MA 205 Complex Analysis: Real Integrals

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We'll continue to work out some integrals using the Residue Theorem. Recall the theorem. We'll start with

Theorem (Jordan's Lemma)

Let f be a continuous function defined on the semicircular contour $C_R = \{Re^{\imath\theta} \mid \theta \in [0,\pi]\}$ of the form

$$f(z) = e^{\imath az} g(z),$$

with $a > 0$. Then,

$$\left| \int_{C_R} f(z)dz \right| \leq \frac{\pi}{a} \max_{\theta\in[0,\pi]} |g(Re^{\imath\theta})|.$$

Proof:

$$\int_{C_R} f(z)dz = \int_0^{\pi} g(Re^{\imath\theta})e^{\imath aR(\cos\theta+\imath\sin\theta)}\imath Re^{\imath\theta}d\theta.$$

Therefore,

$$\begin{aligned}
\left|\int_{C_R} f(z)dz\right| &\leq R\int_0^{\pi}\left|g(Re^{\imath\theta})e^{aR(\imath\cos\theta-\sin\theta)}\imath e^{\imath\theta}\right|d\theta \\
&= R\int_0^{\pi}\left|g(Re^{\imath\theta})\right|e^{-aR\sin\theta}d\theta \\
&\leq 2RM_R\int_0^{\frac{\pi}{2}} e^{-aR\sin\theta}d\theta \\
&\leq 2RM_R\int_0^{\frac{\pi}{2}} e^{\frac{-2aR\theta}{\pi}}d\theta = \frac{\pi}{a}(1-e^{-aR})M_R \leq \frac{\pi}{a}M_R,
\end{aligned}$$

since $\sin\theta \geq \frac{2\theta}{\pi}$ for $\theta \in [0, \frac{\pi}{2}]$.

1. Compute $\int_0^\infty \frac{\sin x}{x} dx$.
We'll consider the function

$$f(z) = \frac{e^{\imath z}}{z}.$$

Let γ be the boundary of the upper part of the annulus $A(0; r, R)$.
Then, $\int_\gamma f(z)dz = 0$, by Cauchy's theorem.

But,

$$\int_{\gamma} f(z)dz = \int_r^R \frac{e^{\imath x}}{x}dx + \int_{\gamma_R} \frac{e^{\imath z}}{z}dz + \int_{-R}^{-r} \frac{e^{\imath x}}{x}dx + \int_{\gamma_r} \frac{e^{\imath z}}{z}dz.$$

Now,

$$\begin{aligned}\int_r^R \frac{\sin x}{x}dx &= \frac{1}{2\imath}\int_r^R \frac{e^{\imath x} - e^{-\imath x}}{x}dx \\ &= \frac{1}{2\imath}\int_r^R \frac{e^{\imath x}}{x}dx + \frac{1}{2\imath}\int_{-R}^{-r} \frac{e^{\imath x}}{x}dx.\end{aligned}$$

Thus, we only need to compute

$$\lim_{R\to\infty}\int_{\gamma_R} \frac{e^{\imath z}}{z}dz \quad \& \quad \lim_{r\to 0}\int_{\gamma_r} \frac{e^{\imath z}}{z}dz.$$

Now,

$$\lim_{R \to \infty} \int_{\gamma_R} \frac{e^{\imath z}}{z} dz = 0,$$

by Jordan's lemma. (Why?) On the other hand, note that $\frac{e^{\imath z}-1}{z}$ has a removable singularity at 0. Thus, there is $M > 0$ such that

$$\left| \frac{e^{\imath z} - 1}{z} \right| \leq M,$$

for $|z| \leq 1$. Thus,

$$\lim_{r \to 0} \int_{\gamma_r} \frac{e^{\imath z} - 1}{z} dz = 0,$$

by appealing to ML inequality.

Therefore,

$$\lim_{r\to 0}\int_{\gamma_r}\frac{e^{\imath z}}{z}dz=-\pi\imath.$$

Thus,

$$\int_0^\infty \frac{\sin x}{x}dx=\frac{\pi}{2}.$$

2. Show that $\int_0^\infty \frac{\log x}{1+x^2}dx = 0$.
We'll work with γ as in the previous problem. We take

$$f(z) = \frac{\log z}{1+z^2},$$

where $\log z$ is a branch of the logarithm which is defined on the x-axis, so that $\int_r^R$ and $\int_{-R}^{-r}$ make sense. For instance, we can take the branch with negative y-axis as the branch cut. Then,

$$\log x = \begin{cases} \log x & \text{if } x > 0, \\ \log|x| + \imath\pi & \text{if } x < 0. \end{cases}$$

Now,

$$\int_\gamma \frac{\log z}{1+z^2}dz = \int_r^R \frac{\log x}{1+x^2}dx + \int_{\gamma_R} \frac{\log z}{1+z^2}dz + \int_{-R}^{-r} \frac{\log|x| + \imath\pi}{1+x^2}dx + \int_{\gamma_r} \frac{\log z}{1+z^2}dz.$$

The lhs is $2\pi\imath \cdot \mathrm{Res}(f;\imath)$ which is $2\pi\imath \cdot \frac{\log \imath}{2\imath} = \frac{\pi^2\imath}{2}$. Also,

$$\int_r^R \frac{\log x}{1+x^2}dx + \int_{-R}^{-r} \frac{\log|x| + \imath\pi}{1+x^2}dx = 2\int_r^R \frac{\log x}{1+x^2}dx + \imath\pi \int_r^R \frac{dx}{1+x}$$

$$= 2\int_r^R \frac{\log x}{1+x^2}dx + \frac{\pi^2\imath}{2}.$$

Thus,

$$\int_r^R \frac{\log x}{1+x^2}dx = -\frac{1}{2}\left[\int_{\gamma_R} \frac{\log z}{1+z^2}dz + \int_{\gamma_r} \frac{\log z}{1+z^2}dz\right].$$

Note that

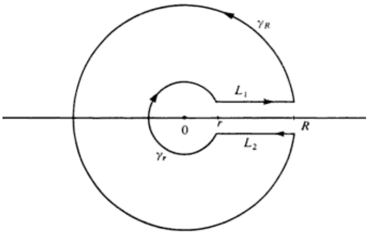
$$\begin{aligned}\left|\int_{\gamma_\rho} \frac{\log z}{1+z^2}dz\right| &= \left|\rho\int_0^\pi \frac{\log\rho + \imath\theta}{1+\rho^2 e^{\imath\theta}}e^{\imath\theta}d\theta\right| \\ &\leq \frac{\rho|\log\rho|}{|1-\rho^2|}\int_0^\pi d\theta + \frac{\rho}{|1-\rho^2|}\int_0^\pi \theta d\theta \\ &= \frac{\pi\rho|\log\rho|}{|1-\rho^2|} + \frac{\rho\pi^2}{|1-\rho^2|}.\end{aligned}$$

This is zero in the limit if $\rho \to 0+$ or $\rho \to \infty$. Thus, the given integral is zero.

3. Show that $\int_0^\infty \frac{x^{-c}}{1+x} dx = \frac{\pi}{\sin \pi c}$ if $0 < c < 1$.
We'll integrate

$$f(z) = \frac{z^{-c}}{1+z},$$

where z^{-c} is the branch corresponding to branch cut being the positive real axis, and integrate along the contour:



Why this particular choice?

By residue theorem,

$$\int_{\gamma} \frac{z^{-c}}{1+z} dz = 2\pi\imath e^{\imath\pi c}.$$

The integral is the sum of four integrals; one on L_1, one on γ_R, one on L_2, one on γ_r. Note that

$$\int_r^R \frac{t^{-c}}{1+t} dt = \lim_{\delta\to 0} \int_{L_1} \frac{z^{-c}}{1+z} dz.$$

Similarly,

$$\lim_{\delta\to 0} \int_{L_2} \frac{z^{-c}}{1+z} dz = -e^{-2\pi\imath c} \int_r^R \frac{t^{-c}}{1+t} dt.$$

Also,

$$\left| \int_{\gamma_\rho} \frac{z^{-c}}{1+z} dz \right| \leq \frac{\rho^{-c}}{|1-\rho|} 2\pi\rho.$$

This is zero in the limit as $\rho \to 0$ or $\rho \to \infty$. Thus we get:

$$2\pi\imath e^{-\imath\pi c} = (1 - e^{-2\imath\pi c}) \int_0^\infty \frac{t^{-c}}{1+t} dt.$$

Thus,

$$\int_0^\infty \frac{t^{-c}}{1+t} dt = \frac{2\pi\imath e^{-\imath\pi c}}{1 - e^{-2\imath\pi c}} = \frac{\pi}{\sin \pi c}.$$

Now, let's introduce ourselves to a very important function; the Gamma function. It's defined as:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt,$$

for $\mathrm{Re}(z) > 0$. Does the integral make sense? Check: it does converge absolutely in the right half plane. Therefore, it is a holomorphic function in the right half plane. (Why?) What's $\Gamma(1)$?

Integration by parts in the region of convergence gives:

$$\Gamma(z+1) = z\Gamma(z).$$

Thus,

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1)\ldots\Gamma(1) = n!.$$

The Gamma function interpolates the factorial function!

We would like to define the Gamma function throughout $\mathbb{C}$. Right now, it's defined only on the right half plane. Any ideas? Use the identity

$$\Gamma(z+1) = z\Gamma(z).$$

Is $\Gamma(z)$ holomorphic throughout? Plug in $z = 0$, and we get:

$$0 \cdot \Gamma(0) = 1.$$

Thus 0 must be a pole for the extended Gamma function. Similarly, all negative integers are also poles. And these are the only poles. Exercise: Check that these poles are simple and

$$\mathrm{Res}(\Gamma; -n) = \frac{(-1)^n}{n!}.$$

What is $\Gamma(x)\Gamma(y)$?

$$\Gamma(x)\Gamma(y) = \int_0^\infty \int_0^\infty e^{-u-v} u^{x-1} v^{y-1} du dv.$$

Put

$$u = zt; \;\; v = z(1-t),$$

and apply the change of variables formula from MA 105. Check:

$$\Gamma(x)\Gamma(y) = \Gamma(x+y) B(x,y),$$

where

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt.$$

Put $t = \frac{s}{s+1}$ to get:

$$B(1-c,c) = \int_0^\infty \frac{s^{-c}}{1+s} ds,$$

for $0 < c < 1$.

Thus, for $0 < x < 1$,

$$\Gamma(x)\Gamma(1-x) = \int_0^\infty \frac{t^{-x}}{1+t}dt = \frac{\pi}{\sin \pi x}.$$

In particular,

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}.$$

Can we say that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

for all $z \in \mathbb{C}$?

YES. By identity theorem. Introduce meromorphic functions. Why is the identity theorem valid for meromorphic functions?