MA 205 Complex Analysis: Examples

August 12, 2017

Recall

Remark: If Ω is a domain, $z_0 \in \Omega$ is any point and γ is any curve not passing through z_0, then $\int_\gamma \frac{1}{z-z_0} dz$ is an integer multiple of $2\pi i$. This integer is (suggestively) called the **winding number** of γ around z_0 and counts the number of times the curves winds around z_0. Note that this intger could be negative which happens when the curve winds around with clockwise orientation.
Last time we saw some nice theoretical applications of Cauchy Integral Formula. Let us begin this talk by seeing some computational applications of Cauchy Integral Formula.

Recall that if Ω is a domain in $\mathbb{C}$ and f is a holomorphic function on and inside a simply closed contour γ and z_0 is an interior point of γ, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi \imath} \int_\gamma \frac{f(z)}{(z-z_0)^{n+1}} dz$$

Remark: If Ω is a domain, $z_0 \in \Omega$ is any point and γ is any curve not passing through z_0, then $\int_\gamma \frac{1}{z-z_0} dz$ is an integer multiple of $2\pi i$. This integer is (suggestively) called the **winding number** of γ around z_0 and counts the number of times the curves winds around z_0. Note that this intger could be negative which happens when the curve winds around with clockwise orientation.
Last time we saw some nice theoretical applications of Cauchy Integral Formula. Let us begin this talk by seeing some computational applications of Cauchy Integral Formula.

Recall that if Ω is a domain in $\mathbb{C}$ and f is a holomorphic function on and inside a simply closed contour γ and z_0 is an interior point of γ, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi\imath} \int_\gamma \frac{f(z)}{(z-z_0)^{n+1}} dz$$

Example 1:

$$\int_{|z|=2} \frac{z^2}{z-1} dz$$
$$= 2\pi i[z^2]|_{z=1} = 2\pi i$$

Example 2:

$$\int_{|z|=2} \frac{e^z}{z^2(z-1)} dz$$
$$= \int_{|z|=\epsilon} \frac{e^z/z-1}{z^2} + \int_{|z-1|=\epsilon} \frac{e^z/z^2}{z-1} dz$$
$$= 2\pi i[\frac{d}{dz}(\frac{e^z}{z-1})]_{z=0} + 2\pi i[\frac{e^z}{z^2}]_{z=1}$$

n=1

$$= -4\pi i + (2\pi i)e = 2\pi i(e-2)$$

Example 3

$$\int_{|z-1|=1} \frac{z^2-4z+3}{z^2-z-1} dz$$

Example 3

$$\int_{|z-1|=1} \frac{z^2-4z+3}{z^2-z-1} dz$$

$$= \int_{|z-1=1} \frac{z^2-z-1-3z+4}{z^2-z-1} dz$$

Example 3

$$\int_{|z-1|=1} \frac{z^2-4z+3}{z^2-z-1} dz$$

$$= \int_{|z-1=1} \frac{z^2-z-1-3z+4}{z^2-z-1} dz$$

$$= \int_{|z-1|=1} [1 - \frac{3z-4}{z^2-z-1}] dz$$

Example 3

$$\int_{|z-1|=1} \frac{z^2-4z+3}{z^2-z-1} dz$$

$$= \int_{|z-1=1} \frac{z^2-z-1-3z+4}{z^2-z-1} dz$$

$$= \int_{|z-1|=1} [1 - \frac{3z-4}{z^2-z-1}] dz$$

$$= \int_{|z-1|=1} 1 dz - \int_{|z-1|=1} \frac{3z-4}{(z-\frac{1+\sqrt{5}}{2})(z-\frac{1-\sqrt{5}}{2})}$$

Example 3

$$\int_{|z-1|=1} \frac{z^2-4z+3}{z^2-z-1} dz$$

$$= \int_{|z-1=1} \frac{z^2-z-1-3z+4}{z^2-z-1} dz$$

$$= \int_{|z-1|=1} [1 - \frac{3z-4}{z^2-z-1}] dz$$

$$= \int_{|z-1|=1} 1 dz - \int_{|z-1|=1} \frac{3z-4}{(z-\frac{1+\sqrt{5}}{2})(z-\frac{1-\sqrt{5}}{2})}$$

$$= 0 \text{ - } \int_{|z-1|=1} \frac{(\frac{3z-4}{z-\frac{1-\sqrt{5}}{2}})}{(z-\frac{1+\sqrt{5}}{2})}$$

Example 3

$$\int_{|z-1|=1} \frac{z^2-4z+3}{z^2-z-1} dz$$

$$= \int_{|z-1=1} \frac{z^2-z-1-3z+4}{z^2-z-1} dz$$

$$= \int_{|z-1|=1} [1 - \frac{3z-4}{z^2-z-1}] dz$$

$$= \int_{|z-1|=1} 1 dz - \int_{|z-1|=1} \frac{3z-4}{(z-\frac{1+\sqrt{5}}{2})(z-\frac{1-\sqrt{5}}{2})}$$

$$= 0 \text{ - } \int_{|z-1|=1} \frac{(\frac{3z-4}{z-\frac{1-\sqrt{5}}{2}})}{(z-\frac{1+\sqrt{5}}{2})}$$

$$= 0 \text{ - } [(3z\text{-}4)/(z\text{-}(1\text{-}\sqrt{5})/2)]_{z=(1+\sqrt{5})/2}$$

$$= (\sqrt{5} - 3)/2$$

Example 4

$$
\begin{aligned}
I &= \int_{|z|=3} \frac{z^9+1}{z^6-1} \\
&= \int_{|z|=3} \frac{z^3(z^6-1)+z^3+1}{z^6-1} \\
&= \int_{|z|=3} z^3 + \int_{|z|=3} \frac{z^3+1}{z^6-1} \\
&= 0 + \int_{|z|=3} \frac{1}{z^3-1} \\
&= 0 \text{ (by an earlier exercise)}
\end{aligned}
$$

We also discussed $\int_{|z=1} \frac{1}{z(z^3+3z-7)}$ on the board.

Singularities

Many times, one has a situation where Ω is an open set and f is a holomorphic function on the complement of a certain subset. The points of this subset are called **singularities** of the function. Given the rigid nature of holomorphic functions, we can get a lot of information on the nature of the singularities; essentially by looking at the function in small punctured neighborhoods of those points. Let us see this in more detail.

Definitions

Singularity of a function: The set of points in Ω where f is not defined or not holomorphic are called the singularities of Ω.
For example $1/z$ has a singularity at 0.

Definitions

Singularity of a function: The set of points in Ω where f is not defined or not holomorphic are called the singularities of Ω.
For example $1/z$ has a singularity at 0.

Singularities are of 2 types, isolated and non-isolated singularities.
A singular point is said to be isolated if the function is holomorphic in a punctured disc around that point.
For example $1/z$ is holomorphic in any punctured disc around 0.
$\frac{1}{z(z-1)}$ has 2 singular points 0 and 1, both of which are isolated singularities; the function is holomorphic in a punctured disc of radius 1 around both of them.

Definitions

Singularity of a function: The set of points in Ω where f is not defined or not holomorphic are called the singularities of Ω.
For example $1/z$ has a singularity at 0.

Singularities are of 2 types, isolated and non-isolated singularities.
A singular point is said to be isolated if the function is holomorphic in a punctured disc around that point.
For example $1/z$ is holomorphic in any punctured disc around 0.
$\frac{1}{z(z-1)}$ has 2 singular points 0 and 1, both of which are isolated singularities; the function is holomorphic in a punctured disc of radius 1 around both of them.

A singularity is non-isolated if it is not isolated ! That is, in no punctured neighborhood of the singularity is the function holomorphic.
For example $f(z) = |z|$ has all points as singularities and hence no point is an isolated singularity.

Removable and Non-Remonavle Singularities

Isolated singularities are of two types; removable and non-removable singularities.

Removable and Non-Remonavle Singularities

Isolated singularities are of two types; removable and non-removable singularities.

If an isolated singularity can be removed by defining the function by assigning a certain value at that point, we say that the singularity is removable. If not we say it is non-removable. For instance, the function $f(z) = \frac{sin(z)}{z}$ is a removable singularity.

Removable and Non-Remonavle Singularities

Isolated singularities are of two types; removable and non-removable singularities.

If an isolated singularity can be removed by defining the function by assigning a certain value at that point, we say that the singularity is removable. If not we say it is non-removable. For instance, the function $f(z) = \frac{sin(z)}{z}$ is a removable singularity. By redefining the function to be $f(z) = \frac{sin(z)}{z}$ for $z \neq 0$ and 1 for $z = 0$, we get a function which is holomorphic holomorphic even at 0. On the other hand $1/z$ has a non-removable singularity at 0.

Removable and Non-Remonavle Singularities

Isolated singularities are of two types; removable and non-removable singularities.

If an isolated singularity can be removed by defining the function by assigning a certain value at that point, we say that the singularity is removable. If not we say it is non-removable. For instance, the function $f(z) = \frac{sin(z)}{z}$ is a removable singularity. By redefining the function to be $f(z) = \frac{sin(z)}{z}$ for $z \neq 0$ and 1 for $z = 0$, we get a function which is holomorphic holomorphic even at 0. On the other hand $1/z$ has a non-removable singularity at 0.

Note that if an isolated singularity at z_0 is removable, then $\lim_{z \to z_0} f(z)$ exists. The converse is also true and that is the Riemann's Removable Singularity Theorem.

Riemann's Removable Singularity Theorem

Theorem: z_0 is removable iff $\lim_{z \to z_0} f(z)$ exists. Clearly removable singularity implies this limit exists. For the converse, suppose this limit exists. Then $\lim_{z \to z_0} (z - z_0) f(z) = 0$. Then define

$$g(z) = \begin{cases} (z - z_0)^2 f(z) & \text{if } z \neq z_0 \\ 0 & \text{if } z = z_0. \end{cases}$$

If f is analytic in a punctured neighbourhood of z_0, then clearly g is analytic throughout that neighbourhood. Write

$$g(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \ldots.$$

Note that $c_0 = g(z_0) = 0$ and $c_1 = g'(z_0) = 0$.

Riemann's Removable Singularity Theorem

Theorem: z_0 is removable iff $\lim_{z \to z_0} f(z)$ exists. Clearly removable singularity implies this limit exists. For the converse, suppose this limit exists. Then $\lim_{z \to z_0} (z - z_0) f(z) = 0$. Then define

$$g(z) = \begin{cases} (z - z_0)^2 f(z) & \text{if } z \neq z_0 \\ 0 & \text{if } z = z_0. \end{cases}$$

If f is analytic in a punctured neighbourhood of z_0, then clearly g is analytic throughout that neighbourhood. Write

$$g(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \dots.$$

Note that $c_0 = g(z_0) = 0$ and $c_1 = g'(z_0) = 0$. Thus,

$$g(z) = c_2(z - z_0)^2 + c_3(z - z_0)^3 + \dots.$$

If we define $f(z_0) = c_2$, then f is holomorphic throughout. i.e., z_0 is a removable singularity.

Pole

Intuitively a pole is a point at which the function blows up from all directions. An isolated singularity z_0 is said to be a pole if $\lim_{z\to z_0} f(z)$ is ∞ (that is the function takes values outside any bounded set in any small punctured neighborhood of z_0). In this case the function $\frac{1}{g(z)}$ is holomorphic at z_0 with $g(0) = 0$. (Why ?).

Pole

Intuitively a pole is a point at which the function blows up from all directions. An isolated singularity z_0 is said to be a pole if $\lim_{z \to z_0} f(z)$ is ∞ (that is the function takes values outside any bounded set in any small punctured neighborhood of z_0). In this case the function $\frac{1}{g(z)}$ is holomorphic at z_0 with $g(0) = 0$. (Why ?). Since $g(z)$ is not identically equal to zero (Why ?), it follows that there exists a positive integer m such that $g(z) = (z - z_0)^m h(z)$ for some holomorphic function $h(z)$ defined in a neighborhood of z_0 with $h(z_0) \neq 0$. Note that such an m and therefore such a $h(z)$ is uniquely defined.

Pole

Intuitively a pole is a point at which the function blows up from all directions. An isolated singularity z_0 is said to be a pole if $\lim_{z \to z_0} f(z)$ is ∞ (that is the function takes values outside any bounded set in any small punctured neighborhood of z_0). In this case the function $\frac{1}{g(z)}$ is holomorphic at z_0 with $g(0) = 0$. (Why ?). Since $g(z)$ is not identically equal to zero (Why ?), it follows that there exists a positive integer m such that $g(z) = (z - z_0)^m h(z)$ for some holomorphic function $h(z)$ defined in a neighborhood of z_0 with $h(z_0) \neq 0$. Note that such an m and therefore such a $h(z)$ is uniquely defined. Thus $f(z) = (z - z_0)^{-m} f_1(z)$ for some holomorphic function $f_1(z)$ in a punctured neighborhood of z_0. m is called the order of the pole and is a measure of how fast the function blows up at z_0. If m is one, we say that the pole is a **simple pole**.

Casorati-Weierstrass Theorem

A function $f(z)$ defined on an open set except at all the poles is called a **meromorphic function**. An isolated singularity that is neither a pole nor a removable singularity is called an **essentially singularity**. These are the most interesting to understand. Like before we have an important theorem on the values attained by a function near an essential singularity.

Casorati-Weierstrass Theorem

A function $f(z)$ defined on an open set except at all the poles is called a **meromorphic function**. An isolated singularity that is neither a pole nor a removable singularity is called an **essentially singularity**. These are the most interesting to understand. Like before we have an important theorem on the values attained by a function near an essential singularity.

Theorem: If z_0 is an isolated singularity, then it essential if and only if the values of f come arbitrarily close to every complex number in a neighborhood of z_0.

Casorati-Weierstrass Theorem

A function $f(z)$ defined on an open set except at all the poles is called a **meromorphic function**. An isolated singularity that is neither a pole nor a removable singularity is called an **essentially singularity**. These are the most interesting to understand. Like before we have an important theorem on the values attained by a function near an essential singularity.

Theorem: If z_0 is an isolated singularity, then it essential if and only if the values of f come arbitrarily close to every complex number in a neighborhood of z_0.

The if part if obvious. For the only if part, suppose f has an essential singularity. Let a be any complex number. Suppose f does not attain values arbitrarily close to a, then
$\lim_{z \to z_0} (z - z_0) \dfrac{1}{(f(z) - a)} = 0$. Hence by Riemann's theorem above, it has a removable singularity at z_0.

Proof cont ..

Depending on whether the singularity can be removed by assigning the value to be zero or a non-zero value, $f(z)$ will have a pole or a removable singularity at z_0. In either case we have a contradiction.

Proof cont ..

Depending on whether the singularity can be removed by assigning the value to be zero or a non-zero value, $f(z)$ will have a pole or a removable singularity at z_0. In either case we have a contradiction. For example, the function $e^{1/z}$ has an essential singularity at 0.

(Check !)

Quote

A good mathematical joke is better, and better mathematics, than a dozen mediocre papers. - J E Littlewood.

Quote

A good mathematical joke is better, and better mathematics, than a dozen mediocre papers. - J E Littlewood.

(Obviously to be taken in jocular vein !)

So time for a mathematical joke ...

Joke

There was a transatlantic flight and the pilot and copilot dropped dead. A desperate flight attendant asked if anyone knew how to fly a plane. An old polish man said: "Well, I used to fly planes in WW II, but nothing like this". When he brought him into the cockpit, his jaw dropped. There were so many buttons, levers, and fancy dials. "What's wrong?" the flight attendant asked.

"I'm just a simple pole in a complex plane", he responded.

Another (non-mathematical) Polish Joke

A Polish immigrant went to the DMV to apply for a driver's license. First, of course, he had to take an eyesight test. The optician showed him a card with the letters:
'C Z W I X N O S T A C Z.'
"Can you read this?" the optician asked.
"Read it?" the Polish guy replied, "I know the guy."