Network Theory Homework 3

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Homework is not to be submitted. If you prepare a solution and upload it to Moodle, you may be eligible for extra credit, as per the course rules.

1. Define the following:
   (a) Homomorphism of graphs
   (b) Isomorphism of graphs
   (c) A cut in a graph
   (d) A tree in a graph
   (e) A spanning tree in a graph
   (f) An Abelian group
   (g) Homomorphism of groups
   (h) Isomorphism of groups
   (i) An injective function
   (j) A surjective function
   (k) A bijective function
   (l) Category
   (m) Initial object in a category
   (n) Terminal object in a category
   (o) Isomorphism of two objects in a category
   (p) Cartesian product of two objects in a category
   (q) Direct sum of two Abelian groups
2. Prove or disprove: The Cartesian product in a category is 'uniquely unique,' i.e., given two distinct Cartesian products (P, π_A, π_B) and (P', π'_A, π'_B) of objects A and B in a category $\mathcal{C}$, there exists a unique isomorphism $f : P \to P'$ such that $\pi'_A \circ f = \pi_A$ and $\pi'_B \circ f = \pi_B$.
3. Prove or disprove: In an Abelian group $(G, 0, +)$, if $g, h, k \in G$ are such that $g + h = h + k = 0$ then $g = k$.

4. An arrow f in a category is **right-cancellable** iff for all arrows h_1, h_2, if $h_1 \circ f = h_2 \circ f$ then $h_1 = h_2$. An arrow $f : A \to B$ in a category is **epi** (or an epimorphism) iff there exists an arrow $g : B \to A$ such that $f \circ g = 1_B$.

   Prove/ Disprove: A function (i.e., an arrow in the category of Sets) is left-cancellable iff it is epi iff it is surjective.

5. An arrow f in a category is **left-cancellable** iff for all arrows h_1, h_2, if $f \circ h_1 = f \circ h_2$ then $h_1 = h_2$. An arrow $f : A \to B$ in a category is **mono** (or a monomorphism) iff there exists an arrow $g : B \to A$ such that $g \circ f = 1_A$.

   Prove/ Disprove: A function (i.e., an arrow in the category of Sets) is left-cancellable iff it is mono iff it is injective.

6. Prove/ Disprove: Given $f : A \to B$, if there exist $g, h : B \to A$ such that $f \circ g = 1_B$ and $h \circ f = 1_A$ then f is an isomorphism and $g = h$. (Hint: Similar to Question 3).

7. Let $G = (N, E)$ be a connected, undirected graph. Let K be a cut of G and let T be a spanning tree of G.

   Prove/ Disprove: There exists an edge $e \in E$ that belongs to both K and T.