Midterm Exam

CS 474: Logic in Computer Science: Spring 2023

Starts Thursday, Mar 30 at 11 AM CDT; due on Friday, Mar 31 at 5 PM CDT

Points: Problem 1: 5x2=10 points, Problem 2: 4x5=20 points; Problem 3: 5+10= 15 points; Problem 4: 15 points, Problem 5: 15 points, Problem 6: 2+15+6+2= 25 points. Total: 100 points

Problem 1. Multiple Choice

(a) Let us consider only models with 10 elements in their universe, and FO logic over some fixed signature. Then the problem of checking validity of a given FO formula over the class of such models is

- ◯ Decidable
- ◯ Undecidable but in r.e.
- ◯ Undecidable and not r.e.

(b) Given a finite model M and a sentence φ, the problem of checking whether $M \models \varphi$ is

- ◯ Decidable
- ◯ Undecidable but in r.e.
- ◯ Undecidable and not r.e.

(c) Given an FO formula φ over a finite signature, the problem of checking whether φ is satisfiable by a finite model is:

- ◯ Decidable
- ◯ Undecidable but in r.e.
- ◯ Undecidable and not r.e.

(d) The problem of checking whether a purely universally quantified formula is valid over a signature that has only the equality relation and no function symbols is:

- ◯ Decidable in P
- ◯ Decidable in NP (but not known to be decidable in P)
- ◯ Undecidable

(e) The problem of checking whether a purely universally quantified formula is valid over a signature that has only the equality relation, no function symbols, no negation, and no implication is:

- ◯ Decidable in P
- ◯ Decidable in NP (but not known to be decidable in P)
- ◯ Undecidable

Problem 2. Show that the following sentences are not valid by constructing a model in which they **do not hold**. For each structure, you must clearly define the universe as well as the interpretation of all constants, functions, and relations occurring in the formula.

(a)
$$(\forall x.\neg(f(x) = x)) \Rightarrow \exists x \exists y.(\ f(x) = y \wedge f(y) = x\ )$$

(b)
$$\forall x.\exists y.P(x, y) \Rightarrow \exists y.\forall x.P(x, y)$$

(c)
$$(\forall x.\exists y.f(y) = x) \Rightarrow (\exists x.f(x) = x)$$

(d)
$$(\forall x.\forall y.\forall z.((f(y) = x \wedge f(z) = x) \Rightarrow y = z))$$
$$\Rightarrow (\forall x.\exists y.f(y) = x)$$

Problem 3. The Clustering Problem in the Land of Choosy People

The Land of Choosy People has a countably infinite number of people living on it. These people have a strong set of likes and dislikes of people (and some for whom they have neither likes nor dislikes). Their leader wants to partition the people into disjoint clusters (sets) such that: (a) no two people in the same cluster dislike each other, and (b) two people who like each other are in the same cluster.

Let us give a unique natural number as an id to each person, and their likes/dislikes using a binary relation L. Let $L(i,j)$ denote that person i likes person j. And let $\neg L(i,j)$ denote that person i does not like person j. Let us model the people's likes and dislikes using an infinite set Γ where each element in Γ is of the form $L(i,j)$ or $\neg L(i,j)$. Assume Γ is infinite. Note that there could be certain pairs i,j such that neither $L(i,j)$ nor $\neg L(i,j)$ belong to Γ.

For any subset $S \subseteq \mathbb{N}$, an equivalence relation $\sim\ \subseteq S \times S$ models clusterings of people. Recall that an equivalence class satisfies three properties: (a) $i \sim i$, (b) If $i \sim j$ then $j \sim i$, and (c) if $i \sim j$ and $j \sim k$, then $i \sim k$, for every $i,j,k \in S$.

We say an equivalence relation $\sim$ over S is a *solution* to the clustering problem if: (a) for every $i,j \in S$ such that $i \sim j$, $\neg L(i,j)$ is not in Γ (i.e., if A does not like B, then A and B are not in the same group), and (b) for every $i,j \in S$ such that $L(i,j) \in \Gamma$, $i \sim j$ (i.e., if A likes B, then A and B are in the same group). An equivalence class of an equivalence relation is a maximal set of elements that are pairwise related to each other.

(a) Consider a subset of people $\{1,2,3,4,5\}$. Let Γ, the set of likes and dislikes, be $\{L(1,3), L(5,3), \neg L(4,5)\}$.

Then one solution is the equivalence relation whose equivalence classes are $\{1,3,5\}$ and $\{2,4\}$. Another solution is $\{1,2,3,5\}$ and $\{4\}$. A third solution is $\{1,3,5\}$, $\{4\}$, and $\{2\}$. In fact, you can verify that these are the only solutions.

Let us model the problem of finding whether a solution exists for this subset of people exists as a propositional satisfiability problem. Model the equivalence class as a set of propositions p_{ij}, which denotes $i \sim j$ holds, and the question of whether a solution exists as a satisfiability problem of a set of propositional formulas. This set models the likes and dislikes of the population and the properties of an equivalence relation using constraints between propositions.

Describe the precise encoding of the above problem as a SAT problem (write math, not Z3 code) of a set of propositional formulae C.

(b) The leader wants to know if every finite subset of her people have a solution (i.e., for every $S \subseteq_{fin} \mathbb{N}$, there is an equivalence relation that is a solution over S), then it is guaranteed that there is a solution for *all* her people (i.e., is there a solution over $\mathbb{N}$)?

Prove to her that the above is indeed true. Your proof should be a formal proof.

Hint: Use the encoding above to encode existence of a solution for all people (i.e., for $\mathbb{N}$) as an infinite set of propositional logic formulae and use the compactness theorem for propositional logic.

Problem 4. Apply quantifier elimination to the formula below stated over the theory of Dense Linear Orders without Endpoints (DLOWE) to obtain an equivalent quantifier-free formula:

$$\forall z.\ \neg\big((z<x\ \vee\ z<y) \wedge (x<z\ \vee\ y<z)\big)$$

Show all steps of the algorithm clearly.

Hint: Informally, the above formula says, "No z is both a lower bound to one among x and y and an upper bound to one among x and y". You can check your answer by informally reasoning whether the quantifier-free formula you obtain is equivalent to this statement.

Problem 5. The theory of rationals with addition $(\mathbb{Q}, 0, 1, +, <, =)$ admits quantifier elimination through the Ferrante-Rackoff method.

Apply this quantifier elimination procedure to find a quantifier-free formula $\varphi(y)$ equivalent to the following formula:

$$\forall x.\ 5 \leq x + 2y \vee x + 3y \leq 10$$

Feel free to simplify any atomic formulae (by simplifying constants using arithmetic), but don't simplify formulae using outside knowledge of rational numbers. Show all steps of the algorithm clearly.

Problem 6. Consider the model of natural numbers with successor: $(\mathbb{N}, 0, s, <, =)$. Here, s is interpreted as the successor relation ($s(i) = i + 1$, for every $i \in \mathbb{N}$).

For a formula $\varphi(x)$ that has only one free variable x (but can have other quantified variables), we say that φ defines the set of natural numbers S, where S is precisely the set of natural numbers such $n \in \mathbb{N}$ such that φ holds when x is interpreted to be n. For example, the formula $\varphi(x) \equiv x > 2 \wedge x < 5$ defines the set $\{3, 4\}$. Note that we have written 2 as a shorthand for $s(s(0))$, similarly 5.

(a) Write some quantifier-free formula without negations, $\varphi_1(x)$, such that the set defined by $\varphi_1(x)$ is finite. Also, write some quantifier-free formula without negations, $\varphi_2(x)$, such that the complement of the set defined by $\varphi_2(x)$ is finite.

(b) Prove that for any *quantifier-free* formula $\varphi(x)$, if S is the set defined by φ, then S is finite or $\mathbb{N} \setminus S$ is finite. Prove this using structural induction.

You can assume the formula uses only the Boolean connectives $\neg$ and $\vee$. Also, in the proof, if certain cases are similar, you can skip them and say which other case they are similar to.

(c) Assume that there is a sublogic L of first-order logic over $(\mathbb{N}, 0, s, <, =)$ admits quantifier elimination. (In fact, the whole logic happens to.)

Argue that any formula $\varphi(x)$ in L with a single free variable x (with quantification) also defines a set S where S is finite or its complement is $\mathbb{N} \setminus S$ finite, using (b) above. Your argument should be crisp but formal.

(d) Using (b) above, show that even-ness is not definable in this logic. Formally, show that there is no formula $\varphi(x)$ that defines the set of even numbers.