MA-207 Differential Equations II

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Theorem

Fix $p \geq 0$ *and* $Z^{(p)} = \{\lambda_{p,1}, \lambda_{p,2}, \ldots\}$ *: zeros of* $J_p(x)$ *on* $(0, \infty)$*. Any square-integrable function* $f(x)$ *on* $[0, 1]$ *can be expanded in a series of scaled Bessel functions* $J_p(\lambda_{p,n} x)$ *as*

$$f(x) = \sum_{n \geq 1} c_n J_p(\lambda_{p,n} x)$$

where

$$c_n = \frac{2}{[J_{p+1}(\lambda_{p,n})]^2} \int_0^1 x\, f(x)\, J_p(\lambda_{p,n} x)\, dx$$

This is Fourier-Bessel series of $f(x)$ *for parameter* p*.*

Proof of orthogonality of scaled Bessel functions

If a, b are positive scalars, then $u(x) = J_p(ax)$ and $v(x) = J_p(bx)$ satisfies

$$u'' + \frac{1}{x}u' + \left(a^2 - \frac{p^2}{x^2}\right)u = 0$$

$$v'' + \frac{1}{x}v' + \left(b^2 - \frac{p^2}{x^2}\right)v = 0$$

Multiply by v and u resp. and subtract, we get

$$(vu'' - uv'') + \frac{1}{x}(vu' - uv') + (a^2 - b^2)uv = 0$$

$$(u'v - v'u)' + \frac{1}{x}(u'v - v'u) = (b^2 - a^2)uv$$

$$(x(u'v - v'u))' = (b^2 - a^2)xuv$$

$$(b^2 - a^2)\int_0^1 xuv\,dx = [x(u'v - v'u)]\Big|_0^1 = (u'v - v'u)(1)$$

$$(b^2 - a^2)\int_0^1 xJ_p(ax)J_p(bx)\,dx = J_p'(a)J_p(b) - J_p'(b)J_p(a)$$

So if $a = \lambda_{p,k}$ and $b = \lambda_{p,l}$ are **distinct**, then

$$\int_0^1 xJ_p(\lambda_{p,k}x)J_p(\lambda_{p,l}x)\,dx = 0$$

To compute the norm of $J_p(\lambda_{p,k}x)$, consider

$$\begin{aligned} 2x^2u'\left[u'' + \frac{1}{x}u' + (a^2 - \frac{p^2}{x^2})u\right] &= 0 \\ = [x^2u'^2 + (a^2x^2 - p^2)u^2]' - 2a^2xu^2 \end{aligned}$$

Integrate on $[0,1]$,

$$2a^2 \int_0^1 xu^2\, dx = [x^2u'^2 + (a^2x^2 - p^2)u^2]\Big|_0^1$$

Since $p \geq 0$, $(pu(0))^2 = (pJ_p(0))^2 = 0$.

Thus, $\big(x^2u'^2 + (a^2x^2 - p^2)u^2\big)(0) = 0$.

Further, $u'(1) = aJ_p'(a)$, so we get

$$\big(x^2u'^2 + (a^2x^2 - p^2)u^2\big)(1) = a^2 J_p'(a)^2 + (a^2 - p^2)J_p(a)^2$$

Put $a = \lambda_{p,k}$ to get

$$2\lambda_{p,k}^2 \int_0^1 xJ_p(\lambda_{p,k}x)^2\, dx = \lambda_{p,k}^2 J_p'(\lambda_{p,k})^2$$

Thus,

$$\int_0^1 xJ_p(\lambda_{p,k}x)^2\, dx = \frac{1}{2}J_p'(\lambda_{p,k})^2 = \frac{1}{2}J_{p+1}(\lambda_{p,k})^2$$

for last equality, use $x = \lambda_{p,k}$ in $J_p'(x) - \dfrac{p}{x}J_p(x) = J_{p+1}(x)$

Eigen Value problems $y'' + \lambda y = 0$

We will develop Fourier series representations of functions that will be used to solve PDE considered later.

Consider the following Boundary Value Problems (BVP), where $\lambda \in \mathbb{R}$ and $L > 0$.

1. Problem 1. $y'' + \lambda y = 0 \qquad y(0) = 0, \qquad y(L) = 0.$
2. Problem 2. $y'' + \lambda y = 0 \qquad y'(0) = 0, \qquad y'(L) = 0.$
3. Problem 3. $y'' + \lambda y = 0 \qquad y(0) = 0, \qquad y'(L) = 0.$
4. Problem 4. $y'' + \lambda y = 0 \qquad y'(0) = 0, \qquad y(L) = 0.$
5. Problem 5. $y'' + \lambda y = 0 \quad y(-L) = y(L),\ y'(-L) = y'(L).$

The boundary condition in problem 5 is called periodic.

Eigenvalue problem $y'' + \lambda y = 0$

Question. For what values of λ does the problem have a non-trivial solutions and what are the solutions?

Any λ for which the problem (1-5) has a non-trivial solution is called an eigenvalue of that problem

Non-trivial solutions for an eigenvalue λ are called λ-eigenfunction, or eigenfunction associated with λ.

A non-zero constant multiple of a λ-eigenfunction is again a λ-eigenfunction.

Problems $1 - 5$ are called eigenvalue problems. Solving an eigenvalue problem means finding all its eigenvalues and associated eigenfunctions.

Theorem

1. *Problems* $1-5$ *have no negative eigenvalues.*
2. $\lambda = 0$ *is an eigenvalue of Problems* 2 *and* 5 *with associated eigenfunctions* $y_0 = 1$.
3. $\lambda = 0$ *is not an eigenvalue of Problems* $1, 3$ *and* 4.

Proof.

Let us prove first two; third is left as an exercise.

Suppose $\lambda < 0$. Let us write $\lambda = -a^2$.

Rewrite the differential equation as $y'' = a^2 y$. The general solution to this is $y(x) = Ce^{ax} + De^{-ax}$. In problem 1 we have the condition $y(0) = y(L) = 0$. This forces that $C + D = 0$ and $Ce^{aL} + De^{-aL} = 0$. One checks easily that this forces $C = D = 0$.

In problem 2 we have the condition that $y'(0) = y'(L) = 0$. This gives $aC - aD = 0$ and $aCe^{aL} - aDe^{-aL} = 0$. Since $a \neq 0$, this forces $C = D = 0$.

Proof.

In problem 3 we have the conditions $y(0) = y'(L) = 0$. This gives $C + D = 0$ and $aCe^{aL} - aDe^{-aL} = 0$. Again this forces $C = D = 0$.

Similarly, do the other problems.

Now consider the second statement in the theorem. If $\lambda = 0$, the clearly, the solution has to be of the form $y(x) = ax + b$.

In problem 2 we have $y'(0) = y'(L) = 0$, and so $a = 0$. Thus, $y(x) = \text{constant}$ is the solution in this case.

In problem 5, we have $y(-L) = y(L)$, that is, $-aL + b = aL + b$. This forces that $a = 0$. Thus, in this case too $y(x) = \text{constant}$. □

Eigenvalue Problem 1

Theorem

The eigenvalue problem

$$y'' + \lambda y = 0 \quad y(0) = 0, \quad y(L) = 0$$

has infinitely many positive eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

with associated eigenfunctions

$$y_n = \sin\frac{n\pi x}{L}, \quad n = 1, 2, \ldots.$$

There are no other eigenvalues.

Proof.

Any eigen value must be positive (by previous theorem).

If y is a solution of $y'' + \lambda y = 0$ with $\lambda > 0$, then

$$y(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x$$

$$y(0) = 0 \implies c_1 = 0$$

$$\implies y(x) = c_2 \sin \sqrt{\lambda} x \quad \text{with} \quad c_2 \neq 0$$

$$y(L) = 0 \implies \sin \sqrt{\lambda} L = 0 \implies \sqrt{\lambda} L = n\pi$$

$$\implies \lambda_n = \frac{n^2 \pi^2}{L^2}$$

is an eigenvalue with an associated eigenfunction

$$y_n = \sin \frac{n \pi x}{L}$$

□

Eigenvalue Problem 2

Theorem

The eigenvalue problem

$$y'' + \lambda y = 0 \quad y'(0) = 0, \quad y'(L) = 0$$

has an eigenvalue $\lambda_0 = 0$ *with eigenfunction* $y_0 = 1$

and infinitely many positive eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

with associated eigenfunctions

$$y_n = \cos\frac{n\pi x}{L} \quad n = 1, 2, \ldots.$$

There are no other eigenvalues.

Proof. Similar to the proof of Problem 1, hence is left as an exercise.

Eigenvalue Problem 3

Theorem

The eigenvalue problem

$$y'' + \lambda y = 0 \quad y(0) = 0, \quad y'(L) = 0$$

has infinitely many positive eigenvalues

$$\lambda_n = \frac{(2n+1)^2\pi^2}{4L^2}$$

with associated eigenfunctions

$$y_n = \sin\frac{(2n+1)\pi x}{2L}, \quad n = 0, 1, 2, \ldots.$$

There are no other eigenvalues.

$y'' + \lambda y = 0 \quad y(0) = 0, \quad y'(L) = 0$

Proof.

Any eigen value must be positive (by previous theorem).

If y is a solution of $y'' + \lambda y = 0$ with $\lambda > 0$, then

$$y(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x$$

$$y(0) = 0 \implies c_1 = 0$$

$$\implies y(x) = c_2 \sin \sqrt{\lambda} x \quad \text{with} \quad c_2 \neq 0$$

$$y'(L) = 0 \implies \sqrt{\lambda} \cos \sqrt{\lambda} L = 0 \implies \sqrt{\lambda} L = \tfrac{2n+1}{2}\pi$$

$$\implies \lambda_n = \frac{(2n+1)^2 \pi^2}{4L^2}$$

is an eigenvalue with an associated eigenfunction

$$y_n = \sin \frac{(2n+1)\pi x}{2L}$$

□

Orthogonality

Definition

We say two integrable functions f and g are orthogonal on an interval $[a, b]$ if

$$\int_a^b f(x)g(x)\,dx = 0$$

More generally, we say functions $\phi_1, \phi_2, \ldots, \phi_n, \ldots$ (finite or infinitely many) are orthogonal on $[a, b]$ if

$$\int_a^b \phi_i(x)\phi_j(x)\,dx = 0 \quad \text{whenever} \quad i \neq j$$

We have already seen orthogonality of Legendre function.
We will study Fourier series w.r.t. different orthogonal systems.

Exercise

Consider the eigenfunctions

1. $\sin\dfrac{\pi x}{L}, \sin\dfrac{2\pi x}{L}, \dots, \sin\dfrac{n\pi x}{L}, \dots$
2. $1, \cos\dfrac{\pi x}{L}, \cos\dfrac{2\pi x}{L}, \dots, \cos\dfrac{n\pi x}{L}, \dots$
3. $\sin\dfrac{\pi x}{2L}, \sin\dfrac{3\pi x}{2L}, \dots, \sin\dfrac{(2n-1)\pi x}{2L}, \dots$
4. $\cos\dfrac{\pi x}{2L}, \cos\dfrac{3\pi x}{2L}, \dots, \cos\dfrac{(2n-1)\pi x}{2L}, \dots$
5. $1, \cos\dfrac{\pi x}{L}, \sin\dfrac{\pi x}{L}, \cos\dfrac{2\pi x}{L}, \sin\dfrac{2\pi x}{L}, \dots, \cos\dfrac{n\pi x}{L}, \sin\dfrac{n\pi x}{L}, \dots$

Show directly that eigenfunctions of (1-4) are orthogonal on $[0, L]$ and of (5) is orthogonal on $[-L, L]$.

We will study series expansions in terms of eigenfunctions. It is used to solve PDEs.

For this we consider the vector space of functions on $[a, b]$ and define an inner product on it

$$\langle f, g\rangle := \int_a^b f(x)g(x)dx$$

Denote by $L^2[a, b]$ the subspace of those functions satisfying $\langle f, f\rangle < \infty$.

To say this is a subspace, one needs to check that if $f, g \in L^2[a, b]$ then $f + g \in L^2[a, b]$. We shall assume this fact.

From now on, we will always be working with functions in some inner product space of the type $L^2[a, b]$. In such a space, the norm of a function is defined to be $\|f\| := \langle f, f\rangle^{1/2}$.

Theorem

Let $f \in L^2[-L, L]$. Then f can be written as a series

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

which is called the Fourier series of f on $[-L, L]$. Here

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx$$

The above series converges to f in norm, that is,

$$\lim_{N \longrightarrow \infty} \left\| f - a_0 - \sum_{n=1}^{N} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \right\| = 0$$

We remark that the formula for the coefficients a_m's can be obtained easy by integrating $f(x)$ with $\cos\frac{m\pi x}{L}$ on $[-L, L]$, and using the facts that (1) we can exchange the integral and the sum, and (2) orthogonality of the different eigenfunctions.

$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx = \int_{-L}^{L}\cos\frac{m\pi x}{L} a_0 +$$

$$+\int_{-L}^{L}\cos\frac{m\pi x}{L}\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$$

$$= \int_{-L}^{L}\cos\frac{m\pi x}{L} a_0 + \sum_{n=1}^{\infty} a_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L} +$$

$$b_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}$$

$$= a_m\int_{-L}^{L}\cos^2\frac{m\pi x}{L}dx$$

Convergence of Fourier series

Qn. What about the convergence of series to $f(x)$?

Definition

A function f is said to be piecewise smooth if

1. f has atmost finitely many points of discontinuity.
2. f' exists and is continuous except at finitely many points.
3. $f(x_0+) = \lim_{x \to x_0^+} f(x)$ and $f'(x_0+) = \lim_{x \to x_0^+} f'(x)$ exists if $a \le x_0 < b$.
4. $f(x_0-) = \lim_{x \to x_0^-} f(x)$ and $f'(x_0-) = \lim_{x \to x_0^-} f'(x)$ exists if $a < x_0 \le b$.

Hence f is piecewise smooth if and only if
f, f' have atmost finitely many jump discontinuity.

Theorem

Let $f(x)$ be a piecewise smooth on $[-L, L]$.
Extend it to all of $\mathbb{R}$ by defining it periodically, that is,
$f(x + 2L) = f(x)$.
Then the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

of f converges to

$$\frac{1}{2}[f(x^+) + f(x^-)]$$

at every point $x \in \mathbb{R}$.
Therefore, at every point x of continuity of f, the Fourier series converges to $f(x)$.

If we re-define $f(x)$ at every point of discontinuity x as

$$\frac{1}{2}[f(x^+) + f(x^-)]$$

then the Fourier series represents the function everywhere.
Thus two functions can have same Fourier series.

Let us now consider a function f such that f has only jump discontinuities, and if x is a such a point of jump discontinuity then $f(x) = \frac{f(x^+)+f(x^-)}{2}$.

The previous theorem tells us that the Fourier series converges to $f(x)$ for all $x \in [-L, L]$,
we may be tempted to infer that the error

$$E_N(x) = \left| F(x) - a_0 - \sum_{n=1}^{N} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \right|$$

can be made as small as we want, for all $x \in [-L, L]$ by choosing N sufficiently large.

However this is NOT true if

- f is discontinuous at some point $\alpha \in (-L, L)$ or
- $f(-L+) \neq f(L-)$

The next result explains this.

Gibbs phenomenon

• If f has a jump discontinuity at $\alpha \in (-L, L)$, then there exists sequence of points $u_N \in (-L, \alpha)$ and $v_N \in (\alpha, L)$ s.t.

$$\lim_{N\to\infty} u_N = \alpha, \quad E_N(u_N) \simeq .09\,|f(\alpha-) - f(\alpha+)|$$

$$\lim_{N\to\infty} v_N = \alpha, \quad E_N(v_N) \simeq .09\,|f(\alpha-) - f(\alpha+)|$$

Maximum of error $E_N(x) \not\to 0$ near α as $N \to \infty$.

• If $f(-L+) \neq f(L-)$, there exists u_N and v_N in $(-L, L)$ s.t.

$$\lim_{N\to\infty} u_N = -L, \qquad E_N(u_N) \simeq .09\,|f(-L+) - f(L-)|$$

$$\lim_{N\to\infty} v_N = \alpha = L, \qquad E_N(v_N) \simeq .09\,|f(-L+) - f(L-)|$$

This is called Gibbs phenomenon.

Example

Let us find the Fourier series of the piecewise smooth function

$$f(x) = \begin{cases} -x, & -2 < x < 0 \\ 1/2, & 0 < x < 2 \end{cases}$$

on $[-2, 2]$.

$$a_0 = \frac{1}{4}\int_{-2}^{2} f(x)\,dx = \frac{1}{4}\left[\int_{-2}^{0}(-x)\,dx + \int_0^2 \frac{1}{2}\,dx\right] = \frac{3}{4}$$

If $n \geq 1$, then

$$\begin{aligned} a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx \\ &= \frac{1}{2}\left[\int_{-2}^{0}(-x)\cos\frac{n\pi x}{2}\,dx + \int_0^2 \frac{1}{2}\cos\frac{n\pi x}{2}\,dx\right] \end{aligned}$$

Example (continued ...)

$$
\begin{aligned}
&= \frac{1}{2}\left[-x\frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_{-2}^{0} + \int_{-2}^{0}\frac{2}{n\pi}\sin\frac{n\pi x}{2}\,dx + 0\right] \\
&= \frac{1}{2}\frac{4}{n^2\pi^2}\left(-\cos\frac{n\pi x}{2}\right)\Big|_{-2}^{0} \\
&= \frac{2}{n^2\pi^2}(\cos n\pi - 1) \\
b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx \\
&= \frac{1}{2}\left[\int_{-2}^{0}(-x)\sin\frac{n\pi x}{2}\,dx + \int_{0}^{2}\frac{1}{2}\sin\frac{n\pi x}{2}\,dx\right] \\
&= \frac{1}{2n\pi}(1 + 3\cos n\pi)
\end{aligned}
$$

Example (continued ...)

Thus, the Fourier series of $f(x)$ is

$$F(x) = \frac{3}{4} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos n\pi - 1}{n^2} \cos \frac{n\pi x}{2} + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1 + 3\cos n\pi}{n} \sin \frac{n\pi x}{2}$$

$\square$

Let us compute $F(x)$ at discontinuous points.

Example (continued ...)

$$F(-2) = F(2) = \frac{1}{2}\left(f(-2+) + f(2-)\right) = \frac{1}{2}\left(2 + \frac{1}{2}\right) = \frac{5}{4}$$

$$F(0) = \frac{1}{2}\left(f(0-) + f(0+)\right) = \frac{1}{2}\left(0 + \frac{1}{2}\right) = \frac{1}{4}$$

To summarize,

$$F(x) = \begin{cases} 5/4, & x = \pm 2 \\ -x, & -2 < x < 0 \\ 1/4, & x = 0 \\ 1/2, & 0 < x < 2 \end{cases}$$