MA 207 - Tutorial - 4

1. Q1. Let $J_p(x)$ and $Y_p(x)$ be two linearly independent solutions of Bessel equation of order p, namely

$$x^2y'' + xy' + (x^2 - p^2)y = 0, \quad p \geq 0$$

   (a)
$$y(x) = c_1 J_p(\lambda x) + c_2 Y_p(\lambda x)$$

   (b)
$$y(x) = x^3 \left(c_1 J_3(x) + c_2 Y_3(x)\right)$$

   (c)
$$y(x) = \sqrt{x}\left(c_1 J_{1/3}\left(\frac{2k}{3}x^{3/2}\right) + c_2 J_{-1/3}\left(\frac{2k}{3}x^{3/2}\right)\right)$$

   (d)
$$y(x) = x^\nu \left[c_1 J_p\left(x^\nu\right) + c_2 Y_p\left(x^\nu\right)\right]$$

2. Q5. Use Q2(f).

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1. Q1. The constant is $\pi/4$.

2. Q4.

   (a) In Q1, take $\alpha = \pi/3$ and $x = \pi/3$.

   (b) In Q1, take $\alpha = \pi/3$ and $x = 2\pi/3$ which is a jump discontinuity.

   (c) take $x = 0$ in Q2.

   (d) take $x = \pi$ in Q2.

   (e) take $x = \pi/2$ in Q3.

   (f) take sum of (c) and (d).

   (g) take $\alpha = \pi/2 = x$ and use $\dfrac{1}{n.(n+2)} = \dfrac{1}{2}\left(\dfrac{1}{n} - \dfrac{1}{n+2}\right)$

3. Q5. Fourier series are given below. For convergence, apply Fourier's theorem.

(a) 5(a).

$$f(x) = -\pi^2 - 12\sum_{n\geq 1}\frac{(-1)^n \cos nx}{n^2} - 4\sum_{n\geq 1}\frac{(-1)^n \sin nx}{n}$$

(b) 5(b)

$$f(x) = \frac{6}{\pi^2}\sum_{n\geq 1}\frac{(-1)^{n+1}\cos n\pi x}{n^2}$$

(c) 5(c)

$$f(x) = \frac{2}{\pi} + \frac{2}{\pi}\sum_{n\geq 2}\frac{1+(-1)^n}{1-n^2}\cos nx$$

(d) 5(f).

$$f(x) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{n\geq 1}\frac{(-1)^{n+1}}{4n^2-1}\cos 2n\pi x, \quad a_{2n-1} = 0 = b_n, \quad n \geq 1$$

(e) 5(g). $x\cos x$ will give sine series and $-\pi\cos$ will give cosine series. Take their sum.

$\sin kx$ is odd, so it will give sine series.

(f) 5(h).

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n\geq 0}\frac{\cos(2n+1)x}{(2n+1)^2} + \frac{2}{\pi}\sum_{n\geq 1}\frac{\sin nx}{n}$$

(g) 5(i).

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n\geq 0}\frac{\cos(2n+1)x}{(2n+1)^2}$$

(h) 5(j).

$$f(x) = \frac{1}{4} - \frac{2}{\pi^2}\sum_{n\geq 0}\frac{\cos(2n+1)\pi x}{(2n+1)^2} + \frac{1}{\pi}\sum_{n\geq 1}(-1)^{n+1}\frac{\sin n\pi x}{n}$$

(i) 5(k).

$$f(x) = \frac{4}{\pi}\sum_{n\geq 0}\frac{\sin(2n+1)\pi x}{(2n+1)}$$

4. Q6.

(a) 6(a).

$$e^{-x} = (1 - \frac{1}{e}) + \sum_{n\geq 1}\frac{2(1-\frac{(-1)^n}{e})}{n^2\pi^2+1}\cos n\pi x$$

(b) 6(b).

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\sum_{k\geq 0}\frac{(-1)^k}{2k+1}\cos\frac{(2k+1)\pi x}{2}$$

(c) 6(c).

$$f(x) = \frac{-8}{\pi}\sum_{k\geq 0}\frac{\cos(2k+1)x}{(2k+1)^2-4}$$

(d) 6(d).

$$f(x) = \frac{-2L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n\geq 1}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{L}$$

(e) 6(e).

$$f(x) = \frac{-7L^4}{5} + \frac{144L^3}{\pi^4}\sum_{n\geq 1}\frac{(-1)^n}{n^4}\cos\frac{n\pi x}{L}$$

(f) 6(f).

$$f(x) = \frac{-2L^4}{5} - 48\frac{L^4}{\pi^4}\sum_{n\geq 1}\frac{2(-1)^n+1}{n^4}\cos\frac{n\pi x}{L}$$

(g) 6(g).

$$f(x) = \frac{3L^4}{5} + 48\frac{L^4}{\pi^4}\sum_{n\geq 1}\frac{(-1)^n+2}{n^4}\cos\frac{n\pi x}{L}$$

5. Q7.

(a) 7(a).

$$f(x) = \sum_{n\geq 1}\frac{2n\pi\left(1-\dfrac{(-1)^n}{e}\right)}{n^2\pi^2+1}\sin n\pi x$$

(b) 7(b).

$$f(x) = \sum_{n\geq 1}\frac{-2a}{n\pi}\sin\frac{2n\pi x}{2a} + \sum_{n\geq 0}\left(\frac{2a}{(2n+1)\pi} + \frac{(-1)^n 4a}{(2n+1)^2\pi^2}\right)\sin\frac{(2n+1)\pi x}{2a}$$

(c) 7(c). $f(x) = \sin 2x$, so $b_2 = 1$, $b_n = 0$, $n \neq 2$

(d) 7(d).

$$f(x) = \frac{8}{\pi}\sum_{n\geq 1}\frac{n\sin 2nx}{4n^2-1}$$

(e) 7(e).

$$f(x) = \frac{-12L^3}{\pi^3}\sum_{n\geq 1}\frac{(-1)^n}{n^3}\sin\frac{n\pi x}{L}$$

(f) 7(f).

$$f(x) = \frac{-48L^4}{\pi^5} \sum_{n\geq 1} \frac{(-1)^n - 1}{n^5} \sin\frac{n\pi x}{L}$$

(g) 7(g).

$$f(x) = \frac{-240L^5}{\pi^5} \sum_{n\geq 1} \frac{(-1)^n + 1}{n^5} \sin\frac{n\pi x}{L}$$

MA 207 - Tutorial - 6

1. Q1.

   (a) Seperates. $a\dfrac{X'(x)}{X(x)} = -b\dfrac{Y(y)}{Y'(y)}$.

   (b) Does not separates if $abc \neq 0$.

   (c) Separates.

   (d) Separates.

   (e) Separates.

2. Q2.

   (a)

$$u(x,t) = \frac{8}{\pi^3} \sum_{n\geq 1} \frac{1}{(2n-1)^3} e^{-n^2\pi^2 t} \sin(2n-1)\pi x$$

   (b)

$$u(x,t) = \frac{\pi}{2} e^{-3t} \sin x + \frac{4}{\pi} \sum_{n=2}^{\infty} \frac{n(-1)^{n-1} - n}{(n^2-1)^2} e^{-3n^2 t} \sin nx$$

   (c)

$$u(x,t) = e^{-\pi^2 t} \cos\frac{\pi x}{2}$$

   (d)

$$u(x,t) = 3/5 + \sum_{n\geq 1} \left(-\frac{48}{n^4\pi^4} \left[(-1)^n + 2\right] \right) \cos(n\pi x)\, \exp(-n^2\pi^2 t).$$

   (e)

$$u(x,t) = \cos(\pi x)\, \exp(-\pi^2 t).$$

3. Q3.

(a)

$$u(x,t) = \sum_{n\geq 1} \left(\frac{768(-1)^n}{n^3\pi^3} - \frac{320(-1)^n}{n^3\pi^3} e^{\frac{-9}{16}n^2\pi^2 t} \right) \sin\frac{n\pi x}{4} + 15x + 1$$

(b)

$$\begin{aligned} u(x,t) &= z(x,t) + 2x + 1, \\ \text{where, } z(x,t) &= \sum_{n\geq 1} Z_n(t)\ \sin(n\pi x) \\ \text{and for } n \geq 1, \quad Z_n(t) &= \frac{4}{n^3\pi^3}\left[(-1)^n - 1\right]\left[1 + \exp(-n^2\pi^2 t)\right]. \end{aligned}$$

(c)

$$u(x,t) = -x - 9t + \sum_{n\geq 1} \frac{12}{n^4\pi^4}\left(1 - (-1)^n\right)\left(1 - e^{-3n^2\pi^2 t}\right)\cos n\pi x$$

(d)

$$\begin{aligned} u(x,t) &= z(x,t) - (\pi/2)x^2, \\ \text{where, } z(x,t) &= \sum_{n\geq 0} Z_n(t)\ \cos(n\pi x) \\ Z_0(t) &= -\pi t + (\pi/6), \\ Z_1(t) &= \left(2 - \frac{2}{\pi}\right)\exp(-3\pi^2 t), \\ \text{and for } n \geq 2, Z_n(t) &= \frac{2[(-1)^n + 1]}{3(1-n^2)n^2\pi} + c_n \exp(-3n^2\pi^2 t), \\ \text{where, } c_n &= \frac{2}{n^2\pi}\left[(-1)^n - \frac{(-1)^n + 1}{3(1-n^2)}\right]. \end{aligned}$$

(e)

$$u(x,t) = 2e^{-4t}\sin 2x + (e^{-t} - e^{-9t})\sin 3x$$

4. Q4. Heat equation is $u_t = ku_{xx}$ for $0 < x < l$.

$$u(x,t) = \frac{400}{\pi}\sum_{n\geq 0}\frac{1}{2n+1}\sin\frac{(2n+1)\pi x}{l} e^{-(2n+1)^2(\pi/l)^2 kt}$$

At the mid point, $u(l/2,t) =$.

$u(l/2,t) \to 0$ exponentially fast as $t \to \infty$.

5. Q5.

$$u(x,t) = \frac{e^{-t} - e^{-4t}}{3}\cos 2x + e^{-t}\sin x$$

6. Q7. Put $u(x,t) = e^{-a^2 t} v(x,t)$. Then v satisfies homogeneous heat equation with Dirichilet boundary conditions. Solve for v.

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Q-1 (b)

$$u(x,t) = \sum_{n\geq 1} \left(\frac{4[1-(-1)^n]}{n^3\pi^3} \right) \left(\frac{1}{3n\pi} \right) \sin(3n\pi t)\, \sin(n\pi x).$$

Q-1 (c)

$$u(x,t) = \sum_{n\geq 1} \left(\frac{48[1-(-1)^n]}{n^5\pi^5} \right) \left(\frac{1}{2n\pi} \right) \sin(2n\pi t)\, \sin(n\pi x).$$

Q-1 (d)

$$u(x,t) = (\pi/2)\cos(\sqrt{5}\ t)\ \sin(x) + \sum_{n\geq 2} \left(\frac{-4n[1+(-1)^n]}{\pi(1-n^2)^2} \right) \cos(n\sqrt{5}\ t)\ \sin(nx)$$

Q-1 (e)

$$u(x,t) = 4 + \sum_{n\geq 1} \left(\frac{192[(-1)^n - 1]}{n^4\pi^4} \right) \cos\left(\frac{n\pi\sqrt{5}}{2}\, t \right) \cos\left(\frac{n\pi x}{2} \right)$$

Q-1 (g)

$$u(x,t) = \frac{\pi^4}{30} + \sum_{n\geq 1} \left(\frac{-24[(-1)^n + 1]}{n^4} \right) \cos(n4t)\ \cos(nx).$$

Q-1 (h)

$$u(x,t) = \frac{\pi^4}{30} t + \sum_{n\geq 1} \left(\frac{-24[(-1)^n + 1]}{n^4} \right) \left(\frac{1}{4n} \right) \sin(n4t)\ \cos(nx).$$

Q-2 (b)

$$u(x,y) = \sum_{n\geq 1} \frac{a_n \sin\left(\frac{n\pi x}{2}\right) \sinh\left(\frac{n\pi(3-y)}{2}\right)}{\sinh\left(\frac{n\pi 3}{2}\right)}$$

where

$$a_n = \frac{-32\,[2(-1)^n + 1]}{n^3\pi^3}, \quad \text{for } n \geq 1.$$

Q-2 (c)

$$u(x,y) = \sum_{n\geq 1} \frac{a_n \sin(nx)\ \sinh(n(\pi - y))}{\sinh(n\pi)}$$

where

$$\begin{aligned} a_n &= \frac{-4n[1+(-1)^n]}{\pi(1-n^2)^2} \quad \text{for } n \geq 2, \\ a_1 &= \pi/2. \end{aligned}$$

Q-2 (e)

$$u(x,y) = \frac{a_0}{a}(a-x) + \sum_{n\geq 1} \left[\frac{a_n}{\sinh\left(\frac{n\pi a}{b}\right)} \right] \sinh\left(\frac{n\pi(a-x)}{b}\right) \cos\left(\frac{n\pi y}{b}\right)$$

Here, $a = 2, b = 1, a_0 = 1/2$ and for $n \geq 1$,

$$a_n = \frac{24\left[(-1)^n - 1\right]}{n^4\pi^4}.$$

Q-2 (f)

$$u(x,y) = \sum_{n\geq 1} \left[\frac{a_n}{\left(\frac{n\pi}{3}\right)\sinh\left(\frac{2n\pi}{3}\right)} \right] \cosh\left(\frac{n\pi x}{3}\right) \sin\left(\frac{n\pi y}{3}\right)$$

where

$$a_n = \frac{36\left[1 - (-1)^n\right]}{n^3\pi^3}.$$

Note: In this example, start with the method of separation of variables to obtain a solution which satisfies the zero boundary conditions $u(x,0) = 0, u(x,3) = 0, u_x(0,y) = 0$ and then obtain the required solution using the remaining non-zero boundary condition.