Well-formed formula

Problem Set 2

1. Let $\mathcal{P}$ denote propositional logic. Suppose we add to $\mathcal{P}$ the axiom schema $(A \to B)$ for wffs A, B of $\mathcal{P}$. Comment on the consistency of the resulting logical system obtained. A logic system $\mathcal{P}$ is inconsistent if it is capable of producing $\perp$ using the rules of natural deduction.

2. An adequate set of connectives is a set such that for every formula there is an equivalent formula with only connectives from that set. For example, $\{\neg, \vee\}$ is adequate for propositional logic since any occurrence of $\wedge$ and $\to$ can be removed using the equivalences

$$\varphi \to \psi \equiv \neg\varphi \vee \psi$$

$$\varphi \wedge \psi \equiv \neg(\neg\varphi \vee \neg\psi)$$

   - Show that $\{\neg, \wedge\}$, $\{\neg, \to\}$ and $\{\to, \perp\}$ are adequate sets of connectives. ($\perp$ treated as a nullary conenctive).
   - Show that if $C \subseteq \{\neg, \wedge, \vee, \to, \perp\}$ is adequate, then $\neg \in C$ or $\perp \in C$.

3. The binary connective *nand*, $F \downarrow G$, is defined by the truth table corresponding to $\neg(F \wedge G)$. Show that *nand* is complete - that is, it can express all binary boolean connectives.

4. The binary connective *xor*, $F \oplus G$ is defined by the truth table corresponding to $(\neg F \wedge G) \vee (F \wedge \neg G)$. Show that *xor* is not complete- that is, it cannot express all binary boolean connectives.

5. If a contradiction can be derived from a set of formulae, then the set of formulae is said to be inconsistent. Otherwise, the set of formulae is consistent. Let $\mathcal{F}$ be a set of formulae. Show that $\mathcal{F}$ is consistent iff it is satisfiable.

6. Suppose $\mathcal{F}$ is an inconsistent set of formulae. For each $G \in \mathcal{F}$, let $\mathcal{F}_G$ be the set obtained by removing G from $\mathcal{F}$.
   (a) Prove that for any $G \in \mathcal{F}$, $\mathcal{F}_G \vdash \neg G$, using the previous question.
   (b) Prove this using a formal proof.