Quantum Mechanics Homework Set 2

Devi Amarsaikhan

May 2023

2.1: Normalizing ket vectors

Normalize the following vectors. Two of the following are orthogonal to each other, indicate which two. ($|x\rangle$, $|y\rangle$, $|z\rangle$ form an orthonormal basis.)

Solution

1. The normalization constant is $\sqrt{1^2+1^2+(-1)^2}=\sqrt{3}$. Therefore, $|\Psi_a\rangle = \frac{1}{\sqrt{3}}\left[|x\rangle + |y\rangle - |z\rangle\right]$

2. A normalized ket vector $|\Psi\rangle$ is one for which $\langle\Psi|\Psi\rangle = 1$.

   Using the fact that $\langle x|y\rangle = 0$, $\langle x|z\rangle = 0$, $\langle y|x\rangle = 0$, $\langle y|z\rangle = 0$, $\langle z|x\rangle = 0$, and $\langle z|y\rangle = 0$, we get $(b*b)\langle\Psi_b|\Psi_b\rangle = (b*b)\left[(3-27i)(3+27i)\langle x|x\rangle + 18^2\langle y|y\rangle + (-4i)(4i)\langle z|z\rangle\right]$.

   $\langle x|x\rangle = 1$, $\langle y|y\rangle = 1$, and $\langle z|z\rangle = 1$, so $(b*b)\left[(3-27i)(3+27i)+18^2+(-4i)(4i)\right] = 1$. We can choose b to be a real number, so we can write it as $b^2(9+729+324+16) = 1$. The normalization factor is $b = \frac{1}{\sqrt{1078}}$ and the normalized vector is $|\Psi_b\rangle = \frac{1}{\sqrt{1078}}\left[(3+27i)|x\rangle + 18|y\rangle + 4i|z\rangle\right]$.

3. $|\Psi_c\rangle = \frac{1}{\sqrt{3}}\left[|x\rangle + |y\rangle + |z\rangle\right]$

4. $|\Psi_d\rangle = \frac{1}{\sqrt{6}}\left[|x\rangle + 2|y\rangle + |z\rangle\right]$

5. $|\Psi_e\rangle = \frac{1}{\sqrt{6}}\left[|x\rangle + |y\rangle + 2|z\rangle\right]$

6. $|\Psi_f\rangle = \frac{1}{\sqrt{2}}\left[|x\rangle - i|y\rangle\right]$

7. $(g*g)\langle\Psi_g|\Psi_g\rangle = (g*g)\left[(e^{-i\beta}\cdot e^{i\beta})[\langle x|x\rangle + (-i\cdot i)\langle y|y\rangle]\right] = 1$. Again chosing the normalization constant to be real, we get $g = \frac{1}{\sqrt{2}}$. Therefore, the normalized vector is $|\Psi_g\rangle = \frac{1}{\sqrt{2}}\left[e^{i\beta}[|x\rangle - i|y\rangle]\right]$.

8. $|\Psi_h\rangle = \frac{1}{\sqrt{3}}\left[|x\rangle + |y\rangle + e^{i\alpha}|z\rangle\right]$

$|\psi_a\rangle$ and $|\psi_e\rangle$ are orthogonal. $\langle\psi_a|\psi_e\rangle = \frac{1}{\sqrt{18}}\left[\langle x|x\rangle + \langle y|y\rangle - 2\langle z|z\rangle\right] = 0$

2.2: Inner products

Given the ket vectors shown below (with the orthonormal basis $|x\rangle$, $|y\rangle$, $|z\rangle$), find the following inner products.

$$
\begin{aligned}
|\Psi_a\rangle &= |x\rangle + |y\rangle - |z\rangle \\
|\Psi_b\rangle &= (3+27i)|x\rangle + |y\rangle - |z\rangle \\
|\Psi_c\rangle &= |x\rangle - i|y\rangle \\
|\Psi_d\rangle &= e^{i\beta}[|x\rangle - i|y\rangle] \\
|\Psi_e\rangle &= |x\rangle + |y\rangle + e^{i\alpha}|z\rangle.
\end{aligned}
$$

Solution

1. $\langle\Psi_a|\Psi_b\rangle = \big[\langle x| + \langle y| - \langle z|\big]\big[(3+27i)|x\rangle + |y\rangle - |z\rangle\big] = (3+27i)\langle x|x\rangle + \langle y|y\rangle + \langle z|z\rangle = 5+27i$

2. $\langle\Psi_b|\Psi_a\rangle = \big[(3-27i)\langle x| + \langle y| - \langle z|\big]\big[|x\rangle + |y\rangle - |z\rangle\big] = (3-27i)\langle x|x\rangle + \langle y|y\rangle + \langle z|z\rangle = 5-27i$

3. $\langle\Psi_d|\Psi_e\rangle = e^{-i\beta}\big[\langle x| + i\langle y| + 0\langle z|\big]\big[|x\rangle + |y\rangle + e^{i\alpha}|z\rangle\big] = e^{-i\beta}\big[\langle x|x\rangle + i\langle y|y\rangle\big] = e^{-i\beta}(1+i)$

4. $\langle\Psi_b|\Psi_e\rangle = \big[(3-27i)\langle x| + \langle y| - \langle z|\big]\big[|x\rangle + |y\rangle + e^{i\alpha}|z\rangle\big] = (3-27i)\langle x|x\rangle + \langle y|y\rangle - e^{i\alpha}\langle z|z\rangle = 4-27i-e^{i\alpha}$

2.3: Outer products

With the column/row vector representation of the HV, PM, and RL bases, find the matrix representation of the following operators (outer products):

1. $|H\rangle\langle V|$
2. $|R\rangle\langle H|$
3. $|M\rangle\langle M|$
4. $|L\rangle\langle R|$

Solution

1. $|H\rangle\langle V| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
2. $|R\rangle\langle H| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 \end{pmatrix}$
3. $|M\rangle\langle M| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$
4. $|L\rangle\langle R| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix}$

2.4: Changing bases

Given the bases for polarization, make the following transformations:

1. $|x_1\rangle = 3|H\rangle - |V\rangle \rightarrow$ PM basis

2. $|x_1\rangle = 3|H\rangle - |V\rangle \rightarrow$ RL basis

3. $|x_2\rangle = 3|R\rangle - |L\rangle \rightarrow$ HV basis

4. $|x_2\rangle = 3|R\rangle - |L\rangle \rightarrow$ PM basis

Solution

1. $|x_1\rangle = 3|H\rangle - |V\rangle = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

   $c_1 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} + c_2 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

   $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

   $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 3 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -1 \end{pmatrix} \sim \begin{pmatrix} \sqrt{2} & 0 & 2 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \sqrt{2} \\ 0 & -\frac{1}{\sqrt{2}} & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \sqrt{2} \\ 0 & 1 & 2\sqrt{2} \end{pmatrix}.$

   Therefore, $c_1 = \sqrt{2}$ and $c_2 = 2\sqrt{2}$ and $|x_1\rangle = \sqrt{2}|P\rangle + 2\sqrt{2}|M\rangle$

2. $|x_1\rangle = 3|H\rangle - |V\rangle \rightarrow$ RL basis

   $d_1 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} + d_2 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

   $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

   $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 3 \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & -1 \end{pmatrix} \sim \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 3 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -i \end{pmatrix} \sim \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 3 \\ 0 & \sqrt{2} & 3-i \end{pmatrix} \sim \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 3 \\ 0 & 1 & \frac{3-i}{\sqrt{2}} \end{pmatrix}$

   $\sim \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 3 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{i-3}{2} \end{pmatrix} \sim \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{i+3}{2} \\ 0 & 1 & \frac{3-i}{\sqrt{2}} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{i+3}{\sqrt{2}} \\ 0 & 1 & \frac{3-i}{\sqrt{2}} \end{pmatrix}.$

   Therefore, $c_1 = \frac{i+3}{\sqrt{2}}$ and $c_2 = \frac{3-i}{\sqrt{2}}$ and $|x_1\rangle = \frac{i+3}{\sqrt{2}}|R\rangle + \frac{3-i}{\sqrt{2}}|L\rangle$

3. $|x_2\rangle = 3|R\rangle - |L\rangle = 3\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} - \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 2\sqrt{2}i \end{pmatrix} = \sqrt{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\sqrt{2}i\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sqrt{2}|H\rangle + 2\sqrt{2}i|V\rangle$

4. $|x_2\rangle = 3|R\rangle - |L\rangle = \begin{pmatrix} \sqrt{2} \\ 2\sqrt{2}i \end{pmatrix}$

$$a\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} + b\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 2\sqrt{2}i \end{pmatrix}$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \sqrt{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 2\sqrt{2}i \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \sqrt{2} \\ \sqrt{2} & 0 & \sqrt{2} + 2\sqrt{2}i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1+2i \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1+2i \\ 0 & \frac{1}{\sqrt{2}} & \frac{1-2i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1+2i \\ 0 & 1 & 1-2i \end{bmatrix}$$

$$|x_2\rangle = 3|R\rangle - |L\rangle = (1+2i)\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} + (1-2i)\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = (1+2i)|P\rangle + (1-2i)|M\rangle$$

2.5: Bell basis

In relation to quantum computation there is a canonical system that is the stepping stone to more sophisticated systems, and this is the Bell basis. In most quantum computation scenarios you deal with two state observable, with eigenvalue (outcomes) labeled 0 and 1. In addition, you often deal with entangled states of these two state observables. In terms of two entangled, two state observables one defines the Bell basis of which any other two state entangled state can be expressed. The state vectors representing the Bell basis are (as is usual in quantum computation, normalization is assumed, i.e. multiply each right hand state by $\frac{1}{\sqrt{2}}$)

$$
\begin{aligned}
|\Psi_+\rangle &= \frac{1}{\sqrt{2}}\left[|1\rangle|0\rangle + |0\rangle|1\rangle\right] = \frac{1}{\sqrt{2}}\left[|10\rangle + |01\rangle\right] \\
|\Psi_-\rangle &= \frac{1}{\sqrt{2}}\left[|1\rangle|0\rangle - |0\rangle|1\rangle\right] = \frac{1}{\sqrt{2}}\left[|10\rangle - |01\rangle\right] \\
|\Phi_+\rangle &= \frac{1}{\sqrt{2}}\left[|0\rangle|0\rangle + |1\rangle|1\rangle\right] = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right] \\
|\Phi_-\rangle &= \frac{1}{\sqrt{2}}\left[|0\rangle|0\rangle - |1\rangle|1\rangle\right] = \frac{1}{\sqrt{2}}\left[|00\rangle - |11\rangle\right]
\end{aligned}
$$

Demonstrate explicitly (carrying out all of the inner products) that this is an orthonormal complete basis for two entangled particles. Show that any general vector in this 4D complex Hilbert space, $|\Psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ can be represented as a combination of these Bell basis states.

Solution

The Hilbert space is 4 dimensional, so we need four orthonormal vectors to form the basis. Then, any vector in the 4D complex vector space can be represented as a combination of these four vectors.

First, let us show these vectors are normalized.

$$\langle\Psi_+|\Psi_+\rangle = \frac{1}{\sqrt{2}}\left[\langle 10| + \langle 01|\right]\frac{1}{\sqrt{2}}\left[|10\rangle + |01\rangle\right] = \frac{1}{2}\left[\langle 10|10\rangle + \langle 10|01\rangle + \langle 01|10\rangle + \langle 01|01\rangle\right] = \frac{1}{2}\left[1+0+0+1\right] = 1$$
$$\langle\Psi_-|\Psi_-\rangle = \frac{1}{\sqrt{2}}\left[\langle 10| - \langle 01|\right]\frac{1}{\sqrt{2}}\left[|10\rangle - |01\rangle\right] = \frac{1}{2}\left[\langle 10|10\rangle - \langle 10|01\rangle - \langle 01|10\rangle + \langle 01|01\rangle\right] = \frac{1}{2}\left[1-0-0+1\right] = 1$$
$$\langle\Phi_+|\Phi_+\rangle = \frac{1}{\sqrt{2}}\left[\langle 00| + \langle 11|\right]\frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right] = \frac{1}{2}\left[\langle 00|00\rangle + \langle 00|11\rangle + \langle 11|00\rangle + \langle 11|11\rangle\right] = \frac{1}{2}\left[1+0+0+1\right] = 1$$
$$\langle\Phi_-|\Phi_-\rangle = \frac{1}{\sqrt{2}}\left[\langle 00| - \langle 11|\right]\frac{1}{\sqrt{2}}\left[|00\rangle - |11\rangle\right] = \frac{1}{2}\left[\langle 00|00\rangle - \langle 00|11\rangle - \langle 11|00\rangle + \langle 11|11\rangle\right] = \frac{1}{2}\left[1-0-0+1\right] = 1$$

Now, let us show all other combinations have probability 0 by showing their inner product is 0.

$$\langle \Psi_+|\Psi_-\rangle = \frac{1}{\sqrt{2}}\left[\langle 10| + \langle 01|\right]\frac{1}{\sqrt{2}}\left[|10\rangle - |01\rangle\right] = \frac{1}{2}\left[\langle 10|10\rangle - \langle 10|01\rangle + \langle 01|10\rangle - \langle 01|01\rangle\right] = \frac{1}{2}\left[1 - 0 + 0 - 1\right] = 0$$
$$\langle \Psi_+|\Phi_+\rangle = \frac{1}{\sqrt{2}}\left[\langle 10| + \langle 01|\right]\frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right] = \frac{1}{2}\left[\langle 10|00\rangle + \langle 10|11\rangle + \langle 01|00\rangle + \langle 01|11\rangle\right] = \frac{1}{2}\left[0 + 0 + 0 + 0\right] = 0$$
$$\langle \Psi_+|\Phi_-\rangle = \frac{1}{\sqrt{2}}\left[\langle 10| + \langle 01|\right]\frac{1}{\sqrt{2}}\left[|00\rangle - |11\rangle\right] = \frac{1}{2}\left[\langle 10|00\rangle - \langle 10|11\rangle + \langle 01|00\rangle - \langle 01|11\rangle\right] = \frac{1}{2}\left[0 - 0 + 0 - 0\right] = 0$$
$$\langle \Psi_-|\Psi_+\rangle = \frac{1}{\sqrt{2}}\left[\langle 10| - \langle 01|\right]\frac{1}{\sqrt{2}}\left[|10\rangle + |01\rangle\right] = \frac{1}{2}\left[\langle 10|10\rangle + \langle 10|01\rangle - \langle 01|10\rangle - \langle 01|01\rangle\right] = \frac{1}{2}\left[1 + 0 - 0 - 1\right] = 0$$
$$\langle \Psi_-|\Phi_+\rangle = \frac{1}{\sqrt{2}}\left[\langle 10| - \langle 01|\right]\frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right] = \frac{1}{2}\left[\langle 10|00\rangle + \langle 10|11\rangle - \langle 01|00\rangle - \langle 01|11\rangle\right] = \frac{1}{2}\left[0 + 0 - 0 - 0\right] = 0$$
$$\langle \Psi_-|\Phi_-\rangle = \frac{1}{\sqrt{2}}\left[\langle 10| - \langle 01|\right]\frac{1}{\sqrt{2}}\left[|00\rangle - |11\rangle\right] = \frac{1}{2}\left[\langle 10|00\rangle - \langle 10|11\rangle - \langle 01|00\rangle + \langle 01|11\rangle\right] = \frac{1}{2}\left[0 - 0 - 0 + 0\right] = 0$$
$$\langle \Phi_+|\Psi_+\rangle = \frac{1}{\sqrt{2}}\left[\langle 00| + \langle 11|\right]\frac{1}{\sqrt{2}}\left[|10\rangle + |01\rangle\right] = \frac{1}{2}\left[\langle 00|10\rangle + \langle 00|01\rangle + \langle 11|10\rangle + \langle 11|01\rangle\right] = \frac{1}{2}\left[0 + 0 + 0 + 0\right] = 0$$
$$\langle \Phi_+|\Psi_-\rangle = \frac{1}{\sqrt{2}}\left[\langle 00| + \langle 11|\right]\frac{1}{\sqrt{2}}\left[|10\rangle - |01\rangle\right] = \frac{1}{2}\left[\langle 00|10\rangle - \langle 00|01\rangle + \langle 11|10\rangle - \langle 11|01\rangle\right] = \frac{1}{2}\left[0 - 0 + 0 - 0\right] = 0$$
$$\langle \Phi_+|\Phi_-\rangle = \frac{1}{\sqrt{2}}\left[\langle 00| + \langle 11|\right]\frac{1}{\sqrt{2}}\left[|00\rangle - |11\rangle\right] = \frac{1}{2}\left[\langle 00|00\rangle - \langle 00|11\rangle + \langle 11|00\rangle - \langle 11|11\rangle\right] = \frac{1}{2}\left[1 - 0 + 0 - 1\right] = 0$$
$$\langle \Phi_-|\Psi_+\rangle = \frac{1}{\sqrt{2}}\left[\langle 00| - \langle 11|\right]\frac{1}{\sqrt{2}}\left[|10\rangle + |01\rangle\right] = \frac{1}{2}\left[\langle 00|10\rangle + \langle 00|01\rangle - \langle 11|10\rangle - \langle 11|01\rangle\right] = \frac{1}{2}\left[0 + 0 - 0 - 0\right] = 0$$
$$\langle \Phi_-|\Psi_-\rangle = \frac{1}{\sqrt{2}}\left[\langle 00| - \langle 11|\right]\frac{1}{\sqrt{2}}\left[|10\rangle - |01\rangle\right] = \frac{1}{2}\left[\langle 00|10\rangle - \langle 00|01\rangle - \langle 11|10\rangle + \langle 11|01\rangle\right] = \frac{1}{2}\left[0 - 0 - 0 + 0\right] = 0$$
$$\langle \Phi_-|\Phi_+\rangle = \frac{1}{\sqrt{2}}\left[\langle 00| - \langle 11|\right]\frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right] = \frac{1}{2}\left[\langle 00|00\rangle + \langle 00|11\rangle - \langle 11|00\rangle - \langle 11|11\rangle\right] = \frac{1}{2}\left[1 + 0 - 0 - 1\right] = 0$$

Thus, these four vectors form an orthonormal basis for the 4D Complex Hilbert space.

2.6: Converting matrix to Dirac form

Consider the following operator expressed as a matrix in canonical form, $\hat{B} = \begin{pmatrix} 1 & -3i \\ 3i & 4 \end{pmatrix}$.
Express this as an outer product form in the HV basis (i.e. in terms of $|H\rangle\langle H|$, $|H\rangle\langle V|$, etc.)

Solution

$$|H\rangle\langle H| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$|H\rangle\langle V| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$|V\rangle\langle H| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$|V\rangle\langle V| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{B} = \begin{pmatrix} 1 & -3i \\ 3i & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -3i \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 3i & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$$

$$\hat{B} = |H\rangle\langle H| - 3i|H\rangle\langle V| + 3i|V\rangle\langle H| + 4|V\rangle\langle V|$$

2.7: GHZ jpd analysis

Consider three observers (Alice, Bob, and Chuck) who receive a series of objects that can be measured in various ways. (Think photons and measuring their polarizations in different bases) Consider a subset of possible polarization measurements:

X Measuring in the PM basis.

Y Measuring in the RL basis.

Z Measuring in the HV basis. (Will not be considered in this problem)

The system can be modeled by random variables (± 1 with 0 mean) for the measurements made by the three observers who choose either X or Y in any one run of the experiment. Consider the outcomes for 4 different cases satisfy the following correlations perfectly (read as $\langle O_A O_B O_C \rangle$):

$$\langle XXX \rangle = +1$$
$$\langle XYY \rangle = +1$$
$$\langle YXY \rangle = +1$$
$$\langle YYX \rangle = -1$$

Show that there does not exist a jpd for the above (three-point) correlations. It is worthwhile to do this in two ways.

2.5a) Fix the outcomes of X_A and Y_A and then use the first three correlations above to determine the possible outcomes of the other random variables. Show that the last correlation can not exist if all outcomes are predetermined.

2.5b) Construct the jpd from the first three correlations. As we are dealing with perfect correlations, determine which elementary probabilities must be 0. Show that no joint can support the last correlation. Use the table below to fill in the elementary probabilities and find what the value of $\langle YYX \rangle$ is from the joint you constructed from the first three values.

Solution

2.5a)

XXX	XYY	YXY	YYX
+++	+++	+++	- - -
- - +	- - +	- - +	++ -
- + -	- + -	- + -	+ - +
+ - -	+ - -	+ - -	- ++

Let us say $X_A = +$ and $Y_A = +$.

According to the table above, the possible values for the remaining when $X_A = +$ are shown below.

$$X_B = +, X_C = +$$

$$Y_B = +, Y_C = +$$

$$X_B = -, X_C = -$$

$$Y_B = -, Y_C = -$$

The possible values for the remaining when $Y_A = +$ are shown below.

$$X_B = +, Y_C = +$$

$$X_B = -, Y_C = -$$

$$Y_B = +, X_C = -$$

$$Y_B = -, X_C = +$$

Possible values for $X_A Y_A X_B Y_B X_C Y_C$ are $++++++$, $++-+-+$, $+++-+-$, $++----$, $++++-+$, $+++-++$, $++-+--$, $++--+-$

According to the table above, the possible values for the remaining when $X_A = +$ are shown below.

$$X_B = +, X_C = +$$

$$Y_B = +, Y_C = +$$

$$X_B = -, X_C = -$$

$$Y_B = -, Y_C = -$$

The possible values for the remaining when $Y_A = -$ are shown below.

$$X_B = -, Y_C = +$$

$$X_B = +, Y_C = -$$

$$Y_B = -, X_C = -$$

$$Y_B = +, X_C = +$$

Possible values for $X_A Y_A X_B Y_B X_C Y_C$ are $+-++++$, $+--+-+$, $+-+-+-$, $+-----$, $+----+$, $+--+++$, $+-+---$, $+-+++-$

According to the table above, the possible values for the remaining when $X_A = -$ are shown below.

$$X_B = -, X_C = +$$

$$X_B = +, X_C = -$$

$$Y_B = -, Y_C = +$$

$$Y_B = +, Y_C = -$$

The possible values for the remaining when $Y_A = +$ are shown below.

$$X_B = +, Y_C = +$$

$$X_B = -, Y_C = -$$

$$Y_B = +, X_C = -$$

$$Y_B = -, X_C = +$$

Possible values for $X_AY_AX_BY_BX_CY_C$ are $-+--++$, $-+-++-$, $-++--+$, $-+++--$, $-+++-+$, $-++-++$, $-+-+--$, $-+--+-$

According to the table above, the possible values for the remaining when $X_A = -$ are shown below.

$$X_B = -, X_C = +$$

$$X_B = +, X_C = -$$

$$Y_B = -, Y_C = +$$

$$Y_B = +, Y_C = -$$

The possible values for the remaining when $Y_A = -$ are shown below.

$$X_B = -, Y_C = +$$

$$X_B = +, Y_C = -$$

$$Y_B = -, X_C = -$$

$$Y_B = +, X_C = +$$

Possible values for $X_AY_AX_BY_BX_CY_C$ are $----++$, $---++-$, $--+--+$, $--++--$, $-----+$, $---+++$, $--+---$, $--+++-$.

Putting them all together, we get $++++++$, $++-+-+$, $+++-+-$, $++----$, $++++-+$, $+++-++$, $++-+--$, $++--+-$, $+-++++$, $+--+-+$, $+-+-+-$, $+-----$, $+----+$, $+--+++$, $+-+---$, $+-+++-$,

− + − − ++, − + − ++ −, − ++ − − +, − +++ − −, − +++ − +, − ++ − ++, − + − + − −, − + − − + −. − − − − ++, − − − ++ −, − − + − − +, − − ++ − −, − − − − − +, − − − +++, − − + − − −, − − +++ −.

Each of them do not work because they do not satisfy one of the correlations. + + + + ++ YYX, ++ − + − + YXY, +++ − + − YXY, ++ − − − − YYX, ++++ − + XXX, +++ − ++ XYY, ++ − + − − XYY, ++ − − + − XXX, + − ++++ YXY, + − − + − + YYX, + − + − + − YYX, + − − − − − YXY, + − − − − + XYY, + − − +++ XXX, + − + − − − XXX, + − +++ − XYY, − + − − ++ YXY, − + − ++ − YYX, − ++ − − + YYX, − +++ − −YXY, − +++ − + XYY, − ++ − ++ XXX, − + − + − − XXX, − + − − + − XYY. − − − − ++ YYX, − − − ++ − YXY, − − + − − + YXY, − − ++ − − YYX, − − − − − + XXX, − − − +++ XYY, − − + − − − XYY, − − +++ − XXX. Therefore, none of these combinations work, showing there is no joint probability distribution.

X_AY_A	$X_BY_BX_CY_C$															
	++++	+++-	++-+	++- -	+-++	+-+-	+- -+	+- - -	-+++	-++-	-+-+	-+- -	- -++	- - +-	- - -+	- - - -
++	YYX	XYY	XXX	XXX	XYY	YXY	XXX	XXX	XXX	XXX	YXY	XYY	XXX	XXX	XYY	YYX
+ -	YXY	XYY	XXX	XXX	XYY	YYX	XXX	XXX	XXX	XXX	YYX	XYY	XXX	XXX	XYY	YXY
- +	XXX	XXX	XYY	YXY	XXX	XXX	YYX	XYY	XYY	YYX	XXX	XXX	YXY	XYY	XXX	XXX
- -	XXX	XXX	XYY	YYX	XXX	XXX	YXY	XYY	XYY	YXY	XXX	XXX	YYX	XYY	XXX	XXX

Figure 1: Table showing which correlation is violated

2.5b)

2.8: GHZ quantum mechanical analysis

Consider the 3-particle entangled state (called a GHZ for Greenberger-Horne-Zeilinger), $|\psi\rangle = \frac{1}{\sqrt{2}}[|+1\rangle|+1\rangle|-1\rangle + |-1\rangle|-1\rangle|+1\rangle]$, where the kets are in the Z basis. Use the form of the operators of X (PM basis), Y (RL basis), and Z (HV basis) to find the three point correlations in the previous problem. You can think of the vectors as $|+1\rangle = |H\rangle$, $|-1\rangle = |V\rangle$. Show that this quantum state can achieve the (three-point) correlations of the GHZ system. You really only need to find the action of the observables on the GHZ ket state. (Review the action of X on the $|\pm 1\rangle$ state and Y on $|\pm 1\rangle$ and then simply find the actions on ABC. Then simply act through the bra vector.)

Note well, $X_A X_B X_C$ etc. is shorthand for $X \oplus X \oplus X$ as these are three different objects.

$$\langle\psi|X_A X_B X_C|\psi\rangle$$

$$\langle\psi|X_A Y_B Y_C|\psi\rangle$$

$$\langle\psi|Y_A X_B Y_C|\psi\rangle$$

$$\langle\psi|Y_A Y_B X_C|\psi\rangle$$

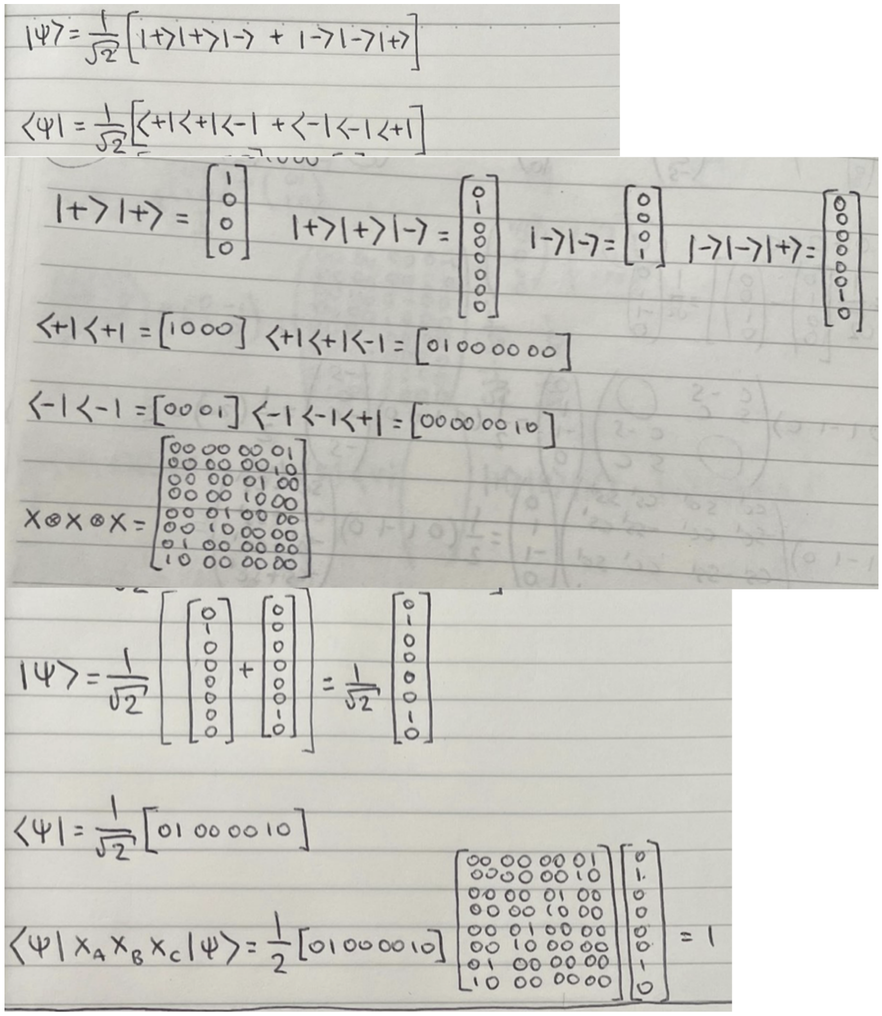


Figure 2: $\langle\psi|X_A X_B X_C|\psi\rangle$

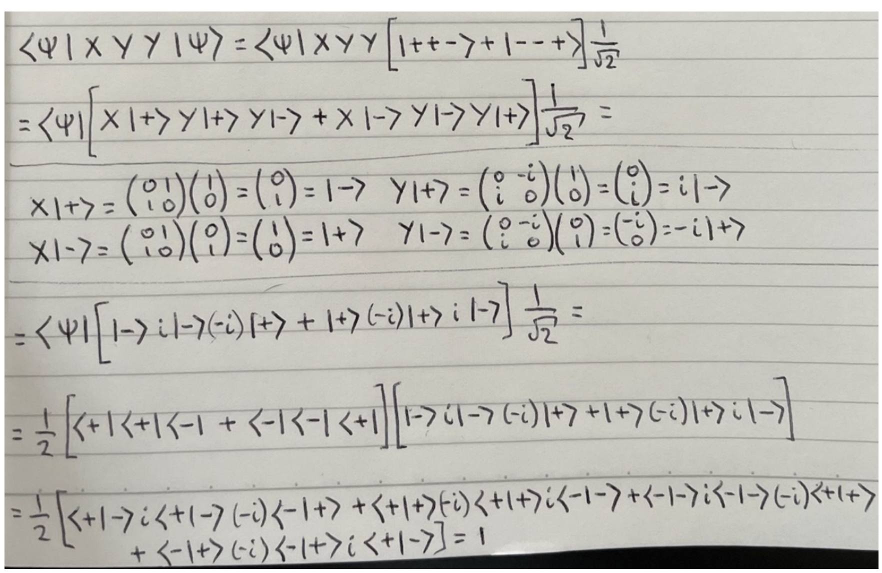
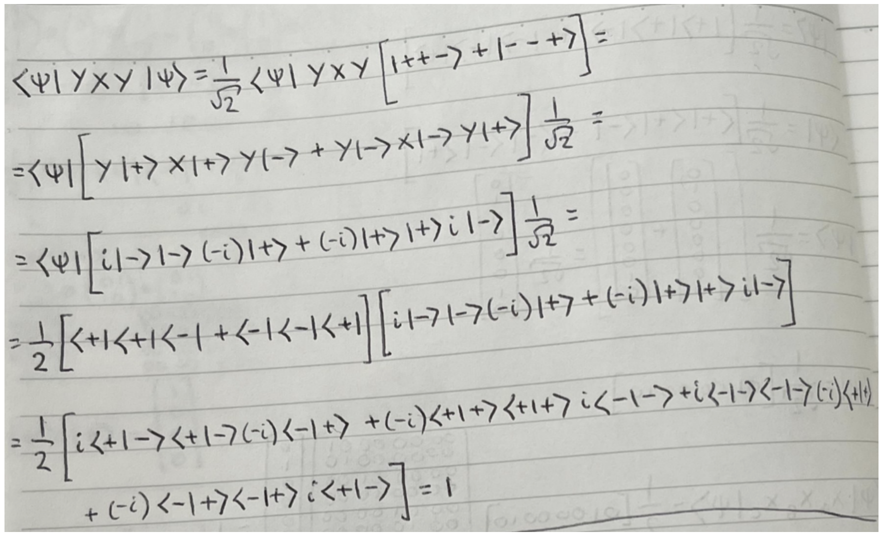


Figure 3: $\langle\psi|X_A Y_B Y_C|\psi\rangle$



$$\langle\psi| YXY |\psi\rangle = \frac{1}{\sqrt{2}}\langle\psi| YXY \left[|{+}{+}{-}\rangle + |{-}{-}{+}\rangle \right] =$$

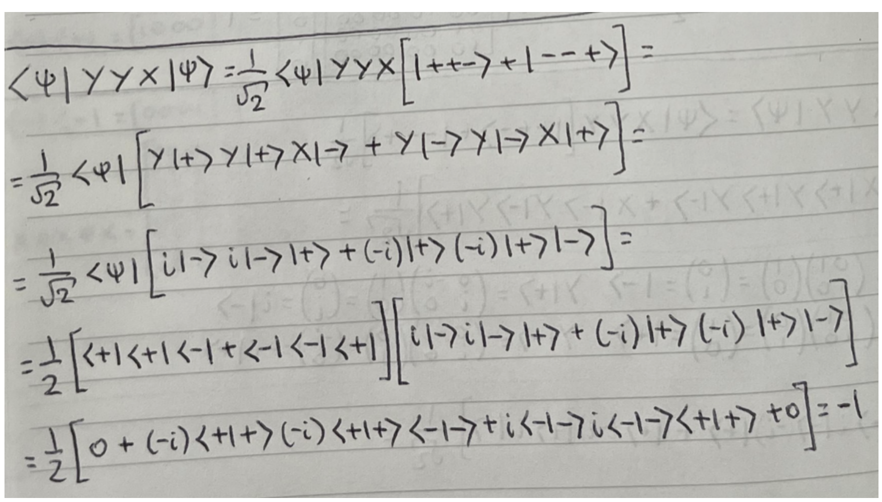
$$= \langle\psi| \left[Y|+\rangle X|+\rangle Y|-\rangle + Y|-\rangle X|-\rangle Y|+\rangle \right] \frac{1}{\sqrt{2}} =$$

$$= \langle\psi| \left[i|-\rangle |-\rangle (-i)|+\rangle + (-i)|+\rangle |+\rangle i|-\rangle \right] \frac{1}{\sqrt{2}} =$$

$$= \frac{1}{2}\left[\langle+|\langle+|\langle-| + \langle-|\langle-|\langle+| \right]\left[i|-\rangle|-\rangle(-i)|+\rangle + (-i)|+\rangle|+\rangle i|-\rangle \right]$$

$$= \frac{1}{2}\Big[i\langle+|-\rangle\langle+|-\rangle(-i)\langle-|+\rangle + (-i)\langle+|+\rangle\langle+|+\rangle i\langle-|-\rangle + i\langle-|-\rangle\langle-|-\rangle(-i)\langle+|+\rangle + (-i)\langle-|+\rangle\langle-|+\rangle i\langle+|-\rangle \Big] = 1$$

Figure 4: $\langle\psi|Y_A X_B Y_C|\psi\rangle$



$$\langle\psi| YYX |\psi\rangle = \frac{1}{\sqrt{2}}\langle\psi| YYX \left[|{+}{+}{-}\rangle + |{-}{-}{+}\rangle \right] =$$

$$= \frac{1}{\sqrt{2}}\langle\psi| \left[Y|+\rangle Y|+\rangle X|-\rangle + Y|-\rangle Y|-\rangle X|+\rangle \right] =$$

$$= \frac{1}{\sqrt{2}}\langle\psi| \left[i|-\rangle i|-\rangle |+\rangle + (-i)|+\rangle (-i)|+\rangle |-\rangle \right] =$$

$$= \frac{1}{2}\left[\langle+|\langle+|\langle-| + \langle-|\langle-|\langle+| \right]\left[i|-\rangle i|-\rangle |+\rangle + (-i)|+\rangle (-i)|+\rangle |-\rangle \right]$$

$$= \frac{1}{2}\left[0 + (-i)\langle+|+\rangle(-i)\langle+|+\rangle\langle-|-\rangle + i\langle-|-\rangle i\langle-|-\rangle\langle+|+\rangle + 0 \right] = -1$$

Figure 5: $\langle\psi|Y_A Y_B X_C|\psi\rangle$