# Special Relativity Homework set 1

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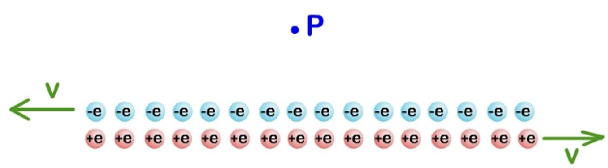
### 1.1 Inconsistency of Galilean Relativity and Maxwell's Equations

A very long beam of electrons is formed at SLAC consisting of $1.0 \times 10^{10}$ electrons per meter, all in a line, traveling at $2.0 \times 10^8 m/s$ to the left. In the opposite direction (to the right) is another beam of positrons with the same density, traveling at the same speed, and very close to the electron beam. A positron is the anti particle of the electron. It is identical in every way except it has a charge of +e. Alice stands next to the beam and measures the current.



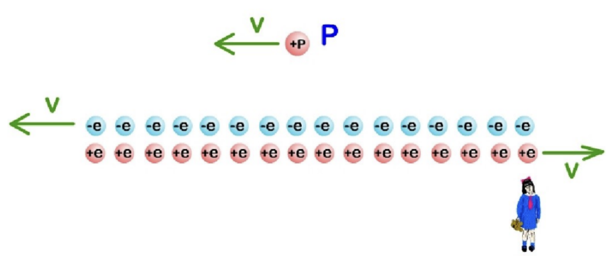

a) Does Alice observe any net electric charge of the beam? No

b) Does Alice observe any electric field at the point P? No

c) In which direction does Alice observe the net (conventional) current of the beam? To the right

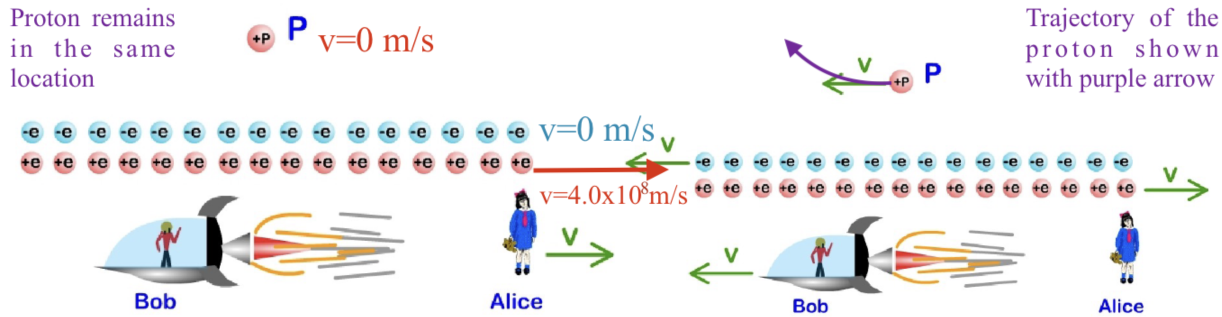
d) In which direction does Alice observe the $\vec{B}$ field, if any, to be in at point P? Out of the page

A stray proton is traveling near the beam. At a specific time, the proton is at point P, 1.0m above the beam and traveling parallel to the beam at $2.0 \times 10^8 m/s$ to the left.




e) What direction is the net force, if any, exerted on the proton as seen by Alice?
The magnetic field is pointing out of the page and the proton is traveling to the left, so the force is exerted in the upward direction.

Now consider Bob who is riding on a very (VERY) fast ship traveling to the left and parallel to the beam at a speed of $2.0 \times 10^8 m/s$. He views the same situation and measures the current of the beam. The image on the left below represents what Bob sees in his reference frame where he sees himself to be at rest. Label on the diagram the directions and magnitudes of the velocities of the proton, electron beam, and positron beam as perceived by Bob.




f) Does Bob observe any net electric charge of the beam? No

g) Does Bob observe any electric field at the point P? No

h) In which direction does Bob observe the net (conventional) current of the beam? To the right

i) In which direction does Bob observe the $\vec{B}$ field, if any, to be in at point P? Out of the page

j) What direction is the net force, if any, exerted on the proton as seen by Bob? Because the proton is not moving in this frame of reference, there is no force exerted on it. $\vec{F} = q(\vec{0} \times \vec{B}) = \vec{0}$

k) Compare the motion of the proton as viewed by Alice and Bob. On each figure above draw the trajectory of the proton after it goes by point P as it appears to each of them. Discuss the problem with this result - provide an example where this result can lead to a logical inconsistency. (Use a trigger that sets of a bomb for dramatic effect.)

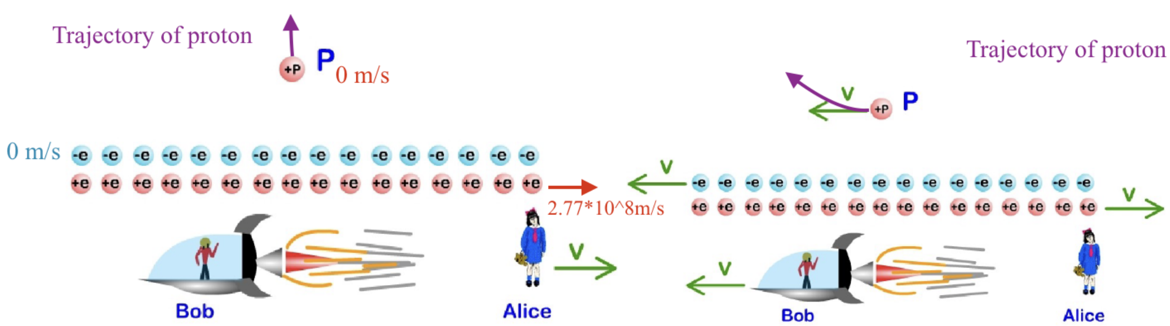
Let us assume there was a device that would explode if it came into contact with a proton. Let us also assume Schrödinger's cat is next to the bomb. In Alice's frame of reference, the upward electric force on the proton will cause it to accelerate upward and collide with the device, resulting in an explosion. Alice will see Shrödinger's cat die. However, in Bob's frame of reference, because the proton is at rest, there is no force due to the magnetic field of the beam. Therefore, the proton does not collide with the device and there is no explosion. Bob sees Shrödinger's cat continue living. However, this is a logical contradiction because two observers at one instant of time at rest next to each other will observe two very different scenarios. As this is not Quantum physics, Shrödinger's cat cannot be both dead and alive at the same time.

**Problem 1.2 : Resolving the previous inconsistency with EM and Gal. Rel.**

In the previous problem involving Alice and Bob observing a proton near counter-propagating beams of electrons and positrons. You should have found that Alice sees the proton to accelerate away from the beam while Bob observes that the proton stays in place. This was because Alice observed a moving charged particle traveling through a magnetic field yielding a magnetic force while Bob observed a stationary proton in a magnetic field. Since the magnetic force is proportional to the velocity, there was no force on the proton in Bob's frame. The inconsistency results from the two different motions in the two different frames. (For e.g., place a trigger of a bomb just above the proton. In Alice's frame the bomb explodes yet no explosion occurs in Bob's frame. A logical inconsistency can arise by having the bomb mortally wound Bob. In one frame he is alive yet in the other he is dead -this is a logical inconsistency.)

Because Bob is traveling at the same velocity as the electrons and proton, they are all at rest in Bob's reference frame. Alice is moving $2.0 \times 10^8 m/s$ to the right and the positrons are moving at a greater speed to the right. Nothing can travel faster than the speed of light, so the positrons do not travel at $4.0 \times 10^8 m/s$. Instead, their speed can be calculated as follows $v = \frac{2\times10^8 m/s + 2\times10^8 m/s}{1 - \frac{(2\times10^8 m/s)^2}{c^2}} = 2.77 \times 10^8 m/s$ because they are traveling at speed $2.0 \times 10^8 m/s$ with respect to Alice who is traveling $2.0 \times 10^8 m/s$ with respect to Bob.

More importantly, because the positrons are moving at such a high speed, the distance between two consecutive positrons is less than the distance between two electrons as a result of length contraction. This means the positive charge density is larger than the negative charge density and there is a net positive charge and an outward electric field. Because the proton is positively charged, it will experience an upward force whose magnitude is equal to the upward force in Alice's frame of reference.



**Problem 1.3 : Sideways light clock**

Consider our light clock gedankenexperiment that yielded our time dilation formula. In that case the light clock had the light travel perpendicular to the relative motion of Alice and Bob. Let's consider now if we lay both clocks on their side so the light travels back and forth along the relative direction of travel. For simplicity we will only consider the scenario from Alice's frame (we know it gives the same result in Bob's frame).

Again, the clocks have the emitter and mirror a distance D apart and Bob travels from right to left at speed v. Break the problem into two parts, the forward part where the light beam first travels in the same direction of travel (right to left) and takes a time $\Delta t_f$, and the backward part where it travels in the opposite direction and takes a time $\Delta t_b$.

a) First, without considering length contraction, show that you do not obtain the same relation between the clocks that we found in class.

b) By considering that Bob's light clock is length contracted, show that you do obtain the same time dilation relation we found in class.

**Solution**

a) In Bob's frame of reference, it takes time $\Delta t = \frac{D}{c}$ for the light to travel from the emitter to the mirror. In Alice's frame of reference, the light travels an additional distance in the forward phase. If the position is taken to be 0m at the emitter at $t = 0s$, the distance from $x = 0m$ of the light beam can be calculated as $ct$ and the distance of the mirror can be calculated as $vt + D$. When the light collides with the mirror, we get $vt + D = ct$ from which we can calculate $\Delta t_f = \frac{D}{c-v}$. After being reflected, the light travels in the opposite direction with respect to the direction of travel of the frame. Therefore, we can write the distance as $-vt + D = ct$ which means $\Delta t_b = \frac{D}{c+v}$. Without considering length contraction, the total time it takes is $\frac{D}{c-v} + \frac{D}{c+v} = \frac{2Dc}{c^2-v^2}$.

b) Now, let us consider length contraction. In 1 second, the light clock travels a distance

D in Bob's frame of reference. However, Bob is traveling at speed v with respect to Alice, so she will observe the distance traveled to be $D\sqrt{1-\frac{v^2}{c^2}}$. Substituting this value for D in the previous equation results in $\Delta t_f + \Delta t_b = \frac{2D\sqrt{\frac{c^2-v^2}{c^2}}c}{c^2-v^2} = \frac{2D}{\sqrt{c^2-v^2}} = \gamma\Delta t$, which is the time dilation result we found in class.

**STP Problem 1.9 Traveling to the Andromeda Galaxy I**

The Andromeda galaxy is approximately 2 million light years distant from Earth as measured in the Earth-linked frame. Is it possible for you to travel to Andromeda in your lifetime? Let's explore this in steps. In the following consider the distance from Earth to Andromeda to be exactly $2 \times 10^6 ly$ and neglect any relative motion of the Earth and Andromeda.

a) Trip 1: Your one-way trip takes $2.01 \times 10^6$ years (as measured in the Earth-linked frame) to cover the distance of $2.00 \times 10^6 ly$. How long does the trip last as measured in your rocket frame?

b) What is your speed on Trip 1 as measured in the Earth-linked frame? Express this speed as a fraction of the speed of light, $\beta = \frac{v}{c}$

c) Trip 2: Your one-way trip to Andromeda takes a time of $2.001 \times 10^6 y$ (using $c = 1\frac{ly}{y}4$), How long does this trip last as measured in your rocket frame? What is the speed of the rocket ship in this case (expressed as before)?

d) Trip 3: Now set the rocket time for the one-way trip to take 20 years, you are in a hurry to get to Andromeda. What is the speed of your rocket (as measured in the Earth-linked frame) in this case? [You will need the binomial expansion.]

**Solution**

a) First, we need to calculate the speed of the rocket with respect to the earth. It travels $2.00 \times 10^6 ly \times 3 \times 10^8 \frac{m}{s} \times 3600 \frac{s}{h} \times 24 \frac{h}{d} \times 365 \frac{d}{y} = 1.89 \times 10^{22} m$ in $2.01 \times 10^6 \times 3600 \frac{s}{h} \times 24 \frac{h}{d} \times 365 \frac{d}{y} = 6.34 \times 10^{13} s$ resulting in a speed of $2.985 \times 10^8 m/s$ with respect to 3 significant figures. Because of time dilation, the time according to the Earth's reference frame will be larger and the time measured by the rocket is $(2.01 \times 10^6 y)\sqrt{1 - \frac{(2.985 \times 10^8 m/s)^2}{(3.0 \times 10^8 m/s)^2}} = 2.00 \times 10^5$ years.

b) $\frac{2.99 \times 10^8 m/s}{3.0 \times 10^8 m/s} = 0.995$

c) The speed of the rocket ship is $\frac{2 \times 10^6 \times 3 \times 10^8}{2.001 \times 10^6} = 2.9985 \times 10^8 m/s$. $\frac{2.9985 \times 10^8 m/s}{3.0 \times 10^8 m/s} = 0.9995$. $(2.001 \times 10^6 y)\sqrt{1 - \frac{(2.9985 \times 10^8 m/s)^2}{(3.0 \times 10^8 m/s)^2}} = 6.33 \times 10^4$ years.

d) Rearranging the time dilation equation, we get $v = c\sqrt{1 - \left(\frac{t'}{t}\right)^2}$. Here, $t' = 20$ years and $t = 2.0 \times 10^6$ years. $v = \left(1 - \frac{20}{2000000}\right)^{\frac{1}{2}} c$. Using the binomial expansion, we can express this value as $(1 - \frac{1}{200000})c = 0.999995c$.

**Problem 1.5 STP 1.11: Experimental Confirmation 1: $\mu$decay.**

At heights of 10 to 60 kilometers above Earth, cosmic rays continually strike nuclei of oxygen and nitrogen atoms and produce muons ($\mu$) (elementary particles about 207 times more massive as electrons). Some of the muons move vertically downward with a speed nearly that of light. Consider one muon as it descends through the atmosphere. In a given sample of muons, half of them decay to other elementary particles in $1.5 \times 10^{-6} sec$, measured with respect to a reference frame in which they are at rest. Half of the remainder decay in the next 1.5 microseconds, and so on. Analyze the results of this decay as observed in two different frames, Idealize the rather complicated actual experiment to the following roughly equivalent situation: All of the muons are produced at the same height of 60 kilometers; all have the same speed; all travel straight down; none are lost to collisions with air molecules on the way down.

a) Approximately how long a time will it take these muons to reach the surface of the Earth, as measured in the Earth frame?

b) If the decay time were the same for Earth observers as for an observer traveling along with the muons, approximately how many half-lives would have passed? Therefore what fraction of those created at 60 kilometers would remain when they reached sea level on Earth? Express your answer as a power of the fraction $\frac{1}{2}$.

c) An experiment determines that the fraction $\frac{1}{8}$ of the muons reaches sea level. Call the rest frame of the muons the rocket frame. In this rocket frame, how many half-lifes have passed between creation of a given $\mu$ and its arrival as a survivor at sea level?

d) In the rocket frame, what is the space separation between birth of a survivor muon and its arrival at the surface of Earth? (Careful!)

Solution

a) The muons are traveling at the speed of light and traveling 60km. Therefore, it will take approximately $2 \times 10^{-4}s$ for them to reach the surface of the Earth as measured in the Earth frame.

b) $\frac{2\times10^{-4}s}{1.5\times10^{-6}s} = 133$ half lives will have passed. Therefore, $\frac{1}{2^{133}}$ of those created at 60km would remain.

c) For an $\frac{1}{8}$ of the muons to reach sea level, 3 half lives would have passed.

d) The rocket frame is the rest frame of the muons. Therefore, the muons do not move and the space separation between the creation of the muon and its arrival at the earth's surface is 0.

**Problem 1.6 STP 1.12: Experimental Confirmation 2: $\pi$+decay.**

Laboratory experiments on particle decay are much more conveniently done with $\pi+$ mesons than with $\mu$mesons, as is seen in the table.

| Time dilation with $\pi^+$ mesons | | |
|---|---|---|
| Particle | time for $\frac{1}{2}$ to decay (in rest frame) | "Characteristic distance" ($c\times$ foregoing time) |
| $\mu$ meson ($207 \times m_e$) | $1.5 \times 10^{-6}s$ | $450m$ |
| $\pi^+$ meson ($273 \times m_e$) | $18 \times 10^{-9}s$ | $5.4m$ |

In a given sample of $\pi+$ mesons half will decay to other elementary particles in 18 nanoseconds ($18 \times 10^{-9}$s) measured in a reference frame in which the $\pi+$ mesons are at rest. Half of the remainder will decay in the next 18 nanoseconds, and so on.

a) In a particle accelerator $\pi+$s are produced when a proton beam strikes an aluminum target in side the accelerator. $\pi+$s leave this target with nearly the speed of light. If there were no time dilation and if no mesons were removed from the resulting beam by collisions, what would be the greatest distance from the target at which half of the $\pi+$s would remain undecayed?

b) The $\pi+$s of interest in a particular experiment have a speed of 0.9978 that of light. By what factor is the predicted distance from the target for half-decay increased by time dilation over the previous prediction – that is, by what factor does this dilation effect allow one to increase the separation between the detecting equipment and target?

**Solution**

a) Assuming the $\pi+$ mesons are traveling at the speed of light, the greatest distance from target at which half of them would remain undecayed is $c(18 \times 10^{-9}s) = 5.4m$ using the half life in rest frame because there is no time dilation.

b) Using the proper time of $18 \times 10^{-9}s$ and speed of 0.9978c, the factor gamma is equal to $\frac{1}{\sqrt{1-0.9978^2}} = 15.08$ and $t' = \gamma t = 2.7 \times 10^{-7}s$. Moreover, the predicted distance will become $15.08 \times 5.4m = 81m$.

# Special Relativity Homework set 2

Devi Amarsaikhan

6 January 2023

**2.1 Relativistic Doppler Effect.**

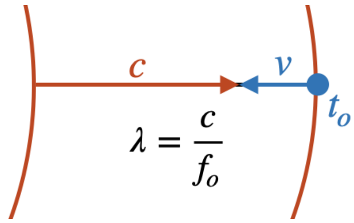
In frame S' a light of frequency $f_o$ is sent radially outwards in all directions at $ct' = 0m$. Another frame, S, moves from right to left at speed v with respect to the light's source.

a) Without using any relativistic effects derive a formula for the frequency measured in the frame S. Consider both cases where the light source is approaching the observer and moving away.

b) Calculate the relativistic Doppler effect with relativity via the causal diamond. It is advantageous to use lightcone coordinates, u, v. Find both when the source is approaching and receding from the observer. [First relate the sides of the causal diamond in the source's frame. Place two observers, one to the left and one to the right of the source's worldline (in S) and determine the rate at which they receive pulses.]

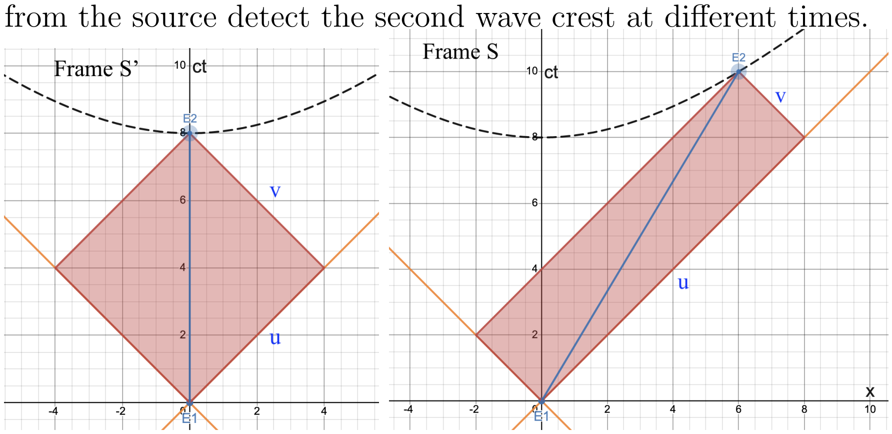
**Solution**

a) In frame S', the wavelength of the wave is $\lambda = \frac{c}{f_o}$. However, the observer is moving at speed v to the left. First, let us consider the case when the observer is approaching the light source. Let $t_0$ be the time when the observer detects the first wave crest. The time it takes for the observer to detect the next wave crest $t_1$ is less than $\frac{1}{f_o}$ because the observer is moving toward the point source. Light travels at speed c and the observer travels at speed v. $\lambda = ct_1 + vt_1$, which means $t_1 = \frac{\lambda}{c+v}$, which means $f_1 = \frac{c+v}{\lambda}$ which is equal to $f_1 = \frac{(c+v)}{c} f_o$.




Now let us consider the case when the observer is moving away from the light source. The wave crest will need to travel a larger distance to be detected by the observer. Let $t_2$ be the time it takes to detect the next wave crest. $\lambda = ct_2 - vt_2$, which means $t_2 = \frac{\lambda}{c-v}$, which means $f_2 = \frac{c-v}{\lambda} = \frac{(c-v)}{c} f_o$

b) Let us construct a causal diamond with side lengths u and v as shown below in frame S'. In the proper frame, the two events are separated by an interval $\Delta s' = c\Delta t'$. The area of the causal diamond is $2 \times \left(\frac{c\Delta t'}{2}\right)^2 = \frac{c^2 \Delta t'^2}{2}$. Because the area of the causal diamond stays constant, the change in u and v are inversely proportional. This means observers moving in opposite directions with the same speed will see the values of u and v to be reversed. Therefore, we need only do one calculation to determine the Doppler effect in two directions. In the moving frame of reference S, if S is moving to the right with positive beta, $u = \frac{1}{\sqrt{2}}(c\Delta t + \Delta x)$ and $v = \frac{1}{\sqrt{2}}(c\Delta t - \Delta x)$ where $\Delta x$ is the horizontal distance between the origin and event E2. We can rewrite u and v as follows $u = \frac{1}{\sqrt{2}}(c\Delta t + \Delta x) = \frac{c\Delta t}{\sqrt{2}}(1 + \beta) = \frac{\Delta s}{\sqrt{2}}\gamma(1 + \beta) = \frac{\Delta s}{\sqrt{2}}\sqrt{\frac{1+\beta}{1-\beta}}$ and $v = \frac{1}{\sqrt{2}}(c\Delta t - \Delta x) = \frac{c\Delta t}{\sqrt{2}}(1 - \beta) = \frac{\Delta s}{\sqrt{2}}\gamma(1 - \beta) = \frac{\Delta s}{\sqrt{2}}\sqrt{\frac{1-\beta}{1+\beta}}$. The change in distance shows us that the observer moving toward the source and the observer moving away from the source detect the second wave crest at different times.



### 2.2 Accelerating observers: horizons

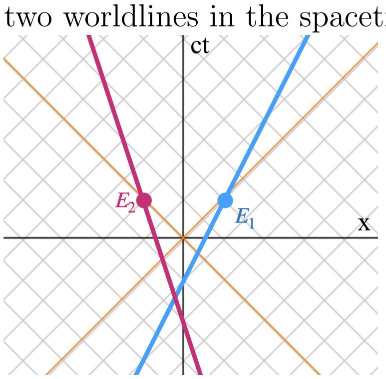
a) Show that for two observers on two infinitely-extendible, non-accelerating, worldlines that there always exists a causal diamond between some event on one worldline and any second, timelike separated, event on the other. This is in flat spacetime (no gravity). Argue via causal diamonds only (no math).

b) Now consider one of the observers to be on board a continually accelerating spaceship. (I.e. on board the ship you can determine the acceleration is constant, how?). Show now that if the two worldlines share a causal diamond near the beginning of the launch, at some time in the future there can no longer be causal diamonds between them. Just consider that the accelerating observer will eventually be causally disconnected from the observer at rest in S. Argue via causal diamonds only (no math).

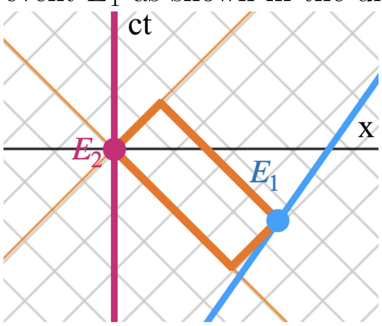
c) What does this have to say about the nature of the reference frame on board the ship? Considering what principles or criteria we've introduced that are now violated, what conclusion might we make about spacetime as experienced in the ship's reference frame.

### Solution

a) Because the events are time-like separated, they can be causally connected. The worldlines are not accelerating, so they are straight lines on the spacetime diagram. Consider two worldlines in the spacetime diagram below and two events $E_1$ and $E_2$.




While these two events are simultaneous in this frame of reference, they are not in the pink frame of reference. First, let us translate the spacetime such that $E_2$ is at the origin. Now, let us perform Lorentz transformations such that the pink worldline is vertical, meaning the object is at rest. The slope of the blue line will be smaller than it was before because it will be traveling faster in this reference frame. Because both objects are traveling at a speed less than the speed of light, their slope as shown on the diagram will be greater than 1. Because they are time-like separated, they can be causally related. We can extend a line segment from $E_2$ to the bottom right and extend a second line segment perpendicular to the light cone side such that it passes through event $E_1$ as shown in the diagram below (not drawn to scale).




b) On board the ship, we can measure the weight of an object as it accelerates. If the weight stays constant, the acceleration is constant. At first, the spaceship has a speed less than the speed of light. At that instant, we can construct a causal diamond because the two events are time-like separated. As the spaceship accelerates, its velocity will get closer and closer to the speed of light.

c) The spaceship's frame of reference is not an inertial reference frame. From the frame of reference of one spaceship, the rocket will first start with speed 0 and accelerate until it reaches half the speed of light. Let us say there is a second space ship moves at constant velocity such that it travels at half the speed of light with reference to the first spaceship at the same time as the rocket. Now, consider the second spaceship's frame of reference. The rocket again starts at speed 0 and accelerated up to half the speed of light. We can set up an infinite number of spaceships where they all experience the rocket accelerate from 0 to 0.5c but their speeds are asymptotically approaching the speed of light. The acceleration approaches zero because it asymptotically approaches the constant speed of light. Newton's Laws are valid in all inertial reference frames. However, because the spaceship is accelerating, the frame of reference is not inertial.

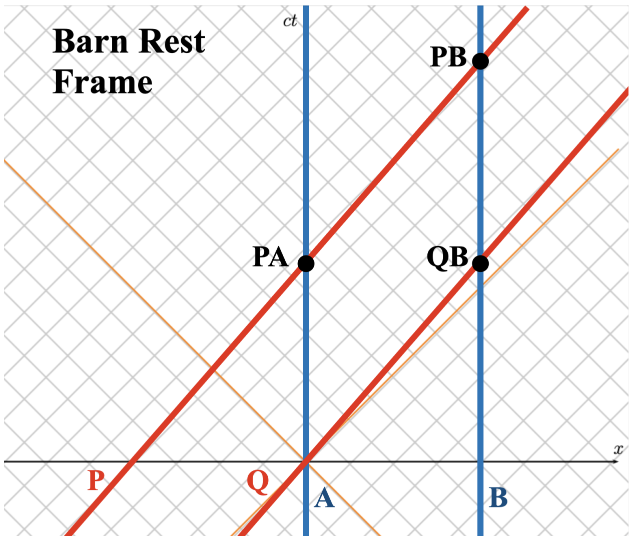
### 2.3 Pole and Barn Paradox

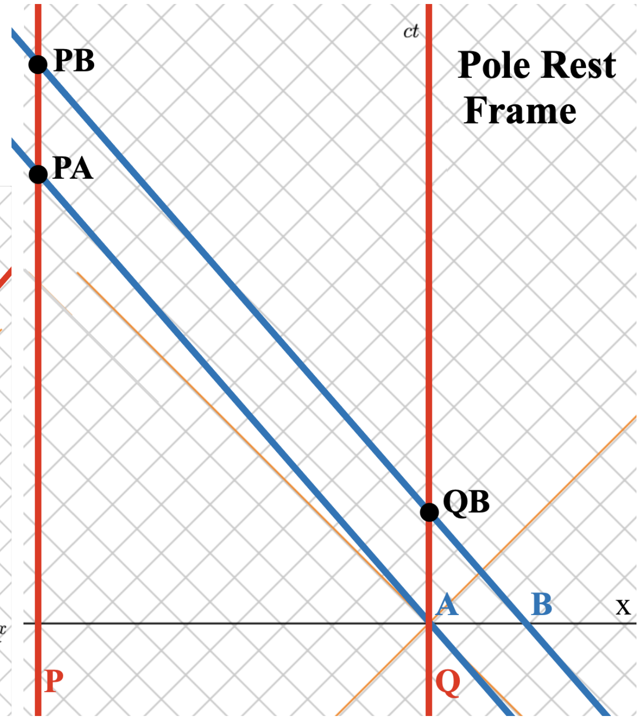
A worried student writes, "Relativity must be wrong. Consider a 20-meter pole carried so fast in the direction of length that it appears to be only 10 meters long in the laboratory frame of reference ( $= 2$). Let the runner who carries the pole enter a barn 10 meters long, as shown in the figure. At some instant the farmer can close the front door and the pole will be entirely enclosed in the barn. However, look at the same situation from the frame of reference of the runner. To him the barn appears contracted to half its length. How can a 20 meter pole possibly fit into a 5-meter barn? Does not this unbelievable conclusion prove that relativity contains somewhere a fundamental logical inconsistency? Examine this situation by making two large spacetime diagrams from both points of view. Explain how the pole and the barn are treated by relativity without internal inconsistency.

a) Make two carefully labeled spacetime diagrams, one an x-ct diagram for the barn rest frame, the other an x'-ct' diagram for the runner rest frame. Referring to the figure, take the event "Q coincides with A" to be the origin of both diagrams. In both plot the worldlines of A,B, P, and Q. Pay attention to the scale of both diagrams. Make sure the slope of the lines corresponds to their velocity. Label the event "Q coincides with B" as QB (derived from Lorentz transformation equations or otherwise). Do the same for the events PA and PB.

b) Give a brief explanation of what would happen if the back door were replaced with steel-reinforced concrete. What happens after the farmer closes the front door on the pole?

**Solution**

a) 




b) Immediately after the farmer closes the front door, the pole will collide with the concrete wall and break. In the barn's frame of reference, the pole will fit exactly inside the 10m long barn. Moreover, the time at which the front end of the pole Q reaches the back wall B coincides with the moment the back end of the pole P reaches the front door of the barn A. There is no contradiction from Alice's frame of reference because the two events are not simultaneous. The front of the pole Q will reach the back end of the barn B and after a short amount of time, the back end of the pole P will reach the front door of the barn A.

### 2.4 T and U Bar Paradox

A U-shaped structure made of the strongest steel contains a detonator switch connected by wire to one metric ton of the explosive TNT. A T-shaped structure made of the same strong steel fits inside the U, with the long arm of the T not quite long enough to reach the detonator switch when both structures are at rest in the laboratory.

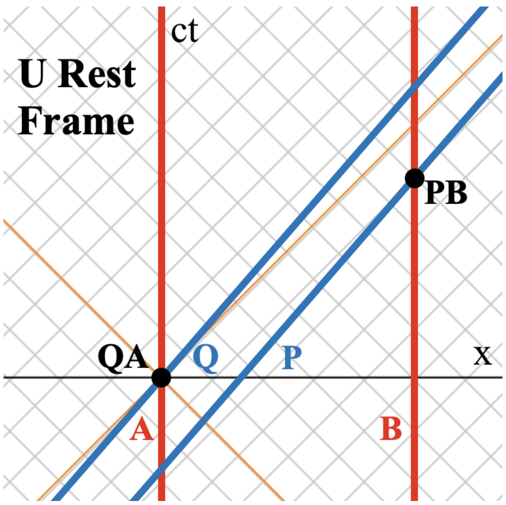
Now the T structure is removed far to the left and accelerated to high speed (with $\gamma = 2$). It is Lorentz-contracted along its direction of motion. As a result, its long arm is not long enough to reach the detonator switch when the two collide. Therefore there will be no explosion.

However, look at the same situation in the rest frame of the T-structure. In this frame the arm of the T has its rest length, while the two arms of the U-structure are Lorentz-contracted. Therefore the arm of the T will certainly strike the detonator switch and there will be a terrible explosion.

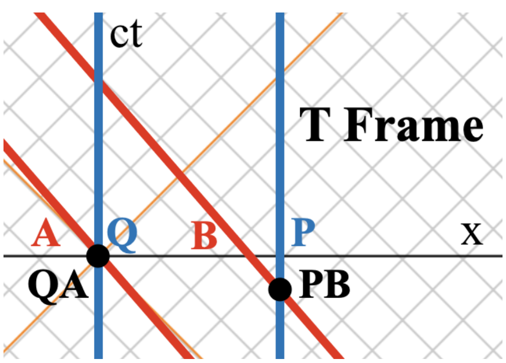
As in the previous problem make two large spacetime diagrams representing the motion of the objects in the T frame and in the U frame. Take the relative speed to be 87% of c, then the contraction is half of the rest length ($\gamma = 2$). Label the front and back of the T structure P and Q respectively and A and B the front and back of the U structure (see figure). Choose the event QA as the origin in both. Determine from your diagram whether there will be an explosion or not. Explain why your diagram gives your answer.

**Solution**

Because the U and T structures are made of the strongest steel, we can assume they will not break. In U's reference frame, when Q and A coincide, the two metals collide. If we assume they do not break, they both travel at the same speed but because QP is shorter than AB when both are traveling at the same speed, P and B never coincide and the bomb does not go off.




In T's reference frame, P and B coincide before Q and A coincide, so the bomb goes off. Because the order of events QA and PB changes, these two events are spacelike separated.



There will be no explosion because these events are space like separated. We discussed that space like separated events cannot be causally related so information cannot pass between them. The information of the explosion can travel at the speed of light. Therefore, it would never reach the observers.

# Special Relativity Homework set 3

Devi Amarsaikhan

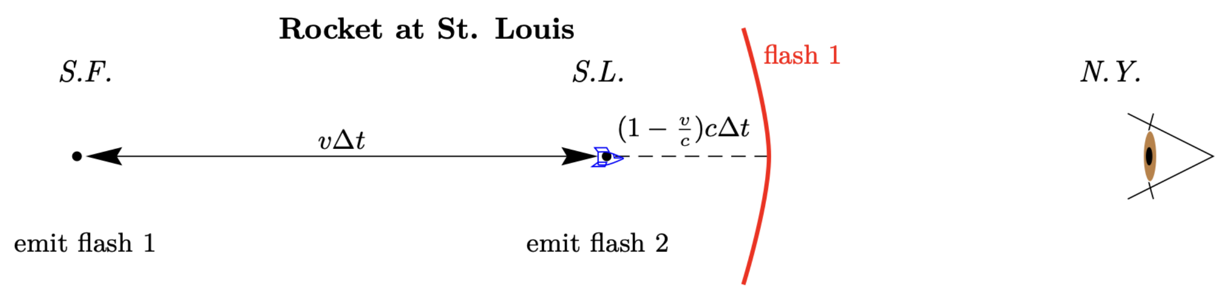
February 2023

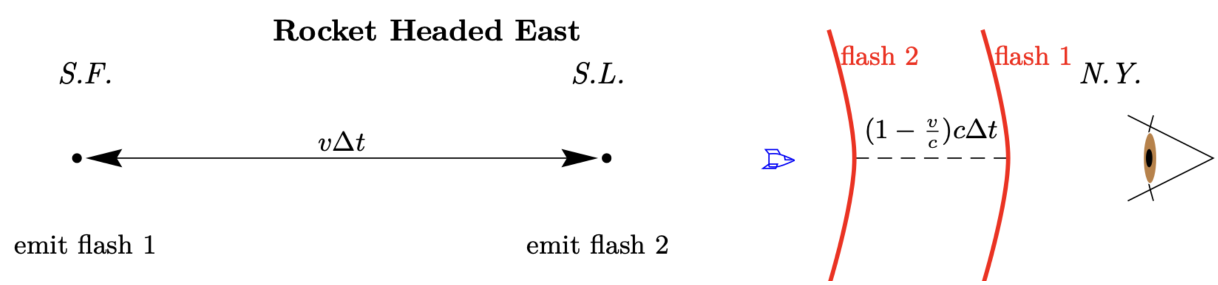
**Problem 3.1 Four times the speed of light?**

We look westward across the United States and see the rocket approaching us at four times the speed of light. [How can this be? We did not say the rocket moves faster than light; we said only that we see it moving faster than light.]

Here is what happens: The rocket streaks under the Golden Gate Bridge in San Francisco, emitting a flash of light that illuminates the rocket, the bridge, and the surroundings. At time $\Delta t$ later the rocket threads the Gateway Arch in St. Louis. The top figure is a visual summary of measurements from our continent-spanning latticework of clocks taken at this moment.

Now the rocket continues toward us as we stand in New York City. The center figure summarizes data taken as the first flash is abut to enter our eye. Flash 1 shows us the rocket passing under the Golden Gate Bridge. An instant later flash 2 shows us the rocket passing through the Gateway Arch.



**Answer the following questions using the symbols from the last two figures.**

a) **The images carried by the two flashes show the rocket how far apart in space?** $(1 - \frac{v}{c})c\Delta t$

b) **What is the time lapse between our reception of these two images?** $(1-\frac{v}{c})\Delta t$**.**

c) **Therefore, what is the apparent speed of the approaching rocket we see?** $\frac{v\Delta t}{(1-\frac{v}{c})\Delta t} = \frac{v}{1-\frac{v}{c}} = \frac{vc}{c-v}$

d) **For what speed v of the rocket does the apparent speed of approach equal four times the speed of light?** $\frac{vc}{c-v} = 4c$, which means $\frac{v}{c-v} = 4$. Rearranging, we get $v = \frac{4c}{5}$.

e) **For what rocket speed do we see the approaching rocket to be moving at 99 times the speed of light?** $\frac{vc}{c-v} = 99c$**.** $\frac{v}{c-v} = 99$**.** $v = 99c - 99v$**.** $100v = 99c$**.** $v = \frac{99c}{100}$**.**

### 3.2 Transformation of y-velocity

The rocket frame ($S'$) is moving in the $x$ direction with speed $v_{rel}$. Now start with an object that is moving in the $y'$ direction, $v'_y = \frac{\Delta y'}{\Delta t'}$ . Show that the x and y components of the velocity, as observed in the S frame are

$$v_x = v_{rel} \qquad \beta_x = \beta_{rel}$$
$$v_y = v'_y\sqrt{1 - \frac{v_{rel}^2}{c^2}} \qquad \beta_y = \beta'_y\sqrt{1 - \beta_{rel}^2}$$

**Solution**

When Frame S is stationary, Frame S' is moving with speed $v_{rel}$ to the right. In Frame S', there is an object moving vertically with speed $v'_y$. Using the Lorentz transformations, we get

$$\Delta t = \gamma(\Delta t' + v_{rel}\frac{\Delta x'}{c^2})$$
$$\Delta x = \gamma(\Delta x' + v_{rel}\Delta t')$$
$$\Delta y = \Delta y'$$
$$\Delta z = \Delta z'$$

$$v_x = \frac{\Delta x}{\Delta t} = \frac{\gamma(\Delta x' + v_{rel}\Delta t')}{\gamma(\Delta t' + v_{rel}\frac{\Delta x'}{c^2})} = \frac{\frac{\Delta x'}{\Delta t'} + v_{rel}}{1 + \frac{v_{rel}}{c^2}\frac{\Delta x'}{\Delta t'}} = \frac{v'_x + v_{rel}}{1 + \frac{v_{rel}}{c^2}v'_x}$$

$$v_y = \frac{\Delta y}{\Delta t} = \frac{\Delta y'}{\gamma(\Delta t' + v_{rel}\frac{\Delta x'}{c^2})} = \frac{\frac{\Delta y'}{\Delta t'}}{\gamma(1 + \frac{v_{rel}}{c^2}\frac{\Delta x'}{\Delta t'})} = \frac{v'_y}{\gamma(1 + \frac{v_{rel}}{c^2}v'_x)}$$

Because the object does not have horizontal velocity in the S' frame, $v_x = v_{rel}$ and $v_y = \frac{v_{rel}}{\gamma} = v_{rel}\sqrt{1 - \frac{v_{rel}^2}{c^2}}$. Dividing both sides of both equations by c, we get the corresponding equations with $\beta$.

### 3.3 The tilted meter stick

A meter stick lying parallel to the x-axis moves in the y-direction in the laboratory frame with speed $v_y$ as shown in the figure. The rocket frame moves at speed $v_{rel}$ in the x direction.

a) In the rocket frame the stick is tilted upward in the positive $x'$-direction as shown in the figure. Explain why this is, first without using any equations.

b) Let the center of the meter stick pass the point $x = y = x' = y' = 0$at time $t = t' = 0$. Calculate the angle $\phi'$ at which the the meter stick is inclined to the $x'$-axis as observed in the rocket frame.

**Solution**

a) Let us set up three synchronized clocks in the laboratory's frame of reference moving with the same velocity as the meter stick. Let us place clock $L_1$ at the left end, $L_2$ at the center, and $L_3$ at the right end of the meter stick. Because the whole stick is on the x-axis at one instance in time, let us call this time 0. We can see that the events when the three clocks $L_1$, $L_2$, and $L_3$ are at the x axis and display time 0 are synchronous in the rocket's frame of reference. Let us set up three other synchronized clocks that are stationary in the rocket's frame of reference. Let us say the location of these three clocks coincide with the location of clocks $L_1$, $L_2$, and $L_3$ in the laboratory's frame of reference at events $E_1$, $E_2$, and $E_3$. Looking at this from the rocket's frame of reference, the meter stick is half as long due to length contraction and is traveling with velocity $-v_{rel}$. For the events to be the same as in the laboratory's frame of reference, the three events will not be simultaneous in the rocket's frame. Here, $E_3$ occurs first followed by $E_2$ and $E_1$. This means the meter stick had more time to travel upward when $E_3$ occurs with respect to when $E_2$ occurs in the rocket's frame of reference. Therefore, the meter stick appears to be tilted.

b) The velocity in the laboratory frame of reference is straight up with magnitude $v_y$. The velocity in the rocket's frame of reference is straight up with magnitude $v_y' = v_y\sqrt{1-\frac{v_{rel}^2}{c^2}}$ and to the left with magnitude $v_{rel}$. $v_x' = -v_{rel}$ and $v_y' = v_y\sqrt{1-\frac{v_{rel}^2}{c^2}}$. This is a meter stick, so in the laboratory frame of reference, event $E_3 = (ct, x, y) = (0, \frac{1}{2}, 0)$. In the rocket's frame of reference, $E_3 = (ct', x', y') = (ct', x', 0)$. Using Lorentz transformation, $t' = \gamma(t' - \frac{v_{rel}x}{c^2}) = \gamma(0 - \frac{v_{rel}}{2c^2}) = -\frac{\gamma v_{rel}}{2c^2}$ and $x = \gamma(x - v_{rel}t) = \frac{\gamma}{2}$. After time $\frac{\gamma v_{rel}}{2c^2}$ passes, we have $t' = 0$ and $x' = y' = 0$. The right hand end will be $(x', y') = (\frac{\gamma}{2} + v_x\Delta t', 0 + v_y\Delta t') = (\frac{\gamma}{2} - v_{rel}(\frac{\gamma v_{rel}}{2c^2}), 0 + \frac{v_y}{\gamma}\frac{\gamma v_{rel}}{2c^2}) = (\frac{L}{2\gamma}, \frac{v_y v_{rel}}{2c^2})$. The angle $\phi'$ will be given by $\phi' = \arctan\left(\frac{y'}{x'}\right) = \arctan\left(\frac{\frac{v_y v_{rel}}{2c^2}}{\frac{1}{2\gamma}}\right) = \arctan\left(\frac{v_y v_{rel}\gamma}{c^2}\right)$.

### 4.1 STP 7-6 Fast Electrons

The Two-Mile Stanford Linear Accelerator accelerates electrons to a final kinetic energy of 47 GeV ($47 \times 10^9$ electron-volts; one electron-volt $=1.6 \times 10^{-19}$ joule). The resulting high-energy electrons are used for experiments with elementary particles. Electromagnetic waves produced in large vacuum tubes ("klystron tubes") accelerate the electrons along a straight pipelike structure $10,000$ feet long (approximately 3000 meters long). Take the rest energy of an electron to be $m0.5MeV = 0.5 \times 10^6$ electron volts.

(a) Electrons increase their kinetic energy by approximately equal amounts for every meter traveled along the accelerator pipe as observed in the laboratory frame. What is this energy gain in MeV/meter? Suppose the Newtonian expression for kinetic energy were correct. In this case how far would the electron travel along the accelerator before its speed were equal to the speed of light?

(b) In reality, of course, even the 47-GeV electrons that emerge from the end of the accelerator have a speed $\beta$ that is less than the speed of light. W hat is the value of the difference $(1-\beta)$ between the speed of light and the speed of these electrons as measured in the laboratory frame? [Hint: For $\beta$ very near the value unity, $1-\beta^2 = (1+\beta)(1-\beta)2(1-\beta)$.Let a 47-GeV electron from this accelerator race a flash of light along an evacuated tube straight through Earth from one side to the other (Earth diameter 12,740 kilometers). How far ahead of the electron is the light flash at the end of this race? Express your answer in millimeters.

(c) How long is the "3000-meter" accelerator tube as recorded on the latticework of rocket clocks moving along with a 47-GeV electron emerging from the acceletator?

### Solution

(a) The rest energy of the electron is $5 \times 10^5$ eV and the final kinetic energy of the electron is $4.7 \times 10^{10}$ eV. The electron travels approximately 3000 meters. Assuming the kinetic energy of the electron increases by equal amounts per meter, we get that the energy gain is $\frac{4.7\times10^{10}eV-5\times10^{5}eV}{3000m} = 1.6\times10^{7}eV/m = 16MeV/m$.

Supposing the Newtonian kinetic energy is correct, the kinetic energy of the electron would be 0 when the electron is at rest and will increase at the rate of $16MeV/m$. When traveling at the speed of light, its kinetic energy will be equal to $\frac{mc^2}{2} = 2.5\times10^{5}eV$. Therefore, it would take $\frac{2.5\times10^{5}eV}{1.6\times10^{7}eV/m} = 1.56\times10^{-2}m$

(b) $K = 4.7\times10^{10} = (\gamma-1)mc^2$, so $\gamma = 94001$. From $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, we get $1-\beta^2 = \frac{1}{\gamma^2}$. Using the approximation $1-\beta^2 2(1-\beta)$, we get $\beta = 1-\frac{1}{2\gamma^2}$. Plugging in the value of $\gamma$, we get $\beta = 0.99999999994$.

With this speed, the electron takes $\frac{12{,}740{,}000m}{0.99999999994c} = 0.042467$ seconds to travel a distance equal to the earth's diameter. In the same amount of time, light travels $\frac{12{,}740{,}000m}{0.99999999994}$. The light travels $0.721mm$ more than the electron.

(c) Due to length contraction, the accelerator tube will seem to be $\frac{3000m}{\gamma} = 3.19cm$.

### 4.2 STP 7-7 Super Cosmic Rays

The Haverah Park extensive air shower array near Leeds, England, detects the energy of individual cosmic ray particles indirectly by the resulting shower of particles this cosmic ray creates in the atmosphere. Between 1968 and 1987 the Haverah Park array detected more than $25,000$ cosmic rays with energies greater than $4\times10^{17}$ electron-volts, including 5 with an energy of approximately $10^{20}$ electron-volts, (rest energy of the proton $= 10^{19}$ electron-volts $= 1.6\times10^{-10}$ joule)

1. Suppose a cosmic ray is a proton of energy $10^{20}$ electron-volts. How long would it take this proton to cross our galaxy as measured on the proton's wristwatch? The diameter of our galaxy is approximately $10^5$ light years. How many centuries would this trip take as observed in our Earth-linked frame?

2. The research workers at Haverah Park find no evidence of an upper limit to cosmic

ray energies. A proton must have an energy of how many times its rest energy for the diameter of our galaxy to appear to it Lorentz-contracted to the diameter of the proton (about 1 femtometer, which is equal to $10^{-15}$ meters)? How many metric tons of mass would have to be converted to energy with 100 percent efficiency in order to give a proton this energy? One metric ton equals 1000 kilograms.

**Solution**

(a) $K_p = (\gamma - 1)mc^2$, so $\gamma = \frac{K_p}{mc^2} + 1$. We were given the rest energy of the proton is approximately $10^9 eV$. Therefore, $\gamma = \frac{10^{20}eV}{10^9 eV} + 1 = 10^11 + 1$. Due to length contraction, the diameter of our galaxy will seem to be $\frac{10^5 lightyears}{10^11+1} = 1.0 \times 10^{-6}$ light-years, which is equal to $9.46 \times 10^9 m$. Therefore, the trip will take $\frac{9.46 \times 10^9 m}{(10^11+1)c} s = 31.5s$.
In our earth-linked frame, due to time dilation, this trip will take $31.5\gamma s = 3.15 \times 10^{12} s =$ 999 centuries.

(b) For the galaxy with length $10^5 lightyears = 9.46 \times 10^{20} m$ to be Lorentz-contracted to $10^{-15} m$, $\gamma = \frac{9.46 \times 10^{20} m}{10^{-15} m} = 9.46 \times 10^{35}$. This is $\gamma - 1 = 9.46 \times 10^{35}$ times its rest energy. The proton's rest energy was given as $1.6 \times 10^{-10} J$, so the total Kinetic energy is $1.6 \times 10^{-10} J \times 9.46 \times 10^{35} = 1.51 \times 10^{26} J$. Dividing this quantity by $c^2$, we get $m_0 = \frac{1.51 \times 10^{26} J}{c^2} = 1.68 \times 10^9 kg = 1.68 \times 10^6$ tons.

### 4.3 STP 8-11 Pair production by a lonely photon?

A gamma ray (high-energy photon, zero mass) can carry an energy greater than the rest energy of an electron- positron pair. (Remember that a positron has the same mass as the electron but opposite charge.) Nevertheless the process

$$\text{(energetic gamma ray)} \rightarrow \text{(electron)} + \text{(positron)}$$

cannot occur in the absence of other matter or radiation.

(a) Prove that this process is incompatible with the laws of conservation of momentum and energy as employed in the laboratory frame of reference. Analyze the alleged creation in the frame in which electron and positron go off at equal but opposite angles $\pm\phi$ with the extended path of the incoming gamma ray.

(b) Repeat the demonstration-which then becomes much more impressive-in the center-of-momentum frame of the alleged pair, the frame of reference in which the total momentum of the two resulting particles is zero.

**Solution**

(a) Because momentum and energy are conserved, $\begin{pmatrix} E_g \\ p_g c \end{pmatrix} = \begin{pmatrix} E_e \\ p_e c \end{pmatrix} + \begin{pmatrix} E_p \\ p_p c \end{pmatrix}$. A gamma ray has no mass, so it must travel at the speed of light. The electron and positron have equal mass. We can orient the system such that the gamma ray is traveling in the horizontal direction so there is no initial vertical momentum. The momentum of the electron and positron are equal but in opposite directions. $E^2 = (pc)^2 + (mc^2)^2$, so we can write $p_g c = \sqrt{(p_e c)^2 + (0.511MeV)^2} + \sqrt{(p_p c)^2 + (0.511MeV)^2}$. From the conservation of momentum, $p_g c = p_e c + p_p c$. Combining the two previous equations, we get $p_e c + p_p c = \sqrt{(p_e c)^2 + (0.511MeV)^2} + \sqrt{(p_p c)^2 + (0.511MeV)^2}$. Because momentum cannot be conserved, a lone gamma ray cannot produce a positron and electron pair.

(b) In the center of momentum frame of reference of the alleged pair, the electron and positron have zero total momentum. According to the law of conservation of momentum, the system must have started with zero momentum. However, because the gamma ray has no mass, it travels at the speed of light and has non zero momentum. Repeat the demonstration-which then becomes much more impressive-in the center-of-momentum frame of the alleged pair, the frame of reference in which the total momentum of the two resulting particles is zero.

### 4.4 Do you weigh mass or energy?

**Consider a special mass-less container that holds two space ships, each of mass m, that can travel at relativistic speeds. The container itself is at rest in the frame under consideration, even though the ships may be moving within it. There are two main ways to model this system: 1) as two ships and a mass-less container that is always at rest, or 2) as a container (the whole system considered as a single object). Here we examine the energy, momentum, and mass in four different scenarios. Give these properties in terms of the mass of the ships m, speed $\beta$ if they are moving, $\gamma = (1-\beta^2)^{-\frac{1}{2}}$, and c.**

**Scenario A: both ships at rest w.r.t. the container.**

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Model 1: Model 2:

$$\begin{array}{ccc} E_1 = mc^2 & E_2 = mc^2 & E_{system} = 2mc^2 \\ p_1 = 0 & p_2 = 0 & p_{system} = 0 \\ m_1 = m & m_2 = m & M_{system} = 2m \end{array}$$

**Scenario B: ships traveling at $\vec{\beta}_1 = -\vec{\beta}_2$.**

$$\begin{array}{lll} Model1: & & Model2: \\ E_1 = \gamma mc^2 & E_2 = \gamma mc^2 & E_{system} = 2\gamma mc^2 \\ p_1 = \gamma mc\beta & p_2 = -p_1 & p_{system} = 0 \\ m_1 = m & m_2 = m & M_{system} = 2m \end{array}$$

**Scenario C: ships traveling at $\vec{\beta}_1 = \vec{\beta}_2$.**

$$\begin{array}{lll} Model1: & & Model2: \\ E_1 = \gamma mc^2 & E_2 = \gamma mc^2 & E_{system} = 2\gamma mc^2 \\ p_1 = \gamma mc\beta & p_2 = p_1 & p_{system} = 2p_1 \\ m_1 = m & m_2 = m & M_{system} = 2m \end{array}$$

**Scenario D: same as scenario B, but measured in a frame where the system travels at $\beta'$(i.e. $\gamma'$).**

$$\begin{array}{ccc} \text{Model 1:} & & \text{Model 2:} \\ E_1 = \gamma_1 mc^2 & E_2 = \gamma_2 mc^2 & E_{system} = (\gamma_1 + \gamma_2)mc^2 \\ p_1 = \gamma' mc(\beta_1 - \beta') & p_2 = \gamma' mc(\beta_2 + \beta') & p_{system} = 2\beta'\gamma mc \\ m_1 = m & m_2 = m & M_{system} = 2m \end{array}$$

$\gamma_1 = \sqrt{\frac{1}{1-(\beta_1-\beta')^2}}$

$\gamma_2 = \sqrt{\frac{1}{1-(\beta_2+\beta')^2}}$