CS 228 : Logic in Computer Science

Krishna. S

Monadic Second Order Logic (MSO)

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- The symbols (and) called paranthesis

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- In a wff $\varphi = \forall X \psi$, every occurrence of X in ψ is bound
- A sentence is a formula with no free first order and second order variables

Assignments on τ-structures

Assignments

For a τ-structure $\mathcal{A}$, an assignment over $\mathcal{A}$ is a pair of functions (α_1, α_2), where

- $\alpha_1 : \mathcal{V}_1 \to u(\mathcal{A})$ assigns every first order variable $x \in \mathcal{V}_1$ a value $\alpha_1(x) \in u(\mathcal{A})$. If t is a constant symbol c, then $\alpha_1(t)$ is $c^{\mathcal{A}}$.

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Binding on a Variable

For an assignment $\alpha = (\alpha_1, \alpha_2)$ over $\mathcal{A}$, and $x \in \mathcal{V}_i$, $i = 1, 2$, $\alpha_i[x \mapsto a]$ is the assignment $\alpha_i[x \mapsto a](y) = \begin{cases} \alpha_i(y), y \neq x, \\ a, y = x \end{cases}$

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- $\mathcal{A} \models_\alpha (\forall X)\varphi$ iff for every $S \subseteq u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[X \mapsto S]} \varphi$

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Recall the signature for the graph structure, $\tau = \{E\}$

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$$\exists X \exists Y \exists Z (\forall x [X(x) \vee Y(x) \vee Z(x)] \wedge$$

$$\forall x \forall y [E(x,y) \rightarrow \{\neg(X(x) \wedge X(y)) \wedge \neg(Y(x) \wedge Y(y)) \wedge \neg(Z(x) \wedge Z(y))\}])$$

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$$\exists I\{\forall x \forall y[(\neg(x = y) \wedge I(x) \wedge I(y)) \rightarrow \neg E(x, y)] \wedge$$

$$\exists x_1 \ldots x_k[\bigwedge_{i \neq j} \neg(x_i = x_j) \wedge \bigwedge_i I(x_i)]\}$$

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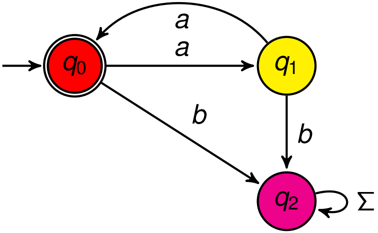
- Given a MSO sentence φ, $L(\varphi)$ defined as usual
- A language $L \subseteq \Sigma^*$ is MSO definable iff there is an MSO formula φ such that $L = L(\varphi)$
- Given an MSO sentence φ, is it satisfiable/valid?

MSO Expressiveness

- Clearly, $FO \subseteq MSO$
- $FO \subset$ Regular
- MSO=Regular

Regular Languages to MSO



Runs of words:

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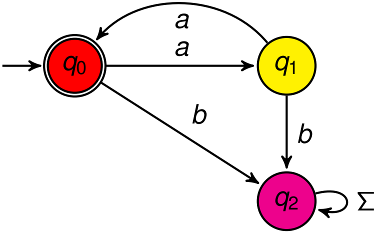

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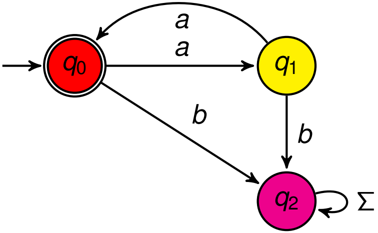

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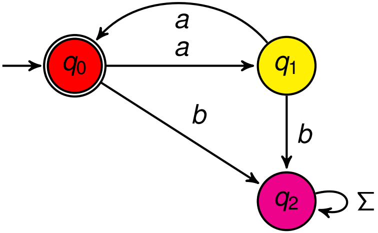

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- For a state $q \in Q$, let X_q=the set of positions of the word where the state is q in the run
- $X_{q_0} = \{0, 2\}, X_{q_1} = \{1\}, X_{q_2} = \{3\}$
- The initial position of any word must belong to X_{q_0} : $0 \in X_{q_0}$

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- If a word *wa* is accepted, then
  - The last position x of the word satisfies $Q_a(x)$
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- $X_{q_0}(0)$, $X_{q_1}(1)$ and $Q_a(0)$. $\delta(q_0, a) = q_1$.
- $X_{q_1}(1)$, $X_{q_0}(2)$ and $Q_a(1)$. $\delta(q_1, a) = q_0$.

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Given a DFA $A = (Q, \Sigma, \delta, q_0, F)$, a word w is accepted iff it satisfies

$$\exists X_0 \exists X_1 \dots X_n \{[\forall x (X_0(x) \vee \cdots \vee X_n(x)) \wedge \forall x \bigwedge_{i \neq j} \neg (X_i(x) \wedge X_j(x))] \wedge$$

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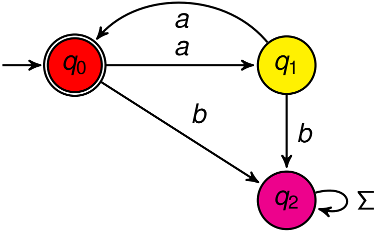
$$\forall x \forall y [S(x, y) \to \bigvee_{\delta(i,a)=j} [X_i(x) \wedge Q_a(x) \wedge X_j(y)]] \wedge$$

$$\exists x [\mathit{last}(x) \wedge \bigvee_{\delta(i,a)=j \in F} [X_i(x) \wedge Q_a(x)]]\}$$

- $w \in L(A)$ iff $w \models \varphi$

Example : Regular to MSO



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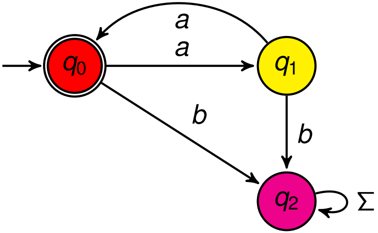

$$\exists X_0 \exists X_1 \exists X_2 \{ [\forall x (X_0(x) \vee X_1(x) \vee X_2(x)) \wedge \forall x [\neg (X_0(x) \wedge X_1(x)) \wedge$$
$$\neg (X_0(x) \wedge X_2(x)) \wedge \neg (X_1(x) \wedge X_2(x))] \wedge [\exists x (\mathit{first}(x) \wedge X_0(x))] \wedge$$

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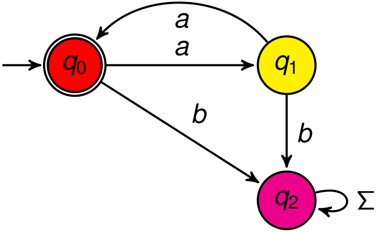

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$$\forall x \forall y[S(x,y) \rightarrow [(X_0(x) \wedge Q_a(x) \wedge X_1(y)) \vee$$
$$(X_0(x) \wedge Q_b(x) \wedge X_2(y)) \vee (X_1(x) \wedge Q_a(x) \wedge X_0(y)) \vee$$
$$(X_1(x) \wedge Q_b(x) \wedge X_2(y)) \vee (X_2(x) \wedge Q_a(x) \wedge X_2(y)) \vee (X_2(x) \wedge Q_b(x) \wedge X_2(y))]]$$

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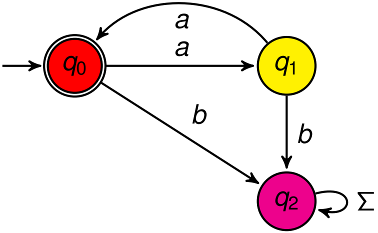

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$$\wedge \exists x[\mathit{last}(x) \wedge (X_1(x) \wedge Q_a(x))]\}$$

MSO to Regular Languages

- Every MSO sentence φ over words can be converted into a DFA A_φ such that $L(\varphi) = L(A_\varphi)$.
- Start with atomic formulae, construct DFA for each of them.
- Conjunctions, Disjunctions, Negation easily handled via union, intersection and complementation of respective DFA
- Handling quantifiers?

Atomic Formulae to DFA

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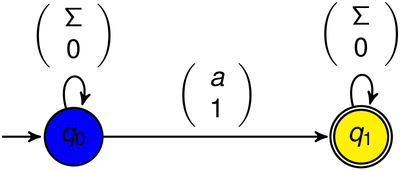
- Deterministic, not complete.



Simple Formulae to DFA

- $Q_a(x) \wedge X(x)$ means that the position x is in the set X, and letter a is true when $x = 1$.

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- Think of a word *baab* which satisfies $Q_a(x) \wedge X(x)$ as

$$\begin{array}{ccc} baab & & baab \\ 0010 & \text{or} & 0100 \\ DD1D & & D1DD \end{array}$$

where D stands for *dont care*. X can have value 0 or 1 at D.

Simple Formulae to DFA

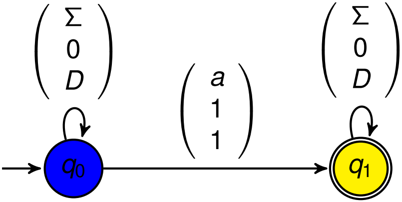
- $Q_a(x) \wedge X(x)$ means that the position x is in the set X, and letter a is true when $x = 1$.
- Think of a word *baab* which satisfies $Q_a(x) \wedge X(x)$ as

baab		*baab*
0010	or	0100
$DD1D$		$D1DD$

  where D stands for *dont care*. X can have value 0 or 1 at D.
- However, the position where $x = 1$ must belong to X.

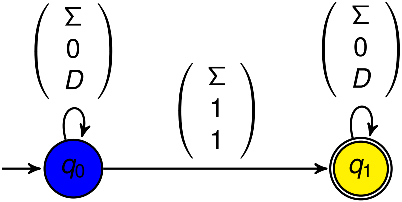
Simple Formulae to DFA

- The first row is over Σ, and the second row captures a possible assignment to x, and the third row captures a possible assignment to X.
- Think of an extended alphabet $\Sigma' = \Sigma \times \{0, 1\} \times \{0, 1\}$, and construct an automaton over Σ'.
- $Q_a(x) \wedge X(x)$: deterministic, not complete



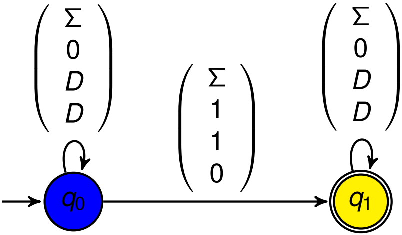
Simple Formulae to DFA

- $X(x)$



Simple Formulae to DFA

- $X(x) \wedge \neg Y(x)$
- $\Sigma' = \Sigma \times \{0,1\} \times \{0,1\} \times \{0,1\}$



Formulae to DFA

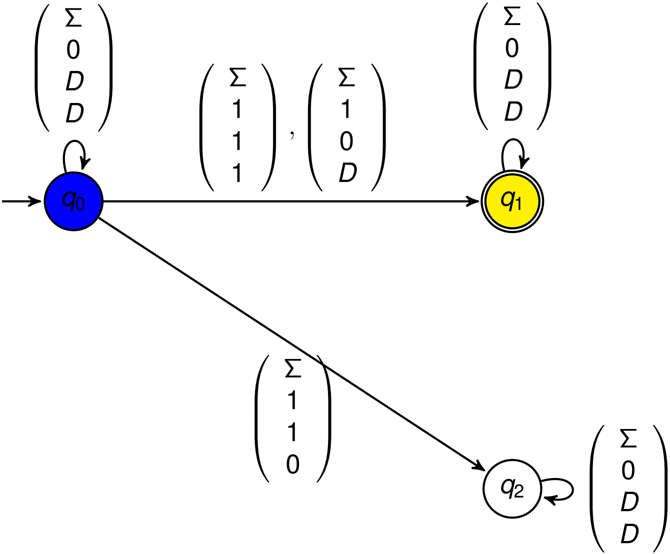
- Given $\varphi(x_1, \dots, x_n, X_1, \dots, X_m)$, an MSO formula over Σ, consider the extended alphabet

$$\Sigma' = \Sigma \times \{0, 1\}^{m+n}$$

- Assign values to x_i, X_j at every position as seen in the cases of atomic formulae
- Keep in mind that every x_i can be assigned 1 at a unique position

Handling Quantifiers

- $\exists X \exists Y \forall x [X(x) \to Y(x)]$



Points to Remember

- Given $\varphi(x_1, \ldots, x_n, X_1, \ldots, X_m)$, construct automaton for atomic MSO formulae over the extended alphabet $\Sigma \times \{0, 1\}^{m+n}$
- Intersect with the regular language where every x_i is assigned 1 exactly at one position
- Given a sentence $Q_{x_1} \ldots Q_{x_n} Q_{X_1} \ldots Q_{X_m} \varphi$, first construct the automaton for the formula $\varphi(x_1, \ldots, x_n, X_1, \ldots, X_m)$
- Replace $\forall$ in terms of $\exists$

Points to Remember

- Given the automaton for $\varphi(x_1, \ldots, x_n, X_1, \ldots, X_n)$, the automaton for $\exists X_i \varphi(x_1, \ldots, x_n, X_1, \ldots, X_n)$ is obtained by projecting out the row of X_i
- This may result in an NFA
- Determinize it and complement it to get a DFA for $\neg \exists x_i \varphi(x_1, \ldots, x_n, X_1, \ldots, X_n)$
- Intersect with the regular language where each of $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ are assigned 1 exactly at one position

The Automaton-Logic Connection

Given any MSO sentence φ, one can construct a DFA A_φ such that $L(\varphi) = L(A_\varphi)$. If a language L is regular, one can construct an MSO sentence φ such that $L = L(\varphi)$.