MA205-4

Zeros of holomorphic functions:

We will show that the zeros of a holomorphic function are isolated i.e, if $f(z_0)=0$ then there exists a neighbourhood of z_0 in which $f(z)\neq 0$ for $z\neq z_0$. Note that this is not true for real differentiable functions; for example consider

$$f(x) = e^{-1/x^2} \quad : x \geq 0$$

$$= 0 \quad : x \leq 0$$

f is infinitely differentiable but

$f(x)=0$, for $x\le 0$, i.e, zeros of f are

not isolated.

Recall:

1. Let $S\subseteq\mathbb{C}$. A point $z_0\in\mathbb{C}$ is a limit point of S if $B_\varepsilon(z_0)$, for each $\varepsilon>0$, contains atleast one point $z\in S$, $z\neq z_0$.

Check: S is closed if and only if S contains all its limit points

2. If $\Omega\subseteq\mathbb{C}$ is connected then it does not have a subset which is both open & closed.

Proof of Morera's theorem :

Let $f \neq 0$ be a holomorphic function & set $Z(f) = \{ z \in \Omega \mid f(z) = 0 \}$.

We will show that $Z(f)$ cannot have a limit point.

Suppose z_0 is a limit point of $Z(f)$.

f continuous $\Rightarrow$ $f(z_0) = 0$

We will first show that : $f^{n}(z_0) = 0 \ \forall n \geq 1$.

Suppose not, i.e., $\exists$ an integer n such that

$f(z_0) = f'(z_0) = \cdots = f^{(n-1)}(z_0) = 0$ &

$f^{n}(z_0) \neq 0$.

Take a small disc around z_0 which is contained in Ω where f has a power series expansion: $f(z) = \sum_{k=n}^{\infty} a_k (z - z_0)^k$

If $g(z) = \sum_{k=n}^{\infty} a_k (z - z_0)^{k-n}$ then

$$f(z) = (z - z_0)^n g(z) \quad \text{with } g(z_0) \neq 0$$

g continuous & $g(z_0) \neq 0 \Rightarrow \exists$ a small neighbourhood of z_0 in which g is never zero.

But z_0 is a limit point of $Z(f)$ & so this neighbourhood will contain a point

$w \neq z_0$ with $f(w) = 0$.

$\Rightarrow$ $g(w) = 0$ (why?)

a contradiction!

Hence $f^n(z_0) = 0$ $\therefore \forall$ $n \geq 1$.

Let $S = \{ z \in \Omega \mid f^n(z) = 0 \ \forall n \geq 0 \}$.

Note $S \neq \phi$ as $z_0 \in S$.

We will show that S is both open & closed

(which implies $S = \Omega$, i.e., $f = 0$ on Ω).

To show S is closed show that all

its limit points lie in S.

Let z be a limit point of S and

$z_k \in S$ be such that $\lim_{k \to \infty} z_k = z$.

Since f^n is continuous, $f^n(z) = 0$, $n \geq 1$

i.e., $z \in S$.

To show S is open:

Let $z \in S$. As Ω is open, there exists

$\varepsilon > 0$ such that $B_\varepsilon(z) \subseteq \Omega$. We choose

ε small enough so that f has a power

series expansion in $B_\varepsilon(z)$ i.e.,

$$f(w) = \sum_{n=0}^{\infty} a_n (w - z)^n$$

with $a_n = \frac{1}{n!} f^n(z) = 0$: for $n \geq 0$

Thus $f = 0$ for every $w \in B_\varepsilon(z)$.

Hence $B_\varepsilon(z) \subseteq S$.

This proves the theorem.

<u>Corollary</u> <u>(Identity theorem)</u> :

If f & g are holomorphic in Ω then

$f \equiv g$ if and only if

$\{z \in \Omega \mid f(z) = g(z)\}$ has a limit

point in Ω.

<u>Examples:</u>

<u>1.</u> Two holomorphic functions which agree

on $\{1/n \; : \; n \geq 1\}$ are the same

provided 0 belongs to the domain of the functions. (eg. $\sin(\frac{2\pi}{z})$)

2: $\exp(z)$ is the only holomorphic function which agrees with e^x on the real line. Similarly for $\sin z$, $\cos z$ ...

3. The identity $\sin^2 z + \cos^2 z = 1$ follows for $\mathbb{C}$ as it holds over $\mathbb{R}$.

Proposition: If f is holomorphic, each zero of f has finite multiplicity, i.e, there exists m such that $f(z) = (z - z_0)^m \cdot g(z)$ with $g(z_0) \neq 0$.

Proof: If there is no such n then we get $f^{n}(z_0) = 0$, for every $n \geq 1$. Arguing as in the earlier proof, we get $f = 0$.