MA205-5

Singularities and Laurent series

We have been studying holomorphic functions and their basic properties. We would like to extend those results, if possible, to more general functions. The easiest case to start will be functions which are not analytic at finitely many points. For example, a function f which is analytic in a punctured disc. Can we get some power series representation of f?

To answer these questions, we start with the following:

Defⁿ: Singular points : The points at which a complex function f is not holomorphic or not defined are called singular points of f or singularities of f.

For example : $f(z) = \frac{1}{z}$: $z \neq 0$

The point 0 is a singularity of f.

Singular points are of two types :

(i) Isolated singularities : A singular point z_0 is isolated if there is a punctured disc , $\{z \in \mathbb{C} \mid 0 < |z - z_0| < \varepsilon\}$ for some

$\varepsilon > 0$ in which f is analytic.

Example: $f(z) = \frac{1}{z}$: $z \neq 0$

0 is an isolated singularity.

(ii) Non-isolated singularities, i.e., a singular

point that is not isolated.

Example: $f(z) = \bar{z}$

As f is not analytic there are many

singularities which are not isolated!.

Isolated singularities are broadly of 3 types:

1. Removable singularity:

In the example $f(z) = \frac{\sin z}{z}$, though 0 is a singularity, the slightly modified function

$$f(z) = \begin{cases} \frac{\sin z}{z} & : z \neq 0 \\ 1 & : z = 0 \end{cases}$$

is holomorphic.

Such a singularity is called a removable singularity. In this case $\lim_{z \to z_0} f(z)$ exists, where z_0 is the singular point.

2. <u>Pole:</u> In the example $f(z) = \frac{1}{z}$, $\lim_{z \to 0} f(z) = \infty$. Such a singularity is called a pole.

3. An isolated singularity which is not removable & not a pole is called an <u>essential singularity</u>.

In this case, $\lim_{z \to z_0} f(z)$ does not exist.

For instance, $\exp\left(\frac{1}{z}\right)$ has limit ∞ as $z \to 0$ along positive x-axis; limit 0 as $z \to 0$ along negative x-axis; further, $|\exp(1/z)| \not\to \infty$ at $z \to 0$.

We will show that a function f with isolated singularities will have a Laurent series expansion. A Laurent series is a generalisation of the Taylor series. Here we allow negative powers. Recall that using CIF we have derived the Taylor series of a holomorphic function.

A Laurent series is derived on an open annulus i.e, the region between 2 concentric circles, on which f is holomorphic.

Note that the inner radius could be 0 & the outer radius could be ∞.

There could also be more than one such annular region for a given singular point & these lead to many LS for a function about a given point, each valid on a different annular region.

Suppose z_0 is an isolated singularity for f. Consider an annulus with radii $r < R$ centered at z_0 such that f is holomorphic there. The main steps in deriving the LS are :

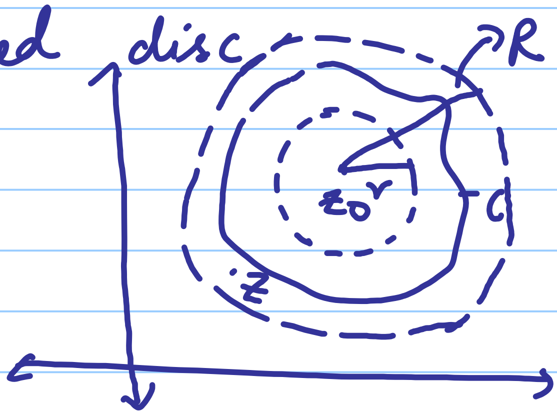
(i) Extend Cauchy-Goursat to multiply connected domains.

(ii) CIF then takes the form:

$$f(z) = \frac{1}{2\pi i} \int_{|w-z_0|=R} \frac{f(w)}{w-z}\, dw - \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{w-z}\, dw$$

where C is any positively oriented simple closed contour around z_0, lying in the punctured disc

$0 < |z - z_0| < R$.



The first integral gives us:

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \quad , \quad \text{where}$$

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} \quad : \quad n \geq 0$$

as earlier.

In the second integral, write

$$\frac{-1}{w-z} = \frac{1}{z-z_0} \cdot \frac{1}{1 - \frac{w-z_0}{z-z_0}}$$

and expand to get $\displaystyle\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$

where $b_n = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(z)\,dz}{(z-z_0)^{-n+1}} \quad : \quad n \geq 1$

i.e., $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$

$: r < |z-z_0| < R.$

This is the L.S around the isolated singularity z_0. The negative part is called the principal part of the L.S

Notation: We will write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad \text{as the L.S.}$$

Residues:

If z_0 is an isolated singularity of f, then f is holomorphic in an annulus $0 < |z - z_0| < R$ for some R. The corresponding L.S is called the Laurent expansion around z_0. Consider the -1^{th} coefficient of this series:

$$a_{-1} = \frac{1}{2\pi i} \int_C f(z)\,dz$$

This is called the residue of f at z_0 written as: $\text{Res}(f; z_0) = a_{-1}$.

Cauchy's Residue theorem:

Let C be a simple, closed contour (oriented positively). Let f be analytic on & inside C except for a finite number of singular points $z_1, \dots, z_n$, inside C. Then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f; z_k)$$

Riemann removable singularity theorem:

Suppose z_0 is an isolated singularity of f and $\lim_{z \to z_0} (z - z_0) f(z) = 0$.

Then z_0 is a removable singularity of f.

Proof: Define

$$g(z) = \begin{cases} (z-a)^2 \cdot f(z) & : z \neq z_0 \\ 0 & : z = z_0 \end{cases}$$

check g is holomorphic in a neighborhood of z_0, using the fact that f is so in a deleted neighborhood of z_0.

Let $g(z) = c_0 + c_1(z - z_0) + c_2(z-z_0)^2 + \cdots$

We have $c_0 = g(z_0) = 0$, $c_1 = g'(z_0) = 0$

i.e. $g(z) = c_2 (z-z_0)^2 + \cdots$

Hence for $z \neq z_0$, we have

$$f(z) = \frac{g(z)}{(z-z_0)^2} = c_2 + c_3 (z-z_0) + \cdots$$

where $g(z)$ is analytic & extends f.

<u>Poles:</u>

Suppose z_0 is a pole of f i.e.,

$$\lim_{z \to z_0} f(z) = \infty \quad \text{-----}\rightarrow \text{(definition)}$$

Then, $\lim_{z \to z_0} (z-z_0) \dfrac{1}{f(z)} = 0$

and by RRST, $\frac{1}{f(z)}$ has a removable

singularity at z_0. That is,

$$g(z) = \begin{cases} \frac{1}{f(z)} & : z \neq z_0 \\ 0 & : z = z_0 \end{cases}$$

is holomorphic in $B_\varepsilon(z_0)$ for some

$\varepsilon > 0$.

So z_0 is a zero of $g(z)$.

Let m = order of zero of z_0 in g

(why is the order finite?)

(This has been proved in last class!)

Then $g(z) = (z - z_0)^m \cdot g_1(z)$

where g_1 is holomorphic & $g_1(z_0) \neq 0$

i.e, $f(z) = \dfrac{h(z)}{(z-z_0)^m}$

where h is holomorphic in $B_\varepsilon(z_0)$.

<u>Note</u>: <u>m</u> is called the order of the pole at z_0. A pole of order 1 is called a simple pole.

Note:

If $\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ is the L.S.

around z_0, then its principal part is

$$\sum_{n=-\infty}^{-1} a_n (z-z_0)^n$$

We have shown above that:

(i) removable singularity iff principal part is zero.

(ii) pole iff principal part is finite.

(iii) essential singularity iff principal part is infinite.

Calculating residues:

- If we have a removable singularity then the residue is 0.

- If z_0 is a pole, then we have

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \cdots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots$$

If $g(z) = (z-z_0)^m f(z) = a_{-m} + \cdots + a_{-1}(z-z_0)^{m-1} + \cdots$

then g is holomorphic &

$$a_{-1} = \frac{g^{m-1}(z_0)}{(m-1)!}.$$

Find the isolated singularities & their residues for $f(z) = \frac{z^2}{1+z^4}$

Singularities are 4^{th} roots of -1:

$$z_n = \exp\left(i\left(\frac{\pi}{4} + \frac{(n-1)\pi}{2}\right)\right) \quad : n = 1,2,3,4.$$

These are all simple poles <u>check</u>

Further, $\operatorname{Res}(f; z_1) = \lim_{z \to z_1} (z - z_1) f(z)$

$$= \frac{z_1^2}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)}$$

$$= \frac{1-i}{4\sqrt{2}}$$

Similarly, $\operatorname{Res}(f; z_2) = \frac{-1-i}{4\sqrt{2}}$ ...

2) Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx$

To do this evaluate $\int_{-r}^{r} \frac{x^2}{1+x^4}\,dx$

and take limit as $r \to \infty$.

Fix $r > 1$. Let C be $[-r, r] \cup \{$upper half of the circle $|z| = r$, oriented counter clockwise$\}$.

Let $f(z) = \frac{z^2}{1+z^4}$

f has 2 poles inside C.

$$\frac{1}{2\pi i} \int_C f(z)\,dz = \text{Res}(f; z_1) + \text{Res}(f; z_2)$$

$$= \frac{-i}{2\sqrt{2}}$$

This is same as

$$\frac{1}{2\pi i}\int_{-r}^{r}\frac{x^2}{1+x^4}\,dx+\frac{1}{2\pi}\int_0^{\pi}\frac{r^3e^{3it}}{1+r^4e^{4it}}\cdot dt$$

i.e., $$\int_{-r}^{r}\frac{x^2}{1+x^4}\,dx=\frac{\pi}{\sqrt{2}}\left(1-ir^3\int_0^{\pi}\frac{e^{3it}}{1+r^4e^{4it}}\,dt\right)$$

As $$\left|ir^3\int_0^{\pi}\frac{e^{3it}}{1+r^4e^{4it}}\,dt\right|\leq\frac{\pi r^3}{r^4-1},$$

in the limit as $r\to\infty$ this is 0.

$$\therefore\int_{-\infty}^{\infty}\frac{x^2}{1+x^4}\,dx=\frac{\pi}{\sqrt{2}}$$