

Q3 Define a mapping $L: V \rightarrow V^*$.

such that $L(f) = f(\alpha) \quad \forall f \in V^*$.

- TPT:- L is linear

Consider $\alpha, \beta \in V$ and $c_1, c_2 \in F$

Let $\gamma = c_1\alpha + c_2\beta$

$$\begin{aligned} L(\gamma) &= L(c_1\alpha + c_2\beta) \\ &= f(c_1\alpha + c_2\beta) \\ &= c_1f(\alpha) + c_2f(\beta) \\ &= c_1L(\alpha) + c_2L(\beta) \end{aligned}$$

- L is non-singular :- $L(\alpha) = 0$ iff $\alpha = 0$.

- Since $\dim V = \dim V^{**} (= \dim V^*)$ and L is non-singular,
 $\therefore L$ is invertible

Hence, L defines an isomorphism.

Q4. $L_1: U \rightarrow V \in \mathcal{L}(U, V)$
 $L_2: U \rightarrow V \in \mathcal{L}(U, V)$

$$\begin{aligned} \rightarrow (L_1 + L_2)(c_1u_1 + c_2u_2) &= L_1(c_1u_1 + c_2u_2) + L_2(c_1u_1 + c_2u_2) \\ &= c_1L_1(u_1) + c_2L_1(u_2) + c_1L_2(u_1) + c_2L_2(u_2) \\ &= c_1(L_1 + L_2)(u_1) + c_2(L_1 + L_2)(u_2) \end{aligned}$$

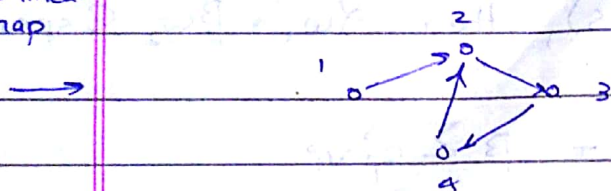
$$\begin{aligned} \rightarrow \text{Zero mapping:- } 0(c_1u_1 + c_2u_2) &= c_10(u_1) + c_20(u_2) \\ &= c_1(0) + c_2(0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \rightarrow \text{Inverse mapping:- } [f + (-f)]u &= f(u) + (-f)(u) \\ &= f(u) + f(-u) \\ &= f(u - u) \\ &= f(0) = 0 \end{aligned}$$

Q5 $\rightarrow S: \mathbb{R}^N \rightarrow \mathbb{R}^E$, where $\mathbb{R}^N = \{f: N \rightarrow \mathbb{R}\}$ is a vector space
 $\mathbb{R}^E = \{f: E \rightarrow \mathbb{R}\}$ " " " "

$$S(n_1, n_2) = f(n_2) - f(n_1)$$

★ $\rightarrow S(af + bg) = aS(f) + bS(g)$
 Prove linear map



$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} = M_S$$

$$\rightarrow S^T: (\mathbb{R}^E)^* \rightarrow (\mathbb{R}^N)^* \quad (\mathbb{R}^E)^* = \{f: E \rightarrow \mathbb{R}\}$$

$$(\mathbb{R}^N)^* = \{f: N \rightarrow \mathbb{R}\}$$

Q6 $W_{KVL} = \text{Im } S \subseteq \mathbb{R}^E$
 $W_{KCL} = \text{Ker } S^T \subseteq (\mathbb{R}^E)^*$

Proof

$$L: V \rightarrow W \quad L^T: W^* \rightarrow V^*$$

$$v_1, Lv_1 \quad \omega_2^*, L^T \omega_2^*$$

$$(\omega_2^*, Lv_1) = (L^T \omega_2^*, v_1)$$

$$\omega_2^*(Lv_1) = (\omega_2^* \circ L)(v_1) = (L^T \omega_2^*)(v_1)$$

$$\text{IF } v_1 \in \text{Ker } L \text{ and } \omega_2^* \in \text{Im } L^T, \text{ then } (\omega_2^*, v_1) = 0$$

$$\rightarrow \text{IF } v \in W_{KVL} \subseteq \mathbb{R}^E \text{ and } i \in W_{KCL} \subseteq (\mathbb{R}^E)^*$$

$$\text{Then } (i, v) = 0$$

$$\sum_{e \in E} v_e i_e = 0$$

Q7 $S = \{v_1, v_2, \dots, v_n\}$ for $v_i \in V$.

$\text{Span} = \{v \mid v = \sum_{i=1}^n c_i v_i \text{ for all combinations of } c_i \in \mathbb{R}\}$

→ If $\alpha, \beta \in \mathbb{R}$, $w_1, w_2 \in \text{span}(S)$, then $\alpha w_1 + \beta w_2 \in \text{span}(S)$

$$\begin{aligned}\alpha w_1 + \beta w_2 &= \alpha \sum c_i v_i + \beta \sum c'_i v_i \\ &= \sum k_i v_i\end{aligned}$$

- Hence, for $\alpha = \beta = 0$, span also contains zero vector.

∴ If is a vector space

Q8 A] Definition

- $\ker L$ contains vectors from domain 'V'

$$\therefore \ker L \subseteq V$$

- $L(0) = 0 \Rightarrow \text{Zero vector} \in \ker L$

- If $v_1, v_2 \in \ker L$, then

$$\begin{aligned}L(\alpha v_1 + \beta v_2) &= \alpha L(v_1) + \beta L(v_2) \\ &= 0\end{aligned}$$

$$\therefore \alpha v_1 + \beta v_2 \in \ker L$$

Hence, $\ker L$ is a subspace of V .

B] Definition $(v_1, \dots, v_n) = (L(v_1), \dots, L(v_n)) = (w_1, \dots, w_n)$

- $\text{Im } L$ takes vectors in W .

$$\therefore \text{Im } L \subseteq W$$

- There exists zero vector in $\text{Im } L$, corresponding to zero vector in V .

- If $v_1, v_2 \in \text{Im } L$, having preimages u_1 and u_2 in V ; then $c_1 v_1 + c_2 v_2 \in \text{Im } L$, with preimage $c_1 u_1 + c_2 u_2$

$$L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2)$$

$$= c_1 v_1 + c_2 v_2 \in \text{Im } L$$

Hence $\text{Im } L$ is a subspace of W .

Q9

$$W^\perp \subseteq V^*$$

$$W^\perp = \{ \omega_2^* \in V^* \mid (\omega_2^*, w_1) = 0 \quad \forall w_1 \in W \}$$

Q10

$$L: V \rightarrow W$$

$$L^T: W^* \rightarrow V^*$$

$$v_1 \rightarrow L v_1$$

$$\omega_2^* \rightarrow L^T \omega_2^*$$

$$\begin{aligned} (\omega_2^*, L v_1) &= \omega_2^* [L(v_1)] = (\omega_2^* \circ L) v_1 = [L^T(\omega_2^*)](v_1) \\ &= (L^T \omega_2^*, v_1) \end{aligned}$$

- Now, $v_1 \in \ker L$ and $\omega_2^* \in \text{Im } L^T$

$$\text{Here } v_2^* \in \mathbb{F} = L^T(\omega_2^*)$$

P.T.O.

$$(v_2^*, v_1) = (L^T w_2^*, v_1) = (w_2^*, L v_1) = (w_2^*, 0)$$

$$(v_2^*, v_1) = 0 \quad \text{for all such } v_2^*, v_1$$

$$\therefore (v_2^*, v_1) = 0 \quad \text{for all such } v_2^*, v_1$$

$$\therefore (\ker L)^\perp = \text{Im } L^T$$

Q 11

A] Let $W = \bigcap_i U_i$ where U_i are vector spaces containing S .

- Zero vector belongs to all $U_i \Rightarrow 0$ belongs to W .
- Vectors $v_1, v_2 \in W$ must also belong to all U_i .

For scalars α and β , αv_1 and βv_2 as well as $\alpha v_1 + \beta v_2$ belong to all U_i .

$$\therefore \alpha v_1 + \beta v_2 \in W$$

$\therefore W$ is a vector space.

B] All vectors v_i in $S = \{v_i\}$ belong to W .
Hence all linear combinations $\sum c_i v_i$ belong to W ($c_i \in \mathbb{R}$).

- The smallest vector space containing S is the span of S .
 $\therefore W$ must be the span of S .