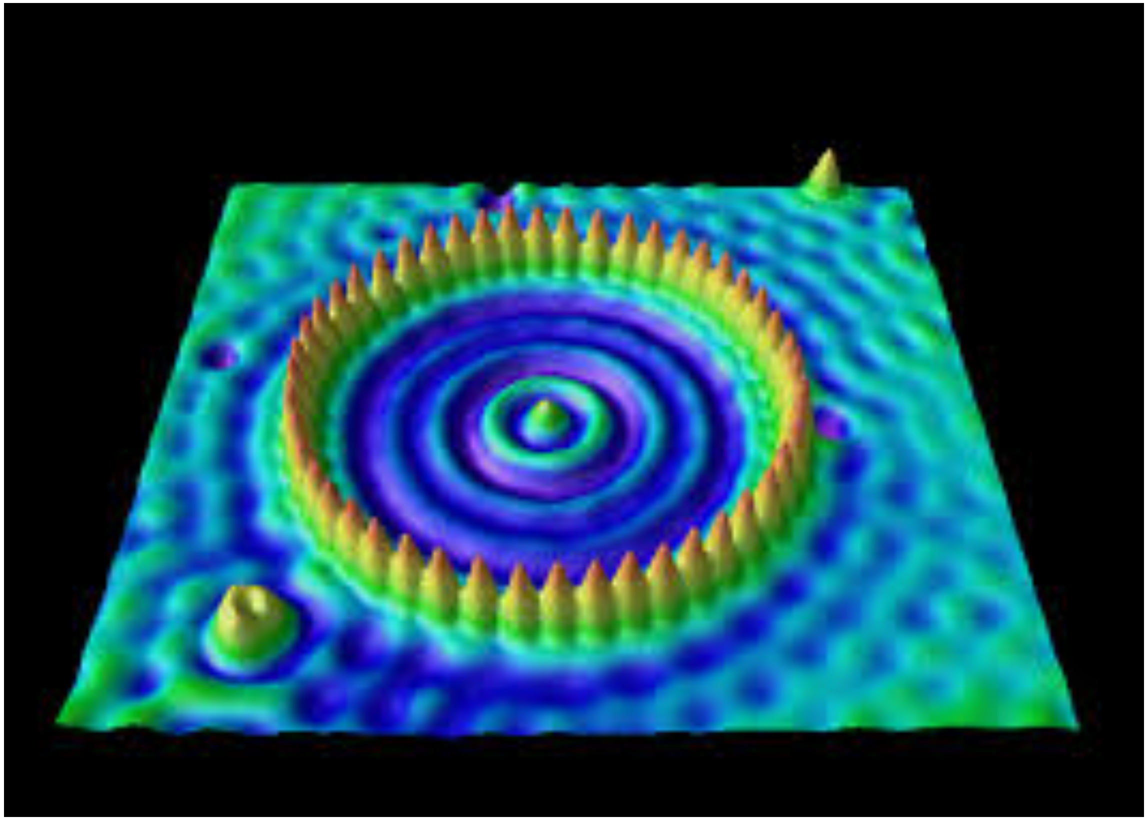
Atomic resolution of surfaces
Nobel Prize in Physics (1986)







Fe atoms on Cu surface STM



Graphite atoms seen by STM