MA 205 Complex Analysis: Introduction

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$\mathbb{R}$ passes both these tests and hence analysis over $\mathbb{R}$ is rich and exciting. But it fails another "algebra test" namely obtaining roots of polynomials.

Fundamental theorem of Algebra

Theorem

Every non-constant polynomial with complex coefficients has a complex root.

This theorem fails over all the other "number systems" we know, namely $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$. Carl Friedrich Gauss (1777-1855) gave some of the earliest proofs of this theorem. Today this theorem has more than a hundred proofs, many of them using complex analysis. We will see at least one proof of this in this course.

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which is another complex number. (Incidentally, if $z = x + \imath y \in \mathbb{C}$, we call x to be $\mathrm{Re}(z)$ and y to be $\mathrm{Im}(z)$.) So coming back to the fundamental theorem of algebra, it is interesting that just adding one root of one real polynomial, namely $X^2 + 1$ gives you all the roots of all the complex polynomials !

Numbers

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Exercise: A complex polynomial of degree n has exactly n roots.

Exercise: Show that a real polynomial that is irreducible has degree at most two. i.e., if

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n, \ a_i \in \mathbb{R},$$

then there are non-constant real polynomials g and h such that $f(x) = g(x)h(x)$ if $n \geq 3$.

Let $\Omega \subseteq \mathbb{C}$ be a subset. We say that Ω is an open subset of $\mathbb{C}$ is given any point $z_0 \in \Omega$, there exists $\delta > 0$ such that the set $\{z \in \mathbb{C} \text{ such that } |z - z_0| < \delta\} \subset \Omega$. Here $|z - z_0|$ is the distance between z and z_0;

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By a δ-neighbourhood of a point $z_0 \in \mathbb{C}$, denoted $B_\delta(z_0)$, we mean the set of points $\{z \in \mathbb{C} \text{ such that } |z - z_0| < \delta\}$.
A subset $Z \subseteq \mathbb{C}$ is said to be **closed** if its complement is open. It is a basic fact that there exists no subset of $\mathbb{C}$ that is both open and closed other than $\emptyset$ and $\mathbb{C}$.
One checks that the following properties hold:
1. $\emptyset$ and $\mathbb{C}$ are both open and closed.
2. Arbitrary unions and finite intersections of open subsets is open.
3. Arbitrary intersections and finite unions of closed sets is closed.

Some More Notions ...

Let $S \subseteq \mathbb{C}$ be any subset.

Interior Point: A point $z_0 \in \mathbb{C}$ is said to be an interior point of S if there exists $\delta > 0$ such that $B_\delta(z_0) \subseteq S$. The union of all interior points is called the Interior of S, denoted $Int(S)$.

Exterior point: A point $z_0 \in \mathbb{C}$ is said to be an exterior point of S if there exists $\delta > 0$ such that $B_\delta(z_0) \cap S = \emptyset$. The union of all exterior points is called Exterior of S, denoted $Ext(S)$

Boundary Point: A point which is neither an interior point nor an exterior point is called a boundary point. The union of all boundary points is called the boundary of S, denoted ∂S

Clearly $\mathbb{C} = Int(S) \cup Ext(S) \cup \partial S$ (Obvious !). Further the intersection of any two of the above subsets is empty. With these notions, one observes that a subset $S \subseteq \mathbb{C}$ is open if and only if $S = Int(S)$.

We define the **closure** of S to be the smallest closed set containing S. It is denoted $\overline{S}$. Equivalently, the closure of S is the union of S together with its limit points.
(In other words a point belongs to the closure of S if it is arbitrarily close to points in S).

Exercises: 1. Show that S is open if and only if $S \cap \partial(S) = \emptyset$.
2. Closure of $S = Int(S) \cup \partial(S)$.

Connectedness

A subset $S \subseteq \mathbb{C}$ is said to be **path-connected** if given any 2 points $z_0, z_1 \in S$, there exists a continuous path joining them. i.e, a continuous function $f : [0,1] \to S$ such that $f(0) = z_0$ and $f(1) = z_1$. An open subset of $\mathbb{C}$ which is path-connected is called a domain. $\emptyset$, open ball in $\mathbb{C}$, $\mathbb{C}$ can be easily seen to be examples of path-connected sets. An arbitrary open set in $\mathbb{C}$ is a disjoint union of domains. In this course we will be mostly interested in complex-valued functions defined over domains.

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3. $\lim_{z \to z_0} [f(z)g(z)] = \lim_{z \to z_0} f(z) \cdot \lim_{z \to z_0} g(z)$
4. $\lim_{z \to z_0} \dfrac{1}{f(z)} = \dfrac{1}{\lim_{z \to z_0} f(z)}$, *if* $\lim_{z \to z_0} f(z) \neq 0$.

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Compactness

A subset $S \subseteq \mathbb{C}$ is said to be **compact** if it is closed and bounded.

Theorem

Any continuous complex valued function on a compact subset $S \subseteq \mathbb{C}$ is bounded. i.e,
$\exists M \in \mathbb{R}$ such that $|f(z)| < M$ for all $z \in S$.

Even the converse is true; if a subset $S \subseteq \mathbb{C}$ has the property that every continuous function on it is bounded, then S is compact. (Exercise !)

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Ditto in the complex case. Let $\Omega \subset \mathbb{C}$ be open. A function $f : \Omega \subset \mathbb{C} \mapsto \mathbb{C}$ is said to be **differentiable**, (sometimes called complex-differentiable) at z_0 if

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Remark: If $f : \Omega \to \mathbb{C}$ is a function, then f can be thought of as a function from $\mathbb{R}^2 \to \mathbb{R}^2$, i.e two functions of two real variables $f(x, y) = (u(x, y), v(x, y))$, where u and v are the real and imaginary parts of f. It can be shown that if f is a holomorphic function, then thought of as a real function from $\mathbb{R}^2$ to itself in the above sense, it is infinitely differentiable; equivalently, both u and v are infinitely differentiable functions of x and y (all partial derivatives exist up all orders). We will soon see that the converse is not true; most infinitely differentiable functions from the real plane ($\mathbb{R}^2$) to itself are **NOT** holomorphic.

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Differentiation

Check for differentiability and holomorphicity:

1. $f(z) = c$ d and h
2. $f(z) = z$ d and h
3. $f(z) = z^n, n \in \mathbb{Z}$ d and h
4. $f(z) = \mathrm{Re}(z)$ d and h
5. $f(z) = |z|$
6. $f(z) = |z|^2$
7. $f(z) = \bar{z}$
8. $f(z) = \begin{cases} \frac{z}{\bar{z}} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$

Syllabus

Definition and properties of analytic functions.
Cauchy-Riemann equations, harmonic functions.
Power series and their properties.
Elementary functions.
Cauchys theorem and its applications.
Taylor series ans Laurent expansions.
Residues and the Cauchy residue formula.
Evaluation of improper integrals.
Conformal mappings.
Inversion of Laplace transforms.

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2. J. M. Howie, Complex analysis, Springer-Verlag (2004).
3. M. J. Ablowitz and A. S. Fokas, Complex Variables-Introduction and Applications, Cambridge University Press, 1998 (Indian Edition).
4. E. Kreyszig, Advanced engineering mathematics (8th Edition), John Wiley (1999).

More advanced references :

1. Lars Ahlfors - Complex Analysis
2. John Conway - Functions of a Complex Variable
3. Serge Lang - Complex Analysis