MA-207 Differential Equations II
Lecture-12 Wave and Laplace Equation

M.K. Keshari

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

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S1 - Lecture 12

Theorem. Let $\nu_n = \dfrac{n\pi}{L}$. Then $u(x,t) =$

$$\sum_{n\geq 1}\left(A_n \cos(a\nu_n t) + \frac{B_n}{a\nu_n}\sin(a\nu_n t)\right)\sin \nu_n x$$

is a formal solution of Dirichlet IBV wave equation

$$u_{tt}(x,t) = a^2 u_{xx}(x,t), \qquad 0 < x < L,\ t > 0;$$
$$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0; \qquad \text{with}$$
$$u(x,0) = f(x) = \sum_{n\geq 1} A_n \sin \nu_n x \qquad \text{and}$$
$$u_t(x,0) = g(x) = \sum_{n\geq 1} B_n \sin \nu_n x$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin \nu_n x\, dx,\ B_n = \frac{2}{L}\int_0^L g(x)\sin \nu_n x dx$$

Theorem. Let $\nu_n = \dfrac{n\pi}{L}$. Then $u(x,t) = (A_0 + B_0 t)$

$$+\sum_{n\geq 1}\left(A_n\cos(a\nu_n t) + \frac{B_n}{a\nu_n}\sin(a\nu_n t)\right)\cos\nu_n x$$

is a formal solution of Neumann IBVP

$$\begin{aligned}
&u_{tt}(x,t) = a^2 u_{xx}(x,t), \qquad 0 < x < L,\ t > 0;\\
&u_x(0,t) = 0, \quad u_x(L,t) = 0, \quad t > 0; \qquad \text{with}
\end{aligned}$$

$$u(x,0) = f(x) = A_0 + \sum_{n\geq 1} A_n\cos\nu_n x \qquad \text{and}$$

$$u_t(x,0) = g(x) = B_0 + \sum_{n\geq 1} B_n\ \cos\nu_n x$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\nu_n\, dx,\ B_n = \frac{2}{L}\int_0^L g(x)\cos\nu_n x\, dx$$

Non homogeneous Wave Equation: Dirichlet boundary condition

The following model describes the vibrations of a string with an external force that depends on time.

$$\begin{aligned}
&u_{tt} - k^2 u_{xx} = F(x,t), \qquad 0 < x < L, \quad t > 0\\
&u(0,t) = f_1(t),\ u(L,t) = f_2(t), \qquad t > 0\\
&u(x,0) = f(x),\ u_t(x,0) = g(x), \qquad 0 \leq x \leq L
\end{aligned}$$

We will first reduce the problem so that boundary conditions are homogeneous. Note that

$$\widetilde{w}(x,t) = \left(1 - \frac{x}{L}\right) f_1(t) + \frac{x}{L} f_2(t)$$

$$\implies \widetilde{w}(0,t) = f_1(t), \quad \widetilde{w}(L,t) = f_2(t)$$

So let us first make the substitution

$$z(x,t) = u(x,t) - \widetilde{w}(x,t)$$

Then clearly

$$\begin{aligned}
&z_{tt} - k^2 z_{xx} = G(x,t),\\
&z(0,t) = 0, \qquad z(L,t) = 0,\\
&z(x,0) = v(x), \qquad z_t(x,0) = w(x).
\end{aligned}$$

To solve for $u(x,t)$, it is enough to solve for $z(x,t)$.

Since the boundary conditions are Dirichlet type, we assume the solution is given by

$$z(x,t) = \sum_{n\geq 1} Z_n(t)\sin(\nu_n x)$$

where $\nu_n = \dfrac{n\pi}{L}$, and solve for $Z_n(t)$'s.

Differentiating $z(x,t)$ term by term, we get that it satisfies the equation

$$z_{tt} - k^2 z_{xx} = \sum_{n\geq 1} \left(Z_n''(t) + k^2\nu_n^2\, Z_n(t)\right)\sin(\nu_n x)$$

Let us write

$$G(x,t) = \sum_{n\geq 1} G_n(t)\sin(\nu_n x)$$

where

$$G_n(t) = \frac{2}{L}\int_0^L G(x,t)\sin(\nu_n x)\,dx$$

Thus, $z_t - k^2 z_{xx} = G(x,t)$ gives

$$Z_n''(t) + k^2\nu_n^2\, Z_n(t) = G_n(t)$$

$$z(x,0) = \sum_{n\geq 1} Z_n(0)\sin(\nu_n x) = v(x)$$

and

$$z_t(x,0) = \sum_{n\geq 1} Z_n'(0)\sin(\nu_n x) = w(x)$$

gives

$$Z_n(0) = \frac{2}{L}\int_0^L v(x)\sin(\nu_n x)\,dx := b_n$$

is the Fourier sine coefficient of $v(x)$ and

$$Z_n'(0) = \frac{2}{L}\int_0^L w(x)\sin(\nu_n x)\,dx := c_n$$

is the Fourier sine coefficient of $w(x)$.

We can solve

$$Z_n''(t) + k^2\nu_n^2\, Z_n(t) = G_n(t)$$

uniquely with given initial conditions

$$Z_n(0) = b_n, \quad Z_n'(0) = c_n$$

where b_n and c_n are Fourier sine coefficients of $v(x)$ and $w(x)$ respectively.

If $Z_n(t)$ is this unique solution, then the series

$$z(x,t) = \sum_{n\geq 1} Z_n(t)\sin(\nu_n x)$$

solves our non homogeneous wave equation with Dirichlet boundary conditions for z.

Example. Consider the following PDE

$$u_{tt} - u_{xx} = e^t, \qquad 0 < x < 1, \quad t > 0$$
$$u(0,t) = 0, \quad u(1,t) = 0, \qquad t > 0$$
$$u(x,0) = x(x-1), \; u_t(x,0) = 0, \qquad 0 \le x \le 1$$

The boundary conditions are Dirichlet type, so we find solution in Fourier sine series. Assume (here $\nu_n = n\pi$)

$$u(x,t) = \sum_{n\ge1} u_n(t)\sin(\nu_n x)$$

The Fourier sine series for $u(x,0) = x(x-1)$ is

$$x(x-1) = \sum_{n\ge1} \frac{-8}{(\nu_{2n-1})^3}\sin(\nu_{2n-1}x)$$

Therefore, we get for $n \ge 1$,

$$u_{2n}(0) = 0, \quad u_{2n-1}(0) = \frac{-8}{(\nu_{2n-1})^3}, \quad u_n'(0) = 0$$

The Fourier sine series for $G(x,t) = e^t$ is given by

$$e^t = \sum_{n\ge1} \frac{4}{\nu_{2n-1}}\sin(\nu_{2n-1}x)\, e^t$$

Substitute

$$u(x,t) = \sum_{n\ge1} u_n(t)\sin(\nu_n x)$$

into the equation $u_{tt} - u_{xx} = e^t$, we get

$$\sum_{n\geq 1}\left(u_n''(t)+\nu_n^2 u_n(t)\right)\sin(\nu_n x) = \sum_{n\geq 1}\frac{4e^t}{\nu_{2n-1}}\sin(\nu_{2n-1}x)$$

For even n, we get

$$u_{2n}''(t) + \nu_{2n}^2 u_{2n}(t) = 0$$

$$\implies u_{2n}(t) = C_{2n}\cos(\nu_{2n}t) + D_{2n}\sin(\nu_{2n}t)$$

Since $u_{2n}(0) = 0$, we get $C_{2n} = 0$.

Further, $u_{2n}'(0) = 0$, we get $D_n = 0$.

Therefore $\boxed{u_{2n}(t) = 0}$.

For odd n,

$$u_{2n-1}''(t) + \nu_{2n-1}^2 u_{2n-1}(t) = \frac{4}{\nu_{2n-1}}e^t$$

To find a particular solution, put $u_{2n-1}(t) = ce^t$,

$$ce^t + \nu_{2n-1}^2 ce^t = \frac{4e^t}{\nu_{2n-1}} \implies c = \frac{4}{\nu_{2n-1}(1+\nu_{2n-1}^2)}$$

The general solution is $u_{2n-1}(t) =$

$$\frac{4e^t}{\nu_{2n-1}(1+\nu_{2n-1}^2)} + C_{2n-1}\cos\nu_{2n-1}t + D_{2n-1}\sin\nu_{2n-1}t$$

Initial conditions are

$$u_{2n-1}(0) = \frac{-8}{\nu_{2n-1}^3}, \quad u_{2n-1}'(0) = 0$$

$$u_{2n-1}(0) = C_{2n-1} + \frac{4}{\nu_{2n-1}(1+\nu_{2n-1}^2)} = \frac{-8}{\nu_{2n-1}^3}$$

$$\implies C_{2n-1} = \frac{-4(2+3\nu_{2n-1}^2)}{\nu_{2n-1}^3(1+\nu_{2n-1}^2)}$$

$$u'_{2n-1}(0) = \frac{4}{\nu_{2n-1}(1+\nu_{2n-1}^2)} + \nu_{2n-1}D_{2n-1} = 0$$

$$\implies D_{2n-1} = \frac{-4}{\nu_{2n-1}^2(1+\nu_{2n-1}^2)}$$

Thus, the solution is given by

$$u(x,t) = \sum_{n\geq 1} u_{2n-1}(t)\sin(\nu_{2n-1}x)$$

where $u_{2n-1}(t)$ is defined above.

Non homogeneous Wave equation: Neumann boundary condition

The following model describes the vibrations of a string with an external force that depends on time.

$u_{tt} - k^2 u_{xx} = F(x,t), \qquad 0 < x < L, \quad t > 0$
$u_x(0,t) = f_1(t), \;\; u_x(L,t) = f_2(t), \qquad t > 0$
$u(x,0) = f(x), \;\; u_t(x,0) = g(x), \qquad 0 \leq x \leq L$

We will first reduce the problem so that boundary conditions are homogeneous. Note that

$$\widetilde{w}(x,t) = \left(x - \frac{x^2}{2L}\right) f_1(t) + \frac{x^2}{2L} f_2(t)$$

$$\implies \widetilde{w}_x(0,t) = f_1(t), \quad \widetilde{w}_x(L,t) = f_2(t)$$

So let us first make the substitution

$$z(x,t) = u(x,t) - \widetilde{w}(x,t)$$

Then clearly

$$\begin{aligned}&z_{tt} - k^2 z_{xx} = G(x,t),\\ &z_x(0,t) = 0, \qquad z_x(L,t) = 0,\\ &z(x,0) = v(x), \qquad z_t(x,0) = w(x).\end{aligned}$$

To solve for $u(x,t)$, it is enough to solve for $z(x,t)$.

Since the boundary conditions are Neumann type, we assume the solution is given by

$$z(x,t) = \sum_{n\geq 1} Z_n(t)\cos(\nu_n x)$$

where $\nu_n = \dfrac{n\pi}{L}$ and solve for $Z_n(t)$'s.

Differentiating $z(x,t)$ term by term, we get that it satisfies the equation

$$z_{tt} - k^2 z_{xx} = \sum_{n\geq 1} \big(Z_n''(t) + k^2\nu_n^2\, Z_n(t)\big)\cos(\nu_n x)$$

Let us write

$$G(x,t) = \sum_{n\geq 1} G_n(t)\cos(\nu_n x)$$

where

$$G_n(t) = \frac{2}{L}\int_0^L G(x,t)\cos(\nu_n x)\,dx$$

Thus, $z_t - k^2 z_{xx} = G(x,t)$ gives

$$Z_n''(t) + k^2\nu_n^2\, Z_n(t) = G_n(t)$$

$$z(x,0) = \sum_{n\geq 1} Z_n(0)\cos(\nu_n x) = v(x)$$

and

$$z_t(x,0) = \sum_{n\geq 1} Z_n'(0)\sin(\nu_n x) = w(x)$$

gives

$$Z_n(0) = \frac{2}{L}\int_0^L v(x)\cos(\nu_n x)\,dx := b_n$$

is the Fourier cosine coefficient of $v(x)$ and

$$Z_n'(0) = \frac{2}{L}\int_0^L w(x)\cos(\nu_n x)\,dx := c_n$$

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We can solve

$$Z_n''(t) + k^2\nu_n^2\, Z_n(t) = G_n(t)$$

uniquely with given initial conditions

$$Z_n(0) = b_n, \quad Z_n'(0) = c_n$$

where b_n and c_n are Fourier cosine coefficients of $v(x)$ and $w(x)$ respectively.

If $Z_n(t)$ is this unique solution, then the series

$$z(x,t) = \sum_{n\geq 1} Z_n(t)\cos(\nu_n x)$$

solves our non homogeneous wave equation with Neumann boundary conditions for z.

Example. Consider the following PDE

$$u_{tt} - u_{xx} = e^t, \qquad 0 < x < 1, \quad t > 0$$
$$u_x(0,t) = 0, \quad u_x(1,t) = 0, \qquad t > 0$$
$$u(x,0) = x(x-1), \; u_t(x,0) = 0, \qquad 0 \le x \le 1$$

The boundary conditions are Neumann type, so we find solution in Fourier cosine series. Assume (here $\nu_n = n\pi$)

$$u(x,t) = \sum_{n\ge 0} u_n(t)\cos(\nu_n x)$$

The Fourier cosine series for $u(x,0) = x(x-1)$ is

$$x(x-1) = \frac{-1}{6} + \sum_{n\ge 1} \frac{4}{(\nu_{2n})^2}\cos(\nu_{2n}x)$$

Therefore, we get $u_0(0) = \dfrac{-1}{6}$ and for $n \ge 1$,

$$u_{2n-1}(0) = 0, \quad u_{2n}(0) = \frac{4}{(\nu_{2n})^2}, \quad u_n'(0) = 0$$

The Fourier cosine series for e^t is given by

$$e^t = e^t$$

Substitute

$$u(x,t) = \sum_{n\ge 1} u_n(t)\cos(\nu_n x)$$

into the equation $u_{tt} - u_{xx} = e^t$, we get

$$\sum_{n\geq 0} \left(u_n''(t) + (\nu_n)^2 u_n(t)\right) \cos(\nu_n x) = e^t$$

For $n = 0$, we get

$$u_0''(t) = e^t \implies u_0(t) = e^t + A_0 + B_0 t$$

Since $u_0(0) = \dfrac{-1}{6}$, we get $A_0 = \dfrac{-7}{6}$.

Further, $u_0'(0) = 0$, we get $B_0 = -1$.

Therefore $\boxed{u_0(t) = e^t - \dfrac{7}{6} - t}$.

For $n \geq 1$, $\quad u_n''(t) + (\nu_n)^2 u_n(t) = 0$

$$\implies u_n(t) = A_n \cos(\nu_n t) + B_n \sin(\nu_n t)$$

Initial conditions are

$$u_{2n-1}(0) = 0, \; u_{2n}(0) = \frac{4}{(\nu_{2n})^2}, \quad u_n'(0) = 0$$

$$\implies A_{2n-1} = 0, \; B_{2n-1} = 0, \; A_{2n} = \frac{4}{(\nu_{2n})^2}, \; B_{2n} = 0$$

$$u_{2n-1}(t) = 0, \quad u_{2n}(t) = \frac{4}{(\nu_{2n})^2} \cos(\nu_{2n} t)$$

Therefore,

$$u(x,t) = (e^t - \frac{7}{6} - t) + \sum_{n\geq 1} \frac{4}{(\nu_{2n})^2} \cos(\nu_{2n} t) \cos(\nu_{2n} x)$$

Now we will start the study of Laplace equation.

The two dimensional Laplace (or potential) equation in $\mathbb{R}^2$ is

$$\Delta u = u_{xx} + u_{yy} = 0$$

The solutions u of the Laplace equation are called harmonic functions.

It is associated with the gravitational and electric fields.
The following are typical problems associated with the Laplace operator.

Dirichlet Problem.

The problem is to find a harmonic function u inside a domain D so that the values of u are prescribed on the boundary ∂D of D,
(i.e. $u = f$ is given on the boundary ∂D).

Neumann Problem.

The problem is to find a harmonic function u inside the domain D so that the normal derivative of u, i.e.

$$(\text{grade}\, u).n(x, y) = g$$

is given on the boundary ∂D, where $n(x, y)$ is the exterior unit normal at the point (x, y).

Steady-State Temperature Problems.

A steady-state or equilibrium function u of 2 dimensional heat equation is a function that is independent of time t, i.e. $u_t = 0$

Thus if u satisfies heat equation $u_t = \Delta u$ and u is steady-state, then it satisfies Laplace equation

$$\Delta u = 0$$

Example. Write the BVP for the steady state temperature $u(x,y)$ in a 1×2 rectangular plate if the bottom horizontal side is kept at 0^0, top horizontal side at 100^0, left vertical side at -10^0 and right vertical side at 200^0.

The equation is

$$\begin{aligned}
&\Delta u = u_{xx} + u_{yy} = 0, \qquad 0 < x < 1, \;\; 0 < y < 2, \\
&u(x,0) = 0, \quad u(x,2) = 100, \quad 0 < x < 1, \\
&u(0,y) = -10, \quad u(1,y) = 200, \quad 0 < y < 2.
\end{aligned}$$

Let us consider the Laplace equation with boundary conditions

$$\begin{array}{lll}
u_{xx}+u_{yy} = 0, & 0 < x < a, & 0 < y < b \\
u(x,0) = f(x), & u(x,b) = 0, & 0 \le x \le a \\
u(0,y) = 0, & u(a,y) = 0, & 0 \le y \le b
\end{array}$$

Let $u(x,y) = X(x)Y(y)$. Then the differential equation becomes

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

Thus,

$$\frac{-X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} = \text{constant} = \lambda$$

$$u(0,y) = X(0)Y(y) = 0 = u(a,y) = X(a)Y(y)$$

Since $Y(y) \neq 0$ identically, so we get

$$X(0) = 0 = X(a)$$

Since $X(x) \neq 0$ identically and

$$u(x,b) = X(x)Y(b) = 0 \implies Y(b) = 0$$

Thus we need to solve the eigen-value problem

$$X''(x) + \lambda X(x) = 0, \quad X(0) = 0, \quad X(a) = 0$$

and

$$Y''(y) - \lambda Y(y) = 0, \quad Y(b) = 0$$

There are infinitely many positive eigenvalues for $n \geq 1$

$$\lambda_n = \nu_n^2, \quad \nu_n = \frac{n\pi}{a}$$

with eigenfunctions $\quad X_n(x) = \sin(\nu_n x)$

Before solving for Y problem, let us recall some generality about hyperbolic functions.

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh' x = \sinh x, \quad \sinh' x = \cosh x$$

$$\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$$

$$\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$$

The $Y''(y) - \lambda_n Y(y) = 0$ has general solution

$$Y_n(y) = A\cosh(\nu_n y) + B\sinh(\nu_n y)$$

$Y(b) = 0$ gives

$$A\cosh(\nu_n b) + B\sinh(\nu_n b) = 0$$

Thus $\quad Y_n(y) =$

$$\frac{-B\left(\sinh(\nu_n b)\cosh(\nu_n y) - \sinh(\nu_n y)\cosh(\nu_n b)\right)}{\cosh(\nu_n b)}$$

$$Y_n(y) = C\sinh(\nu_n(b-y)), \qquad C = \frac{-B}{\cosh(\nu_n b)}$$

Hence the solution with the separated variables is

$$u_n(x,y) = \sinh(\nu_n(b-y))\,\sin(\nu_n x)$$

satisfying

$$u(x,b) = 0,\ \ u(0,y) = 0,\ \ u(a,y) = 0$$

A series solution is therefore

$$u(x,y) = \sum_{n\geq 1} C_n \sinh(\nu_n(b-y))\,\sin(\nu_n x)$$

This satisfies

$$u(x,0) = \sum_{n\geq 1} C_n \sinh(\nu_n b)\sin(\nu_n x) = f(x)$$

$$\implies C_n\sinh(\nu_n b) = b_n = \frac{2}{a}\int_0^a f(x)\sin(\nu_n x)dx$$

Definition.

$$u(x,y)=\sum_{n\geq 1}\frac{b_n}{\sinh(\nu_n b)}\sinh(\nu_n(b-y))\ \sin(\nu_n x)$$

is a (formal) solution of

$$\begin{array}{lll} u_{xx}+u_{yy}=0, & 0<x<a, & 0<y<b \\ u(x,0)=f(x), & u(x,b)=0, & 0\leq x\leq a \\ u(0,y)=0, & u(a,y)=0, & 0\leq y\leq b \end{array}$$

where b_n are Fourier sine coefficients of $f(x)$ on $[0,a]$.

Example. Let $\nu_n=\frac{n\pi}{a}$. Consider

$$\begin{array}{ll} u_{tt}+u_{xx}=0, & 0<x<a,\quad 0<y<b \\ u(x,0)=\sin(\nu_5 x)-3\sin(\nu_9 x), & 0\leq x\leq a \\ u(x,b)=0, & 0\leq x\leq a \\ u(0,y)=0=u(a,y)=0, & 0\leq y\leq b \end{array}$$

Since $f(x)=u(x,0)$ is given in Fourier sine series,

$$b_5=1,\quad b_9=-3,$$

Thus, the solution to the above problem is given by

$$u(x,t)=\frac{1}{\sinh(\nu_5 b)}\sinh(\nu_5(b-y))\ \sin(\nu_5 x)$$

$$+\frac{-3}{\sinh(\nu_9 b)}\sinh(\nu_9(b-y))\ \sin(\nu_9 x)$$

Neumann boundary condition

Consider the following differential equation

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < a,\ 0 < y < b,$$

with boundary conditions

$$\begin{array}{lll} u(x,0) = f(x) & u(x,b) = 0 & 0 \le x \le a \\ u_x(0,y) = 0 & u_x(a,y) = 0 & 0 \le y \le b \end{array}$$

Let $u(x,y) = X(x)Y(y)$. Then the differential equation becomes

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

Thus, we have

$$\frac{-X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} = \text{constant} = \lambda$$

Since

$$u_x(0,y) = X'(0)Y(y) = 0 = u_x(a,y) = X'(a)Y(y) = 0$$

and we do not want Y to be identically zero, we get

$$X'(0) = 0, \quad X'(a) = 0$$

Since $X(x) \neq 0$ identically and

$$u(x,b) = X(x)Y(b) = 0 \implies Y(b) = 0$$

We need to solve the eigenvalue problem

$$X''(x) + \lambda X(x) = 0, \quad X'(0) = 0, \quad X'(a) = 0$$

and

$$Y''(y) - \lambda Y(y) = 0, \quad Y(b) = 0$$

There are infinitely many positive eigenvalues for $n \geq 0$

$$\lambda_n = \nu_n^2, \quad \nu_n = \frac{n\pi}{a}$$

with eigenfunctions $X_n(x) = \cos(\nu_n x)$

For $n = 0$, $\boxed{Y''(y) = 0,\ Y(b) = 0}$ gives $Y_0(y) = C_0(b - y)$.

For $n \geq 1$, $Y''(y) - \lambda Y(y) = 0$ has general solution

$$Y_n(y) = A\cosh(\nu_n y) + B\sinh(\nu_n y)$$

$Y(b) = 0$ gives $A\cosh(\nu_n b) + B\sinh(\nu_n b) = 0.$

Thus $Y_n(y) =$

$$\frac{-B\left(\sinh(\nu_n b)\cosh(\nu_n y) - \sinh(\nu_n y)\cosh(\nu_n b)\right)}{\cosh(\nu_n b)}$$

$$Y_n(y) = C\sinh(\nu_n(b - y)), \quad C = \frac{-B}{\cosh(\nu_n b)}$$

Hence the solution with separated variables is $u_0(x, y) = (b - y)$ and for $n \geq 1$

$$u_n(x, y) = \sinh(\nu_n(b - y))\ \cos(\nu_n x)$$

satisfying $u(x, b) = 0,\ u_x(0, y) = 0,\ u_x(a, y) = 0$

A series solution is therefore

$$u(x, y) = C_0(b - y) + \sum_{n\geq 1} C_n \sinh(\nu_n(b - y))\ \cos(\nu_n x)$$

$$u(x, 0) = C_0 b + \sum_{n\geq 1} C_n \sinh(\nu_n b)\cos(\nu_n x) = f(x)$$

where $C_0 b = a_0$ and $C_n\sinh(\nu_n b) = a_n$ are the Fourier cosine coefficients of $f(x)$ on $[0, a]$.

Definition. $u(x,y) =$

$$\frac{a_0}{b}(b-y) + \sum_{n\geq 1} \frac{a_n}{\sinh(\nu_n b)} \sinh(\nu_n(b-y)) \cos(\nu_n x)$$

is a (formal) solution of

$$\begin{aligned} u_{xx}+u_{yy} &= 0, & 0<x<a, && 0<y<b \\ u(x,0) = f(x), && u(x,b) = 0, && 0\leq x\leq a \\ u_x(0,y) = 0, && u_x(a,y) = 0, && 0\leq y\leq b \end{aligned}$$

where a_n, $n\geq 0$, are Fourier cosine coefficients of $f(x)$ on $[0,a]$.

Example. Consider the Laplace equation with boundary conditions given by (here $\nu_n = \frac{n\pi}{a}$)

$u_{tt} + u_{xx} = 0, \qquad 0<x<a, \quad 0<y<b$

$u(x,b) = 0,$

$u(x,0) = \cos(\nu_5 x) - 3\cos(\nu_9 x), \qquad 0\leq x\leq a$

$u_x(0,y) = 0 = u_x(a,y), \qquad 0\leq y\leq b$

Since f is given by its Fourier cosine series

$$a_5 = 1, \quad a_9 = -3$$

Thus, the solution to the above problem is

$$u(x,t) = \frac{1}{\sinh(\nu_5 b)} \sinh(\nu_5(b-y)) \cos(\nu_5 x)$$

$$+\frac{-3}{\sinh(\nu_9 b)} \sinh(\nu_9(b-y)) \cos(\nu_9 x)$$