Boolean Algebra Problem-set Solutions

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February 12, 2020

The solutions use some of the theorems proved in class.

1. The 0 and 1 elements in a Boolean algebra are unique.

   - Let u, v be two $0-$ elements. Then
   $$u = (u + v) = v$$
   Thus the $0-$element is unique. If p, q are two $1-$elements, then
   $$p = (p.q) = q$$
   and thus the $1-$element is unique.

2. $a.0 = 0$ for each element a.

   - $a.0 + 0 = a.0 + 1.0 = (a + 1).0 = 1.0 = 0$ (we have used $1.u = u$ here).

3. For each element a, $a + a = a$.

   - $a = a + 0 = a + a.\overline{a} = (a + \overline{a}).(a + a) = 1.(a + a) = a + a.$

4. Prove the second De Morgan law:
   $$\overline{a.b} \;\; = \;\; \overline{a} + \overline{b}$$

   - $(\overline{a} + \overline{b}).(a.b) = \overline{a}.a.b + \overline{b}.a.b = 0 + 0 = 0$. Further, $\overline{a} + \overline{b} + a.b = \overline{a} + a.b + \overline{b} + a.b$. Now $\overline{a} + a.b = (\overline{a} + a).(\overline{a} + b) = \overline{a} + b$. Similarly, $\overline{b} + a.b = \overline{b} + a$. Thus,
   $$\overline{a} + \overline{b} + a.b = \overline{a} + a + \overline{b} + b = 1 + 1 = 1.$$
   where the last equality used $a + a = a$, proved above.

5. If $a \leq b$ and $a \neq b$, then $\overline{a}.b \neq 0$.

   - $a \leq b$ implies $a + b = b$. Assume that $\overline{a}.b = 0$. Then $a = a + \overline{a}.b = (a + \overline{a}).(a + b) = 1.b = b$, a contradiction. Thus, $\overline{a}.b \neq 0$.

6. If $a \leq b$ and $c \leq b$, then $(a + c) \leq b$.

   - $(a + c).b = a.b + c.b = a + c$.

7. If $a \leq b$ then $\overline{b} \leq \overline{a}$.

   - $a \leq b$ implies that $a.b = a$ (also $a + b = b$). Thus $\overline{a} + \overline{b} = \overline{a}$. This implies that $\overline{b} \leq \overline{a}$.

8. If $a \leq b$ and $a \leq c$, then $a \leq b.c$.

   - $a.(b.c) = (a.b).c = a.c = a$.

9. Prove that $\overline{a}.b + a.b = b$.

   - $\overline{a}.b + a.b = (\overline{a} + a).b = 1.b = b$.

10. Prove that

$$a.\overline{b}.c + a.\overline{b}.\overline{c} + \overline{a}.\overline{b} = \overline{b}$$

   - Combine $a.\overline{b}.c + a.\overline{b}.\overline{c} = a.\overline{b}.(c + \overline{c}) = a.\overline{b}$, and then $a.\overline{b} + \overline{a}.\overline{b} = (a + \overline{a}).\overline{b} = 1.\overline{b} = \overline{b}$.