MA 205 Complex Analysis: Winding Up

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September 03, 2015

1. Show that $\int_0^\pi \log\sin\theta d\theta = -\pi\log 2$.
What should be the complex function? We've noticed that if trigonometric functions are involved, it's better to work with the exponential function. Note that

$$1 - e^{2\imath z} = -2\imath e^{\imath z}\sin z.$$

So we'll work with $\log(1 - e^{2\imath z})$. When is $1 - e^{2\imath z}$ real and negative? Exactly when $x = n\pi$ and $y \leq 0$. We'll delete these half lines, and consider the corresponding holomorphic branch of $\log(1 - e^{2\imath z})$. We'll integrate this function along the rectangular contour with vertices $0, \pi, \pi + \imath R, \imath R$. Since 0 and π are problematic points, we'll make indentations at these two points. So integration is from r to $\pi - r$ on the x-axis, from $\pi - r$ to $\pi + r\imath$ via the circular quadrant with center π and radius r, from $\pi + r\imath$ to $\pi + R\imath$, from $\pi + R\imath$ to $R\imath$, from $R\imath$ to $r\imath$, and from there to r via the circular quadrant of radius r and center 0.

The complex integral on the whole contour is zero by Cauchy's theorem. The integrals on the two vertical lines are

$$\int_R^r \imath \log(1 - e^{-2y}) dy \quad \& \quad \int_r^R \imath \log(1 - e^{2\imath(\pi + \imath y)}) dy,$$

and these cancel with each other. Integral on the upper horizontal line is

$$\int_\pi^0 \log(1 - e^{2\imath(x + \imath R)}) dx,$$

and this goes to 0 as $R \to \infty$. What about the integrals on the circular quadrants? We claim that they also go to 0 in the limit as $r \to 0$. Since $\log z = \log|z| + \imath\theta$, and since θ is bounded, we only need to show that

$$\int_{\gamma_r} \log|1 - e^{2\imath z}| dz$$

is 0 in the limit.

Real Integrals

Note that as $z \to 0$, $|1 - e^{2\imath z}|$ and $|z|$ go to 0 with the same speed; i.e., $\lim_{z \to 0} \frac{|1 - e^{2\imath z}|}{|z|}$ is finite. Therefore, $\int_{\gamma_r} \log|1 - e^{2\imath z}| dz$ behaves like $\int_{\gamma_r} \log|z| dz$ which is nothing but $\frac{\pi}{2} r \log r$ which goes to 0 as $r \to 0$. Similarly, the integral on the other circular quadrant goes to 0 in the limit. Thus, we get $\int_r^{\pi - r} \log(-2\imath e^{\imath x} \sin x) dx = 0$. In the limit, this gives

$$\int_0^{\pi} \log(2 \cdot -\imath \cdot e^{\imath x} \cdot \sin x) dx = 0.$$

i.e.,

$$\pi \log 2 + \pi \left(-\frac{\pi \imath}{2} \right) + \imath \left(\frac{\pi^2}{2} \right) + \int_0^{\pi} \log \sin x dx = 0.$$

Thus,

$$\int_0^{\pi} \log \sin x dx = -\pi \log 2.$$

2. Show that $\int_0^\infty \frac{\sin ax}{e^{2\pi x}-1} dx = \frac{1}{4}\coth\left(\frac{a}{2}\right) - \frac{1}{2a}$.

Once again, appearance of sin tells us to look for exp. The $e^{2\pi x}$ in the denominator has period $\imath$, and this is a hint that integrating over a rectangular contour with vertices $0, R, R+\imath, \imath$ is a good idea. We'll integrate

$$\frac{e^{\imath az}}{e^{2\pi z}-1},$$

along this contour. Since 0 and $\imath$ are problematic points, we'll indent these points via circular quadrants. Thus, integration is from r to R, R to $R+\imath$, $R+\imath$ to $r+\imath$, from $r+\imath$ to $(1-r)\imath$, from $(1-r)\imath$ to $r\imath$, and from $r\imath$ to r. The complex integral over the whole contour is zero by Cauchy.

Real Integrals

The two horizontal integrals are

$$\int_r^R \frac{e^{\imath ax}}{e^{2\pi x}-1}dx \quad \& \quad \int_R^r \frac{e^{\imath a(x+\imath)}}{e^{2\pi x}-1}dx,$$

and their sum is

$$(1-e^{-a})\int_r^R \frac{e^{\imath ax}}{e^{2\pi x}-1}dx,$$

whose imaginary part in the limit is

$$(1-e^{-a})\int_0^\infty \frac{\sin ax}{e^{2\pi x}-1}dx.$$

The right side vertical integral is

$$\int_0^1 \frac{e^{\imath a(R+\imath y)}}{e^{2\pi R}e^{2\pi\imath y}-1}\imath dy,$$

and its modulus is less than $\frac{1}{e^{2\pi R}-1}$. Thus, this integral is zero in the limit.

Let's now look at the two circular quadrants. Around $\imath$, the integral is

$$\int_0^{-\pi/2} \frac{e^{\imath a(\imath + re^{\imath\theta})}}{e^{2\pi(\imath + re^{\imath\theta})} - 1} \imath re^{\imath\theta} d\theta,$$

which is

$$\imath re^{-a} \int_0^{-\pi/2} \frac{e^{\imath\theta} e^{\imath are^{\imath\theta}}}{e^{2\pi re^{\imath\theta}} - 1} d\theta.$$

This is a tough one to crack, but then really no need to crack this. We are interested only in the limiting case as $r \to 0$. So can ignore terms with r in it. Thus, it's enough to look at

$$\imath re^{-a} \int_0^{-\pi/2} \frac{e^{\imath\theta}(1 + \imath are^{\imath\theta})}{1 + 2\pi re^{\imath\theta} - 1} d\theta,$$

which is

$$\frac{\imath re^{-a}}{2\pi r} \int_0^{-\pi/2} (1 + r[\cdot]) d\theta = -\frac{\imath e^{-a}}{4},$$

as $r \to 0$.

Ditto ditto, check that the other circular quadrant integral in the limit gives $-\frac{\imath}{4}$. The only remaining integral is the left side vertical one:

$$\imath \cdot \int_{1-r}^{r} \frac{e^{-ay}}{e^{2\pi\imath y} - 1} dy.$$

It turns out that this integral does not converge in the limit! But do not panic! We ask ourselves whether we could integrate

$$\int_{1-r}^{r} \mathrm{Re}\left[\frac{e^{-ay}}{e^{2\pi\imath y} - 1}\right] dy,$$

since this is all that we need for the problem at hand. Luckily, this is quite easy.

$$\mathrm{Re}\left[\frac{1}{e^{2\pi\imath y} - 1}\right] = -\frac{1}{2},$$

and thus, the relevant integral gives

$$\frac{1}{2a}[e^{-ar} - e^{-a(1-r)}].$$

Real Integrals

Ignore higher order terms in r to conclude that this is

$$\frac{1}{2a}(1 - e^{-a}),$$

in the limit. Thus,

$$0 = (1 - e^{-a}) \int_0^\infty \frac{\sin ax}{e^{2\pi x} - 1} dx + \frac{1}{2a}(1 - e^{-a}) - \frac{e^{-a}}{4} - \frac{1}{4},$$

and this gives the required answer.

Maximum Modulus Theorem

An important theorem in Complex Analysis states that a non-constant holomorphic function on an open connected domain never attains its maximum modulus. This is called the maximum modulus theorem. If we include the boundary of the domain, then by EVT, modulus of the function will attain its maximum. By MMT, this has to be on the boundary. Once again, this is vastly different from MA 105. Real differentiable functions could have achieved maximum anywhere inside the interval. We'll use CIF and the identity theorem to prove MMT.

Maximum Modulus Theorem

Proof: Suppose there is $z_0 \in \Omega$ such that $|f(z_0)| \geq |f(z)|$ for all $z \in \Omega$. Then we'll prove that f is a constant. Let γ be a small circle around z_0 with radius r such that the closed disc with boundary γ is contained in Ω. CIF gives,

$$f(z_0) = \frac{1}{2\pi\imath} \int_\gamma \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{\imath\theta}) d\theta.$$

Hence,

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{\imath\theta})| d\theta \leq |f(z_0)|,$$

since $|f(z_0)|$ is assumed to be the maximum value.

Thus,

$$\int_0^{2\pi} \left[|f(z_0)| - |f(z_0 + re^{\imath\theta})| \right] dt = 0.$$

Note that the integrand is non-negative. Therefore it has to be zero; i.e., $|f(z_0)| = |f(z_0 + re^{\imath\theta})|$ for all θ. Since this is true for each small r, we see that $|f(z)|$ is a constant on a small disc around z_0. This means that $f(z)$ is a constant, say c, on this small disc. (Why?) This implies that $f \equiv c$ on Ω by the identity theorem, since a disc has limit points!

I would like to bring your attention to a problem proposed by Riemann 150 years ago, in 1859, which is wide open even today, in spite of many serious attempts by some of the most brilliant minds of the last century. The conjecture is about the so called Riemann zeta function, defined by,

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z},$$

for $\mathrm{Re}(z) > 1$. You could check that the series converges absolutely to the right of one, and that it defines a holomorphic function in that domain. This is quite like our definition of the Gamma function in the last lecture. Just as in the case of $\Gamma(z)$, we extend $\zeta(z)$ to the whole of $\mathbb{C}$ by a certain functional equation. The equation relates $\zeta(z)$ with $\zeta(1-z)$. Thus, the understanding of the behaviour of ζ to the right of one gives us all the info to the left of zero. The extended zeta function is meromorphic on $\mathbb{C}$.

Riemann Hypothesis

The difficult region where the behaviour of zeta is not understood is the vertical strip $0 \leq \mathrm{Re}(z) \leq 1$. Specifically one would like to know the zeros of $\zeta(z)$ in this vertical strip. And the

problem is to either prove or disprove that

> *all the zeros of $\zeta(z)$ in the critical strip $0 \leq \mathrm{Re}(z) \leq 1$ lie on the critical line $Re(z) = \frac{1}{2}$.*

It's one of the seven, million dollar, Clay millennium problems, and a solution to this question will have fantastic consequences in mathematics.

Winding Up

So time has come to say goodbye! I immensely enjoyed teaching this class, and I hope all of you have got a flavour of a little bit different kind of mathematics in this course.

Prepare well for the Quiz tomorrow morning and the Endsem on Friday, September 11. Tomorrow we'll meet in LA 302 at 5 pm. Will freeze the answer key for Quiz II and also discuss the endsem exam pattern.

One more $P + \imath J$

We like to believe that our ancestors knew everything! Heard the story of Wi-Fi? Apparently Indians also knew about the Riemann Zeta and the hypothesis of Zeta vanishing on the critical line, thousands of years ago! The Riemann Zeta is nothing but Raman and Sita. The line $\mathrm{Re}(z) = \frac{1}{2}$ was called the Lakshman Rekha. Of course, Sita vanished exactly when she crossed the Lakshman Rekha![1]

[1] This joke is by a colleague of mine from Bangalore.