MA 205 Complex Analysis: Cauchy Integral Formula and its Beautiful Consequences

August 5, 2017

Recall

Last time we saw Cauchy's theorem which said that if Ω is a domain in $\mathbb{C}$, γ is a simply closed contour in Ω and f is a holomorphic function on an open set containing γ and its interior, then $\int_\gamma f(z)dz = 0$.

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Recall that for a parametrized differentiable curve $\gamma : [a, b] \to \Omega$, we define $\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$.

Basic Example

Consider $f(z) = \int_C \frac{1}{z-z_0} dz$ where C is any circle around z_0.

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Theorem

(More General form of Cauchy's theorem) *Let Ω be a domain in $\mathbb{C}$. If γ and γ' are two closed contours in Ω which can be "continuously deformed" into each other, then $\int_C f(z)dz = \int_{C'} f(z)dz$.*

Remark: A similar computation shows that $\int_C \frac{1}{(z-z_0)^m} = 0$ for all $m \neq 1$. This follows from the fact that $\frac{1}{(z-z_0)^m}$ admits a primitive in $\mathbb{C} - \{z_0\}$ when $m \neq 1$. Note that for $m = 1$, $\frac{1}{(z-z_0)^m}$ does not admit a primitive; $log(z - z_0)$ does not define a holomorphic function on any open set containing C.

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Let f be holomorphic everywhere within and on a simple closed curve γ (oriented positively). If z_0 is interior to γ, then,

$$f(z_0) = \frac{1}{2\pi\imath} \int_\gamma \frac{f(z)dz}{z - z_0}.$$

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We use the following fact in the proof, often called ML inequality : If γ is a contour of length L and $|f(z)| \leq M$ on γ, then $\int_{\gamma}|f(z)|dz \leq ML$.

Proof

Proof: We need to show that

$$\int_\gamma \frac{f(z)dz}{z - z_0} = \int_\gamma \frac{f(z_0)dz}{z - z_0},$$

since the latter integral is $2\pi\imath \cdot f(z_0)$.

Cauchy Integral Formula

Thus, we need to show that the difference

$$\int_\gamma \frac{f(z) - f(z_0)}{z - z_0} dz$$

is zero. Since f is continuous at z_0, given $\epsilon > 0$, there is $\delta > 0$ such that

$$|z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon.$$

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$$\left| \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \leq \int_{C_r} \frac{|f(z) - f(z_0)|}{|z - z_0|} dz \leq \frac{\epsilon}{r} 2\pi r = 2\pi\epsilon.$$

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Thus, $\left| \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0} dz \right|$ can be made arbitrarily small; i.e., it is zero.

Example:

(i) $\int_{|z|=2} \frac{e^z dz}{(z+1)(z-3)^2} = \int_{|z|=2} \frac{f(z)dz}{z+1}$, where $f(z) = \frac{e^z}{(z-3)^2}$. So by CIF, answer is $2\pi\imath f(-1) = \frac{\pi\imath}{8e}$.

(ii)
$\int_{|z|=6} \frac{dz}{z^3-1} = 2\pi\imath \left[\frac{1}{(1-\omega)(1-\omega^2)} + \frac{1}{(\omega-1)(\omega-\omega^2)} + \frac{1}{(\omega^2-\omega)(\omega^2-1)}\right] = 0.$

(iii) $\int_{|z|=3} \frac{\cos\pi z}{z^2-1} dz = \frac{1}{2}\int_{|z|=3} \left[\frac{\cos\pi z}{z-1} - \frac{\cos\pi z}{z+1}\right] dz = 0$

OR

$$\begin{aligned} \int_{|z|=3} \frac{\cos\pi z}{z^2-1} dz &= \int_{|z-1|=\varepsilon} \frac{\frac{\cos\pi z}{z+1}}{z-1} dz + \int_{|z+1|=\varepsilon} \frac{\frac{\cos\pi z}{z-1}}{z+1} dz \\ &= 0. \end{aligned}$$

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Here is an example where the Cauchy Integral formula can be used to compute a seemingly hard real integral.
Let k be a real constant. Show that $\int_0^{2\pi} e^{kcos\theta} sin(ksin\theta) d\theta = 0$ and $\int_0^{2\pi} e^{kcos\theta} cos(ksin\theta) d\theta = 2\pi$.

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Applying CIF to $\int_{|z|=1} \frac{e^{kz}}{z} dz$ gives $(2\pi i)e^{k.0} = 2\pi i$.

Hence $2\pi i = \int_{|z|=1} \frac{e^{k(cos\theta + isin\theta)}}{e^{i\theta}} ie^{i\theta} d\theta$
$= \mathrm{i} \int_0^{2\pi} e^{kcos\theta} [cos(ksin\theta) + isin(ksin\theta)] d\theta$
Equating the real and imaginary parts gives us the answer.

Summing Up

We saw two very important theorems, namely Cauchy's theorem and the Cauchy integral formula. The first said that the integral along a closed curve of a function is zero if the function is holomorphic on and within the curve. The second said:

$$f(z_0) = \frac{1}{2\pi\imath} \int_\gamma \frac{f(z)}{z - z_0} dz$$

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Poincare (1854-1912)

Poincare (1854 - 1912) was a great French mathematician and mathematical-physicist of the 19th and early 20th century. His mathematical contributions are as fundamental as those of Gauss and Riemann. He developed the branch of mathematics called Algebraic Topology; the notion of simply-connectedness being a small aspect of this subject. He also did fundamental work in differential equations, theory of several complex variables, number theory and possibly more. In physics, he made fundamental contributions in optics, electricity, elasticity, thermodynamics, potential theory, cosmology ... He also has his name attached to an important aspect of special relativity. He settled a special case of the 3-body problem in mechanics. One of the famous millenium prize problems, the Poincare conjecture is named after him. It was solved in 2006 by Perelman, roughly hundred years after its formulation.

Holomorphic $\implies$ Analytic

Let f be holomorphic at z_0. This means that f is holomorphic in a neighbourhood of z_0. Let $R > 0$ be such that f is holomorphic in $|z - z_0| < R$. Let γ be a circle of radius r with $r < R$ centered at z_0.

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$$f(z) = \frac{1}{2\pi\imath} \int_\gamma \frac{f(w)}{w - z} dw,$$

for any z such that $|z - z_0| < r$. Note that we have chosen $r < R$ so that f is holomorphic on and within γ.

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Text

$$\frac{1}{w - z}.$$

Also always keep in mind that the only series that we know well is the geometric series! Let's look at it closely.

Now,

$$
\begin{aligned}
\frac{1}{w-z} &= \frac{1}{w-z_0}\cdot\frac{w-z_0}{w-z}\\
&= \frac{1}{w-z_0}\cdot\frac{1}{1-\left[\frac{z-z_0}{w-z_0}\right]}\\
&= \frac{1}{w-z_0}\cdot\left[1+\left(\frac{z-z_0}{w-z_0}\right)+\left(\frac{z-z_0}{w-z_0}\right)^2+\dots,\right]
\end{aligned}
$$

since $\left|\frac{z-z_0}{w-z_0}\right| < 1$ for every $w \in \gamma$. We plug this in CIF.

Holomorphic $\implies$ Analytic

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&= \left[\frac{1}{2\pi\imath}\int_\gamma \frac{f(w)}{(w-z_0)}dw\right]+\left[\frac{1}{2\pi\imath}\int_\gamma \frac{f(w)}{(w-z_0)^2}dw\right](z-z_0) \\
&\quad +\left[\frac{1}{2\pi\imath}\int_\gamma \frac{f(w)}{(w-z_0)^3}dw\right](z-z_0)^2+\dots \\
&= \sum_{n=0}^{\infty} a_n(z-z_0)^n,
\end{aligned}$$

for $|z-z_0|<r$ where

$$a_n = \frac{1}{2\pi\imath}\int_{|w-z_0|=r}\frac{f(w)}{(w-z_0)^{n+1}}dw.$$

Remark: Integral of sum needn't be sum of integrals in general, but in the previous slide it can be justified. The key word is "uniform convergence". We'll skip the details.

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Thus, we have proved that if f is holomorphic in the disc $|z - z_0| < R$, then,

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for any $r < R$. Since the power series converges to $f(z)$ for $|z - z_0| < r$, the radius of convergence is at least r. We also know that

$$a_n = \frac{f^{(n)}(z_0)}{n!},$$

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whenever $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$. In particular, a_n does not depend on r. Any $r < R$ gives same a_n. Thus the radius of convergence is at least R.