MA-207 Differential Equations II

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S2 - Lecture 6

Second solution: $r_1 - r_2 \in \mathbb{Z}$

Theorem (Second solution: $r_1 - r_2 \notin \mathbb{Z}$)

A second solution to the differential equation is given by

$$\sum_{n\geq 0} a_n(r_2)x^{n+r_2}$$

Theorem (Second solution: $r_1 = r_2$)

A second solution to the differential equation is given by

$$\sum_{n\geq 0} a_n'(r_2)x^{n+r_2} + \sum_{n\geq 0} a_n(r_2)x^{n+r_2}\log x$$

Theorem (Second solution: $0 \neq r_1 - r_2 \in \mathbb{Z}$)

A second solution to the differential equation is given by

$$\sum_{n\geq 0} A_n'(r_2)x^{n+r_2} + \sum_{n\geq 0} A_n(r_2)x^{n+r_2}\log x$$

Bessel functions

Bessel equation is the second-order linear ODE

$$x^2y'' + xy' + (x^2 - p^2)y = 0 \qquad p \geq 0 \qquad\qquad (*)$$

The first solution $\qquad y_1(x) = x^p \sum_{n\geq 0} \frac{(-1)^n}{2^{2n}n!(1+p)\dots(n+p)}\, x^{2n}$

converges on $(0,\infty)$.

Multiply $y_1(x)$ by $\frac{1}{2^p\Gamma(1+p)}$ (Caution: This should be a nonzero real number!)

Why

$$J_p(x) := \left(\frac{x}{2}\right)^p \sum_{n\geq 0} \frac{(-1)^n}{n!\,\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n} \qquad x > 0.$$

This is called the Bessel function of first kind of order p.

Case 1: $2p$ is not an integer.

Solving the recursion

$$[(r+n)^2 - p^2]a_n(r) + a_{n-2}(r) = 0 \quad n \geq 2 \qquad a_1(r) = 0.$$

for $r = -p$, we obtain

$$y_2(x) = x^{-p} \sum_{n \geq 0} \frac{(-1)^n}{2^{2n} n! (1-p) \dots (n-p)} x^{2n}$$

Multiplying by $\dfrac{1}{2^{-p}\Gamma(1-p)}$ (Caution: This should be a nonzero real number!)

$$J_{-p}(x) := \left(\frac{x}{2}\right)^{-p} \sum_{n \geq 0} \frac{(-1)^n}{n!\,\Gamma(n-p+1)} \left(\frac{x}{2}\right)^{2n} \qquad x > 0.$$

This is a second solution of the Bessel equation linearly independent of $J_p(x)$.

It is unbounded near $x = 0$.

Second independent solution of Bessel equation

Case 2: $2p$ is a positive integer.

Recall that the second solution is given by

$$y_2(x) = \sum_{n\geq 0} A'_n(-p)x^{n-p} + \sum_{n\geq 0} A_n(-p)x^{n-p}\log x$$

where

$$A_n(r) := (r+p)a_n(r)$$

Case 2(a): $2p$ is an odd positive integer, that is, $p = \frac{2l+1}{2}$ for some $l > 0$

We have seen that $A_{2n+1}(r) = (r+p)a_{2n+1}(r) = 0$

$$a_{2n}(r) = \frac{(-1)^n}{\prod_{i=1}^{n}((r+2i)^2 - p^2)}$$

Since the polynomial $\prod_{i=1}^{n}((r+2i)^2-p^2)$ evaluated at $r=-p$, is $\prod_{i=1}^{n} 4i(i-p) \neq 0$,

the function $a_{2n}(r)$ is analytic in a neighborhood of $-p$.

Thus, $A_{2n}(-p)=0$ and $A'_{2n}(-p)=a_{2n}(-p)$.
Thus, in this case we obtain that the second solution is

$$y_2(x) = \sum_{n\geq 0} \frac{(-1)^n}{2^{2n} n!(1-p)\dots(n-p)}\, x^{2n-p}$$

Multiplying by $\dfrac{1}{2^{-p}\Gamma(1-p)}$

$$J_{-p}(x) := \left(\frac{x}{2}\right)^{-p} \sum_{n\geq 0} \frac{(-1)^n}{n!\,\Gamma(n-p+1)} \left(\frac{x}{2}\right)^{2n} \qquad x>0.$$

Case 2(b): $2p$ is an even positive integer, that is, p is a positive integer.

As before, $A_{2n+1}(r) = 0$.
The polynomial $\prod_{i=1}^{n}((r+2i)^2 - p^2)$ evaluated at $r = -p$, is $\prod_{i=1}^{n} 4i(i-p)$,
Thus, if $n < p$, then $a_{2n}(r)$ is analytic in a neighborhood of $-p$.
Thus, if $n < p$, then $A_{2n}(-p) = 0$ and

$$A'_{2n}(-p) = a_{2n}(-p) = \frac{(-1)^n}{2^{2n} n!(1-p)\dots(n-p)} = \frac{1}{2^{2n} n!(p-n)!}$$

If $n \geq p$, then

$$\begin{aligned} A_{2n}(-p) &= \frac{2(-1)^n}{2^{2n} n!(1-p)\dots(-1)\cdot 1\cdot 2\cdots(n-p)} \\ &= \frac{-2(-1)^{n-p}}{2^{2n} n!(p-1)!(n-p)!} \end{aligned}$$

Define

$$f(r) := \Big(\prod_{i=1}^{p-1}((r+2i)^2 - p^2)\Big)(r+3p)\Big(\prod_{i=p+1}^{n}((r+2i)^2 - p^2)\Big) \quad (*)$$

Then

$$A_{2n}(r)f(r) = (-1)^n$$

Differentiating the above and setting $r = -p$ we get

$$A'_{2n}(-p)f(-p) + A_{2n}(-p)f'(-p) = 0$$

Taking log and differentiating $(*)$ we get

$$f'(-p) = f(-p)\Big(\frac{1}{2p} + \sum_{i\in\{1,2,\ldots,n\}\backslash p} \frac{1}{2i} + \frac{1}{2(i-p)}\Big)$$

$$= f(-p)\Big(\frac{H_n}{2} - \frac{H_{p-1}}{2} + \frac{H_{n-p}}{2}\Big),$$

where

$$H_0 = 0, \qquad H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

Thus,

$$A'_{2n}(-p) = -A_{2n}(-p)\Big(\frac{H_n}{2} - \frac{H_{p-1}}{2} + \frac{H_{n-p}}{2}\Big)$$
$$= \frac{2(-1)^{n-p}}{2^{2n}n!(p-1)!(n-p)!}\Big(\frac{H_n}{2} - \frac{H_{p-1}}{2} + \frac{H_{n-p}}{2}\Big)$$

Thus, we get

$$y_2(x) = \sum_{n=0}^{p-1} \frac{1}{2^{2n}n!(p-n)!}x^{2n-p} +$$
$$\sum_{n\geq p} \frac{(-1)^{n-p}}{2^{2n}n!(p-1)!(n-p)!}\Big(H_n - H_{p-1} + H_{n-p}\Big)x^{2n-p} +$$
$$-\sum_{n\geq p} \frac{2(-1)^{n-p}}{2^{2n}n!(p-1)!(n-p)!}x^{2n-p}\log x$$

is a second solution.

Case 3: $p = 0$ (Repeated root case)
The indicial equation has a repeated root $r_1 = r_2 = 0$,

$$a_{2n}(r) = \frac{(-1)^n}{(r+2)^2(r+4)^2\dots(r+2n)^2} \qquad a_{2n+1}(r) = 0$$

Differentiating $a_{2n}(r)$ with respect to r, we get

$$a'_{2n}(r) = -2a_{2n}(r)\left(\frac{1}{r+2} + \frac{1}{r+4} + \cdots + \frac{1}{r+2n}\right)$$

$$a'_{2n}(0) = \frac{(-1)^{n-1}H_n}{2^{2n}(n!)^2}, \quad H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

By theorem stated earlier, the second solution is

$$y_2(x) = J_0(x)\ln x - \sum_{n\geq 1} \frac{(-1)^n H_n}{2^{2n}(n!)^2} x^{2n} \qquad x > 0$$

where $y_1(x) = J_0(x) = \sum_{n\geq 0} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n}$ is Frobenius solution.

Summary of $p = 0$ and $p = 1/2$

For $p = 0$, two independent solutions are $J_0(x)$, which is a real analytic function for all $\mathbb{R}$, and

$$y_2(x) = J_0(x) \ln x - \sum_{n \geq 1} \frac{(-1)^n H_n}{2^{2n}(n!)^2} x^{2n}$$

For $p = 1/2$, two independent solutions are $J_{1/2}(x)$ and $J_{-1/2}(x)$. These can be expressed in terms of the trigonometric functions (Exercise):

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x \quad \text{and} \quad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

Both exhibit singular behavior at 0. Near 0, $J_{1/2}(x)$ is bounded but does not have a bounded derivative, while $J_{-1/2}(x)$ is unbounded near 0.

For real p, define

$$J_p(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(p+n+1)} \left(\frac{x}{2}\right)^{2n+p}$$

1. The above is a well defined power series once we know that the Gamma function never vanishes.
2. If $p \notin \{0, 1, 2, \ldots\}$ $J_p(x)$ and $J_{-p}(x)$ are the two independent solutions of the Bessel equation.
3. If $p \in \{0, 1, 2, \ldots\}$ then $J_{-p}(x) = (-1)^p J_p(x)$. Thus, in this case the second solution is not $J_{-p}(x)$.

Bessel identities

1. $$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$$
2. $$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$$

   The above two can be obtained by formally differentiating the power series.

3. $$J_p'(x) + \frac{p}{x} J_p(x) = J_{p-1}(x)$$
4. $$J_p'(x) - \frac{p}{x} J_p(x) = -J_{p+1}(x)$$

   These follow from (1) and (2). Expand LHS and divide by $x^{\pm p}$;

5. $$J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x)$$
6. $$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x)$$

Add and subtract (3) and (4) to get (5) and (6).

Consequences of Bessel identities

Problem: Let $p > 0$. Show that between any two consecutive zeros of $J_p(x)$, there exists precisely one zero of $J_{p-1}(x)$ and precisely one zero of $J_{p+1}(x)$

Solution: Let $0 < c < d$ be two consecutive zeros of $J_p(x)$.

So $x^p J_p(x)$ vanishes at c and d. By Rolle's theorem,

$$[x^p J_p(x)]'(b) = 0 \quad \text{for some} \quad b \in (c, d)$$

As

$$[x^p J_p(x)]' = x^p J_{p-1}(x)$$

we get $J_{p-1}(b) = 0$.

Repeating the above argument with the identity $[x^{-p} J_p(x)]' = -x^{-p} J_{p+1}(x)$, we get that $J_{p+1}(x)$ has a root in (c, d).

Thus, we have proved that both $J_{p-1}(x)$ and $J_{p+1}(x)$ have at least one root in (c, d).

If $J_{p-1}(x)$ had two roots in (c, d), then from above, we conclude that $J_p(x)$ would have a root in (c, d). However, this contradicts the assumption that c and d are consecutive roots. Thus, J_{p-1} has exactly one root in (c, d).

Similarly, $J_{p+1}(x)$ has exactly one root in (c, d).

Problem: Find a and c so that $J_2(x) - J_0(x) = aJ_c''(x)$.

Solution: Using $J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x)$ for $p = 1$, we get

$J_0(x) - J_2(x) = 2J_1'(x)$

Now using $[x^{-p}J_p(x)]' = -x^{-p}J_{p+1}$ for $p = 0$, we get

$J_0'(x) = -J_1(x)$.

Therefore,
$J_2(x) - J_0(x) = -2J_1'(x) = 2J_0''(x)$.

Hence $a = 2$ and $c = 0$. □

We can use

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x)$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x \qquad\qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

to see that $J_p(x)$ are elementary functions for $p \in \mathbb{Z} + \frac{1}{2}$.

For example,

- $$J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$$

- $$J_{-3/2}(x) = -\frac{1}{x} J_{-1/2}(x) - J_{1/2}(x) = -\sqrt{\frac{2}{\pi x}} \left(\frac{\cos x}{x} + \sin x \right)$$

- $J_{\frac{5}{2}}(x) = \dfrac{3}{x} J_{\frac{3}{2}}(x) - J_{\frac{1}{2}}(x)$
  $$= \sqrt{\frac{2}{\pi x}} \left(\frac{3 \sin x}{x^2} - \frac{3 \cos x}{x} - \sin x \right)$$

These functions are called spherical Bessel functions as they arise in solving wave equations in spherical coordinates.

Theorem (Liouville)

$J_{m+\frac{1}{2}}(x)$'s are the only elementary Bessel functions.

Remark. Integrating some of the Bessel identities we get

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$$
$$\implies \int_0^x t^p J_{p-1}(t)\,dt = x^p J_p(x) + c$$

$$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$$
$$\implies \int_0^x t^{-p} J_{p+1}(t)\,dt = -x^{-p} J_p(x) + c$$

For example,

$$\int_0^x t J_0(t)\,dt = x J_1(x) + c$$

Qualitative properties of solutions

It is rarely possible to solve 2nd order linear ODE

$$y'' + P(x)y' + Q(x)y = 0$$

in terms of familiar elementary functions.

Then how do we understand the nature and properties of solutions.

It is surprising that we can obtain quite a bit of information about the solution from the ODE itself.

Theorem (Sturm separation theorem)

If $y_1(x)$ and $y_2(x)$ are linearly independent solns of

$$y'' + P(x)y' + Q(x)y = 0$$

P, Q continuous on (a, b). Then
(1) $y_1(x)$ and $y_2(x)$ have no common zero in (a, b).
(2) Between any two successive zeros of $y_1(x)$, there is exactly one zero of $y_2(x)$ and vice versa.

Proof of (1). Consider the Wronskian

$$W(x) := W(y_1, y_2) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

It satisfies the differential equation $W' = -P(x)W$ and so is given by

$$W(x) = C\exp\Big(\int_{a_0}^{x} -P(t)dt\Big) \qquad a_0 \in (a, b)$$

In particular, since y_1 and y_2 are linearly independent, the Wronskian is nonzero and so it never vanishes. This proves (1).

Proof of (2). Let x_1 and x_2 be successive zeros of $y_1(x)$.

First let us show y_2 has a zero in (x_1, x_2).
The Wronskian $W(x)$ has the same sign in the interval (a, b) as it never vanishes. Thus, $W(x_1)$ and $W(x_2)$ have the same sign.

$$0 \neq W(x_1) = -y_1'(x_1)y_2(x_1) \qquad\qquad 0 \neq W(x_2) = -y_1'(x_2)y_2(x_2)$$

We conclude that $y_1'(x_1)$ and $y_1'(x_2)$ are nonzero.

It follows that $y_1'(x_1)$ and $y_1'(x_2)$ have opposite signs since x_1 and x_2 are consecutive zeros of y_1.

It follows that $y_2(x_1)$ and $y_2(x_2)$ have opposite signs. Thus, $y_2(x)$ has a zero in (x_1, x_2).

If $y_2(x)$ had two zeros in the interval $x_1 < \alpha < \beta < x_2$, then by the same reasoning, y_1 will have a zero in (α, β), which contradicts the assumption that x_1 and x_2 are successive zeros of y_1.

As a consequence, if y_1 and y_2 are linearly independent solution of $y'' + P(x)y' + Q(x)y = 0$, P, Q continuous on (a, b) then the number of zeros of y_1 and y_2 on (a, b) differ by at most 1.

In particular, either both have finite number of zeros or both have infinite number of zeros in (a, b).

• For further discussion, we need that any ODE in the "standard" form $y'' + P(x)y' + Q(x)y = 0$ can be written in the "normal" form $u'' + q(x)u = 0$.

Define $v(x) := \exp\Big(\int_{a_0}^{x} -\frac{1}{2}P(t)dt \Big)$ and set $u(x) = \frac{y(x)}{v(x)}$.

One easily checks that $u(x)$ satisfies the differential equation

$$u'' + q(x)u = 0 \qquad\qquad q(x) := Q(x) - \frac{1}{4}P(x)^2 - \frac{1}{2}P'(x)$$

It is clear that the zeros of u are the same as those of y.
Also note that we need $P(x)$ to be once differentiable.

Theorem

Let $u(x)$ be a non-trivial solution of $u'' + q(x)u = 0$ on finite interval (a, b), with $q(x)$ continuous. Then $u(x)$ has at most finite number of zeros in (a, b).

Hence if $u(x)$ has infinitely many zeros on $(0, \infty)$, then the set of zeros of $u(x)$ are not bounded.

Proof. Assume $u(x)$ has infinitely many zeros in (a, b). Then $\exists\, x_0 \in [a, b]$ and a sequence of zeros $x_n \neq x_0$ such that $x_n \to x_0$ as $n \to \infty$.

$u(x_0) = \lim_{x_n \to x_0} u(x_n) = 0$ (u is continuous) and

$$u'(x_0) = \lim_{x_n \to x_0} \frac{u(x_n) - u(x_0)}{x_n - x_0} = 0$$

This contradicts the fact that the Wronskian at x_0 is nonzero. □

Theorem

Let $u(x)$ be a non-trivial solution of $u'' + q(x)u = 0$. If $q(x) < 0$ in (a, b) and continuous then $u(x)$ has atmost one zero in (a, b).

Proof. Assume $u(x_0) = 0$. Then $u'(x_0) \neq 0$, since Wronskian $W(x_0) \neq 0$.

Assume x_1 is next zero of $u(x)$ after x_0.

If necessary, multiply by -1 and assume that $u'(x_0) > 0$.

Then $u(x) > 0$ on (x_0, x_1).

Since $u''(x) = -q(x)u(x) > 0$ on (x_0, x_1), $u'(x)$ is an increasing function on (x_0, x_1).

By Rolle's theorem u' has a zero in (x_0, x_1).

But this is not possible as u' is increasing on (x_0, x_1). $\square$

Added after class

Here is a rigorous proof of the following lemma, which was "intuitively" proved in class.

Lemma

Let $f(x)$ be a function on (a, b), such that $f(x)$ is differentiable and $f'(x)$ is continuous. Let $x_1 < x_2$ be roots of $f(x)$ such that $f(x)$ does not vanish on (x_1, x_2). Also assume that $f'(x_i) \neq 0$. Then $f'(x_i)$ have opposite signs.

Proof.

We may assume that $f'(x_1) > 0$, as if this is not the case, then we can work with $-f$. Since $f(x_1) = 0$ and $f'(x_1) > 0$, it follows from the non-vanishing of f, that $f(x) > 0$ in the interval (x_1, x_2). Assume that $f'(x_2) > 0$. As $f'(x)$ is continuous, there is a small neighborhood of x_2, say $(x_2 - \delta, x_2 + \delta)$, such that $f'(x)$ is positive on this neighborhood. Continued ...

Added after class

Proof.

Let $x_2 - \delta < x < x_2$ and consider the equality

$$f(x_2) - f(x) = \int_x^{x_2} f'(t)dt$$

The RHS is positive since $f'(t)$ is positive on (x, x_2). The LHS is $-f(x)$. Thus, we get that $f(x) < 0$. This contradicts the fact that $f(x) > 0$ on (x_1, x_2). $\square$