MA-207 Differential Equations II

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6th November, 2017
S2 - Lecture 13

Two dimensional Laplace equation

Consider the following differential equation

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < a,\ 0 < y < b,$$

called the Laplace equation in two variables.

We can can ask for solutions to the above equation, which satisfy certain boundary conditions.

For example, in today's lecture we will work out the case where

$$\begin{array}{lll} u(x,0) = f(x) & u(x,b) = 0 & 0 \leq x \leq a \\ u(0,y) = 0 & u(a,y) = 0 & 0 \leq y \leq b \end{array}$$

Let $u(x,y) = X(x)Y(y)$. Then the differential equation becomes

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

Dirichlet boundary conditions: Finding some solutions

Thus, we have

$$\frac{-X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} = \text{constant}$$

Since $u(0, y) = X(0)Y(y) = 0$, $u(a, y) = X(a)Y(y) = 0$ and we do not want Y to be identically zero, we get that $X(0) = 0$ and $X(a) = 0$.

This boundary condition on X forces that the constant above should be positive. Let us denote this positive constant by λ^2.

For every $n \geq 1$, let

$$\lambda_n = \frac{n\pi}{a}$$

Dirichlet boundary conditions: Finding some solutions

For each $n \geq 1$, we have a solution to

$$X''(x) + \lambda_n^2 X(x) = 0$$
$$X(0) = 0 = X(a)$$

given by

$$X_n(x) = \sin\left(\frac{n\pi x}{a}\right)$$

Since we do not want $X(x)$ to be identically 0 and $u(x,b) = X(x)Y(b) = 0$, this forces that $Y(b) = 0$. Let us also impose the condition that $Y(0) = 1$.

Next consider for each λ_n the problem

$$Y''(y) - \lambda_n^2 Y(y) = 0$$
$$Y(0) = 1$$
$$Y(b) = 0$$

The solutions to the above equation are given by

$$Y_n(y) = \sinh\left(\frac{n\pi(b-y)}{a}\right) \Big/ \sinh\left(\frac{n\pi b}{a}\right).$$

Thus, for each $n \geq 1$ we get a solution

$$u_n(x,y) = \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right) \Big/ \sinh\left(\frac{n\pi b}{a}\right)$$

Now consider the series

$$u(x,y) = \sum_{n\geq 1} \alpha_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right) \Big/ \sinh\left(\frac{n\pi b}{a}\right),$$

where α_n are real numbers.

This gives that

$$u(x,0) = f(x) = \sum_{n\geq 1} \alpha_n \sin\left(\frac{n\pi x}{a}\right),$$

Thus, if $f(x)$ has the Fourier expansion

$$f(x) = \sum_{n\geq 1} \alpha_n \sin\frac{n\pi x}{a}$$

then we will have solved our Laplace equation with the given boundary conditions.

Dirichlet boundary conditions: Formal solutions

Definition

Consider the Laplace equation with the boundary conditions

$$
\begin{aligned}
& u_{xx} + u_{yy} = 0 && 0 < x < a, \quad 0 < y < b \\
& u(0, y) = 0 = u(a, y) = 0 && 0 \leq y \leq b \\
& u(x, 0) = f(x) && 0 \leq x \leq a \\
& u(x, b) = 0 &&
\end{aligned}
$$

The formal solution of the above problem is

$$u(x, t) = \sum_{n \geq 1} \alpha_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right) \Big/ \sinh\left(\frac{n\pi b}{a}\right),$$

where

$$\alpha_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

Theorem

Let f be continuous and piecewise smooth on $[0, a]$ such that $f(0) = f(a) = 0$. Consider the Laplace equation with the boundary conditions

$$\begin{array}{ll} u_{xx} + u_{yy} = 0 & 0 < x < a, \quad 0 < y < b \\ u(0, y) = 0 = u(a, y) = 0 & 0 \leq y \leq b \\ u(x, 0) = f(x) & 0 \leq x \leq a \\ u(x, b) = 0 & \end{array}$$

The solution to the above problem is given by

$$u(x, t) = \sum_{n \geq 1} \alpha_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b - y)}{a}\right) \Big/ \sinh\left(\frac{n\pi b}{a}\right),$$

where

$$\alpha_n = \frac{2}{L} \int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$$

Dirichlet boundary condition: Example

Example

Consider the Laplace equation with boundary conditions given by

$$\begin{aligned}
&u_{xx} + u_{yy} = 0 && 0 < x < a, \quad 0 < y < b \\
&u(0, y) = 0 = u(a, y) = 0 && 0 \leq y \leq b \\
&u(x, 0) = \sin\left(\frac{5\pi x}{a}\right) - 3\sin\left(\frac{9\pi x}{a}\right) && 0 \leq x \leq a \\
&u(x, b) = 0
\end{aligned}$$

Since f is given by its Fourier series in the above example, it is clear that

$$\alpha_5 = 1$$

$$\alpha_9 = -3$$

Dirichlet boundary condition: Example

Example (continued)

Thus, the solution to the above problem is given by

$$\begin{aligned} u(x,t) = {} & \sin\left(\frac{5\pi x}{a}\right)\sinh\left(\frac{5\pi(b-y)}{a}\right)\Big/\sinh\left(\frac{5\pi b}{a}\right) \\ & - 3\sin\left(\frac{9\pi x}{a}\right)\sinh\left(\frac{9\pi(b-y)}{a}\right)\Big/\sinh\left(\frac{9\pi b}{a}\right) \end{aligned}$$

Consider the following differential equation

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < a,\ 0 < y < b,$$

called the Laplace equation in two variables.

Consider the boundary conditions

$$\begin{array}{lll} u(x,0) = f(x) & u(x,b) = 0 & 0 \le x \le a \\ u_x(0,y) = 0 & u_x(a,y) = 0 & 0 \le y \le b \end{array}$$

Let $u(x,y) = X(x)Y(y)$. Then the differential equation becomes

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

Neumann boundary conditions: Finding some solutions

Thus, we have

$$\frac{-X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} = \text{constant}$$

Since $u_x(0,y) = X'(0)Y(y) = 0$, $u_x(a,y) = X'(a)Y(y) = 0$ and we do not want Y to be identically zero, we get that $X'(0) = 0$ and $X'(a) = 0$.

This boundary condition on X forces that the constant above should be positive. Let us denote this positive constant by λ^2.

For every $n \geq 0$, let

$$\lambda_n = \frac{n\pi}{a}$$

For each $n \geq 0$, we have a solution to

$$X''(x) + \lambda_n^2 X(x) = 0$$
$$X'(0) = 0 = X'(a)$$

given by

$$X_n(x) = \cos\left(\frac{n\pi x}{a}\right)$$

Since we do not want $X(x)$ to be identically 0 and $u(x, b) = X(x)Y(b) = 0$, this forces that $Y(b) = 0$. Let us also impose the condition that $Y(0) = 1$.

Next consider for each λ_n the problem

$$Y''(y) - \lambda_n^2 Y(y) = 0$$
$$Y(0) = 1$$
$$Y(b) = 0$$

Neumann boundary conditions: Finding some solutions

The solutions to the above equation are given by

For $n \geq 0$

$$Y_0(y) = \frac{-1}{b} y + 1$$

and for $n \geq 1$

$$Y_n(y) = \sinh\left(\frac{n\pi(b-y)}{a}\right) \Big/ \sinh\left(\frac{n\pi b}{a}\right).$$

Thus, for each $n \geq 0$ we get a solution

$$u_n(x,y) = \cos\left(\frac{n\pi x}{a}\right) Y_n(y)$$

Now consider the series

$$u(x,y) = \sum_{n \geq 0} \alpha_n \cos\left(\frac{n\pi x}{a}\right) Y_n(y),$$

where α_n are real numbers.

This gives that

$$u(x,0) = f(x) = \alpha_0 + \sum_{n\geq 1} \alpha_n \cos\left(\frac{n\pi x}{a}\right),$$

Thus, if $f(x)$ has the Fourier expansion

$$f(x) = \alpha_0 + \sum_{n\geq 1} \alpha_n \cos\left(\frac{n\pi x}{a}\right)$$

then we will have solved our Laplace equation with the given boundary conditions.

Neumann boundary conditions: Formal solution

Definition

Consider the Laplace equation with the boundary conditions

$$\begin{array}{ll} u_{xx} + u_{yy} = 0 & 0 < x < a, \quad 0 < y < b \\ u_x(0, y) = 0 = u_x(a, y) = 0 & 0 \leq y \leq b \\ u(x, 0) = f(x) & 0 \leq x \leq a \\ u(x, b) = 0 & 0 \leq x \leq a \end{array}$$

The formal solution of the above problem is

$$u(x, y) = \alpha_0 \Big(\frac{-1}{b} y + 1 \Big) + \sum_{n \geq 1} \alpha_n \cos\Big(\frac{n\pi x}{a}\Big) \sinh\Big(\frac{n\pi(b-y)}{a}\Big) \Big/ \sinh\Big(\frac{n\pi b}{a}\Big),$$

where

$$\alpha_0 = \frac{1}{L} \int_0^L f(x)\, dx \qquad \alpha_n = \frac{2}{L} \int_0^L f(x) \cos\frac{n\pi x}{L}\, dx$$

Neumann boundary conditions: Actual solution

Theorem

Let f be continuous and piecewise smooth on $[0,a]$.
Consider the Laplace equation with the boundary conditions

$$\begin{array}{ll} u_{xx}+u_{yy}=0 & 0<x<a, \quad 0<y<b \\ u_x(0,y)=0=u_x(a,y)=0 & 0\le y\le b \\ u(x,0)=f(x) & 0\le x\le a \\ u(x,b)=0 & 0\le x\le a \end{array}$$

The solution to the above problem is given by

$$u(x,y)=\alpha_0\Big(\frac{-1}{b}y+1\Big)+ \sum_{n\ge 1}\alpha_n\cos\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi(b-y)}{a}\right)\Big/\sinh\left(\frac{n\pi b}{a}\right),$$

where

$$\alpha_0=\frac{1}{L}\int_0^L f(x)\,dx \qquad \alpha_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

Example

Example

Consider the Laplace equation with boundary conditions given by

$$\begin{aligned}
&u_{xx} + u_{yy} = 0 && 0 < x < a, \quad 0 < y < b \\
&u_x(0, y) = 0 = u_x(a, y) = 0 && 0 \le y \le b \\
&u(x, 0) = \cos\left(\frac{5\pi x}{a}\right) - 3\cos\left(\frac{9\pi x}{a}\right) && 0 \le x \le a \\
&u(x, b) = 0
\end{aligned}$$

Since f is given by its Fourier series in the above example, it is clear that

$$\alpha_5 = 1$$
$$\alpha_9 = -3$$

Example (continued)

Thus, the solution to the above problem is given by

$$
\begin{aligned}
u(x,t) = &\cos\left(\frac{5\pi x}{a}\right)\sinh\left(\frac{5\pi(b-y)}{a}\right)\Big/\sinh\left(\frac{5\pi b}{a}\right) \\
&- 3\cos\left(\frac{9\pi x}{a}\right)\sinh\left(\frac{9\pi(b-y)}{a}\right)\Big/\sinh\left(\frac{9\pi b}{a}\right)
\end{aligned}
$$

Laplace equation in polar coordinates

Consider the Dirichlet problem in a disc of radius r

$$u_{xx} + u_{yy} = 0$$

with

$$u = f$$

on the boundary of the disc, which is a circle of radius r.
To solve this problem write the Laplace operator in polar coordinates.

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Laplace equation in polar coordinates

Example. Solve for harmonic function $u(r, \theta)$ in unit disc i.e.

$$\Delta u(r, \theta) = 0, \quad r < 1, \; \theta \in [0, 2\pi]$$

with boundary condition

$$u(1, \theta) = f(\theta) = \begin{cases} \sin\theta, & \theta \in [0, \pi] \\ 0, & \theta \in [\pi, 2\pi] \end{cases}$$

Laplace equation in polar coordinates is

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

Assume $u(r, \theta) = R(r)\Theta(\theta)$. Then

$$R''(r)\Theta(\theta) + \frac{1}{r} R'(r)\Theta(\theta) + \frac{1}{r^2} R(r)\Theta''(\theta) = 0$$

$$\frac{R''(r) + \frac{1}{r}R'(r)}{\frac{1}{r^2}R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda$$

$\boxed{\Theta''(\theta) + \lambda\Theta(\theta) = 0}$, $\boxed{r^2 R''(r) + rR'(r) - \lambda R(r) = 0}$

Since $u(r, \theta + 2\pi) = u(r, \theta)$, the functions Θ and Θ' need to be 2π periodic.

Thus for the ODE for Θ, we need to solve

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0, \quad \Theta(0) = \Theta(2\pi), \quad \Theta'(0) = \Theta'(2\pi)$$

The eigenvalues and eigenfunctions for periodic eigenvalue problem in Θ are

$$\lambda_0 = 0, \quad \Theta_0 = 1$$

and for $n \geq 1$,

$$\lambda_n = n^2, \quad \Theta_{n,1}(\theta) = \cos(n\theta), \quad \Theta_{n,2}(\theta) = \sin(n\theta)$$

The problem for R-function, namely

$$r^2 R''(r) + rR'(r) - \lambda R(r) = 0$$

is Cauchy-Euler equation with solution x^m, where

$$m(m-1) + m - \lambda = m^2 - \lambda = 0$$

$$\implies m = \pm\sqrt{\lambda}$$

For $\lambda = \lambda_0 = 0$, the general solutions are

$$R_{0,1}(r) = 1, \quad R_{0,2}(r) = \ln r$$

For $\lambda = \lambda_n = n^2 > 0$, $m = \pm n$, the general solutions are

$$R_{n,1}(r) = r^n, \quad R_{n,2}(r) = r^{-n}$$

Let us look for a solution of the Laplace equation in the disc which is a linear combinations of

$$1,\ \ln r,\ r^n \cos(n\theta),\ r^n \sin(n\theta),\ r^{-n} \cos(n\theta),\ r^{-n} \sin(n\theta)$$

Since we are looking for solutions that are bounded in the disc, we will discard $\ln r$, $r^{-n} \cos(n\theta)$ and $r^{-n} \sin(n\theta)$.
Thus, the series solution has the form

$$u(r, \theta) = A_0 + \sum_{n \geq 1} \left(A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta)\right)$$

The boundary condition is

$$u(1,\theta) = f(\theta) = A_0 + \sum_{n\geq 1}(A_n \cos(n\theta) + B_n \sin(n\theta))$$

Hence, A_i and B_i are Fourier coefficients of $f(\theta)$.
Check that the Fourier series of $f(\theta)$ is

$$f(\theta) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n\geq 1}\frac{\cos(2n\theta)}{4n^2-1} + \frac{1}{2}\sin\theta$$

Therefore, the solution is

$$u(r,\theta) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n\geq 1}\frac{1}{4n^2-1}r^{2n}\cos(2n\theta) + \frac{1}{2}r\sin\theta$$

Example. Solve for harmonic function $u(r,\theta)$ in an annulus

$$\begin{aligned} \Delta u(r,\theta) = 0, \qquad 1 < r < 2,\ \theta \in [0, 2\pi] \\ u(1,\theta) = \cos\theta, \quad 0 \le \theta \le 2\pi \\ u_r(2,\theta) = \sin 2\theta, \quad 0 \le \theta \le 2\pi \end{aligned}$$

This BVP can be interpreted as that for the steady state temperature distribution in an annular region where on the outer boundary the heat flux is prescribed and on the inner boundary, the temperature is prescribed.

Recall that the Laplace equation in polar coordinates is

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

As the polar coordinates (r, θ) and $(r, \theta + 2\pi)$ represent the same point in the plane, any function u defined in the plane is 2π-periodic in θ. Therefore,

$$u(r, 0) = u(r, 2\pi), \quad u_r(r, 0) = u_r(r, 2\pi)$$

Assume $u(r, \theta) = R(r)\Theta(\theta)$. Then

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$

$$\frac{R''(r) + \frac{1}{r}R'(r)}{\frac{1}{r^2}R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda$$

$\boxed{\Theta''(\theta) + \lambda\Theta(\theta) = 0}$, $\boxed{r^2 R''(r) + rR'(r) - \lambda R(r) = 0}$

Since $u(r, \theta + 2\pi) = u(r, \theta)$, the functions Θ and Θ' need to be 2π periodic.

Thus for the ODE for Θ, we need to solve

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0, \quad \Theta(0) = \Theta(2\pi), \quad \Theta'(0) = \Theta'(2\pi)$$

The eigenvalues and eigenfunctions for periodic eigenvalue problem in Θ are

$$\lambda_0 = 0, \quad \Theta_0 = 1$$

and for $n \geq 1$,

$$\lambda_n = n^2, \quad \Theta_{n,1}(\theta) = \cos(n\theta), \quad \Theta_{n,1}(\theta) = \sin(n\theta)$$

The problem for R-function, namely

$$r^2 R''(r) + rR'(r) - \lambda R(r) = 0$$

is Cauchy-Euler equation with solution x^m, where

$$m(m-1) + m - \lambda = m^2 - \lambda = 0$$

$\implies m = \pm\sqrt{\lambda}$ For $\lambda = \lambda_0 = 0$, the general solutions are

$$R_{0,1}(r) = 1, \quad R_{0,2}(r) = \ln r, \quad u_0(r,\theta) = A_0 + B_0 \ln r$$

For $\lambda = \lambda_n = n^2 > 0$, $m = \pm n$, the general solutions are

$$R_{n,1}(r) = r^n, \quad R_{n,2}(r) = r^{-n}$$

Let us look for a solution of the Laplace equation in the disc which is a linear combinations of

$$1, \ \ln r, \ r^n \cos(n\theta), \ r^n \sin(n\theta), \ r^{-n} \cos(n\theta), \ r^{-n} \sin(n\theta)$$

Hence the general solution is

$$u(r,\theta) = (A_0 + B_0 \ln r) + \sum_{n\geq 1} (A_n r^n \cos(n\theta) + B_n n r^{-n} \cos(n\theta)) + \sum_{n\geq 1} (C_n r^n \sin(n\theta) + D_n r^{-n} \sin(n\theta))$$

Since

$$u(1,\theta) = \cos\theta, \quad u_r(2,\theta) = \sin 2\theta$$

$$u(1,\theta) = A_0 + \sum_{n\geq 1}(A_n + B_n)\cos(n\theta) + (C_n + D_n)\sin(n\theta)$$

Compare with $u(1,\theta) = \cos\theta$, we get $\quad A_0 = 0,$

$$A_1 + B_1 = 1,\ A_n + B_n = 0\ (n \geq 2),\ C_n + D_n = 0\ (n \geq 1)$$

$$u_r(r,\theta) = \frac{B_0}{r} + \sum_{n\geq 1} n(A_n r^{n-1} - B_n r^{-n-1})\cos n\theta$$

$$+n(C_n r^{n-1} - D_n r^{-n-1})\sin n\theta$$

Compare with $u_r(2,\theta) = \sin 2\theta$, we get $\ B_0 = 0,$
$2(2C_2 - 2^{-3}D_2) = 1$

$$A_n 2^{n-1} - B_n 2^{-n-1} = 0\ (n \geq 1),\quad C_n 2^{n-1} - D_n 2^{-n-1} = 0\ (n \neq 2)$$

$$A_0 = 0 = B_0$$

For $n = 1$

$$A_1 + B_1 = 1,\ A_1 - B_1 2^{-2} = 0 \implies A_1 = \frac{1}{5}, B_1 = \frac{4}{5}$$

$$C_1 + D_1 = 0, C_1 - D_1 2^{-2} = 0 \implies C_1 = 0, D_1 = 0$$

For $n = 2$,

$$A_2 + B_2 = 0,\ A_2 2 - B_2 2^{-3} = 0 \implies A_2 = 0 = B_2$$

$$C_2 + D_2 = 0,\ 2C_2 - \frac{1}{2^3} D_2 = \frac{1}{2} \implies C_2 = \frac{4}{17}, D_2 = \frac{-4}{17}$$

For $n > 2$,

$$A_n + B_n = 0,\ A_n 2^{n-1} - B_n 2^{-n-1} = 0 \implies A_n^1 = 0 = B_n^1$$

$$C_n + D_n = 0,\ C_n 2^{n-1} - D_n 2^{-n-1} = 0 \implies C_n = 0 = D_n$$

Thus the solution is

$$u(r,\theta) = (\frac{1}{5}r + \frac{4}{5}r^{-1})\cos\theta + (\frac{4}{17}r^2 + \frac{-4}{17}r^{-2})\sin 2\theta$$