Assume F is sat.

$\exists$ an assignment α s.t. $\alpha \models F$.

If $a = T$ in α, then

$$\alpha \models (\neg a \vee C_{k+1}) \wedge (\neg a \vee C_{k+2}) \wedge \dots \wedge B_1 \wedge \dots \wedge B_m$$

$$\Rightarrow \alpha \models C_{k+1} \wedge C_{k+2} \wedge \dots \wedge C_n \wedge B_1 \wedge \dots \wedge B_m$$

$$\Rightarrow \alpha \models R(F)$$

Else if $a = \perp$ in α, then

$$\alpha \models (a \vee C_1) \wedge (a \vee C_2) \wedge \dots \wedge (a \vee C_k) \wedge B_1 \wedge \dots \wedge B_m$$

$$\Rightarrow \alpha \models C_1 \wedge \dots \wedge C_k \wedge B_1 \wedge \dots \wedge B_m$$

$$\Rightarrow \alpha \models R(F).$$

Conversely, let $R(F)$ be sat.

Then $\exists$ α s.t. $\alpha \models R(F)$.

$\therefore \alpha \models B_1 \wedge \dots \wedge B_m$.

(1) $\alpha \models C_1 \wedge \dots \wedge C_k$. Then $\exists$ an assignment β which agrees with α, and assigns $a = \perp$ st

$$\beta \models B_1 \wedge \dots \wedge B_m \text{ and}$$

$$\beta \models \cancel{aA}\ (a \vee C_1) \wedge (a \vee C_2) \wedge \dots \wedge (a \vee C_k)$$

and $\beta \models (\neg a \vee C_{k+1}) \wedge (\neg a \vee C_{k+2}) \wedge \dots \wedge (\neg a \vee C_n)$

$\therefore \beta \models F$

(2) $\exists\ j \in [1, k]$ s.t. $\alpha \not\models C_j$

But $\alpha \models (C_j \vee C_{k+1}) \wedge (C_j \vee C_{k+2}) \wedge \dots \wedge (C_j \vee C_n)$

$$\Rightarrow \alpha \models C_{k+1} \wedge C_{k+2} \wedge \dots \wedge C_n$$

As in (1), let β agree with α and assign $a = T$. Then

$$\beta \models B_1 \wedge \dots \wedge B_m \text{ and}$$

$$\beta \models (a \vee C_1) \wedge \dots \wedge (a \vee C_k)$$

Thus $\beta \models F$.

Thus, it is enough to maintain clauses in $R(F)$.

$\models F \to F'$ and $\models F' \to H$ gives

$\models F \to H$.

Thus we have $\models F \to H$ and $\models H \to G$.

4. $S_1 = \{p \wedge \neg p\}$ no proper subset

$S_2 = \{p, \neg p\}$

$S_3 = \{p_1, p_2, \neg(p_1 \wedge p_2)\}$

$S_4 = \{p_1, p_2, p_3, \neg(p_1 \wedge p_2 \wedge p_3)\}$

$\vdots$

6. (a) Routine.

(b). The idea is to remove the parents of a resolvent and ask if it will work.

Resolution: Resolving F to $R(F)$ if $\exists$ clauses C_i, C_j and a literal 'a' s.t. $a \in C_i$ and $\neg a \in C_j$

F can be written as a conjunction of clauses

$(a \vee C_1), (a \vee C_2) \ldots (a \vee C_k), (\neg a \vee C_{k+1}),$
$\ldots (\neg a \vee C_n), B_1, \ldots B_m$

s.t B_i has neither $a, \neg a$, and $\forall i,\ a \notin C_i, \neg a \notin C_i$

Define $R(F)$ as a conjunction of clauses

$(C_1 \vee C_{k+1}), (C_2 \vee C_{k+1}), \ldots (C_k \vee C_{k+1}),$

$(C_1 \vee C_{k+2}), (C_2 \vee C_{k+2}), \ldots (C_k \vee C_{k+2}),$

$\vdots$

$(C_1 \vee C_n), (C_2 \vee C_n), \ldots (C_k \vee C_n),$

$B_1, \ldots, B_m$.

Indeed, $R(F)$ contains neither a nor $\neg a$.

P.T. F is sat $\iff R(F)$ is sat.

1) Assume $\mathfrak{F} \equiv \mathfrak{G}$.

Eg $\mathfrak{F} = \{p, q\}$ $\mathfrak{G} = \{p \wedge q\}$.

(2)-false $p \not\equiv p \wedge q$, $q \not\equiv p \wedge q$.

Now consider $\mathfrak{F} = \{p, q, p \vee \neg p\}$

$\mathfrak{G} = \{p \vee q, q \vee \neg q\}$.

Then $\forall F \in \mathfrak{F}$, $F \models q \vee \neg q$

$\forall G \in \mathfrak{G}$, $G \models p \vee \neg p$

However $p \wedge q \wedge (p \vee \neg p) \not\equiv (p \vee q) \wedge (q \vee \neg q)$

2) $\mathfrak{F}$ is of the form $\{F_1, F_2, \ldots, F_n, F_1 \wedge F_2, F_1 \wedge F_3, \ldots F_1 \wedge \ldots \wedge F_n\}$, and is inconsistent. That is $\bigwedge_{F \in \mathfrak{F}} F \vdash \bot$.

Consider any $F \in \mathfrak{F}$.

Since $\bigwedge_{F \in \mathfrak{F}} F \vdash \bot$, and $\bigwedge_{F \in \mathfrak{F}} F \in \mathfrak{F}$

Choose $G = \bigwedge_{F \in \mathfrak{F}} F$. Indeed $G \vdash \bot$, hence $G \vdash \neg F$.

3) $\models F \to G$, F not a contradiction and G not a tautology.

$Prop(F) =$ Prop variables in F

$Prop(G) =$ Prop variables in G

Induct on $|Prop(F) - Prop(G)|$.

Base: If $|Prop(F) - Prop(G)| = 0$ then choose $H = F$. Indeed, $\models F \to F$ and $\models F \to G$ and $Prop(F) \subseteq Prop(F) \cap Prop(G)$ (since $|Prop(F) - Prop(G)| = 0$).

Inductive hypothesis. Assume the result whenever $|Prop(F) - Prop(G)| \leq n$. Let $|Prop(F) - Prop(G)| = n+1$.

Let $q \in Prop(F) - Prop(G)$.

Define $F' = F[q \mapsto \top] \vee F[q \mapsto \bot]$.

Then $\models F' \to G$ (why?), $\models F \to F'$.

Indeed, $|Prop(F') - Prop(G)| = n$ and the inductive hypothesis applies.

Hence, $\exists H$ st. $\models F' \to H$, $\models H \to G$.