MA-207 Differential Equations II
Lecture-10 Heat Equation

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S1 - Lecture 10

Recall, we classified a second order linear PDE in two variables x and y as follows. If

$$L = A\frac{\partial^2}{\partial x^2} + 2B\frac{\partial^2}{\partial x \partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F$$

where $A, B, \ldots$ are functions of x and y, then its discriminant is

$$\mathbb{D}(x,y) = A(x,y)C(x,y) - B^2(x,y)$$

The operator L or the PDE $Lu = f$ is said to be elliptic (hyperbolic or parabolic respectively) at (x_0, y_0), if $\mathbb{D}(x_0,y_0) > 0$ (< 0 or $= 0$ respectively).

We say L is elliptic in $\Omega \subset \mathbb{R}^2$ if L is elliptic at every point of Ω.

When the coefficients of an operator L are not constant functions, then the type of L may vary from point to point.

Example. Consider the Tricomi operator (well known)

$$T = \frac{\partial^2}{\partial x^2} + x\frac{\partial^2}{\partial y^2}$$

The discriminant $\mathbb{D} = x$.
Hence T is elliptic in the half-plane $x > 0$, hyperbolic in the half-plane $x < 0$ and parabolic on the y-axis.

Remark about the terminology

Consider

$$L = A\frac{\partial^2}{\partial x^2} + 2B\frac{\partial^2}{\partial x \partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F$$

at the point (x_0, y_0). If we replace $\partial/\partial x$ by ξ and $\partial/\partial y$ by η and evaluate $A, \ldots, F$ at (x_0, y_0), then L becomes a polynomial in 2 variables

$$P(\xi, \eta) = A\xi^2 + 2B\xi\eta + C\eta^2 + D\xi + E\eta + F$$

Consider the curves in (ξ, η)-plane given by

$$P(\xi, \eta) = \text{constant}$$

then these curves are elliptic if $\mathbb{D}(x_0, y_0) > 0$, hyperbolic if $\mathbb{D}(x_0, y_0) < 0$ and parabolic if $\mathbb{D}(x_0, y_0) = 0$.

Second order linear operators in $\mathbb{R}^3$

The classification is done analogously by associating a polynomial of degree 2 in three variables to L and considering the surfaces defined by level sets of the polynomial.

These surfaces are either ellipsoids, hyperboloids, or paraboloids. The operator L is accordingly labelled as elliptic, hyperbolic or parabolic.

We can also proceed as follows; Consider

$$L = a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x \partial y} + 2c\frac{\partial^2}{\partial x \partial z} + d\frac{\partial^2}{\partial y^2} + 2e\frac{\partial^2}{\partial y \partial z} + f\frac{\partial^2}{\partial z^2}$$

$$+ \text{ lower order terms}$$

where $a, b, \ldots$ are functions of (x, y, z).

To L, we associate the symmetric matrix

$$M(x,y,z) = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

Here the (i,j)-th entry is the coefficient of $\dfrac{\partial^2}{\partial x_i \partial x_j}$.

Since M is symmetric, it has 3 real eigenvalues.

- L is elliptic at $P = (x_0, y_0, z_0)$ if all three eigen values of $M(x_0, y_0, z_0)$ are $\neq 0$ and of same sign.
- L is hyperbolic at P if two eigen values $(\neq 0)$ are of same sign and one $(\neq 0)$ of different sign.
- L is parabolic at P if one of the eigenvalue is zero.

Principle of superposition

Let L be a linear differential operator.
The PDE $Lu = 0$ is called homogeneous and the PDE $Lu = f$, $(f \neq 0)$ is non-homogeneous.

Principle 1. If $u_1, \dots, u_N$ are solutions of $Lu = 0$ and $c_1, \dots, c_N$ are constants, then $\sum_{i=1}^{N} c_i u_i$ is also a solution of $Lu = 0$.

In general, space of solutions of $Lu = 0$ contains infinitely many independent solutions and we may need to use infinite linear combinations of them.

Principle 2.

- Assume $u_1, u_2, \dots$ are infinitely many solutions of $Lu = 0$.
- the series $w = \sum_{i \geq 1} c_i u_i$ with $c_1, c_2, \dots$ constants, converges to a twice differentiable function;
- term by term partial differentiation is valid for the series, i.e. $Dw = \sum_{i \geq 1} c_i D u_i$, D is any partial differentiation of order 1 or 2.

Then w is again a solution of $Lu = 0$.

Principle 3 for non-homogeneous PDE.

If u_i is a solution of $Lu = f_i$, then

$$w = \sum_{i=1}^{N} c_i u_i$$

with constants c_i, is a solution of $Lu = \sum_{i=1}^{N} c_i f_i$.

One-dimensional heat equation

Consider a thin uniform rod of length L with constant cross section area A and placed on the x-axis between 0 and L.
Lateral surface of the rod is perfectly insulated.
Heat flows only in the direction of axis of rod i.e. along x-axis.
So the temperature function u at a time t is a function of x and t only.
Hence u is same in each cross section A. So

$$u = u(x,t), \quad 0 \leq x \leq L, \; t \geq 0$$

The mathematical model describing the Heat flow in the rod is

$$u_t = k^2 u_{xx}, \qquad 0 < x < L,\ t > 0,$$

called the one-dimensional heat equation.
Here k is a positive constant called the thermal diffusivity of the rod. Check that

$$u_1(x,t) = e^{-k^2\omega_1^2 t}\sin(\omega_1 x), \quad u_2(x,t) = e^{-k^2\omega_2^2 t}\cos(\omega_2 x)$$

are solutions of 1-dimensional Heat equation for $\omega_1, \omega_2 \in \mathbb{R}$ arbitrary.
Further, linear combination of solutions is a solution.
Hence the solution space is infinite dimensional.

Two-dimensional Heat equation

Consider a thin plate in $x-y$ plane whose lateral faces are perfectly insulated, so that no heat flows in the direction transversal to the plate.
Mathematical model for Heat flow in the plate is

$$u_t = k^2(u_{xx}+u_{yy}), \quad 0 < x < L,\ 0 < L' < y,\ t > 0,$$

Check $u_1(x,y,t) = e^{-k^2\omega_1^2 t}\sin(\omega_1 x)\, e^{-k^2\omega_2^2 t}\sin(\omega_2 y)$,

$$u_2(x,y,t) = e^{-k^2\omega_1^2 t}\sin(\omega_1 x)\, e^{-k^2\omega_2^2 t}\cos(\omega_2 y)$$

are solutions for $\omega_1, \omega_2 \in \mathbb{R}$ arbitrary.
We can interchange $\sin$ and $\cos$ in above solutions to get another solutions.
We can take linear combination of solutions.

Initial Boundary value problem for Heat Equation.

Example 1. Suppose a laterally insulated rod of length L has initial constant temperature 50^o. Then its left end ($x = 0$) is immersed in a tank of icy water at 0^o and its right end is immersed in a tank of boiling water at 100^o.
The set up for temperature function is

$$u_t(x,t) = ku_{xx}(x,t), \quad 0 < x < L, \ t > 0$$
$$u(x,0) = 50, \quad 0 < x < L$$
$$u(0,t) = 0, \quad t > 0$$
$$u(L,t) = 100, \quad t > 0.$$

Example 2. Suppose a laterally insulated rod of length L has initial temperature given by a function $f(x) = x^2 - 3\sin x$.
Further both ends of the rod are insulated, so there is no exchange of heat between rod and outside. The heat flux (temperature gradient) is zero at $x = 0$ and $x = L$ for all $t > 0$.
The set up for temperature function is

$$u_t(x,t) = ku_{xx}(x,t), \quad 0 < x < L, \ t > 0$$
$$u(x,0) = x^2 - 3\sin x, \quad 0 < x < L$$
$$u_x(0,t) = 0, \quad t > 0$$
$$u_x(L,t) = 0, \quad t > 0.$$

As t gets large, $u(x,t)$ approaches a constant value, i.e. the averege of $x^2 - 3\sin x$.

Example 3. Suppose a laterally insulated rod of length L has initial temperature
$f(x) = x^2 - 3\sin x$.

Left end of the rod is insulated and right end is kept in a tank of boiling water at 100^o.

The set up for temperature function is

$$u_t(x,t) = ku_{xx}(x,t), \quad 0 < x < L, \; t > 0$$
$$u(x,0) = x^2 - 3\sin x, \quad 0 < x < L$$
$$u_x(0,t) = 0, \quad t > 0$$
$$u(L,t) = 100, \quad t > 0.$$

Solving Heat equation $u_t = k^2 u_{xx}$

We use method of separation of variables. Suppose

$$v(x,t) = X(x)\,T(t)$$

Substituting this in the Heat equation

$$T'(t)X(x) = k^2 X''(x)T(t).$$

We can now separate the variables:

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{k^2 T(t)}$$

The equality is between a function of x and a function of t, so both must be constant, say $-\lambda$.

We need to solve

$\boxed{X''(x) + \lambda X(x) = 0}$ and $\boxed{T'(t) = -k^2 \lambda T(t)}$

Dirichlet boundary conditions $u(0,t) = u(L,t) = 0$

Initial-boundary value problem is

$$\begin{aligned} &u_t = k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ &u(0,t) = 0 = u(L,t) && t > 0 \\ &u(x,0) = f(x), && 0 \leq x \leq L \end{aligned}$$

Assuming the solution is $v(x,t) = X(x)T(t)$,

$$v(0,t) = X(0)T(t) = 0 = v(L,t) = X(L)T(t)$$

We don't want T to be identically zero, so

$$X(0) = 0 \quad \text{and} \quad X(L) = 0.$$

We need to solve eigenvalue problem

$$\boxed{X''(x) + \lambda X(x) = 0, \quad X(0) = 0, \quad X(L) = 0}$$

The eigenvalues are $\lambda_n = \dfrac{n^2\pi^2}{L^2}$, $n \geq 1$

with associated eigenfunctions $X_n(x) = \sin \dfrac{n\pi x}{L}$.

$$T'(t) = -k^2 \lambda T(t) \implies T(t) = exp(-k^2 \lambda t)$$

The solutions of BVP for each $n \geq 1$ are

$$v_n(x,t) = T_n(t)X_n(x) = exp\left(\frac{-n^2\pi^2k^2}{L^2} t\right) \sin \frac{n\pi x}{L}$$

Note $v_n(x,0) = \sin \dfrac{n\pi x}{L}$. Therefore

$$v_n(x,t) = exp\left(\frac{-n^2\pi^2k^2}{L^2}t\right)\sin\frac{n\pi x}{L}$$

satisfies the IBVP

$$u_t = k^2u_{xx} \qquad 0 < x < L, \quad t > 0$$

$$u(0,t) = 0 = u(L,t) \qquad t > 0$$

$$u(x,0) = \sin\frac{n\pi x}{L} \qquad 0 \le x \le L$$

More generally, if $\alpha_1, \ldots, \alpha_m$ are constants, then

$$u_m(x,t) := \sum_{n=1}^{m} \alpha_n\, exp\left(\frac{-n^2\pi^2k^2}{L^2}t\right)\sin\frac{n\pi x}{L}$$

satisfies the IBVP with $u_m(x,0) = \sum_{n=1}^{m} \alpha_n \sin\frac{n\pi x}{L}$.

Let us consider the formal series

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n\, exp\left(\frac{-n^2\pi^2k^2}{L^2}t\right)\sin\frac{n\pi x}{L}$$

Setting $t = 0$ we get

$$u(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin\frac{n\pi x}{L}$$

To solve our IBVP we would like to have

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \sin\frac{n\pi x}{L}, \qquad 0 \le x \le L$$

Definition. The formal solution of IBVP

$$\begin{aligned} &u_t = k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ &u(0,t) = 0 = u(L,T) && t > 0 \\ &u(x,0) = f(x) && 0 \le x \le L \end{aligned}$$

is $\displaystyle u(x,t) = \sum_{n=1}^{\infty} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2} t\right) \sin \frac{n\pi x}{L}$

where

$$S(x) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi x}{L}, \ \alpha_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

is the Fourier sine series of f on $[0, L]$.

We say $\displaystyle u(x,t) = \sum_{n=1}^{\infty} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2} t\right) \sin \frac{n\pi x}{L}$ is a formal solution, since the series for $u(x,t)$ may NOT satisfy all the requirements of IBVP.

When it does, we say it is an actual solution.

Because of negative exponent in $u(x,t)$, the series converges for all $t > 0$.

Each term in $u(x,t)$ satisfies the heat equation and boundary condition.

If u_t and u_{xx} can be obtained by differentiating the series term by term, once w.r.t. t and twice w.r.t. x for $t > 0$, then u also satisfies these properties.

Theorem. Assume f is continuous and piecewise smooth on $[0, L]$ and $f(0) = f(L) = 0$. Then the actual solution of IBVP

$$\begin{aligned} &u_t = k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ &u(0,t) = 0 = u(L,t) && t > 0 \\ &u(x,0) = f(x) && 0 \leq x \leq L \end{aligned}$$

is $$u(x,t) = \sum_{n=1}^{\infty} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2} t\right) \sin \frac{n\pi x}{L}$$

where

$$S(x) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi x}{L}, \ \alpha_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

is the Fourier sine series of f on $[0, L]$,

Example. Solve IBVP

$$\begin{aligned} &u_t = k^2 u_{xx}, && 0 < x < L, \quad t > 0 \\ &u(0,t) = 0 = u(L,t), && t > 0 \\ &u(x,0) = f(x) = x(x^2 - 3Lx + 2L^2), && 0 \leq x \leq L \end{aligned}$$

The Fourier sine expansion of $f(x)$ is

$$S(x) = \sum_{n \geq 1} b_n \sin \frac{n\pi x}{L}$$

Since $f(0) = f(L) = 0$ and $f''(x) = 6x - 6L$.
Hence $b_n =$

$$\frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} \, dx = \frac{-2}{L}\frac{L^2}{n^2\pi^2}\int_0^L f''(x) \sin \frac{n\pi x}{L} \, dx$$

$$b_n = \frac{-2L}{n^2\pi^2}\int_0^L 6(x-L)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{-12L^2}{n^3\pi^3}\left[-(x-L)\cos\frac{n\pi x}{L}\Big|_0^L + \int_0^L \cos\frac{n\pi x}{L}\,dx\right]$$

$$= \frac{12L^3}{n^3\pi^3}$$

$$S(x) = \frac{12L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi x}{L}.$$

Therefore, the solution of IBVP is

$$u(x,t) = \frac{12L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}exp\left(\frac{-n^2\pi^2k^2}{L^2}t\right)\sin\frac{n\pi x}{L}.$$

Theorem. Suppose we want to solve the IBVP

$$\begin{aligned} &u_t = k^2u_{xx} && 0<x<L, \quad t>0 \\ &u(0,t) = T_1, \quad u(L,t) = T_2, && t>0 \\ &u(x,0) = f(x), && 0\le x\le L \end{aligned}$$

Let us solve for $s(x)$ so that $s(0) = T_1$, $s(L) = T_2$ and $s''(x) = 0$.

Then $s(x) = (T_2 - T_1)\frac{x}{L} + T_1$.

Write $y(x,t) = u(x,t) - s(x)$. Then $y(x,t)$ satisfies

$y_t - y_{xx} = 0$,

$y(0,t) = 0 = y(L,T)$,

$y(x,0) = f(x) - s(x)$.

We can solve for $y(x,t)$ hence for $u(x,t)$.

Neumann boundary conditions

Consider the Initial-boundary value problem

$$\begin{aligned} &u_t = k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ &u_x(0,t) = 0 = u_x(L,t), && t > 0 \\ &u(x,0) = f(x), && 0 \le x \le L \end{aligned}$$

Assuming the solution $v(x,t) = X(x)T(t)$

$$v_x(0,t) = X'(0)T(t) = 0 = v_x(L,t) = X'(L)T(t)$$

we don't want T to be identically zero, we get

$$X'(0) = 0 \quad \text{and} \quad X'(L) = 0.$$

We need to solve eigenvalue problem

$$X''(x) + \lambda X(x) = 0, \;\; X'(0) = 0, \;\; X'(L) = 0, \qquad (*)$$

$$T'(t) = -k^2 \lambda T(t) \implies T(t) = exp(-k^2 \lambda t)$$

The eigenvalues of $(*)$ are $\quad \lambda_n = \dfrac{n^2\pi^2}{L^2}, \; n \ge 0$

with associated eigenfunctions $\; X_n = \cos \dfrac{n\pi x}{L}.$

We get infinitely many solutions for IBVP for $n \ge 0$

$$v_n(x,t) = T_n(t)X_n(x) = exp\left(\frac{-n^2\pi^2k^2}{L^2} t\right) \cos \frac{n\pi x}{L}$$

Note $v_n(x,t)$ satisfies the IBVP with

$$f(x) = v_n(x,0) = \cos \frac{n\pi x}{L}$$

More generally,

If $\alpha_0, \ldots, \alpha_m$ are constants and

$$u_m(x,t) = \sum_{n=0}^{m} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2} t\right) \cos \frac{n\pi x}{L}$$

then $u_m(x,t)$ satisfies the IBVP with

$$f(x) = u_m(x,0) = \sum_{n=0}^{m} \alpha_n \cos \frac{n\pi x}{L}$$

Let us consider the formal series

$$u(x,t) = \sum_{n=0}^{\infty} \alpha_n \, exp\left(\frac{-n^2\pi^2k^2}{L^2} t\right) \cos \frac{n\pi x}{L}$$

To solve our IBVP we would like to have

$$f(x) = \sum_{n=0}^{\infty} \alpha_n \cos \frac{n\pi x}{L} \qquad 0 \le x \le L$$

Theorem. Let $f(x)$ be continuous and piecewise smooth on $[0, L]$ with $f'(0) = f'(L) = 0$.

$$C(x) = \sum_{n=1}^{\infty} \alpha_n \cos \frac{n\pi x}{L}, \quad \alpha_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx$$

is Fourier cosine series of f on $[0, L]$. Then IBVP

$$\begin{array}{ll} u_t = k^2 u_{xx} & 0 < x < L, \quad t > 0 \\ u_x(0,t) = 0 = u_x(L,t) & t > 0 \\ u(x,0) = f(x) & 0 \le x \le L \end{array}$$

has an actual solution

$$u(x,t) = \sum_{n=0}^{\infty} \alpha_n exp\left(\frac{-n^2\pi^2k^2}{L^2} t\right) \cos \frac{n\pi x}{L}$$

Here u_t and u_{xx} can be obtained by term-wise differentiation for $t > 0$.

Example. Solve IBVP

$$u_t = k^2 u_{xx} \qquad 0 < x < L, \quad t > 0$$
$$u_x(0,t) = 0 = u_x(L,t) \qquad t > 0$$
$$u(x,0) = x \qquad 0 \le x \le L$$

The Fourier cosine expansion of $f(x)$ is

$$C(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

$$a_0 = \frac{1}{L}\int_0^L x\,dx = \frac{L}{2}$$

$$a_n = \frac{2}{L}\int_0^L x \cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\frac{-L}{n\pi}\int_0^L \sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2L}{n^2\pi^2}\cos\frac{n\pi x}{L}\Big|_0^L = \frac{2L}{n^2\pi^2}((-1)^n - 1)$$

So $a_{2n} = 0$, $a_{2n-1} = \dfrac{-4L}{\pi^2(2n-1)^2}$.

The Fourier cosine expansion of $f(x)$ is

$$\frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{L}.$$

Therefore, the solution of IBVP is $\quad u(x,t) =$

$$\frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} exp\left(\frac{-(2n-1)^2\pi^2k^2}{L^2}t\right)\cos\frac{(2n-1)\pi x}{L}.$$

Mixed boundary conditions $u(0,t) = u_x(L,t) = 0$

Theorem. Assume $f(x)$ is defined on $[0, L]$ and

$$S_M(x) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{(2n-1)\pi x}{2L}$$

with $\quad \alpha_n = \frac{2}{L}\int_0^L f(x) \sin \frac{(2n-1)\pi x}{2L}\, dx$

be mixed Fourier sine series of f. Then IBVP

$$\begin{aligned} u_t &= k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ u(0,t) &= 0 = u_x(L,t) && t > 0 \\ u(x,0) &= f(x) && 0 \le x \le L \end{aligned}$$

has a formal solution

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n exp\left(\frac{-(2n-1)^2\pi^2k^2}{4L^2} t\right) \sin \frac{(2n-1)\pi x}{2L}.$$

Example. Let $f(x) = x$ on $[0, L]$. Find a formal solution of

$$\begin{aligned} u_t &= k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ u(0,t) &= 0 = u_x(L,t) && t > 0 \\ u(x,0) &= f(x) && 0 \le x \le L \end{aligned}$$

The mixed Fourier sine expansion of $f(x)$ is

$$S_M(x) = -\frac{8L}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{2L}.$$

Therefore, the formal solution of IBVP is $\quad u(x,t) =$

$$-\frac{8L}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} exp\left(\frac{-(2n-1)^2\pi^2k^2}{4L^2} t\right) \sin \frac{(2n-1)\pi x}{2L}.$$

Mixed boundary conditions $u_x(0,t) = u(L,t) = 0$

Theorem. Assume $f(x)$ defined on $[0, L]$ has mixed Fourier cosine series

$$C_M(x) = \sum_{n=1}^{\infty} \alpha_n \cos \frac{(2n-1)\pi x}{2L}$$

$$\alpha_n = \frac{2}{L} \int_0^L f(x) \cos \frac{(2n-1)\pi x}{2L}\, dx$$

Then the IBVP

$$\begin{aligned} &u_t = k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ &u_x(0,t) = 0 = u(L,t) && t > 0 \\ &u(x,0) = f(x) && 0 \le x \le L \end{aligned}$$

has a formal solution

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n exp\left(\frac{-(2n-1)^2\pi^2 k^2}{4L^2} t\right) \cos \frac{(2n-1)\pi x}{2L}$$

Example. Let $f(x) = x - L$ on $[0, L]$. Find a formal solution of

$$\begin{aligned} &u_t = k^2 u_{xx} && 0 < x < L, \quad t > 0 \\ &u_x(0,t) = 0 = u(L,t) && t > 0 \\ &u(x,0) = f(x) && 0 \le x \le L \end{aligned}$$

The mixed Fourier cosine expansion of $f(x)$ is

$$C_M(x) = -\frac{8L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{2L}.$$

Therefore, the formal solution of IBVP is $u(x,t) =$

$$-\frac{8L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} exp\left(\frac{-(2n-1)^2\pi^2 k^2}{4L^2} t\right) \cos \frac{(2n-1)\pi x}{2L}.$$

Non homogeneous Heat Equation: Dirichlet boundary condition

Let us now consider the following PDE

$$u_t - k^2 u_{xx} = F(x,t) \qquad 0 < x < L, \quad t > 0$$
$$u(0,t) = f_1(t), \; u(L,t) = f_2(t) \qquad t > 0$$
$$u(x,0) = f(x) \qquad 0 \le x \le L$$

How do we solve this?

Let us first make the substitution

$$z(x,t) = u(x,t) - (1 - \frac{x}{L})f_1(t) - \frac{x}{L}f_2(t)$$

Then we get $z_t - k^2 z_{xx} = G(x,t)$

$z(0,t) = 0, \quad z(L,t) = 0$

$z(x,0) = g(x)$

It is enough to solve for $z(x,t)$.

By observing the boundary conditions, we guess that we should try and look for a solution of the type

$$z(x,t) = \sum_{n \ge 1} Z_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

Differentiating the above term by term we get

$$z_t - k^2 z_{xx} = \sum_{n \ge 1} \left(Z_n'(t) + \frac{k^2 n^2 \pi^2}{L^2} Z_n(t) \right) \sin\left(\frac{n\pi x}{L}\right)$$

Let us write the Fourier sine series of G,

$$G(x,t) = \sum_{n \ge 1} G_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

$$G_n(t) = \frac{2}{L} \int_0^L G(x,t) \sin\left(\frac{n\pi x}{L}\right) \, dx$$

$$z_t - k^2 z_{xx} = \sum_{n\geq 1}\left(Z_n'(t) + \frac{k^2n^2\pi^2}{L^2}Z_n(t)\right)\sin\left(\frac{n\pi x}{L}\right)$$

$$G(x,t) = \sum_{n\geq 1} G_n(t)\sin\left(\frac{n\pi x}{L}\right)$$

$$z_t - k^2 z_{xx} = G(x,t) \implies Z_n'(t) + \frac{k^2n^2\pi^2}{L^2}Z_n(t) = G_n(t)$$

We also need that $z(x,0) = g(x)$. If

$$g(x) = \sum_{n\geq 1} b_n \sin\frac{n\pi x}{L}$$

then we shoud have $Z_n(0) = b_n$.
Clearly, there is a unique solution $Z_n(t)$.

The series

$$z(x,t) = \sum_{n\geq 1} Z_n(t)\sin(\frac{n\pi x}{L})$$

solves our non homogeneous PDE with Dirichlet boundary conditions for z.