MA 205 Complex Analysis: Examples of Contour Integration

August 26, 2017

Recall

Last time we discussed singularities. We then derived the Laurent series expansion of a function around an isolated singularity. If z_0 is a point, Ω is an open annulus with radii $r < R$ centred at z_0, f a holomorphic function on Ω, then f can be expanded as

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If z_0 is an isolated singularity of f, then f is holomorphic in an annulus $0 < |z - z_0| < R$ for some R. The corresponding Laurent expansion is called the Laurent expansion around z_0.

Recall

The coefficients of f can be calculated as:

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

for any simple closed contour C lying inside Ω. In case f is holomorphic inside $|z - z_0| < R$, then $a_n = 0$ for all negative n and then we get the usual Taylor series expansion.

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The negative terms of the series are called the **Principal part of the Laurent series**. Note that the singularity at z_0 is

- removable iff principal part is zero.
- pole iff principal part is finite.
- essential iff principal part is infinite.

Singularity at ∞

Isolated Singularity at Infinity: $f(z)$ is said to have an isolated singularity at ∞ if f is holomorphic outside a disc of radius R for some R.
For example entire functions.
Equivalently, $f(1/z)$ has an isolated singularity at 0.

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If f has an isolated singularity at ∞, we can talk about the nature of singularity at ∞.
Definition: f is said to have a zero (resp. removable singularity, pole, essential singularity) at ∞ if $f(1/z)$ has a zero (resp. removable singularity, pole, essential singularity) at 0.

Examples: Polynomials have a pole at ∞. We showed earlier that e^z has an essential singularity at ∞. Liouville's theorem shows that if f is an entire function which has a zero at ∞, then f is identically zero. (Why ??) Of course there are plenty of meromorphic functions which have a zero at ∞, for example $1/z$.

Proper Maps

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Theorem: An entire functions from $\mathbb{C}$ to $\mathbb{C}$ has a pole at ∞ if and only if it is a non-constant polynomial.

I leave this as an exercise; at least for the moment.

if function has a pole at infinity then as z tends to infinity fz also tend to infinity thus fz is a proper function and proper en tire function from c to c is polynomial hens non constant

if entire function is non constant then it is un-bounded

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Alternatively, we note that $f(z)$ has a pole of order 3 at $z = 0$, so we can use the general formula for the residue at a pole:

$res(f;0) = \frac{1}{2!}[\frac{d^2}{dz^2}(z^3 f(z))]_{z=0} = \frac{1}{2}[e^z]_{z=0} = \frac{1}{2}$.

Example

Lets compute the residues of $f(z) = \frac{1}{sinh(\pi z)}$ at its singularities. $\frac{1}{sinh(\pi z)}$ has a simple pole at ni for all $n \in \mathbb{Z}$ (Note : To check this show that $\lim_{z \to ni} \dfrac{z - ni}{sinh(\pi z)}$ is a non-zero number). Thus the residue at ni is given by:

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By L'Hospital's rule $= \lim_{z \to ni} \dfrac{1}{\pi cosh(\pi z)}$

$= \frac{1}{\pi cosh(n\pi i)}$
$= \frac{1}{\pi cos(n\pi)}$
$= \frac{(-1)^n}{\pi}$

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We have seen that $sinh^3(z)$ has a pole of order 3 at πi with Taylor series:
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The coefficient of $(z - \pi i)^{-1}$ in the above expression is $1/2$ which is therefore residue of f at πi.

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Direct: $\left|\frac{dz}{(1-z-z^2)z^n}\right| \leq \frac{C}{|z|^2}$ for some constant C and for sufficiently large $|z|$. Hence by the ML-inequality, the integral is zero in the limit.

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$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Thus this is an example where the integral is easily known due to other reasons and can be used to compute residue at some pole.

Partial Fractions: To integrate a rational function of the form $P(z)/Q(z)$, by Euclid's algorithm, first reduce to the case where degree of $P(z)$ is less than that of $Q(z)$. Then use partial fractions according to the following rules :

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Term in denominator	Term in Partial Fraction Decomposition
$(Az+B)^k$	$\frac{A_1}{Az+B} + \cdots + \frac{A_k}{(Az+B)^k}$
$(Az^2+Bz+C)^k$	$\frac{(A_1z+B_1)}{(Az^2+Bz+C)} + \cdots + \frac{(A_kz+B_k)}{(Az^2+Bz+C)^k}$

and so on ...

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Thus $2A + C = 7$ and $2B + D = -4$ giving $C = 1, D = -4$.

Example contd ..

Thus $I = \int_{|z|=2} \frac{3z}{z^2+2} + \int_{|z|=2} \frac{(z-4)}{(z^2+2)^2}$

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The first integral can be easily computed as:
$\frac{3}{2} \int_{|z|=2} \frac{(z+\sqrt{2}i)+(z-\sqrt{2}i)}{z^2+2}$
$= \frac{3}{2}[\int_{|z|=2} \frac{1}{z-\sqrt{2}i} + \frac{1}{z+\sqrt{2}i}]$
$= 6\pi i.$

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To compute the second integral, we use Cauchy's residue theorem which states that the sum of the integral of a function along a simply closed contour equals $2\pi i$ times the sum of the residues at the singularities inside the contour.

Example

Recall : If $f(z)$ has a pole at z_0 of order m, then the residue of f at z_0 can be computed as :

$$res(f; z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d}{dz^{m-1}} [(z - z_0)^m (f(z))]$$

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$res(f, \sqrt{2}i) = \frac{d}{dz}[(z - \sqrt{2}i)^2 \frac{(z-4)}{(z^2+2)^2}]_{z=\sqrt{2}i}$ and

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Adding the two and multiplying by $2\pi i$, we get the second integral. Adding that to the value of the first integral, we get the final answer. I'll leave the details of the computation to you.

Computing Real Integrals

One of the important computational applications of Complex Analysis is in the computation of real integrals.

Let $f : [0, \infty] \to \mathbb{R}$ be a function such that $\int_0^R f(x)dx$ exists for each $R \geq 0$. One then defines the Improper integral $\int_0^\infty f(x)$ to be $\lim_{R\to\infty} \int_0^R f(x)dx$.

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Similarly if $f : [-\infty, \infty] \to \mathbb{R}$ is a function such that $\int_{-a}^b f(x)dx$ exists for each $a, b \geq 0$, then the improper integral $\int_{-\infty}^\infty f(x)$ is defined as $\lim\limits_{a,b\to\infty} \displaystyle\int_{-a}^b f(x)dx$.

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Improper Intergral

For instance the function $\frac{1}{1+x^2}dx$ is integrable on $\mathbb{R}$ while the integral $\int_{-\infty}^{\infty} sin(x)dx$ does not exist. Intuitively, for such an improper integral to exist, the function has to decay to zero sufficiently rapidly outside a "small set ". (Note that it need not quite tend to zero as $|x| \to \infty$).

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The above integral can be thought of as an integral over a part of a contour γ_R; γ_R being a line segment along the real axis between $-R$ and R together with a semicircle C_R of radius R around 0. We can then evaluate the resulting integral by means of residue theorem, and show that the integral over the extra "added" part of γ_R, samely C_R asymptotically vanishes as $R \to \infty$.

Example

Thus taking the contour integral over γ_R and allowing R to tend to ∞, we get the desired answer.

Compute $\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx$.
You might have seen the computation of this integral in MA 105 but now lets work out this computation using using MA 205 !

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You might have seen the computation of this integral in MA 105 but now lets work out this computation using using MA 205 !
The idea is to compute $\int_{-r}^{r} \frac{x^2}{1+x^4} dx$ and take limit as $r \to \infty$. Fix $r > 1$. Let γ be $[-r, r]$ together with C_R, the upper part of the circle $|z| = r$ oriented counterclockwise. Take $f(z) = \frac{z^2}{1+z^4}$. f has two poles inside γ. Now,

$$\frac{1}{2\pi\imath} \int_{\gamma} f(z) dz = \mathrm{Res}(f; z_1) + \mathrm{Res}(f; z_2) = \frac{-\imath}{2\sqrt{2}}.$$

Example

Thus taking the contour integral over γ_R and allowing R to tend to ∞, we get the desired answer.

Compute $\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx$.
You might have seen the computation of this integral in MA 105 but now lets work out this computation using using MA 205 !
The idea is to compute $\int_{-r}^{r} \frac{x^2}{1+x^4} dx$ and take limit as $r \to \infty$. Fix $r > 1$. Let γ be $[-r, r]$ together with C_R, the upper part of the circle $|z| = r$ oriented counterclockwise. Text Take $f(z) = \frac{z^2}{1+z^4}$. f has two poles inside γ. Now,

$$\frac{1}{2\pi\imath} \int_\gamma f(z) dz = \mathrm{Res}(f; z_1) + \mathrm{Res}(f; z_2) = \frac{-\imath}{2\sqrt{2}}.$$

This is same as

$$\frac{1}{2\pi\imath}\left[\int_{-r}^{r} \frac{x^2}{1+x^4} dx + \int_{C_R} \frac{z^2 dz}{1+z^4}\right].$$

along Yr

Example

By changing to polar coordinates for the second integral this equals,

$$\frac{1}{2\pi\imath}\int_{-r}^{r}\frac{x^2}{1+x^4}dx+\frac{1}{2\pi}\int_0^{\pi}\frac{r^3e^{3\imath t}}{1+r^4e^{4\imath t}}dt.$$

Thus,

$$\int_{-r}^{r}\frac{x^2}{1+x^4}dx=\frac{\pi}{\sqrt{2}}-\imath r^3\int_0^{\pi}\frac{e^{3\imath t}}{1+r^4e^{4\imath t}}dt$$

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Note that,

$$\left|\imath r^3\int_0^{\pi}\frac{e^{3\imath t}}{1+r^4e^{4\imath t}}dt\right|\leq\frac{\pi r^3}{r^4-1}.$$

Thus, in the limit, this integral is zero. Therefore,

$$\int_{-\infty}^{\infty}\frac{x^2}{1+x^4}dx=\frac{\pi}{\sqrt{2}}.$$