MA-207 Differential Equations II
Lecture-9 Eigenvalue Problem and Fourier Expansion

M.K. Keshari

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

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S1 - Lecture 9

- **EVP 1.** $y'' + \lambda y = 0, \; y(0) = 0, \, y(L) = 0$

has infinitely many positive eigenvalues $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ for $n \geq 1$ with associated eigenfunctions

$$y_n = \sin \frac{n\pi x}{L}.$$

- **EVP 2.** $y'' + \lambda y = 0, \; y'(0) = 0, \, y'(L) = 0$

has infinitely many non-negative eigenvalues $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ for $n \geq 0$ with associated eigenfunctions

$$y_n = \cos \frac{n\pi x}{L}.$$

• EVP 3. $y'' + \lambda y = 0,\ \ y(0) = 0,\ y'(L) = 0$
has infinitely many positive eigenvalues
$\lambda_n = \left(\dfrac{(2n-1)\pi}{2L}\right)^2, \quad n = 1, 2, \ldots$
with associated eigenfunctions

$$y_n = \sin \frac{(2n-1)\pi x}{2L}.$$

• EVP 4. $y'' + \lambda y = 0,\ \ y'(0) = 0,\ y(L) = 0$
has infinitely many positive eigenvalues
$\lambda_n = \left(\dfrac{(2n-1)\pi}{2L}\right)^2, \quad n = 1, 2, \ldots$
with eigenfunctions $y_n = \cos \dfrac{(2n-1)\pi x}{2L}$.

• EVP 5. $y'' + \lambda y = 0,\ \ y(-L) = y(L),$
$y'(-L) = y'(L)$
has an eigenvalue $\lambda_0 = 0$ with eigenfunction $y_0 = 1$
and infinitely many positive eigenvalues $\lambda_n = \dfrac{n^2\pi^2}{L^2}$,
$n = 1, 2, \ldots$ with associated eigenfunctions

$$y_{1n} = \cos \frac{n\pi x}{L} \quad \text{and} \quad y_{2n} = \sin \frac{n\pi x}{L}.$$

• Eigenfunctions of EVP (1-4) are orthogonal on
$[0, L]$ w.r.t. inner product $\langle f, g\rangle = \displaystyle\int_0^L f(x)g(x)dx$
• Eigenfunctions of EVP 5 are orthogonal on
$[-L, L]$ wrt inner product $\langle f, g\rangle = \displaystyle\int_{-L}^L f(x)g(x)dx.$

Fourier Series.

Let $f \in L^2([-L, L])$ be piecewise smooth. Extend f to $\mathbb{R}$ as a periodic function of period $2L$.
The Fourier series of f is

$$F(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx, \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx, \quad n > 0$$

- $F(x) = \frac{1}{2}[f(x^+) + f(x^-)]$ for all $x \in \mathbb{R}$.

Let $f \in L^2([0, L])$ be <u>smooth</u> (for simplicity).
Extend f to a piecewise smooth function f_1 on $[-L, L]$.

Let $F(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$

be the Fourier series of $f_1(x)$.
Then restriction of F to $(0, L)$ represents f.

- In particular, if f_1 is odd, then we get Fourier sine series of f on $(0, L)$, $\quad f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$
- If f_1 is even, then we get Fourier cosine series of f on $(0, L)$, $\quad f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$

• Let $f \in L^2([0, L])$. Extend f to f_1 on $[0, 2L]$ as $f_1(x) = f(2L - x)$ for $x \in (L, 2L)$.

Extend f_1 to $[-2L, 2L]$ as an odd function and then to $\mathbb{R}$ as a periodic function of period $\underline{4L}$.

Fourier sine series of f_1 on $[0, 2L]$ is

$$F(x) = \sum_{n \geq 1} b_n \sin \frac{n\pi x}{2L}$$

$$b_n = \frac{2}{2L} \int_0^{2L} f_1(x) \sin \frac{n\pi x}{2L}\, dx$$

$$= \frac{1}{L} \int_0^{L} f(x) \sin \frac{n\pi x}{2L} dx + \frac{1}{L} \int_L^{2L} f(2L - x) \sin \frac{n\pi x}{2L} dx$$

$$\int_L^{2L} f(2L - x) \sin \frac{n\pi x}{2L} dx$$

$$(x' = 2L - x), \qquad = \int_L^{0} f(x') \sin(n\pi - \frac{n\pi x'}{2L})(-dx')$$

$$\int_0^{L} (-1)^{n+1} f(x') \sin \frac{n\pi x'}{2L}\, dx'$$

$$b_n = \frac{1}{L} \int_0^{L} f(x) \sin \frac{n\pi x}{2L} dx + \frac{1}{L} \int_0^{L} (-1)^{n+1} f(x) \sin \frac{n\pi x}{2L} dx$$

So $b_{2n} = 0$, $b_{2n-1} = \dfrac{2}{L} \displaystyle\int_0^{L} f(x) \sin \frac{(2n-1)\pi x}{2L}\, dx$.

Thus $F(x) = \displaystyle\sum_{n \geq 1} b_{2n-1} \sin \frac{(2n-1)\pi x}{2L}$.

The Mixed Fourier sine series of $f \in L^2([0,L])$ is the restriction of Fourier sine series of f_1 to $[0,L]$, i.e.

$$F(x) = \sum_{n\geq 1} c_n \sin\frac{(2n-1)\pi x}{2L}$$

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{(2n-1)\pi x}{2L}\,dx$$

This is the Fourier series of f on $[0,L]$ w.r.t. orthogonal system of eigenfunctions

$$B = \{\sin\frac{\pi x}{2L}, \sin\frac{3\pi x}{2L}, \ldots, \sin\frac{(2n-1)\pi x}{2L}, \ldots\}$$

of EVP 3 : $\boxed{y'' + \lambda y = 0, \quad y(0) = 0 = y'(L)}$.

Mixed Fourier cosine series

Let $f \in L^2([0,L])$. Extend f to f_1 on $[0,2L]$ as $f_1(x) = -f(2L-x)$ for $x \in (L,2L)$.

Extend f_1 to $[-2L,2L]$ as an even function and then to $\mathbb{R}$ as a periodic function of period $4L$. Fourier cosine series of f_1 on $[0,2L]$ is

$$F(x) = \sum_{n=1}^{\infty} d_n \cos\frac{(2n-1)\pi x}{2L},\ d_n = \frac{2}{L}\int_0^L f(x)\cos\frac{(2n-1)\pi x}{2L}\,dx$$

This is the Fourier series of f on $[0,L]$ w.r.t. orthogonal system of eigenfunctions

$$B = \{\cos\frac{\pi x}{2L}, \cos\frac{3\pi x}{2L}, \ldots, \cos\frac{(2n-1)\pi x}{2L}, \ldots\}$$

of EVP 4 : $\boxed{y'' + \lambda y = 0, \ y'(0) = 0 = y(L)}$.

A useful observation

Often we need to find Fourier expansion of polynomial functions in terms of the eigenfunctions of EVP 1-4 satisfying the boundary conditions.

We can use "derivative transfer principle" to find Fourier coefficients.

In EVP 1 with $f(0) = 0 = f(L)$, we get Fourier sine series on $[0, L]$.

$$F(x) = \sum_{n\geq 1} b_n \sin \frac{n\pi x}{L} dx$$

$$b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

$$= \frac{2}{L}\frac{L}{n\pi}\int_0^L f'(x) \cos \frac{n\pi x}{L} dx$$

$$= \frac{-2}{L}\left(\frac{L}{n\pi}\right)^2 \int_0^L f''(x) \sin \frac{n\pi x}{L} dx$$

In EVP (2) with $f'(0) = 0 = f'(L)$, we get Fourier cosine series on $[0, L]$,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad 0 \le x \le L$$

where for $n \ge 1$, $\quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$

$$= \frac{-2}{L} \frac{L}{n\pi} \int_0^L f'(x) \sin \frac{n\pi x}{L}\, dx$$

$$= \frac{-2}{L} \frac{L^2}{n^2\pi^2} \int_0^L f''(x) \cos \frac{n\pi x}{L}$$

$$= \frac{2}{L} \left(\frac{L}{n\pi} \right)^3 \int_0^L f'''(x) \sin \frac{n\pi x}{L}$$

In EVP 3 with $f(0) = 0 = f'(L)$, we get Mixed Fourier sine series on $[0, L]$.

$$F(x) = \sum_{n \ge 1} c_n \sin \frac{(2n-1)\pi x}{2L}\, dx$$

$$c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{(2n-1)\pi x}{2L}\, dx$$

$$= \frac{2}{L} \frac{2L}{(2n-1)\pi} \int_0^L f'(x) \cos \frac{(2n-1)\pi x}{2L}\, dx$$

$$= \frac{-2}{L} \left(\frac{2L}{(2n-1)\pi} \right)^2 \int_0^L f''(x) \sin \frac{(2n-1)\pi x}{2L}\, dx$$

In EVP 4 with $f'(0) = 0 = f(L)$, we get Mixed Fourier cosine series on $[0, L]$.

$$F(x) = \sum_{n\geq 1} d_n \cos \frac{(2n-1)\pi x}{2L} \, dx$$

$$d_n = \frac{-2}{L} \int_0^L f(x) \cos \frac{(2n-1)\pi x}{2L} \, dx$$

$$= \frac{2}{L} \frac{2L}{(2n-1)\pi} \int_0^L f'(x) \sin \frac{(2n-1)\pi x}{2L} \, dx$$

$$= \frac{-2}{L} \left(\frac{2L}{(2n-1)\pi} \right)^2 \int_0^L f''(x) \cos \frac{(2n-1)\pi x}{2L} \, dx$$

Example. Find the Fourier cosine expansion of

$$f(x) = x^2(3L - 2x) \quad \text{on} \quad [0, L]$$

$$a_0 = \frac{1}{L} \int_0^L (3Lx^2 - 2x^3) \, dx$$

$$= \frac{1}{L} \left(Lx^3 - \frac{x^4}{2} \right) \Big|_0^L$$

$$= \frac{L^3}{2}$$

Evaluating a_n directly is laborius.

$$f'(x) = 6Lx - 6x^2 \implies f'(0) = f'(L) = 0$$

Note $f'''(x) = -12$. We get

$$a_n = \frac{2}{L}\left(\frac{L}{n\pi}\right)^3 \int_0^L f'''(x)\sin\frac{n\pi x}{L}\,dx$$
$$= \frac{-24}{L}\left(\frac{L}{n\pi}\right)^3 \int_0^L \sin\frac{n\pi x}{L}\,dx$$
$$= \frac{24}{L}\left(\frac{L}{n\pi}\right)^4 \cos\frac{n\pi x}{L}\Big|_0^L = \frac{24L^3}{n^4\pi^4}\left[(-1)^n - 1\right]$$

Thus $a_{2n} = 0$ and $a_{2n-1} = \dfrac{-48L^3}{(2n-1)^4\pi^4}$.

Thus Fourier cosine expansion of $f(x)$ on $[0, L]$ is

$$\frac{L^3}{2} - \frac{48L^3}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\cos\frac{(2n-1)\pi x}{L}$$

Example. Find the Fourier sine expansion of

$$f(x) = x(x^2 - 3Lx + 2L^2) \quad \text{on} \quad [0, L]$$

Note $f(0) = 0 = f(L)$, $f''(x) = 6(x - L)$. Fourier sine coefficient

$$b_n = \frac{-2}{L}\left(\frac{L}{n\pi}\right)^2 \int_0^L f''(x)\sin\frac{n\pi x}{L}\,dx$$
$$= \frac{-12L}{n^2\pi^2}\int_0^L (x - L)\sin\frac{n\pi x}{L}\,dx$$
$$= \frac{12L^2}{n^3\pi^3}\left[(x-L)\cos\frac{n\pi x}{L}\Big|_0^L - \int_0^L \cos\frac{n\pi x}{L}\,dx\right]$$
$$= \frac{12L^2}{n^3\pi^3}\left[L - \frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_0^L\right] = \frac{12L^3}{n^3\pi^3}$$

Therefore, the Fourier sine expansion of $f(x)$ on $[0, L]$ is

$$\frac{12L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi x}{L} \qquad \square$$

Example. Find the mixed Fourier cosine expansion of $f(x) = 3x^3 - 4Lx^2 + L^3$ on $[0, L]$

Soln. $f'(0) = 0 = f(L)$, $f''(x) = 2(9x - 4L)$, we get

$$d_n = \frac{-2}{L}\left(\frac{2L}{(2n-1)\pi}\right)^2 \int_0^L f''(x)\cos\frac{(2n-1)\pi x}{2L}\,dx$$

$$= \frac{-16L}{(2n-1)^2\pi^2}\int_0^L (9x-4L)\cos\frac{(2n-1)\pi x}{2L}\,dx$$

$$= \frac{-32L^2}{(2n-1)^3\pi^3}\Big[(9x-4L)\sin\frac{(2n-1)\pi x}{2L}\Big|_0^L$$

$$-9\int_0^L \sin\frac{(2n-1)\pi x}{2L}\,dx\Big]$$

$$= \frac{-32L^2}{(2n-1)^3\pi^3}\left[(9x-4L)\sin\frac{(2n-1)\pi x}{2L}\Big|_0^L - 9\int_0^L \sin\frac{(2n-1)\pi x}{2L}\right]$$

$$= \frac{-32L^2}{(2n-1)^3\pi^3}\left[(-1)^{n+1}5L + \frac{18L}{(2n-1)\pi}\cos\frac{(2n-1)\pi x}{2L}\Big|_0^L\right]$$

$$= \frac{-32L^3}{(2n-1)^3\pi^3}\left[(-1)^{n+1}5 - \frac{18}{(2n-1)\pi}\right]$$

Therefore, the Mixed Fourier cosine expansion of $f(x)$ on $[0, L]$ is

$$\frac{32L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[(-1)^n 5 + \frac{18}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi x}{2L}$$

Example Find the mixed Fourier sine expansion of

$$f(x) = x(2x^2 - 9Lx + 12L^2) \quad \text{on} \quad [0, L]$$

Since $f(0) = 0 = f'(L)$ and $f''(x) = 6(2x - 3L)$, we get

$$c_n = \frac{-2}{L}\left(\frac{2L}{(2n-1)\pi}\right)^2 \int_0^L f''(x)\sin\frac{(2n-1)\pi x}{2L}\,dx$$

$$= \frac{-48L}{(2n-1)^2\pi^2}\int_0^L (2x-3L)\sin\frac{(2n-1)\pi x}{2L}\,dx$$

$$= \frac{96L^2}{(2n-1)^3\pi^3}\left[(2x-3L)\cos\frac{(2n-1)\pi x}{2L}\Big|_0^L\right.$$

$$\left. -2\int_0^L \cos\frac{(2n-1)\pi x}{2L}\,dx\right]$$

$$= \frac{96L^2}{(2n-1)^3\pi^3}\left[3L - \frac{4L}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{2L}\Big|_0^L\right]$$

$$= \frac{96L^3}{(2n-1)^3\pi^3}\left[3 - \frac{4(-1)^{n-1}}{(2n-1)\pi}\right]$$

Therefore, the mixed Fourier sine expansion of $f(x)$ on $[0, L]$ is

$$\frac{96L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[3 + \frac{4(-1)^n}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi x}{2L}$$

Now we will start the study of Partial differential equations.

A partial differential equation (PDE) is an equation for an unknown function u that involves independent variables $x, y, \ldots$, the function u and the partial derivatives of u.
The order of the PDE is the order of the highest partial derivative of u in the equation.
Examples of some famous PDEs.

1. $u_t - k(u_{xx} + u_{yy}) = 0$
   two dimensional Heat equation, order 2.
2. $u_{tt} - c^2(u_{xx} + u_{yy}) = 0$
   two dimensional wave equation, order 2.
3. $u_{xx} + u_{yy} = 0$
   two dimensional Laplace equation, order 2.
4. $u_{tt} + u_{xxxx}$ Beam equation, order 4.

Examples of non-famous PDE's (I made it up).

1. $u_x + \sin(u_y) = 0$, order 1.
2. $3x^2 \sin(xy) e^{-xy^2} u_{xx} + \log(x^2 + y^2) u_y = 0$,
   order 2.

Definition. A PDE is said to be linear if it is linear in u and its partial derivatives i.e. it is a degree 1 polynomial in u and its partial derivatives.
Examples. Heat equation, Wave equation, Laplace equation and Beam equation are linear PDEs.
First one above is non-linear and 2nd one is linear.
• The general form of first order linear PDE in two variables x, y is

$$A(x,y)u_x + B(x,y)u_y + C(x,y)u = f(x,y)$$

• The general form of first order linear PDE in three variables x, y, z is

$$Au_x + Bu_y + Cu_z + Du = f$$

where coefficients A, B, C, D and f are functions of x, y and z.

• The general form of second order lineat PDE in two variables x, y is

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = f$$

where coefficients A, B, C, D, E, F and f are functions of x and y.

• When $A \ldots, F$ are all constants, then it is called a linear PDE with constant coefficients.

Linear Partial Differential Operator

Second order linear PDE in two variable can be written as $Lu = f$, where

$$L = A\frac{\partial^2}{\partial x^2} + 2B\frac{\partial^2}{\partial x \partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F$$

is the linear differential operator. It is called linear since the map $u \mapsto Lu$ is a linear map.

Examples. Laplace operator in $\mathbb{R}^2$ is

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Heat and Wave operator in one space variable are

$$H = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}, \qquad \Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$$

Classification of second order linear PDE

Consider the linear differential operator L in $\mathbb{R}^2$.

$$L = A\frac{\partial^2}{\partial x^2} + 2B\frac{\partial^2}{\partial x \partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F$$

where $A, \ldots, F$ are functions of x and y.

To the operator L, we associate the discriminant $\mathbb{D}(x, y)$ given by

$$\mathbb{D}(x, y) = A(x, y)C(x, y) - B^2(x, y)$$

The operator L or the PDE $Lu = f$ is said to be

- elliptic at (x_0, y_0), if $\mathbb{D}(x_0, y_0) > 0$,
- hyperbolic at (x_0, y_0), if $\mathbb{D}(x_0, y_0) < 0$,
- parabolic at (x_0, y_0), if $\mathbb{D}(x_0, y_0) = 0$.

If L is elliptic at each point (x, y) in a domain $\Omega \subset \mathbb{R}^2$, then L is called elliptic in Ω.

Similarly for hyperbolic and parabolic. Recall

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad H = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}, \quad \Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$$

- Two dimensional Laplace operator Δ is elliptic in $\mathbb{R}^2$, since $\mathbb{D} = 1$.
- One dimensional Heat operator H is parabolic in $\mathbb{R}^2$, since $\mathbb{D} = 0$.
- One dimensional Wave operator $\Box$ is hyperbolic in $\mathbb{R}^2$, since $\mathbb{D} = -1$.