Homework 5

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1 Problem 1

1.1 Task 1

To prove this we need to show that if $X \subseteq Y$ then $f(X) \subseteq f(Y)$

let $x \in f(X)$ This means that x is either 2 or 3, or there exists y such that $y - 2 \in X$or $y - 3 \in X$ and x = y.

Case 1: when x = 2 or x = 3. Then x is in f(Y) since 2 and 3 are always in f(Y).

Case 2: $\exists y.(y - 2 \in X \vee y - 3 \in X) \wedge x = y$

since x = y and $X \subseteq Y$ then $y - 2 \in Y$ or $y - 3 \in Y$.

which means $y \in f(Y)$ and because $x = y$ then $x \in f(Y)$

since we staretd from an arbitrary element of $f(X)$ and showed that it is in $f(Y)$ then $f(X) \subseteq f(Y)$

hence the f is monotonic.

1.2 Task 2

to define the set S* corresponding to the LFP. We first start with the empty set S_0

$$S_0 = \{\}$$

then we apply the function f to it to get S_1

$$S_1 = f(S_0) = \{2, 3\}$$

then we apply the function f to it to get S_2

$$S_2 = f(S_1) = \{2, 3, 4, 5\}$$

then we apply the function f to it to get S_3

and we can continue this process to get the set S_*

$$S^* = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16.....\} = N$$

2 Question 2

2.1 Task 1

$\varphi_1 = \forall i(l_a(i) \implies (\exists j succ(i) = j \wedge l_b(j))) \wedge \forall i(\neg l_a(i) \implies l_b(i)) \wedge \exists i.(first(i) \wedge l_a(i))$

This formula says that every a is followed by a b and we cannot have a b without an a(to prevent other symbols).

2.2 Task 2

$\varphi_2(x, y) = \exists P(P(x) \wedge \neg(P(y)) \wedge \forall z(P(z) \implies z = x \vee (\exists u, v(succ(u) = z) \wedge \neg(u = y) \wedge succ(v) = u \wedge \neg(P(v)))))$

This formula states that there exists a set P that contains x but not y, such that every position in P is either x or a position that can be reached from x by a finite sequence of successor operations, excluding y.

2.3 Task 3

$\varphi_2(x) = l_a(x) \wedge \forall i(\varphi_2(x, i) \vee x = i \implies (l_a(i) \implies (\exists j succ(i) = j \wedge l_b(j))) \wedge \forall i(\neg l_a(i) \implies l_b(i)))$

Basically this formula says that the x should be a and the formula in task 1 should hold true for all positions $i >= x$.

3 Question 3

3.1 Task 1

the given literals are:

$$\varphi : y \leq x, x \leq y, f(y) = f(7), x \leq 5$$

applying transformation 1 to the third literal we get 2 formulae:
$\Sigma_1 - formula :$

$$y \leq x \wedge x \leq y \wedge x \leq 5 \wedge w_1 = 7$$

$\Sigma_2 - formula :$

$$f(y) = f(w1)$$

With $\{y, w_1\}$ being the shared variable

3.2 Task 2

for the first formula the arrangement is

$$x = y \wedge w_1 = 7 \wedge x \leq 5$$

one satisfying assignment is $\{y = 5, x = 5, w_1 = 7\}$(note this works for any value of x and y less than or equal to 5)

for the second formula the arrangement is

$$f(y) = f(w_1)$$

one satisfying assignment is $\{y = 5, w_1 = 7\}$ with f being the function that maps every element to 1.

3.3 Task 3

from these two models we get the following model for the original formula:

$$\{y = 5, x = 5, w_1 = 7\}$$

with f being the function that maps every element to 1.