2017 - MA 207 - Tutorial 3
(Solutions)

Problem 1 Attempt a power series solution around $x = 0$ for

$$x^2y'' - (1+x)y = 0.$$

Explain why the procedure does not give any nontrivial solutions.

Solutions: Let $y(x) = \sum_{n\geq 0} a_n x^n$ be a solution. Then we have

$$x^2y''(x) = \sum_{n\geq 2} n(n-1)a_n x^n.$$

Hence we have

$$\begin{aligned} x^2y'' - xy - y &= \sum_{n\geq 2} n(n-1)a_n x^n - \sum_{n\geq 1} a_{n-1}x^n - \sum_{n\geq 0} a_n x^n \\ &= a_0 - (a_0 + a_1)x + \sum_{n\geq 2}\big(n(n-1)a_n - a_{n-1} - a_n\big)x^n. \end{aligned}$$

Hence $a_0 = a_1 = 0$ and for every $n \geq 2$,

$$n(n-1)a_n = a_n + a_{n-1},$$

or

$$a_n = \frac{1}{n^2 - n - 1}\, a_{n-1}.$$

This holds for all $n \geq 2$. This implies

$$a_0 = 0, a_1 = 0, \ldots, a_n = 0, \ldots.$$

Reason: The reason why this method does not give us any non-trivial solution is that the differential equation can be written as $y'' - \frac{1+x}{x^2}y = 0$ and the coefficient $-\frac{1+x}{x^2}$ does not have a power series around $x = 0$. In fact 0 is a regular singular point.

Problem 2 Attempt a Frobenius series solution for the differential equation

$$x^2y'' + (3x-1)y' + y = 0.$$

Why does the method fail?

Solutions: Write

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = x^r y_1(x), \quad a_0 \neq 0.$$

Then

$$y'(x) = rx^{r-1}y_1(x) + x^r y_1'(x) \text{ and } y''(x) = x^r y_1''(x) + 2rx^{r-1}y_1'(x) + r(r-1)x^{r-2}y_1(x).$$

Now if y were to be a solution of the given ODE then the following has to happen:

$$x^{r+2}y_1''(x) + x^r\big((2r+3)x - 1\big)y_1'(x) + x^r(r+1)^2 y_1(x) - rx^{r-1}y_1(x) = 0.$$

This implies $ra_0 = 0$ and hence $r = 0$ since $a_0 \neq 0$. Further with $r = 0$, we get

$$x^2y_1''(x) + \big(3x-1\big)y_1'(x) + y_1(x) = 0.$$

Now noting that $y_1(x) = \sum_{n=0}^{\infty} a_n x^n$, a similar computation as before yields,

$$a_{n+1} = (n+1)a_n.$$

The radius of convergence of the resulting power series is 0.

Reason: The method fails because the existence of a Frobenius series solution around x_0 is guaranteed when x_0 is a regular singular point. Here $x_0 = 0$ and it is not a regular singular point.

Problem 3 Locate and classify the singular points for the following differential equations. (All letters other than x and y such as p, λ, etc are constants.)

Consider the following second order ODE in its standard form:

$$y'' + p(x)y' + q(x)y = 0. \tag{0.1}$$

A real number x_0 is called

(1) an ordinary point of (0.1), if both p and q are analytic at x_0;

(2) a regular singular point if $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are analytic at x_0. This is equivalent to saying that there are functions $b(x)$ and $c(x)$ which are analytic at x_0 such that

$$p(x) = \frac{b(x)}{(x - x_0)} \text{ and } q(x) = \frac{c(x)}{(x - x_0)^2}$$

(3) an irregular singular point, if x_0 is not ordinary or regular singular.

Now let us solve the problems.

(a) Bessel equation:

$$x^2 y'' + xy' + (x^2 - p^2)y = 0.$$

Solutions: $x = 0$ is the only singular point and it is regular singular. We can write

$$y'' + \frac{1}{x}y' + \frac{x^2 - p^2}{x^2}y = 0$$

and both 1 and $(x^2 - p^2)$ are real analytic everywhere, in fact polynomials.

(b) Laguerre equation:

$$xy'' + (1 - x)y' + \lambda y = 0.$$

Solutions: $x = 0$ is the only singular point and it is regular singular.

(e) Associated Legendre equation:

$$(1 - x^2)y'' - 2xy' + \left[n(n + 1) - \frac{m^2}{1 - x^2}\right] y = 0$$

Solutions: $x = \pm 1$ are the singular points and both are regular singular.

(f) $xy'' + (\cot x)y' + xy = 0$.

Solutions: In standard form the above equation looks like

$$y'' + \frac{\cos x}{x \sin x}y' + y$$

The singular points are $x = n\pi$. Of these, $x = 0$ is irregular singular, since

$$x\frac{\cos x}{x \sin x} = \frac{\cos x}{\sin x}$$

is not analytic at $x = 0$. If $n \neq 0$, then

$$(x - n\pi)\frac{\cos x}{x \sin x} = \frac{\cos x}{x\frac{\sin x}{x - n\pi}}$$

Since $x\frac{\sin x}{x-n\pi}$ does not vanish at $n\pi$ when $n \neq 0$, we get that the above is analytic at $x = n\pi$. Also $(x-n\pi)^2 1$ is analytic at $x = n\pi$. Thus, if $n \neq 0$ then $x = n\pi$ is regular singular.

Problem 4 In Problem (3) above find the indicial equations corresponding to all the regular singular points.

Solutions: The basic method is as follows: If x_0 is a regular singular point of a second order linear ODE, first write it in the form

$$y'' + \frac{b(x-x_0)}{(x-x_0)}y' + \frac{c(x-x_0)}{(x-x_0)^2}y = 0.$$

Now the indicial equation for the purpose of expanding in fractional powers of $(x-x_0)$ is

$$r(r-1) + b_0 r + c_0 = 0.$$

(a) Bessel equation: $x^2y'' + xy' + (x^2 - p^2)y = 0.$

We have noticed above that $x_0 = 0$ is the only singular point which is regular and that $b(x) = 1,\ c(x) = x^2 - p^2$. Therefore the indicial equation is $r^2 - p^2 = 0$.

(b) Laguerre equation: $xy'' + (1-x)y' + \lambda y = 0.$

In this case, $x_0 = 0$ is the only singular point which is regular and $b(x) = 1-x,\ c(x) = \lambda x$. Hence the indicial equation is $r^2 = 0$.

(e) Associated Legendre equation: $(1-x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0.$

As analyzed above, $x_0 = \pm 1$ are the regular singular points. For $x_0 = 1$, Clearly, if

$$\frac{b(x-1)}{x-1} = \frac{2x}{x^2-1} \qquad \frac{c(x-1)}{(x-1)^2} = \frac{[n(n+1)(1-x^2) - m^2}{(1-x^2)^2}$$

then

$$b_0 = \lim_{x\to 1}(x-1)\frac{2x}{x^2-1} \qquad c_0 = \lim_{x\to 1}(x-1)^2\frac{[n(n+1)(1-x^2) - m^2]}{(1-x^2)^2}$$

One easily checks that $b_0 = 1$ and $c_0 = -m^2/4$.

The indicial equation is $r(r-1) + r - m^2/4 = 0$, that is, $r^2 - m^2/4 = 0$. By symmetry, the same is true for $x_0 = -1$.

For $x_0 = -1$ Clearly, if

$$\frac{b(x+1)}{x+1} = \frac{2x}{x^2-1} \qquad \frac{c(x+1)}{(x+1)^2} = \frac{[n(n+1)(1-x^2) - m^2}{(1-x^2)^2}$$

then

$$b_0 = \lim_{x\to -1}(x+1)\frac{2x}{x^2-1} \qquad c_0 = \lim_{x\to -1}(x+1)^2\frac{[n(n+1)(1-x^2) - m^2]}{(1-x^2)^2}$$

One easily checks that $b_0 = 1$ and $c_0 = -m^2/4$.

(f) $xy'' + (\cot x)y' + xy = 0$.

The regular singular points are $x = n\pi$ for $n \neq 0$. The equation in standard form is

$$y'' + \frac{\cos x}{x\sin x}y' + y = 0$$

To find b_0 and c_0 at the regular singular point $x = n\pi$ we need to write the coefficient functions as power series in $x - n\pi$. Clearly, if

$$\frac{b(x-n\pi)}{x-n\pi} = \frac{\cos x}{x\sin x} \qquad \frac{c(x-n\pi)}{(x-n\pi)^2} = 1$$

then

$$b_0 = \lim_{x\to n\pi}(x-n\pi)\frac{\cos x}{x\sin x} \qquad c_0 = \lim_{x\to n\pi}(x-n\pi)^2 1$$

One easily checks that $b_0 = \frac{1}{n\pi}$ and $c_0 = 0$.

In the following problems, we would find two independent solutions of an ODE of the following type:

$$x^2y'' + xb(x)y' + c(x)y = 0,$$

with $b(x) = \sum_{j\geq 0} b_j x^j$ and $c(x) = \sum_{j\geq 0} c_j x^j$ are analytic functions in a small neighborhood of 0. Note that $x = 0$ is a regular singular point.

For this we define the indicial equation

$$I(r) := r(r-1) + b_0 r + c_0$$

and look for solution of the type

$$y(x) = \sum_{n\geq 0} a_n(r)x^{n+r} \tag{0.2}$$

by substituting this into the differential equation and setting the coefficient of x^{n+r} to be 0, We get the following

(1) The coefficient of x^r is $I(r)a_0$, thus we need $I(r)a_0 = 0$
(2) The coefficient of x^{n+r} , for $n \geq 1$, is

$$I(n+r)a_n + \sum_{i=0}^{n-1} b_{n-i}(i+r)a_i + \sum_{i=0}^{n-1} c_{n-i}a_i. \tag{0.3}$$

Equating this to 0, we find the coefficients a_n explicitly. The feature of the other (than (0.2)) solution depends on the nature of roots of the indicial equation. We explain this below:

Case 1: The roots r_1 and r_2 are such that $r_1 \neq r_2$ and $r_1 - r_2$ is not an integer. Then the solutions are

$$y_1(x) = \sum_{n\geq 0} a_n(r_1)x^{n+r_1} \text{ and } y_2(x) = \sum_{n\geq 0} a_n(r_2)x^{n+r_2}.$$

Case 2: The roots r_1 and r_2 are such that $r_1 = r_2 = r$(say). Then the solutions are

$$y_1(x) = \sum_{n\geq 0} a_n(r)x^{n+r} \text{ and } y_2(x) = \sum_{n\geq 0} a_n'(r)x^{n+r} + \sum_{n\geq 0} a_n(r)x^{n+r}\log x.$$

Case 3: The roots r_1 and r_2 are such that $r_1 > r_2$ and $r_1 - r_2$ is an integer. Then the solutions are

$$y_1(x) = \sum_{n\geq 0} a_n(r_1)x^{n+r_1} \text{ [Note that } r_1 > r_2\text{] and} \tag{0.4}$$

$$y_2(x) = \sum_{n\geq 0} A_n'(r)x^{n+r} + \sum_{n\geq 0} A_n(r)x^{n+r}\log x, \tag{0.5}$$

where $A_n(r) := (r - r_2)a_n(r)$. It should be noted that the denominator of $a_n(r)$ vanishes at r_2 (of order 1) if and only if $n \geq r_1 - r_2$.

Please see the lecture slides for a beautiful discussion in this direction. Having briefly established the theory, we are now ready to solve the rest of the problems.

Problem 5 Find two linearly independent solutions of the following differential equations:

(a) $x^2y'' + x\frac{2x-1}{2}y' + \frac{1}{2}y = 0.$

Solutions: Let us write the given ODE as:

$$y'' + \frac{1}{x}\frac{2x-1}{2}y' + \frac{1}{2x^2}y = 0.$$

Therefore $b(x) = -\frac{1}{2} + x$ and $c(x) = \frac{1}{2}$. The indicial equation for this ODE is

$$2r^2 - 3r + 1 = 0,$$

which has $r_2 = 1$ and $r_2 = \frac{1}{2}$ as its roots. Note that $r_1 - r_2$ is not an integer. The equation defining a_n, for $n \geq 1$, is

$$a_n = -\frac{(n+r-1)}{I(n+r)}a_{n-1} = -\frac{2}{(2n+2r-1))}a_{n-1}.$$

Thus

$$a_n(r_1) = a_n(1) = -\frac{2}{2n+1}a_{n-1} = (-1)^n\frac{2^n}{(2n+1)\cdots 5\cdot 3}a_0.$$

Hence

$$y_1(x) = x\left(1 + \sum_{n\geq 1}(-1)^n\frac{2^n a_0 x^n}{(2n+1)\cdots 5\cdot 3}\right).$$

Since $r_1 - r_2$ is not an integer, $I(n + r_2)$ is also non-zero. Therefore

$$a_n(r_2) = a_n(1/2) = -\frac{1}{n}a_{n-1} = (-1)^n\frac{1}{n\cdot(n-1)\cdots 2\cdot 1}a_0$$

Hence

$$y_2(x) = x^{\frac{1}{2}}\left(1 + \sum_{n\geq 1}(-1)^n\frac{a_0 x^n}{n\cdot(n-1)\cdots 2\cdot 1}\right).$$

In the above solution one can always assume that $a_0 = 1$.

(b) $x^2y'' + x(x^2 - 3)y' + (4 + x^2)y = 0.$

Solutions: Let us write the given ODE as:

$$y'' + \frac{x^2-3}{x}y' + \frac{x^2+4}{x^2}y = 0.$$

Therefore $b(x) = -3 + x^2$ and $c(x) = 4 + x^2$. The indicial equation for this ODE is

$$I(r) = (r-2)^2 = 0,$$

which has $r_1 = r_2 = 2$ as its only root. Let us find the Frobenius solution directly by putting

$$y(x) = x^r \sum_{n\geq 0} a_n(r) x^n.$$

$$y'(x) = \sum_{n\geq 1} (n+r) a_n(r) x^{n+r-1}.$$

$$y''(x) = \sum_{n\geq 2} (n+r)(n+r-1) a_n(r) x^{n+r-2}$$

When we put these expression in the given ODE we get by equation (0.3) above that $a_1(r) = 0$ and for $n \geq 2$, the co-efficient of x^{n+r} to be

$$I(n+r)a_n + (n-1+r)a_{n-2} = 0.$$

Which is same as

$$a_n = -\frac{(n+r-1)}{(n+r-2)^2} a_{n-2} = (-1)^n \frac{(n+r-1)}{(n+r-2)^2} \frac{(n+r-3)}{(n+r-4)^2} \cdots \frac{r+1}{r^2} a_0. \tag{0.6}$$

Since $r = 2$, we have for $n \geq 2$,

$$a_n = -\frac{(n+1)}{n^2} a_{n-2}$$

This shows that $a_{2n+1} = 0$, for every $n \geq 0$, and

$$a_{2n} = (-1)^n \frac{(2n+1)\cdot(2n-1)\cdots 5\cdot 3}{(2n)^2\cdot(2n-2)^2\cdots 4^2\cdot 2^2} a_0.$$

Therefore

$$y_1(x) = \sum_{n\geq 0} a_{2n} x^{2n+r},$$

where a_{2n}s are as expressed above, is a solution for the given ODE. Since the indicial equation has a double root at 2, the other solution is given by

$$y_2(x) = \sum_{n\geq 0} a'_{2n}(r) x^{2n+r} + \log x \sum_{n\geq 0} a_{2n}(r) x^{2n+r},$$

where $a'_{2n}(r)$ to be found by equation (0.6).

$$a_n(2) = (-1)^n \frac{(2n+1)(2n-1)\ldots 3}{2^n (n!)^2} = (-1)^n \frac{(2n+1)!}{2^{2n}(n!)^3}$$

To compute $a'_{2n}(r)$, note that $a_{2n}(r) = \dfrac{f(r)}{g(r)^2}$, where $f(r) = \prod_{i=1}^n (r+2i-1)$ and $g(r) = \prod_{i=1}^n (r+2i-2)$. Hence

$$a'_{2n}(r) = \frac{f'}{g^2} - \frac{2fg'}{g^3} = \frac{ff'}{fg^2} - \frac{2fg'}{g^3} = a_{2n}(r)\left(\frac{f'}{f} - 2\frac{g'}{g}\right)$$

$$a'_{2n}(r) = a_{2n}(r)\left(\sum_{i=1}^n \frac{1}{r+2i-1} - 2\sum_{i=1}^n \frac{1}{r+2i-2}\right)$$

$$a'_{2n}(2) = a_{2n}(2)\left(\sum_{i=1}^{n}\frac{1}{2i+1} - 2\sum_{i=1}^{n}\frac{1}{2i}\right) = (-1)^{2n}\frac{(4n+1)!}{2^{4n}((2n)!)^3}\left(H_{2n+1} - \frac{1}{2}H_n - H_n\right)$$

$$a'_{2n}(2) = (-1)^{2n}\frac{(4n+1)!}{2^{4n}((2n)!)^3}\left(H_{2n+1} - \frac{3}{2}H_n\right), \qquad H_n = \sum_{i=1}^{n}\frac{1}{i}$$

(d) $x^2y'' - x(2-x^2)y' + (2+x^2)y = 0.$

Solutions:

The indicial equation is $I(r) = r(r-1) - 2r + 2 = r^2 - 3r + 2 = (r-1)(r-2)$.
So the roots are $r_1 = 2, r_2 = 1$.
The coefficient of x^{n+r} gives

$$I(n+r)a_n(r) + (n+r-2)a_{n-2}(r) + a_{n-2}(r) = 0$$
$$\implies a_n(r) = -\frac{n+r-1}{I(n+r)}a_{n-2}(r) = \frac{-1}{n+r-2}a_{n-2}(r)$$

Since $a_{-1}(r) = 0$, we get $a_{2n+1}(r) = 0$, and

$$a_{2n}(r) = \frac{-1}{2n+r-2}a_{2n-2}(r) = \frac{(-1)^n}{\prod_{i=1}^{n}(r+2i-2)}$$
$$a_{2n}(2) = \frac{(-1)^n}{\prod_{i=1}^{n}(2i)} = \frac{(-1)^n}{2^n n!}$$

The first solution is

$$y_1(x) = \sum_{n\geq 0}\frac{(-1)^n}{2^n n!}x^{2n}.$$

For second solution, write $A_{2n}(r) = (r - r_2)a_{2n}(r) = (r-1)a_{2n}(r)$.
The second solution is given by

$$y_2(x) = \sum_{n\geq 0}A'_{2n}(r_2)x^{2n+r_2} + \sum_{n\geq 0}A_{2n}(r_2)x^{2n+r_2}\log x$$

Since $a_{2n}(r)$ is analytic at 1, we get $A_{2n}(1) = 0$. Further,

$$A'_{2n}(1) = a_{2n}(1) = \frac{(-1)^n}{\prod_{i=1}^{n}(2i-1)} = \frac{(-1)^n 2^n n!}{(2n)!}$$

Thus the second solution is

$$y_2(x) = \sum_{n\geq 0}\frac{(-1)^n 2^n n!}{(2n)!}x^{2n+1}$$