Lec - 25

→ Laplace transforms.

$$\mathcal{L}(av + bw) = a\mathcal{L}(v) + b\mathcal{L}(w)$$

$\exists \to M_L$ a matrix basis

(Linear map ≡ Matrix with basis)

$$\frac{d}{dt}\left(a f(t) + b g(t)\right) = a \frac{d f(t)}{dt} + b \frac{d g(t)}{dt}$$

If $\quad f(t) = \sum_{i=0}^{\infty} a_i t_i^{\,i} \quad \{1, t, t^2, \dots\}$

Then $\quad \frac{d f(t)}{dt} = \sum_{i=0}^{\infty} i a_i t_i^{\,i-1}$

term 0 → t^{-1} coefficient

$a_1 \to t^0$

$2a_2 \to t^1$

$3a_3 \to t^2$

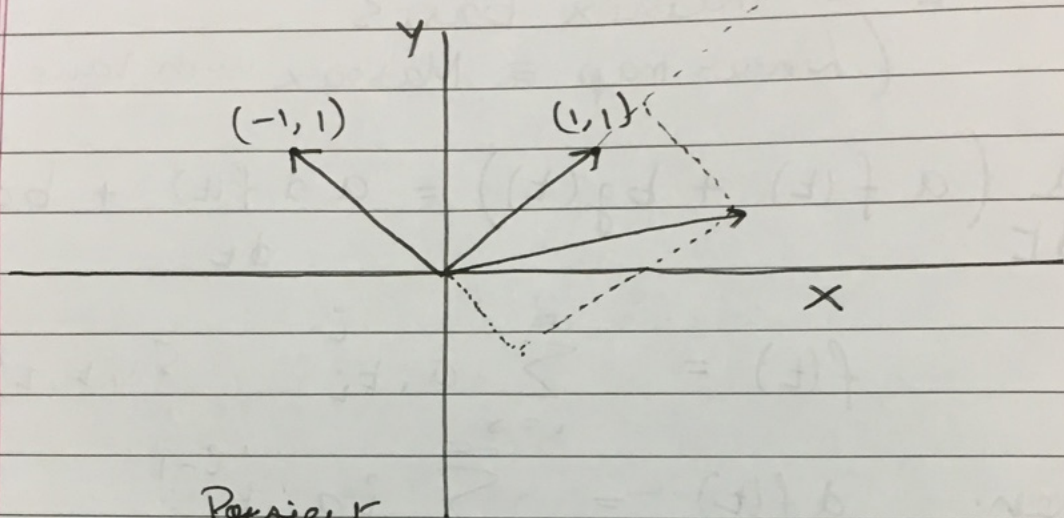
$4a_4 \to t^3$

$$\begin{array}{c} \\ \\ \\ \\ i \\ \\ \end{array}\begin{bmatrix} 0 & 1 & 0 & \cdots & & \\ 0 & 0 & 2 & \cdots & & \\ 0 & 0 & 0 & 3 & \cdots & \\ & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 \quad i+1 \\ & & & & & \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ \cancel{a_i} \\ \\ \end{bmatrix}$$

$\left(\frac{d^2}{dt^2}\right), \left(\frac{d^i}{dt^i}\right) \to$ Also Linear Maps.

→ $\int f(t)\,dt$, $\int (a f(t) + b g(t))\,dt$,

$$a\frac{d}{dt} + b\int dt + c\frac{d^2}{dt^2}$$



$$3\begin{pmatrix}1\\1\end{pmatrix}\overbrace{(1\ \ 1)}^{\text{Project}} - 1\begin{pmatrix}-1\\1\end{pmatrix}(-1\ \ 1)$$

stretch

$$\begin{bmatrix}3&3\\3&3\end{bmatrix} - 1\begin{bmatrix}1&-1\\-1&1\end{bmatrix} = \begin{bmatrix}2&4\\4&2\end{bmatrix}$$

Not obvious what it does

Eigen Value Eqn

If $Mv = \lambda v$

then, $\lambda \rightarrow$ eigen value

$v \rightarrow$ eigen vector $(v \neq 0)$

$\rightarrow \quad L(v) = \lambda v$

$$\frac{df}{dt} = \lambda \, f(t)$$

$$f(t) = c e^{\lambda t} \qquad \{ c e^{\lambda t} \mid \lambda \in \mathbb{C} \}$$

family of solutions for

even for $D^i f = \lambda f$ & $\int f(t)\,dt$ $\cancel{= \lambda f(t)}$

$$g(t) \longmapsto \int g(t) e^{\lambda t} \, dt$$

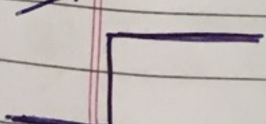
Analogy of dot product with function = Integrals.

$\rightarrow$ Hence the motivation for Laplace tran.

$$\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t) e^{-st} \, dt$$

Laplace examples

ex1)



t=0

$$u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & 1 \text{ o/w} \end{cases}$$

$$\mathcal{L}[u(t)] = \frac{1}{s} = \int_{0^-}^{\infty} e^{-st} \, dt$$

$$= \frac{1}{s}(1 - 0)$$

Ex 2) $f(t) = e^{at}$

$$\mathcal{L}[f(t)] = \int_{0^-}^{\infty} e^{at} e^{-st}\, dt$$

$$= \frac{1}{s-a}, \quad s > a$$

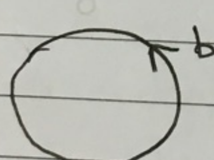
d.n.e, o.w

→ Introduce Complex Numbers

$$j^2 = -1$$

$$e^{(a+jb)t}, \quad a, b \in R$$

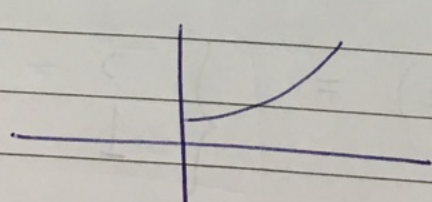
→ $e^{jbt} = \cos(bt) + j\sin(bt)$



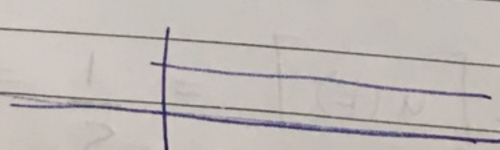
b → Angular velocity

$|z| = 1$

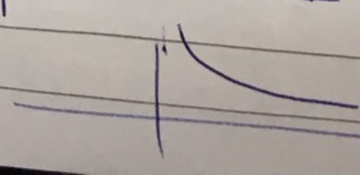
e^{at}



$a > 0$



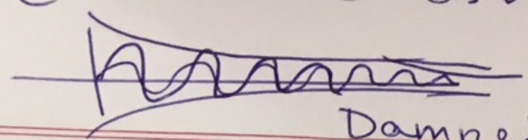
$a = 0$



$a < 0$

$e^{(a+jb)t} = e^{at}\,\mathrm{cis}\,bt$ is just like

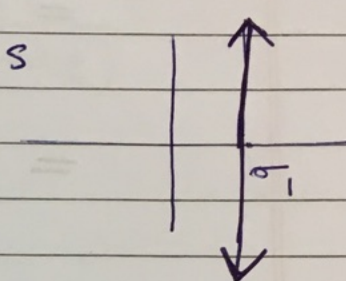
$e^{-t}\sin t$



Damped Oscillation.

→ Inverse Laplace transform

$$f(t) = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)\, e^{st}\, ds$$



Formulas

1) $\mathcal{L}(a f_1(t) + b f_2(t))$

$$= aF_1(s) + bF_2(s)$$

2) $\mathcal{L}[\cos\omega t] = \mathcal{L}\left[\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right]$

$$= \frac{1}{2}\left[\frac{1}{s - j\omega} + \frac{1}{s + j\omega}\right]$$

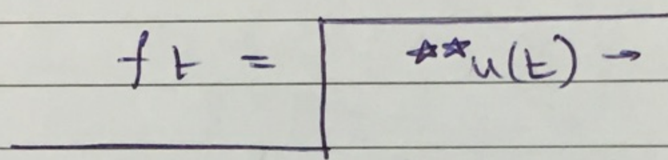
$$\cancel{\tfrac{1}{2}[} = \frac{s}{s^2 + \omega^2}$$

3) $\mathcal{L}[\sin\omega t] = \dfrac{\omega}{s^2 + \omega^2}$

4) $$\mathcal{L}\left[\frac{d}{dt} f(t)\right] = \int_{0^-}^{\infty} \frac{df}{dt} e^{-st} dt$$

$$= e^{-st} f(t) \Big|_{0}^{\infty} + s\int_{0^-}^{\infty} f(t)\, e^{-st} dt$$

$$= {}^{*} sF(s) - f(0^-)^{*}$$



$$f(0^-) = \lim_{t \to 0^-} f(t) \qquad u(0^-) = 0$$

→ $$^{**}\mathcal{L}[\delta(t)] = \mathcal{L}\left[\frac{d\,u(t)}{dt}\right] = 1^{**}$$

5) $$\mathcal{L}\left[\int_{0^-}^{t} f(\tau)\, d\tau\right] = -\frac{e^{-st}}{s}\int_{0^-}^{t} f(\tau)\, d\tau \Big|_{0^-}^{\infty} + \frac{1}{s}\int_{0^-}^{\infty} f(t) e^{-st} dt$$

$$^{**}\boxed{= \frac{F(s)}{s}}^{**}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$$

$$\frac{p(s)}{q(s)} = \sum_i \frac{c_i}{(s-a_i)}$$

where $q(s) = \prod_i (s-a_i)$

$$c_j = \frac{P(a_j)}{\prod_{i \neq j} (a_j - a_i)}$$

Lec - 21

→ Recap,

Laplace transform

$$F(s) = \mathcal{L}[f(t)] = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt$$

→ Stan. formulas

$$\mathcal{L}\left[\frac{df}{dt}\right], \quad \mathcal{L}\left[\int_{0^-}^{t} f(\tau)\, d\tau\right]$$

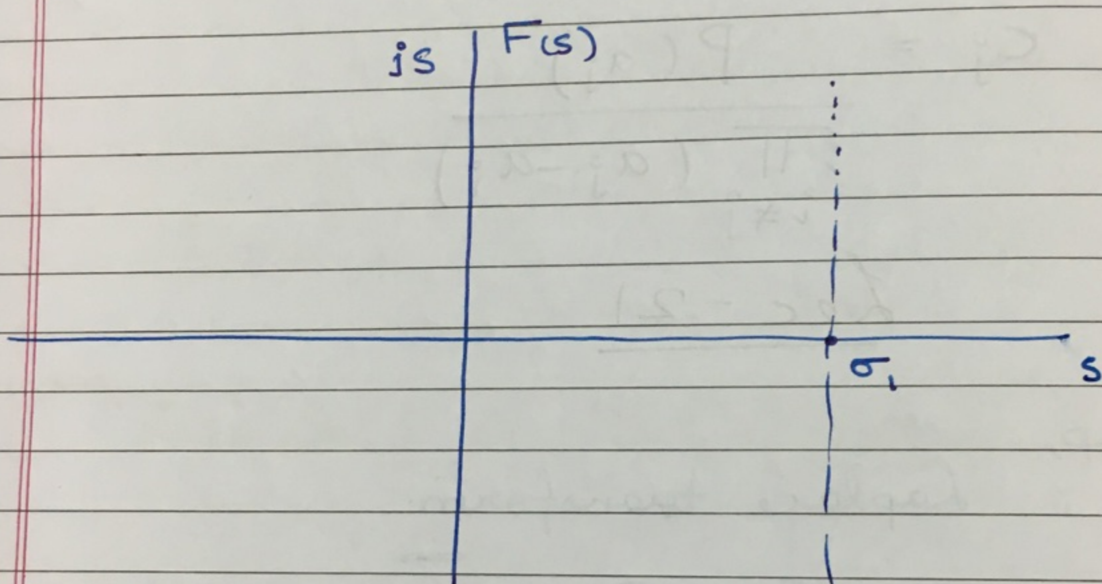
$$\mathcal{L}[e^{at}] = \int_{0^-}^{\infty} e^{-(s-a)t}\, dt = \frac{1}{s-a} \text{ if } s > a$$

$$= \cancel{\frac{1}{s-\alpha}}$$

→ **More on Inverse Laplace**

$$\mathcal{L}^{-1}(F(s)) = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)\, e^{st}\, ds$$

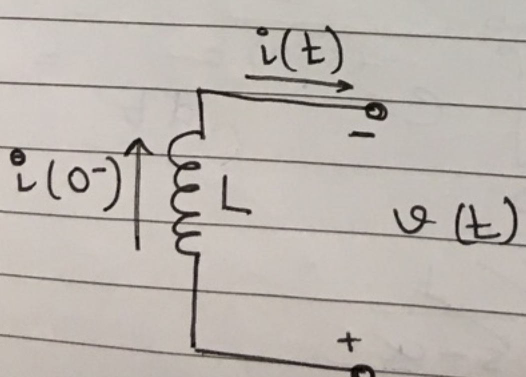
where $\sigma_1 > a$ → leftmost pt. of defn.



→ But if we use

$$\mathcal{L}^{-1}[\mathcal{L}(f(t))] = f(t)$$

→ **Proof of Thevenin / Norton thms**



$$v(t) = L\frac{d\,i(t)}{dt}$$

$$V(s) = L\left\{ s\,I(s) - i(0^-) \right\}$$

Aliter

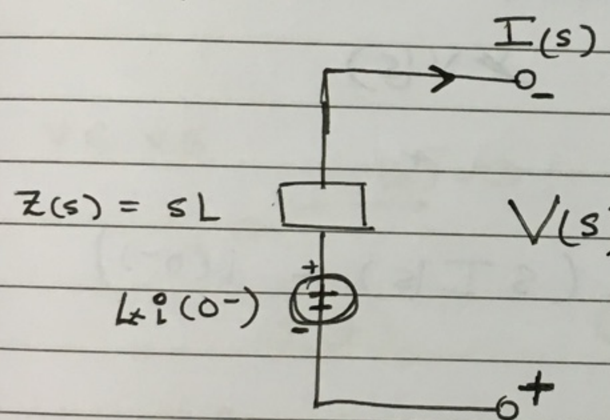
$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)\,d\tau$$

$$= \frac{1}{L}\int_{0^-}^{t} v(\tau)\,d\tau + \overbrace{\cancel{\frac{1}{L}\int_{-\infty}^{0^-} v(\tau)\,d\tau}}^{i(0^-)}$$

$\Rightarrow$

$$I(s) = \frac{V(s)}{sL} + \frac{i(0^-)}{s}$$

→ Transformed ckt.



$$V(s) = \frac{v_0}{s} \quad (\text{If } v(t) = v_0 = \text{const})$$

$$I(s) = \frac{v_0}{s^2 L} + \frac{i(0^-)}{s}$$

$$= \frac{v_0 + i(0^-)\,sL}{s^2 L}$$

$$\mathcal{L}^{-1}[I(s)] = i(t) = i(0^-) + \frac{v_0}{L}\mathcal{L}^{-1}\left[\frac{1}{s^2}\right]$$

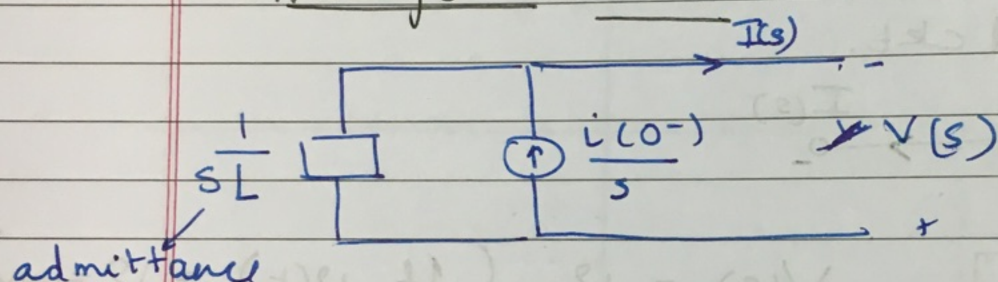
$$\boxed{i(t) = i(0^-) + \frac{v_0}{L}t} \quad \left(\because \mathcal{L}[t] = \frac{1}{s^2}\right)$$

→ Case 2: $v(t) = \sin(\omega t)$

↳ put $\dfrac{e^{j\omega t} - e^{-j\omega t}}{2j}$

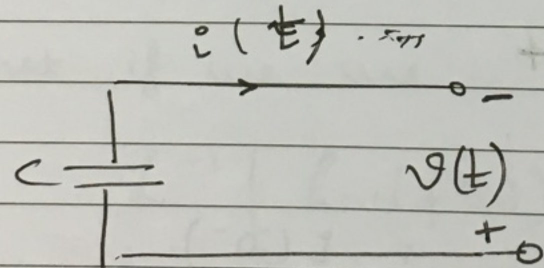
$$\begin{array}{ccc} \text{ODE} & \xrightarrow{\text{LT}} & \text{Laplace Eqn.} \\ \text{Solve ODE} \downarrow & & \downarrow \text{Algebra} \\ \text{Soln. in time} & \xleftarrow{(\text{LT})^{-1}} & \text{Soln. in S} \end{array}$$

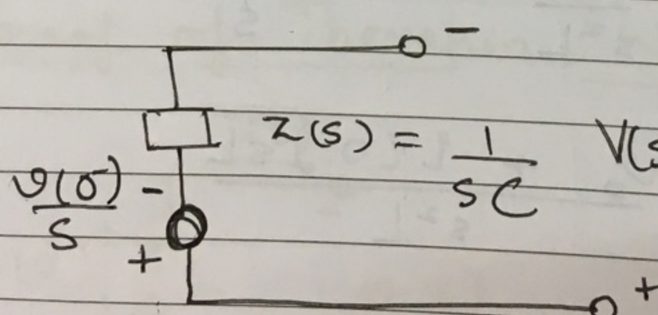
→ Transformed ckt



$$V(s) = L(sI(s) - i(0^-))$$

ex





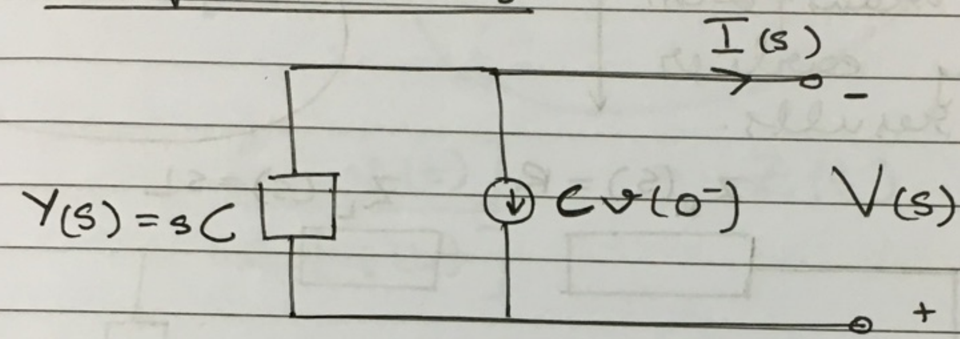
$Z(s) = \dfrac{1}{sC}$ $V(s) = Y(s) = sC$

$$i(t) = C\frac{dv}{dt}$$

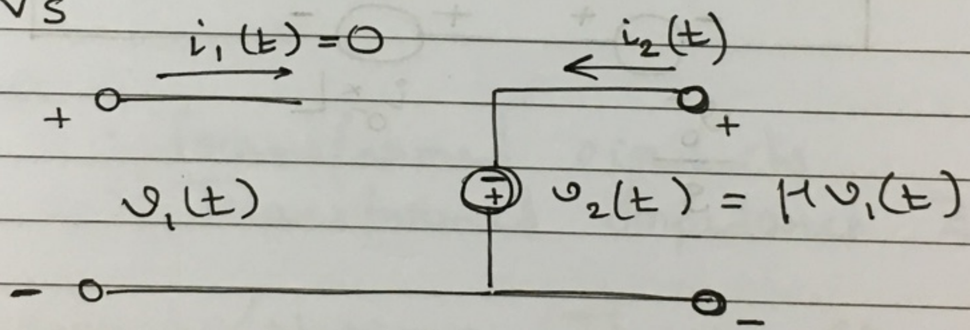
$$I(s) = C\left[sV(s) - v(0^-)\right]$$

$$V(s) - \frac{v(0^-)}{s} = \frac{I(s)}{sC}$$

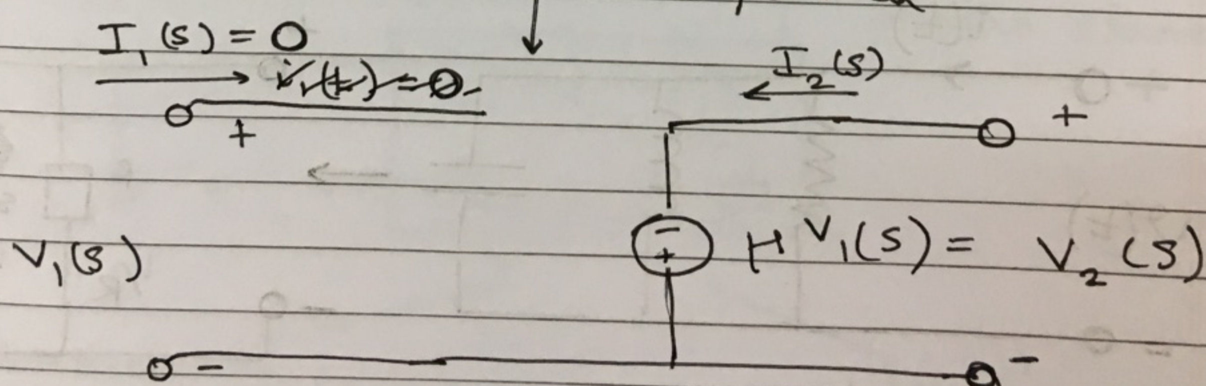
<u>Transformed ckt</u>

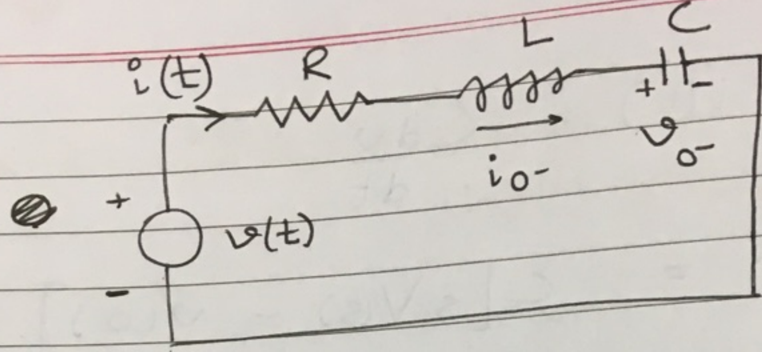


→ VCVS



transformed

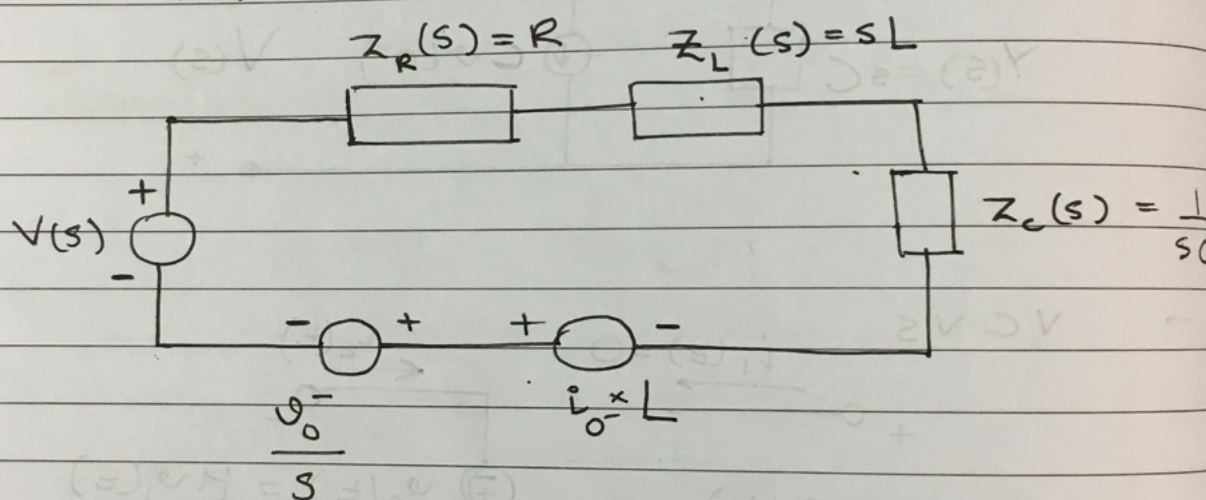


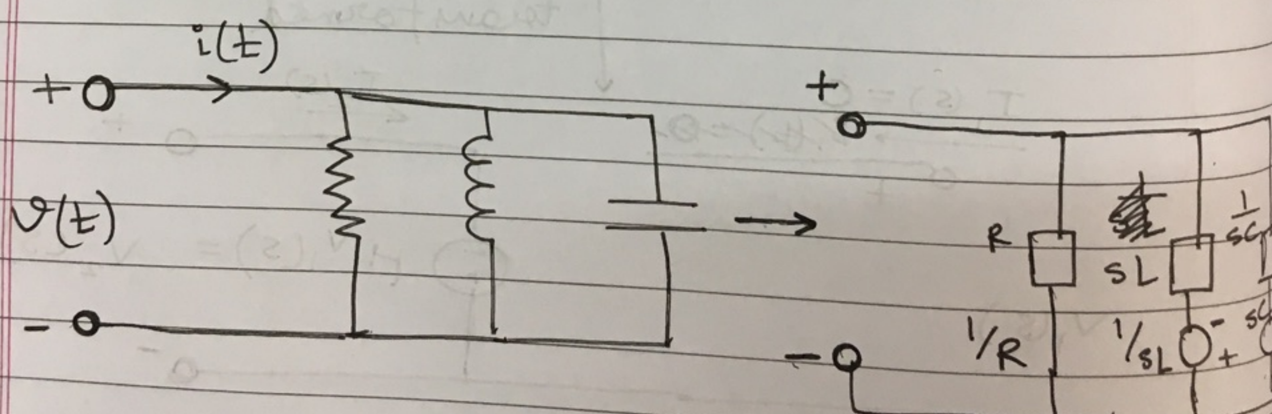


$$\frac{1}{C}\int_{0^-}^{t} i(\tau)\,d\tau$$

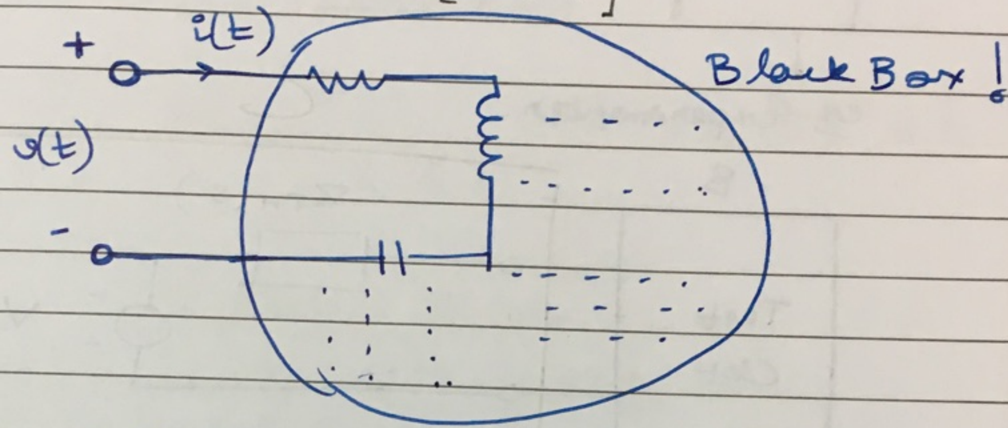
KVL : $v(t) = R\,i(t) + L\frac{di}{dt} + \frac{1}{C}\int_{0^-}^{t} i(\tau)\,d\tau + v_{0^-}$

transform using earlier results.





$$Z(s) = \frac{\mathcal{L}[v(t)]}{\mathcal{L}[i(t)]} = Z(s)$$



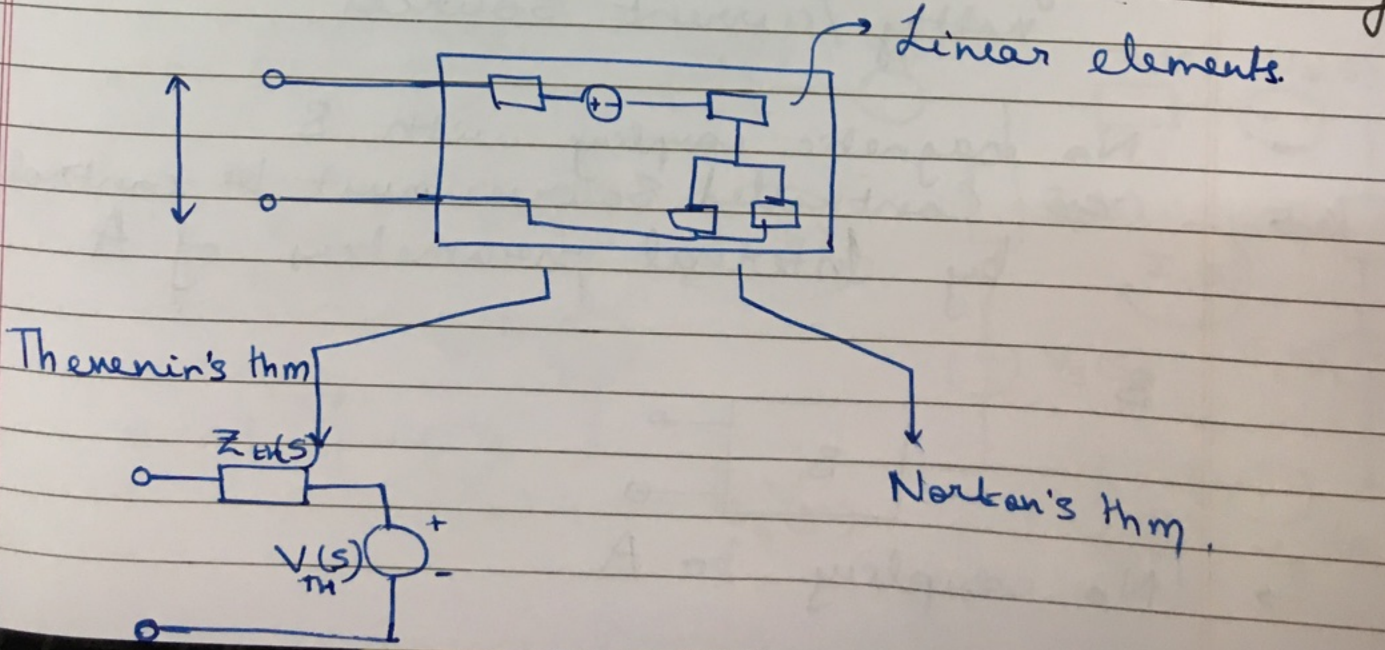
$$\frac{\mathcal{L}[v(t)]}{\mathcal{L}[i(t)]} = Z(s)$$

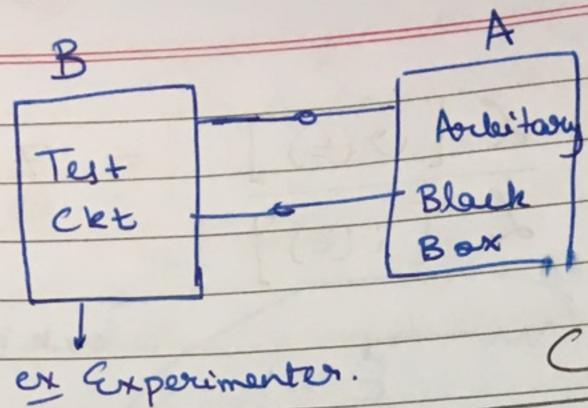
Lec - 22

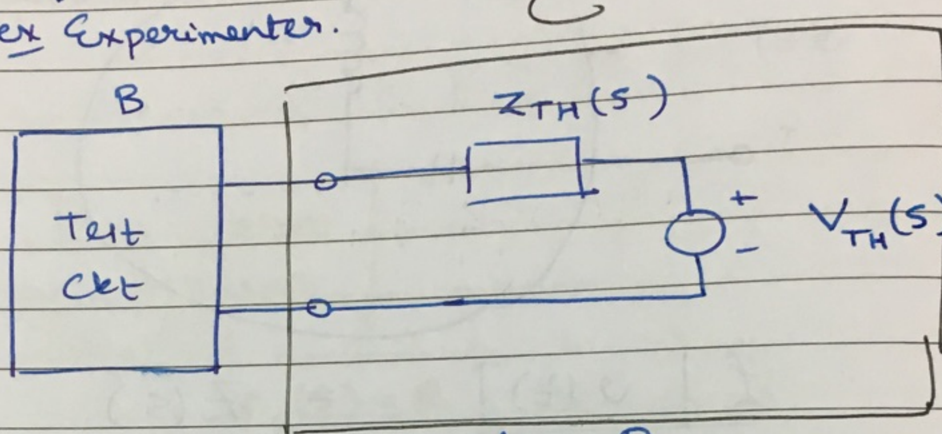
→ Recap

- Transformed circuits
- Transformed Impedance $Z(s)$

Thévenin / Norton's Thm's Chap 9, Valkenburg



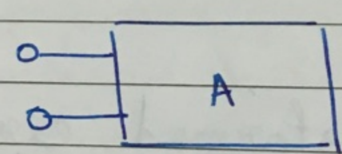




→ A, C indistinguishable for B.

→ Conditions that A and B must satisfy (Valkenberg)

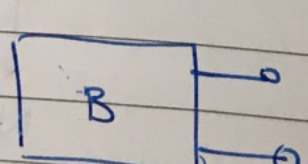
A



→ All elements must be linear
→ May Include Independant / Dependant voltage/current Sources

→ No magnetic coupling with B
→ Controlled Sources must be controlled by Internal parameters of A

B



→ No coupling to A

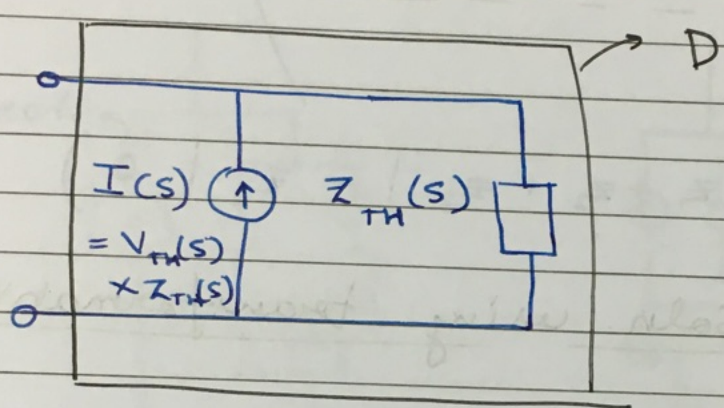
Thm [Thévenin thm]

→ Under above conditions

$\exists$ $Z_{th}(s)$ and $V_{th}(s)$ s.t

$A \equiv C$

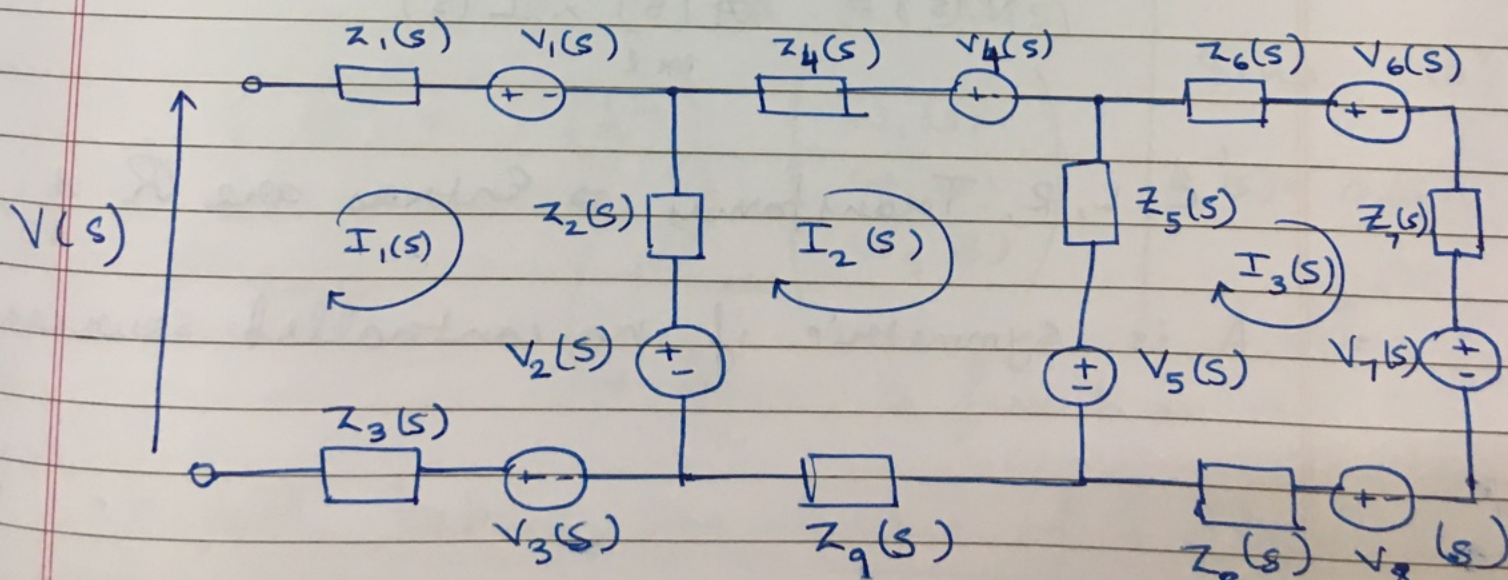
Norton's thm



$\exists$ $Z_{TH}(s)$, $I(s)$ s.t.

$D \equiv A$.

Thm : Thévenin $\Leftrightarrow$ Norton



→ General App.

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} \overbrace{V(s)}^{\text{Gives } Y(s)} - \overbrace{V_1(s) - V_2(s) + V_3(s)}^{\text{set to } 0 \Rightarrow I_{TH}(s) \to 0} \\ V_2(s) - V_4(s) - V_5(s) \\ V_5(s) - V_6(s) - V_7(s) + V_8(s) \end{pmatrix} = A_{3\times 3} \begin{pmatrix} I_1(s) \\ I_2(s) \\ I_3(s) \end{pmatrix}$$

$$a_1 = (Z_1 + Z_2 + Z_3 \mid -Z_2 \mid \overset{\text{share no edges}}{0})$$

→ General soln. using transformations

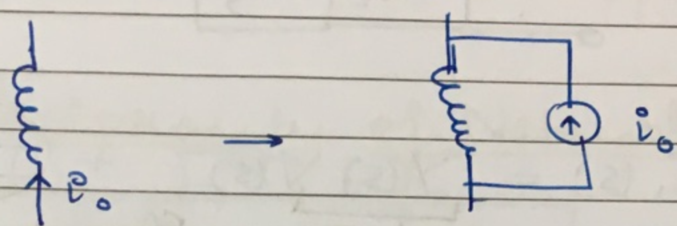
$$\overline{v(t)} = \underset{\substack{l\times l \\ \downarrow \\ |E|-|N|+1}}{A} \overline{i(t)}$$

$$\overline{V(s)} = \underset{l\times l}{A(s)} \times \overline{I(s)}$$

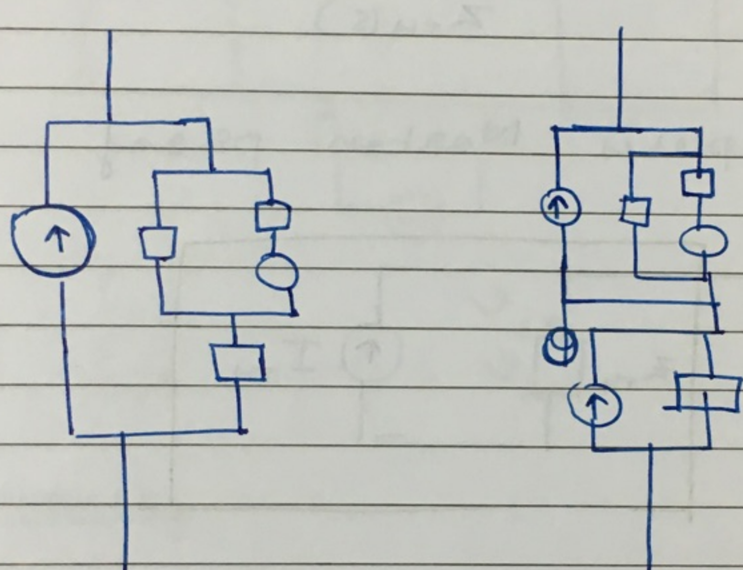
C, L, R, Transformers ⇒ Entries are $R, \frac{d}{dt}, \int$

→ A is symmetric if no controlled sources

→ Inductor with initial conditions



→ Another case.



Conclusions

$$\bar{I}(s) = A^{-1}(s)\,\bar{V}(s)$$

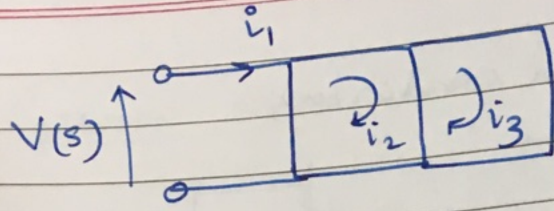
$$\underbrace{I_1(s)}_{1^{st}\text{ row}} = \underbrace{(A^{-1})_1}_{1^{st}\text{ row}} \begin{pmatrix} w_1(s) \\ w_2(s) \\ w_3(s) \end{pmatrix}$$

Since

$$A^{-1}(s) = \frac{\text{adj}\, A\,(s)}{|A|\,(s)}$$

Again, a function of (S)

→ Recall : taking the Inverse of a Matrix



Sources inside the network

Sources give this.

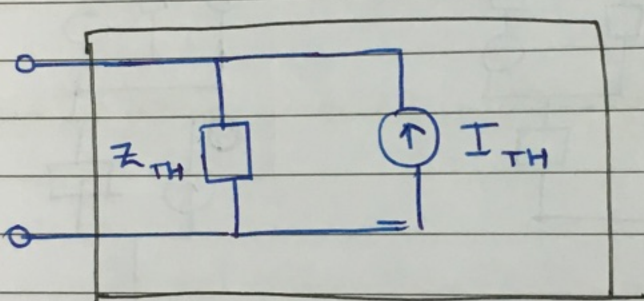
$$I_1(s) = \underbrace{Y(s)\,V(s)}_{\text{Passive Elements give this}} + I_{TH}(s)$$

Reason:

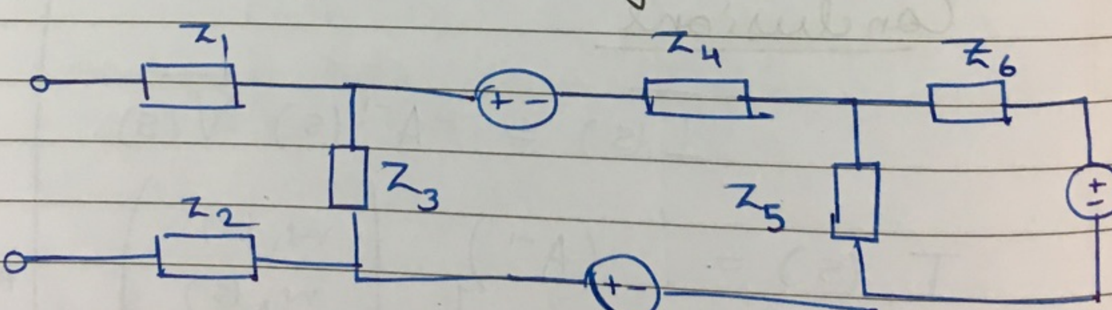
(Passive multiplied by external $V(s)$)

$$Y(s) = \frac{1}{Z_{TH}(s)}$$

→ Completes Norton proof



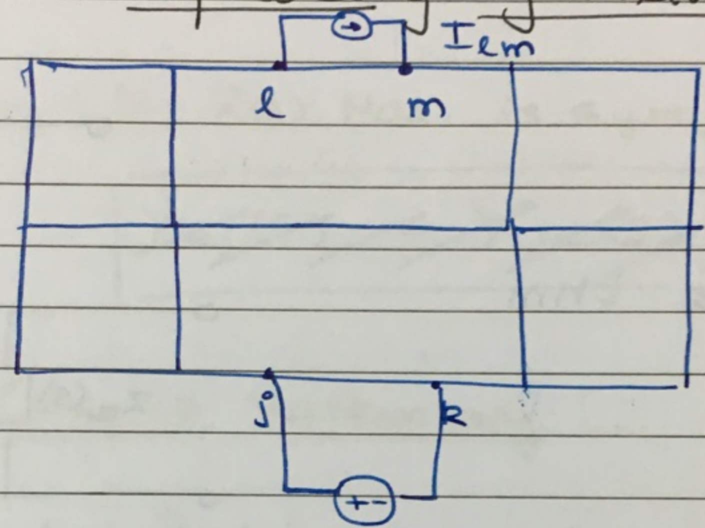
→ Algorithm we get from insight



1) To find $Z_{TH}(s)$ short all voltage sources, open all current sources. Use series-parallel rules for combining Impedances.

$$Z = \left(\left(Z_5 \parallel Z_6 + Z_4 \right) \parallel Z_3 \right) + Z_1 + Z_2$$

Reciprocity of Networks



i_{lm} $\quad\quad$ v_{jk}

i_{jk} $\quad\quad$ v_{lm}

Reciprocity

→ Behaviour of i_{lm} as a function of v_{jk} is the same as behaviour of i_{jk} as a function of v_{lm}

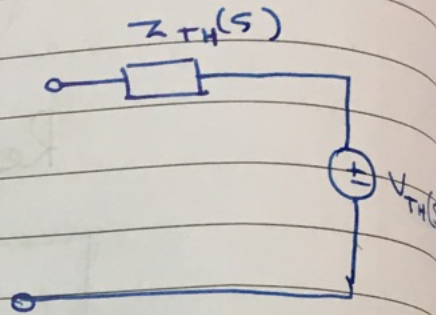
→ Holds if Impedance matrix is symmetric i.e. no controlled sources.

13 Oct

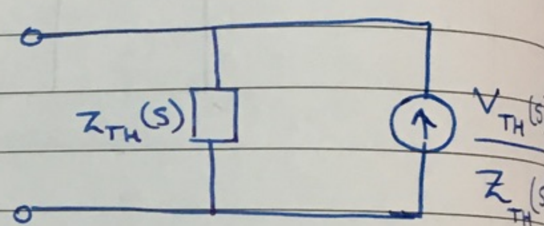
Lec - 23

Recap

Thévenin's thm



Norton's thm



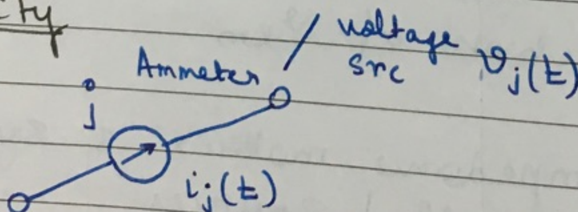
Proof Idea

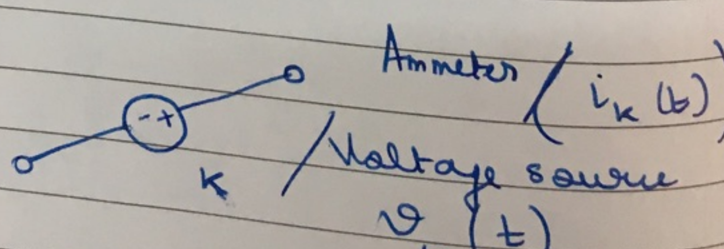
$$\bar{V}(s) = \left(Z_{ij}(s)\right) \bar{I}(s)$$

- Loop Analysis

$$\bar{I}(s) = \left(Z(s)\right)^{-1} \bar{V}(s)$$

Reciprocity





$$\frac{I_j(s)}{V_k(s)} = Y_{jk}(s)$$

↓

Interchange roles of edges

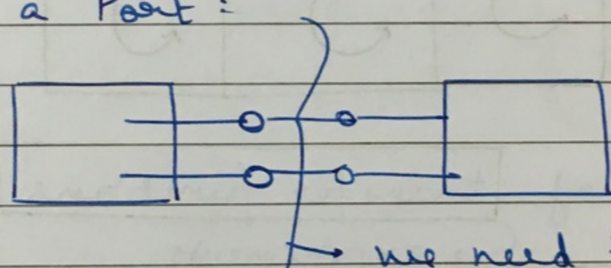
$$\frac{I_k(s)}{V_j(s)} = Y_{kj}(s)$$

Reciprocity (If $Z(s)$ Mat. is sym.) says (~~[illegible]~~)

$\cancel{\boxed{Y_{kj}(s) = Y_{jk}(s)}}$ $\quad Y_{kj}(s) = Y_{jk}(s)$

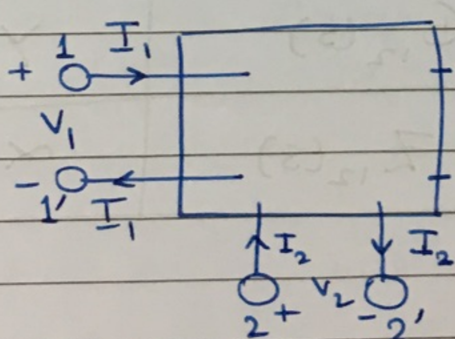
Chap 10 Valkenberg

→ Notion of a Port :



→ we need two branches (1∝) for non zero current to flow b/w

→ Port :



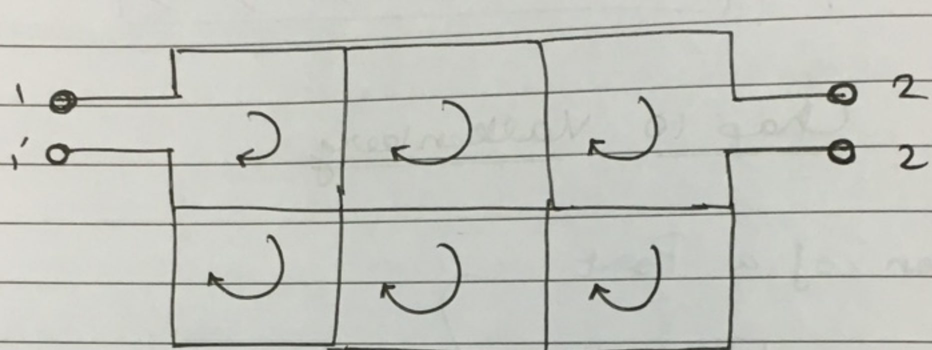
Two terminals with same current going in and out is a port.

→ Thévenin / Norton : Work in 1 port case
single $Z_{TH}(s)$ & $V_{TH}(s)$

→ To Analyse a ckt as a black-box with >1 ports we need to extend it.

Ex 10.

Extended Thévenin / Norton thm



→ Types of $\boxed{\text{transfer functions}}$ possible
↳ experiments

(Inputs) Form Denomin.	(Outputs) Form Numer. $V_2(s)$	$I_2(s)$
$V_1(s)$	$G_{12}(s)$	$Y_{12}(s)$
$I_1(s)$	$Z_{12}(s)$	$\alpha_{12}(s)$

$$Z_{TH} = \frac{V(s)}{I(s)} = Z_{11}(s)$$

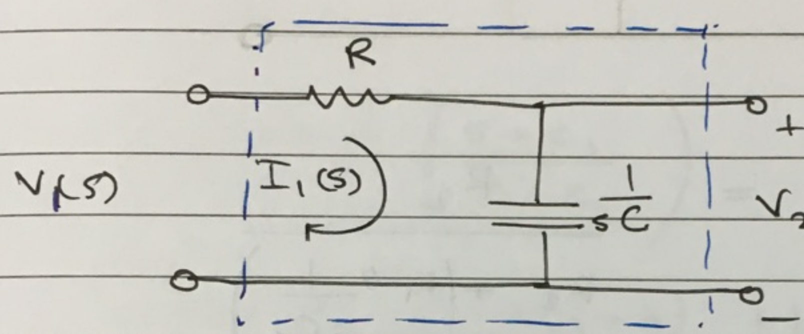
At port 1

→ Indexing :

$$G_{12}(s) = V_2(s)/V_1(s)$$

$$Y_{12}(s) = I_2(s)/V_1(s)$$

Ex 10.3



N.T

$$G_{12}(s) = \frac{1}{g_{21}(s)}$$

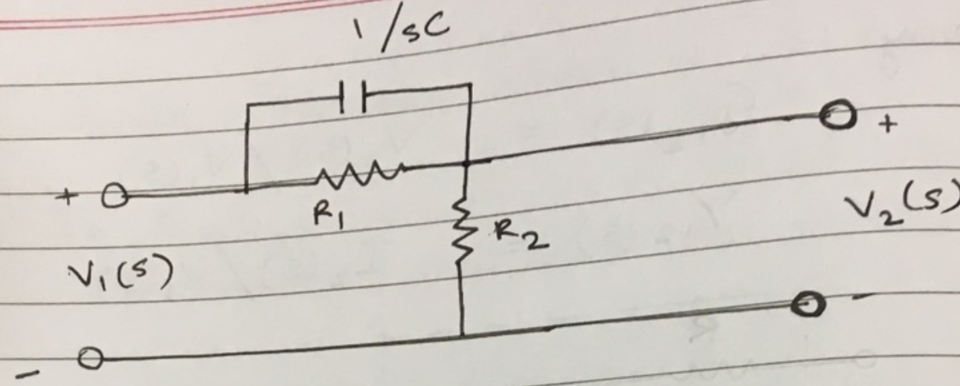
$$G_{12}(s) = \frac{V_2(s)}{V_1(s)} = \frac{\left(\frac{1}{sC}\right) \times \cancel{I_1(s)}}{\left(R + \frac{1}{sC}\right) \times \cancel{I_1(s)}}$$

$$= \frac{\frac{1}{RC}}{\left(s + \frac{1}{RC}\right)}$$

$$Y_{11}(s) = \frac{I_1(s)}{V_1(s)} = \frac{1}{R + \frac{1}{sC}}$$

$$\boxed{= \frac{1 \times s}{R\left(s + \frac{1}{RC}\right)}} = \frac{1}{Z_{TH_1}(s)}$$

Ex5)



$$G_{12}(s) = \frac{R_2}{R_2 + \left(R_1 \parallel \frac{1}{sC}\right)}$$

$$= \frac{R_2}{R_2 + \left(\frac{R_1/sC}{R_1 + \frac{1}{sC}}\right)}$$

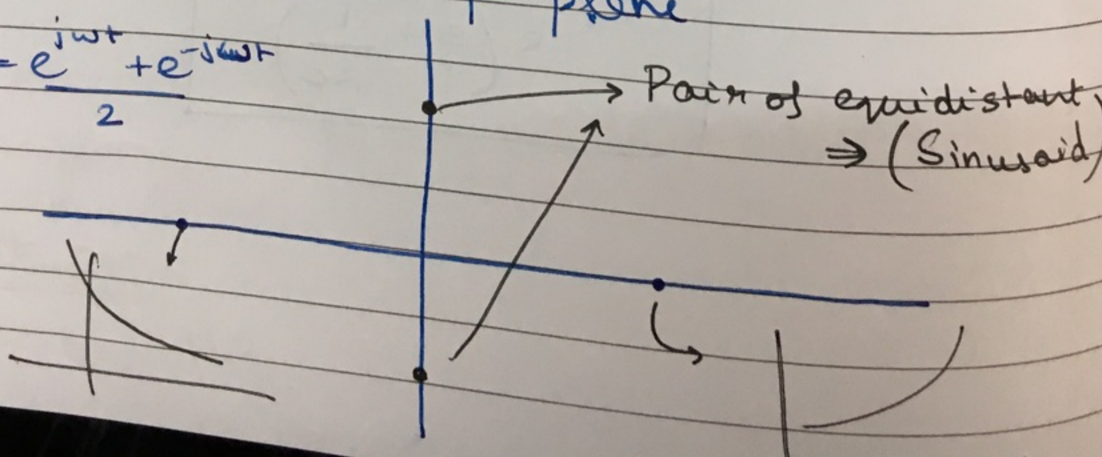
→ Claim: All transfer functions are of the form $\frac{p(s)}{q(s)}$ where p, q are polynomials

→ Taken to be true.

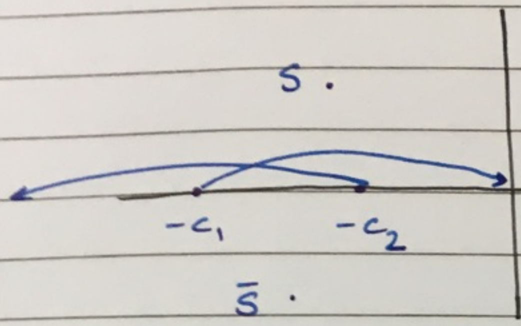
→ What values can s take?

Ans: Complex Numbers. $\mathbb{C}$ plane

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$



Poles & Zeroes



Thinking of transfer functions as conformal mappings.

eg $$G_{12}(s) = \left(\frac{s+c_1}{s+c_2}\right)$$

$$f(z,t) = z(1-t) + tG_{12}(s)$$