The Fast Multipole Method for n-body Problems

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Widely applicable: electrostatics, fluid mechanics, graphics(!)

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  "Multipoles"

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  Quadtree: a trie like data structure

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Stars in a square $=$ cluster

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Time to determine all A_i: $O(p)$ per star.

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Let X be the square of side length s containing star i.

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Consider a star i. We count the number of squares v of side length s such that $f_v(z_i)$ is evaluated.

f_v is evaluated if v is "far enough" from i but the parent of v is not far enough.

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