CS 228 : Logic in Computer Science

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Recap : Semantics

- Each propositional variable is assigned values true/false. Truth tables for each of the operators $\vee, \wedge, \neg, \rightarrow$ to determine truth values of complex formulae.
- $\varphi_1, \ldots, \varphi_n \models \psi$ iff whenever $\varphi_1, \ldots, \varphi_n$ evaluate to true, so does ψ. $\models$ is read as semantically entails
  - Recall $\vdash$, and compare with $\models$
- Formulae φ and ψ are provably equivalent iff $\varphi \vdash \psi$ and $\psi \vdash \varphi$
- Formulae φ and ψ are semantically equivalent iff $\varphi \models \psi$ and $\psi \models \varphi$

Soundness of Propositional Logic

provably implies semantic

$\varphi_1, \ldots, \varphi_n \vdash \psi \Rightarrow \varphi_1, \ldots, \varphi_n \models \psi$

Whenever ψ can be proved from $\varphi_1, \ldots, \varphi_n$, then ψ evaluates to true whenever $\varphi_1, \ldots, \varphi_n$ evauate to true

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- Consider now a proof with k lines.

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- By inductive hypothesis, we have $\varphi_1, \ldots, \varphi_n \models \psi_1$ and $\varphi_1, \ldots, \varphi_n \models \psi_2$. By semantics, we have $\varphi_1, \ldots, \varphi_n \models \psi_1 \wedge \psi_2$.

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- Consider adding ψ_1 in the premises along with $\varphi_1, \ldots, \varphi_n$. Then we will get a proof $\varphi_1, \ldots, \varphi_n, \psi_1 \vdash \psi_2$, of length $k - 1$. By inductive hypothesis, $\varphi_1, \ldots, \varphi_n, \psi_1 \models \psi_2$. By semantics, this is same as $\varphi_1, \ldots, \varphi_n \models \psi_1 \rightarrow \psi_2$

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- The equivalence of $\varphi_1, \ldots, \varphi_n \vdash \psi_1 \rightarrow \psi_2$ and $\varphi_1, \ldots, \varphi_n, \psi_1 \vdash \psi_2$ gives the proof.

Soundness : Other cases

Do this as homework

Completeness

$$\varphi_1, \ldots, \varphi_n \models \psi \Rightarrow \varphi_1, \ldots, \varphi_n \vdash \psi$$

Whenever $\varphi_1, \ldots, \varphi_n$ semantically entail ψ, then ψ can be proved from $\varphi_1, \ldots, \varphi_n$. The proof rules are complete

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- Step 3: Show that $\varphi_1, \ldots, \varphi_n \vdash \psi$

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- Hence, $\models \varphi_1 \to (\varphi_2 \to (\ldots(\varphi_n \to \psi)\ldots))$.

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- Using this insight, we have to give a proof of ψ.

Completeness : Step 2

Truth Table to Proof

Let φ be a formula with variables $p_1, \ldots, p_n$. Let $\mathcal{T}$ be the truth table of φ, and let l be a line number in $\mathcal{T}$. Let $\hat{p}_i$ represent p_i if p_i is assigned true in line l, and let it denote $\neg p_i$ if p_i is assigned false in line l. Then

1. $\hat{p}_1, \ldots, \hat{p}_n \vdash \varphi$ if φ evaluates to true in line l
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- $\hat{p} = p, \hat{q} = q \vdash p \wedge q$
- $\hat{p} = \neg p, \hat{q} = q \vdash \neg(p \wedge q)$
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- Case Negation : $\varphi = \neg\varphi_1$
  - Assume φ evaluates to true in line l of $\mathcal{T}$. Then φ_1 evaluates to false in line l. By inductive hypothesis, $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg\varphi_1$.

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- Case Negation : $\varphi = \neg\varphi_1$
  - Assume φ evaluates to true in line l of $\mathcal{T}$. Then φ_1 evaluates to false in line l. By inductive hypothesis, $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg\varphi_1$.
  - Assume φ evaluates to false in line l of $\mathcal{T}$. Then φ_1 evaluates to true in line l. By inductive hypothesis, $\hat{p}_1, \ldots, \hat{p}_n \vdash \varphi_1$. Use the $\neg\neg i$ rule to obtain a proof of $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg\neg\varphi_1$.

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- Case $\rightarrow$: $\varphi = \varphi_1 \rightarrow \varphi_2$.
  - If φ evaluates to false in line l, then φ_1 evaluates to true and φ_2 to false. Let $\{q_1, \ldots, q_k\}$ be the variables of φ_1 and let $\{r_1, \ldots, r_j\}$ be the variables in φ_2. $\{q_1, \ldots, q_k\} \cup \{r_1, \ldots, r_j\} = \{p_1, \ldots, p_n\}$.

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  - By inductive hypothesis, $\hat{q}_1, \ldots, \hat{q}_k \models \varphi_1$ and $\hat{r}_1, \ldots, \hat{r}_j \models \neg\varphi_2$. Then, $\hat{p}_1, \ldots, \hat{p}_n \models \varphi_1 \wedge \neg\varphi_2$.

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  - By inductive hypothesis, $\hat{q}_1, \ldots, \hat{q}_k \models \varphi_1$ and $\hat{r}_1, \ldots, \hat{r}_j \models \neg\varphi_2$. Then, $\hat{p}_1, \ldots, \hat{p}_n \models \varphi_1 \wedge \neg\varphi_2$.
  - Prove that $\varphi_1 \wedge \neg\varphi_2 \vdash \neg(\varphi_1 \rightarrow \varphi_2)$.

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  - If φ evaluates to true in line l, then there are 3 possibilities. If both φ_1, φ_2 evaluate to true, then we have $\hat{p}_1, \ldots, \hat{p}_n \models \varphi_1 \wedge \varphi_2$. Proving $\varphi_1 \wedge \varphi_2 \vdash \varphi_1 \rightarrow \varphi_2$, we are done.

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  - If both φ_1, φ_2 evaluate to false, then we have $\hat{p}_1, \ldots, \hat{p}_n \models \neg\varphi_1 \wedge \neg\varphi_2$. Proving $\neg\varphi_1 \wedge \neg\varphi_2 \vdash \varphi_1 \to \varphi_2$, we are done.

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  - If both φ_1, φ_2 evaluate to false, then we have $\hat{p}_1, \ldots, \hat{p}_n \models \neg\varphi_1 \wedge \neg\varphi_2$. Proving $\neg\varphi_1 \wedge \neg\varphi_2 \vdash \varphi_1 \to \varphi_2$, we are done.
  - Last, if φ_1 evaluates to false and φ_2 evaluates to true, then we have $\hat{p}_1, \ldots, \hat{p}_n \models \neg\varphi_1 \wedge \varphi_2$. Proving $\neg\varphi_1 \wedge \varphi_2 \vdash \varphi_1 \to \varphi_2$, we are done.

Truth Table to Proof

- Prove the cases $\wedge, \vee$.

On An Example

We know $\models (p \wedge q) \to p$. Using this fact, show that $\vdash (p \wedge q) \to p$.

- $p, q \vdash (p \wedge q) \to p$
- $\neg p, q \vdash (p \wedge q) \to p$
- $p, \neg q \vdash (p \wedge q) \to p$
- $\neg p, \neg q \vdash (p \wedge q) \to p$

Now, combine the 4 proofs above to give a single proof for $\vdash (p \wedge q) \to p$.

Completeness : Steps 2, 3

- Step 2: From $\models \varphi_1 \to (\varphi_2 \to (\ldots(\varphi_n \to \psi)\ldots))$, use LEM on all the propositional variables of $\varphi_1, \ldots, \varphi_n, \psi$ to obtain $\vdash \varphi_1 \to (\varphi_2 \to (\ldots(\varphi_n \to \psi)\ldots))$.

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- Step 3: Take the proof $\vdash \varphi_1 \to (\varphi_2 \to (\dots(\varphi_n \to \psi)\dots))$. This proof has *n* nested boxes, the *i*th box opening with the assumption φ_i. The last box closes with the last line ψ. Hence, the line immediately after the last box is $\varphi_n \to \psi$.

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- In a similar way, the $(n-1)$th box has as its last line $\varphi_n \to \psi$, and hence, the line immediately after this box is $\varphi_{n-1} \to (\varphi_n \to \psi)$ and so on.

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- Step 3: Take the proof $\vdash \varphi_1 \to (\varphi_2 \to (\dots(\varphi_n \to \psi)\dots))$. This proof has n nested boxes, the ith box opening with the assumption φ_i. The last box closes with the last line ψ. Hence, the line immediately after the last box is $\varphi_n \to \psi$.
- In a similar way, the $(n-1)$th box has as its last line $\varphi_n \to \psi$, and hence, the line immediately after this box is $\varphi_{n-1} \to (\varphi_n \to \psi)$ and so on.
- Add premises $\varphi_1, \dots, \varphi_n$ on the top. Use MP on the premises, and the lines after boxes 1 to n in order to obtain ψ.

Summary

Propositional Logic is sound and complete.