MA-207 Differential Equations II
Lecture-8 Eigenvalue Problem and Fourier Expansion

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16th October, 2017
S1 - Lecture 8

Consider the following Eigen Value Problems, where $\lambda \in \mathbb{R}$ and $L > 0$.

1. Problem 1. $y'' + \lambda y = 0$, $y(0) = 0$, $y(L) = 0$.
2. Problem 2. $y'' + \lambda y = 0$, $y'(0) = 0$, $y'(L) = 0$.
3. Problem 3. $y'' + \lambda y = 0$, $y(0) = 0$, $y'(L) = 0$.
4. Problem 4. $y'' + \lambda y = 0$, $y'(0) = 0$, $y(L) = 0$.
5. Problem 5. $y'' + \lambda y = 0$, $y(-L) = y(L)$, $y'(-L) = y'(L)$.

The boundary condition in problem 5 is called periodic. Obviously, $y \equiv 0$ is a (trivial) solution for all 5 problems for all λ.

For most values of λ, there are no other solutions.

Question. For what values of λ does the "EVP" have a non-trivial solutions and what are the solutions?

Any λ for which the EVP 1-5 has a non-trivial solution is called an eigenvalue of that problem and

non-trivial solutions for an eigenvalue λ are called λ-eigenfunction, or eigenfunction associated with λ.

A non-zero constant multiple of a λ-eigenfunction is again a λ-eigenfunction.

Solving an eigenvalue problem means finding all its eigenvalues and associated eigenfunctions.

THEOREM.

1. Problems $1-5$ have no negative eigenvalues.
2. $\lambda = 0$ is an eigenvalue of Problems 2 and 5 with associated eigenfunctions $y_0 = 1$.
3. $\lambda = 0$ is not an eigenvalue of Problems $1, 3, 4$.

Proof. Let us prove first two; third is left as an exercise. Suppose $\lambda < 0$. Write $\lambda = -a^2$, $a > 0$.

The general solution of ODE $y'' = a^2 y$ is $y(x) = Ce^{ax} + De^{-ax}$.

In problem 1, $y(0) = y(L) = 0$ gives $C + D = 0$ and $Ce^{aL} + De^{-aL} = 0$.

Check $\implies C = D = 0$.

In problem 2, $y'(0) = y'(L) = 0$ gives $aC - aD = 0$ and $aCe^{aL} - aDe^{-aL} = 0$.

Since $a \neq 0$, this forces $C = D = 0$.

Similarly, do the other problems.

Now consider the second statement in the theorem. If $\boxed{\lambda = 0}$, the solution of $y'' = 0$ is $y(x) = ax + b$. In problem 2, the derivative vanishes, so $a = 0$. Thus, $y(x) = \text{constant}$ is the solution.

In problem 5, $y(-L) = y(L)$ gives $-aL + b = aL + b$. Thus $a = 0$ and $y(x) = \text{constant}$ is the solution. □

Eigenvalue Problem 1

Theorem

The eigenvalue problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$$

has infinitely many positive eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

with associated eigenfunctions

$$y_n = \sin \frac{n\pi x}{L}, \quad n = 1, 2, \ldots.$$

There are no other eigenvalues.

Proof. Any eigen value must be positive (previous theorem).

If y is a solution of $y'' + \lambda y = 0$ with $\lambda > 0$, then

$$y(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$$

Now $\quad y(0) = 0 \implies c_1 = 0$

$$\implies y(x) = c_2 \sin\sqrt{\lambda}x,\ c_2 \neq 0 \text{ as } y \neq 0.$$

$$y(L) = 0 \implies \sin\sqrt{\lambda}L = 0$$

$\implies \sqrt{\lambda}L = n\pi$. Therefore, for $n \geq 1$, $\lambda_n = \dfrac{n^2\pi^2}{L^2}$

is an eigenvalue with an associated eigenfunction

$$y_n = \sin\frac{n\pi x}{L}. \qquad \square$$

Theorem

The eigenvalue problem 2

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0$$

has an eigenvalue $\lambda_0 = 0$ with eigenfunction $y_0 = 1$ and infinitely many positive eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, \ldots$$

with associated eigenfunctions

$$y_n = \cos\frac{n\pi x}{L}.$$

There are no other eigenvalues.

Proof. Similar to the proof of Problem 1, hence is left as an exercise.

Theorem

The eigenvalue problem 3

$$y'' + \lambda y = 0, \quad y(0) = 0, \ \ y'(L) = 0$$

has infinitely many positive eigenvalues

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4L^2}, \quad n = 1, 2, \ldots$$

with associated eigenfunctions

$$y_n = \sin\frac{(2n-1)\pi x}{2L}.$$

There are no other eigenvalues.

Proof. Any eigenvalue must be positive.

If y is a solution of $y'' + \lambda y = 0$ with $\lambda > 0$, then

$$y(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$$

$$y(0) = 0 \implies c_1 = 0$$

So $y(x) = c_2 \sin\sqrt{\lambda}x$ with $c_2 \neq 0$, as $y \neq 0$.

$$y'(L) = 0 \implies \cos\sqrt{\lambda}L = 0$$

$$\implies \sqrt{\lambda}L = \frac{(2n-1)\pi}{2}.$$

Therefore, for $n \geq 1$, $\lambda_n = \dfrac{(2n-1)^2\pi^2}{4L^2}$ is an eigenvalue with an associated eigenfunction

$$y_n = \sin\frac{(2n-1)\pi x}{2L} \qquad \square$$

Theorem. The eigenvalue problem 4

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(L) = 0$$

has infinitely many positive eigenvalues

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4L^2}, \qquad n = 1, 2, \ldots$$

with associated eigenfunctions

$$y_n = \cos\frac{(2n-1)\pi x}{2L}$$

There are no other eigenvalues.

Proof. Similar to proof of Problem 3, hence is left as an exercise.

Theorem. The eigenvalue problem 5

$$y'' + \lambda y = 0, \quad y(-L) = y(L), \quad y'(-L) = y'(L)$$

has an eigenvalue $\lambda_0 = 0$ with associated eigenfunction $y_0 = 1$
and infinitely many positive eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \qquad n = 1, 2, \ldots$$

with associated eigenfunctions

$$y_{1n} = \cos\frac{n\pi x}{L} \quad \text{and} \quad y_{2n} = \sin\frac{n\pi x}{L}.$$

There are no other eigenvalues.

Proof. We know that $\lambda = 0$ is an eigenvalue with eigenfunction $y_0 = 1$
and any other eigenvalue is positive.
If y is a solution of $y'' + \lambda y = 0$ with $\lambda > 0$, then

$$y(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x$$

The boundary condition $y(-L) = y(L)$ implies

$$c_1 \cos(-\sqrt{\lambda} L) + c_2 \sin(-\sqrt{\lambda} L) =$$
$$c_1 \cos(\sqrt{\lambda} L) + c_2 \sin(\sqrt{\lambda} L)$$

Since $\cos(-\sqrt{\lambda} x) = \cos(\sqrt{\lambda} x)$, and
$\sin(-\sqrt{\lambda} x) = -\sin(\sqrt{\lambda} x)$

We get $\boxed{c_2 \sin \sqrt{\lambda} x = 0}$

Similarly, using $y'(-L) = y'(L)$ for

$$y'(x) = \sqrt{\lambda} \left(-c_1 \sin \sqrt{\lambda} x + c_2 \cos \sqrt{\lambda} x \right)$$

we get $\boxed{c_1 \sin \sqrt{\lambda} L = 0.}$
For a non-trivial solution, $(c_1, c_2) \neq (0, 0)$, we get

$\sin \sqrt{\lambda} L = 0 \implies \sqrt{\lambda} L = n\pi$ for $n \geq 1$.

The eigenvalues are $\lambda_n = \dfrac{n^2 \pi^2}{L^2} \qquad n = 1, 2, \ldots$

and each eigenvalue has two linearly independent associated eigenfunctions

$$\cos \frac{n\pi x}{L} \quad \text{and} \quad \sin \frac{n\pi x}{L}.$$

Definition. Recall that two square integrable functions f and g in $L^2([a,b])$ are **orthogonal** if

$$\langle f, g\rangle := \int_a^b f(x)g(x)\,dx = 0$$

More generally, a set of functions $\{\phi_1, \phi_2, \ldots, \phi_n, \ldots\}$ (finite or infinitely many) are orthogonal on $[a,b]$ if

$$\int_a^b \phi_i(x)\phi_j(x)\,dx = 0, \quad \text{whenever} \quad i \neq j$$

We have seen orthogonality of Legendre function. We will study Fourier series with respect to different orthogonal systems.

Exercise. Consider the eigenfunctions of EVP's 1-5.

1. $\sin\frac{\pi x}{L}, \sin\frac{2\pi x}{L}, \ldots, \sin\frac{n\pi x}{L}, \ldots$
2. $1, \cos\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \ldots, \cos\frac{n\pi x}{L}, \ldots$
3. $\sin\frac{\pi x}{2L}, \sin\frac{3\pi x}{2L}, \ldots, \sin\frac{(2n-1)\pi x}{2L}, \ldots$
4. $\cos\frac{\pi x}{2L}, \cos\frac{3\pi x}{2L}, \ldots, \cos\frac{(2n-1)\pi x}{2L}, \ldots$
5. $$1, \cos\frac{\pi x}{L}, \sin\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \sin\frac{2\pi x}{L}, \ldots, \cos\frac{n\pi x}{L}, \sin\frac{n\pi x}{L}, \ldots$$

Show that eigenfunctions of (1-4) are orthogonal on $[0,L]$ and of (5) is orthogonal on $[-L,L]$.

Theorem. Suppose the functions $\phi_1, \phi_2, \ldots$, are orthogonal on $[a, b]$ and

$$\|\phi_n\|^2 := \int_a^b \phi_n^2(x)\,dx \neq 0 \qquad n = 1, 2, \ldots$$

Let $c_1, c_2, \ldots,$ be constants and $0 < M < \infty$ s.t.

$$\left|\sum_{m=1}^{N} c_m\,\phi_m(x)\right| \leq M, \quad \text{for } a \leq x \leq b \text{ and } N \geq 1.$$

Suppose also that the series $\displaystyle\sum_{m=1}^{\infty} c_m \phi_m(x)$

converges to $f(x)$ and $f(x)$ is integrable on $[a, b]$.
Then

$$c_n = \frac{1}{\|\phi_n\|^2} \int_a^b f(x)\phi_n(x)\,dx, \qquad n = 1, 2, \ldots$$

Proof. $f(x) = \displaystyle\sum_{m=1}^{\infty} c_m \phi_m(x)$ gives

$$\int_a^b f(x)\phi_n(x)\,dx = \int_a^b \left(\sum_{m=1}^{\infty} c_m \phi_m(x)\right) \phi_n(x)\,dx$$

Boundedness of partial sums and integrability of f allows us to interchange the operation of integration and summation, so

$$\int_a^b f(x)\phi_n(x)\,dx = \sum_{m=1}^{\infty} c_m \int_a^b \phi_m(x)\phi_n(x) = c_n \|\phi_n\|^2$$

by orthogonality of $\{\phi_1, \phi_2, \ldots\}$. $\square$

Definition. Suppose $\{\phi_1, \phi_2, \ldots\}$ is orthogonal on $[a, b]$ and $\quad \|\phi_n\|^2 := \int_a^b \phi_n^2(x)\,dx \neq 0,\ n = 1, 2, \ldots$

For $f \in L^2([a, b])$, the series

$$\sum_{n=1}^{\infty} c_n\,\phi_n(x), \text{ where } c_n = \frac{1}{\|\phi_n\|^2}\int_a^b f(x)\phi_n(x)\,dx$$

is Fourier series of f and $c_1, c_2, \ldots$ the Fourier coefficients of f w.r.t. orthogonal set $\{\phi_n\}_{n=1}^{\infty}$.

We write $\quad f(x) \sim \sum_{n=1}^{\infty} c_n\,\phi_n(x), \quad a \leq x \leq b$

Qn. What about the convergence of series to $f(x)$? The answer depends on the orthogonal system $\{\phi_1, \phi_2, \ldots\}$.

Fourier Series.

The set of eigenfunctions $B =$

$$\left\{1, \cos\frac{\pi x}{L}, \sin\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \ldots, \cos\frac{n\pi x}{L}, \sin\frac{n\pi x}{L}, \ldots\right\}$$

of Eigenvalue Problem 5 is orthogonal on $[-L, L]$. The Fourier series of $f \in L^2([-L, L])$ w.r.t. orthogonal set B is

$$a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$\|1\|^2 = 2L,\ \|\cos\frac{n\pi x}{L}\|^2 = L,\ \|\sin\frac{n\pi x}{L}\|^2 = L$$

So the Fourier coefficients are given by

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad n > 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad n > 0$$

Convergence of Fourier Series in norm

For $f \in L^2([-L, L])$, the partial sum of Fourier series of f converges to f in norm, i.e.

$$\|f - a_0 - \sum_{n=1}^{N}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)\|$$

conveges to 0 as $N \to \infty$.

What about pointwise convergence of Fourier series.

Definition. Recall that a function f is said to be piecewise smooth if

1. f has atmost finitely many discontinuity.
2. f' exists and is continuous except at finitely many points.
3. $f(x_0+) = \lim_{x \to x_0^+} f(x)$ and $f'(x_0+) = \lim_{x \to x_0^+} f'(x)$ exists if $a \leq x_0 < b$.
4. $f(x_0-) = \lim_{x \to x_0^-} f(x)$ and $f'(x_0-) = \lim_{x \to x_0^-} f'(x)$ exists if $a < x_0 \leq b$.

Hence f is piecewise smooth if and only if f, f' have atmost finitely many jump discontinuity and f' is piecewise continuous.

Pointwise Convergence of Fourier series

Let $f(x)$ be a piecewise smooth on $[-L, L]$.
Extend $f(x)$ to $\mathbb{R}$ by periodicity $f(x + 2L) = f(x)$.
Then the Fourier series of f on $[-L, L]$,

$$F(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

converges to $\frac{1}{2}[f(x^+) + f(x^-)]$ for all $x \in \mathbb{R}$.
Therefore, $F(x) = f(x)$ if f is continuous at x.

If we change $f(x)$ to $\frac{1}{2}[f(x^+) + f(x^-)]$ at discontinuous points x, then Fourier series of f converges to $f(x)$. Thus two functions can have same Fourier series.

The Fourier series converges to $F(x)$ for all $x \in [-L, L]$, hence for a fixed x, the error

$$E_N(x) = \left| F(x) - a_0 - \sum_{n=1}^{N} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \right|$$

is small if N is large.
But we can not make the error $E_N(x)$ arbitrary small "uniformly" for all $x \in [-L, L]$, by choosing N sufficiently large if

- f is discontinuous at some point $\alpha \in (-L, L)$ or
- $f(-L+) \neq f(L-)$

The next result explains this.

Gibbs Phenomenon.

- If f has a jump discontinuity at $\alpha \in (-L, L)$, then there exist a sequence $u_N \in (-L, \alpha)$ and $v_N \in (\alpha, L)$ s.t.

$$\lim_{N\to\infty} u_N = \alpha, \quad E_N(u_N) \simeq .09\,|f(\alpha-) - f(\alpha+)|$$

$$\lim_{N\to\infty} v_N = \alpha, \quad E_N(v_N) \simeq .09\,|f(\alpha-) - f(\alpha+)|$$

Maximum of error $E_N(x) \not\to 0$ near α as $N \to \infty$.

- If $f(-L+) \neq f(L-)$, $\exists\, u_N, v_N$ in $(-L, L)$ s.t.

$$\lim_{N\to\infty} u_N = -L, \quad E_N(u_N) \simeq .09\,|f(-L+) - f(L-)|$$

$$\lim_{N\to\infty} v_N = L, \quad E_N(v_N) \simeq .09\,|f(-L+) - f(L-)|$$

This is called Gibbs phenomenon.

Example. Find the Fourier series of piecewise smooth function on $[-2,2]$

$$f(x) = \begin{cases} -x, & -2 < x < 0 \\ 1/2, & 0 < x < 2 \end{cases}$$

and determine the sum of the Fourier series.

We need not define f at points of discontinuities, i.e. at points $-2, 0$ and 2, since the coefficients in the Fourier series

$$F(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{2} + b_n \sin \frac{n\pi x}{2} \right)$$

are not affected by them.

Note f is continuous for $x \in (-2,0) \cup (0,2)$.

So $F(x) = f(x)$ for x in $(-2,0) \cup (0,2)$.
So let us compute $F(x)$ at discontinuous points.

$$F(-2) = F(2) = \frac{1}{2}(f(-2+) + f(2-)) = \frac{1}{2}\left(2 + \frac{1}{2}\right) = \frac{5}{4}$$

$$F(0) = \frac{1}{2}(f(0-) + f(0+)) = \frac{1}{2}\left(0 + \frac{1}{2}\right) = \frac{1}{4}$$

To summarize,

$$F(x) = \begin{cases} 5/4, & x = \pm 2 \\ -x, & -2 < x < 0 \\ 1/4, & x = 0 \\ 1/2, & 0 < x < 2 \end{cases}$$

Let us compute the Fourier coefficients now.

$$a_0 = \frac{1}{4}\int_{-2}^{2} f(x)\,dx = \frac{1}{4}\left[\int_{-2}^{0}(-x)\,dx + \int_{0}^{2}\frac{1}{2}\,dx\right] = \frac{3}{4}$$

If $n \geq 1$, then

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0}(-x)\cos\frac{n\pi x}{2}\,dx + \int_{0}^{2}\frac{1}{2}\cos\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[-x\frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_{-2}^{0} + \int_{-2}^{0}\frac{2}{n\pi}\sin\frac{n\pi x}{2}\,dx + 0\right]$$

$$= \frac{1}{2}\frac{4}{n^2\pi^2}\left(-\cos\frac{n\pi x}{2}\right)\Big|_{-2}^{0} = \frac{2}{n^2\pi^2}(\cos n\pi - 1)$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0}(-x)\sin\frac{n\pi x}{2}\,dx + \int_{0}^{2}\frac{1}{2}\sin\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2n\pi}(1 + 3\cos n\pi)$$

Thus, the Fourier series of $f(x)$ is

$$F(x) = \frac{3}{4} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi - 1}{n^2}\cos\frac{n\pi x}{2}$$

$$+\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1 + 3\cos n\pi}{n}\sin\frac{n\pi x}{2}$$

Definition. Recall a function f is odd if $f(-x) = -f(x)$ and even if $f(-x) = f(x)$.

Product of odd and even functions :

- (odd) (odd)=(even)
- (odd) (even)=(odd)
- (even) (even)=(even)

If f is an odd, then $\int_{-L}^{L} f(x)\,dx = 0$,

if f is an even, then $\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$

Note that $\sin\frac{n\pi x}{L}$ is odd and $\cos\frac{n\pi x}{L}$ is even.

Let $f \in L^2([-L, L])$ be piecewise smooth.

• If f is odd, then $f(x)\cos\frac{n\pi x}{L}$ is odd.

Hence $a_n = 0$ and the Fourier series of f is

$$F(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

which has only sine terms.

• If f is even, then $f(x)\sin\frac{n\pi x}{L}$ is odd.

Hence $b_n = 0$ and the Fourier series of f is

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

which has only cosine terms.

Fourier sine series.

If $f \in L^2([0, L])$, then extend f to $[-L, L]$ as an odd function by defining $f(-x) = -f(x)$ for $x \in (-L, 0)$. Then Fourier series of f on $[-L, L]$ is

$$F(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

called Fourier sine series of f on $[0, L]$.
This is the Fourier series of f on $[0, L]$ with respect to orthogonal system of eigen-functions

$$B = \{\sin \frac{\pi x}{L}, \sin \frac{2\pi x}{L}, \ldots, \sin \frac{n\pi x}{L}, \ldots\}$$

of EVP 1: $\quad y'' + \lambda y = 0, \quad y(0) = 0 = y(L).$

Example. Consider the periodic square wave with $L = \pi$.

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < \pi \\ -1 & \text{if } -\pi < x < 0 \end{cases}, \qquad f(x+2\pi) = f(x)$$

Since f is an odd function, Fourier series of f is Fourier sine series

$$F(x) = \sum_{n=1}^{\infty} b_n \sin n\pi$$

Show that

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \begin{cases} \dfrac{4}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Thus the Fourier series of $f(x)$ is

$$\frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right)$$

Since $F(x) = f(x)$ at all points of continuity of f and

$$F(0) = \frac{1}{2}[f(0+) + f(0-)] = 0$$

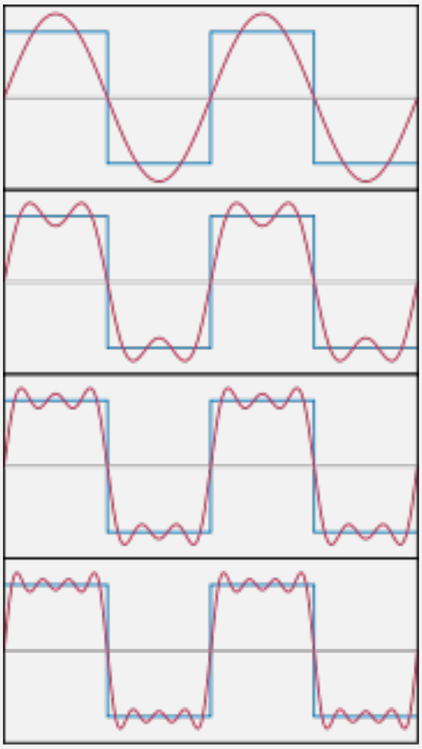
In particular, evaluating at $x = \pi/2$

$$f\left(\frac{\pi}{2}\right) = 1 = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right)$$

$$\Longrightarrow 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$

IMP

The partial sums of the Fourier series wiggle around the square wave.



Fourier cosine series

If $f \in L^2([0, L])$, then extend f to $[-L, L]$ as even function by $f(-x) = f(x)$ for $x \in (-L, 0)$. Then Fourier series of f on $[-L, L]$ is

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad a_0 = \frac{1}{L} \int_0^L f(x)\, dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx, \quad n \geq 1$$

called Fourier cosine series of f on $[0, L]$.

This is the Fourier series of f on $[0, L]$ with respect to orthogonal system of eigen-functions of EVP 2

$$B = \{1, \cos \frac{\pi x}{L}, \cos \frac{2\pi x}{L}, \dots, \cos \frac{n\pi x}{L}, \dots\}.$$

Example. Find the Fourier series of the function

$$f(x) = x^2 \qquad -\pi \leq x \leq \pi$$

Since f is an even, $b_n = 0$.

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2\, dx = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx\, dx = \frac{4}{n^2} \cos n\pi$$

$$= \begin{cases} \dfrac{4}{n^2} & n : \text{is even} \\ -\dfrac{4}{n^2} & n : \text{is odd} \end{cases}$$

Thus the Fourier series of $f(x)$ is

$$F(x) = \frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{4} + \frac{\cos 3x}{9} - \dots\right)$$

Since f is continuous with $f(\pi) = f(-\pi)$, $F(x) = f(x)$ for all $x \in \mathbb{R}$. Hence

$$x^2 = \frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{4} + \frac{\cos 3x}{9} - \dots\right)$$

Evaluating at $x = \pi$,

$$\pi^2 = \frac{\pi^2}{3} + 4\left(1 + \frac{1}{4} + \frac{1}{9} + \dots\right)$$

This yields the identity

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{1}{4}.\frac{2\pi^2}{3} = \frac{\pi^2}{6}$$