MA 205 — 2

Sequences and Series

These are defined similar to the case of real numbers. That is:

Definition: A sequence $\{z_n\}_{n=1}^{\infty}$ of complex numbers has a limit z if for every $\varepsilon > 0$ there exists a positive integer n_0 such that $|z_n - z| < \varepsilon$ whenever $n > n_0$.

This is written as: $\lim\limits_{n\to\infty} z_n = z$

& we say $\{z_n\}_{n=1}^{\infty}$ converges to z. If a sequence does not converge we say it diverges.

Example: ① Let $z_n = \frac{1}{n^3} + i \quad : n = 1, 2, \ldots$

S.T: $\lim_{n\to\infty} z_n = i$

② The sequence $z_n = 1 \quad : n$ - even

$= -1 \quad : n$ - odd

diverges.

Check: ① The limit of a sequence if it exists is unique & the sum & product rules of convergent sequences hold.

② If $z_n = x_n + i y_n : n = 1, 2, \ldots$ & $z = x + iy$, then

$\lim_{n\to\infty} z_n = z$ iff $\lim_{n\to\infty} x_n = x$ & $\lim_{n\to\infty} y_n = y$.

Series:

Def: An infinite series $\sum_{n=1}^{\infty} a_n$ of complex numbers converges to s if the sequence $s_n = \sum_{i=1}^{n} a_i$, of partial sums converges to s. If a series does not converge we say it diverges.

Check: If $z_n = x_n + iy_n$, $n = 1, 2, \cdots$ & $s = x + iy$, then $\sum_{n=1}^{\infty} z_n = s$ iff $\sum_{n=1}^{\infty} x_n = x$ & $\sum_{n=1}^{\infty} y_n = y$,

i.e., $\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} x_n + i \sum_{n=1}^{\infty} y_n$, whenever the 2 series on the R.H.S converge or if the L.H.S converges.

Exercise: If $\sum_{n=1}^{\infty} z_n$ converges then $\lim_{n\to\infty} z_n = 0$.

This implies that the terms of a convergent series in $\mathbb{C}$ are bounded, i.e., $|z_n| \le M$, $\forall\ n \geq 1$ & a fixed real number M.

Absolutely convergent series:

Def^n: A series $\sum_{n=1}^{\infty} z_n$, $z_n = x_n + i y_n$, $n \geq 1$ is said to be absolutely convergent if the series of real numbers:

$$\sum_{n=1}^{\infty} |z_n| = \sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2} \quad \text{converges.}$$

Exercise: Absolute convergence $\Rightarrow$ Convergence.

Solution: Since $|x_n| \leq \sqrt{x_n^2+y_n^2}$ &

$$|y_n| \leq \sqrt{x_n^2+y_n^2}$$

we know from the comparison test in calculus that the 2 series

$\sum_{n=1}^{\infty} |x_n|$ & $\sum_{n=1}^{\infty} |y_n|$ must converge.

Further, over $\mathbb{R}$ if a series converges absolutely then it is convergent.

Hence $\sum_{n=1}^{\infty} x_n = x$ & $\sum_{n=1}^{\infty} y_n = y$ for some $x, y \in \mathbb{R}$. This implies that $\sum_{n=1}^{\infty} z_n$ converges.

Example: Show that $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$: if $|z|<1$.

Sol^n: $S_{n+1} = 1 + z + \dots + z^n = \frac{1-z^{n+1}}{1-z}$: $z \neq 1$.

$$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-z^n}{1-z} = \frac{1}{1-z}$$

(because $\lim_{n\to\infty} z^n = 0$ as $|z|<1$).

Tests for convergence:

Comparison test:

If $\sum_{n=1}^{\infty} b_n$ is absolutely convergent & if $|a_i| \leq |b_i|$ for large i then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Note: This is used in the exercise on page 5; for a real series

- Recall <u>lim sup</u> : Given a sequence of <u>real</u> numbers $x_1, x_2, \dots, x_n, \dots$

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \left[\sup \{x_n, x_{n+1}, \dots\} \right]$$

Note that the sequence in the R.H.S is decreasing. Further, limsup could be $\pm\infty$.

If the $\lim_{n\to\infty} x_n$ exists then it is equal to $\limsup_{n\to\infty} x_n$.

<u>Examples:</u> ① The sequence $1, 2, 3, \dots$ has limsup ∞.

② The sequence $1, 1/2, 1/3, \dots$ has limsup 0.

③ The sequence $1, -1, 1, -1, \ldots$ has limsup 1.

- Similarly define lim inf.

Theorem: <u>Cauchy's root test:</u>

For a series $\sum_{n=1}^{\infty} a_n$ of complex numbers, let $C = \limsup_{n \to \infty} \sqrt[n]{|a_n|}$. Then the series converges absolutely if $C < 1$ & diverges if $C > 1$.

<u>Note:</u> This test is inconclusive for $C = 1$.

Theorem: (Ratio test) : For a series $\sum_{n=1}^{\infty} a_n$,

let $R = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ and

$r = \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. Then if

$R < 1$, the series converges absolutely

$r > 1$, the series diverges

if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for all large n, the series also diverges.

Power Series:

Let $z_0 \in \mathbb{C}$. A series of the form

$\sum_{n=0}^{\infty} c_n (z-z_0)^n$, where $c_n \in \mathbb{C}$ is called a power series around z_0.

Theorem: Existence of Radius of convergence:

For a power series $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ define the number

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|c_n|}},$$

where $R = 0$ or ∞ if $\limsup_{n\to\infty} \sqrt[n]{|c_n|} = \infty$ or 0 respectively. Then

(i) if $|z-z_0| < R$, the series converges absolutely.

(ii) if $|z-z_0| > R$, the series becomes unbounded and so diverges.

(iii) if $0<r<R$ then the series converges uniformly on $\{z \mid |z-z_0|<r\}$.

Moreover, R is the only number having properties (i) & (ii).

Proof: Use the root test.

Remark: The real number R is called the radius of convergence of the power series

Examples: 1. $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$

Apply ratio test. $\displaystyle\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} n = \infty$

i.e, this series converges everywhere.

2. $z - \dfrac{z^3}{3} + \dfrac{z^5}{5} - \cdots$

Show that: Radius of convergence is 1.

(Both tests apply here.

Properties of power series:

These can be added, substracted and multiplied in the obvious way.

It can also be differentiated term by term in its domain of convergence.

If $f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$: $c_n \in \mathbb{C}$, $n = 1, 2, \ldots$

then, $\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$

$$= \sum_{n=0}^{\infty} c_n \left(\lim_{h \to 0} \frac{(z-z_0+h)^n - (z-z_0)^n}{h} \right)$$

$$= \sum_{n=1}^{\infty} n c_n (z-z_0)^{n-1}.$$

<u>Note</u>: Apply the root test to check that the differentiated power series has the same

radius of convergence as the original one.

Proposition: Let $f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$ have radius of convergence $R>0$. Then:

(i) For each $k \geq 1$, the series

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\, c_n (z-z_0)^{n-k}$$

has radius of convergence R.

(ii) The function f is infinitely differentiable on $B_R(z_0)$. Further, $f^{k}(z)$ is given by the series in (i) for all $k \geq 1$ and $|z-z_0|<R$.

continued

(iii) For $n \geq 0$, $c_n = \frac{1}{n!} f^{n}(z_0)$.

<u>Defn</u>: Let $\Omega \subseteq \mathbb{C}$ be a domain. A function $f : \Omega \to \mathbb{C}$ is said to be analytic if it is locally given by a convergent power series, i.e., every $z_0 \in \Omega$ has a neighborhood contained in Ω such that there exists a power series centered at z_0 which converges to $f(z)$ for all z in that neighborhood.

By the earlier proposition, analytic functions are infinitely differentiable.

If $f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$ then

$$c_n = \frac{f^n(z_0)}{n!}.$$

Thus, an analytic function is given by its Taylor series.

We will see later that

holomorphic $\Rightarrow$ analytic.

This will imply that once differentiable in $\mathbb{C}$ is always differentiable!

Special functions

I Exponential function:

Recall that in the last lecture we have seen that the power series in $\mathbb{C}$:

$$\exp(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots$$

converges for every $z \in \mathbb{C}$. In analogy with $\mathbb{R}$ we sometimes denote this by e^z, i.e.,

$$\exp(z) = e^z \quad : z \in \mathbb{C}.$$

Properties of the exponential function:

1. $\frac{d}{dz}(e^z) = e^z$

Term by term differentiation of the power series gives:

$$\frac{d}{dz}(e^z) = \sum_{n=1}^{\infty} n \cdot \frac{z^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = e^z.$$

2. $\exp(0) = e^0 = 1.$

3. Fix $\underline{a \in \mathbb{C}}$. Let $g(z) := e^z \cdot e^{a-z}$.

Then $g'(z) = e^z \cdot e^{a-z} + e^z(-e^{a-z}) = 0.$

$\Rightarrow$ $g(z)$ = a constant function (why?)

Put $z=0$, to get: $g(0) = e^a$

i.e. $g(z) = e^z \cdot e^{a-z} = e^a \quad ; \forall z \in \mathbb{C}.$

$\Rightarrow \quad e^{(z_1+z_2)} = e^{z_1} \cdot e^{z_2} \quad ; z_1, z_2 \in \mathbb{C}$

In particular, $1 = e^0 = e^{z-z} = e^z \cdot e^{-z}$: $z \in \mathbb{C}$

$$\Rightarrow \qquad e^{-z} = \frac{1}{e^z} \quad \& \quad e^z \neq 0 \ : \forall z \in \mathbb{C}.$$

Exercise: Consider the function $\mathbb{C} \xrightarrow{\exp} \mathbb{C} \setminus \{0\}$

$$z \longmapsto \exp(z) \;(= e^z)$$

Is exp one-one? onto?

4. $\exp(\bar{z}) = \overline{\exp(z)}$

i.e., $e^{\bar{z}} = \overline{e^z}$

This follows from the fact that all the coefficients in the power series of e^z real numbers.

Hence for any $z \in \mathbb{C}$,

$$|e^z|^2 = e^z \cdot e^{\bar{z}} = e^{z+\bar{z}} = \exp(2 \cdot \mathrm{Re}(z)).$$

In particular for $\theta \in \mathbb{R}$,

$$|e^{i\theta}|^2 = 1 \quad : \text{as } \mathrm{Re}(i\theta) = 0.$$

Analogous to $\mathbb{R}$, we define:

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + \frac{(-1)^n z^{2n}}{(2n)!} + \cdots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + \frac{(-1)^n z^{2n+1}}{(2n+1)!} + \cdots$$

<u>Exercises:</u> 1. Show that $\sin z$ & $\cos z$ converge at every point in $\mathbb{C}$.

2. Show that: $\frac{d}{dz}(\sin z) = \cos z$ & $\frac{d}{dz}(\cos z) = -\sin z$

(Use term by term differentiation of the Power Series).

3. Show that : $\sin^2 z + \cos^2 z = 1$.

<u>Solution</u> : $\frac{d}{dz}(\sin^2 z + \cos^2 z) = 2\sin z \cdot \cos z - 2\cos z \cdot \sin z$

$$= 0$$

$\therefore$ $\sin^2 z + \cos^2 z$ = a constant function.

Put $z = 0$ to get :

$$\sin^2 z + \cos^2 z = 1.$$

4. Show that :

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2 \quad \&$$

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \cdot \sin z_2$$

5. Show that: $\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$ &

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

Use this to get another proof that

$\sin^2 z + \cos^2 z = 1.$

6. Define the hyperbolic sine & cosine functions:

$$\sinh z = z + \frac{z^3}{3!} + \cdots + \frac{z^{2n+1}}{(2n+1)!} + \cdots$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + \frac{z^{2n}}{(2n)!} + \cdots$$

Find the radius of convergence of the above power series. Show that:

$$\cos(iz) = \cosh z$$

$$\sin(iz) = i \sinh z$$

7. Show that : $e^{iz} = \cos z + i \sin z$: $z \in \mathbb{C}$

(use exercise 5!).

Note By exercise 7. above, we see that there is a natural relation between the complex exponential function & the complex trigonometric functions sin & cos. Such a relation is not detected in real analysis!

8. Let $z = x+iy \in \mathbb{C}$. Show that

$$|e^z| = \exp(\mathrm{Re}(z)) \quad \& \quad \arg(e^z) = \mathrm{Im}(z).$$

Use: $e^z = e^{x+iy} = e^x \cdot e^{iy}$

Periodicity of e^z

For $\theta \in \mathbb{R}$, we have:

$$e^{i\theta} = e^{i(\theta+2\pi n)} \quad : \text{ for any integer } n.$$

$$(\because \quad e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{i(\theta+2\pi n)} = \cos(\theta+2\pi n) + i\sin(\theta+2\pi n)).$$

i.e., $e^{i(2\pi n)} = 1$

Conversely, if $e^z = e^{z+c}$: for some $c \in \mathbb{C}$ & $\forall z \in \mathbb{C}$

then, $e^c = 1$.

$$\Rightarrow \quad 1 = |e^c| = \exp(\mathrm{Re}(c))$$

$$\Rightarrow \quad \mathrm{Re}(c) = 0$$ (why? because exp on the real line is one-one)

$\Rightarrow \quad c = i\theta$ for some $\theta \in \mathbb{R}$.

Further, $1 = e^c = e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow \quad \theta = 2\pi n$, for some integer n.

Note : This periodicity of the complex exp function is not present in the real exponential function.

Exercises: Show that $\sin z$ is unbounded on $\mathbb{C}$. Compare with the real case!

Sin — is bounded on the real axis (recall MA105 Calculus)

— is unbounded on the imaginary axis.

(see below for a proof)

For $y \in \mathbb{R}$ consider $\sin(iy) = \dfrac{e^{-y} - e^{y}}{2i}$.

For $y < 0$, show that $e^{-y} - e^{y}$ is unbounded.

Definition: A power series $\sum_{n=0}^{\infty} a_n z^n$ which converges for all values $z \in \mathbb{C}$ defines an analytic function on $\mathbb{C}$ called an entire function.

Example: Polynomials with complex coefficients, e^z, $\sin z$, $\cos z$ are all entire functions.

Theorem: (Liouville) A bounded entire function is a constant function.

Proof: Will be proved later using "integration".

Exercises: 1. Use Liouville's theorem to prove the Fundamental theorem of algebra:

A non-constant polynomial with complex coefficients has a complex root.

Solution: Let $p(t)$ be such a polynomial.

Suppose $p(z) \neq 0 \quad \forall \; z \in \mathbb{C}$.

$$\Rightarrow \quad f(z) = \frac{1}{p(z)} \quad \text{is entire}$$

Check: $p(t)$ non constant $\Rightarrow \lim\limits_{z \to \infty} p(z) = \infty$

$$\Rightarrow \lim_{z \to \infty} f(z) = 0$$

Check: $\Rightarrow f$ is bounded & hence a constant by Liouville's theorem. $\Rightarrow\!\!\!\!\!\times$ Contradiction.

Exercise: A non-constant complex polynomial assumes all complex values i.e., its range $= \mathbb{C}$.

Q] What is the range of the map

$$\begin{aligned} \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto e^z \end{aligned}$$

? Is it $\mathbb{C}\setminus\{0\}$?

Theorem: (Little Picard's Theorem):

An entire function which misses two complex values is a constant.

Example: Show that $\sin z$ does not miss any complex value.

use $\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$

Exercise: Determine the zeroes of $\sin z$ & $\cos z$.

Logarithm

We have seen earlier that e^z is not a one-one function on $\mathbb{C}$. Even so, we know that it is so, on some subsets like $\mathbb{R} \subseteq \mathbb{C}$. We would like to construct an inverse function viz a "log" on such subsets (if possible) which is analytic, i.e, we want to define $\log w$ so that whenever

$$z = \log w \quad \text{then} \quad w = e^z.$$

Clearly, as $e^z \neq 0$, we cannot define $\log 0$!

$z = x + iy \Rightarrow w = e^z = e^{x+iy} = e^x . e^{iy}$

$\Rightarrow |w| = e^x$ & $y = \arg(w) + 2\pi n$;

for some integer n.

Hence solutions for $w = e^z$ are given by,

$$\left\{ \log|w| + i(\arg(w) + 2\pi n) \;\middle|\; n \text{ is any integer} \right\}.$$

Here $\log|w|$ is the usual real logarithm defined using the real power series for instance.

<u>Definition:</u> If Ω is an open connected subset of $\mathbb{C}$ & $f : \Omega \to \mathbb{C}$ is a continuous function such that $z = \exp f(z)$; $\forall z \in \Omega$

then f is said to be a branch of the logarithm.

Note: $0 \notin \Omega$ (why?)

Proposition: If $\Omega \subseteq \mathbb{C}$ is open & connected & f is a branch of $\log z$ on Ω then the other branches of $\log z$ are the functions

$f(z) + 2\pi n i$: n - an integer.

Proof: First note that if f is a given branch of $\log z$ & n is any integer then the function $g(z) := f(z) + 2\pi n i$ is also a branch as $\exp(g(z)) = \exp(f(z)) = z$.

Conversely, if f & g are branches of $\log z$ then for each $z \in \Omega$;

$$f(z) = g(z) + 2\pi i \, n_z \quad : \text{for some}$$

integer n_z depending on z.

Clearly, the function

$$h(z) := n_z = \frac{f(z) - g(z)}{2\pi i} \quad \text{is}$$

continuous on Ω and its image

$h(\Omega) \subseteq \mathbb{Z}$ - the set of integers.

As Ω is connected, $h(\Omega)$ is connected & hence a constant.

(Use: Continuous image of a connected set is connected).

Hence $\exists$ a fixed integer n such that

$f(z) = g(z) + 2\pi i n$: for every $z \in \Omega$

as required.

- We will usually work with a fixed branch of $\log z$ called the Principal branch defined as follows:

Let $\Omega = \mathbb{C} \setminus \{z \in \mathbb{R} \mid z \leq 0\}$

i.e., complex plane minus the negative real axis.

Clearly, Ω is open & connected.

For $z \in \Omega$, $z = |z| \cdot e^{i\theta}$: $-\pi < \theta < \pi$

$$= r \cdot e^{i\theta}$$

Define: $f(re^{i\theta}) = \log r + i\theta \quad : -\pi < \theta < \pi$

$= \log|z| + i\,\mathrm{Arg}(z)$

Check: f is a branch of $\log z$ on Ω.

Infact, f is analytic & its derivative is $\frac{1}{z}$.

We can use log to define some other functions like the inverse trigonometric functions.

<u>Example:</u> Let $w = \sin^{-1} z$

Then $z = \sin w = \dfrac{e^{iw} - e^{-iw}}{2i}$

Solving this quadratic equation, we get

$$e^{iw} = iz + \sqrt{1-z^2}.$$

Thus, $\sin^{-1} z = \dfrac{1}{i} \log\left(iz + \sqrt{1-z^2}\right).$

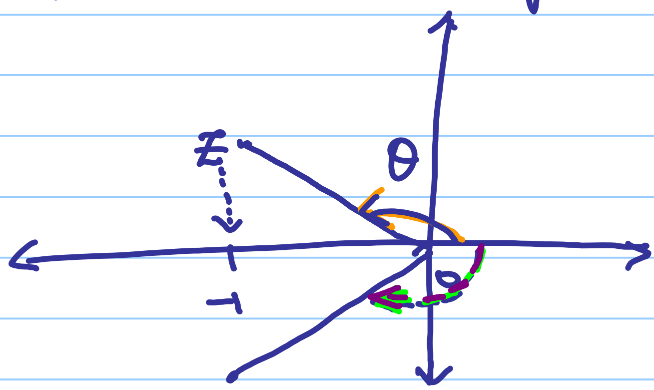
Q] Determine $\cos^{-1} z$ & $\cosh^{-1} z$

Exercises: Recall:

1. $$\mathbb{C}\setminus\{0\} \longrightarrow (-\pi, \pi]$$
$$z \longmapsto \mathrm{Arg}(z)$$

Show that $\mathrm{Arg}(z)$ is discontinuous at every point on the negative real axis.

Choose any point, say -1, in the negative real axis.



As $z \to -1$ from the upper half plane, we have $\underline{\mathrm{Arg}(z) \to \pi}$.

As $z \to -1$ from the lower half plane, we have $\underline{\mathrm{Arg}(z) \to -\pi}$.

Hence $\operatorname{Arg}(z)$ is discontinuous on the negative real axis $= \{ x+iy \mid x \le 0 \ \& \ y=0 \}$.

Check:

$$\operatorname{Arg}(z) = \begin{cases} \text{relate to } \arctan(\ ) \\ \\ \\ \end{cases}$$

2. Show that $\log|z|$ is a harmonic function on the plane except at $z=0$. Find a harmonic conjugate to $\log|z|$ in $\mathbb{C} \setminus \{ x+iy \mid x \le 0 \ \& \ y=0 \}$. Can we find such a harmonic conjugate on $\mathbb{C} \setminus \{0\}$?

Sol^n: $\log|z| = \frac{1}{2}\log(x^2+y^2)$: $z = x+iy$

Let $u(x,y) = \frac{\log(x^2+y^2)}{2}$

$u_x = \frac{x}{x^2+y^2}$; $u_{xx} = \frac{-x^2+y^2}{(x^2+y^2)^2}$: $(x,y) \neq (0,0)$

$u_y = \frac{y}{x^2+y^2}$; $u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}$: $(x,y) \neq (0,0)$

$\Rightarrow$ $u_{xx} + u_{yy} = 0$: $(x,y) \neq (0,0)$

$\Rightarrow$ u is harmonic.

i.e., $\log|z|$ is harmonic in $\mathbb{C} \setminus \{0\}$.

To find its conjugate use CR-equations.

We have: $v_y = \dfrac{x}{x^2+y^2}$ & $v_x = \dfrac{-y}{x^2+y^2}$

Integrate with respect to y to get:

$$v(x,y) = \arctan\left(\frac{y}{x}\right) + f(x) \;:\; \text{for some function } f.$$

Differentiate with respect to x to get:

$$v_x = \left[\frac{1}{\left(1+\left(\frac{y}{x}\right)^2\right)} \cdot \left(\frac{-y}{x^2}\right)\right] + f'(x)$$

$$\Rightarrow \frac{-y}{x^2+y^2} = \frac{-y}{x^2+y^2} + f'(x)$$

i.e., $f'(x) = 0$ i.e., $f(x) = c$, a constant.

$\Rightarrow v(x,y) = \arctan\left(\frac{y}{x}\right) + C$ is a

harmonic conjugate of u.

Note: The harmonic conjugate v in the earlier page is nothing but $Arg(z)$.

Thus $\log z = \log|z| + i\,Arg\, z$ is holomorphic on $\mathbb{C} \setminus \{x+iy \,/\, x \le 0 \;\&\; y=0\}$.

3. Show that $\frac{d}{dz}(\log z) = \frac{1}{z}$.

Sol^n: $\log z = \log|z| + i\,Arg\, z = u + iv$

$$\Rightarrow \frac{d}{dz}(\log z) = u_x + iv_x$$

$$= u_x - iu_y$$

$$u(x,y) = \log|z| = \frac{1}{2}\ln(x^2+y^2)$$

$$\Rightarrow u_x = \frac{x}{x^2+y^2} \; ; \; u_y = \frac{y}{x^2+y^2}$$

$$\therefore \frac{d}{dz}(\log z) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \frac{x-iy}{x^2+y^2}$$

$$= \frac{1}{x+iy}$$

$$= \frac{1}{z}$$

Power functions:

For $a \in \mathbb{C}$ define $z^a := \exp(a \log z)$.

$\log z$ being multivalued $\Rightarrow z^a$ is multivalued.

Taking the principal branch of $\log z$ in the above definition, we get the principal branch of z^a.

This is holomorphic on $\mathbb{C} \setminus \{x+iy \,/\, x \leq 0 \,\&\, y=0\}$

$$\frac{d}{dz}(z^a) = \frac{d}{dz}(\exp(a \log z))$$

$$= \exp(a \log z) \cdot \frac{d}{dz}(a \log z)$$

$$= \exp(a \log z) \cdot \frac{a}{z}.$$

$$= a \cdot \exp(a \log z) \cdot \exp(-\log z)$$

$$= a \cdot \exp((a-1)\log z).$$

Thus, $\frac{d}{dz}(z^a) = a \cdot z^{a-1}$.

where both sides are the principal branches.

<u>Example:</u> If $\log$ is the principal branch

then $i^i = \exp(i \log i) = \exp(i \cdot i\frac{\pi}{2})$

$$= \exp\left(-\frac{\pi}{2}\right) = e^{-\frac{\pi}{2}} \in \mathbb{R}!$$