MA 205 Complex Analysis: Introduction

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Welcome to the first lecture of MA 205! It's a great course, and I hope all of you will find it fun. I taught this course in 2011 and I had a great time! Here's a very brief summary of all that's waiting for us in the next two months.

In this course, we'll study $\mathbb{C}$, and functions from $\mathbb{C}$ to $\mathbb{C}$, just as you studied $\mathbb{R}$, and functions from $\mathbb{R}$ to $\mathbb{R}$ in the early part of MA 105. But you didn't study arbitrary functions from $\mathbb{R}$ to $\mathbb{R}$. First you looked at continuous functions, then even better, differentiable, functions. Later in MA 105, you even studied $f : \mathbb{R}^m \mapsto \mathbb{R}^n$, their continuity, differentiability (both partial and total). Similarly, we'll be interested in $f : \mathbb{C} \mapsto \mathbb{C}$ which are differentiable. If f fails to be differentiable at some points, we'll also investigate such failure.

Thus, in a way,

> *in going from MA 105 to MA 205, we're just going from $\mathbb{R}$ to $\mathbb{C}$.*

But, as you'll see, the tone of trivialization in the above sentence is quite unjust. $\mathbb{C}$ is a thing of beauty, and analysis/calculus over here is immensely charming as well as extremely useful, that you do want to go from $\mathbb{R}$ to $\mathbb{C}$, and you don't want to go any further! In other words, there is MA 205 after MA 105, but no MA 305 or 405! Today, I'll try to illustrate this in two distinct ways; I'll give you two surprises!

First, let's ask ourselves: why did we come to $\mathbb{R}$ in MA 105? Why not $\mathbb{N}, \mathbb{Z}$, or $\mathbb{Q}$? Yes, $\mathbb{N}$ and $\mathbb{Z}$ are ruled out at the very start; they fail the "algebra test". $\mathbb{Q}$ passes the "algebra test" alright, after all it was constructed only to pass this test, but it fails the "analysis test"! Can someone tell me the only question that was asked in the analysis test?

$\mathbb{R}$ passes both these tests. But it fails another test. The test asks: do your polynomials have roots? (Won't you agree that this is an important test as well? Indeed the earlier algebra test was a specific case of this test. It picked a y and asked the family of y: does the polynomial $yx - 1$ has a root in you?)

Now, the simplest real polynomial that does not have a root in $\mathbb{R}$ is $x^2 + 1 = 0$. Now, suppose it has a root somewhere, and suppose we denote it by $\imath$, then of course $-\imath$ is also a root. (Why?) In other words, you are imagining an $\imath$, which has this property that $\imath^2 = -1$. And then we can write $x^2 + 1 = (x - \imath)(x + \imath)$.

Numbers

To pass this test, we imagine $\mathbb{C}$. We imagine $\mathbb{C}$ as

$$\mathbb{R} + \imath\mathbb{R}.$$

i.e., every element z of $\mathbb{C}$ is of the form $a + \imath b$, where a and b are in $\mathbb{R}$. Then we add:

$$(a + \imath b) + (c + \imath d) = (a + c) + \imath(b + d).$$

If $a + \imath b$ of $\mathbb{C}$ is identified with the vector (a, b) of $\mathbb{R}^2$, this is nothing but vector addition. But, we also multiply, which we didn't do with vectors in $\mathbb{R}^2$:

$$(a+\imath b)(c+\imath d) = ac+a\imath d+\imath bc+(\imath b)\cdot(\imath d) = (ac-bd)+\imath(ad+bc),$$

which is another complex number. (Incidentally, if $z = x + \imath y \in \mathbb{C}$, we call x to be $\mathrm{Re}(z)$ and y to be $\mathrm{Im}(z)$.) Now, $\mathbb{C}$ does pass the algebra and analysis tests. Is that clear to you? Does it pass our polynomial test?

YES, IT DOES!

Numbers

This is called the fundamental theorem of algebra: every complex polynomial has a root in $\mathbb{C}$. By the way, what other fundamental theorems do you have?

This is such a neat fact. Adjoin to $\mathbb{R}$ just one root of the simplest real polynomial which did not have a root in $\mathbb{R}$, call it $\mathbb{C}$, and you get all the roots of all the complex polynomials in $\mathbb{C}$! Wonderful!

If you look back, life wasn't this easy earlier. $\mathbb{Q}$ failed the analysis test. Take a simple instance of the failure, say, the Cauchy sequence $1, 1.4, 1.41, 1.414, 1.4142, \ldots$ converges to $\sqrt{2}$ which is not in $\mathbb{Q}$. Adjoining $\sqrt{2}$ to $\mathbb{Q}$ doesn't solve the problem at all! The Cauchy sequence $1, 1.7, 1.73, 1.732, \ldots$ converges to $\sqrt{3}$, which is not there in $\mathbb{Q} + \mathbb{Q}\sqrt{2}$.

Numbers

We'll prove this beautiful theorem later. For now, remember the statement, and remember that it's far from obvious! This is the first promised surprise - the algebra surprise.

Assume FTA for now.

Exercise: A complex polynomial of degree n has exactly n roots.

Exercise: Show that a real polynomial that is irreducible has degree at most two. i.e., if

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n, \; a_i \in \mathbb{R},$$

then there are real polynomials g and h such that $f(x) = g(x)h(x)$ if $n \geq 3$.

Limit

Now to the second promised surprise - the analysis surprise. Let's start with recalling the notions of limit, continuity, and differentiation.

Let f be a real valued function defined on some subset of $\mathbb{R}$.

$$\begin{aligned}
\lim_{x \to a} f(x) = L \quad &\iff \quad \text{values of } f(x) \text{ can be made as close to } L \\
&\qquad \text{as we like, by taking } x \text{ close enough to } a, \\
&\qquad \text{on either side of } a, \text{ but not equal to } a. \\
&\iff \quad |f(x) - L| \text{ can be made as small as we like} \\
&\qquad \text{by taking } |x - a| \text{ sufficiently small, for } x \neq a. \\
&\iff \quad \text{for every } \epsilon > 0, \text{ there is a number } \delta > 0 \text{ such} \\
&\qquad \text{that } |f(x) - L| < \epsilon \text{ whenever } 0 < |x - a| < \delta.
\end{aligned}$$

Thus, for every $\epsilon > 0$, however small it is, we can find $\delta > 0$, such that, if $x \in (a - \delta, a + \delta)$, then, $f(x) \in (L - \epsilon, L + \epsilon)$. If $f : \Omega \subset \mathbb{C} \mapsto \mathbb{C}$, then $L \in \mathbb{C}$ is the limit of f as $z \mapsto z_0$, $z_0 \in \mathbb{C}$, if for every $\epsilon > 0$, there is a number $\delta > 0$ such that $|f(z) - L| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

Limit

Of course, $|z - z_0|$ is the distance between z and z_0; thus, if $z = x + \imath y$ and $z_0 = a + \imath b$, then $|z - z_0| = \sqrt{(x-a)^2 + (y-b)^2}$. Thus, z is in the δ neighbourhood of z_0 means that z is inside the circle with center z_0 and radius δ. Thus, notationally there's no difference between the definitions of limit in the real and the complex variable cases, but while unravelling the definition, we do see a major difference. Remember this! Exactly as in the real case:

Theorem (Limit Laws)

Suppose $c \in \mathbb{C}$ *and* $\lim\limits_{z \to z_0} f(z)$ *and* $\lim\limits_{z \to z_0} g(z)$ *exist. Then,*

1. $\lim\limits_{z \to z_0} [f(z) + g(z)] = \lim\limits_{z \to z_0} f(z) + \lim\limits_{z \to z_0} g(z)$
2. $\lim\limits_{z \to z_0} [cf(z)] = c \lim\limits_{z \to z_0} f(z)$
3. $\lim\limits_{z \to z_0} [f(z)g(z)] = \lim\limits_{z \to z_0} f(z) \cdot \lim\limits_{z \to z_0} g(z)$
4. $\lim\limits_{z \to z_0} \dfrac{1}{f(z)} = \dfrac{1}{\lim\limits_{z \to z_0} f(z)}$, *if* $\lim\limits_{z \to z_0} f(z) \neq 0$.

In the real case:

A function f is continuous at a if

$$\lim_{x \to a} f(x) = f(a).$$

The function f is continuous on an interval if it is continuous at every point in the interval.

Similarly,

A function $f : \Omega \subset \mathbb{C} \mapsto \mathbb{C}$ *is continuous at* $z_0 \in \Omega$ *if*

$$\lim_{z \to z_0} f(z) = f(z_0).$$

The function f is continuous on a domain if it is continuous at every point in the domain.

Check the obvious results for continuous functions. Recall these.

Differentiation

In the real case: The derivative of a function f at a point a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists. Equivalently,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Ditto in the complex case. A function $f : \Omega \subset \mathbb{C} \mapsto \mathbb{C}$ is differentiable at z_0 if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. You can of course prove product rule, chain rule, etc.

Differentiation

We say that f is holomorphic on Ω if f is differentiable at each point of Ω. f is holomorphic at z_0 if it is holomorphic in some neighbourhood of z_0. Similarly, holomorphicity on a non open set.

Remark: Note that differentiability at a point does not imply holomorphicity at that point.

Now, the second surprise. If f is holomorphic in a domain, then f' is also holomorphic there. Thus, in a domain,

Once differentiable, always differentiable.

We'll prove this too, but later. As you know from MA 105, this is far from true in the real variable case. f' needn't even be continuous. Examples?

Can someone venture a guess as to why identical definitions lead to such extreme scenarios?

Differentiation

Check for differentiability and holomorphicity:

1. $f(z) = c$
2. $f(z) = z$
3. $f(z) = z^n, n \in \mathbb{Z}$
4. $f(z) = \mathrm{Re}(z)$
5. $f(z) = |z|$
6. $f(z) = |z|^2$
7. $f(z) = \bar{z}$
8. $$f(z) = \begin{cases} \frac{z}{\bar{z}} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$$