MA-207 Differential Equations II

Ronnie Sebastian

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

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S2 - Lecture 11

Definition (Formal solution for Dirichlet boundary)

The formal solution of IBVP

$$\begin{aligned} u_t &= k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ u(0,t) &= 0 && t > 0 \\ u(L,t) &= 0 && t > 0 \\ u(x,0) &= f(x) && 0 \leq x \leq L \end{aligned}$$

is

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2}\, t\right) \sin\frac{n\pi x}{L}$$

where

$$S(x) = \sum_{n=1}^{\infty} \alpha_n \sin\frac{n\pi x}{L}$$

is the Fourier sine series of f on $[0, L]$ i.e.

$$\alpha_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\, dx.$$

Definition (Formal solution for Neumann boundary condition)

The formal solution of IBVP

$$\begin{aligned} u_t &= k^2 u_{xx} & 0 < x < L, \quad t > 0 \\ u_x(0,t) &= 0 & t > 0 \\ u_x(L,t) &= 0 & t > 0 \\ u(x,0) &= f(x) & 0 \le x \le L \end{aligned}$$

is

$$u(x,t) = \sum_{n=0}^{\infty} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2} t \right) \cos \frac{n\pi x}{L}$$

where

$$S(x) = \sum_{n=0}^{\infty} \alpha_n \cos \frac{n\pi x}{L}$$

is the Fourier cosine series of f on $[0, L]$ i.e.

$$\alpha_0 = \frac{1}{L} \int_0^L f(x)\, dx \qquad \alpha_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx.$$

Non homogeneous PDE: Dirichlet boundary condition

Let us now consider the following PDE

$$\begin{aligned} &u_t - k^2 u_{xx} = F(x,t) && 0 < x < L, \quad t > 0 \\ &u(0,t) = f_1(t) && t > 0 \\ &u(L,t) = f_2(t) && t > 0 \\ &u(x,0) = f(x) && 0 \le x \le L \end{aligned}$$

How do we solve this?

Let us first make the substitution

$$z(x,t) = u(x,t) - (1 - \frac{x}{L}) f_1(t) - \frac{x}{L} f_2(t)$$

Then clearly

- $z_t - k^2 z_{xx} = G(x,t)$
- $z(0,t) = 0$
- $z(L,t) = 0$
- $z(x,0) = g(x)$

Non homogeneous PDE: Dirichlet boundary condition

It is clear that we would have solved for u iff we have solved for z. In view of this observation, let us try and solve the problem for z.

By observing the boundary conditions, we guess that we should try and look for a solution of the type

$$z(x,t) = \sum_{n\geq 1} Z_n(t)\sin(\frac{n\pi x}{L})$$

Differentiating the above term by term we get that is satisfies the equation

$$z_t - k^2 z_{xx} = \sum_{n\geq 1} \big(Z_n'(t) + \frac{k^2 n^2 \pi^2}{L^2} Z_n(t)\big)\sin(\frac{n\pi x}{L})$$

Let us write

$$G(x,t) = \sum_{n\geq 1} G_n(t)\sin(\frac{n\pi x}{L})$$

Non homogeneous PDE: Dirichlet boundary condition

Thus, if we need $z_t - k^2 z_{xx} = G(x,t)$ then we should have that

$$G_n(t) = Z_n'(t) + \frac{k^2 n^2 \pi^2}{L^2} Z_n(t) \qquad (*)$$

We also need that $z(x,0) = g(x)$.
If

$$g(x) = \sum_{n \geq 1} b_n \sin \frac{n\pi x}{L}$$

then we should have that

$$Z_n(0) = b_n \qquad (!)$$

Clearly, there is a unique solution to the differential equation $(*)$ with initial condition $(!)$.

Non homogeneous PDE: Dirichlet boundary condition

The solution to the above equation is given by

$$Z_n(t) = Ce^{-\frac{k^2 n^2 \pi^2}{L^2} t} + e^{-\frac{k^2 n^2 \pi^2}{L^2} t} \int_0^t G_n(s) e^{\frac{k^2 n^2 \pi^2}{L^2} s} ds$$

We can find the constant using the initial condition.

Thus, we let $Z_n(t)$ be this unique solution, then the series

$$z(x,t) = \sum_{n \geq 1} Z_n(t) \sin(\frac{n\pi x}{L})$$

solves our non homogeneous PDE with Dirichlet boundary conditions for z.

Non homogeneous PDE: Dirichlet boundary condition

Example

Let us now consider the following PDE

$$\begin{aligned} &u_t - u_{xx} = e^t && 0 < x < 1, \quad t > 0 \\ &u(0,t) = 0 && t > 0 \\ &u(1,t) = 0 && t > 0 \\ &u(x,0) = x(x-1) && 0 \le x \le 1 \end{aligned}$$

From the boundary conditions $u(0,t) = u(1,t) = 0$ it is clear that we should look for solution in terms of Fourier sine series.

The Fourier sine series of $F(x,t)$ is given by (for $n \ge 1$)

$$\begin{aligned} F_n(t) &= 2\int_0^1 F(x,t) \sin n\pi x \, dx \\ &= 2\int_0^1 e^t \sin n\pi x \, dx \\ &= \frac{2(1-(-1)^n)e^t}{n\pi} \end{aligned}$$

Example (continued ...)

Thus, the Fourier series for e^t is given by

$$e^t = \sum_{n\geq 1} \frac{2(1-(-1)^n)}{n\pi} e^t \sin n\pi x$$

The Fourier sine series for $f(x) = x(x-1)$ is given by

$$x(x-1) = \sum_{n\geq 1} \frac{4((-1)^n - 1)}{(n\pi)^3} \sin n\pi x$$

Substitute $u(x,t) = \sum_{n\geq 1} u_n(t) \sin n\pi x$ into the equation $u_t - u_{xx} = e^t$

$$\sum_{n\geq 1} \left(u_n'(t) + n^2\pi^2 u_n(t)\right) \sin n\pi x = \sum_{n\geq 1} \frac{2(1-(-1)^n)}{n\pi} e^t \sin n\pi x$$

Non homogeneous PDE: Dirichlet boundary condition

Example (continued ...)

Thus, for $n \geq 1$ and even we get

$$u_n'(t) + n^2\pi^2 u_n(t) = 0$$

that is,

$$u_n(t) = C_n e^{-n^2\pi^2 t}$$

If $n \geq 1$ and even, we have that the Fourier coefficient of $x(x-1)$ is 0. Thus, when we put $u_n(0) = 0$ we get $C_n = 0$.

For $n \geq 1$ odd we get

$$u_n'(t) + n^2\pi^2 u_n(t) = \frac{4}{n\pi} e^t$$

that is,

$$u_n(t) = e^{-n^2\pi^2 t} \int_0^t \frac{4}{n\pi} e^s e^{n^2\pi^2 s} ds + C_n e^{-n^2\pi^2 t}$$

Example (continued ...)

If $n \geq 1$ and odd, we have the Fourier coefficient of $x(x-1)$ is $\frac{-8}{(n\pi)^3}$. Thus, we get

$$u_n(0) = C_n = \frac{-8}{(n\pi)^3}$$

Thus, the solution we are looking for is

$$u(x,t) = \sum_{n \geq 0} \left(e^{-(2n+1)^2\pi^2 t} \int_0^t \frac{4}{(2n+1)\pi} e^s e^{(2n+1)^2\pi^2 s} ds + \frac{-8}{((2n+1)\pi)^3} e^{-n^2\pi^2 t} \right) \sin(2n+1)\pi x$$

Non homogeneous PDE: Neumann boundary condition

Let us now consider the following PDE

$$
\begin{aligned}
&u_t - k^2 u_{xx} = F(x,t) \qquad && 0 < x < L, \quad t > 0 \\
&u_x(0,t) = f_1(t) && t > 0 \\
&u_x(L,t) = f_2(t) && t > 0 \\
&u(x,0) = f(x) && 0 \le x \le L
\end{aligned}
$$

How do we solve this?Let us first make the substitution

$$z(x,t) = u(x,t) - (x - \frac{x^2}{2L})f_1(t) - \frac{x^2}{2L}f_2(t)$$

Then clearly

- $z_t - k^2 z_{xx} = G(x,t)$
- $z_x(0,t) = 0$
- $z_x(L,t) = 0$
- $z(x,0) = g(x)$

Non homogeneous PDE: Neumann boundary condition

It is clear that we would have solved for u iff we have solved for z. In view of this observation, let us try and solve the problem for z.

By observing the boundary conditions, we guess that we should try and look for a solution of the type

$$z(x,t) = \sum_{n\geq 0} Z_n(t)\cos(\frac{n\pi x}{L})$$

Differentiating the above term by term we get that is satisfies the equation

$$z_t - k^2 z_{xx} = \sum_{n\geq 0} \left(Z_n'(t) + \frac{k^2n^2\pi^2}{L^2}Z_n(t)\right)\cos(\frac{n\pi x}{L})$$

Let us write

$$G(x,t) = \sum_{n\geq 0} G_n(t)\cos(\frac{n\pi x}{L})$$

Thus, if we need $z_t - k^2 z_{xx} = G(x,t)$ then we should have that

$$G_n(t) = Z_n'(t) + \frac{k^2 n^2 \pi^2}{L^2} Z_n(t) \qquad (*)$$

We also need that $z(x,0) = g(x)$.
If

$$g(x) = \sum_{n \geq 0} b_n \cos \frac{n\pi x}{L}$$

then we should have that

$$Z_n(0) = b_n \qquad (!)$$

Clearly, there is a unique solution to the differential equation $(*)$ with initial condition $(!)$.

Non homogeneous PDE: Neumann boundary condition

The solution to the above equation is given by

$$Z_n(t) = Ce^{-\frac{k^2n^2\pi^2}{L^2}t} + e^{-\frac{k^2n^2\pi^2}{L^2}t}\int_0^t G_n(s)e^{\frac{k^2n^2\pi^2}{L^2}s}ds$$

We can find the constant using the initial condition.

Thus, we let $Z_n(t)$ be this unique solution, then the series

$$z(x,t) = \sum_{n\geq 0} Z_n(t)\cos(\frac{n\pi x}{L})$$

solves our non homogeneous PDE with Dirichlet boundary conditions for z.

Non homogeneous PDE: Neumann boundary condition

Example

Let us now consider the following PDE

$$\begin{aligned}
&u_t - u_{xx} = e^t && 0 < x < 1, \quad t > 0 \\
&u_x(0,t) = 0 && t > 0 \\
&u_x(1,t) = 0 && t > 0 \\
&u(x,0) = x(x-1) && 0 \le x \le 1
\end{aligned}$$

From the boundary conditions $u_x(0,t) = u_x(1,t) = 0$ it is clear that we should look for solution in terms of Fourier cosine series.

The Fourier cosine series of $F(x,t)$ is given by (for $n \ge 0$)

$$F_0(t) = \int_0^1 F(x,t)\,dx = \int_0^1 e^t dx = e^t$$

$$F_n(t) = 2\int_0^1 F(x,t)\cos n\pi x\,dx = 2\int_0^1 e^t \cos n\pi x\,dx = 0 \quad n > 0$$

Example (continued ...)

Thus, the Fourier series for e^t is simply e^t.

The Fourier cosine series for $f(x) = x(x-1)$ is given by

$$x(x-1) = -\frac{1}{6} + \sum_{n\geq 1} \frac{2((-1)^n + 1)}{(n\pi)^2} \cos n\pi x$$

Substitute $u(x,t) = \sum_{n\geq 0} u_n(t) \cos n\pi x$ into the equation $u_t - u_{xx} = e^t$

$$\sum_{n\geq 0} \left(u_n'(t) + n^2\pi^2 u_n(t)\right) \cos n\pi x = e^t$$

Example (continued ...)

Thus, for $n = 0$ we get

$$u_0'(t) = e^t$$

that is,

$$u_0(t) = e^t + C_0$$

In the case $n = 0$, we have that the Fourier coefficient of $x(x-1)$ is $\frac{-1}{6}$. Thus, when we put $u_0(0) = -\frac{1}{6}$ we get $C = -\frac{7}{6}$.
For $n \geq 1$

$$u_n'(t) + n^2\pi^2 u_n(t) = 0$$

that is,

$$u_n(t) = C_n e^{-n^2\pi^2 t}$$

Let us now use the initial condition to determine the constants.

Example (continued ...)

In the case $n \geq 1$ and odd, we have that the Fourier coefficient of $x(x-1)$ is 0. Thus, when we put $u_n(0) = 0$ we get $C_n = 0$.

In the case $n \geq 1$ even, we have the Fourier coefficient of $x(x-1)$ is $\frac{4}{(n\pi)^2}$. Thus, we get

$$C_n = \frac{4}{(n\pi)^2}$$

Thus, the solution we are looking for is

$$u(x,t) = e^t - \frac{7}{6} + \sum_{n\geq 1} \left(\frac{1}{(n\pi)^2} e^{-4n^2\pi^2 t} \right) \cos(2n\pi x)$$

One-dimensional wave equation

Consider the following differential equation

$$u_{tt} = k^2 u_{xx}, \quad 0 < x < L, \ t > 0,$$

called one-dimensional wave equation.

Here k^2 is a positive constant, x is the space variable and t is the time variable.

We wish to find solutions of the above PDE which satisfy the following initial and boundary conditions
The initial conditions are

$$u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = g(x).$$

The (Dirichlet) boundary conditions are

$$u(0,t) = u(L,t) = 0.$$

Dirichlet boundary conditions: Getting some solutions

We will use the method of separation of variables to first find some solutions to the wave equation with boundary conditions. That is, we forget about the initial conditions for now.

Suppose

$$u(x,t) = X(x)\,T(t)$$

Substituting this in wave equation $\boxed{u_{tt} = k^2 u_{xx}}$

$$X(x)T''(t) = k^2 X''(x)T(t).$$

We can now separate the variables:

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{k^2 T(t)}$$

The equality is between a function of x and a function of t, so both must be constant, say $-\lambda$.

Dirichlet boundary conditions: Getting some solutions

Thus, we get the conditions

$$X''(x) + \lambda X(x) = 0 \quad \text{and} \quad T''(t) + k^2 \lambda T(t) = 0.$$

We also have the boundary conditions

$$u(0,t) = X(0)T(t) = 0 \quad \text{and} \quad u(L,t) = X(L)T(t) = 0.$$

Since we don't want T to be identically zero, we get

$$X(0) = 0 \quad \text{and} \quad X(L) = 0.$$

First let us solve the eigenvalue problem

$$\begin{aligned} X''(x) + \lambda X(x) &= 0 \\ X(0) = X(L) &= 0, \end{aligned}$$

The eigenvalues and eigenfunctions are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \qquad\qquad X_n = \sin\frac{n\pi x}{L}, \; n \geq 1.$$

For each λ_n we consider the equation in the t variable

$$T''(t) + k^2 \lambda T(t) = 0$$

Thus, for each λ_n we get a solution for T given by

$$T_n(t) = \alpha_n \cos\left(\frac{kn\pi}{L} t\right) + \frac{\beta_n L}{kn\pi} \sin\left(\frac{kn\pi}{L} t\right),$$

where α_n and β_n are real numbers.

Thus, we get a solution for each $n \geq 1$

$$u_n(x,t) = T_n(t) X_n(x) = \left(\alpha_n \cos\left(\frac{kn\pi}{L} t\right) + \frac{\beta_n L}{kn\pi} \sin\left(\frac{kn\pi}{L} t\right)\right) \sin \frac{n\pi x}{L}$$

Dirichlet boundary conditions: Formal solution

From the above we conclude that one possible solution of the wave equation with boundary conditions is,

$$u(x,t) = \sum_{n\geq 1} \left(\alpha_n \cos\left(\frac{kn\pi}{L} t\right) + \frac{\beta_n L}{kn\pi} \sin\left(\frac{kn\pi}{L} t\right)\right) \sin \frac{n\pi x}{L}.$$

This function satisfies

$$u(x,0) = \sum_{n\geq 1} \alpha_n \sin \frac{n\pi x}{L} \quad \text{and}$$

$$u_t(x,0) = \sum_{n\geq 1} \beta_n \sin \frac{n\pi x}{L}.$$

Dirichlet boundary conditions: Formal solution

Thus, if $f(x)$ and $g(x)$ have Fourier expansions given by

$$f(x) = \sum_{n \geq 1} \alpha_n \sin \frac{n\pi x}{L} \quad \text{and}$$

$$g(x) = \sum_{n \geq 1} \beta_n \sin \frac{n\pi x}{L}.$$

then we will have solved our wave equation with the given boundary and initial conditions.

Definition

Consider the wave equation with initial and boundary values given by

$$\begin{array}{lll} u_{tt} = k^2 u_{xx} & 0 < x < L, & t > 0 \\ u(0,t) = u(L,t) = 0 & & t > 0 \\ u(x,0) = f(x) & 0 \leq x \leq L & \\ u_t(x,0) = g(x) & 0 \leq x \leq L & \end{array}$$

Dirichlet boundary conditions: Formal solution

Definition (continued)

The formal solution of the above problem is

$$u(x,t) = \sum_{n\geq 1} \left(\alpha_n \cos\left(\frac{kn\pi}{L} t\right) + \frac{\beta_n L}{kn\pi} \sin\left(\frac{kn\pi}{L} t\right) \right) \sin \frac{n\pi x}{L}.$$

where

$$\alpha_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx \quad \text{and}$$
$$\beta_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx.$$

We say $u(x,t)$ is a formal solution, since the series for $u(x,t)$ may NOT make sense, or it may not make sense to differentiate it term wise.

Dirichlet boundary conditions: Actual solution

Theorem

Let f and g be continuous and piecewise smooth functions on $[0,L]$ such that $f(0)=f(L)=0$. Then the problem given by

$$\begin{aligned}
&u_{tt} = k^2 u_{xx} && 0 < x < L, \quad t > 0\\
&u(0,t) = u(L,t) = 0 && t > 0\\
&u(x,0) = f(x) && 0 \le x \le L\\
&u_t(x,0) = g(x) && 0 \le x \le L
\end{aligned}$$

has an actual solution, which is given by

$$u(x,t) = \sum_{n\ge 1} \left(\alpha_n \cos\left(\frac{kn\pi}{L} t\right) + \frac{\beta_n L}{kn\pi} \sin\left(\frac{kn\pi}{L} t\right)\right) \sin\frac{n\pi x}{L}.$$

where

$$\alpha_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \quad \textit{and} \quad \beta_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx.$$

Dirichlet boundary conditions: Example

Example

Consider the wave equation with initial and boundary value given by

$$\begin{aligned}
&u_{tt} = 5u_{xx} && 0 < x < 1, \quad t > 0 \\
&u(0,t) = u(L,t) = 0 && t > 0 \\
&u(x,0) = \sin \pi x + 3 \sin 5\pi x && 0 \le x \le 1 \\
&u_t(x,0) = \sin 5\pi x - 26 \sin 9\pi x && 0 \le x \le 1
\end{aligned}$$

Since both f and g are given by their Fourier series in the above example, it is clear that

$$\begin{aligned}
\alpha_1 &= 1 & \beta_1 &= 0 \\
\alpha_5 &= 3 & \beta_5 &= 1 \\
\alpha_9 &= 0 & \beta_9 &= -26
\end{aligned}$$

Example (continued)

Thus, the solution to the above problem is given by

$$u(x,t) = \cos(\sqrt{5}\pi t)\sin(\pi x) + (3\cos(\sqrt{5}\pi t) + \frac{1}{5\pi\sqrt{5}}\sin(\sqrt{5}\pi t))\sin(5\pi x) + \frac{-26}{9\pi\sqrt{5}}\sin(\sqrt{9}\pi t)\sin(9\pi x)$$