Network Theory Homework 5

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Homework is not to be submitted. If you prepare a solution and upload it to Moodle, you may be eligible for extra credit, as per the course rules.

1. Define the following:
   (a) Field
   (b) Vector Space over a field
   (c) Linear Map $L : U \to V$ between vector spaces U and V
   (d) Dual vector space V^*
   (e) Transpose L^T of a linear map L.
   (f) Subspace of a vector space
   (g) A linearly independent set in a vector space
2. Prove or disprove: Let $(F, 0, +, 1, *)$ be a field. Then $(F, 0, +)$ is a vector space over $(F, 0, +, 1, *)$.
3. Prove or disprove: Let F be a field, and V a vector space over F. Then the dual of the dual $(V^*)^*$ is isomorphic to V.
4. Let F be a field. Let U, V be vector spaces over F. Let $\mathcal{L}(U, V)$ be the set of all linear maps from U to V. Give a vector space structure on $\mathcal{L}(U, V)$.
5. Let (N, E) be a graph. Define the map δ. What are the domain and codomain of this map? Prove/ disprove: δ is a linear map. If $N = \{1, 2, 3, 4\}$ and $E = \{(1, 2), (2, 3), (3, 4), (4, 2)\}$ then write down the matrix corresponding to δ. Define the map δ^T. What are the domain and codomain of this map?
6. For a given graph G, define W_{KCL} and W_{KVL}. Prove/ disprove: if $v \in W_{KVL}$ and $i \in W_{KCL}$ then the power $\sum_{e \in E} i_e v_e = 0$.
7. Let V be a vector space and let $S \subseteq V$ be a finite subset of V. Define the span of S. Prove/ disprove: The span is a vector space.

8. Let $L : V \to W$ be a linear map from vector spaces V and W. Define the kernel or nullspace $\ker L$ of L. Prove or disprove: The kernel is a subspace of V. Define the image $\operatorname{im} L$. Prove/ disprove: The image is a subspace of W.

9. Let $W \subseteq V$ be a subspace of a vector space V. Define $W^{\perp}$.

10. Fix a linear map $L : V \to W$. Prove/ disprove: $(\ker L)^{\perp} = \operatorname{im} L^T$.

11. Let $S \subseteq V$ be a finite subset of a vector space V. Let W_S be the intersection of all vector spaces containing S.

    (a) Prove/ disprove: W_S is a vector space.

    (b) Prove/ disprove: W_S is the span of S.