# Homework #3

CS 474: Spring 2023

Due on Friday, Mar 10, 2023 11:59 PM Central

**Homework Policy:** You are allowed to collaborate with your classmates, but declare it in your submission. Even if you work on problems together, each student must write up their solution in their own words in their submission. The CS 473 course's page on academic integrity is a handy reference: https://courses.engr.illinois.edu/cs473/sp2023/integrity.html. Late submissions are not allowed. If you have personal circumstances that will prevent you from submitting on time, write to the course staff as soon as possible and we can try to work with you.

**General Instructions:** (1) For any problem where you are asked to come up with an encoding, do not simply give the final answer. Provide a high-level explanation for the choices made in your encoding and develop it step-by-step. Your solution must be easily understandable. (2) Whenever asked to encode a problem in Z3, describe the encoding at a high level if you have not already done so elsewhere in another problem/sub-problem. Do not copy-paste Z3 code into your submission. Upload the file(s) you write to a public GitHub repository and provide a link in your submission. Ensure that the time on the last commit does not exceed the deadline.

**Points: Problem 1: 10+20+10 points, Problem 2: 20+20 points, Problem 3: 20 points; Total: 100 points**

**Theme** This assignment is about First-Order Logic semantics and Quantifier Elimination.

**Problem 1.** In this problem we consider first-order logic over three models: that of natural numbers, rational numbers, and real numbers, each with addition and multiplication. They are defined by the signature $(\mathrm{U}, 0, 1, +, *)$ where U is $\mathbb{N}, \mathbb{Q}$, and $\mathbb{R}$ respectively. Formulas in each logic are interpreted, i.e., evaluated on the respective structures.

**(a)** For each formula below, state whether they are true or false over integers, rationals, and reals, with one sentence describing why.

1. $\exists y.\,(y * y = 1 + 1)$
2. $\forall x.\,\exists y.\,(x + y = 0)$
3. $\forall x.\,\forall y.\,\big(\neg(y = 0) \Rightarrow (\exists z.\, x * y = x + z)\big)$
4. $\exists x.\,\exists y.\,(x + 1 = 0\ \wedge\ y * y = x)$

**(b)** A formula $\varphi(x_1, \ldots, x_n)$ describes a relation on a model, namely the tuples that satisfy the formula (you can see this formula as a kind of macro).

The following formula expresses on natural numbers that $y$ is a factor of $x$:

$$\mathit{factor}(y, x) := \exists z.\,(y * z = x)$$

Similarly, we can write another formula, again on natural numbers, which expresses that $x$ is prime:

$$prime(x) := \neg(x = 0 \wedge x = 1) \wedge (\forall y.\, factor(y, x) \Rightarrow (y = 1 \vee y = x))$$

where we have used the formula *factor* that we defined earlier. We can *inline* the definition of *factor* to get a pure FOL expression for primality that does not have any macros.

Macros can be combined and composed to write bigger formulas, just like we write smaller functions and compose them in programming.

**Task 1.** Write a formula that captures the greater-than relation on natural numbers. You should write a FOL formula $gt_{\mathbb{N}}(x, y)$ such that it is true precisely when $x$ and $y$ are substituted by natural numbers $m$ and $n$ such that $m > n$.

**Task 2.** Do the same task, but for real numbers. Write a FOL formula $gt_{\mathbb{R}}(x, y)$ that captures the greater-than relation on real numbers. Note that this is not trivial as you only have $+$ and $*$ in the signature.

**Extra Credit.** Do the same task for rational numbers.

**(c)** Professor Moriarty thinks that it's a waste of time to study naturals, rationals, and reals separately. He says, "After all, real numbers include natural numbers. Whatever is true about natural numbers is *obviously* true about the reals too!". Prove that he is wrong. You must give an example of a formula that is true when interpreted over natural numbers but false when interpreted over real numbers.

**Problem 2.**

**(a)**

**Task 1.** Perform quantifier elimination (q.e.) on the following formula $\varphi$ for the logic $(\mathbb{R}, 0, 1, <)$, i.e., real numbers with order. Use the q.e. procedure covered in class for Dense Linear Orders Without Endpoints (DLOWE).

$$\forall z. \Big( l_1 < z \ \land \ z < u_1 \ \land \ l_2 < z \ \land \ z < u_2$$

$$\Rightarrow (\exists w.\ l_1 < w \ \land \ w < u_1 \ \land \ l_2 < w \ \land \ w < u_2 \ \land \ w \neq z) \Big)$$

$\varphi$ says that if there are two open intervals $(l_1, u_1)$ and $(l_2, u_2)$ that intersect, they must intersect at more than one point.

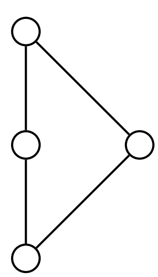
Your solution must show all steps of the elimination, i.e., rewriting $\neq$ using $>$, conversion to DNF, as well as each elimination step. You may simplify intermediate formulae using trivial rewrites (eliminating double negation, rewriting Boolean operations using De Morgan's laws, eliminating atoms that are clearly tautologies or falsities like $x = x, x \neq x$, *True*, *False*).

*Hint: Quantifier Elimination yields equivalent formulas. Don't transform the entire formula at first. Perform q.e. for the formula under the $\exists w$ quantifier first to rewrite it to an equivalent quantifier-free formula, then eliminate the $\exists z$.*

**Task 2.** Z3 supports quantifier elimination for many theories using the tactic `qe` described here: https://microsoft.github.io/z3guide/docs/strategies/summary#tactic-qe. Use Z3 to eliminate quantifiers in $\varphi$ and check that your solution is correct. Report the results of your experiment.

**(b)** Interval graphs (https://en.wikipedia.org/wiki/Interval_graph) are those graphs for which you can map each vertex in the graph to an interval on the real line such that two distinct vertices have an (undirected) edge between them if and only if the corresponding intervals intersect.

**Task 1.** Write a formula $\alpha_G$ over $(\mathbb{R}, 0, 1, <)$ to determine whether the following graph $G$ is an interval graph:



The formula $\alpha_G$ must be such that it is valid if and only if $G$ is an interval graph. You must describe your solution in a mathematically clean and easily readable manner rather than simply stating the formula.

**Task 2.** Use Z3 to determine the validity of $\alpha_G$ and report the result.

**Problem 3. Task 1.** Eliminate quantifiers in the following formula $\psi$, this time over $(\mathbb{Q}, 0, 1, <, +)$. Use the q.e. procedure due to Ferrante and Rackoff covered in class.

$$\psi \equiv \forall x.\, \exists y.\, ((2y > 3x) \wedge (4y < 8x + 10))$$

Note that you are only allowed to make minor boolean rewrites. You must not perform any 'arithmetic' simplifications.

**Task 2.** Use Z3 to check the validity of $\psi$ and report results as you did for the earlier problem.