BOOLEAN ALGEBRA

Additive identity — Complement operator.

$$\{\Omega, +, \cdot, 0 \in \Omega, 1 \in \Omega, \sim\}$$

Multiplicative identity

- If $a, b \in \Omega$, $a+b \in \Omega$, $a \cdot b \in \Omega$
- $a+b = b+a$ | $a \cdot b = b \cdot a$

  $a+(b+c) = (a+b)+c$ | $a\cdot(b\cdot c) = (a\cdot b)\cdot c$.
- $a\cdot(b+c) = a\cdot b + a\cdot c$

  $a+(b\cdot c) = (a+b)\cdot(a+c)$
- $a+0 = a$ | $a\cdot 1 = a$

8/1

- $\forall a$, there exist an $\bar{a}$ ($\sim a$), such that

  $a\cdot\bar{a} = 0$ and $a+\bar{a} = 1$

I IDENTITIES

1 ~~a · a = a~~ Zero element, one element and complement of every element are unique.

$0_1 + 0_2 = 0_1$ $\because 0_2 \in \Omega$

Also $0_1 + 0_2 = 0_2$ $\because 0_1 \in \Omega$

Hence proved.

2 $a\cdot a = a$, $a+a = a$

$a + (\underbrace{\bar{a}\cdot a}_{0}) = (a+\bar{a})\cdot(a+a)$

$a = 1\cdot(a+a)$

$a = a+a$

$a+\bar{a} = 1$

$(a+\bar{a})a = a$

$a\cdot a = a$

3 $1+a = 1$ and $0\cdot a = 0$

$a+\bar{a}+a$

$= \bar{a}+a+a$

$= \bar{a}+a$

$= 1$

Proof For uniqueness of complement,

$$\bar{a}_1 = \bar{a}_1 \cdot 1 = \bar{a}_1(a + \bar{a}_2) = \bar{a}_1 \cdot a + \bar{a}_1 \cdot \bar{a}_2 = 0 + \bar{a}_1 \cdot \bar{a}_2$$
$$= \bar{a}_1 \cdot \bar{a}_2$$

Similarly, $\bar{a}_2 = \bar{a}_1 \cdot \bar{a}_2$

$\therefore \bar{a}_1 = \bar{a}_2$

4 $a + \bar{a} \cdot b = a + b$ ... BODMAS is always followed

LHS $= (a + \bar{a}) \cdot (a + b)$

$= 1 \cdot (a+b)$

5 De Morgan's Laws :- $\overline{a+b} = \bar{a} \cdot \bar{b}$

Show :- $(a+b) \cdot (\bar{a} \cdot \bar{b}) = 0$ $\qquad (a+b) + (\bar{a} \cdot \bar{b}) = 1$

LHS $= a + \bar{a}\bar{b} + b + \bar{a}\bar{b}$

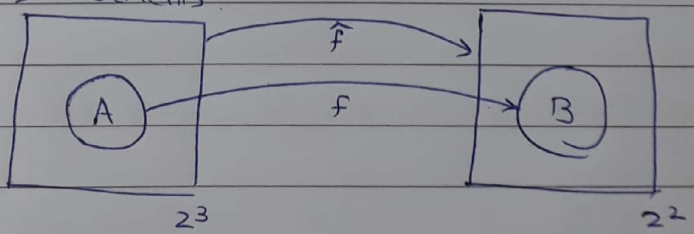
$= a + \bar{b} + b + \bar{a}$

$= 1$

$$\overline{a \cdot b} = \bar{a} + \bar{b}$$

→ To define :- A function $f: A \to B$

If A has 7 elements and B has 3 elements, we do this by by defining a function $\hat{f}$ from a box of 2^3 elements to a box of 2^2 elements



- A is encoded to a 2^3 box. B is encoded to a 2^2 box.
- Elements from 2^3 box not in A have arbitrary images of $\hat{f}$ (does not matter)
- By encoding, Ω can be represent as a set of binary-string elements

Minimal = Atom ?

Theorem Cardinality of Ω in Boolean algebra is a power of 2.

Proof {

Definition If $a \le b$, if $a \cdot b = a$

- $0 \le a \quad \forall a$.
- Minimal element : (u) :- ~~$u \le a$~~ $\forall$ ~~$a \ne 0$~~

Only 0 and u are $\le u$

Definition Atom :- ① $a \ne 0$ and $a \cdot b = a$ or $a \cdot b = 0$

Theorem $|\Omega| \geq 2$

Theorem For any Ω, the set of atoms (A) is never empty.

Proof {

Theorem. If $a \le b$ and $b \le c$, then $a \le c$

$ac = (ab)c = ab = a$

- Hence, there can never be a chain $a < b < c < d < a$

}

③

9/1

- In set algebra, $\le \to \subseteq$ and atom
- Alternative definition of $u \le v$:-

$$u \le v \iff u + v = v$$

Proof $u \cdot v = u$

$u \cdot v + v = u + v$

$(u+1) \cdot v = u + v$

$\therefore v = u + v$

Theorem $u < v \Rightarrow \bar{u} \cdot v \ne 0$

Proof Contradiction :- $u < v$, but $\bar{u} \cdot v = 0$

$u + \bar{u}v = u$

$u + v = u$

$\therefore v = u$

Page No.:

Theorem: A is the set of atoms and $p \in \Omega$

$$p = \sum_{\substack{a \in A \\ a \leq p}} a$$

- Any element of Ω can be constructed from atoms that are $\leq$ that element

Proof

- Let $\omega = \sum_{\substack{a \in A \\ a \leq p}} a \implies \omega \leq p$

$$(a_1 + a_2)p = a_1 p + a_2 p = a_1 + a_2$$
$$\therefore \quad a_1 + a_2 \leq p$$

- Suppose $\omega < p$

$$\implies \bar{\omega} p \neq 0$$

$\therefore \; \exists \; a' \in A$ such that $a' \leq \bar{\omega} p$

$$\therefore \; a' \leq p$$

Then a' should have been included in ω already.

Contradiction

$$\omega = p$$

- There is a bijection between ~~all atoms~~ subsets of A and elements of Ω.
  - Every $p \in \Omega$ can be constructed from atoms
  - Every subset of A constructs some $p \in \Omega$.

Proof: Let a_i and b_j be atoms such that $\sum a_p = \sum b_j$

$$a_1 \cdot \left(\sum a_i\right) = a_1 \cdot \left(\sum b_j\right)$$
$$\implies \quad a_1 = 0$$

Contradiction.

ie ~~Hence,~~ There is a bijection between 2^A and Ω

2^A ← Power set of A.

- If $b \in \Omega \leftrightarrow A_b \in 2^A$ and $c \in \Omega \leftrightarrow A_c \in 2^A$
then $b + c \leftrightarrow A_b \cup A_c$
$b \cdot c \leftrightarrow A_b \cap A_c$
$\overline{b} \leftrightarrow A_b'$

- Hence, cardinality of Ω is a power of 2

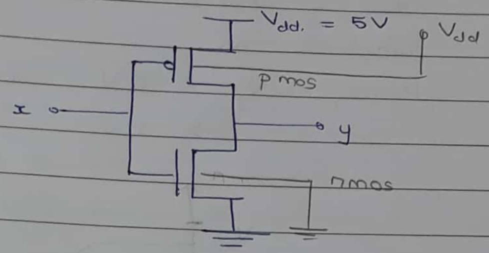
- $\Omega = \{ \text{`}x_1 x_2 x_3 \cdots x_n\text{'} \mid x_i \in \{0, 1\} \}$

④

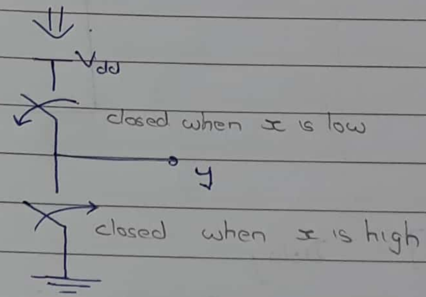
11/1 # CMOS Logic Gates

- 2-input XOR gate :- $(x+y)(\bar{x}+\bar{y})$

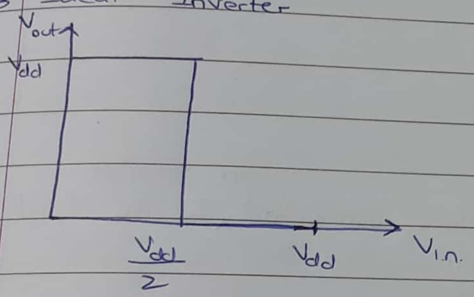
I CMOS Inverter



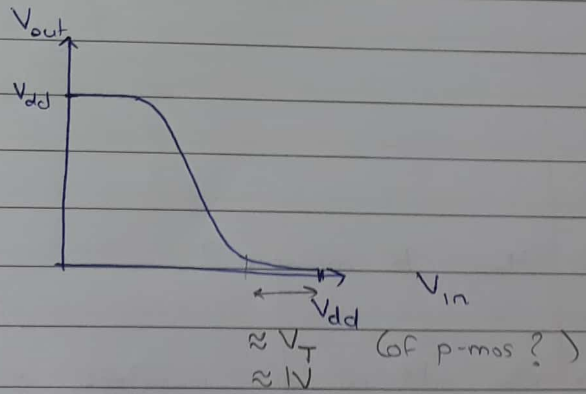
y = V_{dd} when x is low
= 0 high



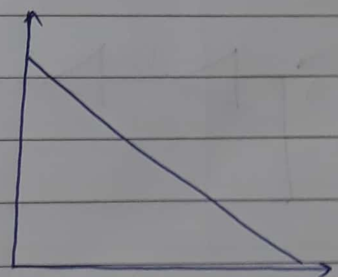
→ Ideal Inverter



→ Actual characteristics



→ Undesirable characteristics
- Poor noise rejection.

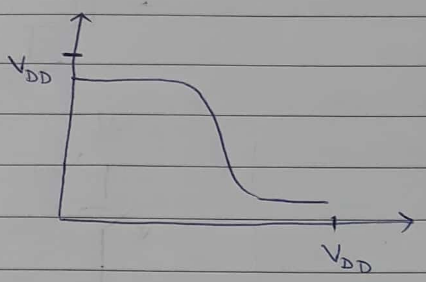


- The transition time should be as small as possible.

A] Output characteristics

When we connect a load to 'y', current flows through it. There is also a (resistive) voltage drop across the transistor which is on.

Hence, $V_y < 5\,V$ when p-mos is on
$> 0\,V$ n-mos



B] Delay

When an inverter has K inverters connected in parallel to its load,

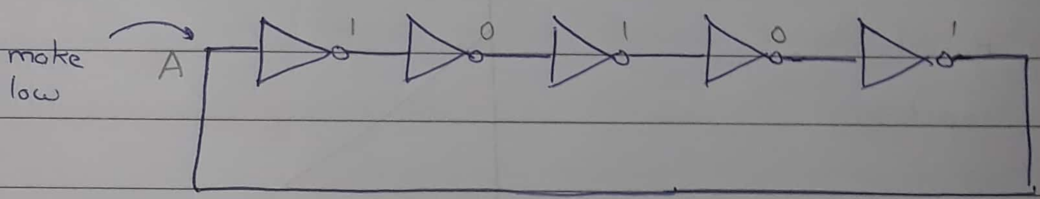
$$\text{Delay} = p + Kg$$

p comes from self-capacitance

Kg comes from capacitances of load inverters.

A How to measure?

- Use odd no. of inverters in series. Supply LOW to first inverter

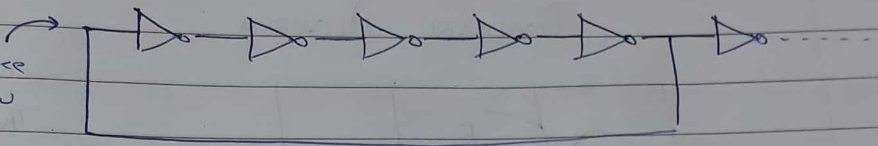


V_A oscillates b/w high and low with period $10 \times$ (Delay of one inverter)

$= 10\,(p + q)$

• Step 2 :-

make low



New period = $8(p+g) + 2(p+2g)$

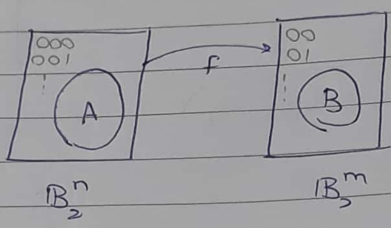
c] Current

Current is drawn only when ~~a~~ ~~t~~ the inverter is switching.

(5)

18/1

BOOLEAN ALGEBRA (Contd)



$\mathbb{B}_2^n \to \mathbb{B}_2^m$

The function f has 'm' different functions, one to independently determine each output bit. These functions are $\mathbb{B}_2^n \to \mathbb{B}_2$

→ $\mathbb{B}_2^n = \{ (x_1 x_2 x_3 \dots x_n) \mid x_i \in \{0,1\} \}$

- $0, 1, x_1, x_2 \dots x_n$ are 'formula's
- Production Rules :-

Rules used to produce new strings of formulas from given strings

eg $A \to (\sim A)$ Inverter

eg $A, B \to A + B$ OR

eg $A, B \to A \cdot B$ AND

- All 'legal' strings can be constructed from formulas using above rules.

eg $(x_1 + x_2) \cdot x_3$ is a legal string

→ Minterms

eg

x_1	x_2	x_3	f (function)
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
⋮	⋮	⋮	0
			0
1	1	1	0

Construct atoms $m_1\ m_2\ m_3$ such that $f = m_1 + m_2 + m_3$

$$\begin{array}{c} 0\\1\\1\\0\\1\\0\\0\\0 \end{array} = \begin{array}{c} 0\\1\\0\\0\\0\\0\\0\\0 \end{array} + \begin{array}{c} 0\\0\\1\\0\\0\\0\\0\\0 \end{array} + \begin{array}{c} 0\\0\\0\\0\\1\\0\\0\\0 \end{array}$$

$f \qquad m_1 \qquad m_2 \qquad m_3$

Minterms

(6)

22/1

$\therefore \quad f = \bar{x}_1 \bar{x}_2 x_3 + \bar{x}_1 x_2 \bar{x}_2 + x_1 \bar{x}_2 \bar{x}_3$

- From 'n' formulas $x_1 \dots x_n$, we can create infinite functions using production rules.

eg- $x_1 \cdot x_1 \cdot x_1 + x_1$ is also a function.

- No. of distinct functions we can create $= 2^{2^n}$

(6)

22/1

→ Any Boolean function can be written as

$$f = \sum_{\substack{a \in A \\ a \leq f}} a \quad \text{... Union of atoms}$$

- By De Morgan's Law,

$$f = \prod_{\substack{a \in A \\ a \leq f}} \bar{a} \quad \text{.... Intersection of complements of atoms}$$

- If a is an atom, $\bar{a}$ is a coatom.

→ If a_1 has formula m_1 and a_2 has formula m_2, then $a_1 + a_2$ has formula $m_1 + m_2$.

Proof: Truth table

→ The formulae $\bar{x}_1 \cdot (\bar{x}_2 \cdot x_3)$ and $(\bar{x}_1 \cdot \bar{x}_2) \cdot x_3$ evaluate to the same function, represented as $\bar{x}_1 \cdot \bar{x}_2 \cdot x_3$

→ In Boolean functional algebra

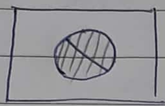
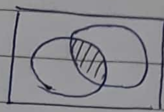
eg

x_1	x_2	F
0	0	0
0	1	1
1	1	0
1	0	1

$f = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \bar{x}_1 x_2 + x_1 \bar{x}_2$ 'Sum-of-products'

$\bar{f} = \begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \bar{x}_1 \bar{x}_2 + x_1 x_2$

$\Rightarrow f = (x_1 + x_2) \cdot (\bar{x}_1 + \bar{x}_2)$ 'Product-of-sums'

- Sum-of-products  Product-of-sums 

→ Minimization of formulae :-

- Use of Boolean Algebra.

eg $\bar{x}_1 \cdot x_2 + x_1 \cdot x_2 = x_2$

$(\bar{x}_1 + x_2) \cdot (x_1 + x_2) = x_2$

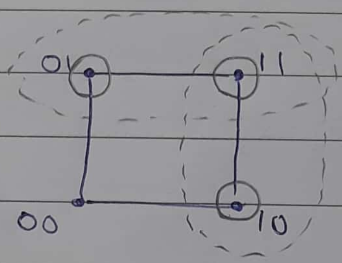
eg $x_1 x_2 + \bar{x}_1 x_3 + x_2 x_3 = x_1 x_2 + \bar{x}_1 x_3$

eg

x_1	x_2	f
0	0	0
0	1	1
1	1	1
1	0	1

$f = \underbrace{\bar{x}_1 x_2 + x_1 x_2} + x_1 \bar{x}_2$ ——— (i)

$= x_2 + x_1 \bar{x}_2 = x_1 + x_2$ ——— (ii)



——— (i)

- - - - (ii)

23/1

I 王 Sum Of Products Minimization

$f = m_1 + m_2 \dots m_k \longrightarrow f = p_1 + p_2 \dots p_r$ where $r < k$

- Literals :-

eg $f = \underset{1\ 2\ 3}{x_1 x_2 x_3} + \underset{4\ 5\ 6}{\bar{x}_1 x_2 x_3}$ has 6 literals

- No. of 2-input gates required (except inverters) = (No. of literals) − 1

- By combining terms, no. of literals ↓

eg In above example, $f = \underset{1\ 2}{x_2 x_3}$ --- requires 1 gate.

- Start with 0 dimensional terms, combine them to form as many 1D squares as possible, combine those to form as many 2D squares as possible, and so on.

Q $f(x_1, x_2, x_3, x_4) = 1$ if binary no. $x_1 x_2 x_3 x_4$ is prime
$\qquad\qquad\qquad\quad = 0$ otherwise.

Meth K-Maps

$x_3x_4 \backslash x_1x_2$	00	01	11	10
00	0	0	0	0
01	0	1	1	0
11	1	1	0	1
10	1	0	0	0

a, b, c

Step 1 Recognize prime implicants − 2D squares that are essential --- a, b, c

Step 2 Now only one 1 remains. Use any one of two possible 2D squares containing this 1.

Meth 2	0D	1D	2D
✓ 2	0010	(2,3) 001_	No
✓ 3	0011	(3,7) 0_11	possible
✓ 5	0101	(3,11) _011	combinations
✓ 7	0111	(5,7) 01_1	No 1D used
✓ 11	1011	(5,13) _101	
✓ 13	1101	All 0D used	

- Combine those that differ in only one variable

Step 2

	✓ 2	✓ 3	✓ 5	7	✓ 11	✓ 13
✓ (2,3)	×	×				
(3,7)		×		×		
✓ (3,11)		×			×	
(5,7)			×	×		
✓ (5,13)			×			×

Choose any one term for 7

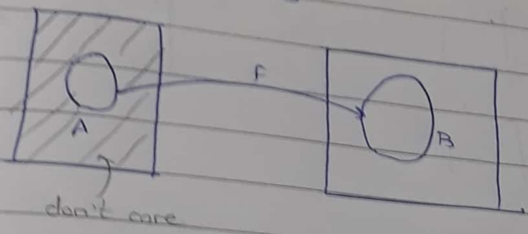
25/1

→ Consensus Equality :-

$$\bar{a}b + ac + bc = \bar{a}b + ac$$

Scanned by CamScanner

A] → 'Don't Care' Conditions



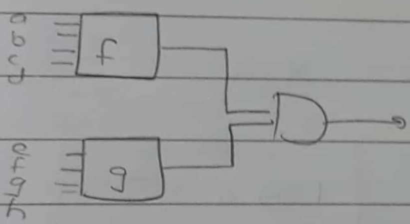
eg $x_1x_2x_3x_4$ in binary from 0 to 9

$f = 1$ if $x_1x_2x_3x_4$ is odd

$= 0$ otherwise

$x_3x_4 \backslash x_1x_2$	00	01	11	10
00			d	
01	1	1	d	1
11	1	1	d	d
10			d	d

• 'Observability' Don't Cares



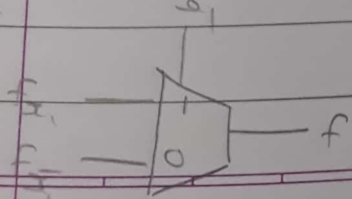
We don't care output of f when g produces zero

B] Shannon's Formula

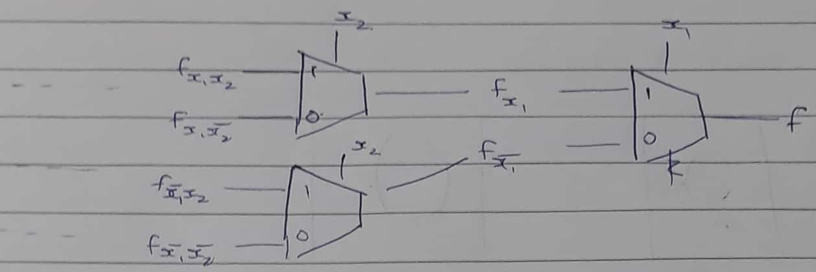
$$f(x_1, x_2 \ldots x_n) = x_1 \cdot \underbrace{f(1, x_2, \ldots x_n)}_{f_{x_1}} + \bar{x}_1 \cdot \underbrace{f(0, x_2 \ldots x_n)}_{f_{\bar{x}_1}}$$

'Cofactors'

$$= x_1 \; ? \; f_{x_1} : f_{\bar{x}_1}$$



Cofactors are simpler expressions than original function (one variable less)



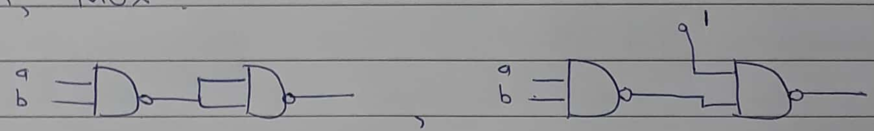
2^n MUXes required

29/1

II UNIVERSAL GATES

= NAND, NOR, MUX.

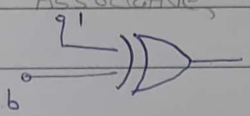
eg AND =



* $a \oplus b = a\bar{b} + \bar{a}b$ ----- XOR

Prove :- Associative, Commutative

eg ~~AND~~ NOT =



$$\boxed{\begin{aligned} 1 \oplus a &= \bar{a} \\ 0 \oplus a &= a \end{aligned}}$$

eg $f_{x_1} \oplus f_{\bar{x}_1} = 1$ if $f(1, x_2 \dots x_n) \neq f(0, x_2 \dots x_n)$

$= 0$ otherwise

This function gives '1' if value of x_1 is affecting the end result.

* $f_{x_1} + f_{\bar{x}_1} = 1$ if $\exists x_1$ such that $f(x_1, x_2 \dots x_n) = 1$

$= 0$ otherwise.

'Existential Quantifier'

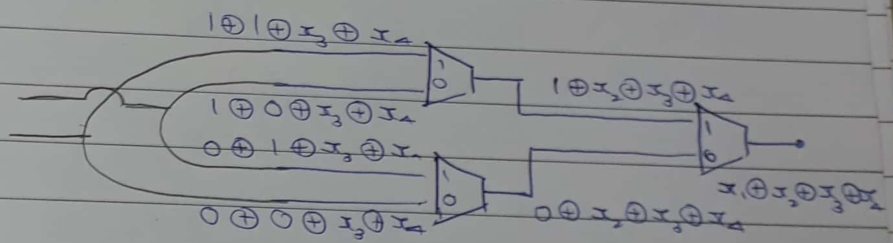
* $f_{x_1} \cdot f_{\bar{x}_1} = 1$ if $\forall x_1$, $f(x_1 \ldots x_n) = 1$
$= 0$ otherwise
'Universal Quantifier'

* $x_1 \oplus x_2 \oplus x_3 \oplus x_4 \ldots\ldots \oplus x_n = 1$ if odd no. of x_i's are 1
'Odd parity function'

eg $x_1 \oplus x_2 \oplus x_3 \oplus x_4$

$x_3x_4 \backslash x_1x_2$	00	01	11	10
00	0	1	0	1
01	1	0	1	0
11	0	1	0	1
10	1	0	1	0

• Shannon's Decomposition :-



- Using logic gates, 2^n gates will be required.
- Using Shannon's method (MUX), $2n$ MUXes are required where every MUX is built using constant no. of gates itself.

Minimization by factorization

1. $pq + pr = p(q+r)$
2. $pq + pr + sq + sqr = (p+s)(q+r)$
3. $tr + uv = f$

$wt + rs = g$

$p + q = t$

A $p\bar{q} + \bar{p}q = (p+q)(\bar{p}+\bar{q})$

30/1

* We want to write $f = q \cdot g$

eg $f = \bar{x}_1 x_2 + x_1 \bar{x}_2$

$g = x_1 + x_2$

TFT q

$g = (x_1+x_2)$ \ x_1x_2	00	01	11	10
0	0	X	X	X
1	X	1	0	1

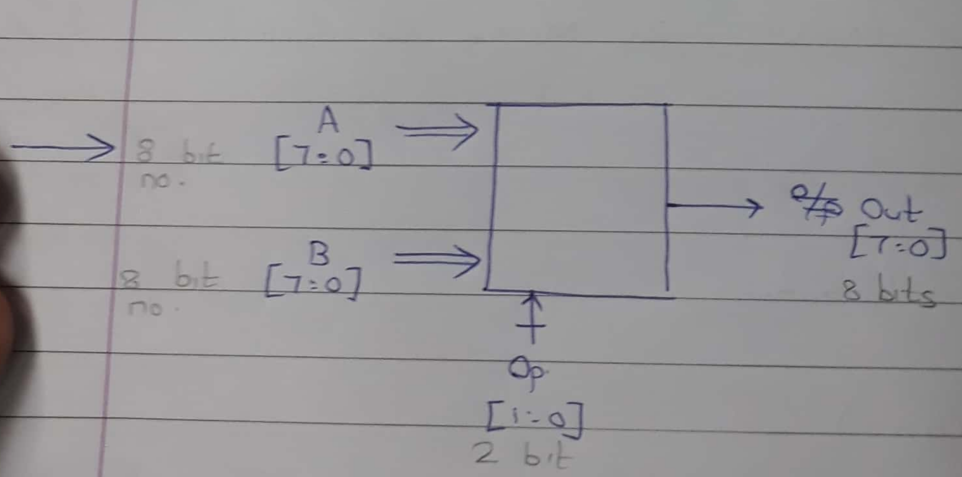
X = not possible (consider as don't care)

Choose prime implicants such that we can take 'g' common --- don't choose verticle PI's

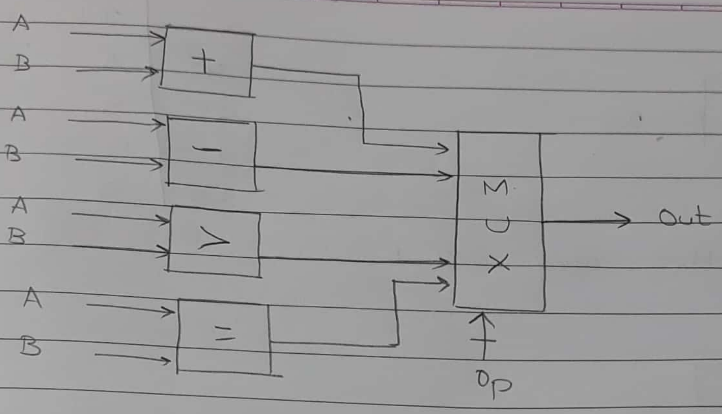
$f = g\bar{x}_1 + g\bar{x}_2$

$= g(\bar{x}_1 + \bar{x}_2)$

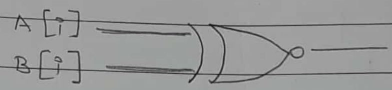
$\boxed{q = \bar{x}_1 + \bar{x}_2}$



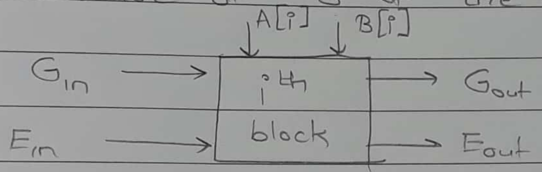
Op	Out
0	A + B
1	A − B
2	A > B
3	A == B



= Use XNOR gate



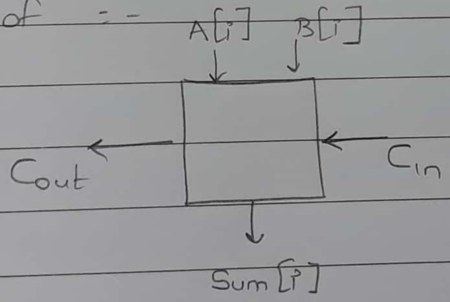
> We will checks from most significant bit
Make a cascade of 8 of the following

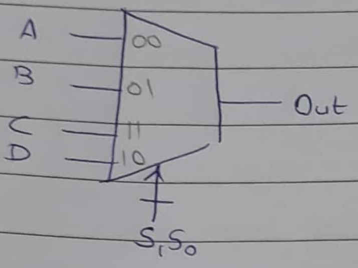


G = 1 ⇒ Until now, A[i] > B[i]

E = 1 ⇒ =

+ Cascade of :-





$S_1 S_0$	Out
00	A
01	B
11	C
10	D

Build 4:1 MUX using 2:1 MUXes