CHAPTER 14

# Prototype and Frequency-Transformed Ladders

The most ancient of wave filters, built late in the last century, were made with inductors and capacitors arranged in ladder form, as shown in Fig. 14.1. Even with roots in antiquity, the lossless ladder remains useful to this day. It is still used in high-frequency applications and as a model for simulation. In this chapter we introduce some of the key concepts in the design of ladder filters. We also show the usefulness of tables, once we have some understanding of how such tables are generated.

## 14.1 SOME PROPERTIES OF LOSSLESS LADDERS

There are various ways to describe the circuit of Fig. 14.1. It is made up of inductors and capacitors, and so can be described as lossless or as reactance. It is simple in the sense that each branch contains only one element. It is arranged such that all series elements are inductors and all shunt elements are capacitors. We will show that it is a lowpass filter and so may be regarded as the prototype circuit to which the frequency transformations of Chapter 11 may be applied to obtain nonsimple ladder structures.

To begin, we see that since at zero frequency the capacitors are open circuits and the inductors are short circuits, the zero-frequency equivalent of the circuit of Fig. 14.1 is two wires, such that

$$\frac{V_2}{V_1}(0) = 1 \tag{14.1}$$

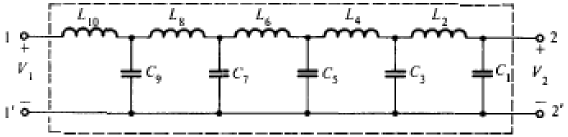



FIGURE 14-1

The general relationship is found by starting at side 2 and working toward side 1. The current in $C_1$ is

$$I_{C_1} = Y_{C_1}V_2 = C_1 s V_2 \tag{14.2}$$

The current in inductor $L_2$ is the same as $I_{C_1}$ so that the voltage across $L_2$ is

$$V_{L_2} = Z_{L_2}I_{L_2} = L_2 s(C_1 s V_2) \tag{14.3}$$

By Kirchhoff's voltage law the voltage across capacitor $C_3$ is

$$V_{C_3} = V_{L_2} + V_2 = (C_1 L_2 s^2 + 1)V_2 \tag{14.4}$$

Then the current in $C_3$ is

$$I_{C_3} = Y_{C_3}V_{C_3} = C_3 s(C_1 L_2 s^2 + 1)V_2 \tag{14.5}$$

If this process of analysis is continued until side 1 is reached, it is found that

$$V_1(s) = Q(s)\, V_2(s) \tag{14.6}$$

or

$$\frac{V_2(s)}{V_1(s)} = \frac{1}{Q(s)} \tag{14.7}$$

where $Q(s)$ is an even polynomial of degree $n$, $n$ being the total number of inductors plus capacitors. Such a function is known as an *all-pole function*, since all zeros are at infinity. At high frequencies the last equation will have the asymptotic form

$$\frac{V_2(s)}{V_1(s)} \approx \frac{K}{s^n} \tag{14.8}$$

meaning that the transfer function will roll off at the rate of $-6n$ dB per octave. Since the zero-frequency value is 1 and the roll off rate is negative, the transfer function represents a lowpass filter.

Two specific circuits of the general form of the ladder of Fig. 14.1 are shown in Fig. 14.2. For the circuit in Fig. 14.2a the transfer function is

$$T(s) = \frac{V_2}{V_1} = \frac{1}{s^2 + 1} \tag{14.9}$$

Similarly, for Fig. 14.2b it is

$$T(s) = \frac{V_2}{V_1} = \frac{1}{s^4 + 3s^2 + 1} \tag{14.10}$$



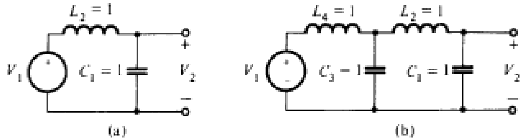



FIGURE 14-2

Both are all-pole functions, and for both the degree of the denominator polynomial is equal to the number of elements in the circuit.

The ladder circuit of Fig. 14.1 is also known as a lossless *coupling circuit*, that usage implying a load resistor at the output side. If there is a resistor at the input side, representing the resistance of the source, for example, then the structure as shown in Fig. 14.3 is known as a *doubly terminated* structure. The input resistor is designated as $R_1$ and the load resistor as $R_2$. To distinguish the voltage at the input to the lossless circuit from that of the source, one is designated as $V_1$ and the other as $V_s$.

A specific example of a lossless coupling circuit with double terminations is shown in Fig. 14.4. It is made from the circuit of Fig. 14.2a with $R_1 = R_2 = 1\ \Omega$. For that circuit the current in the inductor is simply related to the output voltage:

$$I_L = (s + 1)V_2 \tag{14.11}$$

The source voltage is, by Kirchhoff's voltage law,

$$V_s = (s + 1)I_L + V_2 = (s + 1)^2 V_2 + V_2 \tag{14.12}$$

Then

$$\frac{V_2}{V_s} = T = \frac{1}{s^2 + 2s + 2} \tag{14.13}$$

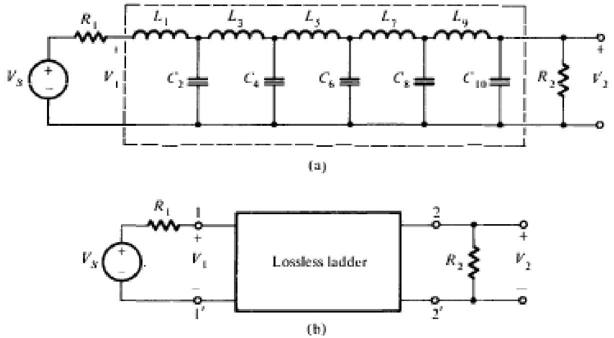



FIGURE 14.3

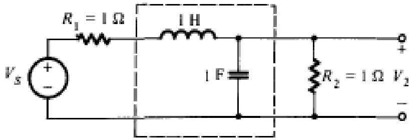



FIGURE 14.4

where T is used to distinguish the new transfer function from $T = V_2/V_1$. Comparing this equation with Eq. (14.8), we see that the addition of two resistors to the circuit has not increased the degree of the denominator polynomial. However, the polynomial is no longer even, but has both even and odd parts. Further, $\mathbf{T}(0) = \frac{1}{2}$, indicating the voltage-divider action of the circuit at zero frequency.

The transfer function for a general doubly terminated circuit such as that shown in Fig. 14.3a has the form

$$\mathbf{T} = \frac{V_2}{V_s} = \frac{K}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0} = \frac{K}{Q(s)} \tag{14.14}$$

For the doubly terminated circuit of Fig. 14.3a it is necessary that

$$\frac{V_2}{V_s}(0) = \frac{R_2}{R_1 + R_2} \tag{14.15}$$

For the special case where $R_1 = R_2$, the ratio of Eq. (14.15) has the value of $\frac{1}{2}$, as was the case for Eq. (14.13). For the last two equations to be compatible, it is necessary that

$$\frac{K}{b_0} = \frac{R_2}{R_1 + R_2} \tag{14.16}$$

This amounts to a constraint on the value of $K$, and tells us that the gain $K$ of the transfer function of Eq. (14.14) has a specific limit. It should be noted that the condition of Eq. (14.16) assumes that $\mathbf{T}(0) = \mathbf{T}_{max}$ as is the case for the Butterworth response. If this maximum does not occur at $\omega = 0$, then we require that

$$|\mathbf{T}(j\omega)| \le \mathbf{T}_{max} \qquad \text{all} \quad \omega \tag{14.17}$$

and that the $K$ in Eq. (14.14) be adjusted such that this is the case.

Comparing the denominator of Eq. (14.14) with the $Q(s)$ in Eq. (14.6), we see that both the lossless circuit and the doubly terminated lossless circuit are represented by transfer functions having the same denominator degree, which is $n$. Further, for both circuits $n$ is the total number of capacitors and inductors. Thus both transfer functions have the same asymptotic rolloff rate of $-6n$ dB/octave.

In arriving at Eq. (14.13) we analyzed the circuit given in Fig. 14.4 and obtained the transfer function $\mathbf{T}(s)$. The synthesis problem is the inverse: Given the values of $R_1$ and $R_2$ and a prescribed form for the transfer function $\mathbf{T}(s)$, we are required to find the lossless ladder circuit element values.



## 14.2 A SYNTHESIS STRATEGY

The doubly terminated circuit already considered is shown again in Fig. 14.5 with two additional features identified. The current from the source is designated $I_1$, and its reference direction is shown in the figure. In addition, the input impedance $Z_{11}$ is identified as the impedance of the $RLC$ circuit made up of the lossless ladder and the terminating resistor $R_2$. We assume that the circuit is operating in the sinusoidal steady state. The input impedance has both a real and an imaginary component

$$Z_{11} = R_{11} + jX_{11} \tag{14.18}$$

The current at the input is

$$I_1 = \frac{V_s}{R_1 + Z_{11}} \tag{14.19}$$

Now since the $LC$ circuit is lossless, we may equate the average power into the circuit to that in the load. Thus,

$$R_{11}|I_1(j\omega)|^2 = \frac{|V_2(j\omega)|^2}{R_2} \tag{14.20}$$

Substituting Eq. (14.19) for $I_1$ into this equation, gives us

$$\frac{R_{11}|V_s(j\omega)|^2}{|R_1 + Z_{11}|^2} = \frac{|V_2(j\omega)|^2}{R_2} \tag{14.21}$$

From this equation we determine the magnitude squared of the desired transfer function

$$\left|\frac{V_2(j\omega)}{V_s(j\omega)}\right|^2 = |\mathbf{T}(j\omega)|^2 = \frac{R_2 R_{11}}{|R_1 + Z_{11}|^2} \tag{14.22}$$

Our end objective is to determine $Z_{11}$ as a function of the $|\mathbf{T}(j\omega)|^2$ specification. To accomplish this, we seemingly go a roundabout way by introducing an auxiliary function:

$$|A(j\omega)|^2 = 1 - 4\frac{R_1}{R_2}|\mathbf{T}(j\omega)|^2 \tag{14.23}$$

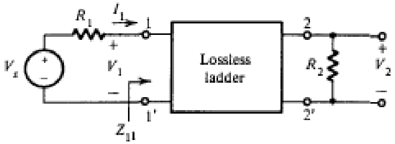



FIGURE 14.5

Into this equation we substitute Eq. (14.22) to give, after some algebraic manipulation,

$$|A(j\omega)|^2 = A(s)A(-s)|_{s=j\omega} = \frac{|R_1 - Z_{11}|^2}{|R_1 + Z_{11}|^2} \tag{14.24}$$

From this equation we see that $A(s)$ may be separated from $A(-s)$ to give

$$A(s) = \pm \frac{R_1 - Z_{11}}{R_1 + Z_{11}} \tag{14.25}$$

Clearly, this equation may be solved for $Z_{11}$ in terms of $A(s)$ and $R_1$. The result is

$$Z_{11} = R_1 \frac{1 - A(s)}{1 + A(s)} \tag{14.26}$$

or

$$Z_{11} = R_1 \frac{1 + A(s)}{1 - A(s)} \tag{14.27}$$

Here we have reduced the problem to determining a lossless circuit terminated in a resistor $R_2$ from a specified $Z_{11}$. Sidney Darlington showed that this was always possible in his classic 1939 paper,* and so the circuit may be found.

To illustrate the application of the strategy, consider the problem of determining circuits having a Butterworth response for which there are equal terminations with $R_1 = R_2 = 1$, a normalized value that may later be magnitude scaled. The Butterworth response is

$$|T_n(j\omega)|^2 = \frac{1}{1 + \omega^{2n}} \tag{14.28}$$

However, this magnitude-squared function must be reduced to satisfy the condition of Eq. (14.16). Since the magnitude-squared function is

$$|T(j\omega)|^2 = T(s)T(-s)|_{s=j\omega} \tag{14.29}$$

then from Eq. (14.14) we have

$$|T(j\omega)|^2 = \frac{K^2}{|Q(j\omega)|^2} = \frac{K^2}{Q(s)Q(-s)}\bigg|_{s=j\omega} \tag{14.30}$$

These quantities are known from our study of the Butterworth functions in Chapter 6, and are shown in the following table:

| *Order* | $\lvert Q(j\omega)\rvert^2$ | $Q(s)$ |
|---|---|---|
| 1 | $1 + \omega^2$ | $s + 1$ |
| 2 | $1 + \omega^4$ | $s^2 + \sqrt{2}s + 1$ |
| 3 | $1 + \omega^6$ | $s^3 + 2s^2 + 2s + 1$ |

* S. Darlington, "Synthesis of Reactance 4-Poles which Produce Prescribed Insertion Loss Characteristics, *J. Math. Phys.*, pp. 257–353, Sept. 1939.



So we see that $b_0$ in Eq. (14.14) is 1 for all $n$, and with equal terminations it is then necessary that $K = \frac{1}{2}$. Then $|T(j\omega)|^2$ in Eq. (14.28) must be multiplied by $K^2 = \frac{1}{4}$. With that accomplished, we may determine the auxiliary function of Eq. (14.23):

$$|A(j\omega)|^2 = 1 - \frac{1}{1 + \omega^{2n}} = \frac{\omega^{2n}}{1 + \omega^{2n}} \tag{14.31}$$

Since

$$\omega^{2n}|_{\omega^2=-s^2} = s^n(-s)^n \tag{14.32}$$

we see that

$$A(s) = \frac{s^n}{Q(s)} \tag{14.33}$$

Substituting this value of $A(s)$ into Eqs. (14.26) and (14.27), we obtain

$$Z_{11} = R_1\left(\frac{1 - A}{1 + A}\right)^{\pm 1} = R_1\left(\frac{Q - s^n}{Q + s^n}\right)^{\pm 1} \tag{14.34}$$

It is conventional to normalize the impedance level of the circuit by letting $R_1 = 1$. If we do that in the last equation, and then let $n$ have several values, we will see the pattern that is followed in obtaining the circuits.

***n* = 1**

For $n = 1$, $Q = s + 1$, and $A(s) = s/(s + 1)$. Hence the impedance $Z_{11}$ is

$$Z_{11} = \frac{1 - s/(s + 1)}{1 + s/(s + 1)} = \frac{1}{2s + 1} \tag{14.35}$$

or

$$Z_{11} = 2s + 1 \tag{14.36}$$

The circuit realizations of these two $Z_{11}$ functions are shown in Fig. 14.6 with the section enclosed by the dashed lines being the lossless coupling circuit. In this case the element values are evident from the direct inspection of the expressions for $Z_{11}$.

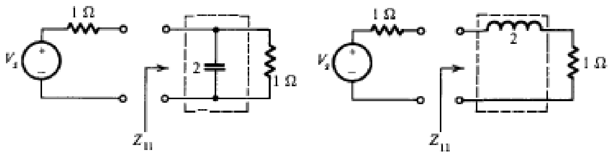



FIGURE 14.6

**$n = 2$**

As already tabulated, $Q = s^2 + \sqrt{2}s + 1$ for the case $n = 2$, and hence

$$A(s) = \frac{s^2}{s^2 + \sqrt{2}s + 1} \tag{14.37}$$

Then we find that $Z_{11}$ is

$$Z_{11} = \frac{\sqrt{2}s + 1}{2s^2 + \sqrt{2}s + 1} \tag{14.38}$$

or the reciprocal of this quantity. This time we cannot determine the circuit represented by this $Z_{11}$ directly by inspection. Instead, we expand this function, or its reciprocal, as a continued fraction. This is always done by dividing the polynomial of higher degree by the polynomial of lower degree. The continued fraction is carried out by synthetic division in these steps:

$$\begin{array}{l} \sqrt{2}s + 1\ )\,2s^2 + \sqrt{2}s + 1\ (\ \sqrt{2}s \leftarrow C_1 s \\ \qquad\qquad \underline{2s^2 + \sqrt{2}s\qquad} \\ \qquad\qquad\qquad\qquad 1\ )\ \sqrt{2}s + 1\ (\ \sqrt{2}s \leftarrow L_2 s \\ \qquad\qquad\qquad\qquad\quad \underline{\sqrt{2}s\qquad} \\ \qquad\qquad\qquad\qquad\qquad\quad 1\ )\ 1\ (\ 1 \leftarrow R_2 \\ \qquad\qquad\qquad\qquad\qquad\qquad\ \ \underline{1} \\ \qquad\qquad\qquad\qquad\qquad\qquad\ \ 0 \end{array} \tag{14.39}$$

Thus the continued fraction is

$$Z_{11} = \cfrac{1}{\sqrt{2}s + \cfrac{1}{\sqrt{2}s + \cfrac{1}{1}}} \tag{14.40}$$

or $1/Z_{11}$ is the same continued fraction. The two circuits that have been found by this process are shown in Fig. 14.7, Fig 14.7a being the direct representation of Eq. (14.40) and Fig. 14.7b being the representation of the reciprocal of this function.

**$n = 3$**

For $n = 3$ we have found that $Q(s) = s^3 + 2s^2 + 2s + 1$, so $A(s)$ is found directly

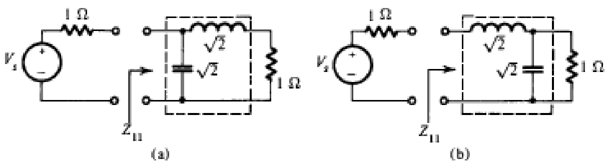



FIGURE 14.7



from Eq. (14.33) as

$$A(s) = \frac{s^3}{s^3 + 2s^2 + 2s + 1} \tag{14.41}$$

So we determine $Z_{11}$ and from that its reciprocal:

$$Z_{11} = \frac{2s^2 + 2s + 1}{2s^3 + 2s^2 + 2s + 1} \tag{14.42}$$

We next expand $Z_{11}$ in a continued fraction by dividing the denominator polynomial by the numerator polynomial in the pattern of dividing one step, inverting, dividing one step, and so on. Thus

$$\begin{array}{l} 2s^2 + 2s + 1\ )\ 2s^3 + 2s^2 + 2s + 1\ (\ s \leftarrow L_1 s \\ \qquad\qquad\quad \underline{2s^3 + 2s^2 + \ s\qquad} \\ \qquad\qquad\qquad\qquad\quad s + 1\ )\,2s^2 + 2s + 1\ (\ 2s \leftarrow C_2 s \\ \qquad\qquad\qquad\qquad\qquad\qquad \underline{2s^2 + 2s\qquad} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 1\ )\ \ s + 1\ (\ s \leftarrow L_3 s \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \ \underline{s\qquad} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 1\ )\ 1\ (\ 1 \leftarrow R_2 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \ \underline{1} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \ 0 \end{array} \tag{14.43}$$

or

$$Z_{11} = \cfrac{1}{s + \cfrac{1}{2s + \cfrac{1}{s + \cfrac{1}{1}}}} \tag{14.44}$$

From this continued fraction expansion, and from its reciprocal, we recognize the two circuits shown in Fig. 14.8, each consisting of a lossless coupling circuit terminated in a 1-Ω resistor. The steps are summarized in Fig. 14.9.

The circuits we have found and their element values are summarized in Table 14.1 for the cases $n = 2$ and $n = 3$ that we have just studied. If this process is continued until $n = 10$, then all of the values shown in the table are obtained. These element values are for the Butterworth case, and once found they may be stored in the table and regarded as completed forever. If necessary, we could derive any one of the circuits with little trouble, but it will never be necessary as long as you have the table.

Comparing the two coupling circuits of Fig. 14.8, we first observe that the impedances of the two circuits have reciprocal relationships and are said to be *duals* of each other. Observe that one has two inductors and one capacitor, while the other has two capacitors and one inductor. Since the two realizations are completely equivalent, there may be some basis for choice depending on whether inductors or capacitors are most easily obtained. One might be called a minimum-capacitance realization, and the other the minimum-inductance realization. These properties hold in general as illustrated by Table 14.2. The first and last element structures are different, depending upon whether $n$ is even or odd, and of

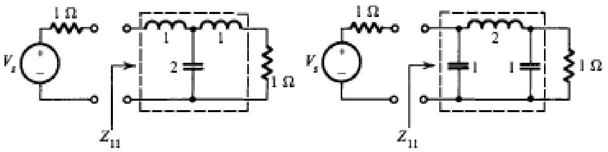



FIGURE 14.8

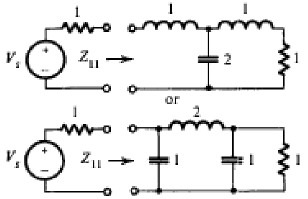
| Procedure to obtain doubly terminated lowpass prototype | Example: Butterworth $n = 3$ $R_1 = R_2 = 1\ \Omega$ |
|---|---|
| $\lvert T(j\omega)\rvert^2, R_1, R_2$ | $\lvert T(j\omega)\rvert^2 = \dfrac{1/4}{1+\omega^6}$ |
| ↓ $\lvert A(j\omega)\rvert^2$ | $\lvert A(j\omega)\rvert^2 = \dfrac{\omega^6}{1+\omega^6}$ |
| ↓ $A(s)$ | $A(s) = \dfrac{\pm s^3}{s^3+2s^2+2s+1}$ |
| ↓ $Z_{11}(s)$ | $Z_{11} = \left(\dfrac{2s^2+2s+1}{2s^3+2s^2+2s+1}\right)^{\pm 1}$ |
| ↓ Continued fraction expansion | $\left.\begin{matrix} Z_{11} \\ \text{and} \\ 1/Z_{11}\end{matrix}\right\} = s + \cfrac{1}{2s + \cfrac{1}{s + \cfrac{1}{1}}}$ |
| ↓ Two circuit realizations |  |



FIGURE 14.9



the two realizations one will always be minimum inductance and the other minimum capacitance.

There are many other interesting properties of ladder circuits which are treated by advanced treatises on the subject.* The general problem of finding a ladder realization is not as simple as might be suggested by the special case of Butterworth filters. In some cases the lossless terminated circuit is found only by including ideal transformers in the realization. Often it is possible to realize different values of $R_1$ and $R_2$, and sometimes it is not possible to find a circuit with equal terminations without the ideal transformer.

One additional example will illustrate the case of unequal terminations. Consider the realization of a filter with a Butterworth response for which $n = 3$, $R_1 = 1\Omega$, and $R_2 = 2\Omega$. From Eq. (14.16) we see that

$$\frac{K}{b_0} = \frac{R_2}{R_1+R_2} = \frac{2}{3} \qquad \text{or} \qquad K = \frac{2}{3}b_0 = \frac{2}{3} \tag{14.45}$$

Then

$$|A(j\omega)|^2 = 1 - 2\,\frac{(2/3)^2}{1+\omega^6} = 1 - \frac{8/9}{1+\omega^6} \tag{14.46}$$

or

$$A|(j\omega)^2| = \frac{\omega^6 + 1/9}{\omega^6+1} \tag{14.47}$$

$$A(s)A(-s) = \frac{(s^3+1/3)(-s^3+1/3)}{(s^3+2s^2+2s+1)(-s^3+2s^2-2s+1)} \tag{14.48}$$

Then

$$A(s) = \frac{s^3+1/3}{s^3+2s^2+2s+1} \tag{14.49}$$

From this we find

$$Z_{11} = \frac{2s^2+2s+2/3}{2s^3+2s^2+2s+4/3} \tag{14.50}$$

The continued fraction expansion of this function gives

$$Z_{11} = \cfrac{1}{s + \cfrac{1}{1.5s + \cfrac{1}{2s + \cfrac{1}{0.5}}}} \tag{14.51}$$

The circuit realizations of this $Z_{11}$ and for its reciprocal are shown in Fig. 14.10.

* D. S. Humphreys, *The Analysis, Design and Synthesis of Electrical Filters*, Prentice-Hall, Englewood Cliffs, N.J., 1970.

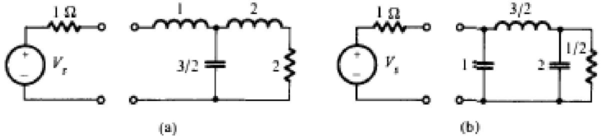



FIGURE 14.10

## 14.3 TABLES FOR OTHER RESPONSES

The last section introduced the philosophy of the generation of tables by a study of the Butterworth response case. In Table 14.1 we have all of the Butterworth circuits from $n = 2$ to $n = 10$.* This table was derived for the half-power frequency of $\omega_0 = 1$ rad/s, normalized in magnitude such that $|T(j0)| = 1$ or $\alpha(0) = 0$ dB. We recognize that the derivations which were illustrated in the last section need not be repeated—we have done it once and for all. If we should need a Butterworth filter of order $n$, we simply consult Table 14.1 which gives the required circuit. We may then frequency scale and magnitude scale as required.

Since the form of the transfer function $|T(j\omega)|^2$ is known for many other kinds of responses, the general procedure given in the last section can be applied to any responses and thus determine ladder circuits for any specified form of magnitude response. For example, we know from our studies in Chapter 8 that the Chebyshev response has the form

$$|T(j\omega)|^2 = \frac{1}{1 + \epsilon^2 C_n^2(\omega)} \tag{14.52}$$

This equation reminds us that the Chebyshev response must be tabulated for different values of $n$, but also some factor that is equivalent to $\epsilon$, usually $\alpha_{max}$. This makes the tabulation of information for the Chebyshev response more complex than it was for the Butterworth response.

Table 14.3 is typical of Chebyshev response tables. In Table 14.3 a passband ripple of $\alpha_{max} = 0.1$ dB is used with $n$ ranging from 2 to 8.

If you object to letting someone else do your work, you can follow the procedure outlined for the Butterworth response and derive your own circuits. Another alternative is to make use of explicit formulas that have been derived for the Butterworth and Chebyshev cases.† The derivations leading to these formulas are very clever and are recommended for the serious reader. The formulas may be used to derive your own tables, or to solve problems for which no convenient tables can be found.

In Chapter 13 we learned of the Cauer (or elliptic) form of response, and found that it was a very efficient magnitude response. Cauer circuits realize a given set of specifications with a lower value of $n$, in general, and this is accom-

* For larger $n$ consult Weinberg, listed in Appendix B.

† A. S. Sedra and P. O. Brackett, *Filter Theory and Design: Active and Passive*, Matrix Publishers, Portland, Ore., pp. 208–212.