MA 205 Complex Analysis: Examples of Contour Integration

August 24, 2017

Recall

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We then looked at various examples of computing residues and contour integrals. Let us begin by looking at some more today.

Improper Integral

Let $f : [0, \infty] \to \mathbb{R}$ be a function such that $\int_0^R f(x)dx$ exists for each $R \geq 0$. One then defines the Improper integral $\int_0^\infty f(x)$ to be

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Similarly if $f : [-\infty, \infty] \to \mathbb{R}$ is a function such that $\int_{-a}^b f(x)dx$ exists for each $a, b \geq 0$, then the improper integral $\int_{-\infty}^\infty f(x)$ is defined as $\lim_{a,b\to\infty} \int_{-a}^b f(x)dx$.

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For instance the function $\frac{1}{1+x^2}dx$ is integrable on $\mathbb{R}$ while the integral $\int_{-\infty}^\infty sin(x)dx$ does not exist. Intuitively, for such an improper integral to exist, the function has to decay to zero sufficiently rapidly outside "small set ".

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If $P(z)/Q(z)$ is a rational function such that with deg $Q(z) \geq$ deg $P(z) + 2$. Then there exists a constant C such that for $|P(z)/Q(z)| \leq C/|z^2|$ for $|z|$ sufficiently large. Thus for a large real number R, $|P(z)/Q(z)| \leq \frac{C}{R^2}$ on the circle of radius R.

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$\int_\gamma \frac{dz}{(1+z^2)^n} = \int_{-R}^{R} \frac{dx}{(1+x^2)^n} + \int_{C_R} \frac{dz}{(1+z^2)^n}$
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By the earlier remark, there exists a constant C , such that $|\frac{1}{(1+z^2)^n}| \leq \frac{C}{R^2}$ on C_R.

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By the earlier remark, there exists a constant C , such that $|\frac{1}{(1+z^2)^n}| \leq \frac{C}{R^2}$ on C_R. Thus by ML inequality the second integral tends to zero as $R \to \infty$. Thus, the answer is $\frac{\pi}{4^{n-1}} \begin{pmatrix} 2n-2 \\ n-1 \end{pmatrix}$.

Jordan's lemma

Theorem (Jordan's Lemma)

Let f be a continuous function defined on the semicircular contour $C_R = \{Re^{\imath\theta} \mid \theta \in [0, \pi]\}$ of the form

$$f(z) = e^{\imath az} g(z),$$

where $g(z)$ is a continuous function and with $a > 0$. Then,

$$\left| \int_{C_R} f(z)dz \right| \leq \frac{\pi}{a} \max_{\theta \in [0,\pi]} |g(Re^{\imath\theta})|.$$

Proof:

$$\int_{C_R} f(z)dz = \int_0^\pi g(Re^{\imath\theta})e^{\imath aR(\cos\theta+\imath\sin\theta)}\imath Re^{\imath\theta}d\theta.$$

Real Integrals

Proof:

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Therefore,

$$\left|\int_{C_R} f(z)dz\right| \quad \leq \quad R\int_0^{\pi}\left|g(Re^{\imath\theta})e^{aR(\imath\cos\theta - \sin\theta)}\imath e^{\imath\theta}\right|d\theta$$

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&= R\int_0^{\pi}\left|g(Re^{\imath\theta})\right|e^{-aR\sin\theta}d\theta \\
&\leq 2RM_R\int_0^{\frac{\pi}{2}} e^{-aR\sin\theta}d\theta \quad \text{where } M_R = sup|g(Re^{i\theta})|
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&\leq 2RM_R\int_0^{\frac{\pi}{2}} e^{\frac{-2aR\theta}{\pi}}d\theta = \frac{\pi}{a}(1-e^{-aR})M_R \leq \frac{\pi}{a}M_R,
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since $\sin\theta \geq \frac{2\theta}{\pi}$ for $\theta \in [0, \frac{\pi}{2}]$.

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$$\int_{\gamma} f(z)dz = \int_{r}^{R} \frac{e^{\imath x}}{x} dx + \int_{\gamma_R} \frac{e^{\imath z}}{z} dz + \int_{-R}^{-r} \frac{e^{\imath x}}{x} dx + \int_{\gamma_r} \frac{e^{\imath z}}{z} dz.$$

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$$\left| \frac{e^{\imath z} - 1}{z} \right| \leq M,$$

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$$\lim_{r\to 0}\int_{\gamma_r}\frac{e^{\imath z}-1}{z}dz = 0,$$

by appealing to ML inequality.

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$$\log x = \begin{cases} \log x & \text{if } x > 0, \\ \log|x| + \imath\pi & \text{if } x < 0. \end{cases}$$

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$$\int_{\gamma} \frac{\log z}{1+z^2}\,dz \quad = \quad \int_{r}^{R} \frac{\log x}{1+x^2}\,dx + \int_{\gamma_R} \frac{\log z}{1+z^2}\,dz$$

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$$\begin{aligned}\int_{\gamma} \frac{\log z}{1+z^2}\,dz &= \int_r^R \frac{\log x}{1+x^2}\,dx + \int_{\gamma_R} \frac{\log z}{1+z^2}\,dz \\ &+ \int_{-R}^{-r} \frac{\log|x| + \imath\pi}{1+x^2}\,dx + \int_{\gamma_r} \frac{\log z}{1+z^2}\,dz.\end{aligned}$$

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Lhs is $2\pi\imath \cdot \mathrm{Res}(f;\imath)$ which is $2\pi\imath \cdot \frac{\log \imath}{2\imath} = \frac{\pi^2 \imath}{2}$. Also,

$$\int_r^R \frac{\log x}{1+x^2}dx + \int_{-R}^{-r} \frac{\log|x| + \imath\pi}{1+x^2}dx = 2\int_r^R \frac{\log x}{1+x^2}dx + \imath\pi \int_r^R \frac{dx}{1+x$$

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(In the Limit)

Real Integrals

Thus,

$$\int_r^R \frac{\log x}{1+x^2}dx = -\frac{1}{2}\left[\int_{\gamma_R} \frac{\log z}{1+z^2}dz + \int_{\gamma_r} \frac{\log z}{1+z^2}dz\right].$$

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Note that

$$\left|\int_{\gamma_\rho} \frac{\log z}{1+z^2}dz\right| \quad = \quad \left|\rho\int_0^\pi \frac{\log\rho + \imath\theta}{1+\rho^2 e^{\imath\theta}} e^{\imath\theta}\, d\theta\right|$$

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Note that

$$\begin{aligned}
\left|\int_{\gamma_\rho} \frac{\log z}{1+z^2}dz\right| &= \left|\rho\int_0^\pi \frac{\log\rho + \imath\theta}{1+\rho^2 e^{\imath\theta}}e^{\imath\theta}d\theta\right| \\
&\leq \frac{\rho|\log\rho|}{|1-\rho^2|}\int_0^\pi d\theta + \frac{\rho}{|1-\rho^2|}\int_0^\pi \theta d\theta \\
&= \frac{\pi\rho|\log\rho|}{|1-\rho^2|} + \frac{\rho\pi^2}{2|1-\rho^2|}.
\end{aligned}$$

This is zero in the limit if $\rho \to 0+$ or $\rho \to \infty$. Thus, the given integral is zero.

Maximum Modulus Theorem

An important theorem in Complex Analysis states that a non-constant holomorphic function on an open connected domain never attains its maximum modulus at any point in the domain. This is called the maximum modulus theorem. Once again, this is vastly different from what happens to real differentiable functions; in fact even for real analytic functions. Real analytic functions can achieve maximum anywhere inside the interval. We'll use CIF and the identity theorem to prove MMT.

Maximum Modulus Theorem

Proof: Suppose there is $z_0 \in \Omega$ such that $|f(z_0)| \geq |f(z)|$ for all $z \in \Omega$. Then we'll prove that f is a constant.

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$$f(z_0) = \frac{1}{2\pi\imath} \int_\gamma \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{\imath\theta}) d\theta.$$

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Hence,

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{\imath\theta})| d\theta \leq |f(z_0)|,$$

since $|f(z_0)|$ is assumed to be the maximum value.

Thus,

$$\int_0^{2\pi} \left[|f(z_0)| - |f(z_0 + re^{\imath\theta})| \right] dt = 0.$$

Note that the integrand is non-negative. Therefore it has to be zero; i.e., $|f(z_0)| = |f(z_0 + re^{\imath\theta})|$ for all θ. Since this is true for each small r, we see that $|f(z)|$ is a constant on a small disc around z_0. This means that $f(z)$ is a constant, say c, on this small disc. (Why?) This implies that $f \equiv c$ on Ω by the identity theorem, since a disc has limit points!

A nice consequence of the Maximum modulus principle is the following lemma of Schwartz.

Schwarz Lemma : Let $\mathbb{D} = \{z : |z| < 1\}$ be the open unit disk and let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic map such that $f(0) = 0$ and $|f(z)| \leq 1$ on $\mathbb{D}$.
Then, $|f(z)| \leq |z| \; \forall z \in \mathbb{D}$ and $|f'(0)| \leq 1$.
Moreover, if $|f(z)| = |z|$ for some non-zero z or $|f'(0)| = 1$, then $f(z) = az$ for some $a \in \mathbb{C}$ with $|a| = 1$.