MA 205 Complex Analysis: Laurent Series and Examples

August 19, 2017

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Isolated singularities are of 3 types: Removable singularity, Pole and Essential Singularity.

A singularity at z_0 is **removable** if $\lim\limits_{z\to z_0} f(z)$ exists. In particular $f(z)$ is bounded in a neighborhood of z_0.
A singularity at z_0 is a **pole** if $f(z) \to \infty$ as $z \to z_0$. In particular the function takes unbounded values in any punctured neighborhood of z_0.
A singularity at z_0 is an **essential singularity** if it is neither a removable singularity nor a pole.

Casorati-Weirestrass Theorem

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We will see little later that this is a special case of a much more general and very beautiful theorem. I will keep the suspense for now !
We also discussed an example : $e^{1/z}$ has an essential singularity at 0.
Let us discuss some more examples (on the board).

Recall how we derived the power series representation of a holomorphic function on a disc centered around z_0. We used

$$f(z) = \frac{1}{2\pi\imath}\int_{|w-z_0|=r}\frac{f(w)}{w-z}dw,$$

and manipulated $\frac{1}{w-z}$ as

$$\frac{1}{w-z_0}\cdot\frac{1}{1-\frac{z-z_0}{w-z_0}}.$$

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The first integral gives rise to $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ with

$$a_n = \frac{1}{2\pi\imath}\int_{|z-z_0|=R}\frac{f(z)}{(z-z_0)^{n+1}}dz,$$

exactly as before.

In the second integral, write

$$\frac{-1}{w-z} = \frac{1}{z-z_0} \cdot \frac{1}{1-\frac{w-z_0}{z-z_0}},$$

.

Note that $|\frac{w-z_0}{z-z_0}| < 1$ for all w with $|w-z_0| = r$.

Expand to get $\displaystyle\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ with :

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We write both together as $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$. This is the Laurent series around the isolated singularity z_0. The negative part is called the principal part of the Laurent series.

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$$a_{-1} = \frac{1}{2\pi\imath} \int_{\gamma} f(z)dz.$$

If you integrate a Laurent series, only a_{-1} remains; other terms vanish. What remains is usually called a residue.

$$a_{-1} = \mathrm{Res}(f; z_0).$$

Often a_{-1} is easy to compute from $f(z)$ and if that's the case integration has become easy.

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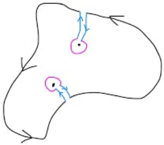
Theorem (Cauchy Residue Theorem)

$$\int_\gamma f(z)dz = 2\pi\imath \cdot \sum_{i=1}^{n} \mathrm{Res}(f, z_i).$$

Proof : We have already seen the proof in the previous lectures. The following figure should remind you of the proof. (Here the case of 1 singularity is considered; similarly one can handle more singular points)



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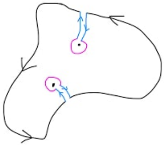

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B'coz it left a residue at every pole!

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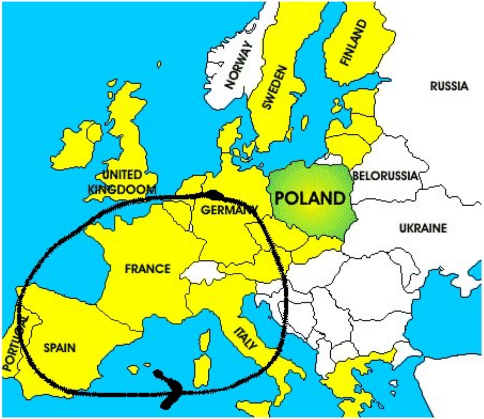

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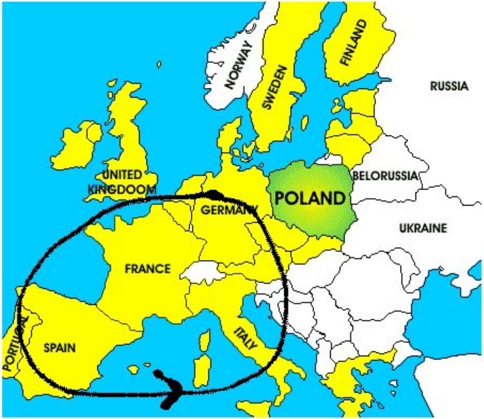

Zero. All the poles are in Eastern Europe.

Modification: Actually there are poles in Western Europe but they are all removable !!

Principal Part of the Laurent Series

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Note that:

- removable iff principal part is zero.
- pole iff principal part is finite.
- essential iff principal part is infinite.

Proof (Easy exercise using previous slides).

Residue at a Pole

If the isolated singularity is removable, then the residue is trivial. If the isolated singularity is a pole, then the residue is trivial to compute. If z_0 is a pole, can write

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \ldots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots.$$

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Thus,

$$g(z) = (z-z_0)^m f(z) = a_{-m} + \ldots + a_{-1}(z-z_0)^{m-1} + a_0(z-z_0)^m + \ldots.$$

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Thus, g is holomorphic and

$$a_{-1} = \frac{g^{(m-1)}(z_0)}{(m-1)!}.$$

Example

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$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$.

For the first term, $\frac{1}{z-2} = -\frac{1}{2}(\frac{1}{1-\frac{z}{2}}) = -\frac{1}{2}\sum_{0}^{\infty}(\frac{z}{2})^n$.

For the second term, $\frac{1}{z-1} = \frac{1}{z}(\frac{1}{1-1/z}) = \sum_{1}^{\infty}\frac{1}{z^n}$.

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Putting the two together we get the desired Laurent Series Expansion.

Example

Determine the Laurent series of $e^{1/z}$ around the point 0.

$$e^{1/z} = \sum_{0}^{\infty} \frac{1}{n!} \frac{1}{z^n}$$

$$= \sum_{-\infty}^{0} \frac{z^n}{(-n)!}$$

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$= \frac{1}{z} + \frac{1}{z^2} + \frac{5}{6}\frac{1}{z^3} + \frac{1}{2}\frac{1}{z^4} + \cdots$