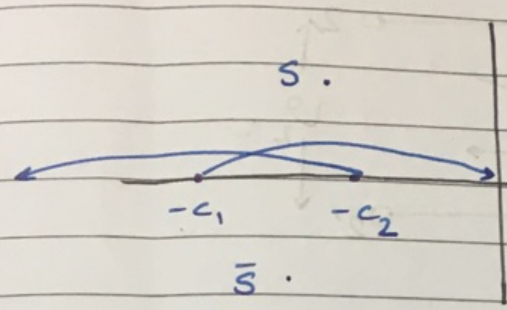
Poles & Zeroes



Thinking of transfer functions as conformal mappings.

or $$G_{12}(s) = \left(\frac{s+c_1}{s+c_2}\right)$$

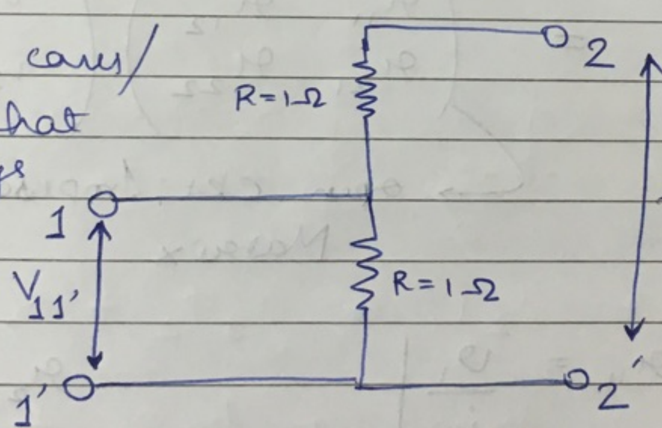
$$f(z,t) = z(1-t) + tG_{12}(s)$$

25-10-17

Lec - 24

→ Last time: Two-port Systems, Reciprocity thm.

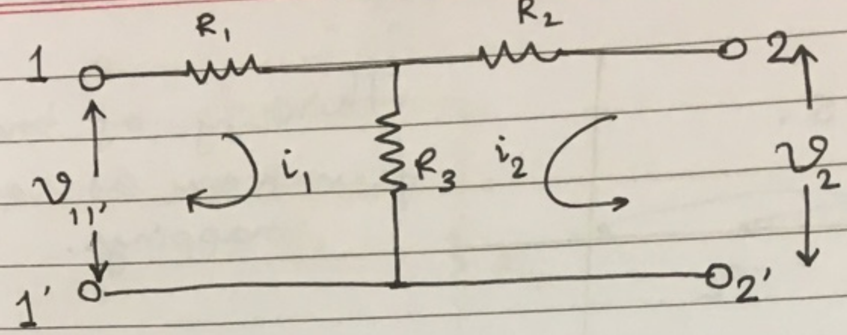
→ Exception case/ this is not what Reciprocity says



→ Check here → get $V_{11'}$

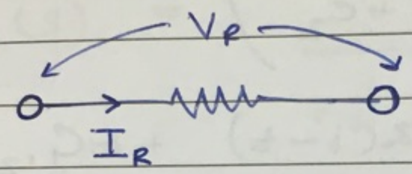
Apply here check at 1 1' → $\frac{V_{22'}}{2}$

→ Today: Examples of 2 port Networks



2)

v_1, v_2, i_1, i_2



$V_R - RI_R = 0$

$v_1 = i_1(R_1 + R_3) + i_2 R_3$ 3)

$v_2 = i_2(R_2 + R_3) + i_1 R_3$

1) $$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$$

→ open ckt Impedance Matrix

$$z_{11} = \left.\frac{v_1}{i_1}\right|_{i_2 = 0} \qquad z_{21} = \left.\frac{v_2}{i_1}\right|_{i_2 =}$$ 4)

$$z_{12} = \left.\frac{v_1}{i_2}\right|_{i_1 = 0} \qquad z_{22} = \left.\frac{v_2}{i_2}\right|_{i_1 =}$$

Short ckt admittance

2) $$\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\left.\frac{i_1}{v_1}\right|_{v_2=0} = \frac{R_2 + R_3}{R_2 R_3 + R_1 (R_2 + R_3)}$$

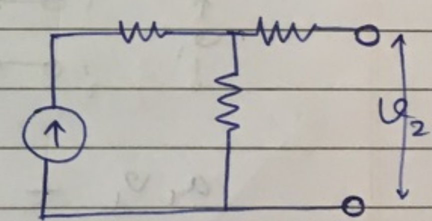
$$\frac{1}{R_2 \parallel R_3 + R_1} = \frac{1}{R_1 + \frac{R_2 R_3}{R_2 + R_3}} = \frac{R_2 + R_3}{R_2 R_3 + R_1 (R_2 + R_3)}$$

?? Transmission Mat.

3) $$\begin{pmatrix} v_1 \\ i_1 \end{pmatrix} = \begin{pmatrix} \bigcirc & \\ & \end{pmatrix} \begin{pmatrix} v_2 \\ i_2 \end{pmatrix}$$

~~Trans~~ $$\left.\frac{i_1}{v_2}\right|_{i_2=0} = \frac{1}{R_3}$$

$\bigcirc v_2 + x i_2$



4) <u>Inverse transmission</u>

~~Hybrid~~ para

$$\begin{pmatrix} v_2 \\ i_2 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix} \begin{pmatrix} v_1 \\ i_1 \end{pmatrix}$$

5) Hybrid

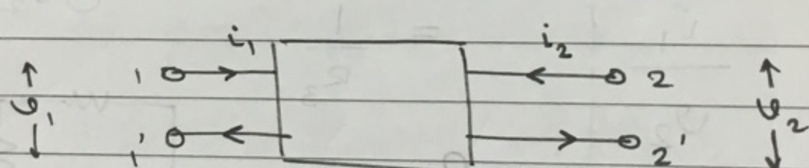
$$\begin{pmatrix} v_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} i_1 \\ v_2 \end{pmatrix}$$

6)

$$\begin{pmatrix} i_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ i_2 \end{pmatrix}$$

$$g_{12} = \left. \frac{i_1}{i_2} \right|_{v_1 = 0}$$

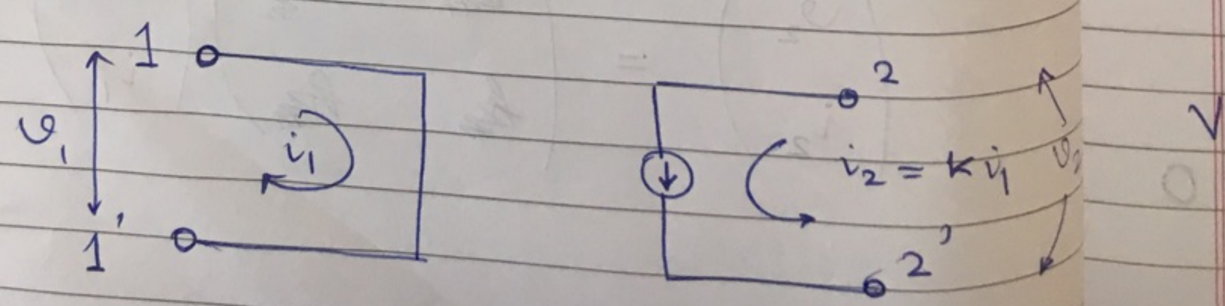
Linear two port (Resistor)



$$a_1 v_1 + a_2 i_1 + a_3 v_2 + a_4 i_2 = 0$$

$$b_1 v_1 + b_2 i_1 + b_3 v_2 + b_4 i_2 = 0$$

Current - controlled current source



We have Equations

$$v_1 = 0 \quad \text{(Shorted)}$$
$$i_2 = k i_1$$

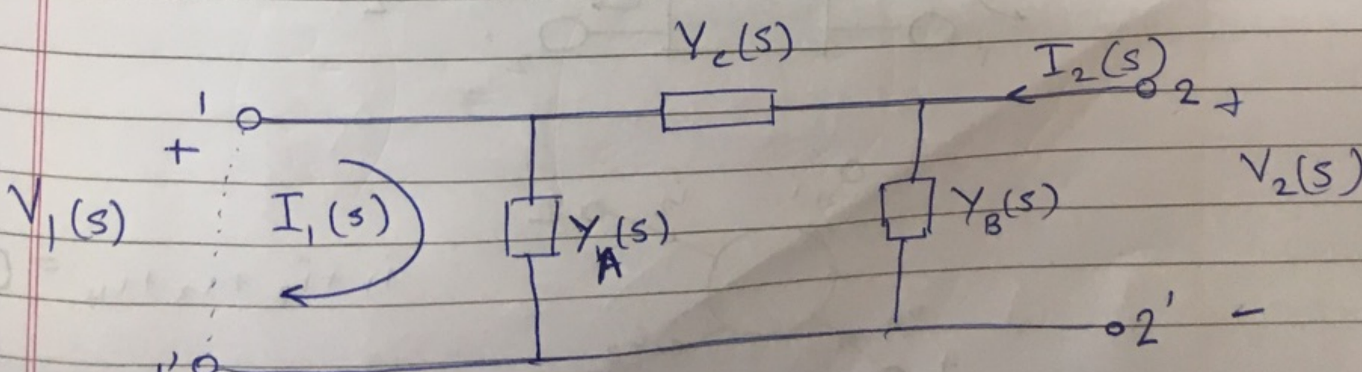
$$\begin{array}{c c} & \begin{array}{cccc} v_1 & i_1 & v_2 & i_2 \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -k & 0 & 1 \end{pmatrix} \end{array} \begin{pmatrix} v_1 \\ i_1 \\ v_2 \\ i_2 \end{pmatrix} = 0$$

$$-s \begin{bmatrix} 0 & 0 \\ 0 & 1/k \end{bmatrix} \begin{bmatrix} v_2 \\ i_2 \end{bmatrix}$$

$$\begin{pmatrix} v_1 \\ i_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -k \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v_2 \\ i_2 \end{pmatrix}$$

$$\cancel{\begin{bmatrix} 1 & 0 \\ 0 & -k \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -k \end{bmatrix} \begin{pmatrix} v_1 \\ i_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v_2 \\ i_2 \end{pmatrix}$$



$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \quad \text{(1)}$$

$$y_{11} = \left. \frac{I_1}{V_1} \right|_{V_2 = 0} = Y_A + Y_C$$

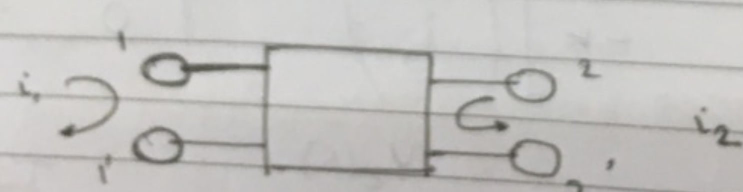
Admittance add in parallel

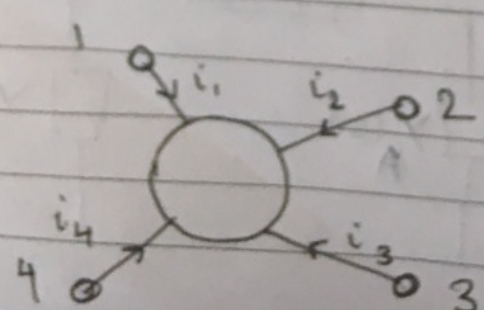
$$y_{12} = \left. \frac{I_1}{V_2} \right|_{V_1 = 0} = -Y_C$$

$$y_{21} = \left. \frac{I_2}{V_1} \right|_{V_2 = 0} = -Y_C$$

$$y_{22} = Y_B + Y_C$$

2 port Network Vs 4 port 4 terminal device

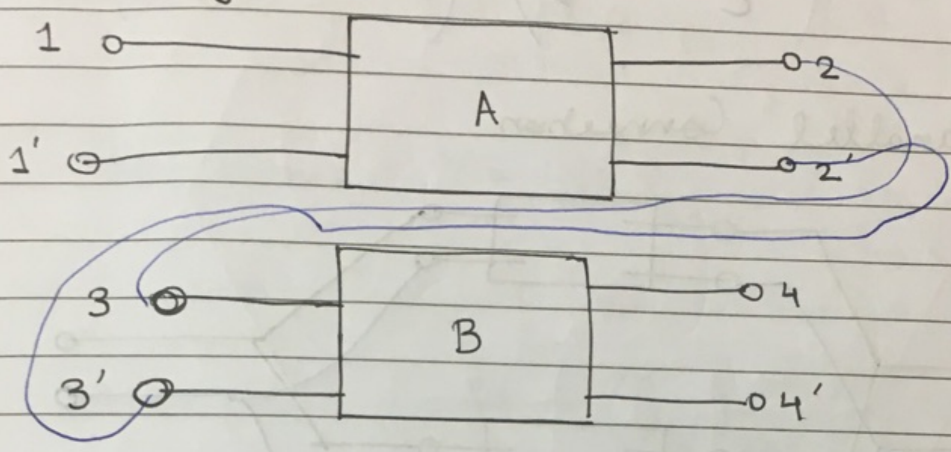




$i_1 + i_2 + i_3 + i_4 = 0$

→ Combination of 2 port Networks

(1) Cascading



$2 \leftrightarrow 3$
$2' \leftrightarrow 3'$

→ Above arrangement is called "cascading"

Cascade (A, B, 2~3, 1,4)

Hint: Transmission Matrix

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} V_2 \\ I_2 \end{pmatrix}$$

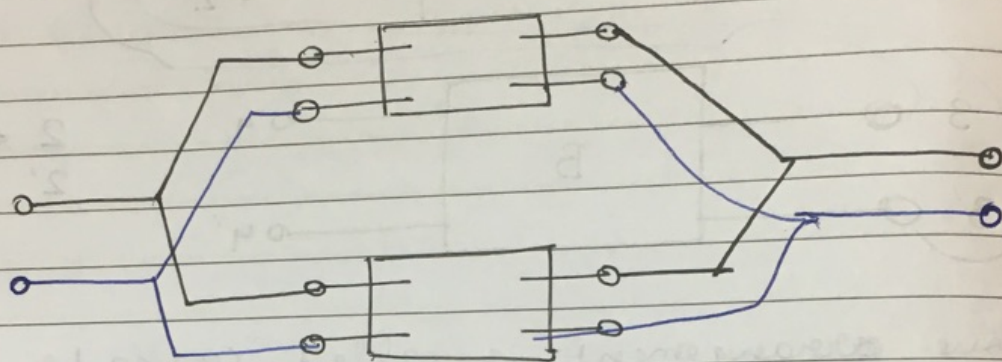
$$\begin{pmatrix} V_2 \\ I_2 \end{pmatrix} = \begin{pmatrix} V_3 \\ -I_3 \end{pmatrix}$$

$$\begin{pmatrix} V_3 \\ I_3 \end{pmatrix} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} V_4 \\ I_4 \end{pmatrix}$$

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix} \begin{pmatrix} V_4 \\ I_4 \end{pmatrix}$$

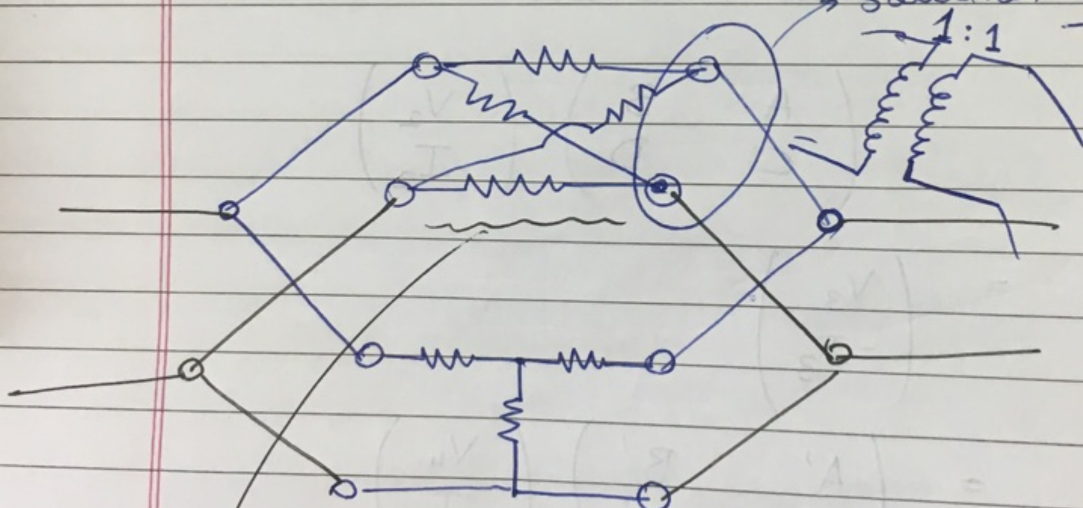
$$= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A' & B' \\ -C' & -D' \end{pmatrix} \begin{pmatrix} V_4 \\ I_4 \end{pmatrix} \rightarrow$$

(2) Parallel Connection



→ Behaviour not always predictable :

solution use transfor



Current will never flow, condition for port violated

Last time

→ <u>Linear 2-port</u>

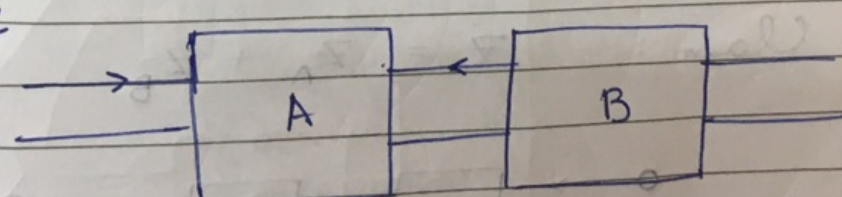
Impedance $Z = (z_{ij})_{2\times 2}$, $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = Z \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$

Admittance $Y = (y_{ij})_{2\times 2}$, $\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = Y \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$

Hybrid
$$\begin{cases} H = (h_{ij})_{2\times 2}, & \begin{pmatrix} V_1 \\ I_2 \end{pmatrix} = H \begin{pmatrix} I_1 \\ V_2 \end{pmatrix} \\ H' = (h'_{ij})_{2\times 2}, & \begin{pmatrix} I_1 \\ V_2 \end{pmatrix} = H' \begin{pmatrix} V_1 \\ I_2 \end{pmatrix} \end{cases}$$

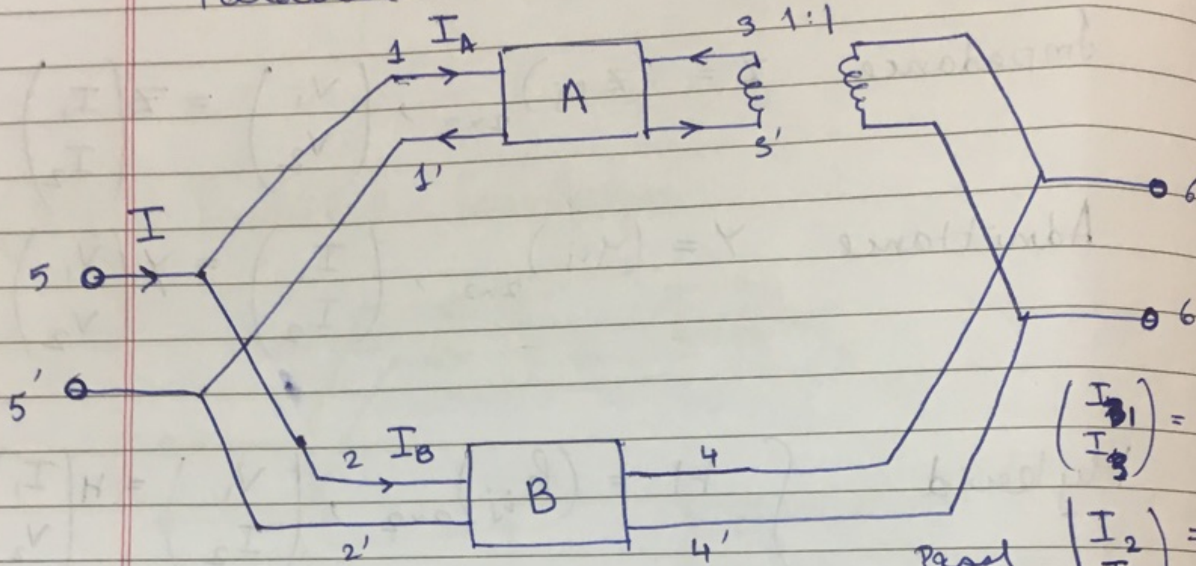
Transmission
$$\begin{cases} T = (t_{ij})_{2\times 2}, & \begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = T \begin{pmatrix} V_2 \\ I_2 \end{pmatrix} \\ T' = (t'_{ij})_{2\times 2}, & \begin{pmatrix} V_2 \\ I_2 \end{pmatrix} = T' \begin{pmatrix} V_1 \\ I_1 \end{pmatrix} \end{cases}$$

→ Cascading



$\boxed{T = T_A \cdot T_B}$ → {Modulo signs}

→ Parallel



$\begin{pmatrix} I_1 \\ I_3 \end{pmatrix} = Y_A$

Proof $\begin{pmatrix} I_2 \\ I_4 \end{pmatrix} = Y_B$

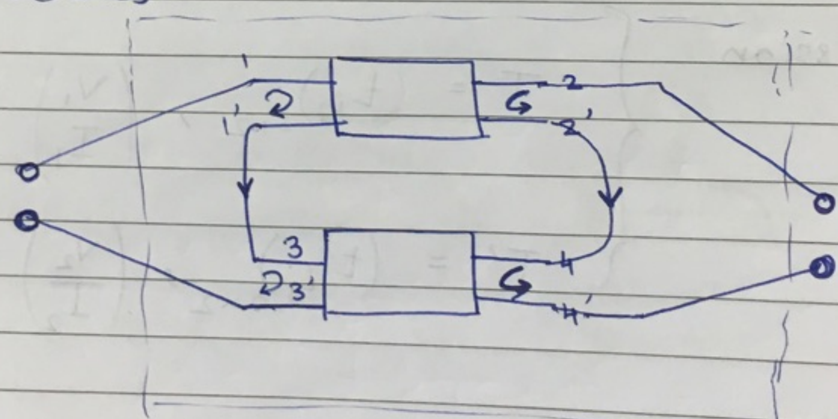
$I_5 = I_1 + I_2$

$I_6 = I_3 + I_4$

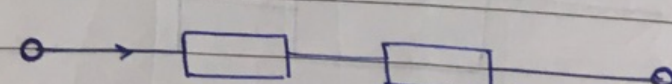
Claim : $Y = Y_A + Y_B$

$$\begin{pmatrix} I_5 \\ I_6 \end{pmatrix} = (Y_A + Y_B)\begin{pmatrix} V_5 \\ V_6 \end{pmatrix}$$

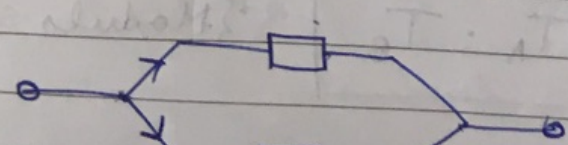
→ Series



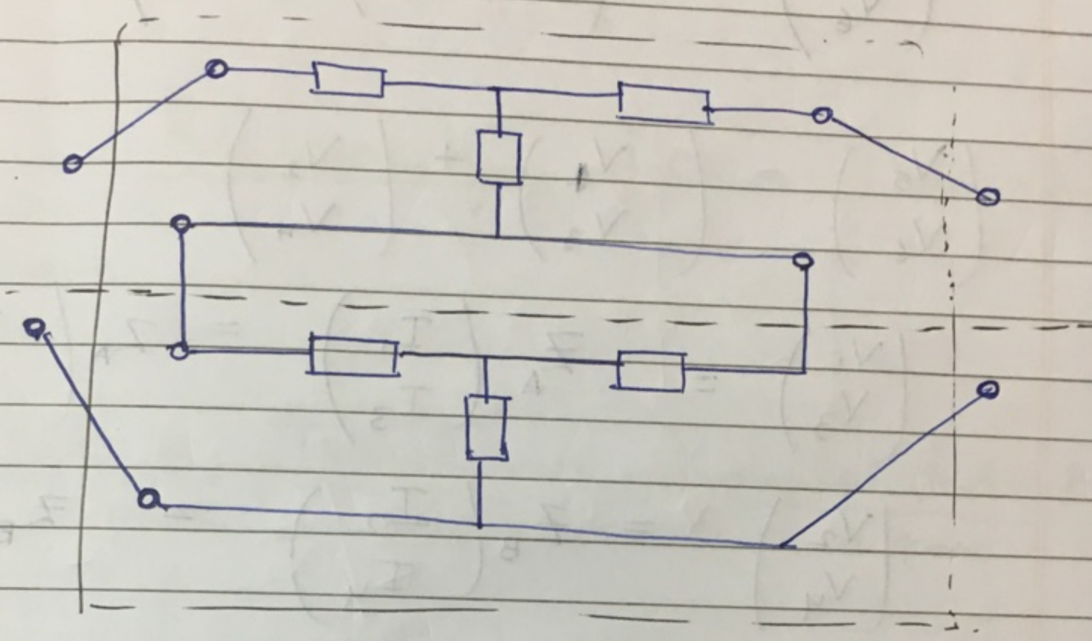
Claim : $Z = Z_A + Z_B$



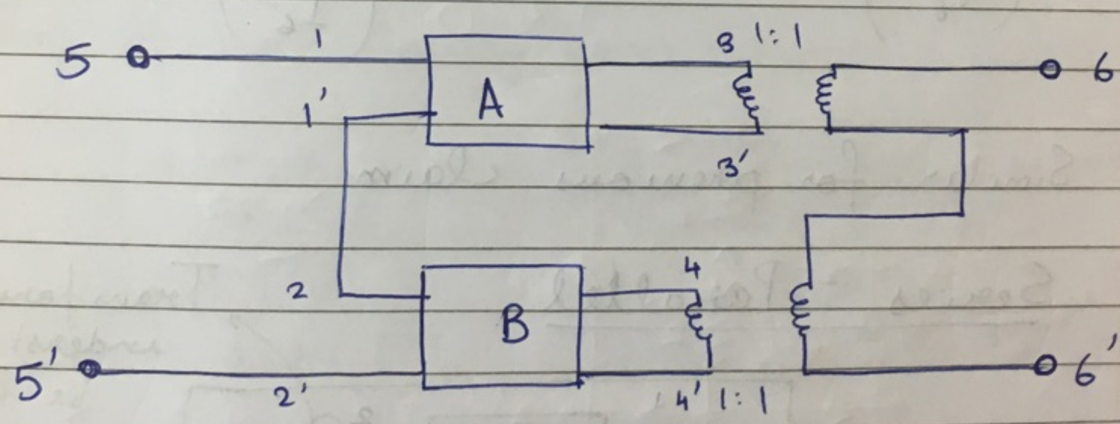
Series parts



→ Case when port Nature breaks down



<u>Solution</u> : Use transformers



$$I_5 = I_1 = I_2$$

$$I_6 = I_3 = I_4$$

$$V_5 = V_1 + V_2 \qquad V_6 = V_3 + V_4$$

$$\begin{pmatrix} V_5 \\ V_6 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix} \begin{pmatrix} I_5 \\ I_6 \end{pmatrix}$$

$$\begin{pmatrix} V_5 \\ V_6 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_3 \end{pmatrix} + \begin{pmatrix} V_2 \\ V_4 \end{pmatrix}$$

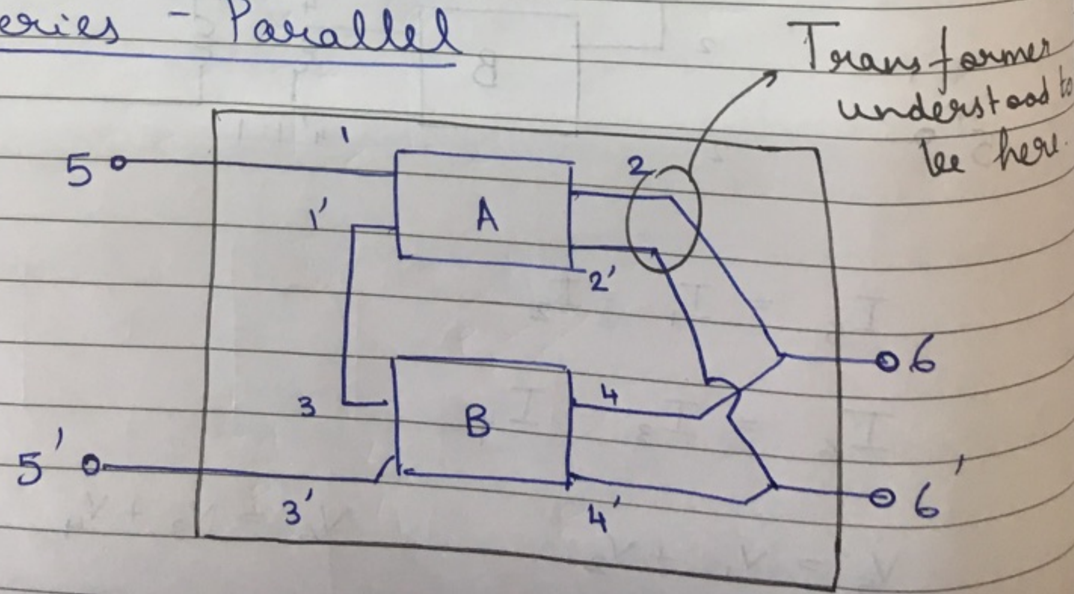
$$\begin{pmatrix} V_1 \\ V_3 \end{pmatrix} = Z_A \begin{pmatrix} I_1 \\ I_3 \end{pmatrix} = Z_A \begin{pmatrix} I_5 \\ I_6 \end{pmatrix}$$

$$\begin{pmatrix} V_2 \\ V_4 \end{pmatrix} = Z_B \begin{pmatrix} I_2 \\ I_4 \end{pmatrix} = Z_B \begin{pmatrix} I_5 \\ I_6 \end{pmatrix}$$

$$\begin{pmatrix} V_5 \\ V_6 \end{pmatrix} = (Z_A + Z_B) \begin{pmatrix} I_5 \\ I_6 \end{pmatrix}$$

→ Similar for previous claim

→ <u>Series - Parallel</u>



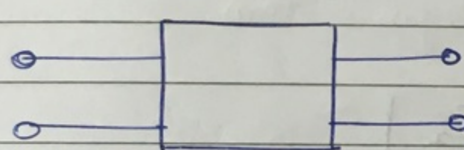
$$\begin{pmatrix} V_5 \\ I_6 \end{pmatrix} = (H_A + H_B) \begin{pmatrix} I_5 \\ V_6 \end{pmatrix}$$

Reciprocity [Ref: Chua : Chap 13.6]

~~Theorem~~

→ Valid for,

$\underbrace{R,\ C,\ L}_{\text{1-port}}$, $\underbrace{\text{Transformer}}_{\text{2-port}}$, $\underbrace{\text{Coupled Inductor}}_{\text{2-port}}$



Thm: If the following exist, then;

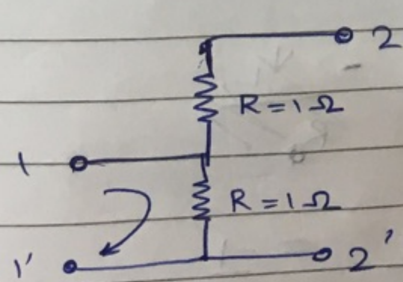
(1) $z_{12} = z_{21}$ (2) $\cancel{Y_{12} = Y_{21}}$ $y_{12} = y_{21}$

(3) $h_{21} = -h_{12}$ (4) $h'_{21} = -h'_{12}$

(5) $\det(T) = 1$

$= \det(T')$

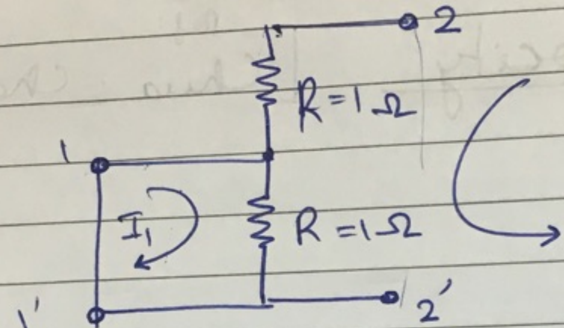
→ Example



$$z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = R$$

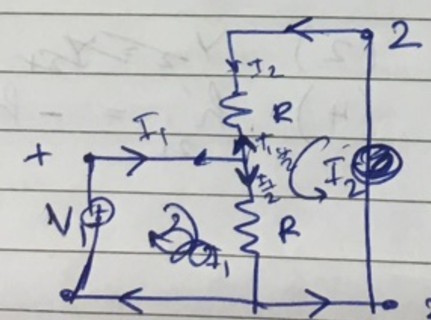
$$z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = \frac{R \times I_2}{I_2} = R$$

2) $y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0}$ =



$I_1 = -\frac{V_2}{R} / V_2 = \frac{-1}{R}$

$y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0}$



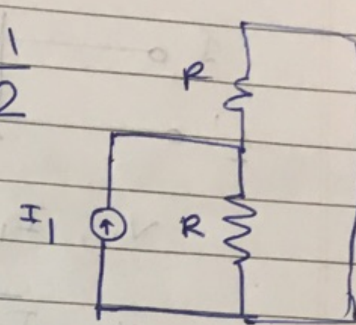
$\frac{V_1}{R}$

$V_1 = I_2 R$

$I_2 - \frac{V_1}{R}$

$-\frac{1}{R} \frac{V_1}{V_1} = \frac{-1}{R}$

3) $h_{21} = \left.\dfrac{I_2}{I_1}\right|_{V_2=0} = \dfrac{-1}{2}$



$h_{12} = \left.\dfrac{V_1}{V_2}\right|_{I_1=0} = \dfrac{1}{2}$

4) $h'_{21} = \left.\dfrac{V_2}{V_1}\right|_{I_2=0} = 1$

$h'_{12} = \left.\dfrac{I_1}{I_2}\right|_{V_1=0} = -1$

5) $t_{11} = \left.\dfrac{V_1}{V_2}\right|_{I_2=0} = 1$

$t_{12} = \left.\dfrac{-V_1}{I_2}\right|_{V_2=0} = R$

$t_{21} = \left.\dfrac{I_1}{V_2}\right|_{I_2=0} = \dfrac{1}{R}$

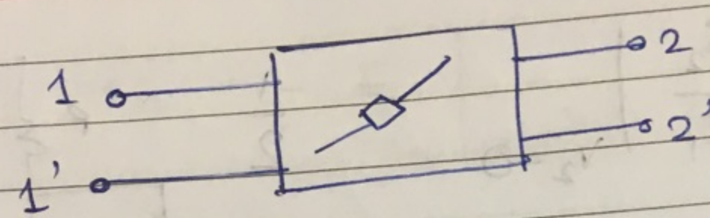
$t_{22} = \left.-\dfrac{I_1}{I_2}\right|_{V_2=0} = -2$

$\det(T) = 1$

$\det T = 1$

$(2-1) = 1$

$$\begin{matrix} 1 & R \\ \frac{1}{R} & 2 \end{matrix}$$



V_1, I_1, V_2, I_2

$\bar{V}_1, \bar{I}_1, \bar{V}_2, \bar{I}_2$

<u>Claim</u> :

$$V_1 \bar{I}_1 + V_2 \bar{I}_2$$

$$= \bar{V}_1 I_1 + \bar{V}_2 I_2$$

→ Prove using Tellegen's thm

1.11.17

<u>Recap</u>

- Series / Parallel / Cascade connection of 2 ports.

- <u>Reciprocity thm</u>

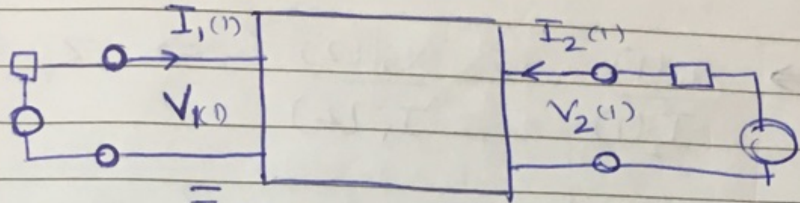
→ For a 2-port made up of R, L, C & coupled Inductors

$Z_{12} = Z_{21}$ $\quad Y_{21} = Y_{12}$, $\det(T) = 1$

$h_{12} = -h_{21}$ $\quad h'_{12} = -h'_{21}$, $\det(T')$

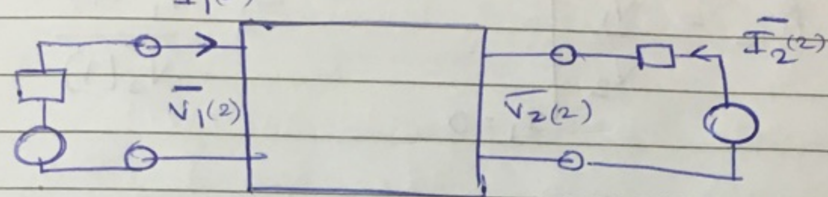
Claim:

(1)



Connection Method - 1

(2)



$$V_1 \bar{I}_1 + V_2 \bar{I}_2 = \bar{V}_1 I_1 + \bar{V}_2 I_2$$

$$\Rightarrow \quad V_1 I_1(2) + V_2(1) I_2(2) = V_1(2) I_1(1) + V_2(2) I_2(1)$$

Lemma

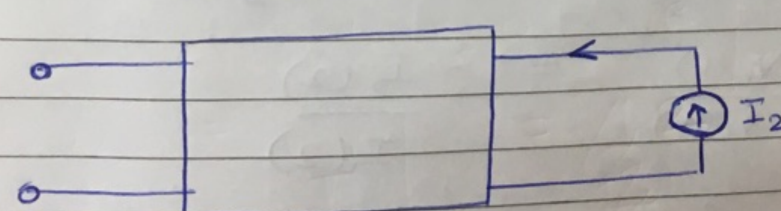
Claim: Reciprocity theorem

Proof:

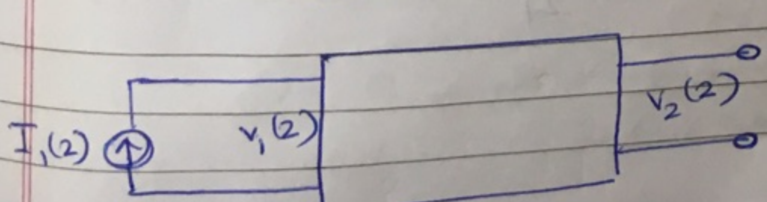
$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = \frac{V_1(1)}{I_2(1)}$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = \frac{V_2(2)}{I_1(2)}$$

Case 1:



Case 2:



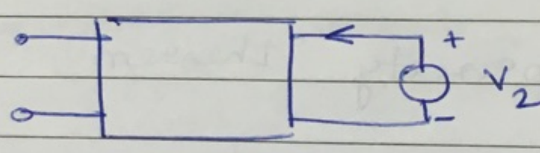
$$\Rightarrow V_1(1)\, I_1(2) = V_2(2)\, I_2(1)$$

$$\Rightarrow \frac{V_1(1)}{I_2(1)} = \frac{V_2(2)}{I_1(2)} \Rightarrow Z_{12} = Z_{21}$$

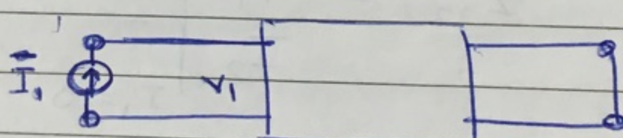
$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1 = 0} = \frac{V_1(1)}{V_2(1)}$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2 = 0} = \frac{I_2(2)}{I_1(2)}$$

Case 1:



Case 2:



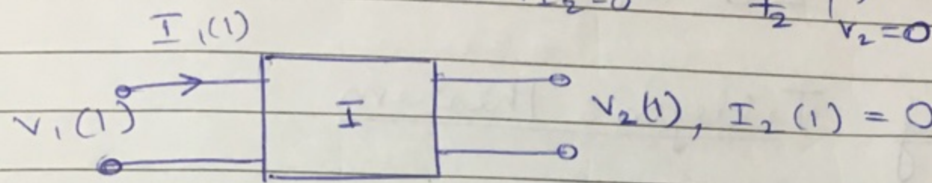
from claim

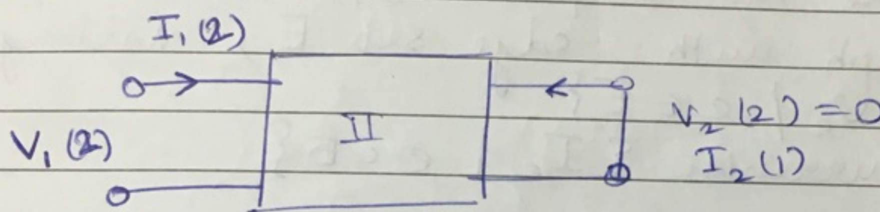
$$V_1(1)\, I_1(2) = -V_2(1)\, I_2(2)$$

$$\frac{V_1(1)}{V_2(1)} = -\frac{I_2(2)}{I_1(2)}$$ using

$$\Rightarrow h_{12} = -h_{21}$$

$$T = \begin{pmatrix} \left.\frac{V_1}{V_2}\right|_{I_2=0} & \left.\frac{-V_1}{I_2}\right|_{V_2=0} \\ \left.\frac{I_1}{V_2}\right|_{I_2=0} & \left.\frac{-I_1}{I_2}\right|_{V_2=0} \end{pmatrix}$$





$$T = \begin{pmatrix} \frac{V_1(1)}{V_2(\cancel{1})} & \frac{-V_1(2)}{I_2(2)} \\ \frac{I_1(1)}{V_2(1)} & \frac{-I_1(2)}{I_2(2)} \end{pmatrix}$$

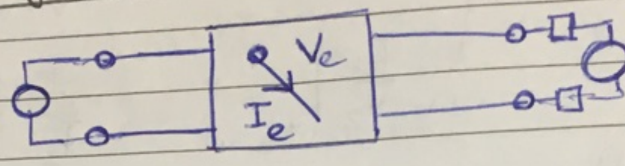
$$\det T = \frac{-V_1(1)\, I_1(2)}{V_2(1)\, I_2(2)} + \frac{V_1(2)\, I_1(1)}{I_2(2)\, V_2(1)} = 1$$

$$= \frac{V_1(2)\, I_1(1) - V_1(1)\, I_1(2)}{V_2(1)\, I_2(2)}$$

why Claim; $V_1^{(1)} I_1(2) + V_2(1)\, I_2(2) = V_1(2)\, I_1(1)$

$$\Rightarrow \frac{V_1(2)\, I_1(1) - V_1(1)\, I_1(2)}{V_2(1)\, I_2(2)} = \cancel{2}\ 1$$

→ Proof of Claim



Using Tellegen's theorem

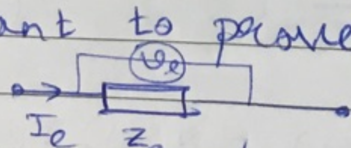
→ Suppose the 2-port circuit corresponds a graph with edge set E, voltages $\{V_e \mid e \in E\}$ and currents $\{I_e \mid e \in E\}$

- Applying Tellegen's thm:

$$V_1(1) I_1(2) + V_2(1) I_2(2) + \sum_{e \in E} V_e(1) I_e(2)$$

$$= 0 = V_1(2) I_1(1) + V_2(2) I_2(1) + \sum_{e \in E} V_e(2) I_e(1)$$

- We want to prove our claim.



↳ General edge

$v_e = I_e Z_e$

Claim: $\forall e \in E$

$$V_e(1) I_e(2) = V_e(2) I_e(1)$$

3.11.17

→ Pf

→

• Si

→ Fi

Pf $v_e(1) = I_e(1) z_e$, $v_e(2) = I_e(2) z_e$

$I_e(1) z_e I_e(2) = L.H.S = I_e(2) z_e I_e(1)$
$= R.H.S$

Hence proved

Chap 12: Valkenburg

Sinusoidal Steady State Analysis

$$\dot{x} = f(x)$$

Steady State $\Rightarrow \{x \mid f(x) = 0\}$

$$V(s) = \frac{A\omega}{s^2+\omega^2} + \frac{B_1}{s+p_1} + \frac{B_2}{s+p_2}$$

$$v(t) = \underbrace{A\sin(\omega t)}_{\text{Recurrent}} + \underbrace{B_1 e^{-p_1 t} + B_2 e^{-p_2 t}}_{\text{transient}}$$

$t > T$, $v(t) \sim ? = A\sin(\omega t)$

~~$V(s) = \frac{A\omega}{s^2+\omega^2} + \frac{B_1}{}$~~

3.11.17 Recap

→ Reciprocity theorem

- Sinusoidal Steady State Analysis

$$A_m \cos(\omega t + \phi)$$

$A_m \in R>0$, $\omega \in R>0$, $\phi \in [0, 2\pi)$

→ Fix ω:

$$A \equiv A_m e^{j\phi}$$

$$A_m \cos(\omega t + \phi) = Re\left(A e^{j\omega t}\right)$$

1) $Re(Ae^{j\omega t}) = Re(Be^{j\omega t})$

$$\Updownarrow$$

$$A = B$$

2) $a_1 Re(A_1 e^{j\omega t}) + a_2 Re(A_2 e^{j\omega t})$

$= Re((a_1 A_1 + a_2 A_2) e^{j\omega t}) \ \forall a_1, a_2 \in$ →

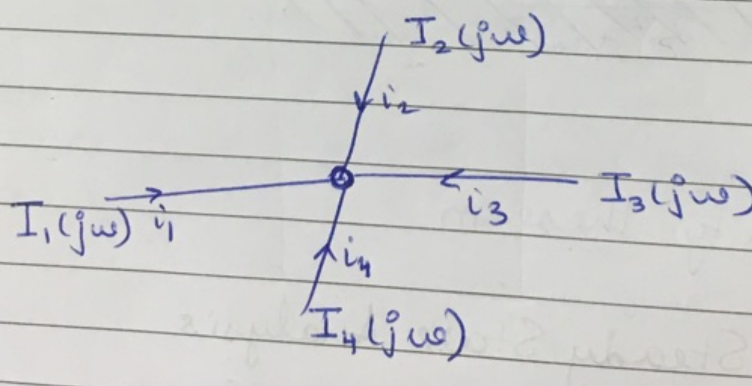
3) $\frac{d}{dt}(Re(Ae^{j\omega t})) = Re(j\omega A e^{j\omega t})$

Interchangable.

<u>Sinusoidal</u> <u>Steady State a frequency</u>

- All $v_e(t)$, $i_e(t)$, $v_n(t)$ are sinusoids at the same ω →
  $e \equiv$ edge $n \equiv$ node

Q→



KCL: $\forall t$ $\quad i_1(t) + i_2(t) + i_3(t) + i_4(t) = 0$ using

$$\Rightarrow I_1 + I_2 + I_3 + I_4 = 0$$

$$V = Z I$$

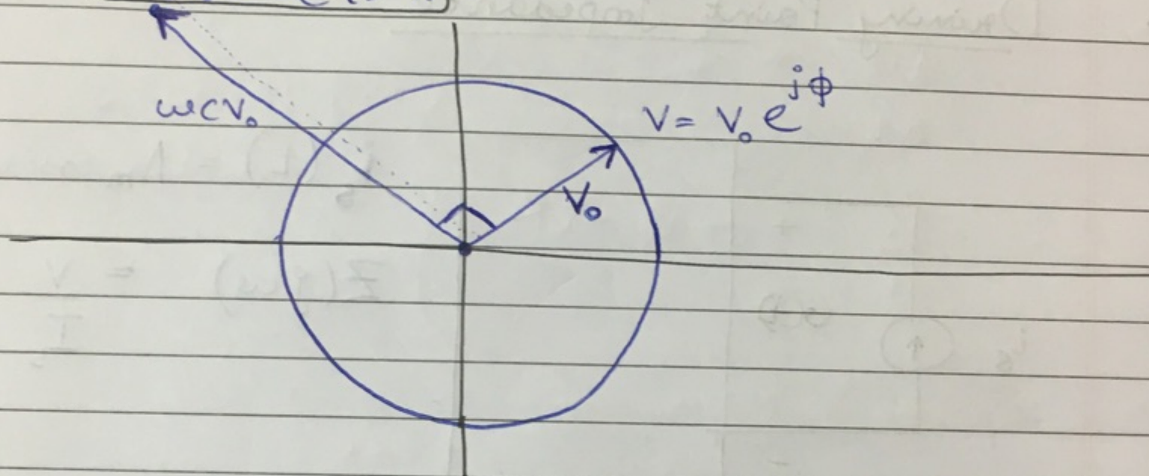
$$I = Y V$$

$$i = C\frac{dv}{dt}$$

$$I = (j\omega C)V \quad \Rightarrow \quad Y = j\omega C$$

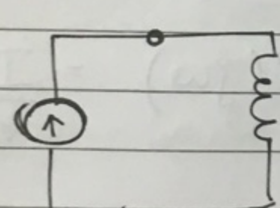
→ I is V rotated by $j\omega C$

$$\boxed{I = V_0 \omega C e^{j\left(\frac{\pi}{2}+\phi\right)}}$$



→ Admittances : $\frac{1}{j\omega C}$, $j\omega L$, R

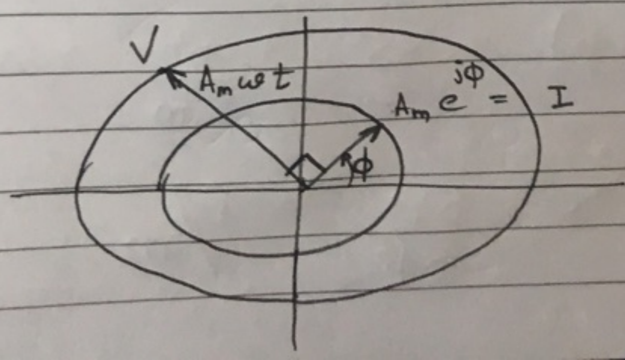
Q→



$$v = L\frac{di}{dt}$$

$$i(t) = A_m \cos(\omega t + \phi)$$
$$v(t) = -(\omega L) A_m \sin(\omega t + \phi)$$

using Phasors

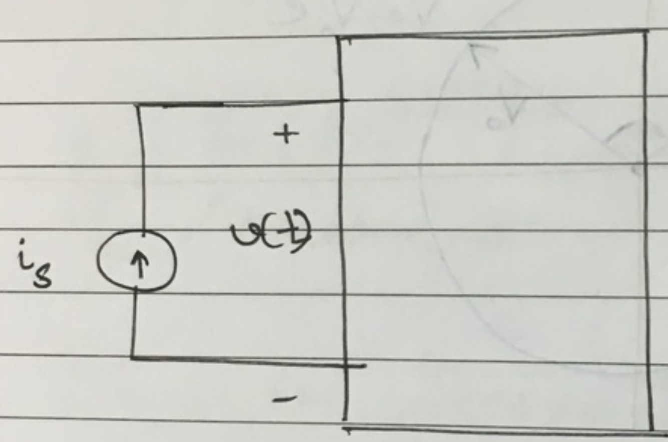


$$v(t) = \mathrm{Re}\left(A_m \omega L \, e^{j(\frac{\pi}{2} + \phi)} e^{j\omega t} \right)$$

$$= \omega L A_m \cos\left(\omega t + \frac{\pi}{2} + \phi \right)$$

$$= -\omega L A_m \sin(\omega t + \phi)$$

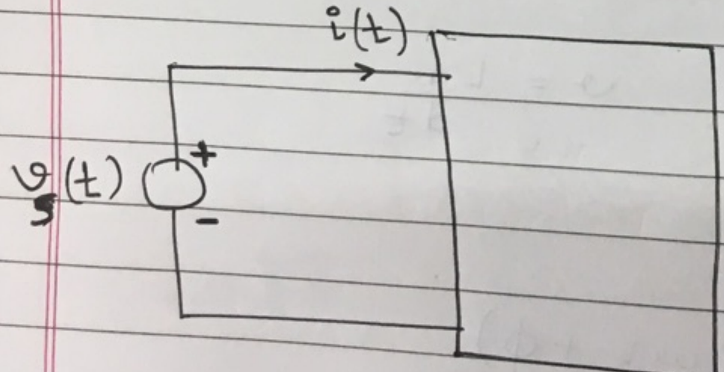
→ Driving Point Impedance



$$i_s(t) = A_m \cos(\omega t$$

$$Z(j\omega) = \frac{V}{I_s}$$

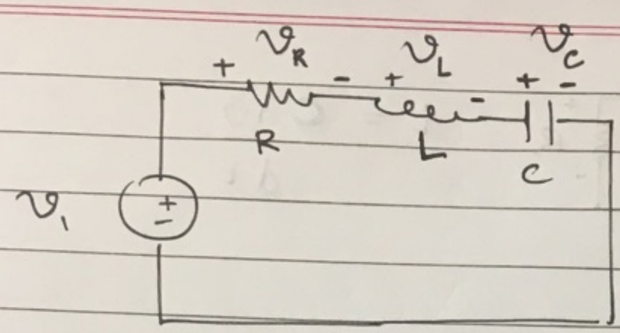
→ Driving Point Admittance



$$Y(j\omega) = \frac{I}{V_s}$$

$$= \frac{1}{Z(j\omega)}$$

Q→



$v_1 = \cos(\omega t)$

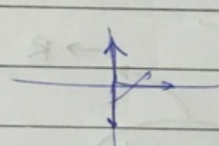
$V_1 = 1$

$$Z(j\omega) = R + j\omega L + \frac{1}{j\omega C}$$

$$I_1 = \frac{V_1}{Z} = \frac{\cancel{\phi}\; V_1}{R + j\omega L - \frac{j}{\omega C}}$$

$$i_1(t) = A\cos(\omega t + \theta_1)$$

$$A = \frac{1}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

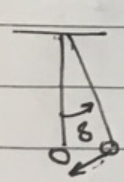


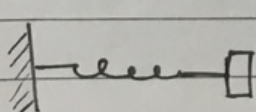
$$\theta = -\tan^{-1}\left(\frac{\omega L - 1/\omega C}{R}\right)$$

→ <u>Analogy with Mechanics</u>

<u>SHM</u>

$m\ddot{\theta} = -\theta$

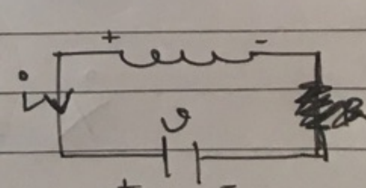




$m\ddot{x} = -kx$

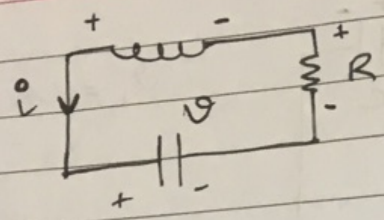
Basic SHM $\approx$

$$\left\{\begin{array}{l} \ddot{x} = -x = \dot{y} \\ y = \dot{x} \\ \\ \dot{x} = y \\ \dot{y} = -x \end{array}\right\}$$



(No R)

$C = L = R = 1$



$$i = C\frac{dv}{dt}$$

$$v = -L\frac{di}{dt} - iR$$

$$\frac{dv}{dt} = i$$

$$\frac{di}{dt} = -v - iR$$

just Like Damped Oscillat

→ Position & Momentum

$$\boxed{\begin{aligned}\dot{x} &= Y\\ \dot{Y} &= -X\end{aligned}}\; -\gamma Y$$

→ R

Damping cost.

x = position
y = momentum

$$U(x) = \frac{1}{2}kx^2$$

$$K(x) = \frac{m\dot{x}^2}{2} = \frac{p^2}{2m}$$

$$H(x, p) = \frac{kx^2}{2} + \frac{p^2}{2m} = \text{const.}$$

$$\dot{x} = \frac{\delta H}{\delta p} = P/m$$

$$\dot{Y} = -\frac{\delta H}{\delta x} = -k \cdot X$$

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -\gamma \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$sX = Y$$
$$sY = -X - \gamma Y$$

$$\dot{x} = Ax \Rightarrow x(t) = e^{At} x(0)$$

$$e^{A} = I + A + \frac{A^2}{2!} + \ldots\ldots$$

$$s^2 X = -X - \gamma s X$$

$$s^2 = -1 - \gamma s \Rightarrow s = \frac{-\gamma \pm \sqrt{\gamma^2 - 4}}{2}$$

→ Forced oscillations → Voltage source $v_s(t)$