MA205-7

Rectangular contours:

This is used usually when the integrand is periodic, eg. with trigonometric functions. The period decides the height of the rectangle.

Evaluate $I = \int_{-\infty}^{\infty} \frac{\sin x}{\sinh x} \, dx$

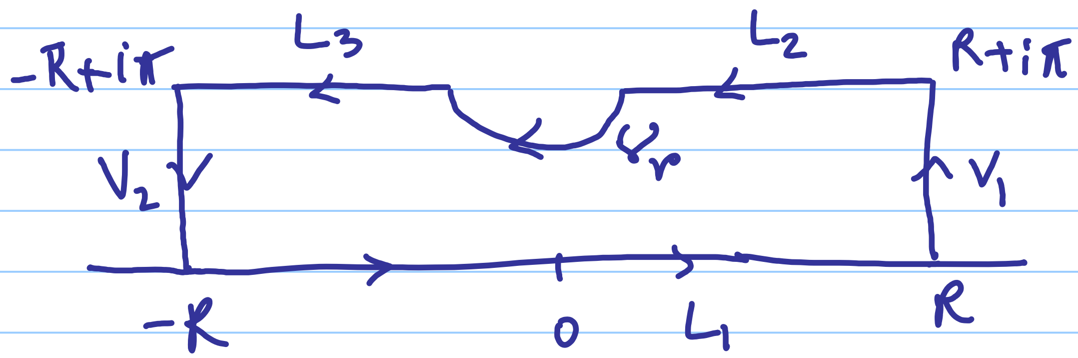
Let $f(z) = \frac{\sin z}{\sinh z}$.

Then f is holomorphic in a neighbourhood of 0.

Take a rectangle with vertices

$-R, R, R+i\pi, R-i\pi$ indented at $i\pi$

to avoid the pole.



Note: $\displaystyle\int_{L_1} f(z)\,dz \longrightarrow I$ as $R\to\infty$

Along V_1 : $z = R+it$: $0 \le t \le \pi$

$$dz = i\,dt$$

$$\sin z = \sin R \cosh t + i \cos R \sinh t$$

Let M be such that $|\sin z| \le M$

(why does such a M exist?)

$$\sinh z = -i\sin(iz) = -i\sin(-t+iR)$$

$$\Rightarrow |\sinh z|^2 = |\sin t\cosh R - i\cos t\sinh R|^2$$

$$= \sin^2 t \cdot \cosh^2 R + \cos^2 t\sinh^2 R$$

$$= \sin^2 t\,(1+\sinh^2 R) + \cos^2 t\sinh^2 R$$

$$= \sin^2 t + \sinh^2 R \geq \sinh^2 R$$

$$\Rightarrow \left|\frac{\sin z}{\sinh z}\right| \leq \frac{M}{\sinh R}$$

$$\left|\int_{V_1} f(z)\,dz\right| \leq \int_0^{\pi} \frac{M}{\sinh R}\,dt = \frac{M\pi}{\sinh R} \to 0$$

as $R \to \infty$

Similarly, $\left|\int_{V_2} f(z)\,dz\right| \to 0$ as $R \to \infty$

$$\int_{L_2} f(z)\,dz + \int_{L_3} f(z)\,dz =$$

$$-\left\{\int_{-R}^{-r} f(t+i\pi)\,dt + \int_{r}^{R} f(t+i\pi)\,dt\right\}$$

Now $f(t+i\pi) = \dfrac{\sin(t+i\pi)}{\sinh(t+i\pi)}$

$$= \frac{\sin t \cosh\pi + i \cos t \cdot \sinh\pi}{-\sinh t}$$

$$= (-\cosh\pi)\frac{\sin t}{\sinh t} - i\sinh\pi \cdot \frac{\cos t}{\sinh t}$$

$$\int_{L_2} f(z)\,dz + \int_{L_3} f(z)\,dz$$

$$= (\cosh\pi)\left\{\int_{-R}^{-r} \frac{\sin t}{\sinh t}\,dt + \int_{r}^{R} \frac{\sin t}{\sinh t}\,dt\right\} \to \cosh\pi \cdot I$$

as $R \to \infty$ & $r \to 0$

$$\lim_{r\to 0} \int_{\gamma_r} f(z)\,dz = -i\pi \operatorname{Res}(f; i\pi)$$

Taking limits as $R \to \infty$ & $r \to 0$, we get

$$I(1+\cosh\pi) - i\pi \operatorname{Res}(f; i\pi) = 0$$

As $i\pi$ is a simple pole, we get :

$$\operatorname{Res}(f; i\pi) = \lim_{z\to i\pi} \frac{(z-i\pi)}{\sinh z} \sin z$$

$$= \frac{\sin(i\pi)}{\cosh i\pi} = i\,\frac{\sinh\pi}{\cos\pi} = -i\sinh\pi$$

$$\therefore\ I(1+\cosh\pi) - \pi\sinh\pi = 0$$

$$I = \frac{\pi\sinh\pi}{1+\cosh\pi}$$

Key hole contours:

This is usually used to avoid a branch cut.

Evaluate: $I = \int_0^\infty \frac{x^{a-1}}{1+x}\, dx \;:\; 0<a<1$

Let $f(z) = \frac{z^{a-1}}{1+z} = \frac{\exp (a-1) \log z}{1+z}$

$\log z = \ln|z| + i \arg z$ defined on

$\mathbb{C} \setminus [0,\infty) \;:\; 0 < \arg z < 2\pi$

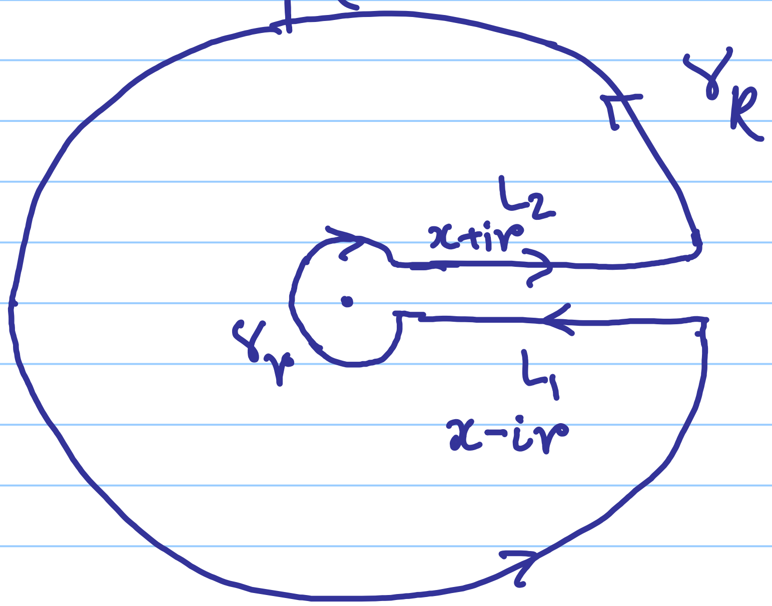
f has a simple pole at $z=-1$.

$\text{Res}(f;-1) = \lim_{z\to -1} \exp (a-1) \log z$

$= \exp\big((a-1)(i\arg(-1))\big)$

$$= \exp\left[(a-1)i\pi\right]$$

$$= -\exp(ai\pi)$$



By Cauchy's theorem :

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz + \int_{\gamma_r} f(z)\,dz + \int_{\gamma_R} f(z)\,dz$$

$$= -2\pi i \exp(ia\pi)$$

$$\int_{\gamma_R} f(z)\,dz = \int_{r}^{2\pi - r} \frac{R^{a-1} e^{(a-1)i\theta} \cdot Rie^{i\theta}\,d\theta}{1+Re^{i\theta}}$$

$$= i \int_{r}^{2\pi-\varepsilon} \frac{R^a}{1+Re^{i\theta}} \cdot e^{ia\theta} \, d\theta$$

As $0<a<1$, this integral $\to 0$ as $R \to \infty$

Show that $\int_{\gamma_r} f(z)\,dz \to 0$ as $r \to 0$

Now $\int_{L_2} f(z)\,dz = \int_{r}^{\infty} \frac{(x+ir)^{a-1}}{1+x+ir} \, dx$

$$\longrightarrow \int_{0}^{\infty} \frac{x^{a-1}}{1+x} \, dx \quad \text{as } r \to 0.$$

Along L_1, $f(z) = \frac{(x-ir)^{a-1}}{1+x-ir} \longrightarrow \frac{|x|^{a-1} \cdot e^{2\pi i a}}{1+x}$

as $r \to 0$

$$\Rightarrow \int_{L_1} f(z)\,dz \longrightarrow -\int_{0}^{\infty} \frac{|x|^{a-1} e^{2\pi i a}}{1+x} \, dx = -e^{2\pi i a} I$$

So $\lim\limits_{\substack{R\to\infty \\ r\to 0}} \left(\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz \right)$

$= (1-e^{2\pi i a}) I$

So letting $R\to\infty$ & $r\to 0$ in the equation $(*)$, we get:

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz + \int_{\gamma_r} f(z)\,dz + \int_{\gamma_R} f(z)\,dz$$

$$= 2\pi i \operatorname{Res}(f; -1)$$, we get

$$(1-e^{2\pi i a}) I = -e^{i\pi a} \cdot 2\pi i$$

ie $\quad I = \dfrac{2\pi i}{e^{i\pi a} - e^{-i\pi a}}$

$$= \pi \operatorname{cosec} \pi a$$

Gamma function :

The above integral is used in the study of the Gamma functions. This function interpolates the $n!$ function. It is defined as:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \quad : \text{for } \mathrm{Re}(z) > 0$$

Check this integral converges absolutely in the right half plane. So its a holomorphic function in the right half plane. Clearly, $\Gamma(1) = 1$.

Using integration by parts in the region of convergence, we get

$$\Gamma(z+1) = z \cdot \Gamma(z)$$

Thus, $\Gamma(n+1) = n \cdot \Gamma(n) = n(n-1)\Gamma(n-1) \dots = n!$

ie, $\Gamma(z)$ interpolates the $n!$ function.

Using $\Gamma(z+1) = z \cdot \Gamma(z)$ we extend Gamma to the whole complex plane.

Again, as $\Gamma(0+1) = 0 \cdot \Gamma(0)$, we see that 0 must be a pole for the extended Gamma function. Similarly, all the negative integers are also poles.

<u>Check:</u> These are the only poles & that these are simple poles.

Show that : $\operatorname{Res}(\Gamma; -n) = \frac{(-1)^n}{n!}$.

Calculate $\Gamma(x) \cdot \Gamma(y)$:

$$\Gamma(x) \cdot \Gamma(y) = \int_0^\infty \int_0^\infty e^{-u-v} u^{x-1} v^{y-1} \, du \, dv$$

Put $u = zt$; $v = z(1-t)$ & apply the change of variables formula to get : $\Gamma(x) \cdot \Gamma(y) = \Gamma(x+y) \cdot B(x,y)$

where $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \, dt$.

Put $t = \frac{s}{s+1}$ to get :

$$B(1-c,c) = \int_0^\infty \frac{s^{-c}}{1+s}\,ds \qquad : \text{for } 0<c<1$$

Thus for $0<x<1$,

$$\Gamma(x)\cdot\Gamma(1-x) = \int_0^\infty \frac{t^{-x}}{1+t}\,dt = \pi\cdot\operatorname{cosec}(\pi x)$$

In particular, $\Gamma(1/2) = \sqrt{\pi}$

In fact, $\Gamma(z)\,\Gamma(1-z) = \pi\cdot\operatorname{cosec}(\pi z)$

for all $z\in\mathbb{C}$ using the identity theorem on the region where Γ is holomorphic & extending this to $\mathbb{C}$.

(check).

Maximum modulus theorem:

A non-constant holomorphic function on a domain never attains its maximum modulus.

Proof: Suppose there is $z_0 \in \Omega$ such that $|f(z_0)| \geq |f(z)| \ : \forall \ z \in \Omega$. We will show that : then f is a constant function.

Let $B_\varepsilon(z_0) \subseteq \Omega$ be such that its closed disc with boundary γ is contained in Ω.

CIF gives:

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-z_0}\, dz$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta$$

Hence $$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})|\, d\theta$$

$$\leq |f(z_0)$$

as $|f(z_0)|$ is assumed to be the maximum value.

Thus $$\int_0^{2\pi} \left[|f(z_0)| - |f(z_0 + re^{i\theta})| \right] dt = 0$$

As the integrand is non-negative, it has to be zero.

That is, $|f(z_0)| = |f(z_0 + re^{i\theta})| \ \forall \theta.$

As this holds $\forall \ r \to 0$, we have

$f(z)$ is a constant on a small disc

around z_0. This implies that f is

a constant on Ω by the identity

theorem as the disc has limit points!

<u>Remark</u>: If we include the boundary

of Ω then f attains its maximum.

Combining this with the above Theorem

says f attains its maximum on the

boundary.