MA 2017, Tutorial Sheet-5
Boundary value problem and Fourier expansion

1. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \sin nx \sin^2 n\alpha = \begin{cases} \text{constant} & (0 < x < 2\alpha) \\ 0 & (2\alpha < x < \pi) \end{cases}$

2. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cos nx}{n^2} = \frac{\pi^2}{12} - \frac{x^2}{4}, \quad (-\pi \le x \le \pi).$

3. Show that $\displaystyle\sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{(2n+1)^3} = \frac{1}{8}\pi x(\pi - x), \quad (0 \le x \le \pi).$

4. Use the Fourier expansions given in problems (1), (2) and (3) along with Fourier's Theorem to deduce the following results.

   (a) $1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \ldots = \frac{2\pi}{3\sqrt{3}}$

   (b) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \frac{1}{10} - \frac{1}{11} + \ldots = \frac{\pi}{3\sqrt{3}}$

   (c) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \ldots = \frac{\pi^2}{12}$

   (d) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \ldots = \frac{\pi^2}{6}$ (Euler's formula)

   (e) $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - + \ldots = \frac{\pi^3}{32}$

   (f) $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \ldots = \frac{\pi^2}{8}$

   (g) $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \ldots = \frac{\pi}{4} - \frac{1}{2}$

5. Find the Fourier series of $f(x)$ on $[-L, L]$ and determine the value that the series takes for $-L \le x \le L$.

   (a) $L = \pi,\ f(x) = 2x - 3x^2$, (b) $L = 1,\ f(x) = 1 - 3x^2$,

   (c) $L = \pi,\ f(x) = |\sin x|$,

   (f) $L = 1,\ f(x) = \begin{cases} 0, & -1 < x < -1/2,\ \ 1/2 < x < 1 \\ \cos \pi x, & -1/2 < x < 1/2, \end{cases}$,

   (g) $L = \pi$, and $f(x)$ is one of the following functions (i) e^x, (ii) $(x - \pi)\cos x$, (iii) $\sin kx$, k not an integer.

   (h) $L = \pi,\ f(x) = x + |x|$ (i) $L = \pi,\ f(x) = \begin{cases} -x & -\pi \le x < 0 \\ x & 0 \le x < \pi \end{cases}$

   (j) $L = 1,\ f(x) = \begin{cases} 0 & -1 < x < 0, \\ x & 0 < x < 1. \end{cases}$, (k) $L = 1,\ f(x) = \begin{cases} -1, & -1 \le x < 0 \\ 1, & 0 \le x \le 1 \end{cases}$,

6. Expand each of the following functions in a Fourier cosine series on $[0, L]$.

(a) $L = 1,\ f(x) = e^{-x}$,

(b) $L = 2,\ f(x) = \begin{cases} 0, & 0 \le x \le 1 \\ 1, & 1 \le x \le 2 \end{cases}$,

(c) $L = \pi,\ f(x) = 2\sin x \cos x$,

(d) $f(x) = x^2 - L^2$.

(e) $f(x) = 3x^2(x^2 - 2L^2)$,

(f) $f(x) = x^3(3x - 4L)$,

(g) $x^2(3x^2 - 8Lx + 6L^2)$.

7. Expand each of the following functions in a Fourier sine series on $[0, L]$.

(a) $L = 1,\ f(x) = e^{-x}$,

(b) $L = 2a,\ f(x) = \begin{cases} x, & 0 < x < a \\ a, & a \le x \le 2a \end{cases}$,

(c) $L = \pi\ f(x) = 2\sin x \cos x$,

(d) $L = \pi\ f(x) = \cos x$.

(e) $f(x) = x(L^2 - x^2)$

(f) $f(x) = x(x^3 - 2Lx^2 + L^3)$,

(g) $f(x) = x(3x^4 - 5Lx^3 + 2L^4)$.