MA 205 Complex Analysis: Review

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1. The function $u(x,y) = \sqrt{\sqrt{x^2+y^2} - x}$ is the real part of

(a) $\sqrt{z}$ (b) $-\sqrt{2z}$ (c) $\sqrt{2z}$ (d) $-\imath\sqrt{2z}$ (e) $\imath\sqrt{2z}$,

where $\sqrt{z}$ denotes the principal branch.

Justification:

In polar coordinates, $u(x,y) = \sqrt{2r}\sin\frac{\theta}{2}$ and
$-\imath\sqrt{2z} = \sqrt{2r}(\sin\frac{\theta}{2} - \imath\cos\frac{\theta}{2})$.

OR

Assume answer is of the form $k\sqrt{z}$ and solve for k.

(a) $\sqrt{z}$ (b) $-\sqrt{2z}$ (c) $\sqrt{2z}$ (♣) $-\imath\sqrt{2z}$ (e) $\imath\sqrt{2z}$.

2. Which among the following functions is not the real part of a holomorphic function:

(a) xy (b) $x^2 + y^2$ (c) $\frac{x}{x^2+y^2}$ (d) $e^x \cos y$ (e) $e^x \sin y$.

Justification:

$u(x, y) = x^2 + y^2$ is the only one in the above which is not harmonic. So it's not the real part of a holomorphic function in any domain.

OR

Check case by case to see that all others come as real part of some holomorphic function.

(a) xy (♣) $x^2 + y^2$ (c) $\frac{x}{x^2+y^2}$ (d) $e^x \cos y$ (e) $e^x \sin y$.

3. The radius of convergence of the power series $\sum_{n=0}^{\infty}(\cos \imath n)z^n$ is

(a) 1 (b) π (c) $\frac{1}{\pi}$ (d) $\frac{1}{e}$ (e) e.

Justification:

The given series is clearly the sum of two power series with one having radius of convergence e and the other with radius of convergence $\frac{1}{e}$. So the answer is $\frac{1}{e}$.

OR

Compute the relevant limit in the formula for radius of convergence.

(a) 1 (b) π (c) $\frac{1}{\pi}$ (♣) $\frac{1}{e}$ (e) e.

4. Let γ be the closed curve consisting of upper parts of the circles $|z| = 1$ and $|z| = 2$ and parts of the x-axis, starting and ending at -2. Then, $\int_\gamma \frac{z}{\bar{z}} dz$ is

(a) $\frac{4}{3}$ (b) 2 (c) $2\left(1 - \frac{\imath}{3}\right)$ (d) $2\imath$ (e) $\frac{4\imath}{3}$.

Justification:

Direct calculation gives
$\int_{-2}^{-1} dx + \imath \int_{\pi}^{0} e^{3\imath\theta} d\theta + \int_1^2 dx + 2\imath \int_0^{\pi} e^{3\imath\theta} d\theta = \frac{4}{3}$.

(♣) $\frac{4}{3}$ (b) 2 (c) $2\left(1 - \frac{\imath}{3}\right)$ (d) $2\imath$ (e) $\frac{4\imath}{3}$.

5. Let γ be the upper part of the ellipse $4x^2 - 4x + y^2 = 0$, oriented from 0 to 1. Then, $\int_\gamma \frac{dz}{1+z^2}$ is

(a) 0 (b) 1 (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$ (e) $\frac{2\pi\imath}{1+\imath}$.

Justification:

$\int_\gamma \frac{dz}{1+z^2} = \int_0^1 \frac{dx}{1+x^2}$ by Cauchy's theorem, and this is $\frac{\pi}{4}$.

(a) 0 (b) 1 (♣) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$ (e) $\frac{2\pi\imath}{1+\imath}$.

6. Given that f is entire and $|f(z)| \leq 10|z|^5$ for all $|z| > 2$. Then a candidate for f is[1]

(a) $z^{\frac{5}{2}}$ (b) $z^{\frac{2}{5}}$ (c) $z^5 + 11z^2 + 1$ (d) $z^{10} + 8z^2 + 1$ (e) $\sin z$.

Justification:

Let f be entire such that there exist constants $M, K > 0$ and a positive integer n such that $|f(z)| \leq K|z|^n$ for all $|z| > M$. Apply Cauchy's estimate for $f^{(n+1)}(z)$ and take limit as $R \to \infty$ to conclude that $f^{(n+1)}(z) = 0$ for all z. Thus f is a polynomial of degree $\leq n$.

(a) $z^{\frac{5}{2}}$ (b) $z^{\frac{2}{5}}$ (♣) $z^5 + 11z^2 + 1$ (d) $z^{10} + 8z^2 + 1$ (e) $\sin z$.

[1] Thanks to Shaurya and Reebhu for their comments.

7. Branch points of $\sin^{-1} z$ are
(a) $\{0, \pm 1\}$ (b) $\{0, \infty\}$ (c) $\{\pm 1\}$ (d) $\{\pm 1, \infty\}$ (e) $\{0, \pm 1, \infty\}$.

Justification:

$$\sin^{-1} z = -\imath \log(\sqrt{1 - z^2} + \imath z).$$

$\{\pm 1\}$ are branch points of $\sqrt{1 - z^2}$ and they continue to be branch points for $\sin^{-1} z$. By changing $z \mapsto \frac{1}{z}$, verify that 0 is a branch point for $\sin^{-1} \left(\frac{1}{z}\right)$.

(a) $\{0, \pm 1\}$ (b) $\{0, \infty\}$ (c) $\{\pm 1\}$ ($\clubsuit$) $\{\pm 1, \infty\}$ (e) $\{0, \pm 1, \infty\}$.

8. There exists $z_0 \in \mathbb{C}$ such that $e^{z_0} = z_0$.
(a) True (b) False.

Justification:

If $f(z) = e^z - z$ misses 0, then it misses $2\pi\imath$ as well, contradicting little Picard.

(♣) True (b) False.

9. The function[2]

$$f(x,y)=\begin{cases}\frac{x\cos x\sinh y-y\sin x\cosh y}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

is harmonic everywhere in $\mathbb{R}^2$.

(a) True (b) False.

Justification:

Consider the entire function

$$h(z)=\begin{cases}\frac{\sin z}{z} & \text{if } z\neq 0\\ 1 & \text{if } z=0.\end{cases}$$

The given function is the imaginary part of $h(z)$, and therefore it is harmonic.

(♣) True (b) False.

[2] Thanks to Reebhu for a correction here.

10. Let

$$f(z) = \begin{cases} \frac{z}{\sin z} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0. \end{cases}$$

Write $f^{(4)}(0) = \frac{p}{q}$, where $p, q \in \mathbb{N}$ with no common factors. Then, $p = \_$, $q = \_$.

Justification:

Expand $f(z)$ as a power series. The coefficient of z^4 is $\frac{7}{360}$. Therefore, $f^{(4)}(0) = \frac{7}{360} \cdot 24 = \frac{7}{15}$.

$p = \underline{\ 7\ }$, $q = \underline{\ 15\ }$.

11. For every $f : \mathbb{C} \to \mathbb{C}$ which is entire, and not of the form $f(z) = z + c$, $c \neq 0$, there exists $z_0 \in \mathbb{C}$ such that $f(z_0) = z_0$.

(a) True (b) False.

Justification:

Give a counter example. For instance, $f(z) = z + e^z$.

(a) True (♣) False.

12. Let f, g be entire functions with $g(z)$ being never equal to 0. Suppose $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Show that $f(z) = cg(z)$ for all $z \in \mathbb{C}$ for some constant c. This true because of

(a) CR (b) FTA (c) Hol $\implies$ Analytic (d) Liouville.

Justification:

$$h = \frac{f}{g}$$

is entire and bounded.

(a) CR (b) FTA (c) Hol $\implies$ Analytic (♣) Liouville.

13. $\displaystyle\int_{|z-1-\imath|=5/4} \frac{\log z}{(z-1)^2}\,dz$ is

(a) $\log 2\pi$ (b) $\log 2\pi\imath$ (c) $\frac{1}{2\pi\imath}$ (d) $2\pi\imath$ (e) $\frac{1}{2\pi}$.

Justification:

$$f^{(n)}(z_0) = \frac{n!}{2\pi\imath}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz.$$

(a) $\log 2\pi$ (b) $\log 2\pi\imath$ (c) $\frac{1}{2\pi\imath}$ (♣) $2\pi\imath$ (e) $\frac{1}{2\pi}$.

14. Which among the following has infinitely many branch points:
(a) $\sin^{-1} z$ (b) $\log \sin^{-1} z$ (c) $\sin z$ (d) $\log \sin z$ (e) $\log \tan^{-1} z$.

Justification:

$\sin z$ has infinitely many zeros.

(a) $\sin^{-1} z$ (b) $\log \sin^{-1} z$ (c) $\sin z$ (♣) $\log \sin z$ (e) $\log \tan^{-1} z$.