An introduction to programming through C++
Ch. 24 : Structural recursion
Layout of mathematical formulae

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Input: A textual description of a formula e.g.

```
\pi = \cfrac{4}{1+\cfrac{1^2}{3+\cfrac{2^2}
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Output: A visually pleasant layout of it:

$$\pi = \cfrac{4}{1+\cfrac{1^2}{3+\cfrac{2^2}{5+\cfrac{3^2}{7+\cfrac{4^2}{9+\ddots}}}}}$$

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- How will we determine the position and the sizes of the different parts of the layout.

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- Eliminating these restrictions: exercises.

Examples of output and input

	Desired output	Input required by our program
0.	a	`a`
1.	$\dfrac{a}{b+c}$	`(a/(b+c))`
2.	$a+\dfrac{b}{c}$	`(a+(b/c))`
3.	$a+b+c+d$	`(((a+b)+c)+d)`
4.	$\dfrac{x+1}{x+3}+\dfrac{x}{5}+6$	`((((x+1)/(x+3))+(x/5))+6)`

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Thus formulae have heirarchical/recursive structure.

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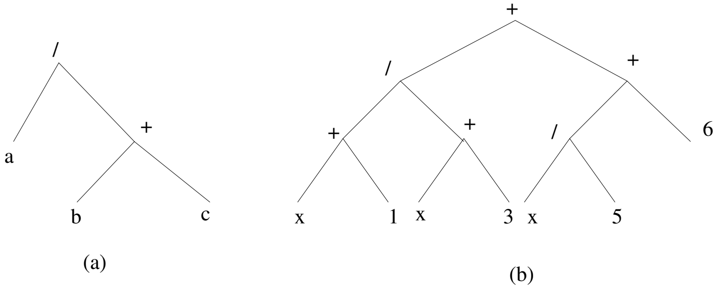
- Sum formulae are represented by a tree: a root node associated with a label "+", left subtree consisting of the representation of the left summand, right subtree consisting of the representation of the right summand.
- Ratio formulae are similarly represented: root node associated with a label "/", left subtree consisting of the representation of the numerator, right subtree consisting of the representation of the denominator.

Examples



(a) Tree for $\frac{a}{b+c}$ (b) Tree for $\frac{x+1}{x+3} + \frac{x}{5} + 6$

Representation in a program

```
struct Node{
  char op;       // operator associated with node if any
  string value;  // value associated with node, if any
  Node* L;       // pointer to left subformula, if any
  Node* R;       // pointer to right subformula, if any
  Node(char op1, Node* L1, Node* L2){
    op = op1;
    L  = L1;
    R  = R1;
  }
  Node(string v){
  // simplified constructor for primitive formulae
    value = v;
    op = 'P';    // 'P' denotes primitive formula
    L = R = NULL // No subformulae.
  }
  // other member functions to be described later
}
```

Creating formulae inside a program

```
Node aexp("a"), bexp("b"), cexp("c");
Node bplusc('+', &bexp, &cexp);
Node f1('/', &aexp, &bexp)

Node f2 = new Node('/', new Node("a"),
                   new Node('+', new Node("b"),
                             new node("c")));
```

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  4. The character $')'$ signifying the end of the sum or the ratio.

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We can put the code for reading in the formula into a constructor.

```
int main(){
  Node formula(cin);  // Described next.  Argument gives
  // the stream from which to read in the formula.
}
```

The code

```
Node::Node(istream &infile){
  char c=infile.get();
                                  // is it a primitive formula?
  if((c >= '0' && c <= '9') ||
     (c >= 'a' && c <= 'z') ||
     (c >= 'A' && c <= 'Z')){
    L=R=NULL; op='P'; value = c;
  }
  else if(c == '('){     // is it a non-primitive formula?
    L = new Node(infile); // recursively get the L formula
    op = infile.get();    // get the operator
    R = new Node(infile); // recursively get the R formula
    if(infile.get() != ')')
      cout << "No matching parenthesis.\n";
  }
  else cout << "Error in input.\n";
}
```

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When we draw the layout, we must specify where to draw it.

```
void Node::draw(double x, double y){ // top left corner
  switch(op){
  case 'P':
    Text(x + textWidth(value)/2,
         y + textHeight()/2, value).imprint();
    break;
  case '+':  ... How do we do this? ...
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To layout a sum formula, we must know how the summands must align with each other.
To layout a ratio formula, we must know how long to draw the line denoting the division.

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Parameters width, height, ascent can be determined recursively.

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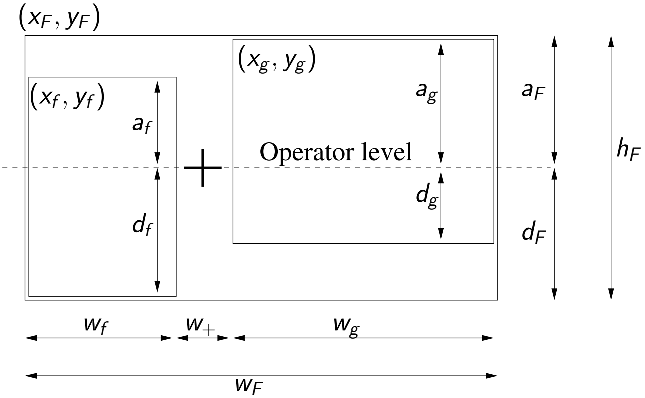

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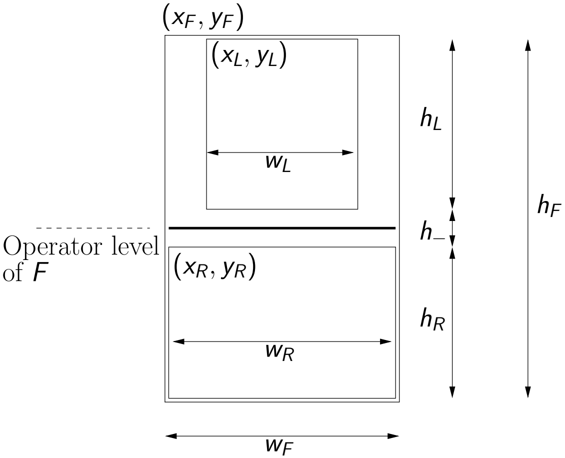

The final code: definition of `Node`

```
struct Node{
  static const int h_bar = 10; // space for horizontal bar
  Node *L, *R;
  char op;
  string value;
  double width, height, ascent, descent;
  Node(string v);
  Node(char op1, Node* L1, Node* R1);
  Node(istream& infile);
  void setSizes();
  void draw(double clx, double y); // to actually draw
};
```

Code to set parameters: function setSizes

```
void Node::setSizes(){
  switch (op){
  case 'P':                       // Primitive formula
    width = textWidth(value);
    height = textHeight(); ascent = descent = height/2;
    break;
  case '+':                         // case L+R
    L->setSizes();
    R->setSizes();
    width = L->width + textWidth(" + ") + R->width;
    descent = max(L->descent, R->descent);
    ascent = max(L->ascent, R->ascent);
    height = ascent + descent;
    break;
  case '/':                        // case L/R
  ...
  }
}
```

Code to set parameters: function setSizes

```
void Node::setSizes(){
  switch (op){
  ...
  case '/':                    // case L/R
    L->setSizes();
    R->setSizes();
    width = max(L->width, R->width);
    ascent = h_bar/2 + L->height;
    descent = h_bar/2 + R->height;
    height = ascent + descent;
    break;
  default: cout << "Invalid input.\n";
  }
}
```

Code for drawing

```
void Node::draw(double x, double y){
  switch(op){  // case 'P' given earlier
  case '+':
    L->draw(x, y + ascent - L->ascent);
    R->draw(x + L->width + textWidth(" + "),
            y + ascent - R->ascent);
    Text(x + L->width + textWidth(" + ")/2, y + ascent,
         string(" + ")).imprint();    // draw the '+'
    break;
  case '/':
    L->draw(x + width/2 - L->width/2, y);
    R->draw(x + width/2 - R->width/2,
            y + L->height + h_bar);
    Line(x, y + ascent, x + width, y + ascent).imprint();
    break;
  default: cout << "Invalid input.\n";
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- Once you represent formulae, you can manipulate them, e.g.
  evaluate them given the values of variables,
  take their derivatives with respect to a variable,
  find the product of formulae.