MA-207 Differential Equations II

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28th September, 2017
S2 - Lecture 4

Recall

The Legendre equation

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0$$

When $p = m$ is an integer then exactly one of the two independent solutions is a polynomial, denoted by $P_m(x)$. This is a polynomial of degree m.

Theorem

We have

$$\langle P_m, P_n \rangle = \int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m = n \end{cases}$$

i.e. Legendre polynomials form an orthogonal basis for the vector space $\mathcal{P}(x)$ and

$$\|P_n(x)\|^2 = \frac{2}{2n+1}$$

Expansion of polynomial in terms of P_n's

Since each $P_n(x)$ is a polynomial of degree n, we see that

$$\{P_0(x), P_1(x), P_2(x), \ldots\}$$

form a basis for the vector space of polynomials $\mathcal{P}(x)$.
If $f(x)$ is a polynomial of degree n, then we can express

$$f(x) = \sum_{k=0}^{n} a_k P_k(x) \qquad a_k \in \mathbb{R}$$

To find a_k, we can use orthogonality of P_n's.

$$\int_{-1}^{1} f(x) P_k(x)\,dx = \int_{-1}^{1} \left(\sum_{i=0}^{n} a_i P_i(x) \right) P_k(x)\,dx$$

$$= \sum_{i=0}^{n} \left(\int_{-1}^{1} a_i P_i(x) P_k(x)\,dx \right) = a_k \int_{-1}^{1} P_k(x) P_k(x)\,dx$$

$$\implies a_k = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_k(x)\,dx$$

Square-integrable functions

A function $f(x)$ on $[-1, 1]$ is square-integrable if

$$\int_{-1}^{1} f(x)f(x)dx < \infty$$

For instance, polynomials, continuous functions, piecewise continuous functions are square-integrable.

The set of all square-integrable functions on $[-1, 1]$ is a vector space and is denoted by $L^2([-1, 1])$.

For square-integrable functions f and g, we define their inner product by

$$\langle f, g \rangle := \int_{-1}^{1} f(x)g(x)dx$$

Fourier-Legendre series

The Legendre polynomials no longer form a basis for the vector space $L^2([-1,1])$ of square-integrable functions.

But they form a maximal orthogonal set in $L^2([-1,1])$.

This means that there is no non-zero square-integrable function which is orthogonal to all Legendre polynomials (a nontrivial fact).

We can expand any square-integrable function $f(x)$ on $[-1,1]$ in a series of Legendre polynomials

$$\sum_{n=0}^{\infty} c_n P_n(x), \quad c_n = \frac{\langle f, P_n \rangle}{\langle P_n, P_n \rangle} = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) dx$$

This is called the Fourier-Legendre series (or simply the Legendre series) of $f(x)$.

Theorem (Convergence in norm)

The Fourier-Legendre series of $f(x) \in L^2([-1,1])$ given by

$$\sum_{n=0}^{\infty} c_n P_n(x), \quad c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) dx$$

converges in L^2 norm to $f(x)$, that is

$$\|f(x) - \sum_{n=0}^{m} c_n P_n(x)\| \to 0 \quad \textit{as} \quad m \to \infty$$

Pointwise convergence of Fourier-Legendre series to $f(x)$ is more delicate.

There are two issues here:

- Does the Fourier-Legendre series converge at x?
- If yes, then does it converge to $f(x)$?

A useful result in this direction is the Legendre expansion theorem:

Theorem

If both $f(x)$ and $f'(x)$ have at most a finite number of jump discontinuities in the interval $[-1, 1]$, then the Legendre series converges to

$$\frac{1}{2}(f(x_-) + f(x_+)) \quad \textit{for } -1 < x < 1$$

$$f(-1_+) \quad \textit{for } x = -1$$

$$f(1_-) \quad \textit{for } x = 1$$

In particular, the series converges to $f(x)$ at every point of continuity x.

Example

Consider the function

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ -1 & \text{if } -1 < x < 0 \end{cases}$$

The Legendre series of $f(x)$ is

$$\sum_{n=0}^{\infty} c_n P_n(x), \quad c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx$$

Since $P_{2n}(x)$ is even function and f is an odd function, we get

$$c_{2n} = 0 \qquad n \geq 0$$

Recall, $P_1(x) = x$, so

$$c_1 = \frac{3}{2} \int_{-1}^{1} f(x) x\, dx = \frac{3}{2}$$

Example (continued . . .)

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$, so

$$c_3 = \frac{7}{2}\int_{-1}^{1} f(x)\frac{1}{2}(5x^3 - 3x)\, dx = \frac{7}{2}(\frac{5}{4}x^4 - \frac{3}{2}x^2)\Big|_0^1 = -\frac{7}{8}$$

Check that the Legendre series of f is

$$\frac{3}{2}P_1(x) - \frac{7}{8}P_3(x) + \frac{11}{16}P_5(x) - \dots$$

By the Legendre expansion theorem, this series converges to $f(x)$ for $x \neq 0$ and to 0 for $x = 0$.

Theorem

Suppose we want to approximate $f \in L^2([-1,1])$ in the sense of least square by polynomials $p(x)$ of degree $\leq n$;
i.e. we want to minimize

$$I = \int_{-1}^{1} [f(x) - p(x)]^2 \, dx$$

Then the minimizing polynomial is precisely the first $n+1$ terms of the Legendre series of $f(x)$, i.e.

$$c_0 P_0(x) + \ldots + c_n P_n(x) \qquad c_k = \frac{2k+1}{2} \int_{-1}^{1} f(x) P_k(x) dx$$

Proof.

Write degree $\leq n$ polynomial $p(x) = \sum_{k=0}^{n} b_k P_k(x)$, then

$$
\begin{aligned}
I &= \int_{-1}^{1} \left[f(x) - \sum_{k=0}^{n} b_k P_k(x) \right]^2 dx \\
&= \int_{-1}^{1} f(x)^2 \, dx + \sum_{k=0}^{n} \frac{2}{2k+1} b_k^2 - 2 \sum_{k=0}^{n} b_k \left[\int_{-1}^{1} f(x) P_k(x) \, dx \right] \\
&= \int_{-1}^{1} f(x)^2 \, dx + \sum_{k=0}^{n} \frac{2}{2k+1} b_k^2 - 2 \sum_{k=0}^{n} b_k \frac{2c_k}{2k+1} \\
&= \int_{-1}^{1} f(x)^2 \, dx + \sum_{k=0}^{n} \frac{2}{2k+1} (b_k - c_k)^2 - \sum_{k=0}^{n} \frac{2}{2k+1} c_k^2
\end{aligned}
$$

Clearly, I is minimum when $b_k = c_k$ for $k = 0, \ldots, n$.

Caution. If f has a power series expansion on $[-1, 1]$, then best "least square polynomial approximation" to $f(x)$ is not the partial sums of the power series, in general.

Some remarks

This brings to an end the discussion of second order linear ODE's which we can solve by power series.

Before we go on to more complicated ODE's, let us review what we have done so far.

1. Given an ODE of the type

$$F_0(x)y'' + F_1(x)y' + F_2(x)y = 0 \qquad (*)$$

first convert it to the standard form

$$y'' + \frac{F_1(x)}{F_0(x)}y' + \frac{F_2(x)}{F_0(x)}y = 0 \qquad (**)$$

Let

$$p(x) := \frac{F_1(x)}{F_0(x)} \qquad q(x) := \frac{F_2(x)}{F_0(x)}$$

2. Now find the set

$$U := \{x_0 \in \mathbb{R} \,|\, p(x), q(x) \text{ are analytic at } x_0\}$$

3. By the existence theorem, for every $x_0 \in U$, there will exist two independent solutions to the above ODE, call them $y_1(x)$ and $y_2(x)$, such that both of them will be analytic in an interval I around x_0.

4. To find the solutions in a neighborhood of x_0, set $y(x) = \sum_{n\geq 0} a_n(x-x_0)^n$ into the ODE $(*)$ or $(**)$ and get recursive relations involving the a_n. Note that when you do this, the coefficient functions $(p(x), q(x), F_0(x), ..)$ have to be written as power series in $x - x_0$. Note that the recursive relation you get, will be same, irrespective of whether you choose equation $(*)$ or $(**)$.

5. Thus, depending on the situation, you may want to choose $(*)$ or $(**)$.
For example, the Legendre equation, in the open interval $(-1,1)$ around $x_0 = 0$, the equation $(*)$ looks like

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0$$

while $(**)$ looks like

$$y'' - 2\Big(\sum_{n\geq 0} x^{2n+1}\Big)y' + p(p+1)\Big(\sum_{n\geq 0} x^{2n}\Big)y = 0$$

In this case it is clear that, we should choose $(*)$, as it will be easier to work with. This is what we did in class.

Ordinary and singular points

Definition

Consider the second-order linear ODE in standard form

$$y'' + p(x)y' + q(x)y = 0 \qquad (*)$$

1. $x_0 \in \mathbb{R}$ is called an ordinary point of $(*)$ if $p(x)$ and $q(x)$ are analytic at x_0
2. $x_0 \in \mathbb{R}$ is called regular singular point if $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are analytic at x_0.
   This is equivalent to saying that there are functions $b(x)$ and $c(x)$ which are analytic at x_0 such that

   $$p(x) = \frac{b(x)}{(x - x_0)} \qquad q(x) = \frac{c(x)}{(x - x_0)^2}$$

3. If $x_0 \in \mathbb{R}$ is not ordinary or regular singular, then we call it irregular singular.

Example

$x = 0$ is an irregular singular point of $x^3y'' + xy' + y = 0$

Let us write the ODE in standard form

$$y'' + \frac{1}{x^2}y' + \frac{1}{x^3}y = 0$$

Then

$$p(x) = \frac{1}{x^2} \qquad q(x) = \frac{1}{x^3}$$

Clearly,

$$xp(x) = \frac{1}{x} \qquad x^2q(x) = \frac{1}{x}$$

are not analytic at 0. Thus, $x = 0$ is an irregular singular point.

Example

Consider the Cauchy-Euler equation

$$x^2y'' + b_0xy' + c_0y = 0 \qquad b_0, c_0 \in \mathbb{R}$$

$x = 0$ is a regular singular point, since we can write the ODE as

$$y'' + \frac{b_0}{x}y' + \frac{c_0}{x^2}y = 0$$

All $x \neq 0$ are ordinary points.

Assume $\boxed{x > 0}$

Note that $y = x^r$ solves the equation iff

$$r(r-1) + b_0r + c_0 = 0$$

$$\iff r^2 + (b_0 - 1)r + c_0 = 0$$

Let r_1 and r_2 denote the roots of this quadratic equation.

Example (continues ...)

- If the roots $r_1 \neq r_2$ are real, then

$$x^{r_1} \quad \text{and} \quad x^{r_2}$$

  are two independent solutions.

- If the roots $r_1 = r_2$ are real, then

$$x^{r_1} \quad \text{and} \quad (\log x) x^{r_1}$$

  are two independent solutions.

- If the roots are complex (written as $a \pm ib$), then

$$x^a \cos(b \log x) \quad \text{and} \quad x^a \sin(b \log x)$$

  are two independent solutions.

This example motivates us to look for solutions which involve x^r.

First solution in regular singular case

Consider $\quad x^2y'' + xb(x)y' + c(x)y = 0$ with

$$b(x) = \sum_{i\geq 0} b_i x^i \qquad c(x) = \sum_{i\geq 0} c_i x^i$$

analytic functions in a small neighborhood of 0.

$x = 0$ is a regular singular point.

Define the indicial equation

$$I(r) := r(r-1) + b_0 r + c_0$$

Look for solution of the type

$$y(x) = \sum_{n\geq 0} a_n x^{n+r}$$

by substituting this into the differential equation and setting the coefficient of x^{n+r} to 0.

First solution in regular singular case

We get the following

1. The coefficient of x^r is $I(r)a_0$, thus we need $I(r)a_0 = 0$
2. The coefficient of x^{n+r}, for $n \geq 1$, is

$$I(n+r)a_n + \sum_{i=0}^{n-1} b_{n-i}(i+r)a_i + \sum_{i=0}^{n-1} c_{n-i}a_i$$

   We need this to be 0

Let r_1 and r_2 be roots of $I(r) = 0$. Assume r_1 and r_2 are real and $r_1 \geq r_2$.

Define $a_0 = 1$.

Set $r = r_1$ in the above equation and define a_n, for $n \geq 1$, inductively by the equation

$$a_n(r_1) = -\frac{\sum_{i=0}^{n-1} b_{n-i}(i+r_1)a_i + \sum_{i=0}^{n-1} c_{n-i}a_i}{I(n+r_1)}$$

Since $I(n+r_1) \neq 0$ for $n \geq 1$, $a_n(r_1)$ is a well defined real number.

Thus,

$$y_1(x) = \sum_{n \geq 0} a_n(r_1) x^{n+r_1}$$

is a possible solution to the above differential equation.

First solution in regular singular case

Theorem

Consider the ODE $x^2y'' + xb(x)\,y' + c(x)\,y = 0 \qquad (*)$

where $b(x)$ *and* $c(x)$ *are analytic at* 0. *Then* $x = 0$ *is a regular singular point of ODE.*

Then $(*)$ *has a solution of the form*

$$y(x) = x^r \sum_{n\geq 0} a_n x^n \quad a_0 \neq 0, \quad r \in \mathbb{C} \qquad (**)$$

The solution $(**)$ *is called Frobenius solution or fractional power series solution.*

The power series $\sum_{n\geq 0} a_n x^n$ *converges on* $(-\rho, \rho)$, *where* ρ *is the minimum of the radius of convergence of* $b(x)$ *and* $c(x)$. *We will consider the solution* $y(x)$ *in the open interval* $(0, \rho)$.

Second solution in regular singular case

The analysis now breaks into the following three cases

- $r_1 - r_2 \notin \mathbb{Z}$
- $r_1 = r_2$
- $0 \neq r_1 - r_2 \in \mathbb{Z}$

In this case, because of the assumption that $r_1 - r_2 \notin \mathbb{Z}$, it follows that $I(n + r_2) \neq 0$ for any $n \geq 1$.

Thus, as before, the second solution is given by

$$y_2(x) = \sum_{n \geq 0} a_n(r_2) x^{n+r_2}$$

Example

Consider the ODE $\quad x^2 y'' - \frac{x}{2} y' + \frac{(1+x)}{2} y = 0$

Observe that $x = 0$ is a regular singular point.

$$\begin{aligned} I(r) &= r(r-1) - \tfrac{1}{2} r + \tfrac{1}{2} \\ &= (2r(r-1) - r + 1)/2 \\ &= (2r^2 - 3r + 1)/2 \\ &= (r-1)(2r-1)/2 \end{aligned}$$

Roots of $I(r) = 0$ are $\boxed{r_1 = 1}$ and $\boxed{r_2 = 1/2}$

Second solution: $r_1 - r_2 \notin \mathbb{Z}$

Example (continues ... $2x^2y'' - xy' + (1+x)y = 0$)

Their difference $r_1 - r_2 = 1/2$ is not an integer.

The equation defining a_n, for $n \geq 1$, is

$$I(n+r)a_n + \frac{1}{2}a_{n-1} = 0$$

Thus,

$$a_n(r) = -\frac{a_{n-1}(r)}{(n+r-1)(2n+2r-1)}$$

Thus,

$$\begin{aligned} a_n(r_1) = a_n(1) &= -\frac{a_{n-1}}{n(2n+1)} \\ &= (-1)^n \frac{1}{n!((2n+1)(2n-1)\ldots 3} \end{aligned}$$

Example (continues ... $2x^2y'' - xy' + (1+x)y = 0$)

$$y_1(x) = x\left(1 + \sum_{n\geq 1} \frac{(-1)^n x^n}{n!(2n+1)(2n-1)\dots 3}\right)$$

$$a_n(r_2) = -\frac{a_{n-1}}{n(2n-1)}$$

$$= (-1)^n \frac{1}{n!(2n-1)(2n-3)\dots 1}$$

$$y_2(x) = x^{1/2}\left(1 + \sum_{n\geq 1} \frac{(-1)^n x^n}{n!(2n-1)(2n-3)\dots 1}\right)$$

Since $|a_n|$ are smaller that $\frac{1}{n!}$, it is clear that both solutions converge on $(0, \infty)$.

Second solution: $r_1 = r_2$

Consider the function of two variables

$$\psi(r,x) := \sum_{n\geq 0} a_n(r) x^{n+r}$$

Consider the differential operator

$$L := x^2 \frac{d^2}{dx^2} + x b(x) \frac{d}{dx} + c(x)$$

We have already computed the coefficient of x^{n+r} in $L\psi(r,x)$. Recall that this is given by

1. The coefficient of x^r is $I(r)a_0$
2. The coefficient of x^{n+r}, for $n \geq 1$, is

$$I(n+r)a_n(r) + \sum_{i=0}^{n-1} b_{n-i}(i+r)a_i(r) + \sum_{i=0}^{n-1} c_{n-i}a_i(r)$$

Second solution: $r_1 = r_2$

Consider the functions $a_n(r)$, defined inductively using the equations

$$a_0(r) := 1$$

and for $n \geq 1$

$$I(n+r)a_n(r) + \sum_{i=0}^{n-1} b_{n-i}(i+r)a_i(r) + \sum_{i=0}^{n-1} c_{n-i}a_i(r) = 0$$

With these definitions, it follows that

$$L\psi(r,x) = I(r)x^r$$