Tutorial - 1

1. Suppose z_1 & z_2 are complex numbers.

Show that : (i) $\big| |z_1| - |z_2| \big| \leq |z_1 - z_2|$

(ii) $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2 \cdot \mathrm{Re}(z_1 \bar{z}_2)$

(iii) $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$

2. Show that : $\dfrac{|x| + |y|}{\sqrt{2}} \leq |z| \leq |x| + |y|$

where $x = \mathrm{Re}(z)$ & $y = \mathrm{Im}(z)$.

3. Suppose z_1 & z_2 are complex numbers regarded as vectors in the plane, the dot & cross products are given by :

(i) $\langle z_1, z_2 \rangle = \mathrm{Re}\,(z_1 \bar{z}_2)$

(ii) $(z_1 \times z_2) = \mathrm{Im}\,(\bar{z}_1 z_2) \cdot \hat{k}$

4. Use : $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$:

for every integer n and prove the following:

(i) Determine the principal argument of

$-1+i$ & compute its 5^{th} power.

(ii) Compute the cube of

$\theta = \cos(2\pi/9) + i\sin(2\pi/9)$ and show

that θ^3 satisfies the equation $z^3 - 1 = 0$.

5. (i) Determine explicitly (ie in terms of the radicals)

the cube roots of unity.

(ii) Similarly, determine the 4^{th} roots of -1.

(iii) Determine the 5^{th} roots of unity.

6. Suppose z_1 & z_2 are complex numbers with positive real parts. Then show that

$$\operatorname{Arg}(z_1 \cdot z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2).$$

Explain why the result fails for general z_1, z_2 and explain how you modify it. Observe the formal analogy with the logarithm.

7. Show that if $|z|=1$, the complex number

$$w = \frac{z-a}{1-z\bar{a}}$$ has unit modulus.

Hint: compute $w\bar{w}$.

8. Let z trace out a circle of unit radius centered at 1 in the complex plane. Determine the curve traced out by z^2 (the image curve under the map $f(z) = z^2$). Discuss & sketch the image under the map $f(z) = z^2 - 1$. Discuss the image of $|z \pm 1| = 1$ under the map $1 - z^2$.

9. Show that if $f(z)$ is a polynomial with $z=a$ as a root of multiplicity n (that is, if $f(z)=(z-a)^n g(z)$ with $g(a) \neq 0$ then,

(i) $f(a) = f'(a) = \cdots = f^{n-1}(a) = 0$.

(ii) $$\frac{f'(z)}{f(z)} = \frac{n}{z-a} + \frac{g'(z)}{g(z)}.$$

Generalize this further.

10. Discuss the image of the circles $|z \pm i|$ under the map $\frac{1+iz}{1-iz}$.

11. Complete the exercises given in the notes!