MA 205 Complex Analysis: Power Series

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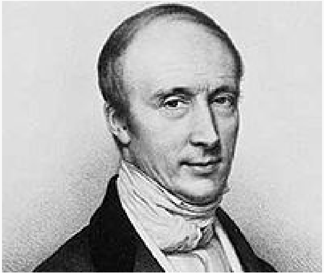
In the last lecture, you saw holomorphic functions in some detail. If $f = u + \imath v$ is holomorphic in Ω, then (i) both u and v satisfy CR equations, and (ii) $f(x, y) = (u(x, y), v(x, y))$ is real differentiable. You also saw that though neither (i) nor (ii) is sufficient to guarantee holomorphicity, both (i) and (ii) together do guarantee holomorphicity of f. You also studied harmonic functions which are closely related to holomorphic functions. The notion of a harmonic conjugate of a given harmonic function was defined, and it was stated that the existence of a harmonic conjugate for every harmonic function is guaranteed if and only if the domain is "simply connected". A proof of this fact, we'll see later.

Questions?

Cauchy (1789-1857) & Riemann (1826-1866); Wiki

Augustin-Louis Cauchy



Bernhard Riemann

"More concepts and theorems have been named for Cauchy than for any other mathematician (in elasticity alone there are sixteen concepts and theorems named for Cauchy)."

"When Riemann was 33, he developed the famous Riemann hypothesis. This was an article that was only 8 pages long. Mathematicians ever since have struggled to try to prove what Riemann wrote."

Laplace (1749-1827); Wiki

"Geometrician of the first rank, Laplace was not long in showing himself a worse than average administrator; from his first actions in office we recognized our mistake. Laplace did not consider any question from the right angle: he sought subtleties everywhere, conceived only problems, and finally carried the spirit of "infinitesimals" into the administration." (Napoleon, 1799)

Polynomials

Today, we'll discuss the so called analytic functions. To warm up, let's first look at the simplest of all functions. What's the most trivial function? $f(z) = a_0$, a constant. Then, may be various powers of z; easily differentiable, integrable etc. Thus the simplest class of functions is polynomials: $f(z) = a_0 + a_1 z + \ldots + a_n z^n$, $a_i \in \mathbb{C}$. These are clearly holomorphic everywhere in $\mathbb{C}$. The same polynomial $f(z)$ can be expanded along any point z_0. What do I mean by this? $f(z)$ can be written as

$$b_0 + b_1(z - z_0) + \ldots + b_n(z - z_0)^n.$$

The complex numbers b_i can be easily calculated. Of course, you could expand powers of $(z - z_0)$ using binomial theorem and compare coefficients to get b_i in terms of a_i. Perhaps a smarter way would be to notice that $b_i = \frac{f^{(i)}(z_0)}{i!}$, which then can be calculated from $f(z) = \sum_{i=0}^{n} a_i z^i$.

A polynomial, by definition, is a finite polynomial; i.e., it comes with a finite degree. As the next simplest class of functions, why can't we consider infinite polynomials? It'll look like:

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots,$$

or more generally,

$$\sum_{i=0}^{\infty} a_i (z - z_0)^i.$$

Of course no one calls this a polynomial; it's called a power series. You'll agree that now we need to be a bit careful. There can be convergence questions. For example, $f(z) = 1 + z + z^2 + \ldots$ makes sense for all z such that $|z| < 1$, but not when $|z| > 1$. (Why?) We say that this power series has radius of convergence $= 1$.

It's a beautiful fact that the radius of convergence exists for any power series; i.e., there exists R such that $\sum_{i=0}^{\infty} a_i(z-z_0)^i$ converges when $|z-z_0| < R$, and diverges when $|z-z_0| > R$. In other words, the radius of convergence is the largest R such that the given power series converges inside a disc of radius R. We'll soon give a formula for R in terms of the coefficients of the given power series.

Before we prove the existence of radius of convergence, let's recall a few definitions and observations.

We write $a = \sum_{n=1}^{\infty} a_i$, $a_i \in \mathbb{C}$, if the sequence of partial sums $\{s_n\}$, where $s_n = a_1 + \ldots + a_n$, converges, and $\lim_{n \to \infty} s_i = a$. The series $\sum_{n=1}^{\infty} a_i$ is said be absolutely convergent if $\sum_{n=1}^{\infty} |a_i|$ is convergent.

Exercise:

1. Absolute convergence $\implies$ convergence.

2. (Comparison Test) If $\sum_{n=1}^{\infty} b_i$ is absolutely convergent, and if $|a_i| \leq |b_i|$ for all large enough i, then $\sum_{n=1}^{\infty} a_i$ is absolutely convergent.

Recall upper limit: for a sequence of real numbers $x_1, x_2, \ldots$, let y_n be the supremum of the set $\{x_n, x_{n+1}, \ldots\}$. Then the sequence $y_1, y_2, \ldots$ either diverges to ∞ or has a finite limit. This is called the upper limit of the sequence $\{x_i\}$. It can be ∞. If limit exists, then the upper limit coincides with the usual limit.

Examples:

1. the sequence $1, 2, 3, \ldots$ has upper limit ∞.
2. the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$ has upper limit 0.
3. the sequence $1, -1, 1, -1, \ldots$ has upper limit 1.

Theorem (Cauchy's Root Test)

For a series $\sum_{n=1}^{\infty} a_i$, *let* $C = \limsup_{i \to \infty} \sqrt[i]{|a_i|}$. *Then the series converges absolutely if* $C < 1$ *and it diverges if* $C > 1$.

Proof: If $C < 1$, then we can choose a k such that $\sqrt[i]{|a_i|} < k < 1$ after a stage (by the definition of the upper limit). Thus, after a stage, $|a_i| < k^i < 1$. Now $\sum_{n=1}^{\infty} k^i$ converges absolutely, and therefore $\sum_{n=1}^{\infty} a_i$ is absolutely convergent (by comparison test). If $C > 1$, then for infinitely many i, $\sqrt[i]{|a_i|} > 1$. Hence $|a_i|$ is bigger than 1 for infinitely many i. Thus, $\lim_{i \to \infty} a_i \neq 0$. So $\sum_{n=1}^{\infty} a_i$ diverges. (Why?)

Theorem (Ratio Test)

For a series $\sum_{n=1}^{\infty} a_i$*, let* $L = \limsup_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right|$*. Then, if* $L < 1$*, the series converges absolutely. The series diverges if there exists* N *such that* $\left|\frac{a_{i+1}}{a_i}\right| > 1$ *for* $i \geq N$*.*

Proof: Let $L < 1$. Let r be such that $L < r < 1$. Then after a stage, say for $i \geq N$, $|a_{i+1}| < r|a_i|$. So $|a_{i+k}| < r^k|a_i|$. Now

$$\sum_{n=1}^{\infty}|a_i| = \sum_{0}^{N}|a_i| + \sum_{N+1}^{\infty}|a_i| = \sum_{0}^{N}|a_i| + \sum_{1}^{\infty}|a_{N+i}|$$

$$< \sum_{0}^{N}|a_i| + |a_N|\sum_{1}^{\infty} r^i = \sum_{0}^{N}|a_i| + |a_N|\frac{r}{1-r} < \infty.$$

In the other case, $|a_{i+1}| > |a_i|$ for all large enough i, so $\lim a_i \neq 0$. Therefore the series diverges.

Theorem (Existence of Radius of Convergence)

For the power series $\sum_{n=1}^{\infty} a_i(z-z_0)^i$*, let* $R = \dfrac{1}{\limsup_{i\to\infty} \sqrt[i]{|a_i|}}$*. Then the power series converges absolutely if* $|z-z_0| < R$ *and diverges if* $|z-z_0| > R$*.*

Proof: Apply root test.

If $\lim_{i\to\infty} \left|\dfrac{a_{i+1}}{a_i}\right|$ exists, then by applying the ratio test instead of the root test, it follows that $R = \lim_{i\to\infty} \left|\dfrac{a_i}{a_{i+1}}\right|$.

Remark: If a series converges by the ratio test, then it converges by the root test as well. But not conversely. Thus the root test is better than the ratio test. But the ratio test is much easier to use whenever it succeeds.

In fact:

$$\limsup_{i\to\infty} \sqrt[i]{|a_i|} \leq \limsup_{i\to\infty} \left| \frac{a_{i+1}}{a_i} \right|$$

Examples:

1. $\sum_{n=1}^{\infty} \frac{z^i}{i!}$. Apply ratio test. $\lim_{i\to\infty} \left| \frac{a_i}{a_{i+1}} \right| = \lim_{i\to\infty} i = \infty$; i.e., the series converges everywhere.
2. $z - \frac{z^3}{3} + \frac{z^5}{5} - \ldots$ Radius of convergence is 1. Both the tests apply here.
3. $\frac{1}{2} + \frac{1}{3}z + \left(\frac{1}{2}\right)^2 z^2 + \left(\frac{1}{3}\right)^2 z^3 + \ldots$. Check that the ratio test fails. Apply root test to show that the radius of convergence is $\frac{1}{\sqrt{2}}$.

Power series can be added, subtracted, and multiplied in the obvious way. It can also be differentiated and integrated term by term, in its domain of convergence. Indeed, $f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$, then,

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \sum_n a_n \left(\lim_{h \to 0} \frac{[(z - z_0 + h)^n - (z - z_0)^n]}{h} \right)$$

$$= \sum_n n a_n (z - z_0)^{n-1}.$$

Similarly, for integration. Apply root test to check that a given power series, the differentiated series and the integrated series, all have the same radius of convergence.

Analytic Functions

A function $f : \Omega \to \mathbb{C}$ is said to be analytic if it is locally given by a convergent power series; i.e., every $z_0 \in \Omega$ has a neighbourhood contained in Ω such that there exists a power series centered at z_0 which converges to $f(z)$ for all z in that neighbourhood. Clearly, analytic functions are infinitely differentiable; you only have to differentiate the power series term by term. Also, if $f(z) = \sum_{n=1}^{\infty} a_i (z - z_0)^i$, then $a_i = \frac{f^{(i)}(z_0)}{i!}$. Thus, an analytic function is given by its Taylor series. We'll later prove:

holomorphic $\implies$ *analytic.*

This would prove our statement from Lecture 1 that once differentiable is always differentiable!

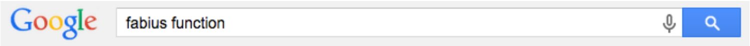
Analytic Functions

Just as in the complex case, power series and analytic functions can be defined in the real case too. But unlike in the complex case, differentiable does not mean real analytic. In fact, even infinitely differentiable does not mean real analytic. For example, $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0, \end{cases}$$

is infinitely differentiable but not real analytic. In this example, $f^{(i)}(0) = 0$ for all i, and thus the Taylor series of f is the zero function.

Build your GK!



Aside

By the way there was a beautiful article that appeared last Friday in The New York Times:

http://www.nytimes.com/2015/07/26/magazine/the-singular-mind-of-terry-tao.html

Have a look! Here's a quote:

"Long ago, mathematicians invented a number that when multiplied by itself equals negative 1, an idea that seemed to break the basic rules of multiplication. It was so far outside what mathematicians were doing at the time that they called it "imaginary". Yet imaginary numbers proved a powerful invention, and modern physics and engineering could not function without them."