HW4

Dev Goyal

April 2023

Problem 1

Task 1:

$$
\begin{aligned}
\varphi(r,s) = (\forall r', s'.\ (r*(90*s+20*(1-s)) + (1-r)*(30*s+60*(1-s))\\
\geq r'*(90*s+20*(1-s)) + (1-r')*(30*s+60*(1-s)))\\
\wedge(s*(r*10+70*(1-r)) + (1-s)*(r*80+(1-r)*40\\
\geq s'*(r*10+70*(1-r)) + (1-s')*(r*80+(1-r)*40)))\\
\wedge(r \geq 0) \wedge (s \geq 0) \wedge (r \leq 1) \wedge (s \leq 1) \wedge (r' \geq 0) \wedge (s' \geq 0) \wedge (r' \leq 1) \wedge (s' \leq 1))
\end{aligned} \tag{1}
$$

Task 2:

the link for the Z3 code is: https://github.com/devg24/CS474/tree/main/HW4
The code outputs sat and gives the equilibrium as r = 0.3 and s = 0.5.

Task 3:

Let r,s be mixed strategies for each player and let F,B denote two outcomes. This is to say that player 1 chooses f with probability r and b with probability 1-r and player 2 chooses f with probability s and b with probability 1-s. let p_{ff} denote the payoff for player 1 when both players choose f and p_{fb} denote the payoff for player 1 when player 1 chooses f and player 2 chooses b. Let p_{bf} denote the payoff for player 1 when player 1 chooses b and player 2 chooses f and p_{bb} denote the payoff for player 1 when both players choose b. Let q_{ff} denote the payoff for player 2 when both players choose f and q_{fb} denote the payoff for player 2 when player 1 chooses f and player 2 chooses b. Let q_{bf} denote the payoff for player 2 when player 1 chooses b and player 2 chooses f and q_{bb} denote the payoff for player 2 when both players choose b.:

$$\begin{aligned}\psi \equiv \exists r, s. \forall p_{ff}, p_{fb}, p_{bf}, p_{bb}, q_{ff}, q_{fb}, q_{bf}, q_{bb}, r', s' \\ (r * (p_{ff} * s + p_{fb} * (1-s)) + (1-r) * (p_{bf} * s + p_{bb} * (1-s)) \\ \geq r' * (p_{ff} * s + p_{fb} * (1-s)) + (1-r') * (p_{bf} * s + p_{bb} * (1-s))) \\ \wedge(s * (r * q_{ff} + q_{fb} * (1-r)) + (1-s) * (r * q_{bf} + (1-r) * q_{bb} \\ \geq s' * (r * q_{ff} + q_{fb} * (1-r)) + (1-s') * (r * q_{bf} + (1-r) * q_{bb}))) \\ \wedge(r \geq 0) \wedge (s \geq 0) \wedge (r \leq 1) \wedge (s \leq 1) \wedge (r' \geq 0) \wedge (s' \geq 0) \wedge (r' \leq 1) \wedge (s' \leq 1)\end{aligned} \tag{2}$$

the link for the Z3 code is: https://github.com/devg24/CS474/tree/main/HW4

Problem 2

The set of terms T is:

{a, f(a), g(a), f(g(a)), f(f(a)), f(f(f(a)))}

the set of equalities E is:

{ (f(f(f(a))), f(a)), (f(a),g(a)), (f(g(a)), a)}

the set of inequalities I is:

{(a,f(a))}

We start with the equivalence relation:

{ {a}, {f(a)}, {g(a)}, {f(g(a))}, {f(f(a))}, {f(f(f(a)))} }

since f(a) and g(a) are equal, we can merge the sets:

{ {a}, {f(a), g(a)}, {f(g(a))}, {f(f(a))}, {f(f(f(a)))} }

since f(g(a)) and a are equal, we can merge the sets:

{ {a, f(g(a))}, {f(a), g(a)}, {f(f(a))}, {f(f(f(a)))} }

since f(f(f(a))) and f(a) are equal, we can merge the sets:

{ {a, f(g(a))}, {f(a), g(a), f(f(f(a)))}, {f(f(a))} }

since f(a) and g(a) are in the same equivalence class, we have to merge the equivalence class of f(f(a)) and f(g(a)):

{ {a, f(g(a)), f(f(a))}, {f(a), g(a), f(f(f(a)))} }

since f(f(a)) and g(a) are in T so we have to merge f(f(f(a))) and f(g(a)) which yields the following equivalence relation:

{ {a, f(g(a)), f(f(a)), f(f(f(a))), f(a), g(a)} }

since a and f(a) are in I so this means that the model is unsatisfiable.

Problem 3

Task 1:

The formula is :

$$\varphi : \forall e'.((\forall x.(f(x,e') = x \wedge f(e',x) = x) \implies (e' = e))) \tag{3}$$

following the steps in the book by bringing the formula to a prenex form and then negating it we get:

$$\neg\varphi : \exists e'.\forall x(f(x,e') = x \wedge f(e',x) = x \wedge \neg(e' = e)) \tag{4}$$

now skolemizing by replacing the quantified variable e' with a new constant c we get:

$$\forall x.(f(x,c) = x \wedge f(c,x) = x \wedge \neg(c = e)) \tag{5}$$

The four universal formulae we get are now :

1. $\forall x, y, z.f(f(x,y),z) = f(x, f(y,z))$
2. $\forall x.f(x,e) = x \wedge f(e,x) = x$
3. $\forall x.f(x,g(x)) = e \wedge f(g(x),x) = e$
4. $\forall x.(f(x,c) = x \wedge f(c,x) = x \wedge \neg(c = e))$

instantiating terms of depth 0 we get the following formulae:

1. $f(f(e,e),e) = f(e,f(e,e))$
2. $f(f(c,e),e) = f(c,f(e,e))$
3. $f(f(e,c),e) = f(e,f(c,e))$
4. $f(f(c,c),e) = f(c,f(c,e))$
5. $f(f(e,e),c) = f(e,f(e,c))$
6. $f(f(c,e),c) = f(c,f(e,c))$
7. $f(f(e,c),c) = f(e,f(c,c))$
8. $f(f(c,c),c) = f(c,f(c,c))$

9. $f(e, e) = e \wedge f(e, e) = e$
10. $f(c, e) = c \wedge f(e, c) = c$
11. $f(e, g(e)) = e \wedge f(g(e), e) = e$
12. $f(c, g(c)) = e \wedge f(g(c), c) = e$
13. $(f(e, c) = e \wedge f(c, e) = e \wedge \neg(c = e))$
14. $(f(c, c) = c \wedge f(c, c) = c \wedge \neg(c = e))$

Task 2:

The formula is :

$$\varphi : \forall x, y.(f(x, y) = e \wedge f(y, x) = e) \implies (y = g(x)) \tag{6}$$

where g(x) is the inverse function.

negating the above and skolemizing using two new constant symbol c and d gives:

$$(f(c, d) = e \wedge f(d, c) = e) \wedge \neg(d = g(c))$$

The four universal formulae we get are now :

1. $\forall x, y, z.f(f(x, y), z) = f(x, f(y, z))$
2. $\forall x.f(x, e) = x \wedge f(e, x) = x$
3. $\forall x.f(x, g(x)) = e \wedge f(g(x), x) = e$
4. $(f(c, d) = e \wedge f(d, c) = e) \wedge \neg(d = g(c))$

instantiating terms of depth 0 we get the following formulae:

1. Substituting e for x, e for y, and e for z: $f(f(e, e), e) = f(e, f(e, e))$
2. Substituting e for x, e for y, and c for z: $f(f(e, e), c) = f(e, f(e, c))$
3. Substituting e for x, e for y, and d for z: $f(f(e, e), d) = f(e, f(e, d))$
4. Substituting e for x, c for y, and e for z: $f(f(e, c), e) = f(e, f(c, e))$
5. Substituting e for x, c for y, and c for z: $f(f(e, c), c) = f(e, f(c, c))$

6. Substituting e for x, c for y, and d for z: $f(f(e,c),d) = f(e,f(c,d))$
7. Substituting e for x, d for y, and e for z: $f(f(e,d),e) = f(e,f(d,e))$
8. Substituting e for x, d for y, and c for z: $f(f(e,d),c) = f(e,f(d,c))$
9. Substituting e for x, d for y, and d for z: $f(f(e,d),d) = f(e,f(d,d))$
10. Substituting c for x, e for y, and e for z: $f(f(c,e),e) = f(c,f(e,e))$
11. Substituting c for x, e for y, and c for z: $f(f(c,e),c) = f(c,f(e,c))$
12. Substituting c for x, e for y, and d for z: $f(f(c,e),d) = f(c,f(e,d))$
13. Substituting c for x, c for y, and e for z: $f(f(c,c),e) = f(c,f(c,e))$
14. Substituting c for x, c for y, and c for z: $f(f(c,c),c) = f(c,f(c,c))$
15. Substituting c for x, c for y, and d for z: $f(f(c,c),d) = f(c,f(c,d))$
16. Substituting c for x, d for y, and e for z: $f(f(c,d),e) = f(c,f(d,e))$
17. Substituting c for x, d for y, and c for z: $f(f(c,d),c) = f(c,f(d,c))$
18. Substituting c for x, d for y, and d for z: $f(f(c,d),d) = f(c,f(d,d))$
19. Substituting d for x, e for y, and e for z: $f(f(d,e),e) = f(d,f(e,e))$
20. Substituting d for x, e for y, and c for z: $f(f(d,e),c) = f(d,f(e,c))$
21. Substituting d for x, e for y, and d for z: $f(f(d,e),d) = f(d,f(e,d))$
22. Substituting d for x, c for y, and e for z: $f(f(d,c),e) = f(d,f(c,e))$
23. Substituting d for x, c for y, and c for z: $f(f(d,c),c) = f(d,f(c,c))$
24. Substituting d for x, c for y, and d for z: $f(f(d,c),d) = f(d,f(c,d))$
25. Substituting d for x, d for y, and e for z: $f(f(d,d),e) = f(d,f(d,e))$
26. Substituting d for x, d for y, and c for z: $f(f(d,d),c) = f(d,f(d,c))$
27. Substituting d for x, d for y, and d for z: $f(f(d,d),d) = f(d,f(d,d))$
28. Substituting e for x: $f(e,e) = e \wedge f(e,e) = e$

29. Substituting c for x: $f(c, e) = c \wedge f(e, c) = c$
30. Substituting d for x: $f(d, e) = d \wedge f(e, d) = d$
31. Substituting e for x: $f(e, g(e)) = e \wedge f(g(e), e) = e$
32. Substituting c for x: $f(c, g(c)) = e \wedge f(g(c), c) = e$
33. Substituting d for x: $f(d, g(d)) = e \wedge f(g(d), d) = e$
34. $(f(c, d) = e \wedge f(d, c) = e) \wedge \neg(d = g(c))$