CS 228 : Logic in Computer Science

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- Given φ, write an algorithm to check $L(\varphi) = \emptyset$?

First-Order Logic over Words

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  - Given a FO formula φ over words, is $L(\varphi)$ non-empty?

A Primer for Words

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- By convention, $\{\}^* = \{\epsilon\}$

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  - $Pref(aaba) = \{\epsilon, a, aa, aab, aaba\}$
  - Proper prefixes = $\{a, aa, aab\}$
  - ϵ, *aaba* improper prefixes

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  - $A = \{a, ba\}, B = \{\epsilon, aa, bb\}$
  - $AB = \{a, a^3, abb, ba, ba^3, babb\}$
  - $BA = \{a, ba, a^3, aaba, bba, bbba\}$

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- Concatenation does not distribute over interesection
  - $A = \{a, ab\}, B = \{b\}, C = \{\epsilon\}$
  - $A(B \cap C) \neq AB \cap AC$

FO for Languages

Formalize in FO

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- L_5 = Words in which whenever an a occurs, it is followed eventually by a b, and no c occurs in between the a and the b
  $aabbabab, aabbcbccaab \in L_5$, $aacaab \notin L_5$.

Satisfiability of FO over Words

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- Given φ, can we easily convert φ into some other mechanism M, which we know how to deal with?