Class Example contd....

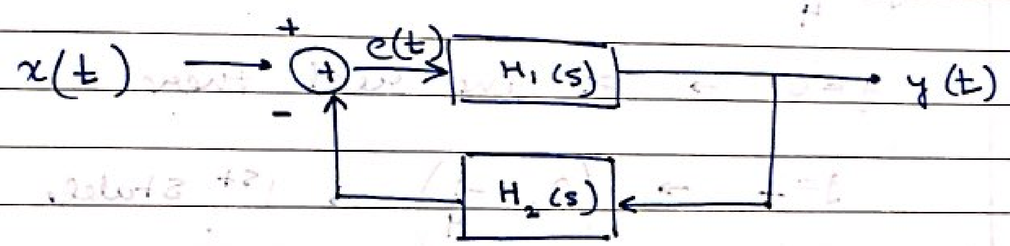
case II $(-2,-1) \rightarrow$ ROC of H of $H(s)$
$\rightarrow$ Contradiction

case III $> -1 \rightarrow$ ROC of $H(s)$
$\rightarrow$ Match ✓

2.4.18

Block Diagram Representations

3) Feedback



$e(t) \leftrightarrow E(s) = X(s) - Y(s)H_2(s)$

$Y(s) = H_1(s)E(s) = H_1(s)[X(s) - Y(s)H_2(s)]$

$\cancel{Y(s)}$

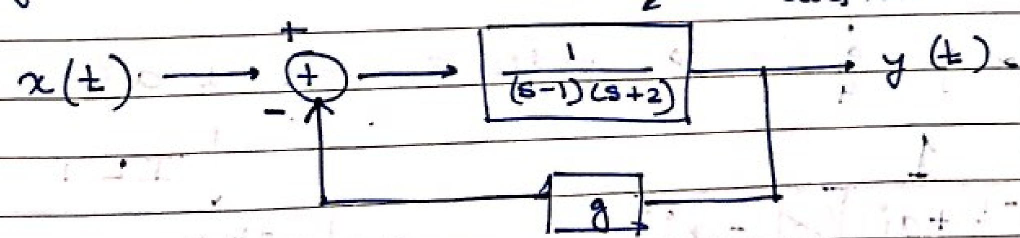
$$\frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

→ ROC of $Y(s)$: is a Right half plane :

Reason: since $X(s)$, $H_1(s)$, $H_2(s)$ all have right half plane ROCs

causal but unstable

eg



$$H(s) = \frac{\frac{1}{(s-1)(s+2)}}{1 + \frac{G(s)}{(s-1)(s+2)}} = \frac{1}{(s-1)(s+2) + G(s)} = \frac{1}{s^2 + s - 2 + G(s)}$$

Roots → In terms of g, pick g s.t roots have -ve real part

So $s^2 + s - 2 + g$

poles at $-\frac{1}{2} \pm \sqrt{\frac{1+8-4g}{4}}$

$= \frac{-1}{2} \pm \sqrt{\frac{9}{4} - g}$

$g > \frac{9}{4}$ → both real

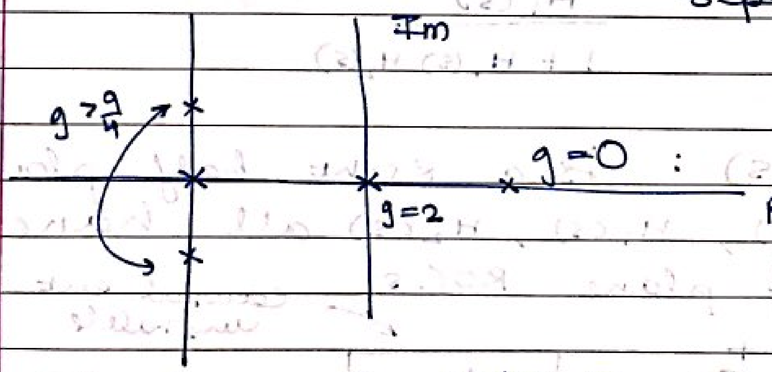
$g = 0$ → α +ve real there

$g = 2$ → $\left(0, -\frac{1}{4}\right)$ → 1st stable case

⇒ $g > 2$ gives stability

$g > \frac{9}{4}$ → two poles (im) vertically seperated

(Right most pole)



$g > \frac{9}{4}$ ⇒ poles $= -\frac{1}{2} \pm aj$

$$\frac{1}{\left(s-\left(-\frac{1}{2}+aj\right)\right)\cancel{\left(-\frac{1}{2}-aj\right)}\left(s-\left(-\frac{1}{2}-aj\right)\right)}$$

↓ ILT

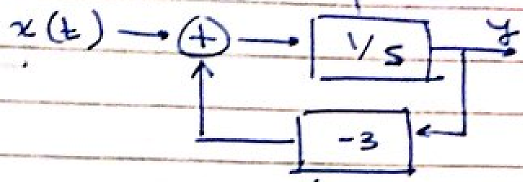
$\frac{1}{a}\boxed{e^{-t/2}\sin(at)}$ → Even though input does not have oscillations output does → undesirable eg $\quad H(s) = \dfrac{1}{(s+3)}$ (Assume RHP ROC),

$\Updownarrow$

$y' + 3y = x$

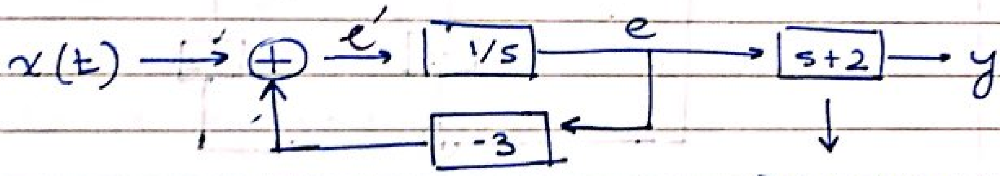
$H(s) = \dfrac{1/s}{1 + 3/s}$

Integrator, unstable

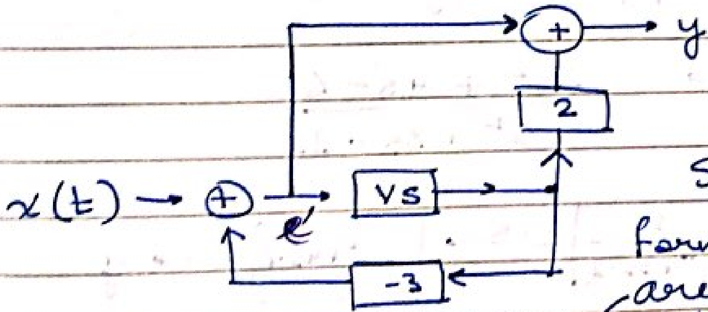


feedback stabilizes the system.

eg $\quad H(s) = \left(\dfrac{s+2}{s+3}\right)$ → Many decompositions are possible



Derivative type operation ⇒ Reuse derivative of e at LHS

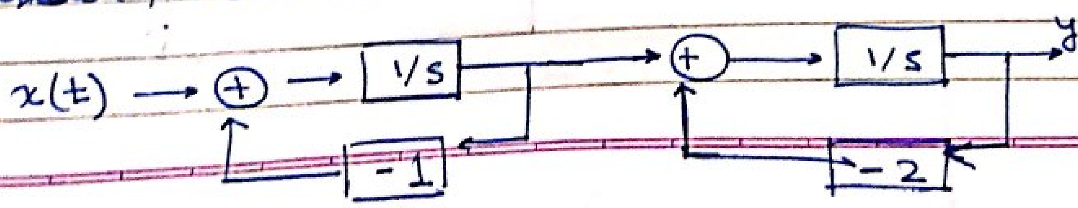


Std./Canonical form since all constants are seperate

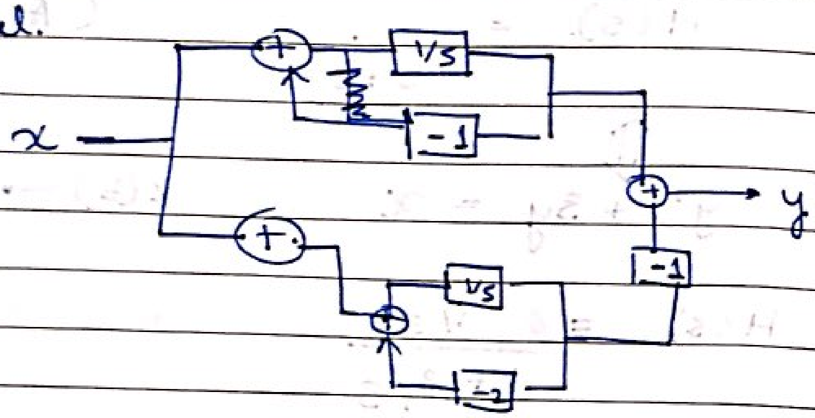
More

eg $\quad H(s) = \dfrac{1}{(s+1)(s+2)} = \dfrac{1}{s^2+3s+2} = \dfrac{1}{(s+1)} - \dfrac{1}{(s+2)}$

I Cascade / Series



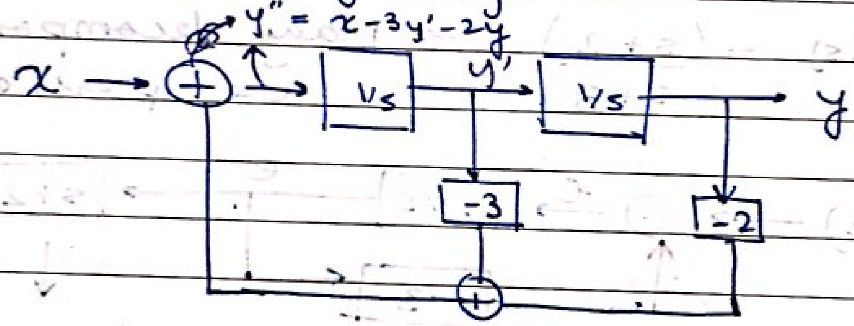
Parallel.



Std. form :

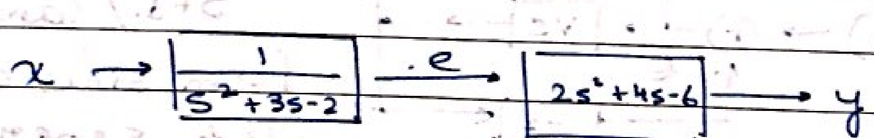
① Write ODE

$$y'' + 3y' + 2y = x$$

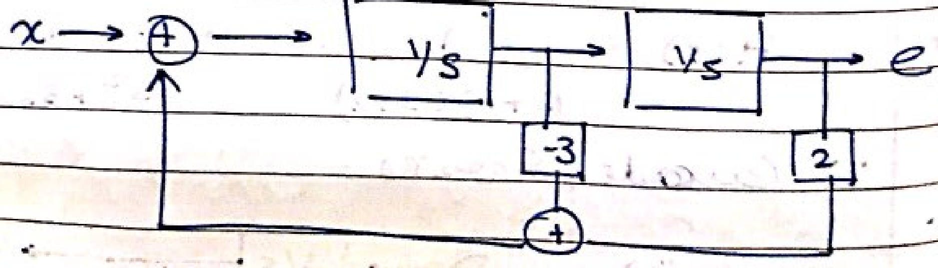


More : Std. form: Sequence of Integrators with values tapped off in between.

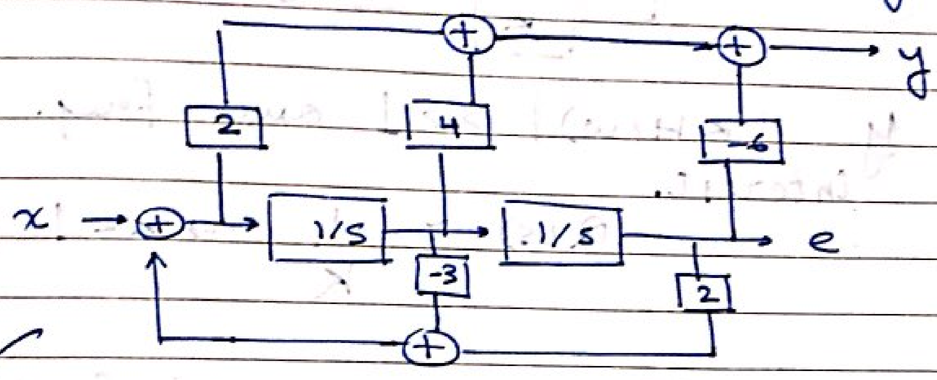
eg $$H(s) = \frac{2s^2 + 4s - 6}{s^2 + 3s - 2}$$



① Get, $e'' + 3e' - 2e = x \Rightarrow e'' = x - 3e' + 2e$



(2) Get $y = \underbrace{2e'' + 4e' - 6e}_{\text{tap from existing}}$

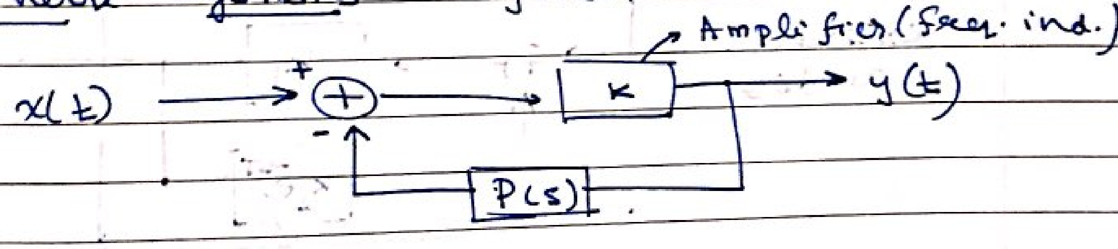


Std. form

3.4.18 **Inverted Pendulum** : Slides

Feedback Systems

a) **Inverse Systems** : Given $P(s)$



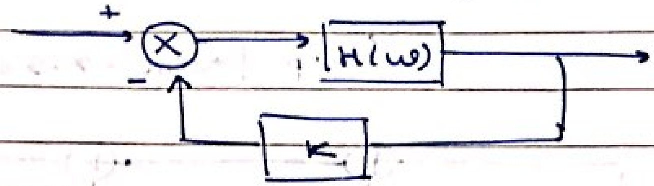
$$Q(s) = \frac{K}{1 + K P(s)} \quad , \text{ if } |K P(s)| >> 1$$

then $Q(s) \approx \dfrac{1}{P(s)}$

b) Compensation for non-ideal elements

– H.S. Black (1920's)

Given, Filter $H(\omega)$ given high, non unifo[rm] gain over some freq. band.



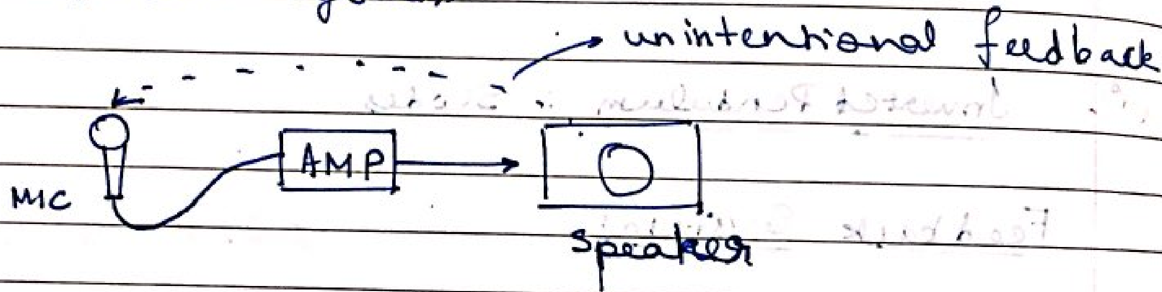
$$Q(s) = \frac{H(\omega)}{1+KH(\omega)}$$

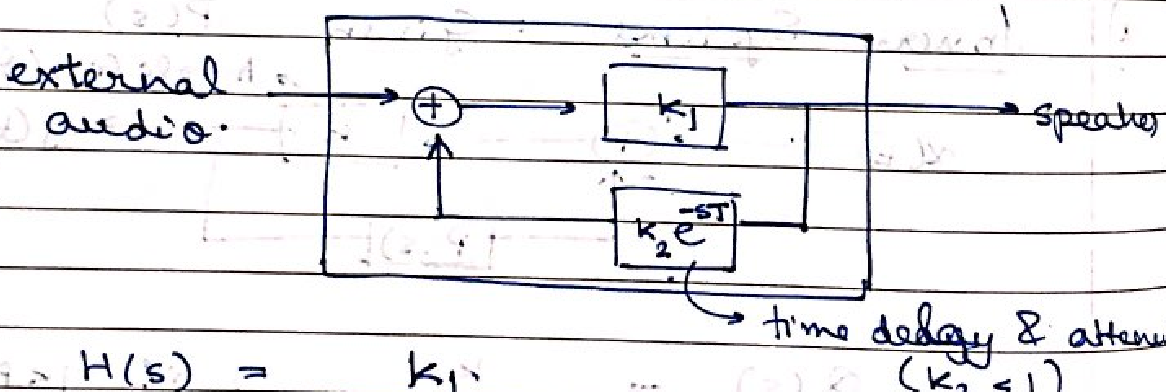
If $|KH(\omega)| >> 1$ over freq. range of interest.

$$Q(s) \approx \frac{1}{K} \Rightarrow K < 1 \text{ for } \underline{\text{amplifi[er]}}$$

c) Undesirable consequences of feedback

eg Audio system





$$H(s) = \frac{k_1}{1 - k_1 k_2 e^{-sT}}$$

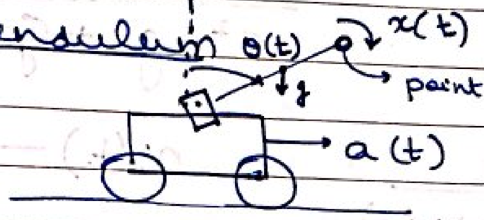
pole at $k_1 k_2 e^{-sT} = 1$

$$s = \frac{1}{T} \ln(k_1 k_2)$$

→ For stability $\text{Re}\{\text{pole}\} < 0 \Rightarrow \boxed{k_1 k_2 < 1}$

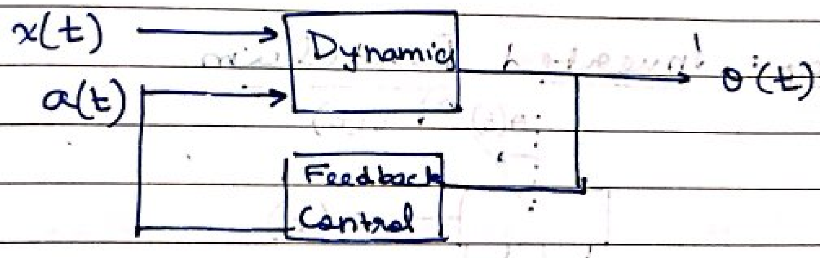
→ <u>Nyquist</u> <u>Stability</u> <u>Criterion</u> : Soln.

<u>Inverted</u> Pendulum



m = 1, L = 1
for rod

$x(t)$ → External Disturbance / Torque
$a(t)$ → accn. of cart (controlled



<u>Dynamics</u>

$$\frac{d^2\theta(t)}{dt^2} = x(t) + g(t)\sin(\theta(t)) - a(t)$$

→ Linearize System around $\theta = 0$.

$$\frac{d^2\theta(t)}{dt^2} = x(t) + g\,\theta(t) - a(t)$$

$$s^2\,\theta(s) = X(s) + g\,(\theta)(s) - A(s)$$

$$\theta(s)\left[s^2 - g\right] = X(s) - A(s)$$

Un<u>controlled</u> : $(a(t) \equiv 0)$

$$\frac{\theta(s)}{X(s)} = \frac{1}{s^2 - g} = H(s)$$

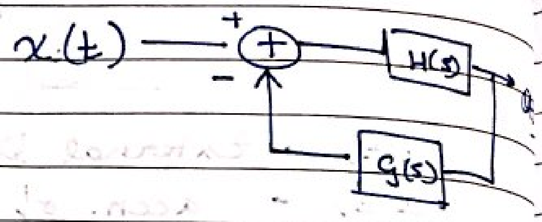
↳ unstable

$\rightarrow$ LSI system

Control: $A(s) = G(s)\,\theta(s)$

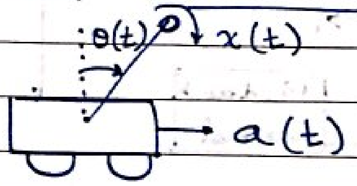
$$\frac{\theta(s)}{X(s)} = \frac{1}{s^2 - g + G(s)}$$

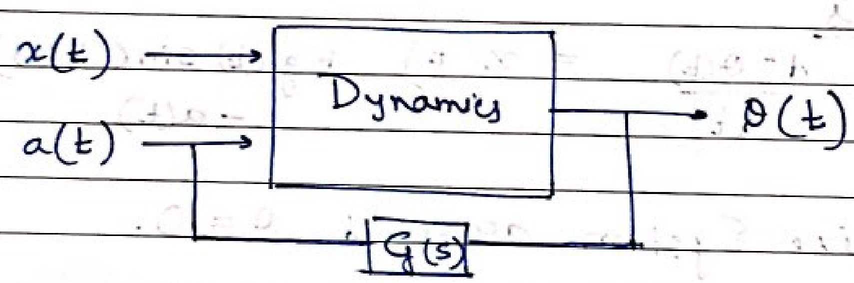
$$= \frac{H(s)}{1 + G(s)H(s)}$$

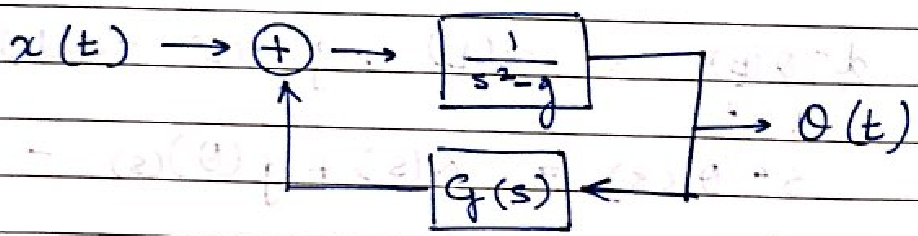


5.3.18

Recap: Inverted Pendulum







$$H(s) = \frac{1}{s^2 - g + G(s)}$$

$\rightarrow$ Find $G(s)$:

1) Proportional Controllers

$a(t) = k\,\theta(t)$ (Proportional control)

$\Rightarrow G(s) = k$

$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{(\pi t)}$

$$H(s) = \frac{1}{s^2 - g + K}$$ → Can never be stable
→ opp sign real poles
→ or 2x purely im.

(ii) Derivative controller

$$a(t) = K\dot{\theta}(t)$$
$$G(s) = Ks$$
$$H(s) = \frac{1}{s^2 + Ks - g}$$

poles : $\frac{-K \pm \sqrt{K^2 + 4g}}{2}$ - Cannot be stabilized

(iii) Proportional & Derivative control

$$a(t) = K_1\theta(t) + K_2\dot{\theta}(t)$$
$$G(s) = K_1 + K_2 s$$

$$H(s) = \frac{1}{s^2 + K_2 s + (K_1 - g)}$$

$$= \frac{-K_2 \pm \sqrt{K_2^2 - 4(K_1 - g)}}{2K_2}$$

$K_2 > 0$ and $K_1 > g$ ⇒ Stable.

HW 5:

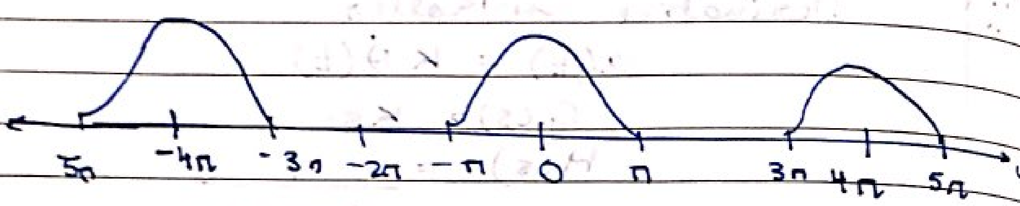
A-1) Key Idea: We can use different low pass filters,

Given, $\omega_c = \pi$

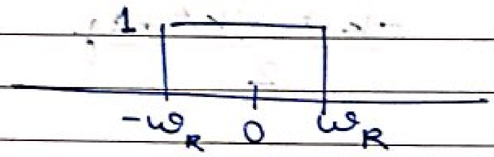
$T_s = \frac{1}{2} \Rightarrow \omega_s = \frac{2\pi}{T_s} = 4\pi$ ↳ Possible to reconstruct original sinc is special, 0 at other points

$$\tilde{x}(t) = T_s \sum x(nT_s)\,\delta(t-nT_s)$$

$\tilde{X}(\omega)$



$\times$



$\omega_R \in (\pi, 3\pi)$

$$x(t) = \tilde{x}(t) * \frac{\omega_R}{\pi}\,\mathrm{sinc}\left(\frac{t\omega_R}{\pi}\right)$$

$$= \frac{T_s\omega_R}{\pi} \sum_n x(nT_s)\,\mathrm{sinc}\left(\frac{\omega_R}{\pi}(t-nT_s)\right)$$

$$\phi = \cancel{\mathrm{sinc}}\ \frac{T_s\omega_R}{\pi}\,\mathrm{sinc}\left(\frac{\omega_R T_s}{\pi}\right)$$

A-S) $$\phi_{xx}(t) = \int_\tau x(\tau)\,x(t+\tau)\,d\tau$$

$$\phi_{xx} = x(t) * z(t)$$

$$z(t) = x(-t)$$

$$x * z = \int_{-\infty}^{\infty} x(\tau)\,x(\tau-t)\,d\tau$$

$$= \phi_{xx}(-t)$$

$$= \phi_{xx}(t)$$

By Definition, ϕ_{xx} is an even function

$$\Phi_{xx}(s) = LT\{x(t) * x(-t)\}$$
$$= X(s)\,X(-s)$$

Unilateral Laplace Transform

→ Unilateral Laplace transform $\mathcal{X}(s)$ of $x(t)$ is defined as

$$\mathcal{X}(s) = \int_{0^-}^{\infty} x(t)\, e^{-st}\, dt$$

Note

- Limit 0^- ⇒ Includes singularities at origin
- If $x(t) = x(t)\,u(t)$, then $\mathcal{X}(s) = X(s)$
- ROC ~~must be~~ is a right half plane.

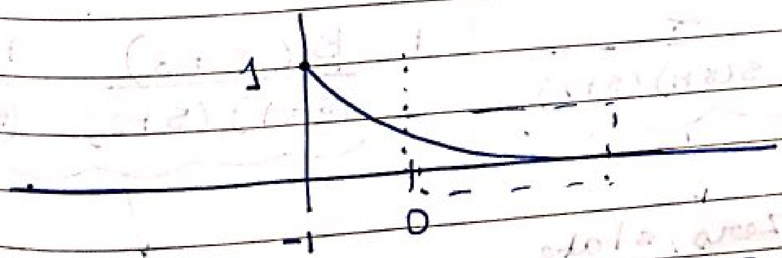
eg

$$x(t) = e^{-a(t+1)}\, u(t+1)$$

→ unilateral

$e^{-a(t+1)} u(t)$

$e^{-a} e^{-at} u(t)$



$$X(s) = \frac{e^{s}}{s+a} \quad (Re(s) > -a)$$

$$\mathcal{X}(s) = \frac{e^{-a}}{s+a} \quad (Re(s) > -a)$$

Properties of unilateral are modified versions of Bilateral LT

eg

1) $x_1(t) = x_2(t) = 0 \quad \forall \; t < 0$

$$(x_1 * x_2)(t) \xrightarrow{ULT} X_1(s) X_2(s)$$

2) $$x(t) \xrightarrow{ULT} X(s)$$

$$x'(t) \xrightarrow{ULT} sX(s) - x(0^-)$$

eg $\quad y'' + 3y' + 2y = x(t)$

$$x(t) = \alpha u(t)$$

$$y(0^-) = \beta, \quad y'(0^-) = \gamma$$

$$y \rightarrow Y(s)$$

$$y' \rightarrow sY(s) - \beta$$

$$y'' \rightarrow s[sY(s) - \beta] - \gamma$$

$$s^2 (Y(s) - \beta s - \gamma + 3(sY(s) - \beta) + 2Y(s) = \frac{\alpha}{s}$$

$$Y(s)[s^2 + 3s + 2] = \frac{\alpha}{s} + \beta(s+3) + \gamma$$

$$Y(s) = \underbrace{\frac{\alpha}{s(s+1)(s+2)}}_{\text{zero state Response/ LSI response}} + \underbrace{\frac{\beta(s+3)}{(s+1)(s+2)} + \frac{\gamma}{(s+1)(s+2)}}_{\text{zero input response}}$$

Discrete time Fourier Transform

→ **Recap: Discrete time Phasors**

$$e^{j\omega n}$$

→ only interval of 2π meaningful for ω

→ Need only consider an interval of length 2π

→ Periodic only if ω is rational numbers multiple of π

i.e. $\boxed{\frac{\omega}{\pi} \in \mathbb{Q}}$

→ $\omega \approx 0, 2\pi$ → slow phasor

$\omega \to \pi$ → fastest phasor

→ FS representation: All phasors periodic with period N

if, $x[n] = x[n+N]$, then

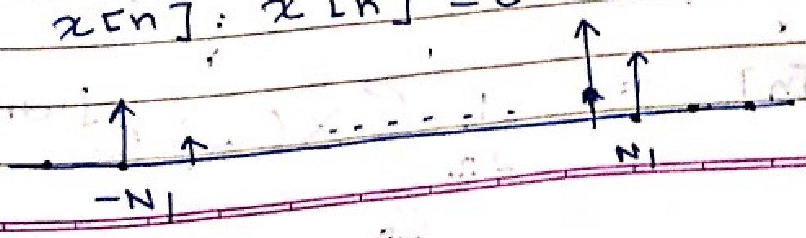
$$x[n] = \sum_{k \in \langle N \rangle} a_k e^{jk\omega_0 n}, \text{ where } \omega_0 = \frac{2\pi}{N}$$

$$a_k = \frac{1}{N} \sum_{n \in \langle N \rangle} x[n] e^{-jk\omega_0 n}$$

~~$x[n]$~~

~~$x[n] = 0 \ \forall\ |n|$~~

$x[n]$: $x[n] = 0 \ \forall\ |n| > N_1$



→ Define $\tilde{x}[n]$, periodic extension s.t. that, period $N > 2N_1 + 1$

$$\tilde{x}[n] = \sum_{k \in \langle N \rangle} a_k \, e^{jk\frac{2\pi}{N}n}$$

where, $$a_k = \frac{1}{N} \sum_{n \in \langle N \rangle} \tilde{x}[n] \, e^{-jk\frac{2\pi}{N}n}.$$

$$a_k = \frac{1}{N} \sum_{-\infty}^{\infty} x[n] \, e^{-jk\frac{2\pi}{N}n}$$

<u>Define</u>,

$$X(\omega) := \sum_{-\infty}^{\infty} x[n] \, e^{-j\omega n}$$

then,

$$a_k = \frac{1}{N} X\left(k \cdot \frac{2\pi}{N}\right)$$

$$\tilde{x}[n] = \sum_{k \in \langle N \rangle} \frac{1}{N} X\left(k \frac{2\pi}{N}\right) e^{jk\frac{2\pi}{N}n}$$

In the limit $N \to \infty$, think of it as Reimann sum $\Rightarrow \delta = \frac{2\pi}{N}$

$$\tilde{x}[n] = \frac{1}{2\pi} \sum_{k \in \langle N \rangle} \frac{2\pi}{N} X\left(k \cdot \frac{2\pi}{N}\right) e^{jk\frac{2\pi}{N}n}$$

$N \uparrow \infty$ ↓ $\qquad$ ↓ $N \uparrow \infty$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\omega) \, e^{j\omega n} \, d\omega$$

→ Duali

10.4.1

→ Synthesis & Analysis Equation

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\omega) e^{j\omega n} d\omega$$

where, $X(\omega)$ → DTFT of x / Spectrum of x

$$X(\omega) = \sum_{n\in -\infty}^{\infty} x[n] e^{-j\omega n}$$

Table

Here time ⇒ Independant variable

	Signal	Transform
FS	Cont. time, periodic	disc time, aperiodic
FT Duality ✓	~~Const.~~ Cont. time, aperiodic	Cont. time, aperiodic
DTFS Duality ✓	Discrete time, periodic	Discrete time, periodic.
DTFS	disc. time aperiodic	Cont. time periodic

Duality

10.4.18

Recap

DTFT

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\omega) \underline{e^{j\omega n}} d\omega$$

Not all of these phasors are periodic

2π → Always over 2π

$$X(\omega) = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n}$$

→ periodic ~~with~~ ~~ω~~ ωp 2π

Also,

$\uparrow \omega \not\Rightarrow \uparrow$ frequency

$\downarrow \omega \not\Rightarrow \downarrow$ frequency

eg $x[n] = a^n u[n]$, $|a| < 1, a \in \mathbb{R}$

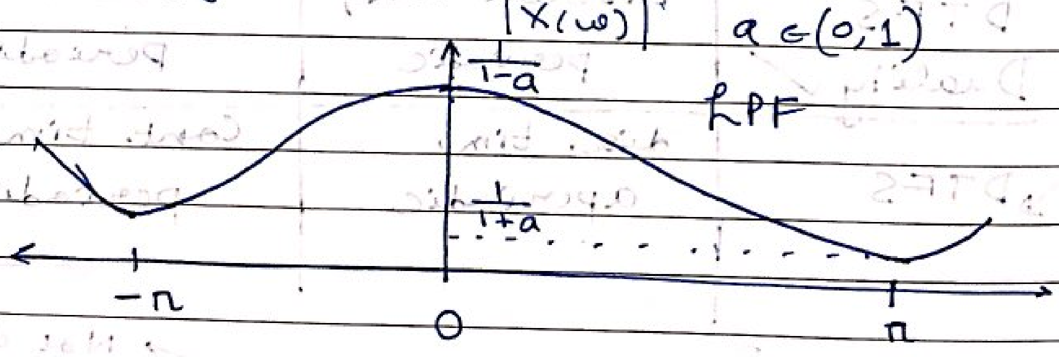
$$X(\omega) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{0}^{\infty} (a e^{-j\omega})^n$$

$$X(\omega) = \boxed{\frac{1}{1 - a e^{-j\omega}}}$$

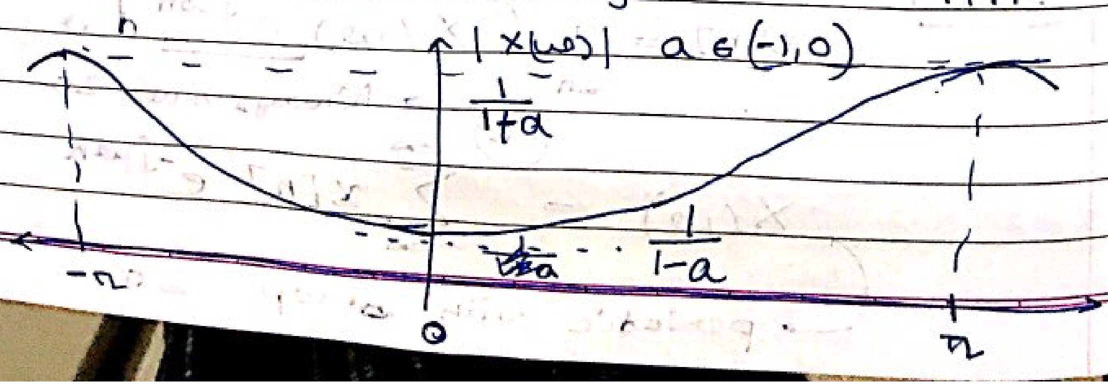
$$X(\omega) = \frac{1}{1 - a(\cos\omega - j\sin\omega)}$$

$$= \frac{1}{(1 - a\cos\omega) + j a \sin\omega}$$

→ Plot Magnitude



→ $a < 0 \Rightarrow$ Shift by π HPF



→ Shifting Changes : LPF $\longleftrightarrow$ HPF

Convergence

$$\sum_{-\infty}^{\infty} |x[n]| < \infty \Rightarrow \text{DTFT Converges}$$

→ Properties : Text Book

DTFT Representation for periodic Signals

$x[n] \longrightarrow$ period N

$$x[n] = \sum_{k \in \langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega$$

$$X(\omega) = 2\pi \sum_{-\infty}^{\infty} a_k \delta\left(\omega - k \cdot \frac{2\pi}{N}\right)$$

Infinite Sum since $X(\omega)$ is periodic in our usual framework

Properties

$x[n] \longrightarrow X(\omega)$,

$y[n] \longrightarrow Y(\omega)$

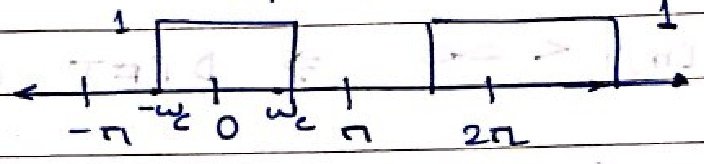
• Time & frequency Shift

$x[n-n_0] \longrightarrow e^{-j\omega n_0} X(\omega)$

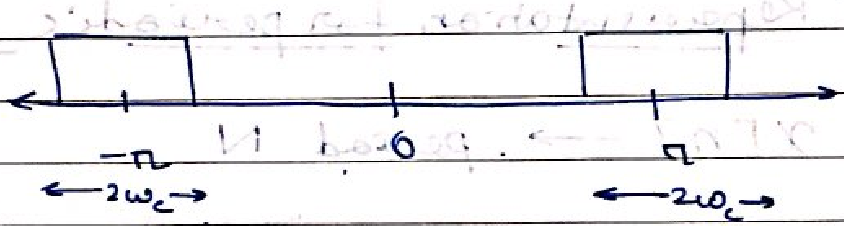
$$e^{j\omega_0 n}\, x[n] \longrightarrow X(\omega - \omega_0)$$

→ <u>Ideal filters in Discrete time</u>

$H_{lp}(\omega)$



$H_{hp}(\omega)$



$$h_{lp}[n] = e^{j\pi n}\, h_{hp}(n)$$

$$\Rightarrow h_{lp}[n] = (-1)^n h_{hp}[n] \quad \text{(DT only!)}$$

$$h_{lp}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega \quad \text{(time domain)}$$

$$= \frac{1}{(jn)\,2\pi}\left[e^{j\omega_c n} - e^{-j\omega_c n}\right]$$

$$= \frac{\sin[\omega_c n]}{\pi n} \longrightarrow \text{DT sinc function}$$

$$x[n] \longrightarrow X(\omega)$$

$$x^*[n] \longrightarrow X^*(-\omega)$$

Differencing & Accumulation

1) $x[n] - x[n-1] \longrightarrow (1-e^{-j\omega})X(\omega)$

2) Running Sum

$$\sum_{-\infty}^{n} x[k] \longrightarrow \frac{X(\omega)}{1-e^{-j\omega}} + \pi X(0) \sum_{-\infty}^{\infty} \delta(\omega - k2\pi)$$

Note:

$$u[n] \longrightarrow \frac{1}{1-e^{-j\omega}} + \pi \sum_{-\infty}^{\infty} \delta(\omega - k2\pi)$$

- $x[n] \longrightarrow X(\omega)$
  $x[-n] \longrightarrow X(-\omega)$

Squeezing → Complicated, information loss.

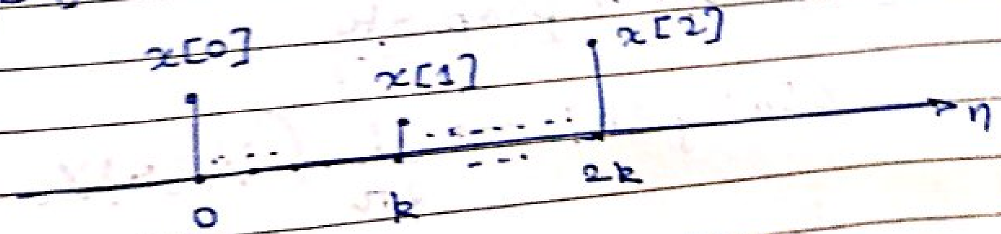
Expansion, $k > 1$

for $k \geq 1$;

$$x_k[n] = \begin{cases} x[n/k] & \text{if } k \text{ divides } n \\ 0 & \text{else} \end{cases}$$

could do interpolation instead

$x_k[k]$



$$X_k(\omega) = \sum_{-\infty}^{\infty} x_k[n] e^{-j\omega n}$$

$$= \sum_{-\infty}^{\infty} x[n] e^{-j\omega(kn)} = \underline{\underline{X(k\omega)}}$$

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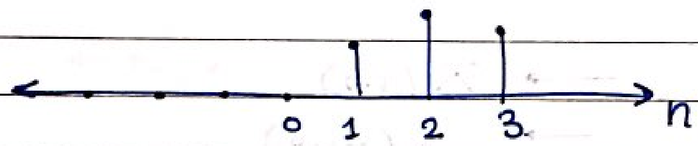
<u>Recap</u> : DTFT

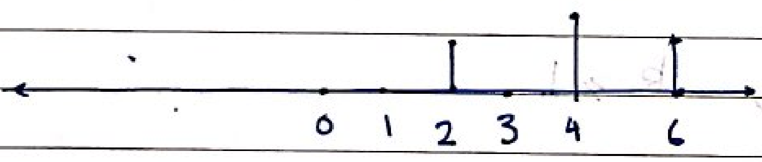
$$x_k[n] = \begin{cases} x[n/k] & \text{if } n \% k = 0 \\ 0 & \text{ow} \end{cases}$$

$x[n]$



$x_2[n]$

$X_k(\omega) = X(k\omega)$



1) <u>Convolution property</u>

$$(x * y)[n] \longrightarrow X(\omega) Y(\omega)$$

2) <u>Multiplication property</u>

$$x[n] \, y[n] \longrightarrow \frac{1}{2\pi} \int_{2\pi} X(\theta) Y(\omega - \theta) \, d\theta$$

3) Parsenal's Relation

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$$\sum_{-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(\omega)|^2 d\omega$$

→ Duality b/w c-t FS and d-t FT

(a) $x[n] \xrightarrow{DTFT} X(\omega) \xrightarrow{FS} a_k$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\omega) e^{j\omega n} d\omega$$

$$a_k = \frac{1}{2\pi}\int_{2\pi} X(\omega) e^{-j\left(\frac{\cancel{2\pi}}{\cancel{2\pi}}\right)k\omega} d\omega$$

$\frac{2\pi}{T}$

$$= x[-k]$$

→ $x[\cdot] \xrightarrow{DTFT} X(\cdot) \xrightarrow{FS} x[-\cdot]$

(b) $x(t) \xrightarrow{FS} a_k \xrightarrow{DTFT} A(\omega)$

periodic w.p T

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$

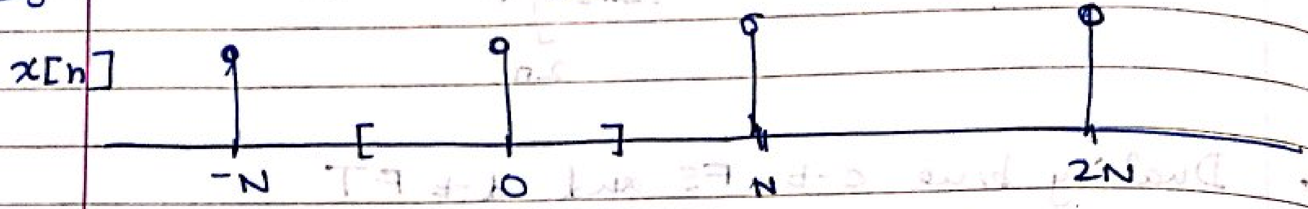
$$A(\omega) = \sum_{-\infty}^{\infty} a_k e^{-j\omega k} = x\left(-\frac{T\omega}{2\pi}\right)$$

→⇒ $x(\cdot) \xrightarrow{FS} a_{\cdot} \xrightarrow{DTFT} x\left(\frac{-T\cdot}{2\pi}\right)$

↓

periodic w.p T

eg $\quad x[n] = \sum_{-\infty}^{\infty} \delta[n - kN]$



$$a_k = \frac{1}{N} \sum x[n] e^{-j\frac{2\pi}{N}kn} = \frac{1}{N}$$

$\Rightarrow$ from formula,

$$X(\omega) = \frac{2\pi}{N} \sum_{-\infty}^{\infty} \delta\left(\omega - \frac{2\pi}{N}k\right)$$

Sampling Theorem

$$x[n] \longrightarrow \otimes \longrightarrow \tilde{x}[n]$$

$$\uparrow$$

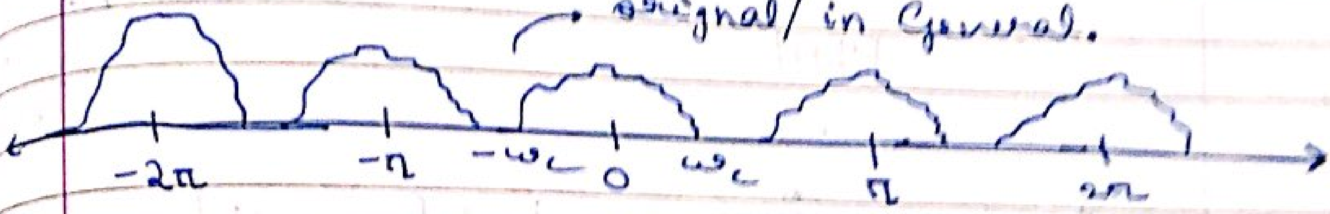
$$p[n] = \sum_{-\infty}^{\infty} \delta[n - kN]$$

$$P(\omega) = \frac{2\pi}{N} \sum_{-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$$

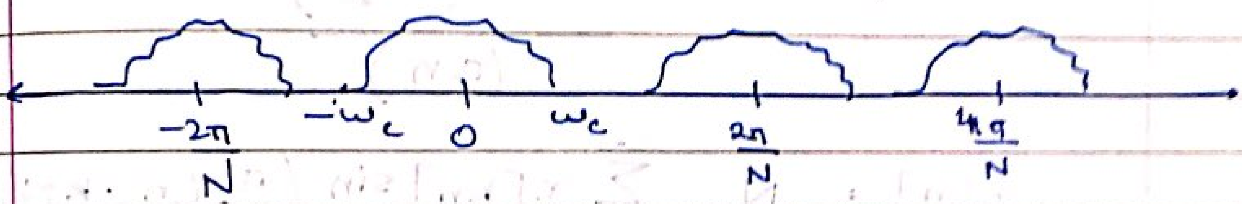
$$\tilde{X}(\omega) = \frac{1}{2\pi} \int_{2\pi} P(\theta) X(\omega - \theta)\, d\theta$$

$$= \frac{1}{N} \int \left\{ \sum_{0}^{N-1} \delta\left(\theta - \frac{2\pi}{N}k\right) \right\} X(\omega - \theta)\, d\theta$$

$$= \frac{1}{N} \sum_{0}^{N-1} X\left(\omega - \frac{2\pi k}{N}\right)$$



$\tilde{X}(\omega)$



→ For perfect reconstruction, we need

$$2\omega_c < \frac{2\pi}{N} = \omega_s$$

$$\Updownarrow$$

$$N < \frac{\pi}{\omega_c}$$

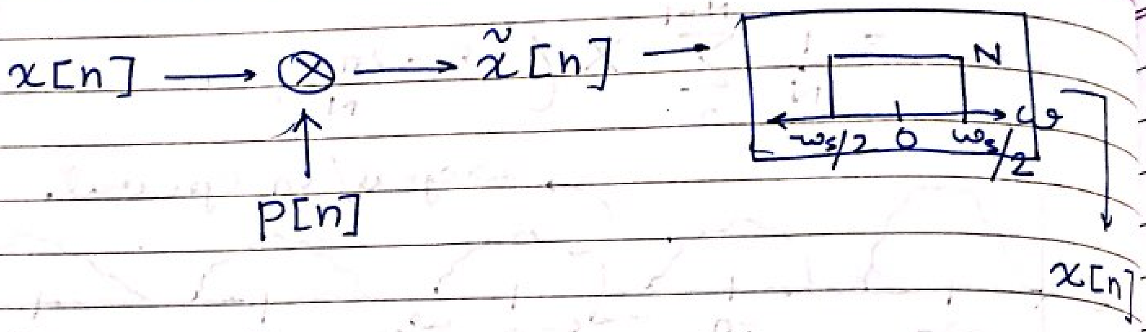
Eg $\omega_c = \frac{2\pi}{9}$ $\quad N < 4.5, \Rightarrow N \leq 4$

Say, $\omega_c = 0.99\pi$, we need

$\omega_c < \frac{\pi}{2}$ for

subsampling feasibility.

↳ $N=1$ is just original.



$$\tilde{x}[n] = \sum_{k=-\infty}^{\infty} x[kN]\, \delta[n-kN]$$

$$h_{lp}[n] = \frac{N \sin\left(\frac{\omega_s n}{2}\right)}{(\pi n)}$$

$$x[n] = \frac{\frac{N}{\pi} \sum_{-\infty}^{\infty} x[kn] \sin\left(\frac{\pi}{N}(n-kN)\right)}{(n-kN)}$$

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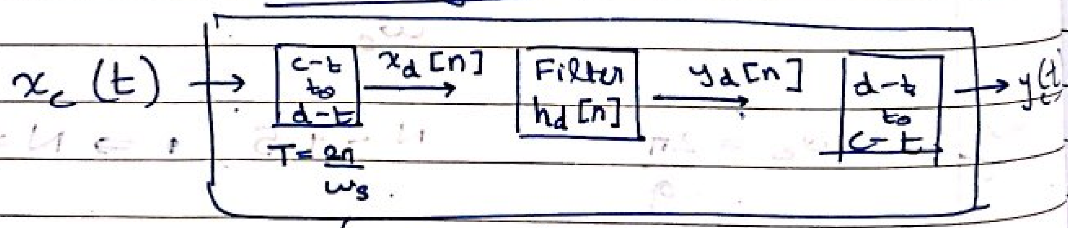
Discrete time processing of Continuous time signals



↳ c-t filter

→ Reason: Work in Digital Domain
- Analog : Hard coded
- Digital : → Dynamic
  → off the Shelf cheap processors

1) Sampling, $T = \frac{2\pi}{\omega_s}$ where, $\omega_s > 2\omega_c$

s.t $x_d[n] = x_c(nT)$

$$\tilde{x}(t) = x_c(t) \times \left(\sum_n \delta(t - nT) \right)$$

$$= \sum x_c(nT)\, \delta(t - nT)$$

$$\tilde{X}(\omega) = \frac{1}{T} \sum_k X_c(\omega - k\omega_s)$$

$$= \int \tilde{x}(t)\, e^{-j\omega t}\, dt$$

$$\boxed{\tilde{X}(\omega) = \sum x_c(nT)\, e^{-j\omega nT}}$$

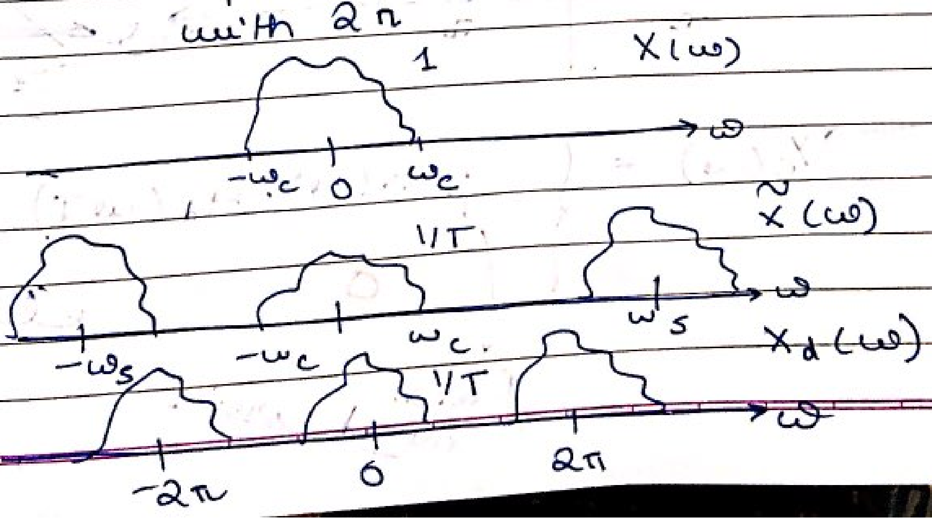
$$X_d(\omega) = \sum x_d[n]\, e^{-j\omega n}$$

$$X_d(\omega) = \sum x_c(nT)\, e^{-j\omega n}$$

$$X_d(\omega T) = \tilde{X}(\omega)$$

periodic with ω_s

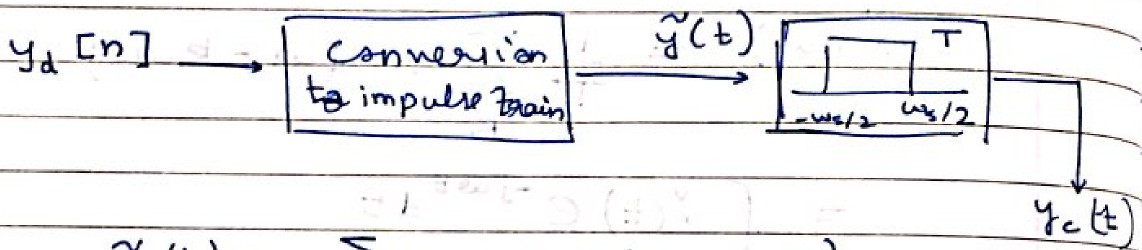
periodic with 2π



$$X_d(\omega) = \tilde{X}\left(\frac{\omega}{T}\right)$$

$$Y_d(\omega) = X_d(\omega)\cdot H_d(\omega)$$

$$= \tilde{X}\left(\frac{\omega}{T}\right)\cdot H_d(\omega)$$



$$\tilde{y}(t) = \sum y_d[n]\,\delta(t - nT)$$

$$Y_d(\omega) = \tilde{Y}\left(\frac{\omega}{T}\right)$$

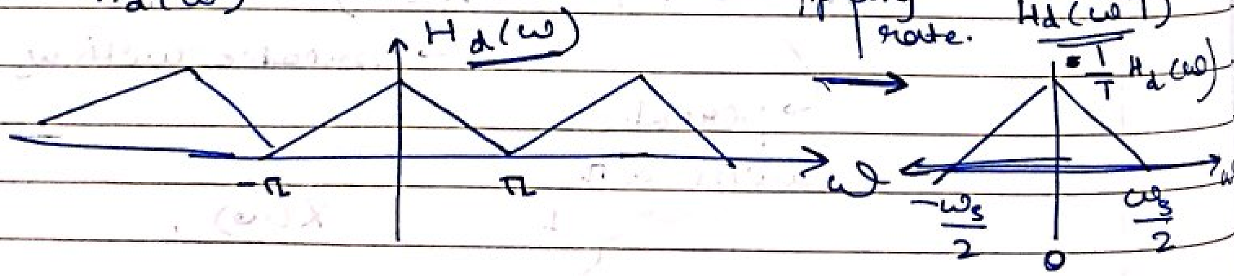
$$\tilde{Y}(\omega) = Y_d(\omega T)$$

Reconstruction

$$\boxed{\tilde{Y}(\omega) = \tilde{X}(\omega)\cdot H_d(\omega T)}$$

d-t filter tuned to sampling rate.

$H_d(\omega)$



$$Y_c(\omega) = \begin{cases} \frac{1}{T} X(\omega)\cdot H_d(\omega T) & |\omega| \le \frac{\omega_s}{2} \\ 0 & \text{else} \end{cases}$$

factor missed earlier

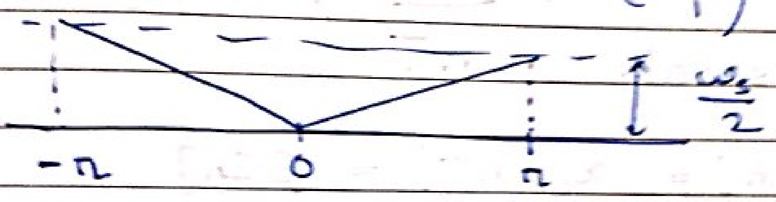
y. Differentiator

$$y_c = x_c'$$
$$Y(\omega) = j\omega \cdot X_c(\omega)$$
$$H_c(\omega) = j\omega$$

→ What is the d-t filter we need?

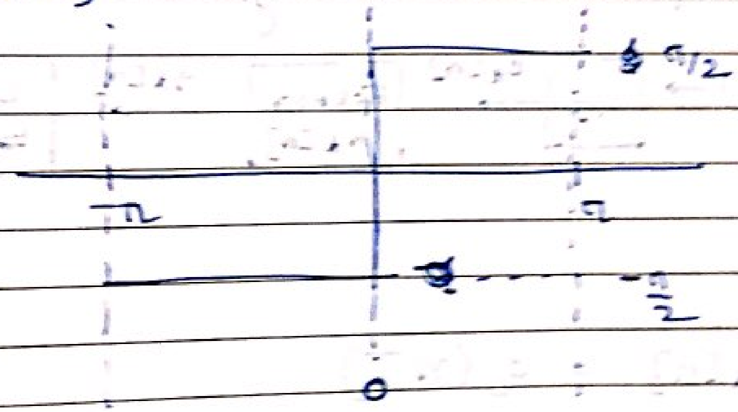
$$H_d(\omega) = H_c\left(\frac{\omega}{T}\right) \qquad |\omega| < \pi$$

↳ d-t FT



$|H_d(\omega)|$

$\measuredangle H_d(\omega)$



$$x_c(t) = \frac{\sin(\pi t/T)}{\pi t} = \frac{1}{T}\,\mathrm{sinc}\left(\frac{t}{T}\right)$$

$$y_c(t) = x_c'(t)$$

$$= \frac{\cos\left(\frac{\pi t}{T}\right)}{tT} - \frac{\sin\left(\frac{\pi t}{T}\right)}{\pi t^2}$$

→ Discrete time samples:

$$y_d[n] = y_c(nT) = \begin{cases} 0, & n=0 \\ \frac{\cos(n\pi)}{nT^2}, & n \neq 0 \end{cases}$$

$$h_d[n] = \begin{cases} 0, & n=0 \\ \frac{(-1)^n}{nT}, & n \neq 0 \end{cases}$$ using convolution

For this signal:

$$x_d[n] = x_c(nT) = \frac{\delta[n]}{T}$$

17.4.18 Recap: d-t Filtering

$$x_c(t) \to \boxed{C/D} \xrightarrow{x_d[n]} \boxed{\text{Filter } h_d[n]} \xrightarrow{y_d[n]} \boxed{\text{Convert to impulse train}} \xrightarrow{\tilde{y}(t)} \boxed{\;\;}_{-\frac{\omega_s}{2}}^{\frac{\omega_s}{2}} \to y_c(t)$$

$$x_d[n] = x_c(nT)$$

$$\tilde{y}(t) = \sum y[n]\,\delta(t-nT)$$

$$\cancel{Y_d(\omega) \neq H_d(\omega)}$$

→ Where,

$$X_d(\omega) = \tilde{X}\left(\frac{\omega}{T}\right), \text{ where}$$

$$\tilde{X}(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$$

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$$\tilde{y}(t) = \sum y[n]\,\delta(t-nT)$$

$$Y_d(\omega) = H_d(\omega)\cdot X_d(\omega)$$

$$Y_d(\omega) = \tilde{Y}\left(\frac{\omega}{T}\right)$$

$$\tilde{Y}(\omega) = Y_d(\omega T) = H_d(\omega T)\,\tilde{X}(\omega)$$

periodic w.p ω_s

$$H_c(\omega) = \begin{cases} H_d(\omega T) & |\omega| < \frac{\omega_s}{2} \\ 0 & \text{else} \end{cases}$$

End-end frequency response

Q→

$$x[n] \longrightarrow X(\omega)$$

$$x[n-n_0] \longrightarrow e^{-j\omega n_0} X(\omega)$$

$\omega \in (-\pi, \pi]$

→ What is the Inverse of : $e^{-j\omega/2} X(\omega)$

must specify over what since $e^{-j\omega/2}$ is not periodic with 2π

$x[n-1/2] \times$

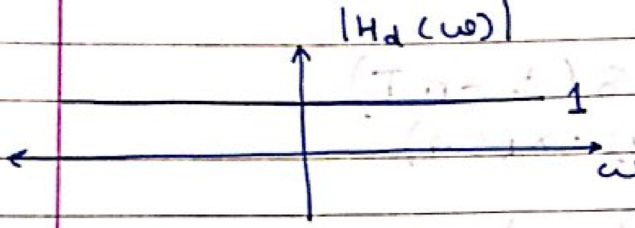
ex

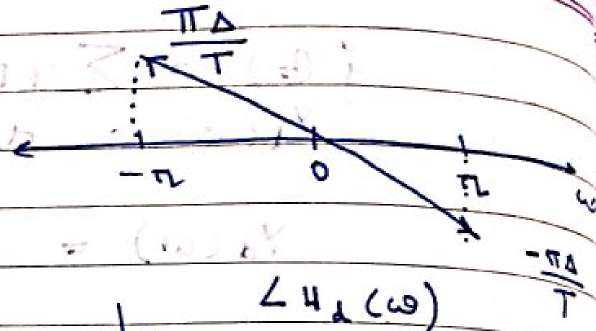
Want to achieve:

$$y_c(t) = x_c(t-\Delta)$$ (Can't do in Analog)

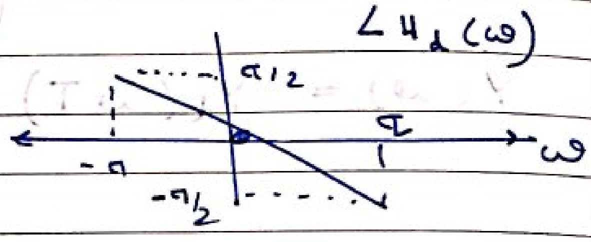
$$H_c(\omega) = e^{-j\omega\Delta}$$

$$H_d(\omega) = H_c\left(\frac{\omega}{T}\right) \quad (|\omega| < \pi)$$





pick $\Delta = \frac{T}{2}$



→ Use the sinc i/p trick:

$$x_c(t) = \frac{\sin\left(\frac{\pi t}{T}\right)}{\pi t}$$

(Chosen in such a way that input to filter is a single d-t impulse)

Satisfies Nyquist & sampled → single impulse.

$$y_c(t) = x_c(t - \Delta)$$

$$x_d[n] = x(nT) = \frac{1}{T}\delta[n]$$

$$h_d[n] = T y_d[n] = T y_c(nT)$$

→ <u>Gaussian Pulse</u>

$$x(t) = e^{-t^2/2}$$

$$X(s) = \int x(t) e^{-st} dt$$

$$= \int e^{-t^2/2} e^{-st} \, dt$$

$$= e^{s^2/2} \int e^{-\frac{1}{2}(t^2 + 2st + s^2)} \, dt$$

$$= e^{s^2/2} \underbrace{\int_{-\infty}^{\infty} e^{-t^2/2} \, dt}_{c} = c e^{s^2/2}$$

→ FT (Different x, Same $X \Rightarrow X(j\omega)$)

$$X(\omega) = c e^{-\omega^2/2}$$

Find c, using duality

$$x(\cdot) \xrightarrow{FT} c x(\cdot) \xrightarrow{FT} c^2 x(\cdot) = 2\pi x(-\cdot)$$

$$\Rightarrow c^2 = 2\pi$$
$$c = \sqrt{2\pi}$$

19.4.18

→ A non-zero signal cannot be time & band-limited

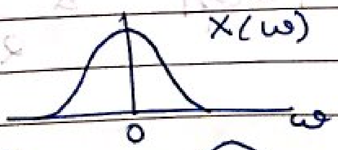
→ ~~Proof~~ Proof: via contradiction:

Assume $\exists \, x(t)$ s.t

$|x(t)| = 0 \;\; \forall \; |t| > t_0$

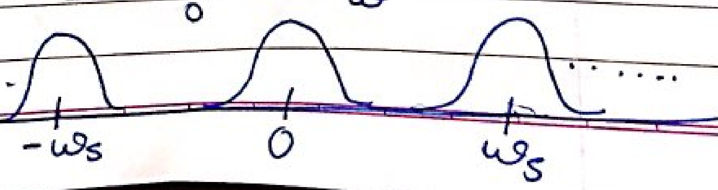
$|X(\omega)| = 0 \;\; \forall \; |\omega| > \omega_0$

→ Say $X(\omega)$ is



then $\tilde{X}(\omega)$ will be

(we can always define.



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→ if, $T = \frac{2\pi}{\omega_s}$

$\tilde{X}(\omega)$: Can be written in terms of sample

$$\tilde{X}(\omega) = T \sum_{-\infty}^{\infty} x(nT) e^{j\left(\frac{2\pi}{\omega_s}\right)k\omega}$$

x time limited

$$\Rightarrow \quad \tilde{X}(\omega) = T \underbrace{\sum_{-n_1}^{n_1} x(nT) e^{jTk\omega}}_{\text{Analytic}}$$

→ RHS cannot be zero over an internal

o/w Identically zero

→ contradiction

→ The Uncertainty Principle

$$\int_{-\infty}^{\infty} |x(t)|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

→ Say, $\int |x(t)|^2 dt = 1$ (like pdf)

→ Hence we can talk about variance

→ Can Interpret

$|x(t)|^2$ & $\frac{1}{2\pi} |X(\omega)|^2$ as pdf in (t & ω)

$x(at)$

$$\mu_{time} = \int t\,|x(t)|^2\,dt$$

$$\sigma^2_{time} = \int (t - \mu_{time})^2\,|x(t)|^2\,dt$$

$$\mu_{freq} = \frac{1}{2\pi}\int \omega\,|X(\omega)|^2\,d\omega$$

$$\sigma^2_{freq} = \frac{1}{2\pi}\int (\omega - \mu_{freq})^2\,|X(\omega)|^2\,d\omega$$

<u>Result</u> : $\sigma^2_{time} \times \sigma^2_{freq} \geq \frac{1}{4}$

<u>H.W - 6</u>

1-a) $X(\omega) = A(\omega) + jB(\omega) \leftrightarrow x[n]$

$B(\omega) + A(\omega)e^{j\omega} \leftrightarrow y[n]$

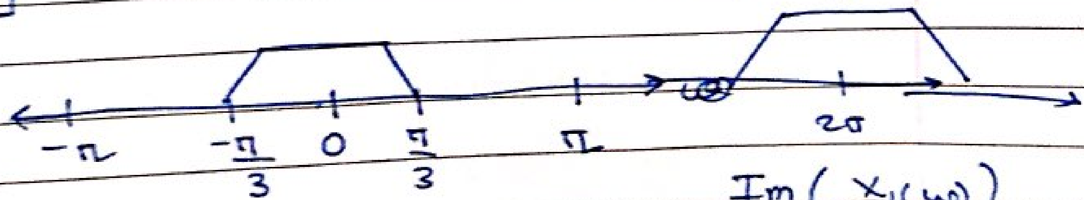
Even part of $x[n]$: $z[n] = \frac{x[n] + x[-n]}{2} \leftrightarrow A(\omega)$

Odd part : $a[n] = \frac{x[n] - x[-n]}{2} \leftrightarrow jB(\omega)$

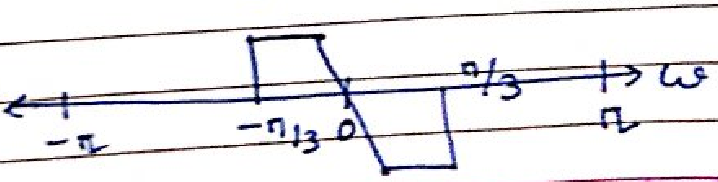
$\Rightarrow\ y[n] = a[n] + z[n+1]$

1-b) $x_1[n]$

$Re(X_1(\omega))$

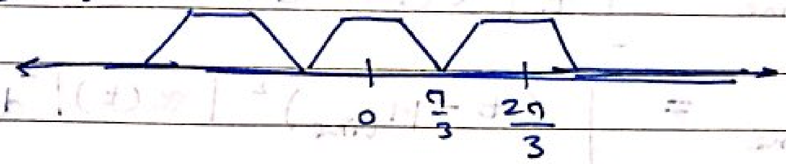


$Im(X_1(\omega))$



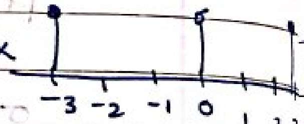
Notice Conjugate symmetric

a) $X_2(\omega)$



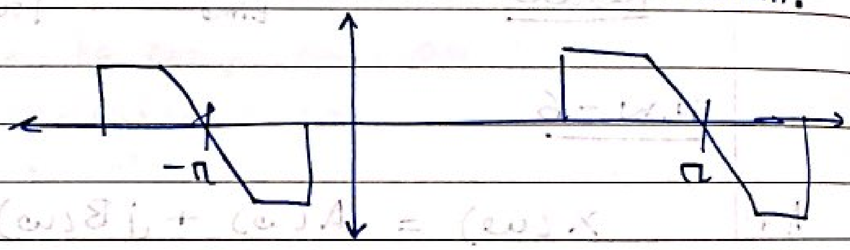
Q→ Write $x_2[n]$ in terms of $x_1[n]$

$$x_2[n] = \left[\frac{x_1[n] + x_1[-n]}{2}\right] \times$$



↳ from sampling theorem.

$X_3(\omega)$ →



~~to~~ ~~xpre~~ express $x_3[n]$ in terms of $x_1[n]$

$$x_3[n] = (-1)^n \left[\frac{x_1[n] - x_1[-n]}{2j}\right]$$