Ciência de Dados - Desafio Matemática
Determinantes

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0.1 Exercício 01

Deduza a determinante 4×4 usando a fórmula:
$\det(A) = \sum_{\sigma \in S_n} (\prod_{i=1}^{n} (-1)^{\text{sgn}(\sigma)} \cdot a_{i\sigma(i)})$

Resolução:

1) Seja a matriz 4x4:

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

2) Sabendo que S_4 é a permutação de "1234", isto é, $P_4 = 4!$, então há 24 elementos nesse conjunto, logo temos que S:

$$\begin{aligned} S_4 = \{\, & 1234, 1432, 1324, 1342, 1243, 1423, \\ & 2314, 2413, 2431, 2134, 2341, 2143, \\ & 3214, 3412, 3421, 3124, 3241, 3142, \\ & 4213, 4312, 4123, 4321, 4132, 4231 \,\} \ . \end{aligned}$$

3) Resolvendo a determinante 4×4 pela fórmula dada:

$\det(A) = \sum_{\sigma \in S_4} (\prod_{i=1}^{4} (-1)^{\text{sgn}(\sigma)} \cdot a_{i\sigma(i)}) = \prod_{i=1}^{4} (-1)^{\text{sgn}(1234)} \cdot a_{i1234(i)} + \prod_{i=1}^{4} (-1)^{\text{sgn}(1432)} \cdot a_{i1432(i)} + \prod_{i=1}^{4} (-1)^{\text{sgn}(1324)} \cdot a_{i1324(i)} + \prod_{i=1}^{4} (-1)^{\text{sgn}(1342)} \cdot a_{i1342(i)} + \prod_{i=1}^{4} (-1)^{\text{sgn}(1243)} \cdot a_{i1243(i)} + \prod_{i=1}^{4} (-1)^{\text{sgn}(1423)} \cdot a_{i1423(i)} + \prod_{i=1}^{4} (-1)^{\text{sgn}(2314)} \cdot a_{i2314(i)} + \prod_{i=1}^{4} (-1)^{\text{sgn}(2413)} \cdot a_{i2413(i)} + \prod_{i=1}^{4} (-1)^{\text{sgn}(2431)} \cdot a_{i2431(i)} + \prod_{i=1}^{4} (-1)^{\text{sgn}(2134)} \cdot a_{i2134(i)} + \prod_{i=1}^{4} (-1)^{\text{sgn}(2341)} \cdot a_{i2341(i)} + \prod_{i=1}^{4} (-1)^{\text{sgn}(2143)} \cdot a_{i2143(i)} + \prod_{i=1}^{4} (-1)^{\text{sgn}(3214)} \cdot a_{i3214(i)} + \prod_{i=1}^{4} (-1)^{\text{sgn}(3412)} \cdot a_{i3412(i)} + \prod_{i=1}^{4} (-1)^{\text{sgn}(3421)} \cdot a_{i3421(i)} + \prod_{i=1}^{4} (-1)^{\text{sgn}(3124)} \cdot a_{i3124(i)} + \prod_{i=1}^{4} (-1)^{\text{sgn}(3241)} \cdot a_{i3241(i)} + \prod_{i=1}^{4} (-1)^{\text{sgn}(3142)} \cdot a_{i3142(i)} + \prod_{i=1}^{4} (-1)^{\text{sgn}(4213)} \cdot a_{i4213(i)} + \prod_{i=1}^{4} (-1)^{\text{sgn}(4312)} \cdot a_{i4312(i)} + \prod_{i=1}^{4} (-1)^{\text{sgn}(4123)} \cdot a_{i4123(i)} + \prod_{i=1}^{4} (-1)^{\text{sgn}(4321)} \cdot a_{i4321(i)} + \prod_{i=1}^{4} (-1)^{\text{sgn}(4132)} \cdot a_{i4132(i)} + \prod_{i=1}^{4} (-1)^{\text{sgn}(4231)} \cdot a_{i4231(i)}$

4) Posto isso, vamos resolver o $\text{sgn}(\sigma)$ para cada σ, a fim de calcularmos os

produtórios dessa determinante:
sgn(1234) = 0, sgn(1432) = 1, sgn(1324) = 1, sgn(1342) = 2, sgn(1243) = 1, sgn(1423) = 2, sgn(2314) = 2, sgn(2413) = 3, sgn(2431) = 2, sgn(2134) = 1, sgn(2341) = 3, sgn(2143) = 2, sgn(3214) = 1, sgn(3412) = 2, sgn(3421) = 3, sgn(3124) = 2, sgn(3241) = 2, sgn(3142) = 3, sgn(4213) = 2, sgn(4312) = 3, sgn(4123) = 3, sgn(4321) = 2, sgn(4132) = 2, sgn(4231) = 1

5) Basta, agora, pegarmos as posições de cada σ da permutação, da seguinte forma:
1234(1) = 1; 1234(2) = 2; 1234(3) = 3; 1234(4) = 4; 1432(1) = 1; 1432(2) = 4; 1432(3) = 3; 1432(4) = 2; 1324(1) = 1; 1324(2) = 3; 1324(3) = 2; 1324(4) = 4; 1342(1) = 1; 1342(2) = 3; 1342(3) = 4; 1342(4) = 2; 1243(1) = 1; 1243(2) = 2; 1243(3) = 4; 1243(4) = 3; 1423(1) = 1; 1423(2) = 4; 1423(3) = 2; 1423(4) = 3; 2314(1) = 2; 2314(2) = 3; 2314(3) = 1; 2314(4) = 4; 2413(1) = 2; 2413(2) = 4; 2413(3) = 1; 2413(4) = 3; 2431(1) = 2; 2431(2) = 4; 2431(3) = 3; 2431(4) = 1; 2134(1) = 2; 2134(2) = 1; 2134(3) = 3; 2134(4) = 4; 2341(1) = 2; 2341(2) = 3; 2341(3) = 4; 2341(4) = 1; 2143(1) = 2; 2143(2) = 1; 2143(3) = 4; 2143(4) = 3; 3214(1) = 3; 3214(2) = 2; 3214(3) = 1; 3214(4) = 4; 3412(1) = 3; 3412(2) = 4; 3412(3) = 1; 3412(4) = 2; 3421(1) = 3; 3421(2) = 4; 3421(3) = 2; 3421(4) = 1; 3124(1) = 3; 3124(2) = 1; 3124(3) = 2; 3124(4) = 4; 3241(1) = 3; 3241(2) = 2; 3241(3) = 4; 3241(4) = 1; 3142(1) = 3; 3142(2) = 1; 3142(3) = 4; 3142(4) = 2; 4213(1) = 4; 4213(2) = 2; 4213(3) = 1; 4213(4) = 3; 4312(1) = 4; 4312(2) = 3; 4312(3) = 1; 4312(4) = 2; 4123(1) = 4; 4123(2) = 1; 4123(3) = 2; 4123(4) = 3; 4321(1) = 4; 4321(2) = 3; 4321(3) = 2; 4321(4) = 1; 4132(1) = 4; 4132(2) = 1; 4132(3) = 3; 4132(4) = 2; 4231(1) = 4; 4231(2) = 2; 4231(3) = 3; 4231(4) = 1;

6) Por fim, utilizando esses resultados, podemos substituí-los no determinante 4×4 a ser resolvido:
$\det(A) = \prod_{i=1}^{4}(-1)^0 \cdot a_{i1234(i)} + \prod_{i=1}^{4}(-1)^1 \cdot a_{i1432(i)} + \prod_{i=1}^{4}(-1)^1 \cdot a_{i1324(i)} + \prod_{i=1}^{4}(-1)^2 \cdot a_{i1342(i)} + \prod_{i=1}^{4}(-1)^1 \cdot a_{i1243(i)} + \prod_{i=1}^{4}(-1)^2 \cdot a_{i1423(i)} + \prod_{i=1}^{4}(-1)^2 \cdot a_{i2314(i)} + \prod_{i=1}^{4}(-1)^3 \cdot a_{i2413(i)} + \prod_{i=1}^{4}(-1)^2 \cdot a_{i2431(i)} + \prod_{i=1}^{4}(-1)^1 \cdot a_{i2134(i)} + \prod_{i=1}^{4}(-1)^3 \cdot a_{i2341(i)} + \prod_{i=1}^{4}(-1)^2 \cdot a_{i2143(i)} + \prod_{i=1}^{4}(-1)^1 \cdot a_{i3214(i)} + \prod_{i=1}^{4}(-1)^2 \cdot a_{i3412(i)} + \prod_{i=1}^{4}(-1)^3 \cdot a_{i3421(i)} + \prod_{i=1}^{4}(-1)^2 \cdot a_{i3124(i)} + \prod_{i=1}^{4}(-1)^2 \cdot a_{i3241(i)} + \prod_{i=1}^{4}(-1)^3 \cdot a_{i3142(i)} + \prod_{i=1}^{4}(-1)^2 \cdot a_{i4213(i)} + \prod_{i=1}^{4}(-1)^3 \cdot a_{i4312(i)} + \prod_{i=1}^{4}(-1)^3 \cdot a_{i4123(i)} + \prod_{i=1}^{4}(-1)^2 \cdot a_{i4321(i)} + \prod_{i=1}^{4}(-1)^2 \cdot a_{i4132(i)} + \prod_{i=1}^{4}(-1)^1 \cdot a_{i4231(i)}$

$\therefore \det(A) = a_{11} \cdot a_{22} \cdot a_{33} \cdot a_{44} - a_{11} \cdot a_{24} \cdot a_{33} \cdot a_{42} - a_{11} \cdot a_{23} \cdot a_{32} \cdot a_{44} + a_{11} \cdot a_{23} \cdot a_{34} \cdot a_{42} - a_{11} \cdot a_{22} \cdot a_{34} \cdot a_{43} + a_{11} \cdot a_{24} \cdot a_{32} \cdot a_{43} + a_{12} \cdot a_{23} \cdot a_{31} \cdot a_{44} - a_{12} \cdot a_{24} \cdot a_{31} \cdot$

$a_{43} + a_{12} \cdot a_{24} \cdot a_{33} \cdot a_{41} - a_{12} \cdot a_{21} \cdot a_{33} \cdot a_{44} - a_{12} \cdot a_{23} \cdot a_{34} \cdot a_{41} + a_{12} \cdot a_{21} \cdot a_{34} \cdot a_{43} - a_{13} \cdot a_{22} \cdot a_{31} \cdot a_{44} + a_{13} \cdot a_{24} \cdot a_{31} \cdot a_{42} - a_{13} \cdot a_{24} \cdot a_{32} \cdot a_{41} + a_{13} \cdot a_{21} \cdot a_{32} \cdot a_{44} + a_{13} \cdot a_{22} \cdot a_{34} \cdot a_{41} - a_{13} \cdot a_{21} \cdot a_{34} \cdot a_{42} + a_{14} \cdot a_{22} \cdot a_{31} \cdot a_{43} - a_{14} \cdot a_{23} \cdot a_{31} \cdot a_{42} - a_{14} \cdot a_{21} \cdot a_{32} \cdot a_{43} + a_{14} \cdot a_{23} \cdot a_{32} \cdot a_{41} + a_{14} \cdot a_{21} \cdot a_{33} \cdot a_{42} - a_{14} \cdot a_{22} \cdot a_{33} \cdot a_{41}$

0.2 Exercício 02

Calcule o determinante, usando o que foi deduzido, de duas matrizes definidas pelo autor:

$$\det(A) = 0$$
$$\det(A) \neq 0$$

Resolução:

1) $\det(A) = 0$:

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

$\det(A) = 0 - 0 - 0 + 0 - 0 + 0 + 0 - 0 + 0 - 0 - 0 + 0 - 0 + 0 - 0 + 0 + 0 - 0 + 0 - 0 - 0 + 0 + 0 - 0 = 0$

2) $\det(A) \neq 0$

$$\det(A) = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$\det(A) = 0 - 0 - 0 + 0 - 0 + 0 + 0 - 0 + 0 - 0 - 0 + 0 - 0 + 0 - 0 + 0 + 0 - 0 + 0 - 0 - 1 + 0 + 0 - 0 = -1$