Partial Differential Equations

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CHAPTER 1

Tutorial Problems

1.1. Power series and series solutions

Problems.

(1) Find the radius of convergence of the following power series:
  (a) $\sum x^n$
  (b) $\sum \frac{x^m}{m!}$
  (c) $\sum m! x^m$
  (d) $\sum_{m=k}^{\infty} m(m-1)\cdots(m-k+1)x^m$
  (e) $\sum \frac{(2n)!}{2^{2n}(n!)^2} x^n$
  (f) $\sum_{1}^{\infty} \frac{x^m}{m(m+1)\cdots(m+k+1)}$
  (g) $\sum_{1}^{\infty} \frac{n^n}{n!} x^n$
  (h) $\sum_{1}^{\infty} \frac{(2n)!}{n^n} x^n$
  (i) $\sum_{1}^{\infty} \frac{(3n)!}{2^n (n!)^3} x^n$

(2) Determine the radius of convergence of
$$\sum n! x^{n^2} \quad \text{and} \quad \sum x^{n!}.$$

(3) Show that if $\sum_{n=1}^{\infty} a_n x^n$ has radius of convergence R, then $\sum_{n=1}^{\infty} a_n x^{2n}$ has radius of convergence $\sqrt{R}$ and $\sum_{n=1}^{\infty} a_n^2 x^n$ has radius of convergence R^2.

(4) Apply the power series method around $x = 0$ to solve the following differential equations.
  (a) $(1-x^2)y' = y$
  (b) $y' = xy,\ y(0) = 1$
  (c) $(1-x^2)y' = 2xy$
  (d) $y' - 2xy = 1,\ y(0) = 0$. Use the solution to deduce the Taylor series for $e^{x^2} \int_0^x e^{-t^2}\, dt$.

(5) Find the power series solutions for the following differential equations around $x = 1$, that is in powers of $(x-1)$.
  (a) $y'' + y = 0$

(b) $y'' - y = 0$

(6) Find the power series solutions for the following differential equations around $x = 0$.

(a) Tchebychev equation:

$$(1 - x^2)y'' - xy' + p^2 y = 0.$$

When do we have polynomial solutions?

(b) Airy equation:

$$y'' - xy = 0.$$

(c) Hermite equation :

$$y'' - x^2 y = 0.$$

(7) Show that the function $(\sin^{-1} x)^2$ satisfies the initial value problem (IVP):

$$(1 - x^2)y'' - xy' = 2, \quad y(0) = 0,\, y'(0) = 0.$$

Hence find the Taylor series for $(\sin^{-1} x)^2$ around 0. What is its radius of convergence ?

(8) Show that the even and odd parts of the binomial series of $(1 - x)^{-m}$ are two linearly independent power series solutions of

$$(1 - x^2)y'' - 2(m + 1)xy' - m(m + 1)y = 0$$

around $x = 0$. Hence deduce that $\{(1 - x)^{-m}, (1 + x)^{-m}\}$ is another linearly independent set of solutions.

1.2. Legendre equation and Legendre polynomials

Problems.

(1) Express x^2, x^3, and x^4 as a linear combination of the Legendre polynomials. (This is possible since the Legendre polynomials form a basis for the vector space of polynomials.)

(2) Show that

$$\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n = \sum_{m=0}^{[n/2]}(-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}\, x^{n-2m}$$

where $[n/2]$ denotes the greatest integer less than or equal to $n/2$.

Both expressions equal $P_n(x)$, the n-th Legendre polynomial. The expression in the lhs is known as the Rodrigues formula.

(3) Show that if $f(x)$ is a polynomial with double roots at a and b then $f''(x)$ vanishes at least twice in (a,b). (This is also true if $f(x)$ is a smooth function.)

Generalize this and show (using Rodrigues' formula) that $P_n(x)$ has n distinct roots in $(-1,1)$.

(4) Take the Rodrigues formula as the definition for $P_n(x)$, and show the following relations.
   (a) $P_n(-x) = (-1)^n P_n(x)$
   (b) $P_n'(-x) = (-1)^{n+1} P_n'(x)$
   (c) $P_n(1) = 1$ and $P_n(-1) = (-1)^n$
   (d) $P_{2n+1}(0) = 0$ and $P_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}$
   (e) $P_n'(1) = \frac{1}{2}n(n+1)$ and $P_n'(-1) = (-1)^{n-1}\frac{1}{2}n(n+1)$
   (f) $P_{2n}'(0) = 0$ and $P_{2n+1}'(0) = (-1)^n \dfrac{(2n+1)!}{2^{2n}(n!)^2}$.

(5) Show that

$$\int_{-1}^{1}(1-x^2)P_m'(x)P_n'(x)\,dx = \begin{cases} \dfrac{2n(n+1)}{2n+1} & \text{if } m=n, \\ 0 & \text{otherwise.} \end{cases}$$

(6) Show the following relations when $n-m$ is even and nonnegative.
   (a) $\displaystyle\int_{-1}^{1} P_m' P_n' dx = m(m+1)$
   (b) $\displaystyle\int_{-1}^{1} x^m P_n'(x)dx = 0$. What is the value of the integral if $n-m$ is odd (instead of even)?

(7) If $x^n = \displaystyle\sum_{r=0}^{n} a_r P_r(x)$, then show that $a_n = \dfrac{2^n(n!)^2}{(2n)!}$.

(8) Expand the following functions $f(x)$ in a series of Legendre polynomials:

$$f(x) \approx \sum_{n\geq 0} c_n P_n \quad \text{with} \quad c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx.$$

The Rodrigues formula is useful to evaluate these integrals. The Legendre expansion theorem (stated in the lecture notes) applies in each case.

(a)
$$f(x) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1. \end{cases}$$

(b)
$$f(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1. \end{cases}$$

(c)
$$f(x) = \begin{cases} -x & \text{if } -1 < x < 0 \\ x & \text{if } 0 < x < 1. \end{cases}$$

(d)
$$f(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ x & \text{if } 0 < x < 1 \end{cases}$$

(9) Consider the *associated Legendre equation*

$$(1 - x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1 - x^2}\right] y = 0 \tag{1}$$

which occurs in quantum physics. Substituting

$$y(x) = (1 - x^2)^{m/2} v(x),$$

show that v satisfies

$$(1 - x^2)v'' - 2(m+1)xv' + [n(n+1) - m(m+1)]v = 0 \tag{2_m}$$

Show that $v = D^m P_n$ satisfies (2_m). Thus

$$y(x) = (1 - x^2)^{m/2} D^m P_n(x)$$

is the bounded solution of (1) and is called an *associated Legendre function*.

1.3. Frobenius method for regular singular equations

Problems.

(1) Attempt a power series solution around $x = 0$ for
$$x^2y'' - (1+x)y = 0.$$
Explain why the procedure does not give any nontrivial solutions.

(2) Attempt a Frobenius series solution for the differential equation
$$x^2y'' + (3x-1)y' + y = 0.$$
Why does the method fail?

(3) Locate and classify the singular points for the following differential equations. (All letters other than x and y such as p, λ, etc are constants.)

(a) Bessel equation:
$$x^2y'' + xy' + (x^2 - p^2)y = 0.$$

(b) Laguerre equation:
$$xy'' + (1-x)y' + \lambda y = 0.$$

(c) Jacobi equation:
$$x(1-x)y'' + (\gamma - (\alpha+1)x)y' + n(n+\alpha)y = 0.$$

(d) Hypergeometric equation:
$$x(1-x)y'' + [c - (a+b+1)x)]y' - aby = 0.$$

(e) Associated Legendre equation:
$$(1-x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0$$

(f)
$$xy'' + (\cot x)y' + xy = 0.$$

(4) In Problem (3) above find the indicial equations corresponding to all the regular singular points.

(5) Find two linearly independent solutions of the following differential equations.

(a) $x(x-1)y'' + (4x-2)y' + 2y = 0.$

(b) $(1-x^2)y'' - 2xy' + 2y = 0.$

(c) $x^2y'' + x^3y' + (x^2-2)y = 0.$

(d) $xy'' + 2y' + xy = 0.$

1.4. Bessel equation and Bessel functions

Problems.

(1) Using the indicated substitutions, reduce the following differential equations to the Bessel equation and find the general solution in term of the Bessel functions.
 (a) $x^2y'' + xy' + (\lambda^2x^2 - p^2)y = 0, \quad (\lambda x = z)$
 (b) $xy'' - 5y' + xy = 0, \quad (y = x^3u)$.
 (c) $y'' + k^2xy = 0, \quad (y = u\sqrt{x}, \ \dfrac{2}{3}kx^{3/2} = z)$.
 (d) $x^2y'' + (1-2p)xy' + p^2(x^{2p} + 1 - p^2)y = 0, \ (y = x^pu, \ \ x^p = z)$.

(2) Show that
 (a) $J_{1/2} = \sqrt{\frac{2}{\pi x}}\sin x$
 (b) $J_{-1/2} = \sqrt{\frac{2}{\pi x}}\cos x$
 (c) $J_{3/2} = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$
 (d) $J_{-3/2} = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)$

(3) For an integer n show that $J_n(x)$ is an even (resp. odd) function if n is even (resp. odd).

(4) Show that between any two consecutive positive zeros of $J_n(x)$ there is precisely one zero of $J_{n+1}(x)$ and one zero of $J_{n-1}(x)$.

(5) Show the following.
 (a) $J_3 + 3J_0' + 4J_0''' = 0$.
 (b) $J_2 - J_0 = aJ_c''$ find a and c.
 (c) $\int J_{p+1}dx = \int J_{p-1}dx - 2J_p$.

(6) If y_1 and y_2 are any two solutions of the Bessel equation of order p, then show that $y_1y_2' - y_1'y_2 = c/x$ for a suitable constant c.

(7) Show that

$$\int x^\mu J_p(x)dx = x^\mu J_{p+1}(x) - (\mu - p - 1)\int x^{\mu-1}J_{p+1}(x)dx.$$

(8) Expand the indicated function in Fourier-Bessel series over the given interval and in terms of the Bessel function of given order. (The Bessel expansion theorem applies in each case.)
 (a) $f(x) = 1$ over $[0,3]$, $p = 0$.
 (b) $f(x) = x$ over $[0,1]$, $p = 1$.
 (c) $f(x) = x^3$ over $[0,3]$, $p = 1$.
 (d) $f(x) = x^2$ over $[0,2]$, $p = 2$.
 (e) $f(x) = \sqrt{x}$ over $[0,\pi]$, $p = \frac{1}{2}$.

(9) Show Schlömilch's formula

$$\exp\left(\frac{tx}{2} - \frac{x}{2t}\right) = \sum_{-\infty}^{\infty} J_n(x)t^n.$$

Use this formula to show that

$$J_0^2 + 2\sum_{n=1}^{\infty} J_n^2 = 1.$$

Deduce that $|J_0| \leq 1$ and $|J_n| \leq \frac{1}{\sqrt{2}}$.

(10) Show that

$$\begin{aligned}
\int J_0(x)dx &= J_1(x) + \int \frac{J_1(x)dx}{x} \\
&= J_1(x) + \frac{J_2(x)}{x} + 1.3 \int \frac{J_2(x)dx}{x^2} \\
&= J_1(x) + \frac{J_2(x)}{x} + \frac{1.3J_3(x)}{x^2} + 1.3.5 \int \frac{J_3(x)dx}{x^3} \\
&. \\
&. \\
&. \\
&= J_1(x) + \frac{J_2(x)}{x} + \frac{1.3J_3(x)}{x^2} + \cdots + \frac{1.3.5\ldots(2n-3)J_n(x)}{x^{n-1}} \\
&\qquad + 1.3.5\ldots(2n-1) \int \frac{J_n(x)dx}{x^n}
\end{aligned}$$

1.5. Fourier series

Problems.

(1) Show that

$$\sum_{n=1}^{\infty} \frac{1}{n} \sin nx \sin^2 n\alpha = \begin{cases} \text{constant} & (0 < x < 2\alpha) \\ 0 & (2\alpha < x < \pi) \end{cases}$$

(2) Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos nx}{n^2} = \frac{\pi^2}{12} - \frac{x^2}{4}, \quad (-\pi \le x \le \pi).$$

(3) Show that

$$\sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{(2n+1)^3} = \frac{1}{8}\pi x(\pi - x), \quad (0 \le x \le \pi).$$

(4) Use the Fourier expansions given in problems (1), (2) and (3) along with Fourier's Theorem to deduce the following results.

(a) $1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \ldots = \frac{2\pi}{3\sqrt{3}}$

(b) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \frac{1}{10} - \frac{1}{11} + \ldots = \frac{\pi}{3\sqrt{3}}$

(c) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$

(d) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6}$ (Euler's formula)

(e) $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - + \ldots = \frac{\pi^3}{32}$

(f) $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} \ldots = \frac{\pi^2}{8}$

(g) $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \ldots = \frac{\pi}{4} - \frac{1}{2}$

(5) Using the Parseval identity, show that

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \ldots = \frac{\pi^4}{96}.$$

Hint: Use

$$f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2, \\ \pi - x & \text{if } \pi/2 < x < 3\pi/2. \end{cases}$$

or problem (2).

(6) Find the Fourier series of the function $f(x)$ which is assumed to have the period 2π, where

(a) $f(x) = x, \quad 0 < x < 2\pi.$

(b) $f(x) = \begin{cases} -x & -\pi \le x < 0 \\ x & 0 \le x < \pi \end{cases}$

(c) $f(x) = x + |x|, \quad -\pi < x < \pi.$

(7) Find the Fourier series of the periodic function $f(x)$ of period $p = 2$ where
$$f(x) = \begin{cases} 0 & -1 < x < 0, \\ x & 0 < x < 1. \end{cases}$$

(8) State whether the given function is even or odd. Find its Fourier series.
(a)
$$f(x) = \begin{cases} k & -\pi/2 < x < \pi/2, \\ 0 & \pi/2 < x < 3\pi/2. \end{cases}$$
(b)
$$f(x) = 3x(\pi^2 - x^2), \quad -\pi < x < \pi.$$

(9) Find the Fourier series for the given functions f on the prescribed interval.
(a)
$$f(x) = \begin{cases} -1 & \text{if } -1 \le x < 0 \\ 1 & \text{if } 0 \le x \le 1 \end{cases}$$
for $|x| \le 1$.
(b)
$$f(x) = \begin{cases} -x, & -1 \le x < 0 \\ x, & 0 \le x \le 1 \end{cases}$$
for $|x| \le 1$.
(c)
$$f(x) = \begin{cases} 0, & -2 \le x < 1 \\ 3, & 1 \le x \le 2 \end{cases}$$
for $|x| \le 2$.
(d)
$$f(x) = e^{x/a}, \quad |x| \le l.$$
(e)
$$f(x) = \sin^2 x, \quad |x| \le \pi.$$

(10) Expand each of the following functions in a Fourier cosine series on the prescribed interval.
(a)
$$f(x) = e^{-x}, \quad 0 \le x \le 1.$$
(b)
$$f(x) = \begin{cases} 0, & 0 \le x \le 1 \\ 1, & 1 \le x \le 2 \end{cases}$$
for $0 \le x \le 2$.
(c)
$$f(x) = 2 \sin x \cos x, \quad 0 \le x \le \pi.$$

(11) Expand each of the following functions in a Fourier sine series on the prescribed interval.
(a)
$$f(x) = e^{-x}, \quad 0 < x < 1.$$

(b)

$$f(x) = \begin{cases} x, & 0 < x < a \\ a, & a \leq x \leq 2a \end{cases}$$

for $0 < x < 2a$.

(c)

$$f(x) = 2\sin x \cos x, \quad 0 < x < \pi.$$

(d)

$$f(x) = \cos x, \quad 0 < x < \pi.$$

1.6. Heat equation by separation of variables

For the two-dimensional heat equation, the following are relevant.

(a) The Laplacian in polar coordinates in the plane is

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

(b) The Laplacian in spherical polar coordinates for the sphere of radius b is

$$\Delta = \frac{1}{b^2}\Big(\frac{\partial^2}{\partial \varphi^2} + \cot\varphi\frac{\partial}{\partial \varphi} + \frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial \theta^2}\Big).$$

Problems.

(1) Which of the following PDEs can be reduced to two or more ODEs by the method of separation of variables?
  (a) $au_{xy} + bu = 0$
  (b) $au_{xx} + 2bu_{xy} + cu_{yy} = 0$
  (c) $au_{xx} + 2bu_{xy} + cu_{y} = 0$
  (d) $z_{xx} + xyz_y = 0$
  (e) $f(x)\theta_{tt} = a^2[f(x)\theta_x]_x$

(2) The curved surface of a thin rod of length ℓ is insulated. The temperature throughout the rod is 100. If at each end of the rod the temperature is suddenly reduced to 0 at time $t = 0$, find the temperature subsequently. What is the explicit temperature at the mid-point of the rod and how does it behave with respect to the time variable t?

(3) Solve the following nonhomogeneous differential equation

$$u_t - u_{xx} = 8e^{-t}\sin 3x$$

with boundary and initial conditions:

$$u(0,t) = 0 = u(\pi,t) \quad \text{and} \quad u(x,0) = 2\sin 2x.$$

(4) Solve

$$u_t - u_{xx} = e^{-t}\cos 2x$$

with boundary and initial conditions:

$$u_x(0,t) = e^{-t},\ u_x(\pi,t) = -e^{-t} \quad \text{and} \quad u(x,0) = \sin x.$$

Hint: Start with $z(x,t) = e^{-t}\sin x$ to homogenize the boundary conditions.

(5) For the heat equation:

$$u_t - ku_{xx} = 0, \quad 0 < x < \ell,\ t > 0$$

with initial condition $u(x,0) = u_0(x)$, and Neumann boundary conditions $u_x(0,t) = u_x(\ell,t) = 0$, show that

$$\int_0^\ell u(x,t)\,dx = C,$$

where C is a constant. In other words, the average temperature stays constant. Further, show that

$$\lim_{t\mapsto\infty} u(x,t) = \frac{1}{\ell}\int_0^\ell u_0(x)\,dx.$$

Compute the solution, when u_0 is:

$$(i)\, u_0(x) = x \quad \text{and} \quad (ii)\, u_0(x) = \sin^2(\frac{\pi x}{\ell}).$$

(6) Show that in a thin rod with insulated ends, the average temperature

$$\bar{u} = \frac{1}{\ell} \int_0^\ell u(x,t) dx$$

remains constant. Show that the temperature distribution coverges to this constant at all points uniformly and exponentially fast as $t \to \infty$. For convenience, assume that $u(x,0) = u_0(x)$ is a Riemann integrable function on $[0,\ell]$.

(7) Compute the solution of

$$u_t - k u_{xx} + a^2 u = 0, \quad 0 < x < \ell,\ t > 0$$

with initial condition $u(x,0) = u_0(x)$, and Dirichlet boundary conditions $u(0,t) = u(\ell,t) = 0$. Find $\lim_{t\mapsto\infty} u(x,t)$.

(8) Solve the following heat equation:

$$u_t - k u_{xx} = 0, \quad 0 < x < \ell,\ t > 0$$

- (a) with zero initial condition $u(x,0) = 0$ and Dirichlet boundary conditions: $u(0,t) = 0$, and $u(\ell,t) = e^{-t}$. Assume $\ell/\pi c$ is not an integer.
- (b) with zero initial condition $u(x,0) = 0$ and Neumann boundary conditions: $u_x(0,t) = 0$ and $u_x(\ell,t) = e^{-t}$. Assume $\ell/\pi\sqrt{k}$ is not an integer.
- (c) with initial condition $u(x,0) = u_0(x)$ and Dirichlet boundary conditions: $u(0,t) = 0$ and $u(\ell,t) = t$. Discuss the behaviour of the solution for large t.

(9) A thin circular disc of radius R whose upper and lower faces are insulated is initially at the temperature $u(r,\theta) = f(r)$.

- (a) If the temperature along the circumference of the disc is suddenly reduced to 0 and maintained at that value, find the temperature in the disc as a function of (r,t).
- (b) For $f(r) = 100(1-r^2/R^2)$, if the temperature along the circumference of the disc is suddenly raised to 100 and maintained at that value, then find the temperature in the disc subsequently.

(10) A thin upper hemisphere of radius R whose outer and inner surfaces are insulated, is initially at temperature $u(\theta,\varphi) = f(\varphi)$, with φ being the polar angle. If the temperature around the boundary of the shell (the equator) is suddenly reduced to 0 and maintained at that value, find the subsequent temperature in the hemisphere as a function of (φ,t).

Find the explicit solution if

(a) $f(\varphi) = \cos 3\varphi$

(b) $f(\varphi) = \cos 2\varphi$.

What is the temperature at its topmost point as a function of t?

1.7. Wave equation by separation of variables

Problems.

(1) Consider the wave equation
$$u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < \ell,\ t > 0$$
with initial position $f(x)$, initial velocity $g(x)$ and Neumann boundary conditions $u_x(0,t) = u_x(\ell,t) = 0$. Compute the solution for:
(a) $f(x) = x^2(x-\ell), \quad g(x) = 0$
(b) $f(x) = \sin^2(\frac{\pi x}{\ell}), \quad g(x) = 0$
(c) $f(x) = 0, \quad g(x) = 1.$

(2) Solve the wave equation
$$u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < \ell,\ t > 0$$
with zero initial conditions and inhomogeneous Neumann boundary conditions: $u_x(0,t) = t$ and $u_x(\ell,t) = 0$.

(3) Solve the wave equation
$$u_{tt} - c^2 u_{xx} = -xe^{-t}/\ell, \quad 0 < x < \ell,\ t > 0$$
with initial conditions: $u(x,0) = u_t(x,0) = 0$ and boundary conditions: $u(0,t) = e^{-t}, u(\ell,t) = 1$.

(4) An elastic membrane in the shape of a plane circular sector of radius R and angle β is clamped along its boundary. Find the fundamental modes of vibrations.

(5) The portion of the cone $x^2 + y^2 = z^2 \tan^2 \alpha$ between its vertex O and the rim $z = R\cos\alpha$ is clamped along its rim. Find the fundamental harmonics. What are the orders of the Bessel equations that you get? What is the connection with the previous question?

(6) Find the azimuthal angle independent pure harmonics and their associated frequencies of a thin hemisphere of unit radius whose equator is clamped.

(7) A polar cap of the standard sphere $\mathbb{S}^2$ between the polar angles $\varphi = 0$ and $\varphi = \varphi_0$ is clamped along its rim. What are the frequencies of its fundamental modes (pure harmonics) which do not depend on the azimuthal angle θ. More precisely, write down the equation whose solutions are these frequecies. (This equation is known as the *characteristic equation* of the vibration problem.)

(8) Find the characteristic equation to determine the frequencies of the pure harmonics of an annulus $\{0 < a \leq r \leq b\}$ which do not depend on the angle θ in polar coordinates.

1.8. Laplace equation by separation of variables

Problems.

(1) Show that a solution of Laplace equation: $\Delta u = 0$ in the disc of radius 1 with the boundary condition : $u(1, \theta) = f(\theta)$ is given by

$$u(r, \theta) = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

where a_n, b_n are the Fourier coefficients of f.

(2) Using separation of variables, solve the Neumann problem for the Laplace equation: $\Delta u = 0$ in the disc of radius 1 with the boundary condition : $\frac{\partial u}{\partial r}(1, \theta) = \sin^3 \theta$.

(3) A thin sheet of metal bounded by x-axis and the lines $x = 0$ and $x = 1$ and extending to infinity in the y direction has its vertical edges maintained at the constant temperature $u = 0$. Over its lower edge the temperature distribution $u(x, 0) = 100$ is maintained. Find the steady-state temperature distribution. Solve the problem when the vertical edges are insulated and the lower edge is maintained at $u(x, 0) = \sin \pi x$.

(4) Use separation of variables to solve the Laplace equation $u_{xx} + u_{yy} = 0$, subject to the homogeneous boundary conditions $u(x, 0) = u_x(\pi, y) = u_x(0, y) = 0$ and the nonhomogeneous boundary condition $u(x, 2) = 4 + 3\cos x - 2\cos 2x$.

(5) A right circular solid cylinder of radius b and height h has its lower base maintained at the constant temperature $u = 100$ and its upper base at $u = 0$. If the curved surface is insulated, then find the steady state temperature distribution in the cylinder. What if the curved surface is maintained at $u = 50$ instead of being insulated?

(6) The upper half of the sphere of radius b is maintained at a temperature $u = 100$, and the lower half is maintained at $u = 0$. Find the steady-state temperature distribution in the solid enclosed by the sphere.

(7) Previous problem if the upper half is maintained at $u = 50 \cos \varphi$ and the lower half at $u = -50 \cos \varphi$.

(8) Find the steady-state temperature distribution in a thin unit spherical frustum between $z = \pm\frac{1}{2}$, whose upper boundary is maintained at the constant temperature T and the lower boundary as per (i) T (ii) $-T$.

(The frustum is the portion of the unit sphere whose z coordinate is between $-1/2$ and $1/2$.)

Generalize to $T = f(\theta)$, instead of being a constant.

(9) Show that in solving for the steady-state temperature distribution in space using the separation of spherical coordinates (r, θ, φ), we are naturally led to solving for the amplitudes of the pure harmonics of the unit sphere.

(10) Find the steady-state temperature function in the shell enclosed between two concentric spheres of radii b_1 and b_2 respectively, if the temperature distributions $u(b_1, \varphi) = f_1(\varphi)$ and $u(b_2, \varphi) = f_2(\varphi)$ are maintained over the inner and outer surfaces, respectively. Solve explicitly when $b_1 = 1$, $b_2 = 2$ and $f_1(\varphi) = \cos \varphi$ and $f_2(\varphi) = 3 \cos 2\varphi$.

(11) What is the gravitational potential of a thin circular disc of radius a and mass M if the potential on the perpendicular axis of the disc at a distance

r from the centre of the disc is $\dfrac{2M}{a^2}\left(\sqrt{r^2+a^2}-r\right)$? (The gravitational potential satisfies the Laplacian in three space.)