MA-207 Differential Equations II

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S2 - Lecture 1

Some Class Policies

EVALUATION: 50 marks are waiting to be earned:

Two In-Tutorial Quizzes	2×2.5 marks
Main Quiz	13 marks
End Semester exam	32 marks
Total	**50** marks

ATTENDANCE:

- Attendance in the first week of classes is mandatory.
- Attendance $< 80\% \implies$ you *may* be awarded a DX grade.

ACADEMIC HONESTY: Be honest. Do not violate the academic integrity of the Institute. Any form of academic dishonesty will invite severe penalties.

Books

Elementary differential equations with boundary value problems
by William F. Trench (available online)

Differential Equations with Applications and Historical Notes
by George F. Simmons

Office hours - Wednesdays 7 to 8 pm

Welcome to MA 207, a sequel to MA 108.

Since most of the functions encountered in MA 108 were elemetary functions, let us begin by recalling their definitions.

An algebraic function is a polynomial function, example,

$$x^3 + 3x + 2,$$

a rational function or equivalently quotient of polynomial functions, example

$$\frac{x^3 + 3x + 2}{x^5 + 2x^3 + 5},$$

or more generally, any function $y = f(x)$ that satisfies an equation of the form

$$P_n(x)y^n + P_{n-1}(x)y^{n-1} + \ldots + P_1(x)y + P_0(x) = 0$$

for some n, where each $P_i(x)$ is a polynomial.

The elementary functions are algebraic functions, trigonometric functions, example

$$\sin x, \quad \cos x, \quad \tan x$$

inverse trigonometric functions, example

$$\sin^{-1} x, \quad \cos^{-1} x, \quad \tan^{-1} x$$

exponential and logarithmic functions, example

$$e^{x^2}, \quad \log(x^2 + x + 1)$$

and all other functions that can be constructed from these functions by adding, subtracting, multiplying, dividing and composition of functions. Thus

$$y = tan \left[\frac{xe^{1/x^2} + \tan^{-1}(1 + x^2) + \sqrt{x^2 + 3}}{\sin x \cos 2x - \sqrt{\log x} + x^{3/2}} \right]^{1/3}$$

is an elementary function.

Beyond elementary functions lie the special functions, example Gamma function, Beta function, Riemann zeta function etc.

Definition

The Riemann zeta function is defined on the set $\{s \in \mathbb{C} \,|\, \mathrm{Re(s)} > 1\}$ by

$$\zeta(s) := \sum_{n \geq 1} \frac{1}{n^s}$$

It is a non-trivial theorem that the zeta function extends to the whole plane as a meromorphic function.
The Riemann hypothesis states that all the nontrivial zeros of the zeta function lie on the line $\mathrm{Re}(s) = \frac{1}{2}$.
This is one of the millennium problems and has a prize of 1 million US dollars.

Large number of special functions arise as solutions of 2nd order linear ODE.

Suppose we want to solve

$$y'' + y = 0$$

Then elementary functions $y = \sin x$ and $y = \cos x$ are solutions.
Suppose we want to solve

$$xy'' + y' + xy = 0$$

This equation can not be solved in terms of elementary functions.
In fact, there is no known kind of 2nd order linear ODE;
apart from those with constant coefficients (considered in MA 108), which can be solved in terms of elementary functions.

Let $y_1(x)$ be one solution of the ODE

$$y'' + p(x)y' + q(x)y = 0$$

with $p(x), q(x)$ continuous, then we can try to use the method of variation of parameters to find another linearly independent solution, that is, put

$$y_2 = u(x)y_1(x)$$

in the ODE and solve for $u(x)$.

Question. How to find the 1st solution?

For this, we will solve our ODE in terms of power series.

Let us review power series, which is used throughout in this course.

Definition

For real numbers $x_0, a_0, a_1, a_2, \ldots$, an infinite series

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n := a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots.$$

is called a power series in $x - x_0$ with center x_0.

For a real number x_1, if the limit

$$\lim_{N \to \infty} \sum_{n=0}^{N} a_n(x_1 - x_0)^n$$

exists and is finite, then we say the power series converges at the point $x = x_1$

In this case, the sum of the series is the value of the limit.

If the series does not converge at x_1, that is either limit does not exist or it is $\pm\infty$, then we say the power series diverges at x_1.

A power series always converges at its center $x = x_0$.

Theorem

For any power series,

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

exactly one of these statements is true.

1. *The power series converges only for $x = x_0$.*
2. *The power series converges for all values of x.*
3. *There is a positive number $0 < R < \infty$ such that the power series converges if $|x - x_0| < R$ and diverges if $|x - x_0| > R$.*

R is called the radius of convergence of the power series.

We define $R = 0$ in case (i)
and $R = \infty$ in case (ii).

Question. How to compute the radius of convergence?

Theorem

- *(Ratio test) If $a_n \neq 0$ for all n and*

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

- *(Root test)* $\limsup_{n \to \infty} |a_n|^{1/n} = L$

Then radius of convergence of the power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ *is*

$\boxed{R = 1/L}$.

For $L = 0$, we get $R = \infty$
and for $L = \infty$, we get $R = 0$.

Theorem

Let $R > 0$ be the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Then the power series converges (absolutely) for all $x \in (x_0 - R, x_0 + R)$.

For $R = \infty$, we write $(x_0 - R, x_0 + R) = (-\infty, \infty) = \mathbb{R}$.

The open interval $(x_0 - R, x_0 + R)$ is called the interval of convergence of the power series.

Example

Find the radius of convergence and interval of convergence (if $R > 0$) of the following three series

$$(i) \ \sum_{0}^{\infty} n! x^n \quad (ii) \ \sum_{10}^{\infty} (-1)^n \frac{x^n}{n^n} \quad (iii) \ \sum_{0}^{\infty} 2^n n^3 (x-1)^n$$

$$(i) \ \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \right| = \lim_{n\to\infty} (n+1) = \infty$$

So $R = 0$ in case (i).

Similarly, in case (ii) $R = \infty$
and in case (iii) $R = 1/2$.

Interval of convergence : in case (ii) $(-\infty, \infty)$
and in case (iii) $(1/2, 3/2)$

Correction: added after the class

In the class, we stated ratio test with $\limsup$ instead of using $\lim$ which we have corrected now.

The $\limsup$ definition in the ratio test does not give radius of convergence, though it gives convergence of the series for $x \in (x_0 - R, x_0 + R)$, where $R = 1/L$ and $L = \limsup |a_{n+1}|/|a_n|$.

For an example, take the series

$$1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + \dots.$$

Here the coefficient of x^n is 1 if n is even and 2 is n is odd. Now

$$\limsup \frac{a_{n+1}}{a_n} = \lim b_n, \qquad b_n = \sup\{\frac{a_{n+1}}{a_n}, \frac{a_{n+2}}{a_{n+1}}, \dots, \}$$

Since all $b_n = 2$ for all n, we get

$$\limsup \frac{a_{n+1}}{a_n} = 2 \implies R = 1/2.$$

Correction : added after the class

Note that the series

$$1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + \dots = \sum_0^\infty x^n + \sum_0^\infty x^{2n+1}$$

Since both series have radius of convergence 1, then sum has radius of convergence atleast 1, whereas we found $R = 1/2$.

This example shows that the ratio test using $\limsup$ does not give radius of convergence, in general.

The root test definition using $\limsup$ is correct.

In our example,

$$\limsup a_n^{1/n} = \limsup\{1, 2^{1/1}, 1, 2^{1/3}, \dots, 1, 2^{1/2n-1}, \dots\} = 1$$

Hence the radius of convergence of the series is exactly 1.

Some remarks made in class

For a sequence $\{a_n\}_{n\geq 1}$ let us recall the definition of

$$\limsup\{a_n\}$$

For every $k \geq 1$ define

$$b_k := \sup_{n\geq k}\{a_n\}$$

Convince yourself that the $\{b_k\}_{k\geq 1}$ is a decreasing sequence

$$b_1 \geq b_2 \geq b_3 \geq \dots$$

Define

$$\limsup\{a_n\} := \lim_{n\to\infty} b_n$$

Similarly, define $\liminf\{a_n\}$, by replacing $\sup$ by $\inf$ in the above discussion.
Note that for a sequence $\{a_n\}_{n\geq 1}$, the limit may not exist.
However, the $\limsup$ and $\liminf$ always exist (possibly $+\infty$ and $-\infty$).

Some remarks made in class

Theorem

Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. Then $\lim_{n\to\infty} a_n$ exists if and only if $\limsup\{a_n\} = \liminf\{a_n\}$.
Further, if $\lim_{n\to\infty} a_n$ exists, then

$$\limsup\{a_n\} = \liminf\{a_n\} = \lim_{n\to\infty} a_n$$

Strictly speaking, when we say that $\lim_{n\to\infty} a_n$ exists, we mean that this limit exists and is finite.
However, sometimes we shall be a little careless and say that $\lim_{n\to\infty} a_n$ exists in the following cases also: if $\lim_{n\to\infty} a_n = \infty$ or $\lim_{n\to\infty} a_n = -\infty$.
Recall, for example, the definition of $\lim_{n\to\infty} a_n = \infty$. For every $N \in \mathbb{R}$, there exists $n(N) \geq 1$ (that is, n depends on N) such that $a_k \geq N$ for all $k \geq n(N)$.
For example, convince yourself that for the sequence defined by $b_{2n-1} := n$ and $b_{2n} := n - 1$ $(n \geq 1)$, we have $\lim_{n\to\infty} b_n = \infty$

Some remarks made in class

An easy proof that

$$\sum_{n\geq 1}\frac{1}{n}=\infty$$

$$\sum_{n\geq 1}\frac{1}{n}>\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}>\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}=\frac{1}{1}+\frac{1}{2}+\frac{1}{2}$$

$$\sum_{n\geq 1}\frac{1}{n}>\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{8}>\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}$$

The RHS is

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{1}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$$

Proceeding in the above fashion we see that

$$\sum_{n\geq 1}\frac{1}{n}>\sum_{n=1}^{2^k}\frac{1}{n}>1+\frac{k}{2}$$

Let $k\to\infty$. This clever proof is due to Euler!