HW1

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1 Question 1

Base Case:

if α is a proposition p

- if α is satisfied by v, then let v' be an extension of v which maps p to true so q_α is true and $p \iff q_\alpha$ is True thus Circuit(α) is satisfied by v'.
- if Circuit(α) and q_α is satisfied by v then since $p \iff q_p$ so p is true and α is satisfied by v' which is an extension of v that maps p to true.

Inductive Case:

if α is a negative of a proposition p

- if α is satisfied by v, then let v' be an extension of v which maps p to false so, q_α is True and $\neg p \iff q_\alpha$ is true thus Circuit(α) is satisfied by v'
- if Circuit(α) and q_α is satisfied by v then since $\neg p \iff q_p$ so $\neg p$ is true and α is satisfied by v' which is an extension of v which maps p to false.

if α is a conjunction of β and γ

- if α is satisfied by v, then let v' be an extension that satisfies Circuit(β) and Circuit(γ) so $q_{\beta\cap\gamma}$ are true and q_β and q_γ are true and hence Circuit(α) is satisfied by v'
- if Circuit(α) and q_α is satisfied by v then from the definintion of a circuit, q_β and q_γ are true and hence α is satisfied by v' which is an extension of v that satisfies β and γ.

if α is a disjunction of β and γ

- if α is satisfied by v, then let v' be an extension that satisfies Circuit(β) or Circuit(γ) so $q_{\beta\cup\gamma}$ are true and q_β or q_γ are true and hence Circuit(α) is satisfied by v'
- if Circuit(α) and q_α is satisfied by v then from the definintion of a circuit, q_β or q_γ are true and hence α is satisfied by v' which is an extension of v that satisfies β or γ.

Hence the proof is complete and by induction we can say that for every propositional formula α, the formula $q_\alpha \cap Circuit(\alpha)$ is satisfied iff α is satisfied by v.

2 Question 2

the set of formulas that represent the constraints of the problem are as follows:

- $\forall i \in \{1, 2, 3....\}, (p_{i,1} \to \neg(p_{i,2} \cup p_{i,3} \cup p_{i,4})) \cap (p_{i,2} \to \neg(p_{i,1} \cup p_{i,3} \cup p_{i,4})) \cap (p_{i,3} \to \neg(p_{i,2} \cup p_{i,1} \cup p_{i,4})) \cap (p_{i,4} \to \neg(p_{i,2} \cup p_{i,3} \cup p_{i,1}))$

  This ensures that no two students are in the same house.

- $\forall (i, j) \in F, \neg(p_{i,1} \cap p_{j,1}) \cap (\neg(p_{i,2} \cap p_{j,2})) \cap \neg(p_{i,3} \cap p_{j,3}) \cap \neg(p_{i,4} \cap p_{j,4})$

  This ensures that no two friends are in the same house.

Since the set of formulas is infinite, we need to prove that every finite subset of these formulas has a model. This follows from the assumption that finite undirected graphs whose vertices have *degree* ≤ 3 are four-colorable, as the graph formed by the friendship relation F is such a graph.

Finally, by the Compactness Theorem of First-Order Logic, the entire set of formulas has a model, and hence it is possible to sort the countably infinite set of students into four houses while avoiding placing any two friends in the same house.

3 Question 3

- Let us introduce some notation to model the problem.

  - let $V \subseteq N$ be the set of vertices of the graph
  - let $E \subseteq VXV$ be the set of edges of the graph
  - let $C \subseteq N$ be the set of colors that can be used to color the graph
  - let $p_{i,c}$ be the proposition that vertex i is colored with color c

  The set of constraints that we need to satisfy are as follows:

  - $|C| = 3$ (since we need to have only three colors)
  - $\forall (i, j) \in E, (p_{i,c} \to \neg p_{j,c})$ for a color $c \in C$(since no two adjacent vertices can have the same color)
  - $\forall i \in V, \exists c \in C, p_{i,c}$ (since every vertex must be colored)