MA 205 Complex Analysis: Exponential Function

July 27, 2017

Cauchy-Maclaurin intergral test

Consider a eventually non-negative, continuous function f defined on the unbounded interval $[1, \infty)$, on which it is eventually monotone decreasing. Then the infinite series $\sum_1^\infty f(n)$ converges if and only the improper integral $\int_1^\infty f(x)dx$ exists (i.e, is finite). If the integral diverges then the series diverges as well.
This test is used for instance to show thre divergence of the harmonic series $\sum_1^\infty 1/n$. It also shows the convergence of the series $\sum_1^\infty 1/n^\alpha$ for any $\alpha > 1$.

Introduction

Today, we'll use our knowledge of power series to construct a few basic functions. Before that let's first recall that the derivative of a constant function is zero. On the other hand, if the derivative is zero throughout its domain, can we say that the function has to be a constant? This is certainly true for functions of a real variable

Introduction

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Now consider the power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots .$$

We've seen that its radius of convergence is ∞; i.e., this function is well-defined for any $z \in \mathbb{C}$. This function will keep our company throughout this course. So let's befriend it a bit more !

Exponential Function

If

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots,$$

then $f(0) = 1$. By term by term differentiation, one observes that

$$f'(z) = f(z).$$

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We'll denote this function by $\exp(z)$, also denoted e^z. Note that

$$(\exp(bz))' = b\exp(bz).$$

Now consider the function

$$h(z) = \exp(z) \cdot \exp(-z).$$

This is defined throughout $\mathbb{C}$. What's $h'(z)$?

$$h'(z) = \exp(z) \cdot (-\exp(-z)) + \exp(-z) \cdot \exp(z) = -h + h = 0.$$

Therefore $h(z) \equiv c$ and since $h(0) = 1$, it is identically equal to 1. Thus, we have proved two things:

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Thus, we have proved two things:
(i) $\exp(z)$ is never vanishing.
(ii) $\exp(-z) = \frac{1}{\exp(z)}$.

Exponential Function

Note that the derivative of $f(z) = a\exp(bz)$ is $f'(z) = bf(z)$. Interestingly, the converse is also true. Thus,

$$f(z) = a\exp(bz) \text{ for } a, b \in \mathbb{C} \iff f'(z) = bf(z).$$

Proof: Assume $f'(z) = bf(z)$ for $b \in \mathbb{C}$. Now consider

$$h(z) = f(z)\exp(-bz).$$

Then, $h'(z) = -bh + bh = 0$, for all z in the domain. So, $h(z) \equiv a$ for some $a \in \mathbb{C}$. Therefore,

$$f(z) = \frac{a}{\exp(-bz)} = a\exp(bz),$$

by what we already know.

Corollary: $f' = f$ and $f(0) = 1$ characterizes the exponential function. The function

$$f(z) = e^x(\cos y + \imath \sin y),$$

is holomorphic throughout $\mathbb{C}$ and $f' = f$. Clearly, $f(0) = 1$ as well. Thus,

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Remark: e^x here is e to the power of x, and e is the number that you know from MA 105 (base of natural logarithm). We'll try and reconstruct everything about logarithm and exponential function from scratch.

Exponential Function

By now we know that exp is defined throughout $\mathbb{C}$ and that 0 is not in the range of $\exp(z)$. , exp is a map from $\mathbb{C} \to \mathbb{C}^{\times}$. Now exp has this wonderful property that it takes the "correct" operation in $\mathbb{C}$ to the "correct" operation in $\mathbb{C}^{\times}$.

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Thus in the language of group theory exp is a homomorphism from $\mathbb{C}$ to $\mathbb{C}^{\times}$.

Proof: Fix $w \in \mathbb{C}$. Then the function $f(z) = \exp(w+z)$ is holomorphic in $\mathbb{C}$ and $f'(z) = f(z)$. So, $f(z) = a\exp(z)$ for some constant a. By evaluating f at 0, see that $a = \exp(w)$. Thus, $f(z) = \exp(w) \cdot \exp(z)$.

Euler (1707-1783); Wiki

In particular:

$$\exp(\imath\theta) = \cos\theta + \imath\sin\theta$$

and

$$\exp(nz) = \exp(z)^n.$$

euler.png

Euler later developed a cataract in his left eye, which was discovered in 1766. Just a few weeks after its discovery, he was rendered almost totally blind. However, his condition appeared to have little effect on his productivity, as he compensated for it with his mental calculation skills and exquisite memory. For example, Euler could repeat the Aeneid of Virgil from beginning to end without hesitation, and for every page in the edition he could indicate which line was the first and which the last. With the aid of his scribes, Euler's productivity on many areas of study actually increased. He produced on average, one mathematical paper every week in the year 1775.

"Read Euler, read Euler, he is the master of us all." (Laplace)

This property of converting addition into multiplication also characterizes $\exp(z)$.

Let $0 \in \Omega$. Suppose $f : \Omega \to \mathbb{C}$ is such that f is differentiable at 0 and $f(0) \neq 0$. Suppose $f(w+z) = f(w)f(z)$ whenever $w, z, w+z \in \Omega$. Then, $f(z) = \exp(bz)$, where $b = f'(0)$.

Now we define e as follows:

$$e = \exp(1).$$

It is an irrational number (slightly hard exercise !) and its approx. value is 2.718. A much deeper fact that e is not the root of any polynomial with rational coefficients. It is an open problem whether $e + \pi$ is irrational !

One can then define e^x for any real number; first for rational powers and then for arbitrary real powers by taking limits. One can then show that this deinition of e^x coincides with the definition of e^x as defined using the power series $1 + x + x^2/2! + \dots$.

Trigonometric Functions

Recall the Taylor expansions:

$$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots = \frac{\exp(\imath y) - \exp(-\imath y)}{2\imath}$$

$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \ldots = \frac{\exp(\imath y) + \exp(-\imath y)}{2}.$$

Motivated by this, we define complex trigonometric functions:

$$\sin z = \frac{\exp(\imath z) - \exp(-\imath z)}{2\imath} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots$$

$$\cos z = \frac{\exp(\imath z) + \exp(-\imath z)}{2} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots.$$

Define hyperbolic sine and hyperbolic cosine by:

$cos(hz) = (e^z + e^{-z})/2$. Its power series is given by

$$coz(hz) = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$$

$sin(hz) = (e^z - e^{-z})/2$. Its power series is given by

$$sin(hz) = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

Cx means complex plane minus 0,0

Exercise:

(i) Define other trigonometric functions.

(ii) Show that $\exp : \mathbb{C} \to \mathbb{C}^{\times}$ is onto. Is it one-to-one ? (Show that $e^z = e^{(z+2\pi i)}$

(iii) Show that $\sin, \cos : \mathbb{C} \to \mathbb{C}$ are surjective. In particular, note the difference with real sine and cosine which were bounded by 1.

(iv) Show that $\sin^2 z + \cos^2 z = 1$,

$\sin(z + w) = \sin z \cos w + \cos z \sin w$,

$\cos(z + w) = \cos z \cos w - \sin z \sin w$.

Logarithms cont...

We have seen that e^z is not a 1-1 function. Hence its inverse is not defined everywhere. Nevertheless we would like to construct an analytic inverse function, called logarithm on certain subsets, i.e, a analytic function $w = log(z)$ such that $e^w = z$. As remarked before, log will be undefined at 0.

Logarithms cont...

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Definition

Let $\Omega \subseteq \mathbb{C}$ be a domain. Let $f(z)$ be a continuous function on Ω such that $\exp(f(z)) = z \ \forall z \in \Omega$. Then f is called a branch of the logarithm.

Lemma

Let $\Omega \subseteq \mathbb{C}$ be a domain and let f be a branch of the logarithm. Then any other branch of the logarith differs from f by a constant multiple multiple of $2\pi i$.

Proof.

Let $g(z)$ be a branch of the logarithm. Then $f(z) - g(z)$ is a constant multiple of $2\pi i$ for all $z \in \Omega$. Since Ω is connected, and $f(z) - g(z)$ is continuous while integral multiples of $2\pi i$ is a discrete set, it follows that $f(z) - g(z)$ is a constant multiple of $2\pi i$

$\square$

Logarithm

We will usually work with a fixed branch of the logarithm called the **Principal Branch**. This is defined as follows:

Let $\Omega \subset \mathbb{C}$ be the open subset defined by $\mathbb{C}$ minus the negative real line. For any $z \in \Omega$, $z = |z|e^{i\theta} : -\pi < \theta < \pi = re^{i\theta}$, define $f(z) = logr + i\theta = log|z| + iArg(z)$.

One checks that $f(z)$ is a branch of $log(z)$ on Ω. In fact, f is analytic and $f'(z) = \frac{1}{z}$.

Zero's of Homolorphic Functions

Let Ω be a domain in $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be a complex analytic function defined on Ω. This means f can be expressed by a power series expanded around any point in Ω.

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$$f(z) = \frac{z^{n+1}}{(n+1)!} f^{(n+1)}(0) + \frac{z^{n+2}}{(n+2)!} f^{(n+2}(0) + \ldots$$

$$= z^{n+1}\left(\frac{f^{(n+1)}(0)}{(n+1)!} + \frac{z}{(n+2)!} f^{n+2}(0) + \ldots\right)$$

Now as $z \to 0$ along the sequence $\{z_n\}$, we see that the lhs is identically zero. Hence the rhs also vanishes identically along this sequence. Hence the term inside the bracket vanishes along $\{z_n\}$ and hence by continuity,vanishes at the limit, namely 0, thereby showing that $f^{(n+1)}(0) = 0$. This contradicts the assumption on n.

Note that this property fails for real differentiable functions. It however works equally wekk for real analytic functions as well (same proof as above).The function

$$f(x) = e^{-1/x} \text{ for } x > 0$$

$$= 0 \text{ for } x < 0$$

is infinitely differentiable but vanishes along the entire negative real line.

Once we prove that holomorphic functions are analytic, the above fact will hold for holomorphic functions as well.

This property allows us to prove identities like $sin^2(z) + cos^2(z) = 1$. (Why ?)