HOMEWORK 5

Q3 Define a ~~mapping~~ mapping $L_\alpha : V \to V^{**}$.
such that $L_\alpha(f) = f(\alpha) \quad \forall \quad f$ in V^*.

- TPT :- L_α is linear
Consider $\alpha, \beta \in V$ and $c_1, c_2 \in F$
Let $\gamma = c_1\alpha + c_2\beta$

$$L(\gamma) = L(c_1\alpha + c_2\beta)$$
$$= f(c_1\alpha + c_2\beta)$$
$$= c_1 f(\alpha) + c_2 f(\beta)$$
$$= c_1 L(\alpha) + c_2 L(\beta)$$

- L is non-singular :- $L(\alpha) = 0$ iff $\alpha = 0$.

- Since ~~V~~ $\dim V = \dim V^{**}$ $(= \dim V^*)$ and L is non-singular,
$\therefore$ L is invertible

Hence, L defines an isomorphism.

Q4. $L_1 : U \to V \quad \in \mathbb{L}(U, V)$
$L_2 : U \to V \quad \in \mathbb{L}(U, V)$

→ $(L_1 + L_2)(c_1u_1 + c_2u_2) = L_1(c_1u_1 + c_2u_2) + L_2(c_1u_1 + c_2u_2)$
$= c_1L_1(u_1) + c_2L_1(u_2) + c_1L_2(u_1) + c_2L_2(u_2)$
$= c_1(L_1 + L_2)(u_1) + c_2(L_1 + L_2)(u_2)$

✓

→ Zero mapping:- $0(c_1u_1 + c_2u_2) = c_1 0(u_1) + c_2 0(u_2)$
$= c_1(0) + c_2(0)$
$= 0$

→ Inverse mapping:- $[f + (-f)]u = f(u) + (-f)(u)$
$= f(u) + f(-u)$
$= f(u - u)$
$= f(0) = 0$

Q5 → $\delta : \mathbb{R}^N \to \mathbb{R}^E$, where $\mathbb{R}^N = \{f : N \to \mathbb{R}\}$ is a vector space

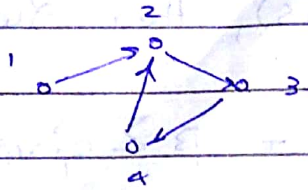
$\mathbb{R}^E = \{f : E \to \mathbb{R}\}$ " " " "

$$\delta(n_1, n_2) = f(n_2) - f(n_1).$$

★ → $\delta(af + bg) = a\,\delta(f) + b\,\delta(g)$

Prove linear map

→



$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} = M_\delta$$

→ $\delta^T : (\mathbb{R}^E)^* \to (\mathbb{R}^N)^*$ $\quad (\mathbb{R}^E)^* = \{f : E \to \mathbb{R}\}$

$(\mathbb{R}^N)^* = \{f : N \to \mathbb{R}\}$

Q6 $W_{KVL} = \operatorname{Im} \delta \subseteq \mathbb{R}^E$

$W_{KCL} = \operatorname{Ker} \delta^T \subseteq (\mathbb{R}^E)^*$

Proof

- $L : V \to W$ $\qquad L^T : W^* \to V^*$

$v_1 \quad Lv_1 \qquad\qquad w_2^* \quad L^T w_2^*$.

$(w_2^*, Lv_1) = (L^T w_2^*, v_1)$

$w_2^*(Lv_1) = (w_2^* \circ L)(v_1) = (L^T w_2^*)(v_1)$

- If $v_1 \in \ker L$ and $\overset{L^T\omega_2^*}{\cancel{\ }} \in \operatorname{Im} L^T$, then $(\overset{L^T\omega_2^*}{\cancel{\ }}, v_1) = 0$

→ ∴ If $v \in W_{KVL} \subseteq \mathbb{R}^E$, $i \in W_{KCL} \subseteq (\mathbb{R}^E)^*$

Then $(i, v) = 0$

$$\therefore \sum_{e \in E} v_e i_e = 0.$$

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Q7 $S = \{v_1, v_2 \ldots\ldots v_n\}$ for $v_i \in V$.

$\text{Span} = \{v \mid v = \sum_{i=1}^{n} c_i v_i$ for all combinations of $c_i \in \mathbb{R}\}$

$\rightarrow$ If $\cancel{v_1}, \cancel{v_2}$ $w_1, w_2 \in \text{span}(S)$, then $\alpha w_1 + \beta w_2 \in \text{span}(S)$

$$\alpha w_1 + \beta w_2 = \alpha \sum c_i v_i + \beta \sum c_i' v_i$$

$$= \sum k_i v_i$$

- Hence, for $\alpha = \beta = 0$, span also contains zero vector.

$\therefore$ It is a vector space

Q8 A] Definition

- ker L contains vectors from domain 'V'

  $\therefore \ker L \subseteq V$

- $L(0) = 0 \Rightarrow$ Zero vector $\in \ker L$

- If $v_1, v_2 \in \ker L$, then

  $L(\alpha v_1 + \cancel{\ } \beta v_2) = \alpha L(v_1) + \beta L(v_2)$

  $= 0$

  $\therefore \alpha v_1 + \beta v_2 \in \ker L$

Hence, ker L is a subspace of V.

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B] Definition

- $\operatorname{im} L$ takes vectors in W.

  $\therefore \quad \operatorname{im} L \subseteq W$

- There exists zero vector in $\operatorname{im} L$, corresponding to zero vector in V.

- If $v_1, v_2 \in \operatorname{im} L$, having preimages u_1 and u_2 in V; then $c_1 v_1 + c_2 v_2 \in \operatorname{im} L$, with preimage $c_1 u_1 + c_2 u_2$

$$L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2)$$
$$= c_1 v_1 + c_2 v_2$$

Hence $\operatorname{im} L$ is a subspace of W.

Q9 $\quad W^{\perp} \subseteq V^{*}$

$$W^{\perp} = \{ \omega_2^{*} \in V^{*} \mid (\omega_2^{*}, \omega_1) = 0 \quad \forall\ \omega_1 \in W \}$$

Q10 $\quad L : V \to W \qquad\qquad L^{T} : W^{*} \to V^{*}$

$\qquad v_1 \to L v_1 \qquad\qquad \omega_2^{*} \to L^{T} \omega_2^{*}$

- $(\omega_2^{*}, L v_1) = \omega_2^{*}[L(v_1)] = (\omega_2^{*} \circ L) v_1 = [L^{T}(\omega_2^{*})](v_1)$

$$= (L^{T} \omega_2^{*}, v_1)$$

- Now, $v_1 \in \ker L$ and $v_2^{*} \in \operatorname{Im} L^{T}$

  Here $v_2^{*} = L^{T}(\omega_2^{*})$

P.T.O.

$(v_2^*, v_1) = (L^T w_2^*, v_1) = (w_2^*, Lv_1) = (w_2^*, 0)$

$= 0$

$\therefore (v_2^*, v_1) = 0$ for all such v_2^*, v_1

$\therefore (\ker L)^{\perp} = \operatorname{im} L^T$

Q11

A] Let $W = \bigcap_i U_i$ where U_i are vector spaces containing S.

- Zero vector belongs to all $U_i \Rightarrow$ It belongs to W.

- Vectors $v_1, v_2 \in W$ must also belong to all U_i

For scalars α and β, αv_1 and βv_2 as well as $\alpha v_1 + \beta v_2$ belong to all U_i

$\therefore \alpha v_1 + \beta v_2 \in W$.

$\therefore$ W is a vector space.

B] All vectors v_i in $S = \{v_i\}$ belong to W.
Hence all linear combinations $\sum c_i v_i$ belong to W $(c_i \in \mathbb{R})$

- The smallest vector space containing S is the span of S.
$\therefore$ W must be the span of S.