CS 228 : Logic in Computer Science

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Basic Rules So Far

- $ightharpoonup \land i, \land e_1, \land e_2$ (and introduction and elimination)
- $\neg \neg e, \neg \neg i$ (double negation elimination and introduction)
- ► MP (Modus Ponens)
- $ightharpoonup \rightarrow i$ (Implies Introduction : remember opening boxes)
- \lor $\lor i_1, \lor i_2, \lor e$ (Or introduction and elimination)

► $(p \lor q) \lor r \vdash p \lor (q \lor r)$ 1. $(p \lor q) \lor r$ premise

 $(p \lor q) \lor r \vdash p \lor (q \lor r)$

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	assumption
3.		

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$$(p \lor q) \lor r \vdash p \lor (q \lor r)$$

- 1. $(p \lor q) \lor r$ premise
- p ∨ q assumption
 p assumption

 $(p \lor q) \lor r \vdash p \lor (q \lor r)$

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	assumption
3.	p	assumption
4.	$p \lor (q \lor r)$	√ <i>i</i> ₁ 3
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 $\blacktriangleright (p \lor q) \lor r \vdash p \lor (q \lor r)$

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	assumption
3.	p	assumption
4.	$p \lor (q \lor r)$	√ <i>i</i> ₁ 3
5.	q	assumption
6.		

 $(p \lor q) \lor r \vdash p \lor (q \lor r)$

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	assumption
3.	р	assumption
4.	$p \lor (q \lor r)$	∨ <i>i</i> ₁ 3
5.	q	assumption
6.	$q \vee r$	∨ <i>i</i> ₁ 5
7.		

 $(p \lor q) \lor r \vdash p \lor (q \lor r)$

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	assumption
3.	p	assumption
4.	$p \lor (q \lor r)$	√ <i>i</i> ₁ 3
5.	q	assumption
6.	$q \vee r$	∨ <i>i</i> ₁ 5
7.	$p \lor (q \lor r)$	∨ <i>i</i> ₂ 6
8.		

 $(p \lor q) \lor r \vdash p \lor (q \lor r)$

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	assumption
3.	p	assumption
4.	$p \lor (q \lor r)$	∨ <i>i</i> ₁ 3
5.	q	assumption
6.	$ q \lor r$	∨ <i>i</i> ₁ 5
7.	$p \lor (q \lor r)$	∨ <i>i</i> ₂ 6
8.	$p \lor (q \lor r)$	∨ <i>e</i> 2, 3-4, 5-7
9.		

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9.	r	assumption
0.		

3/1/

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8.	$p \lor (q \lor r)$	∨ <i>e</i> 2, 3-4, 5-7
9.	r	assumption
0.	$q \lor r$	√ <i>i</i> ₂ 9
1.		

3/1/

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7.	$p \lor (q \lor r)$	∨ <i>i</i> ₂ 6
8.	$p \lor (q \lor r)$	∨ <i>e</i> 2, 3-4, 5-7
9.	r	assumption
١0.	$q \lor r$	∨ <i>i</i> ₂ 9
11.	$p \lor (q \lor r)$	√ <i>i</i> ₂ 10

$$(p \lor q) \lor r \vdash p \lor (q \lor r)$$

Text

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	assumption
3.	p	assumption
4.	$p \lor (q \lor r)$	∨ <i>i</i> ₁ 3
5.	q	assumption
6.	$ q \lor r$	∨ <i>i</i> ₁ 5
7.	$p \lor (q \lor r)$	∨ <i>i</i> ₂ 6
8.	$p \lor (q \lor r)$	∨ <i>e</i> 2, 3-4, 5-7
9.	r	assumption
0.	$q \lor r$	√ <i>i</i> ₂ 9
1.	$p \lor (q \lor r)$	√ <i>i</i> ₂ 10
2.	$p \lor (q \lor r)$	∨ <i>e</i> 1, 2-8, 9-11

▶
$$\vdash p \rightarrow (q \rightarrow p)$$

1. true

premise

$$\blacktriangleright \vdash p \rightarrow (q \rightarrow p)$$

1.	true	premise
2.	р	assumption
2		

4/1/

$$\blacktriangleright \vdash p \rightarrow (q \rightarrow p)$$

1.	true	premise
2.	р	assumption
3.	q	assumption
4.		

▶
$$\vdash p \rightarrow (q \rightarrow p)$$

١.	ırue	premise
2.	р	assumption
3.	q	assumption
4.	р	copy 2
_		

▶
$$\vdash p \rightarrow (q \rightarrow p)$$

1.	true	premise
2.	р	assumption
3.	q	assumption
4.	p	copy 2
5.	$oldsymbol{q} ightarrow oldsymbol{p}$	<i>→ i</i> 3-4

6.

$$\blacktriangleright \vdash p \rightarrow (q \rightarrow p)$$

1.	true	premise
2.	р	assumption
3.	q	assumption
4.	р	copy 2
5.	$oldsymbol{q} ightarrow oldsymbol{p}$	→ <i>i</i> 3-4
6.	$p \rightarrow (q \rightarrow p)$	\rightarrow i 2-5

▶ We have seen $\neg \neg e$ and $\neg \neg i$, the elimination and introduction of double negation.

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- $ightharpoonup \perp \to \varphi$ for any formula φ .

Rules with \bot

The \perp elimination rule $\perp e$

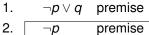
$$\frac{\perp}{\psi}$$

The \perp introduction rule $\perp i$

$$\frac{\varphi \qquad \neg \varphi}{\bot}$$

- - 1. $\neg p \lor q$ premise
 - 2.

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- 2. $\neg p$ premise
- 3.

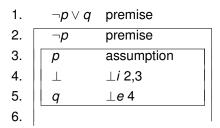


¬p prem
 p assu

3. p assumption4.

1.	$\neg p \lor q$	premise
2.	$\neg p$	premise
3.	р	assumption
4.		<i>⊥i</i> 2,3
5.		

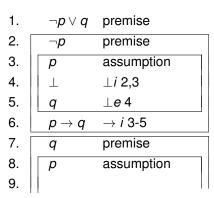
▶
$$\neg p \lor q \vdash p \rightarrow q$$



▶
$$\neg p \lor q \vdash p \rightarrow q$$

1.	$\neg p \lor q$	premise
2.	$\neg p$	premise
3.	р	assumption
4.		<i>⊥i</i> 2,3
5.	q	⊥ <i>e</i> 4
6.	p o q	→ <i>i</i> 3-5
7.	q	premise
8.		

▶
$$\neg p \lor q \vdash p \rightarrow q$$



▶
$$\neg p \lor q \vdash p \rightarrow q$$

1.	$\neg p \lor q$	premise
2.	$\neg p$	premise
3.	р	assumption
4.		<i>⊥i</i> 2,3
5.	q	⊥ <i>e</i> 4
6.	p o q	→ <i>i</i> 3-5
7.	q	premise
8.	р	assumption
9.	q	copy 7
0.	p o q	<i>→ i</i> 8-9
1.	p o q	∨ <i>e</i> 1, 2-6, 7-10

Introducing Negations (PBC)

- In the course of a proof, if you assume φ (by opening a box) and obtain \bot in the box, then we conclude $\neg \varphi$
- ▶ This rule is denoted $\neg i$ and is read as \neg introduction.
- Also known as Proof By Contradiction

- 1. $p \rightarrow \neg p$ premise
- 2.

An Example

1.	p ightarrow eg p	premise

2. *p* assumption3.

An Example

$$\blacktriangleright \ p \to \neg p \vdash \neg p$$

1.	p ightarrow eg p	premise
2.	р	assumption
3.	eg p	MP 1,2
4.		

An Example

1.	p ightarrow eg p	premise
2.	р	assumption
3.	$\neg p$	MP 1,2
4.		<i>⊥i</i> 2,3
5.		¬i 2-4

The Last One

Law of the Excluded Middle (LEM)



Summary of Basic Rules

- \blacktriangleright $\land i$, $\land e_1$, $\land e_2$,
- ¬¬e
- ► MP
- $\rightarrow i$
- $\triangleright \forall i_1, \forall i_2, \forall e$
- ▶ Copy, $\neg i$ or PBC
- ► *⊥e*, *⊥i*

Derived Rules

- ▶ MT (derive using MP, $\perp i$ and $\neg i$)
- ▶ $\neg \neg i$ (derive using $\bot i$ and $\neg i$)
- ▶ LEM (derive using $\forall i_1, \bot i, \neg i, \forall i_2, \neg \neg e$)

The Proofs So Far

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The Proofs So Far

- So far, the "proof" we have seen is purely syntactic, no true/false meanings were attached
- ▶ Intuitively, $p \rightarrow q \vdash \neg p \lor q$ makes sense because you think semantically. However, we never used any semantics so far.
- Now we show that whatever can be proved makes sense semantically too.

▶ Each propositional variable is assigned values true/false. Truth tables for each of the operators $\lor, \land, \neg, \rightarrow$ to determine truth values of complex formulae.

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 - ► Recall ⊢, and compare with ⊨

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- ▶ Formulae φ and ψ are provably equivalent iff $\varphi \vdash \psi$ and $\psi \vdash \varphi$

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Soundness of Propositional Logic

$$\varphi_1, \ldots, \varphi_n \vdash \psi \Rightarrow \varphi_1, \ldots, \varphi_n \models \psi$$

Whenever ψ can be proved from $\varphi_1, \dots, \varphi_n$, then ψ evaluates to true whenever $\varphi_1, \dots, \varphi_n$ evaluate to true