Recall

Last time we saw some examples of computing improper integrals using residue calculus. We ended the lecture with the proof of the maximum modulus theorem which stated that a non-constant holomorphic function on a domain never attains its maximum modulus at any point in the domain. I commented that this fails for real analytic functions. The maximum modulus theorem has a nice consequence, namely Schartz lemma. This theorem once again emphasizes the rigid the nature of holomorphic functions.

Schwartz Lemma

<u>**Schwarz Lemma :**</u> Let $\mathbb{D} = \{z : |z| < 1\}$ be the open unit disk and let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic map such that $f(0) = 0$ and $|f(z)| \leq 1$ on $\mathbb{D}$.
Then, $|f(z)| \leq |z|\ \forall z \in \mathbb{D}$ and $|f'(0)| \leq 1$.
Moreover, if $|f(z)| = |z|$ for some non-zero z or $|f'(0)| = 1$, then $f(z) = az$ for some $a \in \mathbb{C}$ with $|a| = 1$.

Proof

Let $g(z) = \begin{cases} \frac{f(z)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0, \end{cases}.$

Proof

Let $g(z) = \begin{cases} \frac{f(z)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0, \end{cases}$.

Then $g(z)$ is holomorphic on the whole of $\mathbb{D}$. Now if $D_r = \{z : |z| \leq r\}$ denotes the closed disk of radius r centered at the origin, then the maximum modulus principle implies that, for $r < 1$, given any z in D_r, there exists z_r on the boundary of D_r such that

Proof

Let $g(z) = \begin{cases} \frac{f(z)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0, \end{cases}$.
Then $g(z)$ is holomorphic on the whole of $\mathbb{D}$. Now if $D_r = \{z : |z| \leq r\}$ denotes the closed disk of radius r centered at the origin, then the maximum modulus principle implies that, for $r < 1$, given any z in D_r, there exists z_r on the boundary of D_r such that

$$|g(z)| \leq |g(z_r)| = \frac{|f(z_r)|}{|z_r|} \leq \frac{1}{r}.$$

As $r \to 1$ we get $|g(z)| \leq 1$.

Proof

Let $g(z) = \begin{cases} \frac{f(z)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0, \end{cases}$.

Then $g(z)$ is holomorphic on the whole of $\mathbb{D}$. Now if $D_r = \{z : |z| \leq r\}$ denotes the closed disk of radius r centered at the origin, then the maximum modulus principle implies that, for $r < 1$, given any z in D_r, there exists z_r on the boundary of D_r such that

$$|g(z)| \leq |g(z_r)| = \frac{|f(z_r)|}{|z_r|} \leq \frac{1}{r}.$$

As $r \to 1$ we get $|g(z)| \leq 1$.

Moreover, suppose $|f(z)| = |z|$ for some non-zero z in $\mathbb{D}$, or $|f'(0)| = 1$.

Proof

Let $g(z) = \begin{cases} \frac{f(z)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0, \end{cases}$.

Then $g(z)$ is holomorphic on the whole of $\mathbb{D}$. Now if $D_r = \{z : |z| \leq r\}$ denotes the closed disk of radius r centered at the origin, then the maximum modulus principle implies that, for $r < 1$, given any z in D_r, there exists z_r on the boundary of D_r such that

$$|g(z)| \leq |g(z_r)| = \frac{|f(z_r)|}{|z_r|} \leq \frac{1}{r}.$$

As $r \to 1$ we get $|g(z)| \leq 1$.

Moreover, suppose $|f(z)| = |z|$ for some non-zero z in $\mathbb{D}$, or $|f'(0)| = 1$.Then, $|g(z)| = 1$ at some point of $\mathbb{D}$. Hence by Maximum Modulus Principle, $g(z)$ is a constant , say a with $|a| = 1$. Therefore, $f(z) = az$, as desired.

Open Mapping Theorem

The maximum modulus theorem is a special case of a even more powerful theorem called the Open Mapping Theorem.

Theorem: Any non-constant holomorphic function defined on a domain $\Omega \subseteq \mathbb{C}$ is open; i.e, maps open subsets of $\mathbb{C}$ contained in Ω to open subsets of $\mathbb{C}$.

Open Mapping Theorem

The maximum modulus theorem is a special case of a even more powerful theorem called the Open Mapping Theorem.

Theorem: Any non-constant holomorphic function defined on a domain $\Omega \subseteq \mathbb{C}$ is open; i.e, maps open subsets of $\mathbb{C}$ contained in Ω to open subsets of $\mathbb{C}$.

The theorem has an interesting proof which unfortunately we will skip due to lack of time. One of the striking applications of this theorem is another proof of the fundamental theorem of algebra. I will list the key steps here without going into details :
1. Let $f(z)$ be a non-constant polynomial. Then $f(z)$ is a proper map and hence (by problem 1 of the extra problems sheet) a closed map (takes closed sets to closed sets).
2. By open mapping theorem, it is also an open map.

Proof cont..

3. Thus in particular the image of $\mathbb{C}$ under $f(z)$ is both open and closed. Now recall from lecture 1 that the only subsets of $\mathbb{C}$ that are both open and closed are $\emptyset$ and $\mathbb{C}$. Hence image of $f(z)$ is all of $\mathbb{C}$ thereby proving the FTA.

Here is an interesting fact whose content suggests that complex analytic functions are not THAT rigid ! :
Mittag-Leffler's Theorem : Given any discrete sequence of points going to infinity, there exists a meromorphic functions with poles exactly along this sequence and having prescribed principal parts at those poles.

In the mathematical field of complex analysis, a meromorphic function on an open subset D of the complex plane is a function that is holomorphic on all of D except for a set of isolated points, which are poles of the function

Examples

Show that $\int_0^\infty \frac{x^{-c}}{1+x}dx = \frac{\pi}{\sin \pi c}$ if $0 < c < 1$.

Show that $\int_0^\infty \frac{x^{-c}}{1+x}dx = \frac{\pi}{\sin \pi c}$ if $0 < c < 1$.
We'll integrate

$$f(z) = \frac{z^{-c}}{1+z},$$

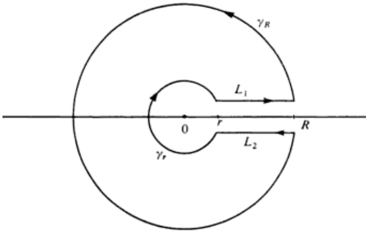
where z^{-c} is the branch corresponding to branch cut being the positive real axis, and integrate along the contour:

Examples

Show that $\int_0^\infty \frac{x^{-c}}{1+x}dx = \frac{\pi}{\sin \pi c}$ if $0 < c < 1$.
We'll integrate

$$f(z) = \frac{z^{-c}}{1+z},$$

where z^{-c} is the branch corresponding to branch cut being the positive real axis, and integrate along the contour:



By residue theorem,

$$\int_\gamma \frac{z^{-c}}{1+z} dz = 2\pi\imath e^{-\imath\pi c}.$$

By residue theorem,

$$\int_\gamma \frac{z^{-c}}{1+z}dz = 2\pi\imath e^{-\imath\pi c}.$$

The integral is the sum of four integrals; one on L_1, one on γ_R, one on L_2, one on γ_r.

By residue theorem,

$$\int_{\gamma} \frac{z^{-c}}{1+z}dz = 2\pi\imath e^{-\imath\pi c}.$$

The integral is the sum of four integrals; one on L_1, one on γ_R, one on L_2, one on γ_r. Note that

$$\int_r^R \frac{t^{-c}}{1+t}dt = \lim_{\delta\to 0}\int_{L_1} \frac{z^{-c}}{1+z}dz.$$

Similarly,

$$\lim_{\delta\to 0}\int_{L_2} \frac{z^{-c}}{1+z}dz = -e^{-2\pi\imath c}\int_r^R \frac{t^{-c}}{1+t}dt.$$

Also,

$$\left|\int_{\gamma_\rho} \frac{z^{-c}}{1+z}dz\right| \leq \frac{\rho^{-c}}{|1-\rho|}2\pi\rho.$$

Real Integrals

Also,

$$\left| \int_{\gamma_\rho} \frac{z^{-c}}{1+z} dz \right| \leq \frac{\rho^{-c}}{|1-\rho|} 2\pi\rho.$$

This is zero in the limit as $\rho \to 0$ or $\rho \to \infty$. Thus we get:

$$2\pi \imath e^{-\imath\pi c} = (1 - e^{-2\imath\pi c}) \int_0^\infty \frac{t^{-c}}{1+t} dt.$$

Also,

$$\left| \int_{\gamma_\rho} \frac{z^{-c}}{1+z} dz \right| \leq \frac{\rho^{-c}}{|1-\rho|} 2\pi\rho.$$

This is zero in the limit as $\rho \to 0$ or $\rho \to \infty$. Thus we get:

$$2\pi\imath e^{-\imath\pi c} = (1 - e^{-2\imath\pi c}) \int_0^\infty \frac{t^{-c}}{1+t} dt.$$

Thus,

$$\int_0^\infty \frac{t^{-c}}{1+t} dt = \frac{2\pi\imath e^{-\imath\pi c}}{1 - e^{-2\imath\pi c}} = \frac{\pi}{\sin \pi c}.$$

Real Integral

Integrate $I = \int_{-\infty}^{\infty} \frac{e^{x/2} dx}{coshx}$

Real Integral

Integrate $I = \int_{-\infty}^{\infty} \frac{e^{x/2}dx}{coshx}$

In this case coshx has infinitely many poles along the imaginary axis, namely at $z = i(\pi/2 + n\pi), n \in \mathbb{Z}$ and so we do not choose the previous kind of contours. Instead we choose a rectangular contour γ consisting of vertices $L, -L, L + i\pi$ and $-L + i\pi$.

Real Integral

Integrate $I = \int_{-\infty}^{\infty} \frac{e^{x/2}dx}{coshx}$

In this case coshx has infinitely many poles along the imaginary axis, namely at $z = i(\pi/2 + n\pi), n \in \mathbb{Z}$ and so we do not choose the previous kind of contours. Instead we choose a rectangular contour γ consisting of vertices $L, -L, L + i\pi$ and $-L + i\pi$.

By residue theorem, $\int_{\gamma} \frac{e^{z/2}dx}{coshz} = 2\pi i Res(f, i\frac{\pi}{2}) = 2\pi e^{i\frac{\pi}{4}}$.

Integrate $I = \int_{-\infty}^{\infty} \frac{e^{x/2}dx}{coshx}$

In this case coshx has infinitely many poles along the imaginary axis, namely at $z = i(\pi/2 + n\pi), n \in \mathbb{Z}$ and so we do not choose the previous kind of contours. Instead we choose a rectangular contour γ consisting of vertices $L, -L, L + i\pi$ and $-L + i\pi$.

By residue theorem, $\int_\gamma \frac{e^{z/2}dx}{coshz} = 2\pi i Res(f, i\frac{\pi}{2}) = 2\pi e^{i\frac{\pi}{4}}$.
Now $|cosh(L + iy)| = |e^{L+iy} + e^{-L-iy}|/2 \geq \frac{1}{2}(|e^{L+iy}| - |e^{-L-iy}|) = (e^L - e^{-L})/2 \geq e^L/4$

Real Integral

Integrate $I = \int_{-\infty}^{\infty} \frac{e^{x/2}dx}{coshx}$

In this case coshx has infinitely many poles along the imaginary axis, namely at $z = i(\pi/2 + n\pi), n \in \mathbb{Z}$ and so we do not choose the previous kind of contours. Instead we choose a rectangular contour γ consisting of vertices $L, -L, L + i\pi$ and $-L + i\pi$.

By residue theorem, $\int_\gamma \frac{e^{z/2}dx}{coshz} = 2\pi i Res(f, i\frac{\pi}{2}) = 2\pi e^{i\frac{\pi}{4}}$.
Now $|cosh(L + iy)| = |e^{L+iy} + e^{-L-iy}|/2 \geq \frac{1}{2}(|e^{L+iy}| - |e^{-L-iy}|) = (e^L - e^{-L})/2 \geq e^L/4$
From this it follows from the ML-inequality that as L tends to ∞, the integral along the right vertical side tends to zero. Similarly one checks that the integral along the left vertical side also tend to zero.

Example cont ..

Now since $cosh(x + i\pi) = -coshx$, the integrals along the horizontal sides are related by

$$\int_{L}^{-L} \frac{e^{(x+i\pi)/2}dx}{cosh(x + i\pi)} = e^{i\pi/2} \int_{-L}^{L} \frac{e^{x^2}dx}{coshx}$$

Example cont ..

Now since $cosh(x + i\pi) = -coshx$, the integrals along the horizontal sides are related by

$$\int_{L}^{-L} \frac{e^{(x+i\pi)/2}dx}{cosh(x + i\pi)} = e^{i\pi/2} \int_{-L}^{L} \frac{e^{x^2}dx}{coshx}$$

Taking L tending to ∞ , we see that
$I = \frac{2\pi e^{i\pi/4}}{(1+e^{i\pi/2})} = \frac{\pi}{cos(\pi/4)} = \pi\sqrt{2}.$