# Lattice Point Geometry Investigations

Stanford University Online High School

UM170 Lattice Point Geometry and the Geometry of Numbers

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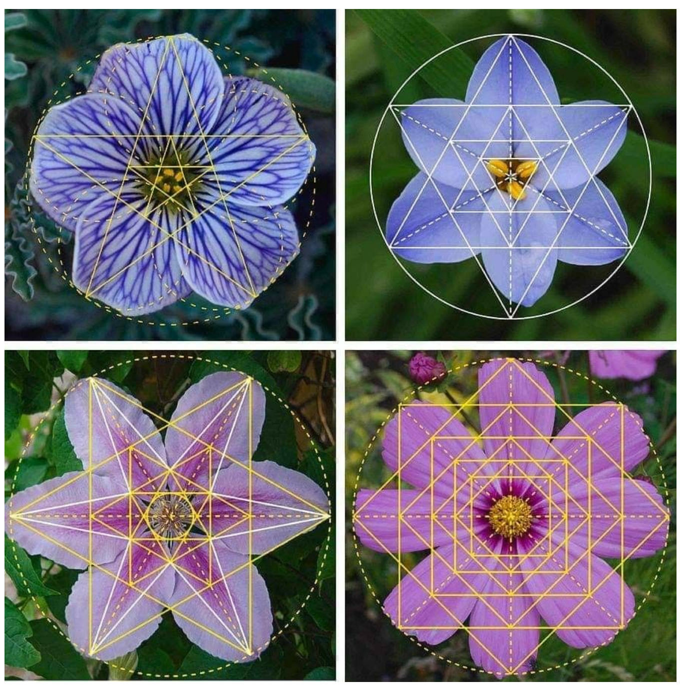

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# 1 Definitions

### Definition 1

A **lattice point** is a point $(x, y) \in \mathbb{R}^2$ such that x and y are both integers.

### Definition 2

The array of points in the plane having both coordinates equal to integers is called the **lattice of integer points in the plane** or **lattice plane**. We will denote the lattice of integer points in the plane $\mathbb{Z}^2$.

### Definition 3

A **lattice line** is a line that contains two distinct lattice points.

### Definition 4

A **lattice line segment** is a line segment that has two distinct lattice points as endpoints.

### Definition 5

A **polygon** is a geometric object consisting of a finite set of ordered points (called **vertices**) and an equal number of line segments (called **sides**). No three successive points are collinear and each side connects two consecutive pairs of points.

### Definition 6

A **simple** polygon (or **Jordan** polygon) is a polygon whose sides only intersect at the vertices. Jordan polygons have a well defined interior and exterior and are topologically equivalent to disks. Throughout this portfolio, unless explicitly stated otherwise, all polygons are assumed to be simple polygons.

### Definition 7

A **lattice polygon** is a polygon whose vertices are all lattice points.

### Definition 8

A polygon is **convex** if the line segment joining any two vertices of the polygon lies inside the polygon. Otherwise, the polygon is **concave**.

### Definition 9

An **n-gon** is a polygon with n sides.

### Definition 10

An **equilateral polygon** is a polygon in which all of the sides have equal length.

### Definition 11

An **equiangular polygon** is a polygon in which all of the angles have equal measure.

### Definition 12

A **regular polygon** is a polygon that is both equilateral and equiangular.

### Definition 13

A **lattice angle** is an angle whose vertices are lattice points and whose sides are lattice lines.

### Definition 14

A **primitive lattice polygon** is a lattice polygon that has no lattice points in its interior, and no lattice points other than vertices on its sides.

### Definition 15

Where P is a lattice polygon,

- **B(P)** is the number of lattice points on the *boundary* (edges) of P.
- **I(P)** is the number of lattice points in the *interior* of P.
- **L(P)** is the total number of lattice points on the *boundary and in the interior* of P. $L(P) = B(P) + I(P)$.

### Definition 16

A **rational number** is a real number that can be expressed in the form $\frac{p}{q}$ where p and q are integers (and $q \neq 0$). The set of all rational numbers is denoted $\mathbb{Q}$.

## 2 Lattice Polygon Investigations

### Exercise 1. Existence of a Regular Lattice 3-gon

Is it possible to construct a regular lattice 3-gon? If so, provide an example. If not, prove it.

### Solution

It is not possible to construct a regular lattice 3-gon. We will prove this statement using proof by contradiction. Let us construct a coordinate system such that one of the vertices of the regular lattice 3-gon, say vertex A, is at the origin. This can be done without loss of generality. Translations preserve the distances between the vertices and the angles between the sides of a polygon, so our polygon remains a regular 3-gon. Moreover, because the location of one of the vertices after translation is a lattice point and the side lengths and angle measurements are preserved, the remaining two vertices will also be lattice points. This means the equilateral triangle will remain a regular lattice 3-gon after the translation.

There are two cases we need to consider:

1. One side of the triangle lies on an axis

2. None of the sides lie on an axis

These two cases are exhaustive because the condition is binary, either one side lies on an axis or none do.

**Case 1:** Let us suppose one of the edges of the 3-gon lies on an axis. Without loss of generality, let us choose the x-axis. If the y-axis was chosen, we can rotate the coordinate plane and triangle by 90° such that the side lies on the x-axis. Let us label the vertices $A = (x_A, y_A)$, $B = (x_B, y_B)$, and $C = (x_C, y_C)$ as shown in the image below.

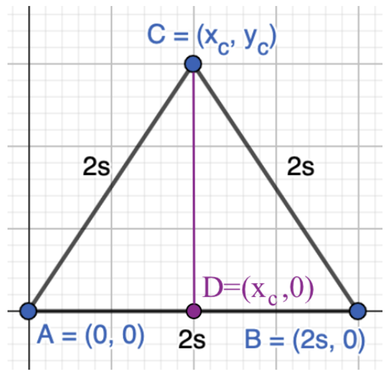


Figure 1: Regular Lattice 3-gon Case 1

For $C$ to be a lattice point, its x and y coordinates need to be integers. $x_C$ is halfway between $x_A$ and $x_B$. To prove this, let us construct a line passing through C perpendicular to the line segment AB. Let us label the point of intersection as D. The length of the line segments AD and DB can be calculated as follows, $||AD|| = ||AC|| \cos(\alpha)$ and $||DB|| = ||BC|| \cos(\beta)$, where $\alpha$ is the angle corresponding to vertex A and $\beta$ is the angle corresponding to vertex B. Because this is an equilateral triangle, $||AC|| = ||BC||$ and $\alpha = \beta$. Therefore, $||AD|| = ||DB||$. Moreover, because $A$ is at the origin, $x_C = ||AD||$ and $||AB|| = x_B = s$, where s is the side length of the triangle. Moreover, $||AB|| = ||AD|| + ||DB||$, so $x_C = \frac{x_B}{2}$. Therefore, for $x_C$ to be an integer, the side length of the triangle needs to be an even integer.

The y coordinate of C, $y_C$, is equal to the length of the line segment $CD$. By the Pythagorean Theorem, $||CD|| = \sqrt{(2s)^2 - (s)^2} = \sqrt{3}s$. Because the side length of the triangle, s, is an even integer, the y coordinate of lattice point C cannot be an integer. Therefore, it is not possible to construct a regular lattice 3-gon with one side on an axis.

**Case 2:** Let us consider the second case where none of the sides lie on an axis. There are two subcases as shown in the two figures below.

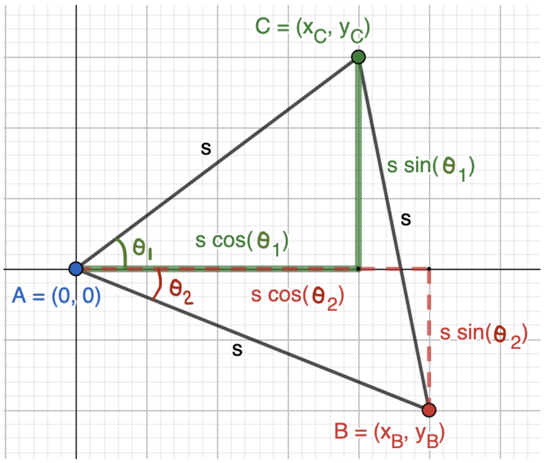


Figure 2: Regular Lattice 3-gon Case 2.1

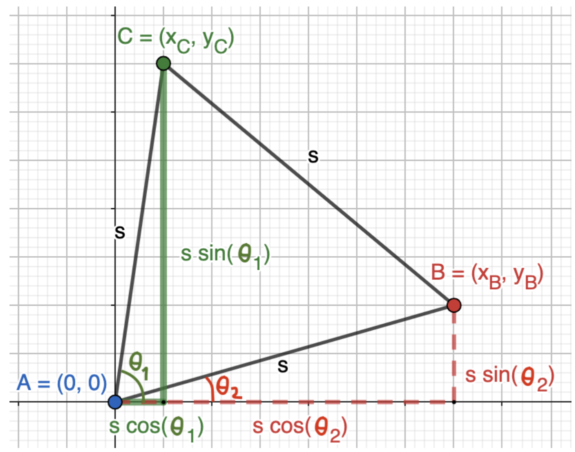


Figure 3: Regular Lattice 3-gon Case 2.2

**Case 2.1:** $\theta_1 + \theta_2 = 60°$. For points B and C to be lattice points, their x and y coordinates need to be integers. By constructing right triangles, we can calculate the coordinates as follows:

$x_C = s\cos(\theta_1)$ and $y_C = s\sin(\theta_1)$

$x_B = s\cos(\theta_2) = s\cos(60° - \theta_1) = s\cos(60°)\cos(\theta_1) + s\sin(60°)\sin(\theta_1) = \frac{s\cos(\theta_1)}{2} + \frac{\sqrt{3}s\sin(\theta_1)}{2}$

$y_B = s\sin(\theta_2) = s\sin(60° - \theta_1) = s\sin(60°)\cos(\theta_1) - s\cos(60°)\sin(\theta_1) = \frac{\sqrt{3}s\cos(\theta_1)}{2} - \frac{s\sin(\theta_1)}{2}$.

If $x_C$ is an integer, $y_B$ is irrational because it has a term that is an integer multiple of $\sqrt{3}$. On the other hand, if $y_B$ is an integer, $s\cos(\theta_1)$ must be an integer multiple of $\sqrt{3}$. Therefore, it is not possible to construct a regular lattice 3-gon of this form.

**Case 2.2**: We arrive at the same contradiction as in Case 2.1 when the whole triangle is contained within one quadrant. In this subcase, $\theta_1 - \theta_2 = 60°$ and the coordinates of the triangles can be calculated as follows.

$x_C = s\cos(60° + \theta_2) = \frac{s\cos(\theta_2)}{2} - \frac{\sqrt{3}s\sin(\theta_2)}{2}$

$y_C = s\sin(60° + \theta_2) = \frac{\sqrt{3}s\cos(\theta_2)}{2} + \frac{s\sin(\theta_2)}{2}$

$x_B = s\cos(\theta_2)$ and $y_B = s\sin(\theta_2)$.

It is not possible for both $x_C$ and $y_B$ to be integers and for both $x_B$ and $y_C$ to be integers.

In each possible case, we have arrived at a contradiction. Therefore, it is not possible to construct a regular lattice 3-gon.

### Exercise 2. Existence of a Regular Lattice 4-gon

Is it possible to construct a regular lattice 4-gon (i.e. a square)? If so, provide an example. If not, prove it.

### Solution

Connect the lattice points (0,0), (1,0), (1,1), and (0,1) with four sides, two of which are parallel to the x axis and two of which are parallel to the y axis. The side lengths are all equal to 1 and the internal angles are all right angles.

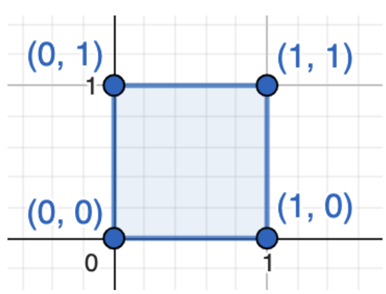


Figure 4: Regular Lattice 4-gon

**Exercise 3. Existence of a Regular Lattice 5-gon**

Is it possible to construct a regular lattice pentagon (5-gon)? If so, provide an example. If not, prove it.

**Solution**

It is not possible to construct a regular lattice pentagon. We will prove this by contradiction by calculating the area of the pentagon using two different methods, one of which results in a rational number and another which results in an irrational number.

First, let us separate the pentagon into three triangles by constructing two diagonals originating from the same vertex as shown below.

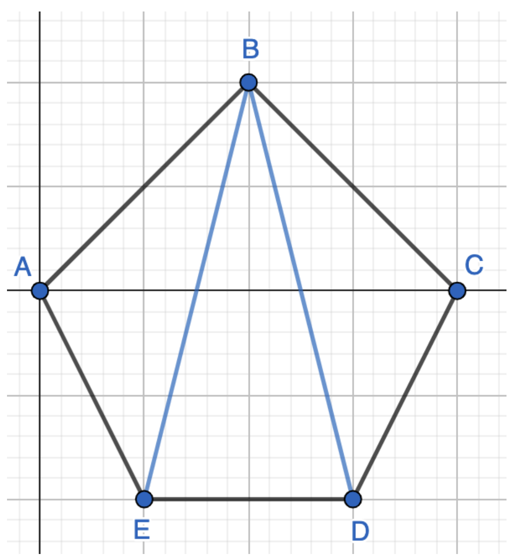


Figure 5: Regular Lattice Pentagon Area Calculation 1

The area of each triangle can be calculated by the formula $\left|\frac{P_x(R_y-Q_y)+Q_x(P_y-R_y)+R_x(Q_y-P_y)}{2}\right|$, where P, Q, and R are vertices of the triangle, see Appendix A. This is a regular lattice pentagon, so the coordinates of each vertex is an integer. The sum, product, division, and subtraction of integers result in a rational number, so the area of each triangle is rational. Because the area of the pentagon is the sum of the area of each triangle, the area of the pentagon must be rational.

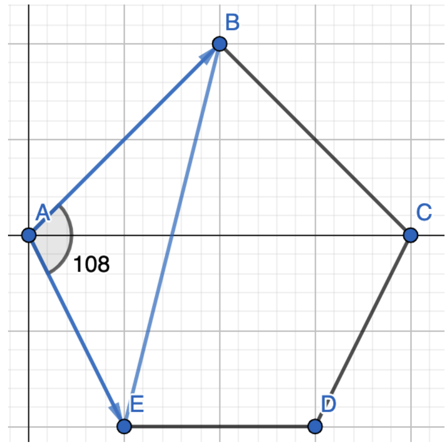


Figure 6: Regular Lattice Pentagon Area Calculation 2

Now, let us calculate the area of triangle with vertices A, B, and E using a different method. Because the pentagon is regular, the internal angle corresponding to vertex A is equal to 108°. Using the cross product, the area can be calculated as $\frac{\vec{AB}\times\vec{AE}}{2} = \frac{s^2\sin(108°)}{2}$, where s is the side length of the pentagon. Without loss of generality, the pentagon can be translated such that the vertex A ends up at the origin. By labeling the coordinates of vertex B as $(x_B, y_B)$, we can calculate the side length by $s = \sqrt{x_B^2 + y_B^2}$. Because B is a lattice point, $x_B$ and $y_B$ are integers. Therefore, $s^2 = x_B^2 + y_B^2$ is rational. sin(108°) is not rational because it contains a term with $\sqrt{5}$ raised to an odd power, see Appendix B. One computation of the area of a regular lattice pentagon results in a rational value while another computation results in an irrational value. Because the area of the regular lattice pentagon cannot be both rational and irrational, it is not possible to construct a regular lattice pentagon.

**Exercise 4**

Show that the cosine of each interior and exterior angle of any regular lattice polygon must be rational.

**Solution**

Construct a lattice plane such that one vertex of the polygon is located at the origin. Let us call that vertex $O$. The two vertices that are adjacent to $O$ have coordinates $A = (x_A, y_A)$ and $B = (x_B, y_B)$. For this n-gon to be a regular lattice polygon, $x_A$, $y_A$, $x_B$, $y_B \in \mathbb{Z}$.

Let us construct two vectors with their tail at O and their tips at A and B respectively. Taking the dot product between them will result in the integer $A \cdot B = x_A x_B + y_A y_B$, which is rational. Let us say the side length of the n-gon is L. Then, we also know the dot product is equal to $|L||L|\cos(\theta)$. Because the dot product is rational, $\cos(\theta)$ needs to be rational. Because $\cos(\pi - \theta) = \cos(\pi)\cos(\theta) + \sin(\pi)\sin(\theta) = -\cos(\theta)$, the cosine of the exterior angles of the polygon also need to be rational.

**Exercise 5**

Use Exercise 4 to show that it is not possible to construct a regular lattice octagon (8-gon).

**Solution**

The interior angle of a regular 8-gon is given by $\frac{(8-2)\pi}{8} = \frac{3\pi}{4}$. Because $\cos(\frac{3\pi}{4}) = -\frac{\sqrt{2}}{2}$ is not rational, by the result of Exercise 4, it is not possible to construct a regular lattice 8-gon.

**Exercise 6**

Show that if $\alpha$ is a lattice angle and if the measure of $\alpha$ is not equal to $\frac{\pi}{2}$ or an odd integer multiple of $\frac{\pi}{2}$, then the tangent of $\alpha$ is a rational number.

**Solution**

Let $\alpha$ be the lattice angle between two vectors with their tail at the origin and their tips at lattice points. Let us label the lattice points $B = (x_1, y_1)$ and $C = (x_2, y_2)$ as shown in the image below.

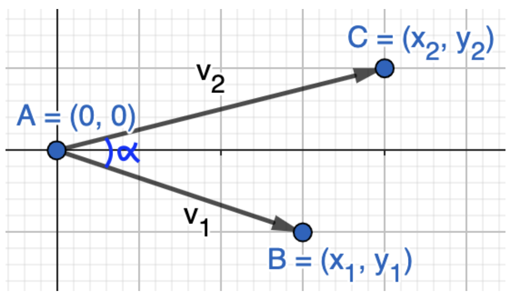


Figure 7: Lattice Angle $\alpha$

From the cross product formula, we get $\sin(\alpha) = \frac{\vec{v_1} \times \vec{v_2}}{||\vec{v_1}||||\vec{v_2}||}$ and from the dot product formula, we get $\cos(\alpha) = \frac{\vec{v_1} \cdot \vec{v_2}}{||\vec{v_1}||||\vec{v_2}||}$. Using these, expressions, $\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)} = \frac{\vec{v_1} \times \vec{v_2}}{\vec{v_1} \cdot \vec{v_2}}$. Because $\alpha$ is not an odd integer multiple of $\frac{\pi}{2}$, $\cos(\alpha) \neq 0$ and the denominator is not equal to 0.

The cross product gives two times the area of the triangle constructed by the two vectors. The area of a triangle is also given by the formula $\left|\frac{P_x(R_y-Q_y)+Q_x(P_y-R_y)+R_x(Q_y-P_y)}{2}\right|$, see Appendix A. Because the coordinates of the three points are integers, the area is rational, which means the cross product must be rational as well. The dot product is also rational because it is the sum of a product of integers. The tangent of the angle $\alpha$ is equal to the division of two rational numbers, so it must itself be rational for lattice angles not equal to an odd integer multiple of $\frac{\pi}{2}$.

**Exercise 7**

Make a conjecture about the positive integers n for which it is possible construct a regular lattice n-gon. Although you do not need to provide a formal proof of your conjecture here, you should provide sufficient justification and reasoning to indicate why you believe your conjecture is valid.

**Solution**

The only integer for which it is possible to construct a regular lattice n-gon is 4. We will prove this by calculating the area of the polygon in two different ways and showing that one method results in a rational value while the other method results in an irrational value for all integers except 4.

Let us pick one vertex and label it R. Its two adjacent vertices are labeled Q and T. By constructing a diagonal from vertex R to every other vertex except Q and T, we can construct $n-2$ triangles. The vertices of each triangle is a lattice point. In Exercise 3, we saw that the area of each triangle is rational because it can be calculated using the sum, product, subtraction, and division of the coordinates of its vertices, which are all integers. The area of the polygon is the sum of the areas of each triangle, so it must also be rational.

We can also calculate the area of one of these triangles using the cross product. Let us choose the triangle whose vertices are Q, R, and T. Let us construct two vectors $\vec{RQ}$ and $\vec{RT}$. Using the cross product formula, $\frac{\vec{RQ}\times\vec{RT}}{2} = \frac{s^2\sin(\theta)}{2}$, where s is the side length and $\theta$ is the internal angle of the polygon. The square of the side length of a regular lattice n-gon is rational because its two endpoints are integers. $s = ||RQ|| = ||RT|| = \sqrt{(T_x - R_x)^2 + (T_y - R_y)^2}$ and $s^2 = (T_x - R_x)^2 + (T_y - R_y)^2$.

Using the first method, the area of the polygon is rational. Because the polygon is regular, all internal angles are equal. The polygon can be divided into $n-2$ triangles, so the internal angle is equal to $\frac{(n-2)\pi}{n}$. Thus, we can rewrite the sin expression as $\sin\left(\frac{(n-2)\pi}{n}\right) = \sin\left(\pi - \frac{2\pi}{n}\right) = \sin(\pi)\cos(\frac{2\pi}{n}) - \cos(\pi)\sin(\frac{2\pi}{n}) = \sin(\frac{2\pi}{n})$.

For the two area calculations to be consistent, $\sin(\frac{2\pi}{n})$ must be rational. Moreover, from Exercise 6, we know that when $\theta \neq \frac{k\pi}{2}$ where k is an odd integer, $\tan(\theta)$ is rational. Because we are considering regular lattice polygons with more than 4 sides, $\frac{2\pi}{n}$ is never an odd integer multiple of $\frac{k\pi}{2}$ and $\tan(\frac{2\pi}{n})$ must be rational. Because $\cos(\frac{2\pi}{n}) = \frac{\sin(\frac{2\pi}{n})}{\tan(\frac{2\pi}{n})}$ and both $\sin(\frac{2\pi}{n})$ and $\tan(\frac{2\pi}{n})$ are rational, $\cos(\frac{2\pi}{n})$ is rational as well.

Using Euler's formula $e^{ix} = \cos(x) + i\sin(x)$, we can write $e^{\frac{2\pi i}{n}} = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$. Because $\cos(\frac{2\pi}{n})$ and $\sin(\frac{2\pi}{n})$ are both rational, $e^{\frac{2\pi i}{n}}$ only has rational coefficients. It is also a root of unity. $e^{-\frac{2\pi i}{n}} = \cos\left(\frac{2\pi}{n}\right) - i\sin\left(\frac{2\pi}{n}\right)$ is also a root of unity that has rational coefficients. Therefore, $(x - e^{\frac{2\pi i}{n}})(x - e^{-\frac{2\pi i}{n}}) = x^2 - 2\cos\left(\frac{2\pi}{n}\right) + 1$ divides the polynomial $x^{n-1} + ... + x + 1$, which is irreducible, resulting in a contradiction. Moreover, if $n = pk$ satisfies these conditions for some prime number p,$n = p$ must also satisfy the conditions, which too is a contradiction. Therefore, it is not possible to contruct a regular lattice n-gon where n is a power of 2 not equal to 4.

In Exercise 2, we showed that we can construct a regular lattice 4-gon and in Exercise 5, we showed that we cannot construct a regular lattice 8-gon. If we are able to construct a regular lattice n-gon where $n = 2^k$, we should be able to construct a regular lattice n-gon where $n = 2^{k-1}$ by numbering the vertices from 1 to n and connecting all the even numbered vertices. Therefore, we can reduce each $n = 2^k$ to $n = 2^3$, which we know is not possible. Therefore, the only value of n for which we can construct a regular lattice n-gon is 4. See Exercise 2 for an example.

**Exercise 8**

For which positive integers n is it possible to construct an equilateral (but not necessarily regular) lattice n-gon?

**Solution**

We will prove that there exists an equilateral lattice n-gon if and only if n is an even integer greater than or equal to 4. For the odd case, we will use proof by contradiction and for the even case, we will provide a method to construct an equilateral lattice n-gon for each even $n = 2k$ where k is an integer.

Let us start by showing that if n is odd, then there does not exist an equilateral lattice n-gon. We will do this using graph theory and proof by contradiction. Let us assume there exists an equilateral lattice polygon $P_n$ with side lengths d. Let $(x_1, y_1)$ and $(x_2, y_2)$ be two adjacent vertices of $P_n$. Then, the square of the side length is equal to $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

There are four possible results of taking the modulus 4 of a number and there are only two possible results of taking the modulus 4 of a square of a number, as shown below.

$$a \equiv 0(\mod 4); a^2 \equiv 0(\mod 4)$$

$$a \equiv 1(\mod 4); a^2 \equiv 1(\mod 4)$$

$$a \equiv 2(\mod 4); a^2 \equiv 2^2 \equiv 0(\mod 4)$$

$$a \equiv 3(\mod 4); a^2 \equiv 3^2 \equiv 1(\mod 4)$$

Coming back to our expression for the distance between the two points, we see that $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ is the sum of two squares. Taking the modulus 4 of both sides, we can see there are three possibilities $d^2 \equiv 0, 1,$ or 2 mod 4. We will consider each case separately.

**Case 1:** $d^2 \equiv 1$ **mod** 4

Let us construct a graph G with all lattice points $(x, y) \in \mathbb{Z}^2$ in the following manner. start by showing that if n is odd, then there does not exist an equilateral lattice n-gon. We will do this using graph theory and proof by contradiction. Let us assume there exists an equilateral lattice polygon $P_n$ with side lengths d. Let $(x_1, y_1)$ and $(x_2, y_2)$ be two adjacent vertices of $P_n$. Then, the square of the side length is equal to $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

Let us construct a graph G with all lattice points $(x, y) \in \mathbb{Z}^2$. Here, the vertices of G are the points $(x, y)$ and we will connect two vertices with an edge if the distance between them is d. This means the edges of our lattice n-gon are edges of this graph. Let us construct two subsets of the vertex set V as follows. If x and y have the same parity, place them in subset X and if x and y have different parity, place them in subset Y. This is a bipartite construction because $X \cap Y = \emptyset$ and $X \cup Y = V$, all vertices in the graph.

It is not possible for an edge in G to join two vertices from the same subset. If an edge connected two vertices $(x_1, y_1) \in X$ and $(x_2, y_2) \in X$, then $x_1$ and $y_1$ would have the same parity and $x_2$ and $y_2$ would have the same parity. Then, $x_2 - x_1$ and $y_2 - y_1$ would have the same parity, which means $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ is even. However, this is a contradiction because $d^2 \equiv 1$ mod 4. The same argument applies to two vertices from the subset Y. Thus, our graph is bipartite. However, each edge of the regular lattice n-gon $P_n$ is an edge in G. Therefore, G has an odd cycle. This is a contradiction because bipartite graphs cannot contain an odd cycle.

**Case 2:** $d^2 \equiv 2$ **mod** 4

Let us construct a graph G with all lattice points $(x, y) \in \mathbb{Z}^2$. Like before, we will connect two vertices with an edge if the distance between them is d. Let us construct two subsets of the vertex set V as follows. If x is even, put it in subset X and if x is odd, put it in subset Y. Once again, this is a bipartite construction because $X \cap Y = \emptyset$ and $X \cup Y = V$.

Let us show that it is bipartite. If there is an edge joining two vertices $(x_1, y_1)$ and $(x_2, y_2)$ from X, $x_1$ and $x_2$ are both even. This means $x_1 - x_2$ is also even, so $(x_1 - x_2)^2 \equiv 0 \bmod 4$. $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ and we have just shown $(x_1 - x_2)^2 \equiv 0 \bmod 4$. Therefore, $d^2 \equiv 0, 1 \bmod 4$. This is a contradiction because we are considering the case where $d^2 \equiv 2 \bmod 4$. Thus, G is bipartite and cannot contain an odd cycle, but $P_n$ was an odd cycle. Thus, no such polygon exists.

**Case 3:** $d^2 \equiv 0$ **mod** 4

$d^2$ is divisible by 4, so d must be even and $\frac{d}{2}$ is an integer. Let us construct a vector for each edge in $P_n$ and translate each vectors to the origin. Let $(p_i, q_i)$ denote the endpoint of the i-th vector upon translation to the origin. $p_i, q_i \in \mathbb{Z}$ and $d^2 = p_i^2 + q_i^2 \equiv 0 \bmod 4$. Both p and q are even, so $\left(\frac{p_i}{2}, \frac{q_i}{2}\right) \in \mathbb{Z}^2$. We can thus construct a new, smaller equilateral lattice n-gon $P_n^{(1)}$ whose side lengths are all $d_1 = \frac{d}{2}$. Now, we begin again with $P_n^{(1)}$. If $d_1^2 \equiv 1$ or $2 \bmod 4$, we obtain the same contradictions as before. If $d_1^2 \equiv 0 \bmod 4$, we can scale the polygon by $\frac{1}{2}$ again. No lattice polygon can have a side length that is smaller than 1, we will eventually reach a polygon that cannot be scaled by a factor of $\frac{1}{2}$ and still be a lattice polygon. At this point, $d_1^2 \equiv 1$ or $2 \bmod 4$ and we arrive at the contradiction. Therefore, it is not possible to construct an equilateral lattice n-gon when n is odd.

Now, let us show that it is possible to construct an equilateral lattice n-gon if n is an even integer greater than or equal to 4. Let $n = 2k$ where $k \in \mathbb{Z}$ and $k \geq 2$. The number of ordered integer pair solutions $(x, y)$ of the equation $x^2 + y^2 = m$, where m is a nonnegative integer, is $4(d_1 - d_3)$, where $d_1$ is the number of divisors of m of the form $4j + 1$ and $d_3$ is the number of divisors of m of the form $4j + 3$.

For example, consider the equation $x^2 + y^2 = 5$. $d_1$ is 1 because there is one divisor of 5 of the form $4j + 1$, namely 1 and $d_3$ is 0 because there are no divisors of 5 of the form $4j + 3$. There are thus $4(d_1 - d_0) = 4$ ordered integer pair solutions, which are $(1, 2)$, $(1, -2)$, $(-1, 2)$, and $(-1, -2)$.

Now, let us consider the equation $x^2+y^2 = 5^{k-1}$. $d_1 = k$ and $d_3 = 0$. Therefore, this equation has $4(d_1 - d_3) = 4(k - 0) = 2n$ integral solutions $(x, y)$. These solutions occur in pairs. When $(x, y)$ is a solution, $(-x, -y)$ is also a solution, so the 4k solutions occur in $2k = n$ pairs. We can construct an equilateral lattice n-gon by choosing any k of these pairs, providing $n = 2k$ lattice points. Each of these vectors will have length $d = \sqrt{x^2 + y^2} = \sqrt{5^{k-1}}$. These vectors form a closed polygon because their sum is equal to 0. Thus, we have shown that it is possible to construct an equilateral polygon with n sides where n is an even integer greater than or equal to 4.

One such example is shown below with $x^2+y^2 = 5^2$. Here, $d_1 = 3$ and we have three divisors $1, 5,$ and $5^2$. $n = 2$ and $k = 6$, which means $k = 3$ pairs. Therefore, we have $4k = 12$ solutions. We can choose three of the pairs to get a 6-gon.

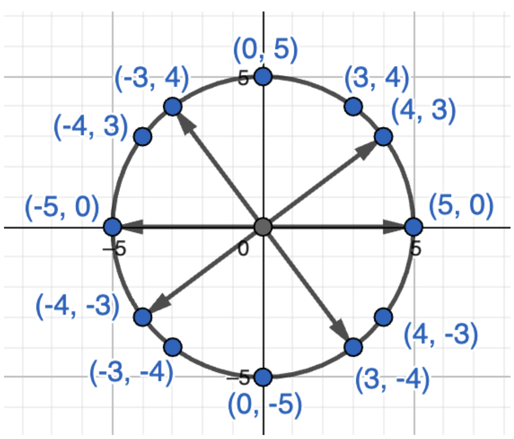


Figure 8: Vectors

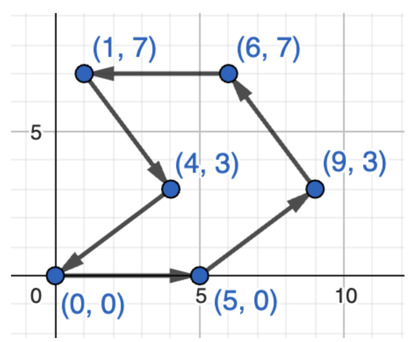


Figure 9: Equilateral 6-gon

**Exercise 9**

Is it possible to construct a lattice square whose area is not a perfect square? If so, provide an example. If not, prove it.

**Solution**

It is possible to construct a lattice square whose area is not a perfect square. An example is a square with coordinates $(0, 0)$, $(1, 2)$, $(3, 1)$, and $(2, -1)$, which has an area of 5.

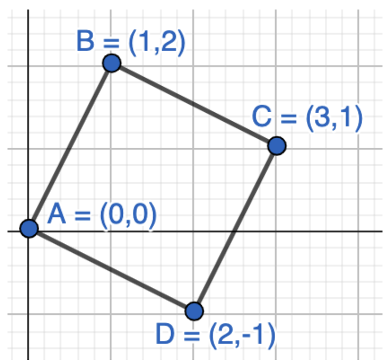


Figure 10: Lattice Square

**Exercise 10**

Is it possible to construct a lattice square whose area is not an integer? If so, provide an example. If not, prove it.

**Solution**

It is not possible to construct a lattice square whose area is not an integer. The vertices of a lattice square are all lattice points. Consider two adjacent vertices with coordinates $(x_1, y_1)$ and $(x_2, y_2)$. The length of the side length is given by $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Because the area is equal to the square of the side length, it is equal to $(x_2 - x_1)^2 + (y_2 - y_1)^2$, which is an integer.

**Exercise 11**

Show that there exists a lattice square with area n, where n is a positive integer, if and only if there exist non-negative integers a and b such that $n = a^2 + b^2$

**Solution**

We need to prove both directions of the bijection. First, let us prove that there exist non-negative integers a and b such that $n = a^2 + b^2$ if there is a lattice square with area n. Next, we will prove there exists a lattice square with area n if there exist non-negative integers a and b such that $n = a^2 + b^2$.

The area of a lattice square is given by the square of its side length. If one side is bounded by two lattice points with coordinates $(x_1, y_1)$ and $(x_2, y_2)$, the side length is given by $s = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ using the Pythagorean Theorem. Therefore, the area is $A = (x_2 - x_1)^2 + (y_2 - y_1)^2$. The two points are lattice points so $x_1$, $y_1$, $x_2$, and $y_2$ are all integers. Therefore, there exist two integers are $a = x_2 - x_1$ and $b = y_2 - y_1$. To ensure these integers are non-negative, we can take the absolute value of the subtractions, $a = |x_2 - x_1|$ and $b = |y_2 - y_1|$.

Given non-negative integers a and b such that the sum of their squares is an integer, we can construct a lattice square. The vertices can have coordinates $(0, 0)$, $(a, b)$, $(-b, a)$, and $(-b + a, a + b)$, as shown in the image below. To prove that this is indeed a lattice square, we need to show all side lengths are equal and all internal angles are equal to $90°$.

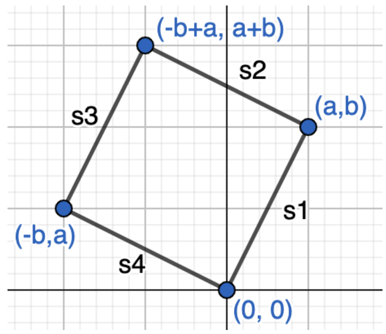


Figure 11: Constructed Lattice Square

Using the pythagorean theorem, $s1 = \sqrt{(a-0)^2 + (b-0)^2}$, $s2 = \sqrt{(-b+a-a)^2 + (a+b-b)^2}$, $s3 = \sqrt{(-b+b-a)^2 + (a-a-b)^2}$, and $s4 = \sqrt{(-b-0)^2 + (a-0)^2}$. All expressions simplify to $\sqrt{a^2+b^2}$. Therefore, all sides have equal length.

To show that all internal angles are equal to 90°, we can show that s1 is parallel to s3, s2 is parallel to s4, and s1 is perpendicular to s2. Then, because corresponding angles are equal and $180° = 2 \times 90°$, all internal angles are right angles. Let us construct vectors parallel to the sides as follows:

$$\vec{s1} = \begin{bmatrix} a-0 \\ b-0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\vec{s2} = \begin{bmatrix} -b+a-a \\ a+b-b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}$$

$$\vec{s3} = \begin{bmatrix} -b+b-a \\ a-a-b \end{bmatrix} = \begin{bmatrix} -a \\ -b \end{bmatrix}$$

$$\vec{s4} = \begin{bmatrix} 0+b \\ 0-a \end{bmatrix} = \begin{bmatrix} b \\ -a \end{bmatrix}$$

$\vec{s1} = -\vec{s3}$ and $\vec{s2} = -\vec{s4}$. Because they are integer multiples of one another, they are parallel. $\vec{s1} \cdot \vec{s2} = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} -b \\ a \end{bmatrix} = -ab + ba = 0$, which means they are perpendicular. Therefore, the constructed polygon has equal sides and all internal angles equal to $90°$, so it is a lattice square. We have proved both directions of the bijection.

**Exercise 12**

Is it possible to construct a lattice triangle whose area is not an integer? If so, provide an example. If not, prove it.

**Solution**

The lattice triangle whose vertices have coordinates $(0, 0)$, $(1, 0)$, and $(1, 1)$ has area equal to $\frac{1}{2}$.

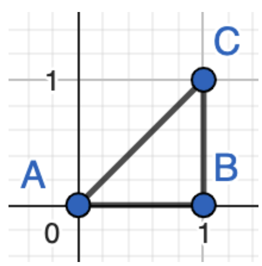


Figure 12: Lattice Triangle with non-integer Area

**Exercise 13**

Construct (at least) 5 lattice polygons with different areas. Find the area of each polygon. Make a conjecture about the possible values of the area of a lattice polygon based on your computations in this problem.

**Solution**

It is possible to construct lattice polygons with all integer areas by constructing rectangles of side lengths 1 and n. The five constructed polygons are shown on the next page. Because each lattice polygon can be divided into triangles whose areas are integer multiples of 0.5, I conjecture the area of lattice polygons are an integer multiple of 0.5.

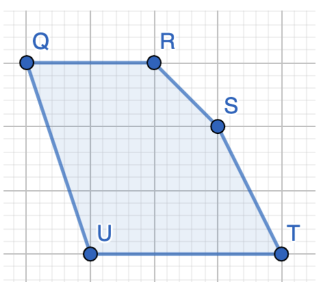


Figure 13: $A(P_1) = 8$

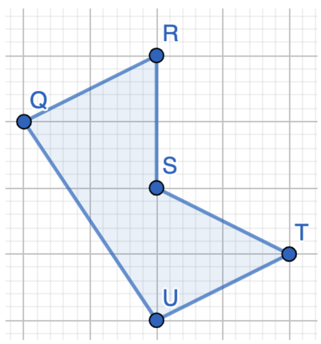


Figure 14: $A(P_2) = 6$

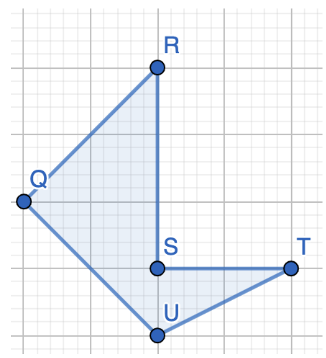


Figure 15: $A(P_3) = 5$

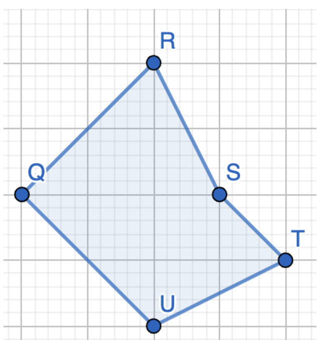


Figure 16: $A(P_4) = 7.5$

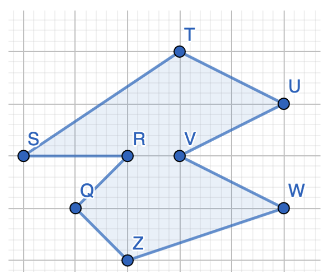


Figure 17: $A(P_4) = 9.5$

**Exercise 14**

Let T be a lattice triangle with $I(T) = 0$. What are the possible values of $B(T)$? Prove your result.

**Solution**

Consider the lattice triangle with one vertex at the origin, one vertex at point $(1, 1)$, and one vertex on the x axis with a positive x coordinate. When $x_B = 1$, there are three lattice points on the boundary, as shown in the first image below. Each time $x_B$ increases by 1, one new boundary point is added. By choosing the third vertex to have coordinates $(n - 2, 0)$, where n is greater than or equal to 3, we can form a lattice triangle with exactly n boundary lattice points

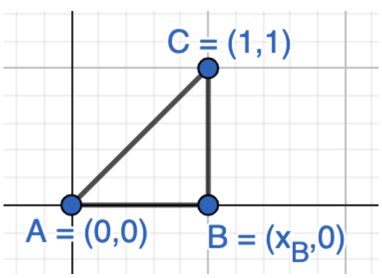


Figure 18: $x_B = 1$ and $B(P) = 3$

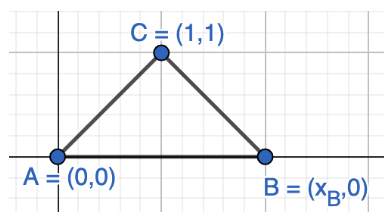


Figure 19: $x_B = 2$ and $B(P) = 4$

**Exercise 15**

Make a conjecture about the possible values of $B(T)$ for a lattice triangle T with $I(T) = 1$.

**Solution**

It is possible for the lattice triangle to have 3, 4, 6, and 8 boundary points and 1 interior lattice point as shown in the four images below. It does not seem to be possible to have more boundary points without having an additional interior lattice point.

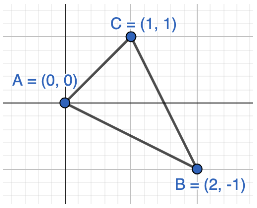


Figure 20: $I(P) = 1$, $B(P) = 3$

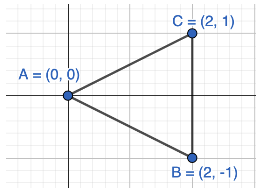


Figure 21: $I(P) = 1$, $B(P) = 4$

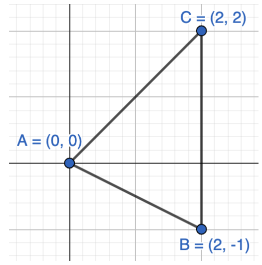


Figure 22: $I(P) = 1$, $B(P) = 6$

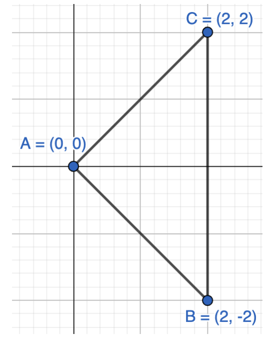


Figure 23: $I(P) = 1$, $B(P) = 8$

### Exercise 16

Construct (at least) 5 different non-congruent primitive lattice triangles, and find their area. Make a conjecture about the value of the area of a primitive lattice triangle.

### Solution

Primitive lattice triangles can be constructed with one vertex on the origin, one vertex with coordinates $(1, 0)$, and a third vertex with coordinates $(x_B, 1)$. We can calculate the area of the triangle using cross products of two vectors $\vec{AB}$ and $\vec{AC}$. $\vec{AB} \times \vec{AC} = |\vec{AB}||\vec{AC}|\sin(\theta)$. $|\vec{AC}| = 1$ and $|\vec{AB}| = \sqrt{x_B^2 + 1}$. $\sin(\theta) = \frac{1}{\sqrt{x_B^2+1}}$. Taking the product of these values and dividing by two, we get that the area of the triangle is equal to 0.5 for all primitive lattice triangles of this form.

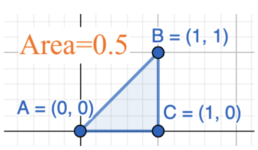


Figure 24: Primitive Lattice Triangle 1

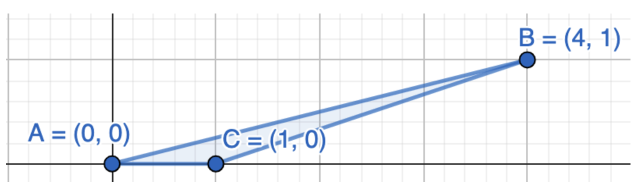


Figure 25: Primitive Lattice Triangle 2

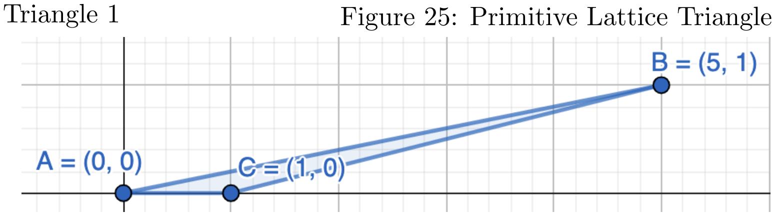


Figure 26: Primitive Lattice Triangle 3

For primitive lattice triangles of a different form, let us calculate their area by constructing an enclosing rectangle and subtracting areas of triangles and rectangles. For the triangle of the left below, its area is $A_1 = 10 - 5 - 1.5 - 2 - 1 = 0.5$. For the triangle of the right below, its area is $A_2 = 6 - 3 - 1 - 1 - 0.5 = 0.5$. It seems as though all primitive lattice triangles has an area equal to 0.5.

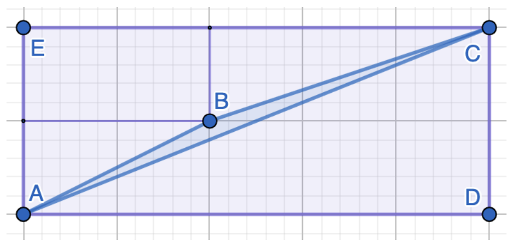

Figure 27: Primitive Lattice Triangle 4

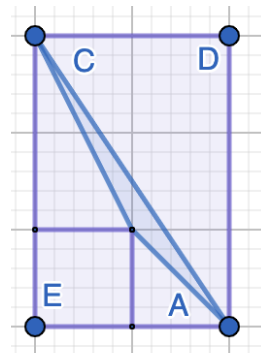

Figure 28: Primitive Lattice Triangle 5

**Exercise 17**

Construct several (at least 5) different polygons that contain 4 boundary lattice points and 6 interior lattice points. Keep in mind that the polygons do not need to be convex! Find the area of each polygon. What do you observe? Make a conjecture based on your observations in this exercise.

**Solution**

Each of the lattice polygons, shown on the next page, has an area of 7 square units. I predict this is true for all lattice polygons with 4 boundary and 6 interior lattice points.

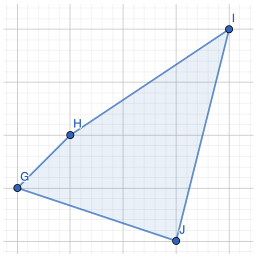


Figure 29: Polygon 1

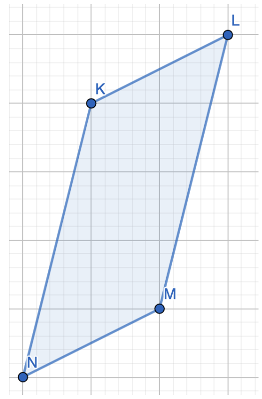


Figure 30: Polygon 2

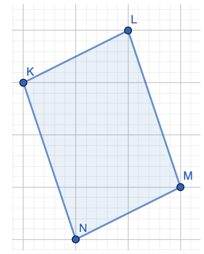


Figure 31: Polygon 3

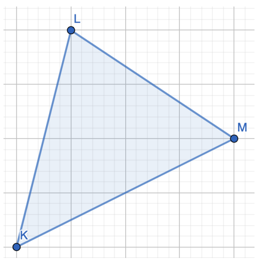


Figure 32: Polygon 4

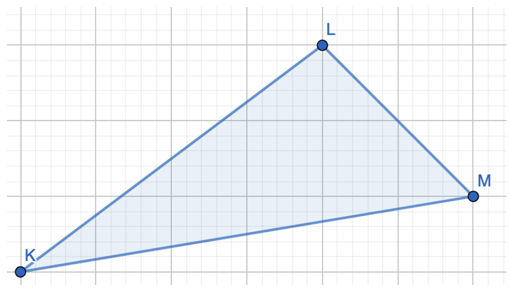


Figure 33: Polygon 5

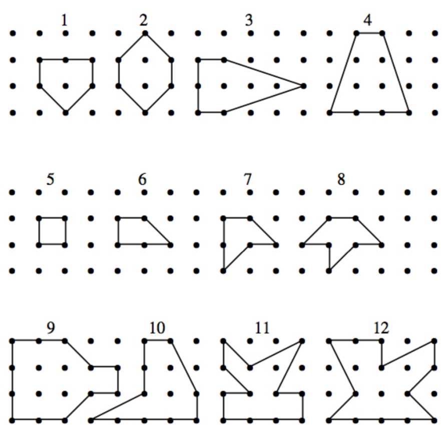


Figure 34: Polygons for Exercise 18

**Exercise 18**

Find the area of each of the lattice polygons in Figure 34. Make a table that contains the following information for each polygon: the area of the polygon, the number of lattice points inside the polygon (I), and the number of lattice points on the boundary of the polygon (B).

**Solution**

| Polygon | Area | I | B |
|---|---|---|---|
| 1 | 3 | 1 | 6 |
| 2 | 4 | 2 | 6 |
| 3 | 5 | 3 | 6 |
| 4 | 6 | 4 | 6 |
| 5 | 1 | 0 | 4 |
| 6 | 1.5 | 0 | 5 |
| 7 | 2 | 0 | 6 |
| 8 | 2.5 | 0 | 7 |
| 9 | 9 | 4 | 12 |
| 10 | 6 | 2 | 10 |
| 11 | 6 | 1 | 12 |
| 12 | 8.5 | 3 | 13 |
| 19.1 | 14 | 9 | 12 |
| 19.2 | 15.5 | 10 | 13 |
| 19.3 | 16.5 | 10 | 15 |
| 19.4 | 16 | 11 | 12 |
| 19.5 | 29 | 18 | 23 |
| 20 | 5.5 | 5 | 3 |

**Exercise 19**

Construct 5 different lattice polygons. To keep this problem interesting, at least 3 of your polygons should be non-convex. All of your polygons should have at least 6 sides, and at least 10 boundary lattice points and at least 8 interior lattice points. For each of these 5 polygons, find the area, the number of lattice points inside the polygon, and the number of lattice points on the boundary of the polygon. Add this information to your table from Exercise 18.

**Solution**

The 5 polygons are inserted in order from left to right below.

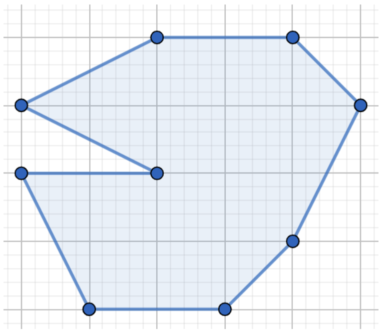

Figure 35: Polygon 19.1

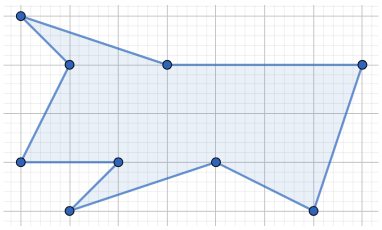

Figure 36: Polygon 19.2

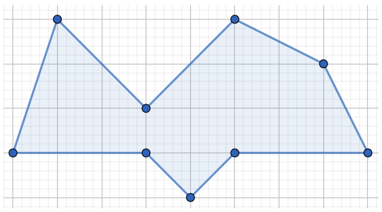

Figure 37: Polygon 19.3

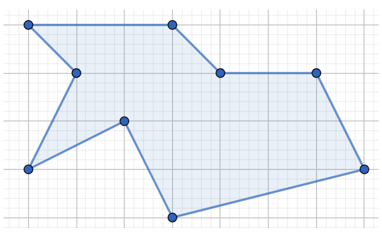

Figure 38: Polygon 19.4

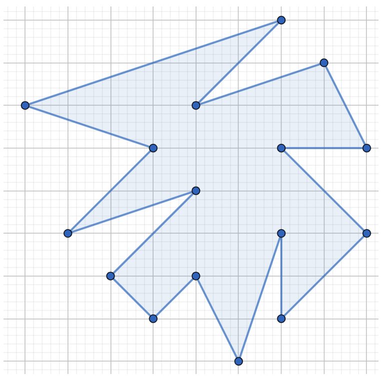

Figure 39: Polygon 19.5

**Exercise 20**

Let P be the triangle with vertices (0, 0), (3, 1), and (1, 4). Find the area of P , the number of lattice points inside the polygon, and the number of lattice points on the boundary of the polygon. Add this information to your table from Exercise 18.

**Solution**

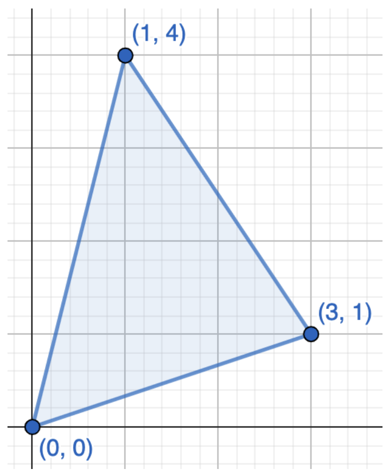


Figure 40: Polygon 20

**Exercise 21**

Based on your work so far, make a conjecture about the area of a lattice pentagon in the case when B (the number of lattice points on the boundary) is even and in the case when B is odd.

**Solution**

When B is even, the area of the lattice polygon is an integer and when B is odd, the area is an integer multiple of 0.5.

**Exercise 22**

Based on your work so far, conjecture a formula that relates the area of a lattice polygon to the number of lattice points inside the polygon and the number of lattice points on the boundary of the polygon. Explain how you obtained your conjecture, and why you think it makes sense (including a proof or partial proof if you have ideas). Although you do not need to provide a formal proof here, you should provide sufficient justification and reasoning to indicate why you believe your conjecture is valid. Please do not try to find the formula online or in another reference–it's so much more fun and interesting if you discover the formula on your own! If you're not sure where to start, start with thinking about a linear relationship. If I increases by 1 and B stays fixed, what happens to the area? Similarly, if B increases by 1 and I stays fixed, what happens to the area? Use these observations to try to find an equation that relates A, B, and I.

**Solution**

The area (A), number of interior lattice points (I), and number of boundary lattice points seem to be related by the formula $A = I + 0.5B - 1$.

Consider the triangle of area 1 shown below. We can increment the number of boundary points by moving vertex B to the right by one unit. This increases the base length (b) by 1 and the area by 0.5 because $A = \frac{b}{2}$. If we move vertex C down by 1 unit, the number of boundary lattice points increase by 2 and the area increases by 1. We can get the -1 value by plugging in values from actual lattice polygons and determining the correct offset.

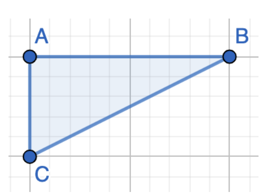


Figure 41: Example

**Exercise 23**

Construct your own lattice polygon, and verify that the formula that you conjectured in Exercise 22 is satisfied. Your polygon should be at least somewhat interesting–for example, make it non-convex, and have at least 10 boundary lattice points and at least 8 interior lattice points.

**Solution**

The polygon below has 35 interior lattice points and 27 boundary lattice points and area 47.5. The equation is satisfied.

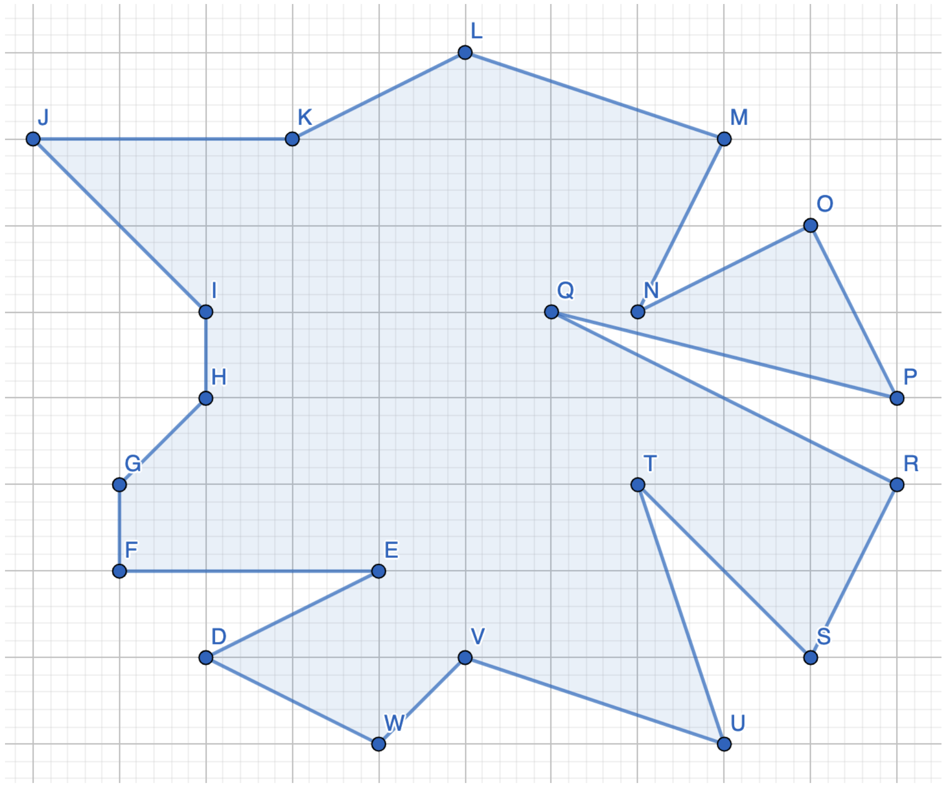


Figure 42: Interesting Polygon

**Exercise 24**

Assuming that your conjecture in Exercise 22 is valid, make a conjecture about the possible values of the area of a lattice polygon. Explain how you obtained your conjecture, and why you think it makes sense (including a proof or partial proof if you have ideas).

**Solution**

If my conjecture in Exercise 22 is valid, the only valid lattice polygon areas are integer multiples of 0.5. This is because the number of internal and boundary lattice points are integers and the only non-integral value is the 0.5 multiplying the number of boundary lattice points.

This makes sense because the area of a polygon can be determined by constructing a circumscribed rectangle and removing smaller rectangles (with sides parallel to the x and y axes) and triangles from the area. All lattice rectangles with sides parallel to the x and y axes will have integer side lengths, so will have integer areas. Right triangles with one side parallel to the x-axis and another side parallel to the y-axis will have an area equal to an integer multiple of 0.5 because the area is equal to the half of base times height and both the base length and height are integers. For regions that cannot be constructed by the removal of right triangles, we can remove a rectangle and add right triangles. For example, consider the circled region in the image below. We can remove a 2 by 2 lattice 4-gon and add two right triangles each with side lengths of 1 and 2.

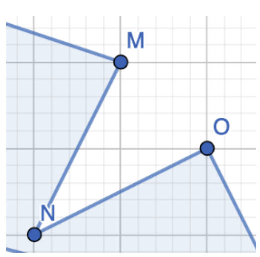


Figure 43: Polygon Section

# 3 Basic Properties of Lattice Points in the Plane

### Exercise 25

Does every lattice line segment have rational length? If so, prove it. If not, provide an example of a lattice line segment with non-rational length.

### Solution

As shown in the image below, the lattice line segment with endpoints at $(0, 0)$ and $(1, 2)$ has non-rational length $\sqrt{1^2 + 2^2} = \sqrt{5}$.

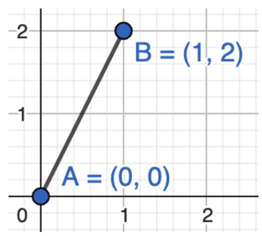


Figure 44: Lattice Line Segment

### Exercise 26

Show that the square of the length of any lattice line segment is an integer.

### Solution

A lattice line segment is bounded by two lattice points with integer coordinates, say $(x_1, y_1)$ and $(x_2, y_2)$. The square of the length of the lattice line segment is given by the following equation $\left(\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\right)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$. The difference of two integers is an integer and the sum of the square of two integers is also an integer, this expression is rational.

**Exercise 27**

Let L be a line with rational slope in the plane. Show that if there is a lattice point on L, then the y-intercept of L is rational.

**Solution**

The line equation can be written as $y = mx + b$ where m is the slope and b is the y-intercept of line L. Because m is rational, we can write it as $\frac{a}{c}$ where $a \in \mathbb{Z}$ and $c \in \mathbb{Z}$. Multiplying both sides of the equation by c, we get $cy = ax + bc$. If there is a lattice point on L, the equation is true for some integers $x_0$ and $y_0$. Because $a, c, x_0, y_0 \in \mathbb{Z}$, $b \in \mathbb{Z}$ because it is not possible for the multiplication, addition, and subtraction of integers to result in a non-integer number.

**Exercise 28**

Let L be a line with rational slope in the plane. Show that if there is one lattice point on L, then there are infinitely many lattice points on L.

**Solution**

Let us say a lattice point on L has coordinates $(x_0, y_0)$. Because the slope of L is rational, we can write it as $\frac{a}{c}$, where $a, c \in \mathbb{Z}$. This means the point $(x_0 + c, y_0 + a)$ is also on the line. Both coordinates are the sum of integers, so this too is a lattice point. The line contains all lattice points $(x_0 + pc, y_0 + pa)$ for all integers p. There are an infinite possible values of p[1], so there are infinitely many lattice points on L.

[1] Suppose there were a finite possible values for p. We can take the maximum possible value and add 1. This results in another integer, which is a contradiction. Thus, there are infinitely many possibilities for p.

**Exercise 29**

Let $p = (m, n)$ be a lattice point in the plane with $\gcd(m, n) = 1$. Show that there are no lattice points strictly between the origin $o = (0, 0)$ and p on the line segment op.

**Solution**

The line passes through the origin and point $p = (m, n)$. Therefore, its slope is $\frac{N}{M}$[2] and its y-intercept is 0. Using this information, the equation of the line can be written as $y = \frac{N}{M}x$. The greatest common divisor of n and m is 1 so we cannot simplify the fraction. For y to be an integer, $M|Nx$. This means $\exists k \in \mathbb{Z}$ such that $Nx = Mk$. This is only possible if $x$ is an integer multiple of M. If we write $x = Mp$, the equation would become $NMp = Mk$. Dividing both sides by M, we get that there is an integer $k = Np$ that makes the equation valid. If x is not a multiple of M, we cannot divide both sides by M which means there is no integer k that can make the equation valid. This means the points on the line that are lattice points occur when x is an integer multiple of M. There are no lattice points between the origin and point p on the line segment op.

**Exercise 30**

Show that if $p = (m, n)$ is a visible point on the lattice line L through the origin (0, 0), then any lattice point on L is of the form (tm, tn) for some integer t.

**Solution**

As before, we can write the line equation for L as $y = \frac{N}{M}x$. $p = (m, n)$ is a visible point, so the greatest common divisor of m and n is 1. Let a point $(a, b)$ be on the line. In Exercise 29, we established that for b to be an integer, a must be an integer multiple of M. If $a = tM$, $b = \frac{N}{M}tM = tN$. Therefore, the points $(tM, tN)$ are on L where $t \in \mathbb{Z}$.

[2] I have written the slope using capital N and M to distinguish from the coordinates of point p.

**Exercise 31**

Let m and n be nonnegative integers. Show that there are exactly $\gcd(m, n) - 1$ lattice points on the line segment between the origin and the point $(m, n)$, not including the endpoints.

**Solution**

The line equation is $y = \frac{N}{M}x$. If the greatest common divisor of n and m is q, we can write $\frac{N}{M} = \frac{a}{b}$, where $N = qa$ and $M = qb$. Therefore, the line equation is $y = \frac{a}{b}x$. The values of x that make y an integer are 0, b, 2b, 3b, ... $qb$. Without considering the boundary points, there are $q - 1$ lattice points between the origin and point $(M, N)$ on the line segment. Because q is the greatest common divisor of m and n, there are $\gcd(m, n) - 1$ lattice points on the line segment between the origin and the point $(m, n)$, not including endpoints.

**Exercise 32**

Let P be a lattice n-gon with vertices

$$p_1 = (a_1, b_1), p_2 = (a_2, b_2), ..., p_n = (a_n, b_n).$$

Let

$$d_i = gcd(a_{i+1} - a_i, b_{i+1} - b_i)$$

for i = 1, 2, . . . , $n - 1$ and let

$$d_n = gcd(a_1 - a_n, b_1 - b_n).$$

Show that the number of lattice points on the boundary of P is given by

$$B(P) = \sum_{i=1}^{n} d_i.$$

**Solution**

Each line segment of the polygon has at least two boundary lattice points, the two vertices it is adjacent to. To determine the total number of boundary lattice points on a polygon with n vertices, we can add the number of boundary lattice points on each line segment excluding the endpoints and add n because each vertex is over-counted. Alternatively, we can include one of the endpoints of each line segment.

To count the number of boundary lattice points on each side of the polygon, we can translate them such that one of the vertices lies on the origin. For example, if the line segment connecting vertices $p_1$ and $p_2$ are translated with $p_1$ at the origin, the coordinates of $p_2$ will become $(a_2 - a_1, b_2 - b_1)$. This can be done without loss of generality because the number of boundary lattice points will remain the same because the endpoints are lattice points and the slope remains the same. To ensure the coordinates of both endpoints are non-negative, we can reflect it across the x or y axis as needed without loss of generality.

From the previous exercise, we know there are $\gcd(m, n) - 1$ lattice points on the line segment between the origin and the point $(m, n)$, excluding endpoints. The sum of $d_i$ where $1 \leq i \leq n-1$ counts the boundary points on the line segments connecting vertices $p_i$ to $p_{i+1}$ and $d_n$ includes the boundary lattice points on the line segment connecting vertices $p_n$ and $p_1$. Because we are not subtracting one, one endpoint of each line segment is included, so the sum is equal to the number of boundary lattice points of the polygon.

# 4 The Algebraic Structure of the Lattice $\mathbb{Z}^2$

**Exercise 33**

Give two examples of vectors in $\mathbb{R}^4$, and find their sum.

**Solution**

The sum of two vectors $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 5 \end{bmatrix}$ is $\vec{u} + \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 3 \\ 9 \end{bmatrix}$

**Exercise 34**

Choose 2 (unequal) vectors in $\mathbb{R}^2$, and illustrate the Parallelogram Rule for your vectors.

**Solution**

$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. As shown in the image below, the top right vertex of the parallelogram formed by these two vectors is $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$, which is equal to the sum of vectors $\vec{u}$ and $\vec{v}$.

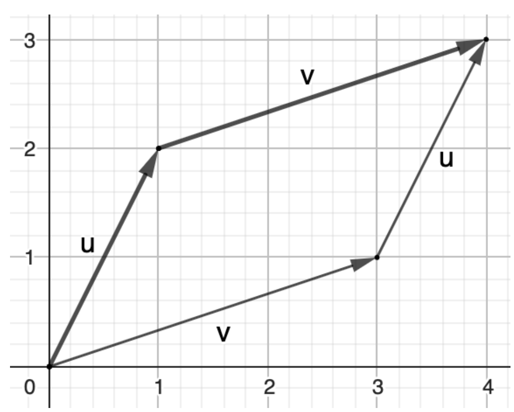


Figure 45: Sum of Vectors Using Parallelogram Rule

### Exercise 35

Give an example of a vector $\vec{u}$ in $\mathbb{R}^2$. Compute $2\vec{u}$,-$2\vec{u}$, and $\frac{1}{2}\vec{u}$. Next, draw each of these vectors. What do you observe? What does the set of all scalar multiples of a fixed nonzero vector form?

### Solution

$\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $2\vec{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $-2\vec{u} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$, $\frac{1}{2}\vec{u} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$. The scalar multiples of a fixed nonzero vector form a line.

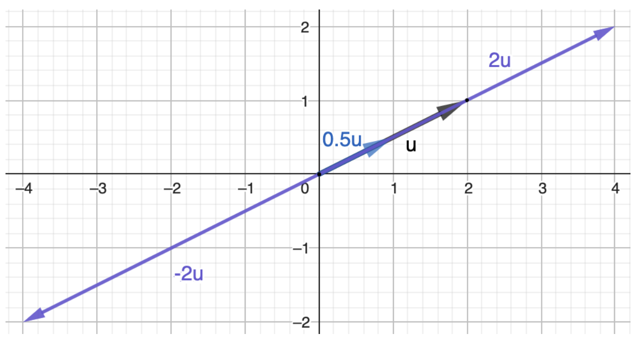


Figure 46: Scalar Multiple of Vectors

**Exercise 36**

Find $\frac{1}{2}\begin{bmatrix}-2\\5\end{bmatrix}+3\begin{bmatrix}4\\1\end{bmatrix}-2\begin{bmatrix}\frac{3}{4}\\8\end{bmatrix}$. This is an example of a linear combination of 3 vectors in $\mathbb{R}^2$.

**Solution**

$$\frac{1}{2}\begin{bmatrix}-2\\5\end{bmatrix}+3\begin{bmatrix}4\\1\end{bmatrix}-2\begin{bmatrix}\frac{3}{4}\\8\end{bmatrix}=\begin{bmatrix}9.5\\1.5\end{bmatrix}$$

**Exercise 37**

Express $\vec{v}=\begin{bmatrix}8\\-12\end{bmatrix}$ as a $\mathbb{Z}$-linear combination of 2 vectors in $\mathbb{Z}^2$.

**Solution**

$$\vec{v}=\begin{bmatrix}8\\-12\end{bmatrix}=2\begin{bmatrix}2\\-2\end{bmatrix}+4\begin{bmatrix}1\\-2\end{bmatrix}$$

**Exercise 38**

Are the vectors $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}$ linearly independent or dependent?

**Solution**

Let us name the vectors $\vec{u}=\begin{bmatrix}1\\2\end{bmatrix}$ and $\vec{v}=\begin{bmatrix}3\\4\end{bmatrix}$. We can write the equation $c_1\begin{bmatrix}1\\2\end{bmatrix}+c_2\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ as a system of two equations as follows,

$$\begin{cases} c_1+3c_2=0\\ 2c_1+4c_2=0\end{cases}$$

We can write this using matrix multiplication $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Row reducing the augmented matrix $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ shows us that the trivial solution is the only solution and the two vectors are linearly independent.

**Exercise 39**

Are the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \end{bmatrix}$ linearly independent or dependent?

**Solution**

Let us name the vectors $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -3 \\ -6 \end{bmatrix}$. Because $3u + v = \vec{0}$, the two vectors are linearly dependent.

**Exercise 40**

(a) Is the set $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^2$? Prove your result.

(b) Is the set $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ a $\mathbb{Z}$-basis for $\mathbb{Z}^2$? Prove your result.

**Solution**

(a) Yes. According to exercise 38, the two vectors are not scalar multiples of on another. Therefore, they are linearly independent. A set of two linearly independent vectors is a basis for $\mathbb{R}^2$.

(b) No. The set is not a $\mathbb{Z}$-basis for $\mathbb{Z}^2$ because there are no integers $c_1$ and $c_2$ such that $c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The row reduced echelon form of the augmented matrix is

calculated as follows. $\begin{bmatrix}1 & 3 & 1\\2 & 4 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 3 & 1\\0 & -2 & -1\end{bmatrix} \sim \begin{bmatrix}1 & 3 & 1\\0 & 1 & 0.5\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -0.5\\0 & 1 & 0.5\end{bmatrix}$.
$c_1 = -0.5$ and $c_2 = 0.5$.

**Exercise 41**

Use Definition 25 to show that the matrix $A = \begin{bmatrix}1 & 3\\4 & 11\end{bmatrix}$ is invertible over $\mathbb{R}$

**Solution**

If matrix A is invertible, there exists a matrix $B = \begin{bmatrix}a & b\\c & d\end{bmatrix}$ such that $AB = I$. This means $AB = \begin{bmatrix}1 & 3\\4 & 11\end{bmatrix}\begin{bmatrix}a & b\\c & d\end{bmatrix} = \begin{bmatrix}a+3c & b+3d\\4a+11c & 4b+11d\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$. We need to solve the following system of equations

$$a + 3c = 1 \tag{1}$$

$$b + 3d = 0 \tag{2}$$

$$4a + 11c = 0 \tag{3}$$

$$4b + 11d = 1 \tag{4}$$

Subtracting 4 times equation (1) from (3), we get $c = 4$ and subtracting 4 times equation (2) from (4), we get $d = -1$. Inserting these values back into equations (1) and (2) respectively, we get $a = -11$ and $b = 3$. Therefore, $B = \begin{bmatrix}-11 & 3\\4 & -1\end{bmatrix}$ and A is invertible.

**Exercise 42**

Use Definition 25 to show that the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is invertible over $\mathbb{R}$

**Solution**

If matrix A is invertible, there exists a matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $AB = I$. This means $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. We need to solve the following system of equations

$$a + 2c = 1 \tag{1}$$

$$b + 2d = 0 \tag{2}$$

$$3a + 4c = 0 \tag{3}$$

$$3b + 4d = 1 \tag{4}$$

Subtracting 3 times equation (1) from (3), we get $c = \frac{3}{2}$ and subtracting 3 times equation (2) from (4), we get $d = -\frac{1}{2}$. Inserting these values back into equations (1) and (2) respectively, we get $a = -2$ and $b = 1$. Therefore, $B = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$ and A is invertible.

**Exercise 43**

Use Definition 25 to show that the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is not invertible over $\mathbb{R}$

**Solution**

If matrix A is invertible, there exists a matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $AB = I$. This means $AB = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 2a+4c & 2b+4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. We need to solve the following system of equations

$$a + 2c = 1 \tag{1}$$

$$b + 2d = 0 \tag{2}$$

$$2a + 4c = 0 \tag{3}$$

$$2b + 4d = 1 \tag{4}$$

Dividing both sides of equation (3) by 2, we can see that $a + 2c = 1$ and $a + 2c = 0$, so there are no solutions to the system of equations and A is not invertible.

## Exercise 44

(a) Construct (at least) 3 different (non-identity) matrices with real entries that are invertible over R. Show that each of your matrices is invertible over R using Definition 25. Then find the determinant of each of your matrices.

(b) Construct (at least) 3 different matrices with real entries that are not invertible over R. Show that each of your matrices is not invertible over R using Definition 25. Then find the determinant of each of your matrices.

(c) Performing additional computations if necessary, make a conjecture about the determinant of a matrix with real entries that is invertible over $\mathbb{R}$.

## Solution

(a) Matrix $\begin{bmatrix} 3 & 5 \\ 2 & 2 \end{bmatrix}$ is invertible. As before, we can get a system of four equations by multiplying by matrix with entries a, b, c, and d.

$$3a + 5c = 1 \tag{1}$$

$$3b + 5d = 0 \tag{2}$$

$$2a + 2c = 0 \tag{3}$$

$$2b + 2d = 1 \tag{4}$$

Solving the system of equations, we get that $B = \begin{bmatrix} -\frac{1}{2} & \frac{5}{4} \\ \frac{1}{2} & -\frac{3}{4} \end{bmatrix}$. Therefore, A is invertible.

The determinant is $6 - 10 = -4$.

Matrix $\begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}$ is invertible. The system of four equations is as follows.

$$a + 3c = 1 \tag{1}$$

$$b + 3d = 0 \tag{2}$$

$$5a + 2c = 0 \tag{3}$$

$$5b + 2d = 1 \tag{4}$$

Solving the system of equations, we get that $B = \begin{bmatrix} -\frac{2}{13} & \frac{3}{13} \\ \frac{5}{13} & -\frac{1}{13} \end{bmatrix}$. Therefore, A is invertible. The determinant is $2 - 15 = -13$.

Matrix $\begin{bmatrix} -1 & 0 \\ 4 & -2 \end{bmatrix}$ is invertible. The system of four equations is as follows.

$$-a = 1 \tag{1}$$

$$-b = 0 \tag{2}$$

$$4a - 2c = 0 \tag{3}$$

$$4b - 2d = 1 \tag{4}$$

Solving the system of equations, we get that $B = \begin{bmatrix} -1 & 0 \\ -2 & -0.5 \end{bmatrix}$. Therefore, A is invertible. The determinant is $2 - 0 = 2$.

(b) Matrix $\begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$ is not invertible because the system of equations below has no solution.

$$3a + 6c = 1 \tag{1}$$
$$3b + 6d = 0 \tag{2}$$
$$a + 2c = 0 \tag{3}$$
$$b + 2d = 1 \tag{4}$$

Subtracting 3 times equation (3) from (1) results in $0 = 1$, which is not possible. The determinant is $4 - 4 = 0$.

Matrix $\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}$ is not invertible because the system of equations below has no solution.

$$-a + 3c = 1 \tag{1}$$
$$-b + 3d = 0 \tag{2}$$
$$a - 3c = 0 \tag{3}$$
$$b - 3d = 1 \tag{4}$$

Adding equation (3) to (1) results in $0 = 1$, which is not possible. The determinant is $3 - 3 = 0$.

Matrix $\begin{bmatrix} 3 & 0 \\ 10 & 0 \end{bmatrix}$ is not invertible because the system of equations below has no solution.

$$3a = 1 \tag{1}$$
$$3b = 0 \tag{2}$$
$$10a = 0 \tag{3}$$
$$10b = 1 \tag{4}$$

$a$ cannot be both $\frac{1}{3}$ and 0 at the same time so the matrix is not invertible. The determinant is 0.

(c) A matrix is invertible over $\mathbb{R}$ if its determinant is nonzero.

**Exercise 45**

Consider the $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that if $ad - bc \neq 0$, then A is invertible and $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Show that if $ad - bc = 0$, then A is not invertible.

**Solution**

Let us say there exists a matrix $A^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ such that $AA^{-1} = I$ and $A^{-1}A = I$. Carrying out the matrix multiplication, we can construct a set of four linear equations as follows. $AA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ec + dg & cf + dh \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$ae + bg = 1 \tag{1}$$

$$af + bh = 0 \tag{2}$$

$$ec + dg = 0 \tag{3}$$

$$cf + dh = 1 \tag{4}$$

We can write this system of linear equations in augmented matrix form and row reduce.

$$\begin{bmatrix} a & 0 & b & 0 & 1 \\ 0 & a & 0 & b & 0 \\ c & 0 & d & 0 & 0 \\ 0 & c & 0 & d & 1 \end{bmatrix}$$

Let us label the rows of the matrix as $R_1$ through $R_4$. Let us multiply $R_1$ and $R_2$ by d and $R_3$ and $R_4$ by b and perform the following row operations: $R_1 - R_3 \to R_1$ and $R_2 - R_4 \to R_2$.

$$\begin{bmatrix} ad & 0 & bd & 0 & d \\ 0 & ad & 0 & bd & 0 \\ bc & 0 & bd & 0 & 0 \\ 0 & bc & 0 & bd & b \end{bmatrix} \sim \begin{bmatrix} ad-bc & 0 & 0 & 0 & d \\ 0 & ad-bc & 0 & 0 & -b \\ bc & 0 & bd & 0 & 0 \\ 0 & bc & 0 & bd & b \end{bmatrix}$$

Let us multiply $R_4$ by $\frac{ad-bc}{bd}$ and do the following row operation $R_4 - \frac{c}{d}R_2 \to R_4$.

$$\begin{bmatrix} ad-bc & 0 & 0 & 0 & d \\ 0 & ad-bc & 0 & 0 & -b \\ bc & 0 & bd & 0 & 0 \\ 0 & \frac{(ad-bc)bc}{bd} & 0 & ad-bc & \frac{(ad-bc)b}{bd} \end{bmatrix} \sim \begin{bmatrix} ad-bc & 0 & 0 & 0 & d \\ 0 & ad-bc & 0 & 0 & -b \\ bc & 0 & bd & 0 & 0 \\ 0 & \frac{(ad-bc)c}{d} & 0 & ad-bc & a-\frac{bc}{d} \end{bmatrix}$$

$$\sim \begin{bmatrix} ad-bc & 0 & 0 & 0 & d \\ 0 & ad-bc & 0 & 0 & -b \\ bc & 0 & bd & 0 & 0 \\ 0 & 0 & 0 & ad-bc & a \end{bmatrix}$$

Let us multiply $R_3$ by $\frac{ad-bc}{bd}$ and do the following row operation $R_3 - \frac{c}{d}R_1 \to R_3$.

$$\sim \begin{bmatrix} ad-bc & 0 & 0 & 0 & d \\ 0 & ad-bc & 0 & 0 & -b \\ \frac{(ad-bc)bc}{bd} & 0 & ad-bc & 0 & 0 \\ 0 & 0 & 0 & ad-bc & a \end{bmatrix} \sim \begin{bmatrix} ad-bc & 0 & 0 & 0 & d \\ 0 & ad-bc & 0 & 0 & -b \\ 0 & 0 & ad-bc & 0 & -c \\ 0 & 0 & 0 & ad-bc & a \end{bmatrix}$$

Finally, dividing each row by $ad-bc$, we get $\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{d}{ad-bc} \\ 0 & 1 & 0 & 0 & \frac{-b}{ad-bc} \\ 0 & 0 & 1 & 0 & \frac{-c}{ad-bc} \\ 0 & 0 & 0 & 1 & \frac{a}{ad-bc} \end{bmatrix}$

Because each matrix has a unique row reduced form, this means the unique solution is

$e = \frac{d}{ad-bc}$, $f = \frac{-b}{ad-bc}$, $g = \frac{-c}{ad-bc}$, and $h = \frac{a}{ad-bc}$ and the inverse matrix is $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Because the term $ad - bc$ is in the denominator, $ad - bc \neq 0$ for the inverse matrix to be defined and if $ad - bc = 0$, the matrix does not have an inverse.

**Exercise 46**

Use Definition 27 to show that the matrix $A = \begin{bmatrix} 1 & 3 \\ 4 & 11 \end{bmatrix}$ is invertible over $\mathbb{Z}$

**Solution**

Using the form of the inverse matrix for 2 by 2 matrices from Exercise 45, we get that $A^{-1} = \frac{1}{11-12}\begin{bmatrix} 11 & -3 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -11 & 3 \\ 4 & -1 \end{bmatrix}$. All entries in this matrix is an integer and the following product is equal to the identity matrix. $AA^{-1} = \begin{bmatrix} 1 & 3 \\ 4 & 11 \end{bmatrix}\begin{bmatrix} -11 & 3 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Therefore, matrix A is invertible over $\mathbb{Z}$.

**Exercise 47**

Use Definition 27 to show that the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is not invertible over $\mathbb{Z}$

**Solution**

Using the form of the inverse matrix for 2 by 2 matrices from Exercise 45, we get that $A^{-1} = \frac{1}{4-6}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$. The inverse matrix form is unique and two entries of the inverse are not integers. Therefore, A is not invertible over $\mathbb{Z}$.

**Exercise 48**

Construct (at least) 3 different (non-identity) matrices with integer entries that are invertible over $\mathbb{Z}$.

(a) Show that each of your matrices is invertible over $\mathbb{Z}$.

(b) Find the determinant of each of your matrices. What do you observe?

(c) Performing additional computations if necessary, make a conjecture about the determinant of a matrix with integer entries that is invertible over $\mathbb{Z}$.

**Solution**

(a) $A = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$. $AA^{-1} = I$ and all entries of matrix $A^{-1}$ are integers, so matrix A is invertible over $\mathbb{Z}$.

$B = \begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 4 & 3 \\ -1 & -1 \end{bmatrix}$. $BB^{-1} = I$ and all entries of matrix $B^{-1}$ are integers, so matrix B is invertible over $\mathbb{Z}$.

$C = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ and $C^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$. $CC^{-1} = I$ and all entries of matrix $C^{-1}$ are integers, so matrix C is invertible over $\mathbb{Z}$.

(b) $det(A) = 1$, $det(B) = -1$, and $det(C) = 1$. The determinant of each matrix is equal to positive or negative 1.

(c) For a matrix to be invertible over $\mathbb{Z}$, all of its entries should be integers. For $2 \times 2$ matrices, the inverse has a specific form. Namely, each integer entry is divided by the determinant. One way for all its entries to remain integers is to have the determinant equal to plus or negative 1. We will prove that these are the only possible values for a 2 by 2 matrix invertible over $\mathbb{Z}$ in Exercise 49.

**Exercise 49**

Let A be a $2 \times 2$ matrix with entries in $\mathbb{Z}$. Show that A is invertible over $\mathbb{Z}$ if and only if $\det(A) = \pm 1$. Note that you only need to prove this result here for $2 \times 2$ matrices, but the same result holds for more general $n \times n$ matrices with entries in $\mathbb{Z}$. (You are of course welcome to prove the more general result here if you wish.)

**Solution**

For a matrix to be invertible over $\mathbb{Z}$, it must have integer entries and there must exist a matrix with integer entries whose product equals the identity matrix. Let our matrix be $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. In Exercise 45, we derived the form of the inverse matrix to be $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. For all entries to be integers, $ad - bc | d, a, -b, -c$. This means there exist integers $e$, $f$, $g$, and $h$ such that $(ad - bc)e = d$, $(ad - bc)f = a$, $(ad - bc)g = -b$, and $(ad-bc)h = -c$. Using these expressions, we can write $ad-bc = (ad-bc)^2ef-(ad-bc)^2gh$. In Exercise 45, we showed that for a matrix to be invertible, its determinant is not equal to 0. Therefore, we can divide both sides of the equation by $ad-bc$ and get $1 = (ad-bc)(ef-gh)$. $ad - bc \in \mathbb{Z}$ and $ef - gh \in \mathbb{Z}$. Therefore, for the equation $1 = (ad - bc)(ef - gh)$ to be true, $ad - bc = \pm 1$ and $ef - gh = \pm 1$.

**Exercise 50**

Which real numbers have a multiplicative inverse in R? Prove your statement.

**Solution**

All real numbers except 0 have a multiplicative inverse in $\mathbb{R}$. If $a = 0$, there does not exist a number $b$ such that $ab = 1$. However, for any other real number $c$, $d = \frac{1}{c} \in \mathbb{R}$ is its multiplicative inverse.

**Exercise 51**

Which integers have a multiplicative inverse that is also an integer? Prove your statement.

**Solution**

1 and $-1$ are the only integers that have a multiplicative inverse that is also an integer. Namely, $1 \times 1 = 1$ and $-1 \times -1 = 1$. There is no multiplicative inverse for 0 and for all other integers, k, the multiplicative inverse is a fraction, $\frac{1}{k}$.

**Exercise 52**

Construct 3 different examples of bases $\{v = \langle v1, v2\rangle, w = \langle w1, w2\rangle\}$ of $\mathbb{R}^2$. Note that your bases do not need to be $\mathbb{Z}$-bases!

(a) Show that each of your examples is actually a basis of $\mathbb{R}^2$.

(b) For each basis, compute the determinant of the matrix $\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$ What do you observe?

(c) Doing more computations if necessary, state and prove a conjecture about the determinant of a matrix whose columns form a basis for $\mathbb{R}^2$.

**Solution**

(a) A basis for $\mathbb{R}^2$ will be composed of two linearly independent vectors. Two vectors are linearly independent if they are not scalar multiples of one another. Therefore, the following three vector pairs are bases for $\mathbb{R}^2$.

$\{\vec{v_1} = \langle 3, 5\rangle, \vec{v_2} = \langle 1, 2\rangle\}$

$\{\vec{u_1} = \langle 2, 3\rangle, \vec{u_2} = \langle 3, 2\rangle\}$

$\{\vec{w_1} = \langle 4, 7\rangle, \vec{w_2} = \langle 0, -1\rangle\}$

(b) $\begin{vmatrix} \vec{v_1} & \vec{v_2} \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = 1$

$\begin{vmatrix} \vec{u_1} & \vec{u_2} \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -5$

$\begin{vmatrix} \vec{w_1} & \vec{w_2} \end{vmatrix} = \begin{vmatrix} 4 & 0 \\ 7 & -1 \end{vmatrix} = -4.$

(c) The determinant of the matrices whose columns form the basis for $\mathbb{R}^2$ are all nonzero in the examples given above. If we view the matrix as a transformation matrix, we should be able to take the inverse of the matrix to get the inverse transformation matrix. Thus, the area of the parallelogram must be nonzero which means the determinant must be nonzero.

**Exercise 53**

Construct 3 different examples of $\mathbb{Z}$-bases $\{v = \langle v_1, v_2\rangle, w = \langle w_1, w_2\rangle\}$ of $\mathbb{Z}^2$.

(a) Show that each of your examples is actually a $\mathbb{Z}$-basis of $\mathbb{Z}^2$.

(b) For each basis, compute the determinant of the matrix $\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$ What do you observe?

(c) Doing more computations if necessary, state and prove a conjecture about the determinant of a matrix whose columns form a basis for $\mathbb{Z}^2$.

**Solution**

(a) $\{\vec{v_1} = \langle 5, 2\rangle, \vec{v_2} = \langle 7, 3\rangle\}$. For these to be a $\mathbb{Z}$-basis of $\mathbb{Z}^2$, they need to be linearly independent and span $\mathbb{Z}^2$. They are linearly independent if the only constants $c_1$ and $c_2$ for which $c_1\vec{v_1} + c_2\vec{v_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is true is $c_1 = 0$ and $c_2 = 0$.

Using Cramer's Rule, $c_1 = \dfrac{\begin{vmatrix} 0 & 7 \\ 0 & 3 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix}} = 0$ and $c_2 = \dfrac{\begin{vmatrix} 5 & 0 \\ 2 & 0 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix}} = 0$. Therefore, the two vectors are linearly independent. They span $\mathbb{Z}^2$ if there are integers $c_3$ and $c_4$ such that $c_3\vec{v_1} + c_4\vec{v_2} = \begin{bmatrix} a \\ b \end{bmatrix}$ for all integers a and b.

Again using Cramer's rule, $c_3 = \dfrac{\begin{vmatrix} a & 7 \\ b & 3 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix}} = 3a - 7b$ and $c_4 = \dfrac{\begin{vmatrix} 5 & a \\ 2 & b \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix}} = 5b - 2a$. Because a and b are integers, $c_3$ and $c_4$ are also integers.

$\{\vec{v_3} = \langle -1, -5\rangle, \vec{v_4} = \langle 1, 4\rangle\}$. For these to be a $\mathbb{Z}$-basis of $\mathbb{Z}^2$, they need to be linearly independent and span $\mathbb{Z}^2$. They are linearly independent if the only constants $k_1$ and $k_2$ for which the following equation is true is $k_1 = 0$ and $k_2 = 0$. $k_1\vec{v_3} + k_2\vec{v_4} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Using Cramer's Rule, $k_1 = \dfrac{\begin{vmatrix} 0 & 1 \\ 0 & 4 \end{vmatrix}}{\begin{vmatrix} -1 & 1 \\ -5 & 4 \end{vmatrix}} = 0$ and $c_2 = \dfrac{\begin{vmatrix} -1 & 0 \\ -5 & 0 \end{vmatrix}}{\begin{vmatrix} -1 & 1 \\ -5 & 4 \end{vmatrix}} = 0$. Therefore, the two vectors are linearly independent. They span $\mathbb{Z}^2$ if there are integers $k_3$ and $k_4$ such that $k_3\vec{v_3} + k_4\vec{v_4} = \begin{bmatrix} a \\ b \end{bmatrix}$ for all integers a and b.

Again using Cramer's rule, $k_3 = \dfrac{\begin{vmatrix} a & 1 \\ b & 4 \end{vmatrix}}{\begin{vmatrix} -1 & 1 \\ -5 & 4 \end{vmatrix}} = 4a - b$ and $k_4 = \dfrac{\begin{vmatrix} -1 & a \\ -5 & b \end{vmatrix}}{\begin{vmatrix} -1 & 1 \\ -5 & 4 \end{vmatrix}} = -b + 5a$. Because a and b are integers, $k_3$ and $k_4$ are integers.

$\{\vec{v_5} = \langle 3, 8\rangle, \vec{v_6} = \langle 1, 3\rangle\}$. For these to be a $\mathbb{Z}$-basis of $\mathbb{Z}^2$, they need to be linearly independent and span $\mathbb{Z}^2$. They are linearly independent if the only constants $p_1$ and $p_2$ for which the following equation is true is $p_1 = 0$ and $p_2 = 0$. $p_1\vec{v_5} + p_2\vec{v_6} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Using Cramer's Rule, $p_1 = \dfrac{\begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 8 & 3 \end{vmatrix}} = 0$ and $p_2 = \dfrac{\begin{vmatrix} 3 & 0 \\ 8 & 0 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 8 & 3 \end{vmatrix}} = 0$. Therefore, the two vectors are linearly independent. They span $\mathbb{Z}^2$ if there are integers $p_3$ and $p_4$ such that $p_3\vec{v_5} + p_4\vec{v_6} = \begin{bmatrix} a \\ b \end{bmatrix}$ for all integers a and b.

Again using Cramer's rule, $p_3 = \dfrac{\begin{vmatrix} a & 1 \\ b & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 8 & 3 \end{vmatrix}} = 3a - b$ and $p_4 = \dfrac{\begin{vmatrix} 3 & a \\ 8 & b \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 8 & 3 \end{vmatrix}} = 3b - 8a$. Because a and b are integers, $p_3$ and $p_4$ are integers.

(b) $\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix} = 1$, $\begin{vmatrix} -1 & 1 \\ -5 & 4 \end{vmatrix} = 1$, and $\begin{vmatrix} 3 & 1 \\ 8 & 3 \end{vmatrix} = 1$. The determinants are all equal to 1.

(c) The determinant of a matrix whose columns form a $\mathbb{Z}$-basis for $\mathbb{Z}^2$ is equal to plus or minus 1. To prove this, let us show that the matrix is invertible over $\mathbb{Z}$. Because the two vectors form a basis for $\mathbb{Z}^2$, some $\mathbb{Z}$-linear combination of them will equal the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and another $\mathbb{Z}$-linear combination of them will equal the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We can express this using the following equations

$$c_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + c_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$c_3 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + c_4 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We can write these two equations as a product of matrices.

$\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Because $v_1$, $v_2$, $w_1$, $w_2$, $c_1$, $c_2$, $d_1$, $d_2 \in \mathbb{Z}$, the matrix is $\mathbb{Z}$-invertible. In Exercise 49, we showed that the determinants of a $\mathbb{Z}$-invertible matrix can be equal to 1 or -1. Therefore, for two vectors to form a $\mathbb{Z}$-basis for $\mathbb{Z}^2$, the determinant of the matrix whose columns are these basis vectors is 1 or -1.

**Exercise 54**

Sketch the parallelogram spanned by each of the $\mathbb{Z}$-bases for $\mathbb{Z}^2$ that you constructed in Exercise 53.

(a) Find the area of the parallelogram spanned by each of the $\mathbb{Z}$-bases for $\mathbb{Z}^2$ that you constructed in Exercise 53.

(b) State and prove a conjecture about the area of a lattice parallelogram $P(\vec{v}, \vec{w})$, where $\vec{v}$ and $\vec{w}$ form a basis of $\mathbb{Z}^2$.

**Solution**

(a) The area of the parallelogram is given by the determinant of the matrix formed by the two basis vectors, see Appendix C.

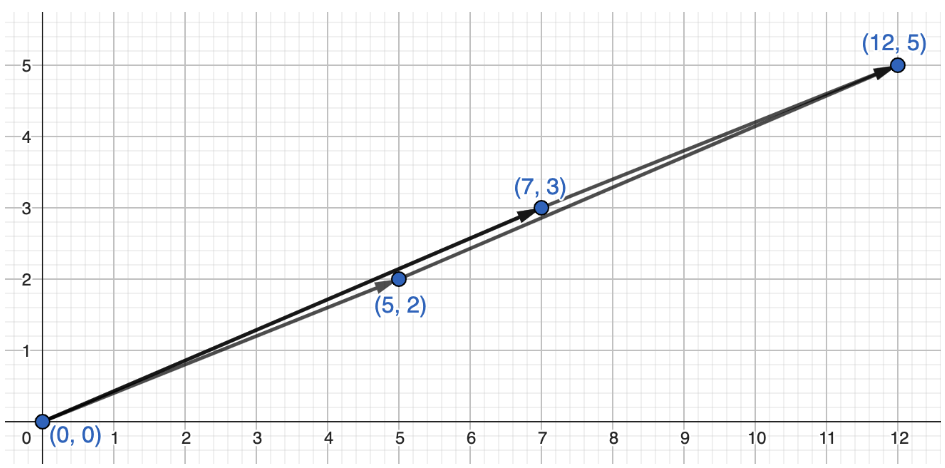


Figure 47: $\{\vec{v_1} = \langle 5, 2\rangle, \vec{v_2} = \langle 7, 3\rangle\}$, $A_1 = \begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix} = 1$

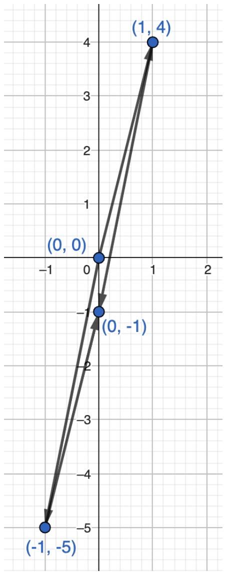


Figure 48: $\{\vec{v_1} = \langle -1, -5\rangle, \vec{v_2} = \langle 1, 4\rangle\}$, $A_2 = \begin{vmatrix} -1 & 1 \\ -5 & 4 \end{vmatrix} = 1$

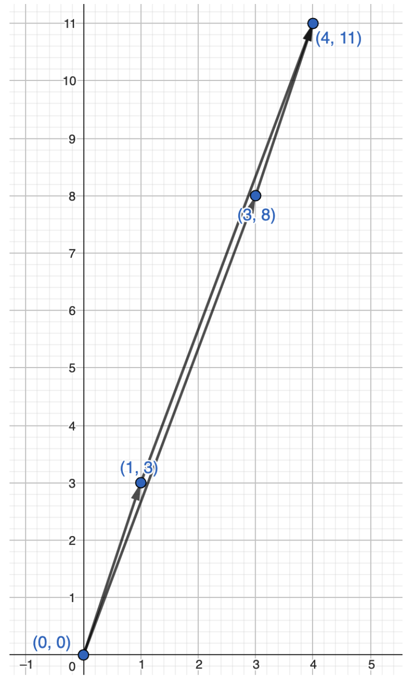


Figure 49: $\{\vec{v_1} = \langle -1, -5\rangle, \vec{v_2} = \langle 1, 4\rangle\}$, $A_3 = \begin{vmatrix} 3 & 1 \\ 8 & 3 \end{vmatrix} = 1$

(b) In all three examples above, the area of a lattice parallelogram is equal to 1. Let us prove this is true for all bases for $\mathbb{Z}^2$. Let us construct a parallelogram using two vectors that form a basis of $\mathbb{Z}^2$. They are necessarily linearly independent and the area of the resulting parallelogram is not 0. Moreover, from the result of Exercise 53, we know their determinant must be equal to 1 or -1. Because the area is always nonnegative, we can take the magnitude, so the area must be equal to 1.

**Exercise 55**

Sketch the parallelogram spanned by each of the $\mathbb{Z}$-bases for $\mathbb{Z}^2$ that you constructed in Exercise 53.

(a) How many lattice points are on the sides of the parallelogram? How many lattice points are in the interior?

(b) Doing more computations if necessary, make and prove a conjecture about the number of lattice points on the sides and in the interior of a lattice parallelogram $P(\vec{v}, \vec{w})$, where $\vec{v}$ and $\vec{w}$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^2$.

**Solution**

(a) In each of the three parallelograms constructed above, there are no boundary lattice points other than the four vertices and no interior lattice points.

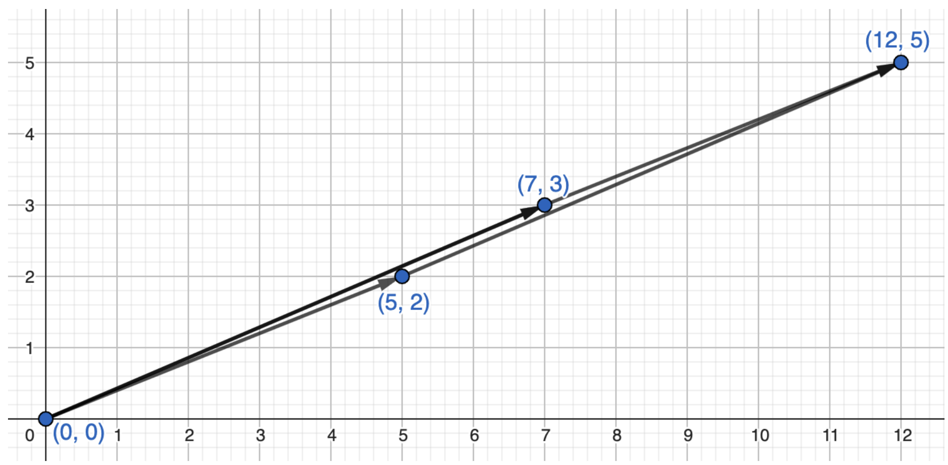


Figure 50: $\{\vec{v_1} = \langle 5, 2\rangle, \vec{v_2} = \langle 7, 3\rangle\}$

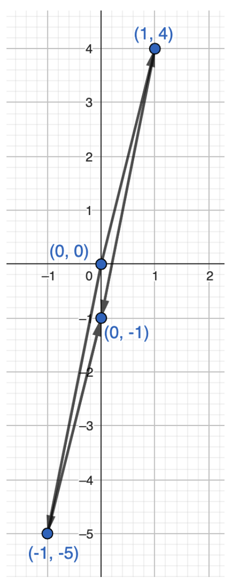


Figure 51: $\{\vec{v_1} = \langle -1, -5\rangle, \vec{v_2} = \langle 1, 4\rangle\}$

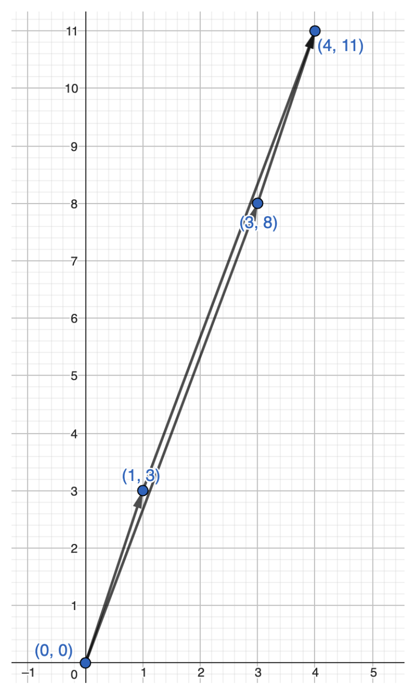


Figure 52: $\{\vec{v_1} = \langle -1, -5\rangle, \vec{v_2} = \langle 1, 4\rangle\}$

(b) Based on the results of the three parallelograms constructed above, I conjecture that there are four lattice points on the sides and no lattice points on the interior of a lattice parallelogram $P(\vec{v}, \vec{w})$, where $\vec{v}$ and $\vec{w}$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^2$.

Let us now prove this statement for all vector pairs that form a $\mathbb{Z}$-basis for $\mathbb{Z}^2$. The two vectors $\vec{v}$ and $\vec{w}$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^2$. Therefore, if there are any lattice points inside or on a side of the parallelogram, we can write it as a linear combination of the two basis vectors. To be bounded by the parallelogram, each basis vector can be multiplied by at most 1 and at least 0. $P(\vec{v}, \vec{w}) = \{a\vec{v} + b\vec{w} : 0 \leq a \leq 1, 0 \leq b \leq 1\}$. There are two possibilities for each a and b. By the multiplicative principle, there are four possible pairs for a and b. When $a = 0$ and $b = 0$, it is the lattice point at the origin. When $a = 1$ and $b = 0$, it is the lattice point at the tail of vector $\vec{v}$. When $a = 0$ and $b = 1$, it is the lattice point at the tail of vector $\vec{w}$. Lastly, when both $a = 1$ and $b = 1$, it is the lattice point at the tail of vector $\vec{v} + \vec{w}$. Therefore, there are four boundary lattice points and no interior lattice points for a lattice parallelogram $P(\vec{v}, \vec{w})$, where $\vec{v}$ and $\vec{w}$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^2$.

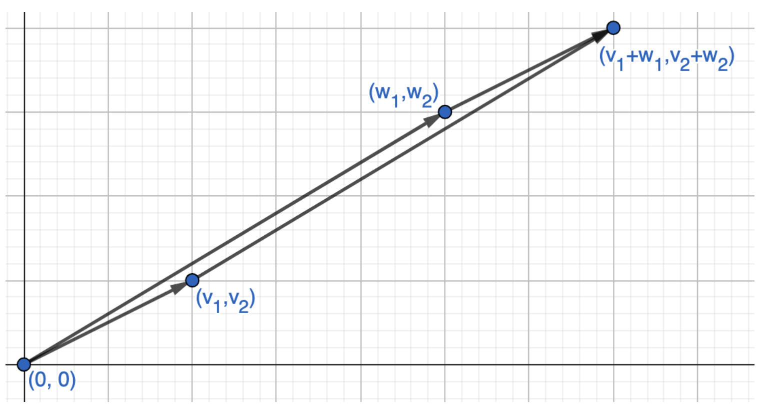


Figure 53: $P(\vec{v}, \vec{w})$

**Exercise 56**

Suppose that the parallelogram $P(\vec{v}, \vec{w})$, where $\vec{v}, \vec{w} \in \mathbb{Z}^2$, is a primitive lattice parallelogram. Show that $\vec{v}$ and $\vec{w}$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^2$.

**Solution**

Let us construct a lattice square $S(\vec{p}, \vec{q})$ where $\vec{p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{q} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We can perform shear transformations in the horizontal and vertical directions. Because all points in the lattice plane are transformed, there are no new boundary or interior lattice point on the parallelogram. Therefore, the result of these transformations is always a primitive lattice parallelogram.

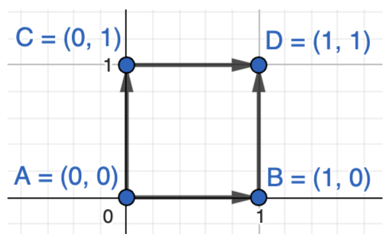


Figure 54: Primitive Lattice Square

The $\mathbb{Z}^2$ vector space is composed of integers and the vertices of lattice parallelograms are lattice points. We can shear transform the lattice square horizontally by 1 unit and vertically by 1 unit. Using a series of horizontal and vertical transformation of 1 unit, it is possible to construct each possible lattice parallelogram in the $\mathbb{Z}^2$ vector space because the final parallelogram must be a lattice parallelogram.

The area of the lattice square above is 1. The area of the lattice parallelogram can be calculated by multiplying the area of the square by the determinant of the transformation matrice(s). The horizontal shear transformation matrix has the form $\begin{bmatrix} 1 & \lambda_x \\ 0 & 1 \end{bmatrix}$ and the vertical shear transformation matrix has the form $\begin{bmatrix} 1 & 0 \\ \lambda_y & 1 \end{bmatrix}$. The determinant of both transformation matrix is 1. $\begin{vmatrix} 1 & \lambda_x \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \lambda_y & 1 \end{vmatrix} = 1.$

In Exercise 54, we showed that the area of the parallelogram $P(\vec{v}, \vec{w})$ is given by the determinant of a matrix whose columns are $\vec{v}$ and $\vec{w}$. The area of each primitive lattice parallelogram is equal to 1, which means the determinant of the matrix expressing the parallelogram is equal to 1. In Exercise 53, we showed that if the determinant of a matrix with integer values is equal to 1 or -1, the columns form a $\mathbb{Z}$-basis for $\mathbb{Z}^2$. This means $\vec{v}$ and $\vec{w}$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^2$.

**Exercise 57**

Let T be a primitive lattice triangle. If $\vec{v}$ and $\vec{w}$ are vectors corresponding to adjacent sides of T, show that $\vec{v}$ and $\vec{w}$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^2$.

**Solution**

We can form a parallelogram by adding a vector $\vec{v}$ to the tail of $\vec{w}$ and adding a vector $\vec{w}$ to the tail of vector $\vec{v}$. T is a primitive lattice triangle, which means there are no internal and boundary lattice points besides the three vertices. Reflecting T across the x axis and translating it such that the three vertices remain lattice points does not add or remove any interior and boundary lattice points. This means the triangle T' shown in the image below is also a primitive lattice triangle. Together, triangles T and T' form a primitive lattice parallelogram. In Exercise 56, we showed that $\vec{r}$ and $\vec{q}$ of a primitive lattice parallelogram $P(\vec{r}, \vec{q})$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^2$. Because we can construct a primitive lattice pentagon with the vectors $\vec{v}$ and $\vec{w}$, they form a $\mathbb{Z}$-basis of $\mathbb{Z}^2$.

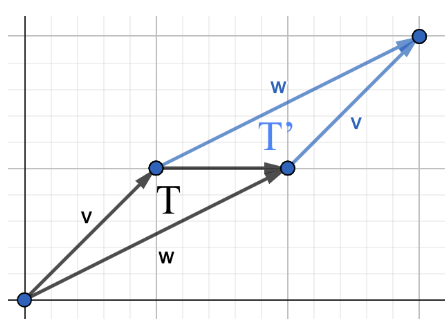


Figure 55: Primitive Lattice Triangle and Corresponding Parallelogram

**Exercise 58**

Prove that the area of a primitive lattice triangle is equal to $\frac{1}{2}$.

**Solution**

In Exercise 54, we proved that the area of a primitive lattice parallelogram is equal to 1. Because there are only four boundary points, which are the vertices of a parallelogram, there are only two ways we can construct two primitive lattice triangles, by connecting two opposite vertices. The opposite sides of a parallelogram have equal length and opposite angles have equal value, so the two triangles have equal area. This means the area of each primitive lattice triangle is equal to $\frac{1}{2}$.

# 5 Pick's Theorem

**Theorem 2. Pick's Theorem.**

Let P be a lattice polygon in $\mathbb{R}^2$. Suppose that there are $B(P)$ lattice points on the boundary of $P$ and $I(P)$ lattice points in the interior of $P$. Then the area $A(P)$ is given by:
$A(P) = \frac{1}{2}B(P) + I(P) - 1$

## 5.1 Proof of Pick's Theorem using Graph Theory

We will need a few classic dissection theorems to prove Pick's Theorem using graph theory.

**Theorem 3**

Every convex n-gon can be dissected into $n-2$ triangles by means of nonintersecting diagonals. The vertices of the triangles in this dissection by diagonals are vertices of the original polygon.

**Exercise 59 Prove Theorem 3.**

Let us construct a diagonal between two arbitrary non-adjacent vertices. This creates two smaller convex polygons that share one side and two vertices. The original polygon is convex, so all inner angles are smaller than 180°. Because the angles corresponding to the two vertices get smaller and all other angles remain the same, the two resulting polygons are both convex.

We can dissect the convex polygon into $n-2$ triangles in the following manner. Label the vertices of the convex polygon from 1 to n in the clockwise direction. Construct a diagonal connecting vertex 1 and all vertices from 3 to n-1. Because all diagonals originate from the same vertex, they do not intersect. Each time we construct a diagonal, we construct a triangle and a smaller convex polygon except when constructing the last diagonal which dissects a quadrilateral into two triangles. There are $n-3$ vertices the original vertex is not adjacent to. Therefore, we can construct $n-3+1 = n-2$ triangles by means of non-intersecting diagonals of the original n-gon.

**Theorem 4**

Every n-gon can be dissected into $n-2$ triangles by means of nonintersecting diagonals. The vertices of the triangles in this dissection by diagonals are vertices of the original polygon.

**Exercise 60 Prove Theorem 4**

Let us prove Theorem 4 using strong induction by showing that it is always possible to construct a diagonal when fewer than $n-2$ triangles have been formed. The base case occurs when $n = 3$ and the three sides of the polygon result in $n - 2 = 1$ triangles. For our inductive step, let us assume every polygon with k vertices, where $3 < k < n$, can be dissected into $k - 2$ triangles by non-intersecting diagonals. To prove this statement, it is sufficient to prove that the polygon has a diagonal.

Suppose a polygon $P$ with $n$ sides, where $n > 3$, has a diagonal. Constructing the diagonal splits the polygon into two smaller polygons, say $P_1$ and $P_2$ with $n_1$ and $n_2$ sides respectively. These two polygons both have at least 3 sides, which is the smallest possible polygon. Because the diagonal is included in both polygons, the sum of the number of sides of the polygon must equal $n + 2$. This means the polygons have at most $n + 2 - 3 = n - 1$ sides. By the inductive hypothesis, because both $P_1$ and $P_2$ have at least 3 and at most n-1 sides, they can both be dissected into $n_1 - 2$ and $n_2 - 2$ sides respectively. Including the original dissection into $P_1$ and $P_2$, these diagonals dissect $P$ into $(n_1 - 2) + (n_2 - 2) + 2 = n - 2$ triangles. Therefore, it is sufficient to prove that any polygon with more than 3 sides has some diagonal d.

We now need to prove it is possible to construct a diagonal for every polygon with greater than 3 sides.

First, let us prove that in each polygon, there is necessarily a convex angle. We will prove this by attempting to construct a polygon whose angles are all concave and show that this is not possible. Let us construct an arbitrary concave angle as shown in Figure 56 below. For the angles corresponding to vertices A and C to also be concave, the next vertices of the polygon need to be chosen from the white region as shown in Figure 57. However, because the polygon is concave, the blue region is the region inside the polygon. To enclose the blue region, there needs to be vertices in the blue region, such as vertex F, which would be adjacent to a convex angle, such as vertex E as shown in Figure 58.

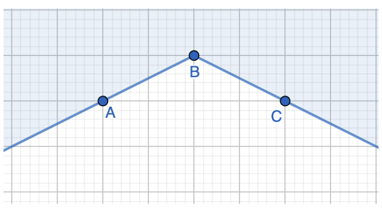


Figure 56: Concave Angle

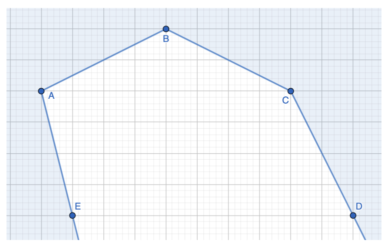


Figure 57: Vertex Constructions in White Region

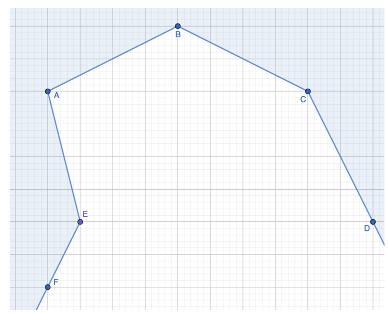


Figure 58: Vertex Constructions in Blue Region

Now that we have shown that every polygon has at least one convex angle, let us choose one at random and show it is possible to construct an internal diagonal. There are two different types of convex angles. In the first case, we can directly construct an internal diagonal. In the example shown in Figure 59, we can connect vertices P and R.

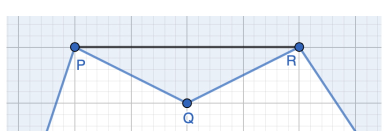


Figure 59: Convex Angle Case 1

In the second case, it is not possible to construct that diagonal because it is not contained within the polygon. In the example shown in Figure 60, the diagonal connecting vertices L and N lies outside the polygon.

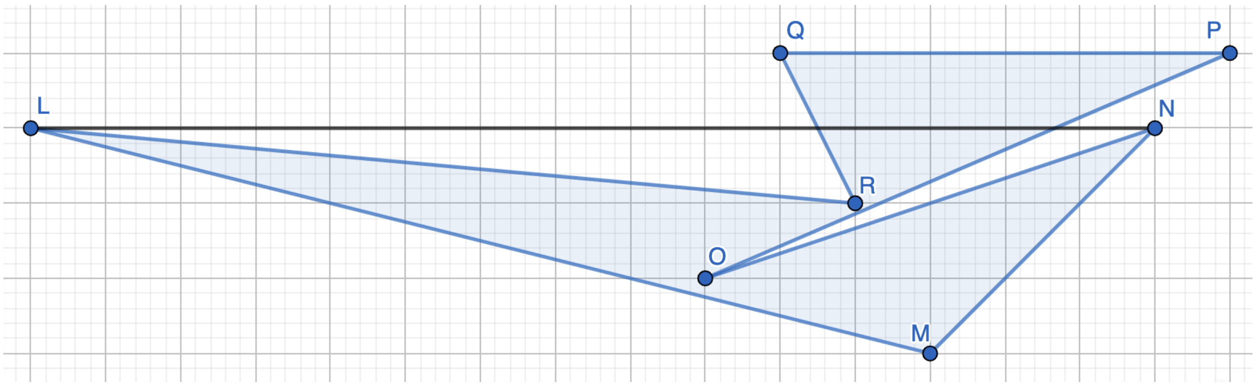


Figure 60: Convex Angle Case 2

Let us construct a line segment connecting vertex M to the closest vertex inside the enclosed region of the triangle LMN. If this line segment is entirely enclosed within the polygon, that is our diagonal and we have finished. However, in the example shown, the closest vertex to M is R and the line segment MR lies outside the polygon. Because this is a Jordan polygon, this means there is another vertex inside the triangle LMN, in this case O. If O was not in the triangle LMN, the line segments OP and ON will have intersected the line segment LM. Because our polygon is finite, there are a finite number of vertices we can check. Because this pattern cannot continue indefinitely there must be a vertex that we can connect with vertex M to construct a diagonal. Therefore, it is possible to construct a diagonal in each polygon, completing the proof.

**Lemma 1**

Every lattice triangle can be dissected into primitive lattice triangles.

**Exercise 61 Prove Lemma 1**

Let us proceed using proof by strong induction on the number of interior lattice points. In the base case, $I(T) = 0$. If the triangle only has three boundary lattice points, it is a primitive lattice triangle and we are done. If the triangle is not a primitive lattice triangle, let us label the vertices as A, B, and C and the boundary lattice points on the side AB as $P_1$ to $P_n$, boundary lattice points on the side BC as $Q_1$ to $Q_n$, and boundary lattice points on the side CA as $R_1$ to $R_n$. One such example is shown in Figure 61 below.

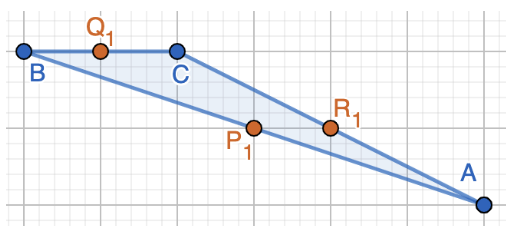


Figure 61: Non-Primitive Lattice Triangle with $I(T) = 0$

Let us construct a line segment from vertex A to all boundary lattice points $Q_1$ to $Q_n$ on the line segment BC. Then, let us construct a line segment from boundary lattice point $Q_1$ to all vertices $P_1$ to $P_n$ on line segment AB and a line segment from boundary lattice point $Q_n$ to all vertices $R_1$ to $R_n$ on line segment CA. This dissects our triangle with no interior points into primitive lattice triangles. A dissection of the triangle in Figure 61 is shown in Figure 62.

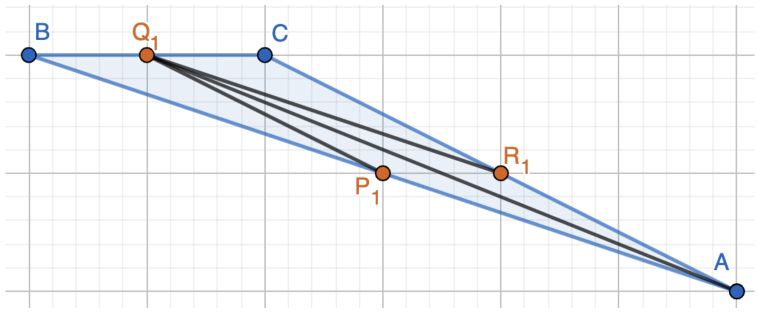


Figure 62: Non-Primitive Lattice Triangle with $I(T) = 0$

For the inductive step while the triangle has interior lattice points, choose one at random and construct three line segments connecting the interior lattice point to the three vertices of the triangle. Repeat this process until we have dissected our initial triangle into smaller triangles with no interior lattice points. Then, we have arrived at our base case and can construct primitive lattice triangles. An example is shown in Figures 63 and 64 below.

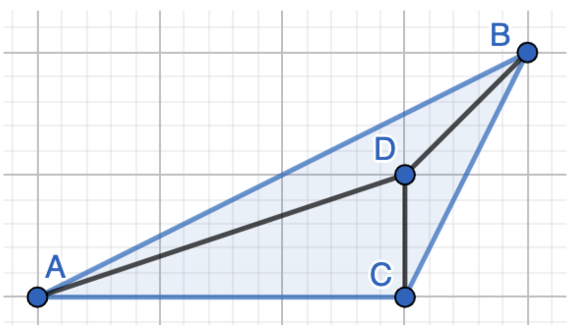


Figure 63: Triangle with $I(T) = 1$

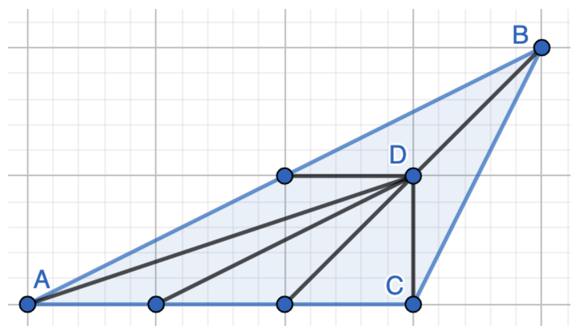


Figure 64: Dissection of Smaller Triangles

**Exercise 62**

Prove Theorem 5, which states that every lattice polygon can be dissected into primitive lattice triangles.

**Solution**

Using the result of Exercise 60, we can dissect a lattice polygon with n sides into $n-2$ triangles. Then, using the result of Exercise 61, we can dissect each lattice triangle into primitive lattice triangles.

**Exercise 63**

Dissect the twelve lattice polygons in Figure 65 into primitive lattice triangles.

**Solution**

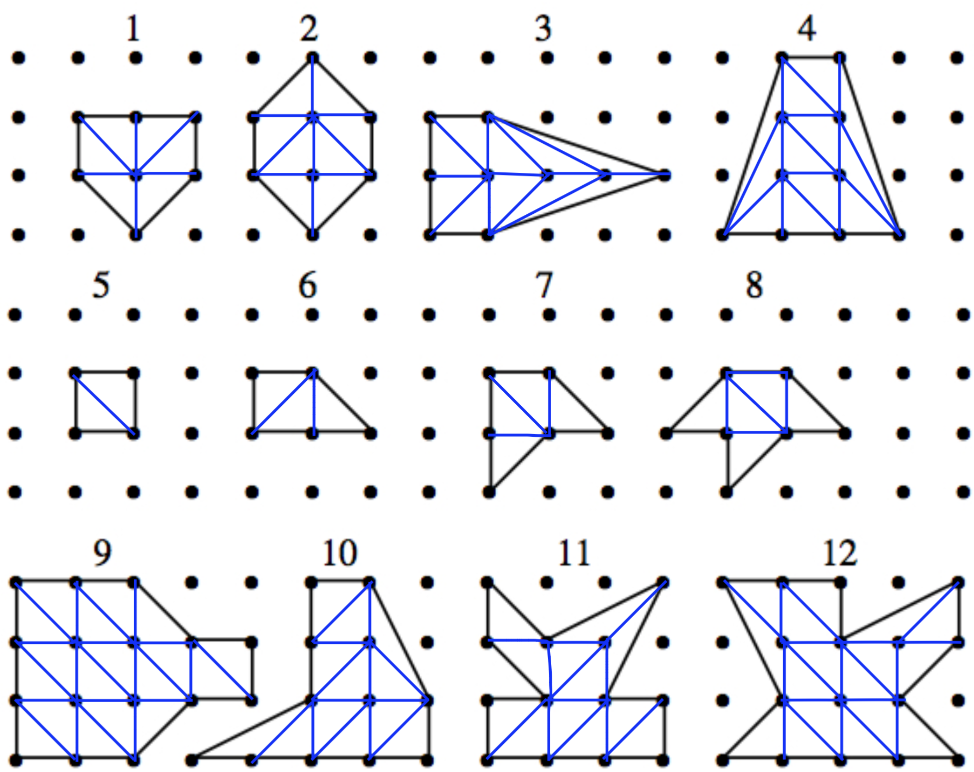


Figure 65: Dissection of Twelve Lattice Polygons

**Exercise 64**

By Theorem 5, we know that P has a dissection into primitive lattice triangles. Observe that since the sides of the triangles in the dissection of P do not intersect, and since the triangles are primitive, each lattice point in P is a vertex of a triangle. This dissection makes P a connected planar graph G as follows. The vertices of the graph G are the lattice points in P and on the sides of P. The edges of the graph G are the sides of the primitive triangles that triangulate P. Demonstrate this construction using the polygons shown in Figure 65. For each of the polygons in Figure 65, use your dissection into primitive lattice triangles from Exercise 63 to construct the graph G. Let $e_i$ denote the number of edges of G inside the polygon P and let $e_b$ denote the number of edges of G that are on the boundary of the original polygon P. Finally, compute the quantity $2v - e_b - 1$. Complete the following table. What do you observe?

**Solution**

| Polygon | Area | f = # of regions of G | 2v-e$_b - 1$ |
|---|---|---|---|
| 1 | 3 | 7 | 7 |
| 2 | 4 | 9 | 9 |
| 3 | 5 | 11 | 11 |
| 4 | 6 | 13 | 13 |
| 5 | 1 | 3 | 3 |
| 6 | 1.5 | 4 | 4 |
| 7 | 2 | 5 | 5 |
| 8 | 2.5 | 6 | 6 |
| 9 | 9 | 19 | 19 |
| 10 | 6 | 13 | 13 |
| 11 | 6 | 13 | 13 |
| 12 | 8.5 | 18 | 18 |

In each of the 12 cases above, $f = 2v - e_b - 1$.

**Exercise 65**

Use Exercise 58 to state and prove an equation that relates the area of P to f.

**Solution**

f is equal to the number of regions of the graph. Of these, $f - 1$ is inside the polygon. From Exercise 58, we know each primitive lattice triangle has an area of $\frac{1}{2}$. Therefore, the area of P is equal to $\frac{f-1}{2}$.

**Exercise 66**

The next step in the proof of Pick's Theorem is to compute $e$ and relate it to $f$. Let $e_i$ denote the number of edges of G inside the polygon P and let $e_b$ denote the number of edges of G that are on the boundary of the original polygon $P$. Show that $f = 2v - e_b - 1$.

**Solution**

Each face inside the polygon is bounded by 3 edges while the exterior face is bounded by $e_b$ edges. Each edge is adjacent to two faces. Therefore, we can write the number of edges as $e = \frac{3(f-1)+e_b}{2}$.

From Euler's formula, we know $v - e + f = 2$. Rearranging this, we get $e = v + f - 2$. Equating this with the expression for e we got before, we get $v + f - 2 = \frac{3(f-1)+e_b}{2}$. Solving for $f$, we get $f = 2v - e_b - 1$.

**Exercise 67**

Finally, show that $A(P) = \frac{1}{2}B(P) + I(P) - 1$. This concludes the proof of Pick's Theorem!

**Solution**

From Exercise 65, we know $A(P) = \frac{f-1}{2}$. Using the result from Exercise 66, we get $A(P) = \frac{2v-e_b-1-1}{2}$. Here, $e_b = B(P)$ because $e_b$ is the number of edges of the graph that are on the boundary of the original polygon and $B(P)$ is equal to the number of boundary points of the polygon. Moreover, $v = B(P) + I(P)$ because $v$ is the total number of vertices of the graph and $B(P)$ is the number of boundary lattice points of the polygon and $I(P)$ is the number of interior lattice points of the polygon. Putting it all together, we get $A(P) = \frac{2(B(P)+I(P))-B(P)-2}{2} = \frac{B(P)+2I(P)-2}{2} = \frac{1}{2}B(P) + I(P) - 1$.

## 5.2 Proof of Pick's Theorem by Additivity

Let P be a lattice polygon in the plane. We will need the following definitions:

**Definition 32**

For each $p \in \mathbb{Z}^2$, we define $a_p(P)$, called the measure of the internal angle of P at p, as follows:

- If p is a vertex of P, then $a_p(P)$ is the measure of the interior angle of the polygon at p.
- If p is on a side of P, but is not a vertex of P, then $a_p(P) = \pi$
- If p is in the interior of P, then $a_p(P) = 2\pi$.
- For all other lattice points $p \in \mathbb{Z}^2$, we set $a_p(P) = 0$.

**Definition 33**

The lattice weight of P at p is defined as $w_p = \frac{1}{2\pi} a_p(P)$.

**Definition 34**

The total weight of P is $W(P) = \sum_{p \in \mathbb{Z}^2} w_p(P)$.

**Exercise 68**

Why is $W(P)$ finite?

**Solution**

As can be seen from Definition 34, the total weight is calculated by adding $w_p(P)$ for all points $p \in \mathbb{Z}$. Because $w_p(P) = 0$ for all points on the exterior of P, we are only considering points on the interior and boundary of P. Because we are only considering lattice points, $W(P)$ is finite.

**Definition 35**

Let P be a lattice polygon, and let Q be a lattice polygon whose intersection with P is a portion s' of a side s of P. Then the join, or sum, of P and Q, denoted $P + Q$, is the lattice polygon obtained by gluing P and Q along s'. The vertices, boundary lattice points, and interior lattice points of P and of Q that do not lie on s' become vertices, boundary lattice points, and interior lattice points, respectively, of $P + Q$, as described in the following way. Let $p'$ be a lattice point on $s'$.

- If $p'$ is a vertex of P but not a vertex of Q, then $p'$ is a vertex of $P + Q$.
- If $p'$ is a vertex of P and a vertex of Q, and if the interior angles of P and Q at p' are not supplementary, then p' is also a vertex of $P + Q$.
- If $p'$ is a vertex of P and a vertex of Q, and if the interior angles of P and Q at p' are

supplementary, then p' is a lattice point on a side of $P+Q$, but is not a vertex of $P+Q$.

- Similar statements hold if the initial hypothesis is that p' is a vertex of Q.
- If p' is a lattice point on s' that is not a vertex of P or Q, then p' is an interior point of $P+Q$.

**Exercise 69**

Construct a lattice polygon P and a lattice polygon Q whose intersection with P is a portion s' of a side s of P. Then construct the join $P+Q$ using Definition 35.

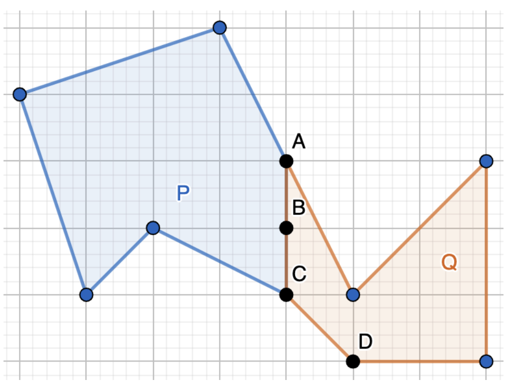


Figure 66: Polygons P and Q

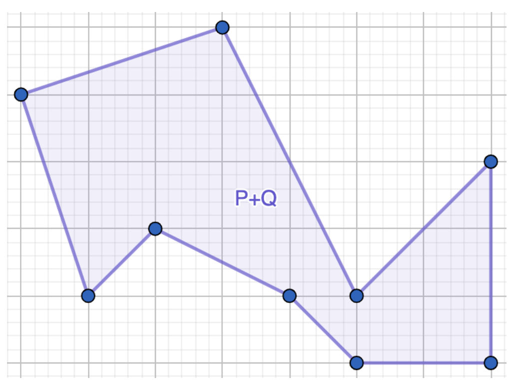


Figure 67: Polygon $P+Q$

To determine the polygon $P+Q$, let us consider vertices A, B, C, and D, where D represents all vertices that belong to only one of the polygons.

Point A is a vertex of both polygons P and Q. Moreover, the interior angles for polygon P and polygon Q at vertex A are supplementary. Therefore, A is a lattice point on the side of $P+Q$.

Point B is on the shared part of the side s' but is not a vertex of P or Q. Therefore, it is an interior point of $P+Q$.

Point C is a vertex of both polygons. However, unlike vertex A, the interior angles are not supplementary. Therefore, point C is a vertex of polygon $P+Q$.

Lastly, point D is a vertex of Q but not a vertex of P. Therefore, it is a vertex of polygon $P+Q$. This is true for all vertices that belong to only one of the polygons.

**Lemma 2**

Let P be a lattice polygon, and let Q be a lattice polygon whose intersection with P is a portion s' of a side s of P. Then $W(P+Q) = W(P) + W(Q)$.

**Lemma 3**

Let P be a lattice polygon. Then $W(P) = A(P)$, where $W(P)$ is the total weight of $P$ and $A(P)$ is its area.

**Exercise 70**

Case 1. Suppose that R is a lattice rectangle of length r and width s with horizontal and vertical sides.

(a) How many lattice points does R have on its sides? How many of these are vertices?

(b) How many interior lattice points does R have?

(c) Compute $W(R)$ and $A(R)$ and show that they are equal.

**Solution**

(a) Let us divide the lengths and sides of R into line segments of length 1. There are an equal number of line segments and lattice points. The number of line segments on one side is equal to the side length. Therefore, because the perimeter of R is $2(r+s)$, there are $2(r+s)$ lattice points on the boundary of R. Of these, four are vertices.

(b) R has $(r-1)(s-1) = rs - r - s + 1$ interior lattice points.

(c) R has $2r + 2s - 4$ lattice points on its side. For each boundary lattice point that is not a vertex, $a_s(R) = \pi$. Therefore, $w_s = \frac{\pi}{2\pi} = \frac{1}{2}$. R has four vertices, for which $a_s(R) = \frac{\pi}{2}$ and $w_s = \frac{\pi}{4\pi} = \frac{1}{4}$. R has $rs - r - s + 1$ interior points, for which $a_s(R) = 2\pi$ and $w_s = \frac{2\pi}{2\pi} = 1$. All other points have a weight equal to 0. Therefore, adding all the weights, we get $W(R) = \frac{1}{2}(2r+2s-4) + \frac{1}{4}(4) + 1(rs-r-s+1) = r+s-2+1+rs-r-s+1 = rs$. This is equal to the area of the rectangle, $A(R) = rs$.

**Exercise 71**

Case 2. Suppose that $T$ is a lattice right triangle with horizontal and vertical legs of length r and s, respectively. Show that $W(T) = A(T)$.

**Solution**

Consider the triangle $T'$ obtained by rotating $T$ by $\pi$ around the midpoint of the hypotenuse of $T$. Combining $T$ and $T'$ forms a rectangle $R$ with side lengths r and s. In Exercise 70, we showed that $W(R) = A(R)$ for a lattice rectangle with horizontal and vertical sides.

Here, triangles $T$ and $T'$ share one side. According to Lemma 2, $W(T+T') = W(T)+W(T')$. Triangle T' was obtained by rotating triangle T by $\pi$. Because they are congruent triangles whose vertices are lattice points, $W(T) = W(T')$. This means $W(T) = W(T') = \frac{W(T+T')}{2}$.

Moreover, we know $W(R) = W(T + T')$. The two shorter sides of each triangle coincide with a side of the rectangle. Moreover, all interior lattice points of the triangles coincide with interior lattice points of the rectangle. Lastly, the lattice points on the hypotenuse of the two triangles each have weight $\frac{\pi}{2}$ and combined have a weight of $\pi$, which is equal to the weight of the interior lattice point of the rectangle. Thus, $W(R) = W(T + T')$. From the result of Exercise 70, we know $W(R) = A(R)$. Because T and T' are congruent, they have equal areas. Thus, $A(T) = A(T') = \frac{A(R)}{2} = \frac{W(R)}{2} = W(T') = W(T)$. Thus, we have shown that $W(T) = A(T)$ where T is a lattice right triangle with horizontal and vertical legs.

**Exercise 72**

Now suppose that T is an arbitrary lattice triangle. We can embed T into the smallest possible lattice rectangle R with horizontal and vertical sides. There are two cases to consider in this scenario:

1. If all of the interior angles of T measure at most $\frac{\pi}{2}$, then R is the union of T and three Case 2 lattice triangles.

2. If T has an interior angle measuring more than $\frac{\pi}{2}$, then R is the union of T, a Case 1 lattice rectangle, and three Case 2 lattice triangles.

Show that $W(T) = A(T)$ in both of the cases described above.

**Solution**

1. As shown in the image below, we have an acute lattice triangle T. We can form triangles T1, T2, and T3 such that together, they form a rectangle R.From Lemma 2, we know $W(T{+}T1{+}T2{+}T3) = W(R) = W(T){+}W(T1){+}W(T2){+}W(T3)$. Moreover, we know $A(R) = R(T) + R(T1) + R(T2) + A(T3)$. From Exercise 70, we know $W(T1) = A(T1)$, $W(T2) = A(T2)$, and $W(T3) = A(T3)$. Therefore, $W(T) = W(R) - W(T1) - W(T2) - W(T3) = A(R) - A(T1) - A(T2) - A(T3) = A(T)$, which simplifies to $W(T) = A(T)$.

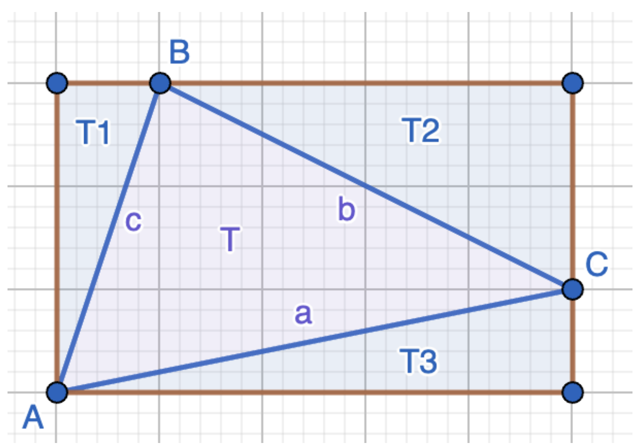


Figure 68: Case 1 Triangles and Rectangle

2. Shown below is an obtuse lattice triangle T, right triangles T1, T2, and T3 and rectangles R1 (larger rectangle) and R2 (smaller rectangle). By Lemma 2, $W(R2 + T1 + T2 + T3 + T) = W(R1) = W(R2) + W(T1) + W(T2) + W(T3) + W(T)$. We also know $A(R2 + T1 + T2 + T3 + T) = A(R1) = A(R2) + A(T1) + A(T2) + A(T3) + A(T)$. From Exercise 70, we know $W(T1) = A(T1)$, $W(T2) = A(T2)$, and $W(T3) = A(T3)$. From Exercise 71, we know $W(R2) = A(R2)$. Therefore, we get $W(T) = W(R1) - W(R2) - W(T1) - W(T2) - W(T3) = A(R1) - A(R2) - A(T1) - A(T2) - A(T3) = A(T)$.

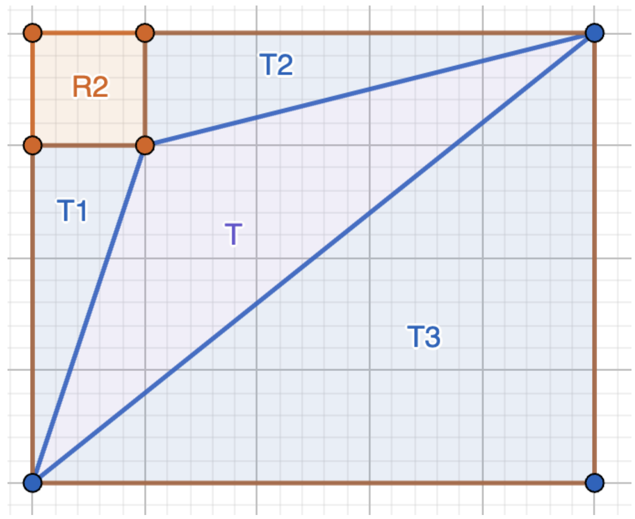


Figure 69: Case 2 Triangles and enclosing Rectangles

Thus, we have shown that $W(T) = A(T)$ for all lattice triangles.

**Exercise 73**

Let P be a lattice n-gon. Let $B(P)$ denote the number of lattice points on the sides (boundary) of P, and let $I(P)$ denote the number of interior lattice points of P.

(a) How many vertices does the n-gon have?

(b) How many lattice points does P have on its sides that are not vertices?

(c) What is the sum of the interior angles of P?

(d) Use (a)-(c) to show that the sum of $w_p$ as p ranges over all points on the sides of P is $\frac{1}{2}B(P) - 1$

(e) Show that the sum of $w_p$ as p ranges over all the lattice points in the interior of P is $I(P)$.

(f) Show that $W(P) = \frac{1}{2}B(P) + I(P) - 1$.

(g) Conclude the proof of Pick's Theorem.

**Solution**

(a) The n-gon has n vertices.

(b) P has $B(P) - n$ lattice points on its sides that are not vertices.

(c) The sum of the interior angles of P is $(n-2)\pi$.

(d) The n-gon has n vertices, whose total weight is $\frac{(n-2)\pi}{2\pi} = \frac{n-2}{2}$. The n-gon also has $B(P)-n$ lattice points on its sides that are not vertices, each with weight $\frac{1}{2}$. Therefore, the sum of $w_p$ as p ranges over all points on the sides of P is $\frac{n-2}{2} + \frac{B(P)-n}{2} = \frac{1}{2}B(P) - 1$.

(e) Each interior point of P has weight 1. Therefore, the sum of $w_p$ as p ranges over all the lattice points in the interior of P is $I(P)$.

(f) The total weight of all lattice points on or inside the polygon is given by the sum of $w_p$ as p ranges over all points on the sides of P and in the interior of P. All other points have weight 0. $W(P) = \frac{1}{2}B(P) + I(P) - 1$.

(g) According to Lemma 3, $A(P) = W(P)$. Therefore, $W(P) = A(P) = \frac{1}{2}B(P) + I(P) - 1$, which concludes our proof of Pick's Theorem.

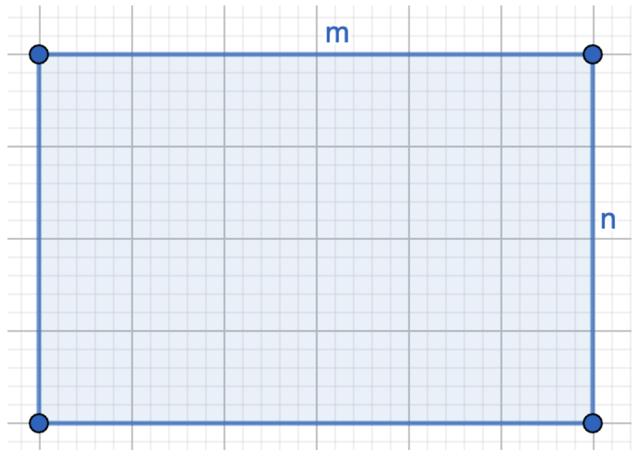


Figure 70: $m \times n$ Lattice Aligned Rectangle

## 5.3 Proof of Pick's Theorem by Induction

In this series of exercises, we will prove Pick's Theorem using induction on the number of sides of the polygon.

### Exercise 74

Consider an $m \times n$ lattice-aligned rectangle. Show that Pick's Theorem holds for a lattice-aligned rectangle.

### Solution

Pick's Theorem states that $A(P) = \frac{1}{2}B(P) + I(P) - 1$, where $A(P)$ is the area, $B(P)$ is the number of boundary lattice points, and $I(P)$ is the number of interior lattice points of the lattice polygon. The area of the rectangle is $mn$.

Let us now determine the number of boundary and interior lattice points. We can divide the boundary of the rectangle into line segments of length one and associate each line segment with the boundary lattice point that is to the right in a counterclockwise manner. Because we can pair each line segment with a boundary lattice point, $B(P)$ is equal to the perimeter of the rectangle, which means $B(P) = 2(m + n)$.

Now, let us consider the number of interior lattice points. Let us construct a line segment $L_m$ parallel to the x axis with one endpoint on the left side of the rectangle and the other endpoint on the right side of the rectangle. $L_m$ has length m. Let us translate $L_m$ such that the left endpoint is at the origin. From Exercise 31, we know the line segment from the origin to point $(x, y)$ has $\gcd(x, y) - 1$ lattice points without considering end points. Therefore, $L_m$ has $\gcd(m, 0) - 1 = m - 1$ lattice points. [3] For each m by n lattice rectangle, we can construct $n - 1$ such line segments on the interior. Because each line segment has $m - 1$ interior lattice points, we get that there are $(m-1)(n-1) = mn - m - n + 1$ interior lattice points.

We can thus calculate $\frac{1}{2}B(P) + I(P) - 1 = m + n + mn - m - n + 1 - 1 = mn = A(P)$. We have shown that Pick's Theorem holds for a lattice-aligned rectangles.

**Exercise 75**

Prove that Pick's Theorem holds for a lattice-aligned right triangle with legs of lengths m and n.

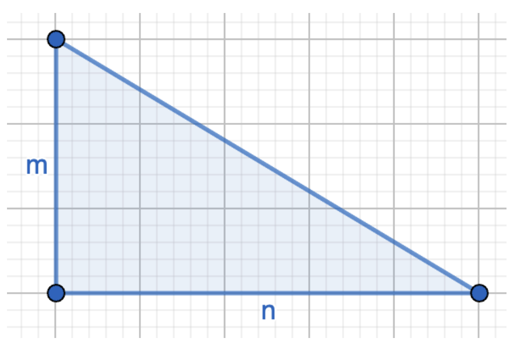


Figure 71: Caption

[3] $\gcd(m, 0)$ is the largest integer k such that $k|m \in \mathbb{Z}$ and $k|0 \in \mathbb{Z}$. Let us consider the second condition. If we choose $n = 0$, k can be arbitrarily large and still fulfill the condition $k|0 \in \mathbb{Z}$ because $k \times n = k \times 0 = 0$. Therefore, we need only consider the second condition, where the largest possible value for k is m.

**Solution**

The area of the lattice-aligned right triangle is $A(T) = \frac{mn}{2}$.

Let us determine the number of boundary and interior lattice points. According to Exercise 31, there are $\gcd(m, n) - 1$ lattice points on the line segment between the origin and the point $(m, n)$, not including endpoints. Without loss of generality, let us translate the right triangle such that the point corresponding to the right angle is at the origin. Next, let us rotate it such that one side lies on the positive x axis and the other side lies on the positive y axis. Using the same reasoning as in the previous problem, there are $m+n+1$ lattice points on the two shorter sides of the triangle including both end points of the hypotenuse. Adding the points on the hypotenuse, we get $B(T) = m+n+1+\gcd(m, n)-1 = m+n+\gcd(m, n)$.

Let us now determine the number of interior lattice points. Let us construct an identical lattice aligned triangle and place it on the coordinate system such that the two triangles form an $m \times n$ lattice aligned rectangle. The two lattice triangles are congruent, so have the same number of internal lattice points. This means the sum of the interior lattice points of the two triangles and the number of lattice points on the hypotenuse of one triangle is equal to the number of interior lattice points on the rectangle. Therefore, $I(T) = \frac{I(R)-B(\text{hypotenuse})}{2} = \frac{mn-m-n+1-\gcd(m,n)+1}{2}$.

Let us now calculate $\frac{1}{2}B(T) + I(T) - 1 = \frac{m+n+\gcd(m,n)}{2} + \frac{mn-m-n+1-\gcd(m,n)+1}{2} - 1 = \frac{mn}{2}$. Thus, we have shown that Pick's Theorem holds for a lattice-aligned right triangle.

**Exercise 76**

The next step is to show that that Pick's Theorem holds for arbitrary lattice triangles. If T is an arbitrary lattice triangle, draw right triangles A, B, C to form a rectangle R, as shown below.

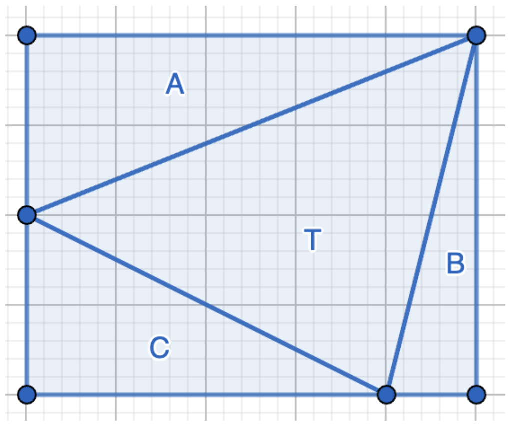


Figure 72: Arbitrary Lattice Triangle T

**Solution**

Let us label the sides of the triangle as $S_{TA}$, $S_{TB}$, and $S_{TC}$. Let us name the remaining sides as $S_{AH}$, $S_{AV}$, etc for all three triangles A, B, and C where H stands for horizontal and V stands for vertical.

We know Pick's Theorem holds for the rectangle R. $A(R) = \frac{1}{2}B(R) + I(R) - 1$. Pick's Theorem also holds for triangles A, B, and C. Therefore, we can write the following equations $A(T_A) = \frac{1}{2}B(T_A)+I(T_A)-1$, $A(T_B) = \frac{1}{2}B(T_B)+I(T_B)-1$, and $A(T_C) = \frac{1}{2}B(T_C)+I(T_C)-1$. The area of triangle T can be found by subtracting the area of triangles A, B, and C from the area of the rectangle, $A(T_T) = A(R) - A(T_A) - A(T_B) - A(T_C)$. In other words, $A(T_T) = \frac{1}{2}B(R)+I(R)-1-(\frac{1}{2}B(T_A)+I(T_A)-1)-(\frac{1}{2}B(T_B)+I(T_B)-1)-(\frac{1}{2}B(T_C)+I(T_C)-1)$. $A(T_T) = \frac{1}{2}\left(B(R) - B(T_A) - B(T_B) - B(T_C)\right) + \left(I(R) - I(T_A) - I(T_B) - I(T_C)\right) + 2$.

The number of boundary lattice points on triangle T is equal to the number of boundary lattice points on triangles A, B, and C minus the number of boundary lattice points on rectangle R. $B(T) = B(T_A) + B(T_B) + B(T_C) - B(R)$. This means, we can rewrite $B(R) - B(T_A) - B(T_B) - B(T_C)$ as $-B(T)$.

The number of interior lattice points on triangle T is equal to the number of interior lattice points on rectangle R minus the interior lattice points in triangles A, B, and C minus the boundary lattice points of triangle T, plus the three triangle T vertices that are on the boundary of R. $I(T_T) = I(R) - I(T_A) - I(T_B) - I(T_C) - B(T_T) + 3$. This means, we can rewrite $I(R) - I(T_A) - I(T_B) - I(T_C)$ as $I(T_T) + B(T_T) - 3$. Therefore, we can write $A(T_T) = \frac{1}{2}\left(-B(T)\right) + \left(I(T_T) + B(T_T) - 3\right) + 2 = \frac{1}{2}B(T_T) + I(T_T) - 1$. Thus, Pick's Theorem holds for the lattice triangle T.

So far, we have shown that Pick's Theorem is true for every polygon with 3 sides. To complete the proof that Pick's Theorem is true for any polygon, we will use induction on the number of sides of the polygon. The base case is $n = 3$ sides, and we have already shown that Pick's Theorem holds for $n = 3$. For the inductive step, assume that Pick's Theorem holds for $n = 3, 4, ..., k - 1$ sides. We must now prove that Pick's Theorem holds for $n = k$ sides to complete the induction.

**Exercise 77**

Using induction, complete the proof that Pick's Theorem is true for any polygon.

**Solution**

Let us use induction on the number of sides of the polygon. The base case is when the polygon has $n = 3$ sides. We have shown this to be true in Exercise 78. For the inductive step, let us assume that Pick's Theorem holds for $n = 4, 5, ..., k-1$ sides. We will now prove that Pick's Theorem holds for $n = k$ sides.

Suppose that P is a polygon with k sides ($k > 3$). By Exercise 60, we know we can always split P into 2 smaller polygons $P_1$ and $P_2$, each with fewer than k sides. By the inductive hypothesis, the two smaller polygons $P_1$ and $P_2$ satisfy Pick's Theorem. Therefore, we have $A(P_1) = I(P_1) + \frac{B(P_1)}{2} - 1$ and $A(P_2) = I(P_2) + \frac{B(P_2)}{2} - 1$. Because $A(P) = A(P_1) + A(P_2)$, we can see that $A(P) = I(P_1) + \frac{B(P_1)}{2} - 1 + I(P_2) + \frac{B(P_2)}{2} - 1$.

**Exercise 78**

Finally, find a relationship between I and $I(P_1)$, $I(P_2)$ and between B and $B(P_1)$, $B(P_2)$ to conclude that $A(P) = I(P) + \frac{1}{2}B(P) - 1$.

**Solution**

The interior lattice points in the larger polygon P is equal to the interior lattice points in $P_1$ and $P_2$ and the boundary lattice points on the shared diagonal. Let us say there are b points on the shared diagonal. We need to add b-2, not including the endpoints. $I(P) = I(P_1) + I(P_2) + b - 2$. The number of boundary lattice points on P is equal to the number of boundary lattice points on the two polygons minus 2b plus 2. $B(P) = B(P_1)+B(P_2)-2b+2$. $A(P) = (I(P_1)+I(P_2))+\frac{B(P_1)+B(P_2)}{2}-2 = (I(P)-b+2)+\frac{B(P)+2b-2}{2}-2 = I(P)+\frac{1}{2}B(P)-1$. This concludes the proof of Pick's Theorem using induction.

## 6 Applications of Pick's Theorem

### Exercise 79

Use Pick's Theorem to prove that it is not possible to construct an equilateral lattice triangle.

### Solution

We will prove this by contradiction. Consider an equilateral triangle with side length s. The height of the triangle is $s\sin(60°) = \frac{s\sqrt{3}}{2}$. This means its area is equal to $\frac{s^2\sqrt{3}}{4}$. Because the triangle is a regular lattice triangle, the vertices are all lattice points. The square of the distance between lattice points is always an integer. Let us say there are two lattice points with coordinates $(x_1, y_1)$ and $(x_2, y_2)$. The distance between them is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ and the square of the distance between them is $(x_2 - x_1)^2 + (y_2 - y_1)^2$, which is an integer. This means the area of the equilateral lattice triangle is equal to an integer times $\frac{\sqrt{3}}{4}$. From Pick's Theorem, we know the area of a lattice polygon is an integer multiple of $\frac{1}{2}$. This means it is not possible to construct an equilateral lattice triangle.

### Exercise 80

Use Pick's Theorem to show that the area of a primitive lattice triangle is equal to $\frac{1}{2}$.

### Solution

A primitive lattice triangle has three boundary lattice points and no interior lattice points. Therefore, according to Pick's Theorem, $A(T) = \frac{1}{2}B(T) + I(T) - 1 = \frac{3}{2} + 0 - 1 = \frac{1}{2}$.

**Exercise 81**

Use Pick's Theorem to show that if it is possible to construct a regular lattice n-gon, then $\tan\left(\frac{\pi}{n}\right)$ is rational.

**Solution**

Let us suppose P is a regular lattice n-gon with side lengths equal to s. According to Pick's Theorem, its area is an integer multiple of $\frac{1}{2}$. We can calculate the area of the polygon geometrically. Let us find the center of the polygon and construct n triangles by connecting the center to each vertex. Each triangle is congruent because this is a regular polygon and the area of the polygon is n times the area of a triangle. The area of a triangle can be calculated using $\frac{b \times h}{2}$ where b is the base length and h is the height. The base length is s, so we need to calculate height h. The angle $\alpha$ shown in the image is equal to $\frac{2\pi}{n}$, which means the angle $\beta$ shown in the image is $\frac{\pi}{n}$. This means $\tan\left(\frac{\pi}{n}\right) = \frac{s/2}{h}$ and $h = \frac{s}{2\tan\left(\frac{\pi}{n}\right)}$. Thus, the area of the polygon is $\frac{ns^2}{4\tan\left(\frac{\pi}{n}\right)}$. From Pick's Theorem, this quantity must be an integer multiple of $\frac{1}{2}$. $n, 4, s^2 \in \mathbb{Z}$. Therefore, $\tan\left(\frac{\pi}{n}\right) \in \mathbb{Q}$.

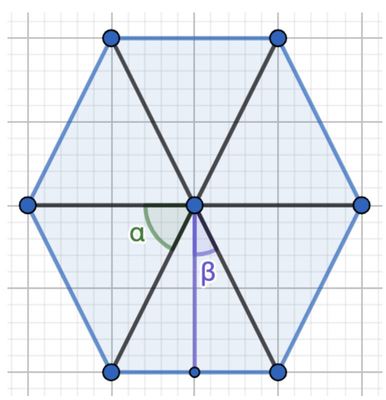


Figure 73: Regular Lattice n-gon

**Exercise 82**

Show that if P is a convex lattice pentagon, then the area of P must be greater than or equal to $\frac{5}{2}$. Is this bound strict? In other words, is it possible to construct a convex lattice pentagon with area equal to $\frac{5}{2}$?

**Solution**

Let P be a convex lattice pentagon. Because P is a pentagon, it has at least 5 boundary lattice points, its vertices. According to Pick's Theorem, $A(P) = \frac{B(P)}{2} + I(P) - 1$. For the area of the pentagon to be greater than $\frac{5}{2}$, the pentagon can either have at least 7 boundary lattice points or at least 1 interior lattice point.

There are two possible parity cases for each integer, even or odd. For a pair of integers, there are four possible parity cases, (even, even), (even, odd), (odd, even), or (odd, odd). The pentagon has five vertices and there are four possible parities for each vertex. By the pigeon hole principle, at least two of the five vertices have the same parity. Let us say these two vertices have coordinates $(x_1, y_1)$ and $(x_2, y_2)$. Because they have the same parity, their midpoint with coordinates $(\frac{x_1+x_2}{2}, \frac{y_2+y_1}{2})$ is a lattice point. Because the polygon is convex, the midpoint is not outside the polygon. There are two possible cases. In the first case, the two vertices are non-adjacent and their midpoint is an interior lattice point. According to Pick's Theorem, the area is $A(P) \geq \frac{5}{2} + 1 - 1 = \frac{5}{2}$, as required.

In the second case, the two vertices are adjacent and their midpoint is a boundary lattice point. Let us apply the pigeon hole principle again to five boundary lattice points. This will include one of the pentagon vertices pair with the same parity and their midpoint. This ensures we will find a different midpoint that is also a lattice point. If this is an interior lattice point, we are done and the area is at least $A(P) \geq \frac{6}{2} + 1 - 1 = 3$. If this is a boundary lattice point, the area is at least $A(P) \geq \frac{7}{2} + 0 - 1 = \frac{5}{2}$. We have shown that the area of the convex lattice pentagon is at least $\frac{5}{2}$ for all possible cases.

It is possible for a convex lattice pentagon to have area equal to $\frac{5}{2}$. One example is shown below.

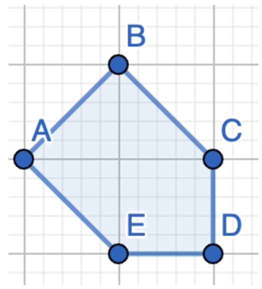


Figure 74: Convex Lattice Pentagon with Area $\frac{5}{2}$

**Exercise 83**

Let A denote the point $(n, 0)$ and let B denote the point $(0, n)$. There are $n-1$ lattice points, each of the form $(i, n-i)$, for $i = 1, 2, 3, ..., n-1$, between A and B. Connect each one of them with the origin $O(0,0)$. The lines divide $\Delta$OAB into n small triangles. It is clear that the 2 triangles next to the axes contain no lattice points in their interior. Prove that if n is prime, then each of the remaining triangles contains exactly the same number of interior lattice points. Find an expression (in terms of n) for the number of interior lattice points in each of these triangles.

**Solution**

Each of these n triangles have the same area. This is because the area of a triangle can be calculated by taking the half of its base times height. Using the Pythagorean Theorem, the base of each triangle is equal to $\sqrt{2}$ and the height of each triangle can be calculated using the equation $n^2 = h^2 + h^2$, which means $h = \frac{n}{\sqrt{2}}$ and the area of each triangle is $A = \frac{n}{2}$.

We can confirm this area by using Pick's Theorem, $A(T) = \frac{B(T)}{2} + I(T) - 1$. The two triangles next to the axes have no interior lattice points and $n+2$ boundary lattice points. Plugging these values into the equation for Pick's Theorem, we get $A(T) = \frac{n+2}{2} + 0 - 1 = \frac{n}{2}$. To determine the number of interior lattice points on the remaining triangles, we need to show that they each have only three boundary lattice points, which means they must have the same number of interior lattice points.

Let us label the boundary lattice points on the line segment AB, not including the endpoints, as $P_i$, where i is the x-coordinate. The coordinates of each point is $P_i = (i, n-i)$. Let us write the coordinates of the point as $P_i = (p_x, p_y)$. $p_x + p_y = n$, where n is a prime number. Because n is prime, $\gcd(p_x, n) = 1$ and $\gcd(p_y, n) = 1$. This means $\gcd(p_x, p_y) = 1$. If $\gcd(p_x, p_y) \neq 1$, then there exists some number a such that $a|p_x$ and $a|p_y$. However, this would mean $a|p_x + p_y$, which means $a|n$. But because n is prime, this results in a contradiction. Therefore, because $p_x = i$ and $p_y = n - i$, $\gcd(i, n-i) = 1$.

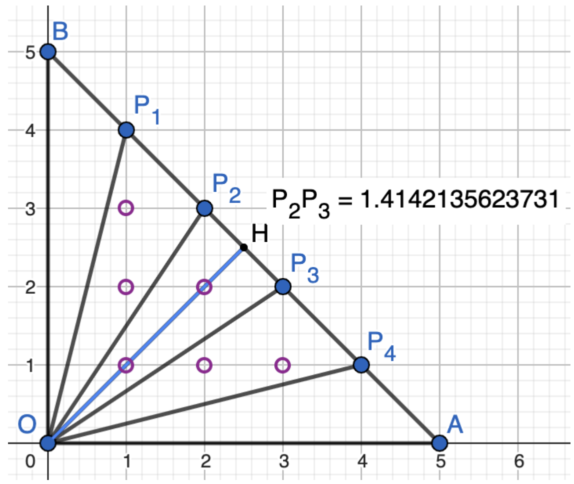


Figure 75: Example with $n = 5$

In Exercise 29, we showed that there are no lattice points strictly between the origin and a point $Q = (m, n)$ if $\gcd(m, n) = 1$. This means each of the triangles other than those next to the axes have only three boundary lattice points. Plugging in $A(T) = \frac{n}{2}$ and $B(T) = 3$ into Pick's Theorem equation, we get $I(T) = A(T) - \frac{B(T)}{2} + 1 = \frac{n}{2} - \frac{3}{2} + 1 = \frac{n-1}{2}$.

**Exercise 84**

Let n be an integer greater than or equal to 3. Prove that there is a set of n points in the plane such that the distance between any 2 points is irrational and each set of three points determines a non-degenerate triangle with rational area.

**Solution**

Let us consider all lattice points with a positive x coordinate on the parabola $y = x^2$ in the plane $z = 0$. We can chose two with coordinates $(x_1, x_1^2)$ and $(x_2, x_2^2)$. The distance between them is $\sqrt{(x_2 - x_1)^2 + (x_2^2 - x_1^2)^2} = \sqrt{(x_2 - x_1)^2 + (x_2 - x_1)^2(x_2 + x_1)^2} = |x_2 - x_1|\sqrt{1 + (x_2 + x_1)^2}$. We are only considering lattice points with positive x coordinates, so $x_1$ and $x_2$ are distinct integers greater than 0. Therefore, $(x_2 + x_1)^2$ is an integer greater than $(1 + 2)^2 = 9$. The difference between two square numbers, both greater than 9, is greater than 1. Therefore, adding 1 to the value $(x_2 + x_1)^2$ does not make it a square number and $\sqrt{1 + (x_2 + x_1)^2}$ is irrational. This means the distance between any two points is irrational.

Triangles whose vertices are lattice points have area given by Pick's theorem. Because there are an integer number of interior and boundary lattice points, this means the area of the triangle is an integer multiple of $\frac{1}{2}$. The set of all points on the parabola we are considering are lattice points. Therefore, the area of the triangle formed between any three different points is rational.

**Exercise 85**

Show that if $T$ is a lattice triangle with $I(T) = 1$, then $B(T) = 3, 4, 6, 8,$ or 9.

**Solution**

Suppose T is a lattice triangle with one interior point. Without loss of generality, let us suppose that the vertices of the triangle T are $(0, 0)$, $(a_1, a_2)$, and $(b_1, b_2)$. If they are not, we can translate the triangle such that one vertex is at the origin. Translation preserves angles and side lengths, so the triangle will not change. According to Pick's Theorem, $A(T) = \frac{1}{2}B(T) + I(T) - 1$. Because $I(T) = 1$, $A(T) = \frac{1}{2}B(T)$, which means $2A(T) = B(T)$.

Let us count the number of boundary lattice points on each of the three sides of the triangle, $OA$, $OB$, and $AB$. According to the results of Exercise 31, we can determine the number of boundary lattice points on each side of the triangle using the gcd. On the side $OA$, there are $\gcd(a_1, a_2) = a$ boundary lattice points including one endpoint. On the side $OB$, there are $\gcd(b_1, b_2) = b$ boundary lattice points and on the side $AB$, there are $\gcd(b_1 - a_1, b_2 - a_1) = c$ boundary lattice points. Therefore, we can calculate the total number of boundary points by the sum $B(T) = gcd(a_1, a_2) + gcd(b_1, b_2) + gcd(b_1 - a_1, b_2 - a_1) = a + b + c$. Because the gcd of two numbers is always greater than or equal to 1, $B(T) \geq 3$.

For the next step of our proof, we need the following Lemma.

**Lemma 1**

For the lattice triangle T with one interior lattice point, ab, ac, and bc are divisors of $B(T)$.

**Proof of Lemma 1**

Let us calculate the area of the triangle using two different methods. By Pick's Theorem, we know $A(T) = \frac{1}{2}B(T)$. We can also calculate the area by using the determinant of the matrix whose columns are two vectors of the triangle. Here, we have chosen the vectors to begin at the origin and have an endpoint at $(a_1, a_2)$ and $(b_1, b_2)$. $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. $A(T) =$



$\frac{1}{2}\left|det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\right| = \frac{1}{2}\left|a_1b_2 - a_2b_1\right|$.

$$\begin{aligned}\frac{1}{2}B(T) = A(T) &= \frac{1}{2}\left|a_1b_2 - a_2b_1\right| \\ &= \frac{1}{2}\left|\frac{a \cdot a_1 b \cdot b_2}{ab} - \frac{a \cdot a_2 b \cdot b_1}{ab}\right| \\ &= \frac{1}{2}(ab)\left|\frac{a_1}{a} \cdot \frac{b_2}{b} - \frac{a_2}{a} \cdot \frac{b_1}{b}\right| \\ &= \frac{1}{2}(ab)k\end{aligned}$$

We have thus arrived at the expression $\frac{1}{2}B(T) = \frac{1}{2}abk$, where $k = \frac{a_1}{a} \cdot \frac{b_2}{b} - \frac{a_2}{a} \cdot \frac{b_1}{b}$. Here, $a = \gcd(a_1, a_2)$, which means $a|a_1$ and $a|a_2$. Similarly, because $b = \gcd(b_1, b_2)$, $b|b_1$ and $b|b_2$. Therefore, k is an integer. Because $\frac{1}{2}B(T) = \frac{1}{2}abk$, $ab|B(T)$.

Let us show this is also true for ac and bc, where $c = \gcd(b_1 - a_1, b_2 - a_2)$.

Let us calculate the area using the vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \end{bmatrix}$. $A(T) = \frac{1}{2}\left|det \begin{bmatrix} a_1 & a_1 - b_1 \\ a_2 & a_2 - b_2 \end{bmatrix}\right| = \frac{1}{2}\left|a_1(a_2 - b_2) - a_2(a_1 - b_1)\right|$. To simplify this expression, let us write $c_1 = a_1 - b_1$ and $c_2 = a_2 - b_2$.

$$\begin{aligned}A(T) &= \frac{1}{2}\left|a_1c_2 - a_2c_1\right| \\ &= \frac{1}{2}\left|\frac{a \cdot a_1 c \cdot c_2}{ac} - \frac{a \cdot a_2 c \cdot c_1}{ac}\right| \\ &= \frac{1}{2}(ac)\left|\frac{a_1}{a} \cdot \frac{c_2}{c} - \frac{a_2}{a} \cdot \frac{c_1}{c}\right| \\ &= \frac{1}{2}(ac)q\end{aligned}$$

$c = \gcd(b_1 - a_1, b_2 - a_2) = \gcd(c_1, c_2)$, which means $c|c_1$ and $c|c_2$. Therefore, q is an integer and $ac|B(T)$.

Finally, let us calculate the area using the vectors $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ and $\begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. $A(T) = \frac{1}{2} \left| det \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix} \right| = \frac{1}{2} \left| b_1c_2 - b_2c_1 \right|$.

$$
\begin{aligned}
A(T) &= \frac{1}{2} \left| b_1c_2 - b_2c_1 \right| \\
&= \frac{1}{2} \left| \frac{b \cdot b_1 c \cdot c_2}{bc} - \frac{b \cdot b_2 c \cdot c_1}{bc} \right| \\
&= \frac{1}{2}(bc) \left| \frac{b_1}{b} \cdot \frac{c_2}{c} - \frac{b_2}{b} \cdot \frac{c_1}{c} \right| \\
&= \frac{1}{2}(bc)r
\end{aligned}
$$

$b|b_1$, $b|b_2$, $c|c_1$, and $c|c_2$. Therefore, r is an integer and $bc|B(T)$. Thus, we have shown that $ab, ac, bc|B(T)$.

For the next step of our proof, let us introduce a second Lemma.

**Lemma 2**

If the lattice triangle T has one interior lattice point, $B(T)$ divides 6, 8, or 9.

**Proof of Lemma 2**

By Exercise 32, we showed that $B(T) = a + b + c$. Without loss of generality, let us assume $a \geq b \geq c$. Then, $a \geq \frac{1}{3}B(T)$. [4] We know $a|B(T)$, which means there is some integer s such that $a \cdot s = B(T)$. $a \geq \frac{1}{3}B(T)$, so we know $s \leq 3$. $b + c \neq 0$, so we know $s \neq 1$ because $a \neq B(T)$. Therefore, there are only two possible values for s, 2 and 3. $a = \frac{1}{2}B(T)$ and $a = \frac{1}{3}B(T)$. Let us consider each case separately.

**Case 1:** $a = \frac{1}{3}B(T)$**.**

$a+b+c = B(T)$, so $b+c = \frac{2}{3}B(T)$. $b|B(T)$, so there is some integer t such that $b \cdot t = B(T)$. Because $b > c$, $b \geq \frac{B(T)}{3}$, which means $m \leq 3$. $m \neq 1$ because then $b + c$ will b greater than $\frac{2}{3}B(T)$ and $m \neq 2$ because then $b > a$, which is not true. Therefore, the only possible value for b is $b = \frac{B(T)}{3}$, which means $c = \frac{B(T)}{3}$.

**Case 2:** $a = \frac{1}{2}B(T)$**.**

$a + b + c = B(T)$, so $b + c = \frac{1}{2}B(T)$. Again, $b|B(T)$, so there is some integer u such that $b \cdot u = B(T)$. Because $b > c$, $b \geq \frac{B(T)}{4}$. Moreover, because $c \neq 0$, $b \neq \frac{B(T)}{2}$. Therefore, $b = \frac{B(T)}{3}$, which means $c = \frac{B(T)}{6}$ or $b = \frac{B(T)}{4}$, which means $c = \frac{B(T)}{4}$.

[4] If $a < \frac{1}{3}B(T)$, then $b < \frac{1}{3}B(T)$ and $c < \frac{1}{3}B(T)$ because $a > b > c$. Thus, their sum cannot equal $B(T)$, $a + b + c < B(T)$.

Thus, the only possible values for a, b, and c are

1. $a = \frac{B(T)}{3}$, $b = \frac{B(T)}{3}$, and $c = \frac{B(T)}{3}$.

2. $a = \frac{B(T)}{2}$, $b = \frac{B(T)}{3}$, and $c = \frac{B(T)}{6}$.

3. $a = \frac{B(T)}{2}$, $b = \frac{B(T)}{4}$, and $c = \frac{B(T)}{4}$.

Recall from Lemma 1 that ab, ac, and bc are divisors of $B(T)$, where $a = \gcd(a_1, a_2)$, $b = \gcd(b_1, b_2)$, and $c = \gcd(a_1 - b_1, a_2 - b_2)$, which means $a, b, c \in \mathbb{Z}$.

$ab|B(T)$. From the first triplet, we get $\left(\frac{B(T)}{3}\right)\left(\frac{B(T)}{3}\right)|B(T)$, which means $\frac{1}{9}(B(T))^2|B(T)$, which means there is some integer u such that $\frac{1}{9}B(T)^2u = B(T)$, $B(T)u = 9$, which means $B(T)|9$. This means $B(T)$ can be 1, 3, or 9 and because $B(T) \geq 3$, $B(T)$ may be 3 or 9.

From the second triplet, we get $\left(\frac{B(T)}{2}\right)\left(\frac{B(T)}{3}\right)|B(T)$, which means $\frac{1}{6}(B(T))^2|B(T)$, which means there is some integer v such that $\frac{1}{6}B(T)^2v = B(T)$, $B(T)v = 6$, which means $B(T)|6$. $B(T)$ can be 1, 2, 3, or 6. But because T is a triangle, $B(T) \geq 3$, which means $B(T)$ may be 3 or 6.

From the third triplet, we get $\left(\frac{B(T)}{2}\right)\left(\frac{B(T)}{4}\right)|B(T)$, which means $\frac{1}{8}(B(T))^2|B(T)$, which means there is some integer w such that $\frac{1}{8}B(T)^2w = B(T)$, $B(T)w = 8$, which means $B(T)|8$. This means $B(T)$ can be 1, 2, 4, or 8 and because $B(T) \geq 3$, $B(T)$ may be 4 or 8.

Now, let us use the conditions $bc|B(T)$.

1. $a = \frac{B(T)}{3}$, $b = \frac{B(T)}{3}$, and $c = \frac{B(T)}{3}$.
2. $a = \frac{B(T)}{2}$, $b = \frac{B(T)}{3}$, and $c = \frac{B(T)}{6}$.
3. $a = \frac{B(T)}{2}$, $b = \frac{B(T)}{4}$, and $c = \frac{B(T)}{4}$.

For the first triplet, $a = b = c = \frac{B(T)}{3}$, so using the argument $\left(\frac{B(T)}{3}\right)\left(\frac{B(T)}{3}\right)|B(T)$ results in the same values we derived using the condition $ab|B(T)$. $B(T)$ may be 3 or 9.

From the second triplet, we get $\left(\frac{B(T)}{3}\right)\left(\frac{B(T)}{6}\right)|B(T)$, which means $\frac{1}{18}(B(T))^2|B(T)$, which means there is some integer v such that $\frac{1}{18}B(T)^2 v = B(T)$, $B(T)v = 18$, which means $B(T)|18$. $B(T)$ can be 1, 2, 3, 6, 9, and 18. But because T is a triangle, $B(T) \geq 3$, which means $B(T)$ may be 3, 6, 9, or 18.

From the third triplet, we get $\left(\frac{B(T)}{4}\right)\left(\frac{B(T)}{4}\right)|B(T)$, which means $\frac{1}{16}(B(T))^2|B(T)$, which means there is some integer w such that $\frac{1}{16}B(T)^2 w = B(T)$, $B(T)w = 16$, which means $B(T)|16$. This means $B(T)$ can be 1, 2, 4, 8, or 16 and because $B(T) \geq 3$, $B(T)$ may be 4, 8, or 16. Thus, we get $B(T)$ can be 3, 4, 6, 8, 9, 16, or 18.

Now, let us use the conditions $ac|B(T)$.

1. $a = \frac{B(T)}{3}$, $b = \frac{B(T)}{3}$, and $c = \frac{B(T)}{3}$.
2. $a = \frac{B(T)}{2}$, $b = \frac{B(T)}{3}$, and $c = \frac{B(T)}{6}$.
3. $a = \frac{B(T)}{2}$, $b = \frac{B(T)}{4}$, and $c = \frac{B(T)}{4}$.

For the first triplet, $a = b = c = \frac{B(T)}{3}$, so we get the same values as before, namely 3 or 9.

From the second triplet, we get $\left(\frac{B(T)}{2}\right)\left(\frac{B(T)}{6}\right)|B(T)$, which means $\frac{1}{12}(B(T))^2|B(T)$, which means there is some integer v such that $\frac{1}{12}B(T)^2v = B(T)$, $B(T)v = 12$, which means $B(T)|12$. $B(T)$ can be 1, 2, 3, 4, 6, or 12. But because T is a triangle, $B(T) \geq 3$, which means $B(T)$ may be 3, 4, 6, or 12.

From the third triplet, we get $\left(\frac{B(T)}{2}\right)\left(\frac{B(T)}{4}\right)|B(T)$, which means $\frac{1}{8}(B(T))^2|B(T)$, which means there is some integer w such that $\frac{1}{8}B(T)^2w = B(T)$, $B(T)w = 8$, which means $B(T)|8$. This means $B(T)$ can be 1, 2, 4, or 8 and because $B(T) \geq 3$, $B(T)$ may be 4 or 8.

We got three different sets of possible values of $B(T)$. Because ab, bc, and ac are all divisors of $B(T)$, we need to take the intersection of the three sets, $3, 4, 6, 8, 9 \cap 3, 4, 6, 8, 9, 16, 18 \cap 3, 4, 6, 8, 9, 12 = 3, 4, 6, 8, 9$.

Thus, the only possible values of $B(T)$ are 3, 4, 6, 8, or 9. An example of a triangle with 1 interior lattice point and each of the possible boundary lattice points are shown below.

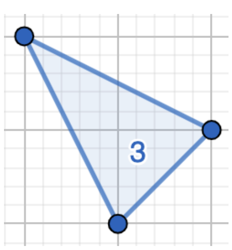

Figure 76: Triangle with $I(T) = 1$ and $B(T) = 3$

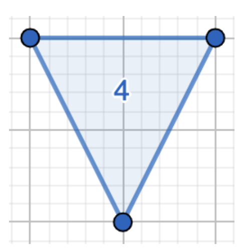

Figure 77: Triangle with $I(T) = 1$ and $B(T) = 4$

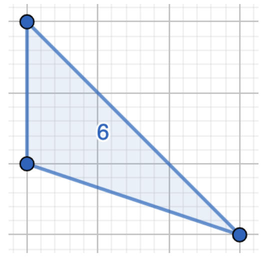


Figure 78: Triangle with $I(T) = 1$ and $B(T) = 6$

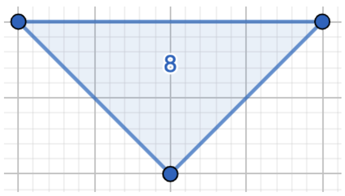


Figure 79: Triangle with $I(T) = 1$ and $B(T) = 8$

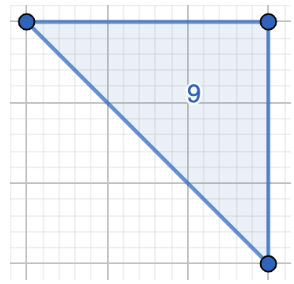


Figure 80: Triangle with $I(T) = 1$ and $B(T) = 9$

# 7 Farey Sequences

**Definition 36 Farey Sequence.**

The Farey sequence of order $n$, denoted $F_n$ is the sequence of completely reduced fractions between 0 and 1 which, in lowest terms, have denominators less than or equal to n, arranged in order of increasing size.

The first five Farey sequences are shown below.

$$F_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}$$
$$F_2 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}$$
$$F_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$$
$$F_4 = \left\{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\right\}$$
$$F_5 = \left\{\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}\right\}$$

**Exercise 86**

Find $F_6$ and $F_7$.

**Solution**

$$F_6 = \left\{\frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1}\right\}$$
$$F_7 = \left\{\frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1}\right\}$$

**Exercise 87 Properties of Farey Sequences.**

Prove each of the following statements.

(a) $F_n$ contains $F_k$ for all $k \leq n$.

(b) Let $|F_n|$ denote the number of fractions in $F_n$. For $n > 1$, $|F_n|$ is odd.

**Solution**

(a) This is true by the definition of the Farey Sequence. Each fraction in $F_k$ is less than 1 and in its reduced form has a denominator less than or equal to k. Each fraction in $F_n$ is less than 1 and in its reduced form has a denominator less than or equal to n. Because $k \leq n$, each fraction in $F_k$ is also in $F_n$.

(b) Let us prove this using induction.

**Base case:** $n = 2$ and $F_2 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}$. $|F_2| = 3$, which is odd.

**Inductive Step:** Suppose $|F_k|$ is odd for $k > 1$. We want to show that $|F_{k+1}|$ is also odd. From part a, we know $F_k$ is contained in $F_{k+1}$. This means we need to show that $|F_{k+1}| - |F_k|$ is even. The set $F_{k+1} \backslash F_k$ contains all non-reducible fractions that are less than 1 and have denominator equal to $k+1$. Let $S_{k+1}$ be the set of all fractions with a denominator equal to $k+1$ including reducible fractions. $S_{k+1} = \left\{\frac{0}{k+1}, \frac{1}{k+1}, ...\frac{k+1}{k+1}\right\}$ and $|S_{k+1}| = k+2$. $S_{k+1}$ contains $F_{k+1} \backslash F_k$ and we can remove the reducible fractions from $S_{k+1}$ to get $F_{k+1} \backslash F_k$. Let $\frac{a}{k+1}$ be a reducible fraction where $0 \leq a \leq k+1$. There exist integers q, r, and t such that $a = qr$ and $k+1 = qt$ for positive integers $r < a$ and $t < k+1$. This means $\frac{(k+1)-a}{k+1}$ is also reducible because $\frac{(k+1)-a}{k+1} = \frac{qt-qr}{qt}$. This means reducible fractions come in pairs. One exception is the fraction $\frac{1}{2}$ because $\frac{1}{2} = \frac{2-1}{2}$.

When $k+2$ is even, $S_{k+1}$ contains an even number of fractions and of those, an even number is reducible. Therefore, $|F_{k+1} \backslash F_k|$ is even.

When $k+2$ is odd, $S_{k+1}$ contains an odd number of fractions. It also has an odd number of reducible fractions because it contains the non reduced form of $\frac{1}{2}$. This means $|F_{k+1} \backslash F_k|$ is even, which means $|F_{k+1}|$ is odd. Thus, by the Principle of Mathematical Induction, $|F_k|$ is odd for all $k > 1$.

**Definition 37 Euler's phi function or Euler's totient function.**

Let $\phi(n)$ denote the number of positive integers less than or equal to n that are relatively prime to n, i.e. the number of integers d such that $a \leq d \leq n$ and $\gcd(d, n) = 1$. For example, $\phi(9) = 6$ because the six integers $1, 2, 4, 5, 7, 8$ are relatively prime to 9 byt 3, 6, and 9 are not. As another example, $\phi(10) = 4$ because the four integers 1,3,7,9 are relatively prime to 10, but 2,4,5,6,8,10 are not.

**Exercise 88**

Show that $|F_n| = |F_{n-1}| + \phi(n)$.

**Solution**

The set $F_n \backslash F_{n-1}$ contains all non-reducible fractions whose denominator is equal to n. For the fraction $\frac{a}{n}$ to be non reducible, $\gcd(a, n) = 1$. Because $\phi(n)$ is the number of integers that are less than n and relatively prime to n, $|F_n| = |F_{n-1}| + \phi(n)$.

**Exercise 89 The Mediant Property.**

Unfortunately, addition of fractions is not as easy as we would like it to be. For example, $\frac{1}{5} + \frac{1}{3} \neq \frac{1+1}{5+3} = \frac{1}{4}$.

(a) Looking at the Farey sequences $F_4$ and $F_5$, how does $\frac{1}{4}$ relate to $\frac{1}{5}$ and $\frac{1}{3}$?

(b) Can you find other Farey sequences in which you observe this phenomena? In particular, choose a Farey sequence $F_n$ and choose 3 consecutive terms of $F_n$, say $\frac{p_1}{q_1}$, $\frac{p_2}{q_2}$, $\frac{p_3}{q_3}$. Compute $\frac{p_1+p_3}{q_1+q_3}$ What do you observe?

**Solution**

(a) In the Farey sequence $F_5$, $\frac{1}{4}$ is between the fractions $\frac{1}{5}$ and $\frac{1}{3}$.

(b) I have chosen the three fractions $\frac{1}{3}$, $\frac{2}{5}$, and $\frac{3}{7}$ in $F_7$. $\frac{1+3}{3+7} = \frac{4}{10} = \frac{2}{5}$. The fraction obtained by adding the numerators and denominators of the two outer fractions and simplifying it results in the middle fraction.

**Exercise 90**

(a) The fractions $\frac{2}{5}$ and $\frac{3}{7}$ are adjacent terms of the Farey sequence $F_7$. Compute $5 \cdot 3 - 2 \cdot 7$.

(b) Choose two other adjacent terms $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ of $F_7$ and compute $p_2 q_1 - p_1 q_2$.

(c) Choose two other adjacent terms $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ of $F_5$ and compute $p_2 q_1 - p_1 q_2$.

(d) Suppose that $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ are two successive terms of a Farey sequence $F_n$. Make a conjecture about the value of $p_2 q_1 - p_1 q_2$. We will use Pick's Theorem to prove this conjecture!

**Solution**

(a) $5 \cdot 3 - 2 \cdot 7 = 1$.

(b) I chose the fractions $\frac{4}{5}$ and $\frac{5}{6}$. $p_2 q_1 - p_1 q_2 = 5 \cdot 5 - 4 \cdot 6 = 1$.

(c) I chose the fractions $\frac{3}{5}$ and $\frac{2}{3}$. $2 \cdot 5 - 3 \cdot 3 = 1$.

(d) I predict that for all successive fractions $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$, $p_2 q_1 - p_1 q_2 = 1$.

**Exercise 91**

Suppose that $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ are two successive terms of $F_n$. In this problem, we will use Pick's Theorem to prove that $p_2 q_1 - p_1 q_2 = 1$. Let T be a triangle with vertices $(0,0)$, $(p_1, q_1)$, and $(p_2, q_2)$.

(a) Show that the only boundary points of T are the vertices of the triangle, i.e. $B(T) = 3$.

(b) Show that T has no lattice points in its interior, i.e. $I(T) = 0$

(c) Conclude, using Pick's Theorem, that $A(T) = \frac{1}{2}$.

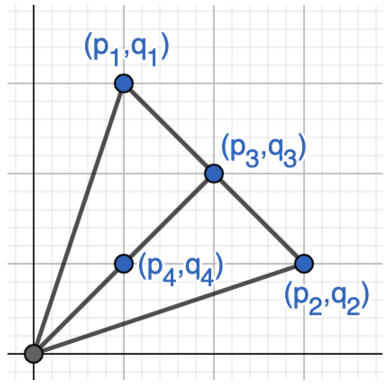


Figure 81: Visual Depiction

(d) Use geometry to show that $A(T) = \frac{1}{2}(p_2q_1 - p_1q_2)$.

(e) Conclude that $p_2q_1 - p_1q_2 = 1$.

**Solution**

(a) $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ are two terms of $F_n$. This means the fractions are in their reduced form and $\gcd(p_1, q_1) = 1$ and $\gcd(p_2, q_2) = 1$. In Exercise 29, we showed that there are no lattice points strictly between the origin and a point $Q = (m, n)$ if $\gcd(m, n) = 1$. This means there are no boundary lattice points on the line segments connecting the origin to points $(p_1, q_1)$ and $(p_2, q_2)$. Now, suppose there is a lattice point on the line segment connecting points $(p_1, q_1)$ and $(p_2, q_2)$. Let us label this point $(p_3, q_3)$. If $\gcd(p_3, q_3) = 1$, because $q_3 < q_1 \leq n$, $\frac{p_3}{q_3}$ should be in the Farey sequence and the point $(p_3, q_3)$ is a boundary lattice point. However, this results in a contradiction. The line segment connecting the origin to point $(p_1, q_1)$ has slope $\frac{q_1}{p_1}$ and the line segment connecting the origin to point $(p_2, q_2)$ has slope $\frac{q_2}{p_2}$. Because $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ are two successive terms of $F_n$, $\frac{p_1}{q_1} < \frac{p_2}{q_2}$. This means $\frac{q_1}{p_1} > \frac{q_2}{p_2}$. If $(p_3, q_3)$ is a boundary lattice point, the slope of the line segment connecting the origin to $(p_3, q_3)$ is between $\frac{q_1}{p_1}$ and $\frac{q_2}{p_2}$. $\frac{p_1}{q_1} < \frac{p_3}{q_3} < \frac{p_2}{q_2}$. Taking the inverse of each fraction, we get $\frac{q_2}{p_2} < \frac{q_3}{p_3} < \frac{q_1}{p_1}$, which means $\frac{q_3}{p_3}$ is in the Farey sequence between $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ which is a contradiction because $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ are two successive terms of $F_n$.

(b) If $\gcd(p_3, q_3) \neq 1$, the point $(p_4, q_4)$ is in the triangle such that the slope of the line segment connecting $(p_4, q_4)$ to the origin is equal to the slope of the line segment connecting $(p_3, q_3)$ to the origin. We have already shown that it is not possible for this reduced fraction to be in the Farey sequence because $\frac{q_1}{p_1}$ and $\frac{q_2}{p_2}$ are successive terms of $F_n$. Therefore, the triangle has no interior lattice points.

(c) Using Pick's Theorem, $A(T) = \frac{B(T)}{2} + I(T) - 1 = \frac{3}{2} + 0 - 1 = \frac{1}{2}$.

(d) We can construct two vectors, one from the origin to point $(p_1, q_1)$ and the other from the origin to point $(p_2, q_2)$. Let us construct the parallelogram used in the Parallelogram Law of Vector Addition. The area of the parallelogram is equal to the determinant of the matrix composed of the two vectors, see Appendix C. The area of the triangle is equal to half of the area of the parallelogram. The matrix is $\begin{bmatrix} p_2 & p_1 \\ q_2 & q_1 \end{bmatrix}$ and the determinant is $\begin{vmatrix} p_2 & p_1 \\ q_2 & q_1 \end{vmatrix} = p_2q_1 - p_1q_2$. This means the area of the triangle is $\frac{1}{2}(p_2q_1 - p_1q_2)$

(e) From part c, we know the area of the triangle is $\frac{1}{2}$ and from part d, we know the area of the triangle is $\frac{1}{2}(p_2q_1 - p_1q_2)$. Equating these two values, we get $p_2q_1 - p_1q_2 = 1$.

**Exercise 92**

Prove that if $0 < \frac{a}{b} < \frac{c}{d} < 1$, then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

**Solution**

From the inequality $\frac{a}{b} < \frac{c}{d}$, we get $ad < bc$. By adding $ab$ to both sides, we get the inequality $ad + ab < bc + ab$, which we can rewrite as $a(b+d) < b(a+c)$ and $\frac{a}{b} < \frac{a+c}{b+d}$, which is the first inequality. Alternatively, we could have added $dc$ to both sides of the inequality $ad < bc$ to get $ad + dc < bc + dc$. We can simplify this expression as $d(a+c) < c(b+d)$ and rewrite it as $\frac{a+c}{b+d} < \frac{c}{d}$, which is the second inequality, completing the proof.

**Exercise 93**

Prove that if $\frac{a}{b}$ and $\frac{c}{d}$ are adjacent in some $F_n$, then $\gcd(a+c, b+d) = 1$.

**Solution**

Let us construct the vectors $\vec{v_1} = \langle a, b\rangle$ and $\vec{v_2} = \langle c, d\rangle$. Because the two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are adjacent in some $F_n$, we know the triangle whose vertices are the origin and the endpoints of the two vectors is a primitive lattice triangle by the results of Exercise 91. If we extend the triangle into a parallelogram with the addition of the vertex $(a+c, b+d)$, it will be a primitive lattice parallelogram as shown in Exercise 58. This means there are no points on the line segment connecting the origin to point $(a+c, b+d)$. Using the result of Exercise 29, because there are no lattice points strictly between the origin and point $(a+c, b+d)$, $\gcd(a+c, b+d) = 1$.

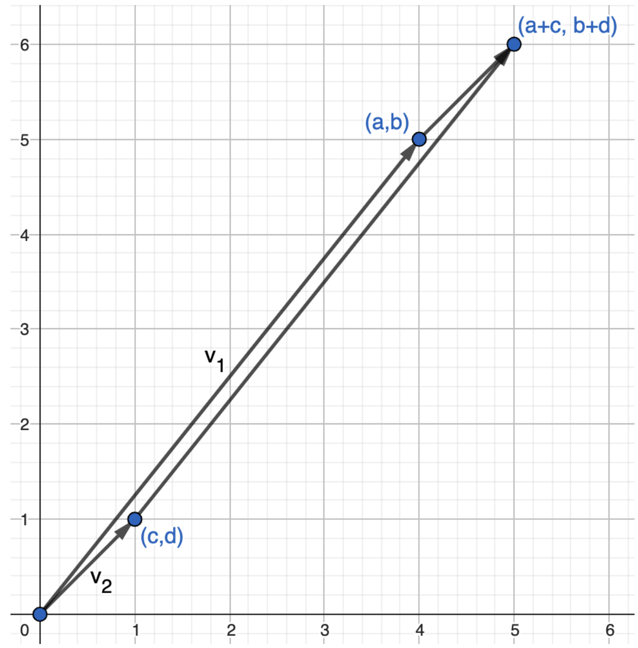


Figure 82: Primitive Lattice Parallelogram

We thus have the following algorithm for computing $F_n$ using $F_{n-1}$:

**Algorithm 1 How to Compute $F_n$ using $F_{n-1}$.**

1. Copy $F_{n-1}$ in order.

2. Insert the mediant fraction $\frac{a+c}{b+d}$ between $\frac{a}{b}$ and $\frac{c}{d}$ if $b+d \leq n$. (If $b+d > n$, the mediant $\frac{a+c}{b+d}$ will appear in a later sequence).

**Exercise 94**

Use Algorithm 1 to compute $F_8$ using $F_7$.

$F_7 : \left\{\frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1}\right\}$

**Solution**

$F_8 : \left\{\frac{0}{1}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{1}{1}\right\}$

The fractions that have been added include: $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, and $\frac{7}{8}$.

**Exercise 95**

Without listing out all of the fractions in $F_{100}$, find the fraction $\frac{a}{b}$ immediately before and the fraction $\frac{c}{d}$ immediately after $\frac{61}{79}$ in $F_{100}$.

**Solution**

According to the Mediant Property, we know $\frac{a+c}{b+d} = \frac{61}{79}$. Because $100(\text{mod} 79) \equiv 21$, which is smaller than 100, we know the fraction $\frac{a+c}{b+d}$ must be in reduced form. Therefore, $a + c = 61$ and $b + d = 79$.

Moreover, from the results of Exercise 91, we know $61b - 79a = 1$ and $79c - 61d = 1$. Therefore, we have four equations with four unknowns. We can write this as a product of a matrix and a vector as follows.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -79 & 61 & 0 & 0 \\ 0 & 0 & 79 & -61 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 61 \\ 79 \\ 1 \\ 1 \end{bmatrix}$$

Let us label the rows of the augmented coefficient matrix as $R_1$ through $R_4$ and perform the following row operations:

$$R_3 \to R_3 + 79 \times R_1$$

$$R_3 \to R_3 - 61 \times R_2$$

$$R_4 \to R_4 - R_3$$

$$R_1 \to R_1 - \frac{R_3}{79}$$

$$R_3 \to \frac{R_3}{79}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 61 \\ 0 & 1 & 0 & 1 & 79 \\ -79 & 61 & 0 & 0 & 1 \\ 0 & 0 & 79 & -61 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 61 \\ 0 & 1 & 0 & 1 & 79 \\ 0 & 61 & 79 & 0 & 4820 \\ 0 & 0 & 79 & -61 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 61 \\ 0 & 1 & 0 & 1 & 79 \\ 0 & 0 & 79 & -61 & 1 \\ 0 & 0 & 79 & -61 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 & 61 \\ 0 & 1 & 0 & 1 & 79 \\ 0 & 0 & 79 & -61 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \frac{61}{79} & \frac{4818}{79} \\ 0 & 1 & 0 & 1 & 79 \\ 0 & 0 & 79 & -61 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \frac{61}{79} & \frac{4818}{79} \\ 0 & 1 & 0 & 1 & 79 \\ 0 & 0 & 79 & -61 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 0 & \frac{61}{79} & \frac{4818}{79} \\ 0 & 1 & 0 & 1 & 79 \\ 0 & 0 & 1 & -\frac{61}{79} & \frac{1}{79} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, there are an infinite number of possible solutions for the system of four equations.

$$a = \frac{4818 - 61d}{79}$$
$$b = 79 - d$$
$$c = \frac{61d + 1}{79}$$

However, there are certain bounds we can consider. First, because there are no terms in $F_{100}$ between the two fractions $\frac{a}{b}$ and $\frac{61}{79}$, we know $b + 79 > 100$ or $b > 21$. Similarly, because there are no terms in $F_{100}$ between the two fractions $\frac{61}{79}$ and $\frac{c}{d}$, $d > 21$. Using the equation $b = 79 - d$, we get a bound for d as follows $22 \leq d \leq 58$. We also know the values a, b, c, and d must be integers because $\frac{a}{b}$ and $\frac{c}{d}$ are fractions in the Farey sequence. Substituting the smallest possible term for $d = 22$, we get the solution $a = 44$, $b = 57$, $c = 17$, and $d = 22$. Because the Farey sequence has a specific order, the solution is unique. $\frac{a}{b} = \frac{44}{57}$ and $\frac{c}{d} = \frac{17}{22}$.

**Exercise 96**

Let $b_j$ be the ordered denominators of $F_n$. Find, with proof, each of the following sums.

(a)
$$\sum_{j=1}^{|F_n|-1} \frac{b_j}{b_{j+1}}$$

(b)
$$\sum_{j=1}^{|F_n|-1} \frac{1}{b_j b_{j+1}}$$

**Solution**

(a) Let us consider three consecutive terms in the Farey sequence. $\frac{a_1}{b_1}$, $\frac{a_2}{b_2}$, and $\frac{a_3}{b_3}$, where $\frac{a_2}{b_2}$ is a newly added term. Before this term was added, we will include the fraction $\frac{b_1}{b_3}$ in the summation. After the fraction $\frac{a_2}{b_2}$ is added, we will include the sum $\frac{b_1}{b_2}+\frac{b_2}{b_3}$. $\frac{a_3}{b_3}$ is the mediant fraction between $\frac{a_1}{b_1}$ and $\frac{a_3}{b_3}$. Therefore, $b_2 = b_1 + b_3$. Let us calculate the sum

$$\begin{aligned}
\frac{b_1}{b_3}+\frac{b_3}{b_1} &= \frac{b_1}{b_2}+\frac{b_2}{b_1}+\frac{b_2}{b_3}+\frac{b_3}{b_2}\\
&= \frac{b_1}{b_1+b_3}+\frac{b_1+b_3}{b_1}+\frac{b_1+b_3}{b_3}+\frac{b_3}{b_1+b_3}\\
&= \frac{b_1+b_3}{b_1+b_3}+\frac{b_1+b_3}{b_1}+\frac{b_1+b_3}{b_3}\\
&= \frac{b_1+b_3}{b_1+b_3}+1+\frac{b_3}{b_1}+\frac{b_1}{b_3}+1\\
&= \frac{b_3}{b_1}+\frac{b_1}{b_3}+3
\end{aligned}$$

This shows us that the addition of each new term increases the sum by 3. Here, we added both fractions $\frac{b_1}{b_3}$ and $\frac{b_3}{b_1}$. Previously, we showed that each term in the Farey sequence $\frac{a}{b}$, except the one in the middle, has a pair, which is $1-\frac{a}{b}=\frac{b-a}{b}$. Therefore, $\sum_{j=1}^{|F_n|-1}\frac{b_j}{b_{j+1}}=\sum_{j=1}^{|F_n|-1}\frac{b_{j+1}}{b_j}$ and by adding both terms, we are adding each term twice. Therefore, we need to divide this sum by 2. Let us label this sum $S_n$.

In Exercise 88, we showed that there are $\psi(n)$ new terms added to expand the Farey sequence $F_{n-1}$ to $F_n$. Therefore, $S_{n+1}=S_n+3\psi(n+1)$.

The Farey sequence $F_1=\left\{\frac{0}{1},\frac{1}{1}\right\}$ has two terms, so $|F_1=2|$ and $S_n=\frac{1}{1}+\frac{1}{1}=2$. Each subsequent Farey sequence has $\psi(n)$ more terms. Therefore, $|F_1=2|=2+\sum_{i=2}^{n}\psi(i)$. Moreover, $S_n=2+3\sum_{i=2}^{n}\psi(i)=2+3(|F_n|-2)=3|F_n|-4$. Dividing this sum by 2, we get $\sum_{j=1}^{|F_n|-1}\frac{b_j}{b_{j+1}}=\frac{3|F_n|-4}{2}$.

(b) In Exercise 91, we showed that for two adjacent fractions in the Farey sequence, $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$, $a_2b_1-a_1b_2=1$. This means $\frac{a_{j+1}}{b_{j+1}}-\frac{a_j}{b_j}=\frac{a_{j+1}b_j-a_jb_{j+1}}{b_jb_{j+1}}=\frac{1}{b_jb_{j+1}}$. Using this, we can rewrite the sum as follows $\sum_{j=1}^{|F_n|-1}\frac{1}{b_jb_{j+1}}=\sum_{j=1}^{|F_n|-1}\left(\frac{a_{j+1}b_j-a_jb_{j+1}}{b_jb_{j+1}}\right)=\frac{a_2}{b_2}-\frac{a_1}{b_1}+\frac{a_3}{b_3}-\frac{a_2}{b_2}+\ldots+\frac{a_{|F_n|}}{b_{|F_n|}}-\frac{a_{|F_n|-1}}{b_{|F_n|-1}}$. All terms except $\frac{a_1}{b_1}$ and $\frac{a_{|F_n|-1}}{b_{|F_n|-1}}$ cancel. Moreover, we know the first term of the Farey sequence is $\frac{a_1}{b_1}=\frac{0}{1}=0$ and the last term of the Farey sequence is

$\frac{a_{|F_n|-1}}{b_{|F_n|-1}} = \frac{1}{1} = 1$. Therefore, $\sum_{j=1}^{|F_n|-1} \frac{1}{b_j b_{j+1}} = 1$.

**Exercise 97**

Show that if $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive terms of $F_n$, then $b + d > n$.

**Solution**

If $b + d \leq n$, the mediant fraction $\frac{a+c}{b+d}$ should be in $F_n$ in between the two fractions $\frac{a}{b}$ and $\frac{c}{d}$ according to Exercise 92. However, this is a contradiction because these are consecutive terms of the Farey sequence.

**Exercise 98**

Show that if $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive terms of $F_n$ and if $n > 1$, then $b + d < 2n$.

**Solution**

Let us suppose $b + d < 2n$ is false. This means $b + d \geq 2n$. There are two possible cases. First, both b and d are equal to n. By Exercise 91, because $\frac{a}{b} = \frac{a}{n}$ and $\frac{c}{d} = \frac{c}{n}$ are consecutive terms of $F_n$, $bc - ad = n(c - a) = 1$. We can rewrite this as $c - a = \frac{1}{n}$. However, because $\frac{a}{b}$ and $\frac{c}{d}$ are irreducible fractions, a, c, b, and d are integers. Therefore, for $c - a = \frac{1}{n}$ to be true, n must be equal to 1, which is a contradiction because we assumed $n > 1$. In the second case, $b + d > 2n$, which means one or both of b and d are greater than n, which is not possible because we are considering fractions that are less than 1 and whose denominators are less than or equal to n.

**Exercise 99. Dirichlet's Theorem on Rational Approximations**

The terms of the Farey sequence $F_n$ partition the interval $[0, 1]$ into subintervals of length at most $\frac{1}{n}$. If $\alpha$ is any real number in $[0, 1]$, then there are consecutive terms $\frac{a}{b}$ and $\frac{c}{d}$ of $F_n$ such that $\alpha \in [\frac{a}{b}, \frac{c}{d}]$. Show that if $\alpha$ is a real number in $[0, 1]$ and if n is a positive integer, then there is a rational number $\frac{h}{k}$ with $0 < k \leq n$ such that $|\alpha - \frac{h}{k}| \leq \frac{1}{k(n+1)}$. This exercise demonstrates that one of $\frac{a}{b}$ or $\frac{c}{d}$ provides a good rational approximation to the real number $\alpha$.

**Solution**

Let us prove the above statement by contradiction. Suppose neither $\frac{a}{b}$ nor $\frac{c}{d}$ satisfy the given property. In other words, $\left|\alpha - \frac{a}{b}\right| > \frac{1}{b(n+1)}$ and $\left|\alpha - \frac{c}{d}\right| > \frac{1}{d(n+1)}$. Using the fact that $\frac{a}{b} \leq \alpha \leq \frac{c}{d}$, we can remove the absolute value bars and write $\alpha - \frac{a}{b} > \frac{1}{b(n+1)}$ and $\frac{c}{d} - \alpha > \frac{1}{d(n+1)}$. We get two different expressions for $\alpha$, $\alpha > \frac{1}{b(n+1)} + \frac{a}{b}$ and $\alpha < \frac{c}{d} - \frac{1}{d(n+1)}$. This tells us $\frac{1}{b(n+1)} + \frac{a}{b} < \frac{c}{d} - \frac{1}{d(n+1)}$. We can rewrite this as $\frac{1}{b(n+1)} + \frac{1}{d(n+1)} < \frac{c}{d} - \frac{a}{b}$. $\frac{(b+d)(n+1)}{bd(n+1)} < \frac{bc-ad}{bd}$. $\frac{b+d}{n+1} < bc - ad$. Because $\frac{a}{b}$ and $\frac{c}{d}$ are two consecutive terms in the Farey sequence, by Exercise 91, $bc - ad = 1$. Therefore, $\frac{b+d}{n+1} < 1$, which means $b + d < n + 1$. Moreover, in Exercise 97, we showed that $n < b + d$. Therefore, $n < b + d < n + 1$, which is not possible because b and d are integers. Thus, one of the two inequalities is not true. This means one of the fractions is a good rational approximation for $\alpha$.

## 8 Ford Circles

**Definition 38 Ford Circle.**

For every rational number $\frac{p}{q}$ in lowest terms, the Ford circle $C(p,q)$ is the circle with center $(\frac{p}{q}, \frac{1}{2q^2})$ and radius $\frac{1}{2q^2}$ . This means that $C(p,q)$ is the circle tangent to the $x$-axis at $x = \frac{p}{q}$ with radius $\frac{1}{2q^2}$. Observe that every small interval of the $x$-axis contains points of tangency of infinitely many Ford circles. Several examples are shown below.

**Exercise 100**

Graph $C(3,4)$ and $C(4,5)$ clearly on the same set of axes. I recommend that you use a graphing software package to do this. What do you observe about these two circles?

**Solution**

$C(3,4)$ is the ford circle with center $(\frac{3}{4}, \frac{1}{32})$ and radius $\frac{1}{32}$ and $C(4,5)$ is the ford circle with center $(\frac{4}{5}, \frac{1}{50})$ and radius $\frac{1}{50}$. The two circles are tangent to each other and tangent to the x-axis.

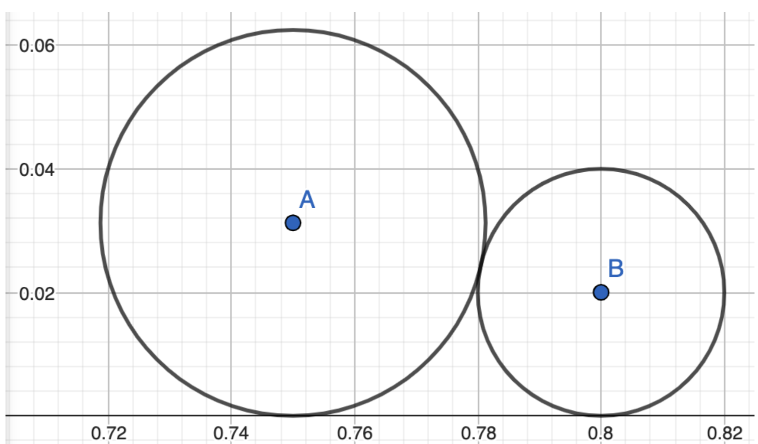


Figure 83: Ford Circles $C(3,4)$ and $C(4,5)$

**Exercise 101**

Choose two fractions $\frac{a}{b}$ and $\frac{c}{d}$ (other than $\frac{3}{4}$ and $\frac{4}{5}$) that are adjacent in $F_6$, and clearly graph $C(a, b)$ and $C(c, d)$ on the same set of axes. What do you observe about the two circles?

**Solution**

I have graphed the Ford circles corresponding to the two fractions $\frac{1}{3}$ and $\frac{2}{5}$, $C(1, 3)$ and $C(2, 5)$. $C(1, 3)$ is the ford circle with center $(\frac{1}{3}, \frac{1}{18})$ and radius $\frac{1}{18}$. $C(2, 5)$ is the ford circle with center $(\frac{2}{5}, \frac{1}{50})$ and radius $\frac{1}{18}$. Like in the previous exercise, the two circles are tangent to each other and to the x-axis.

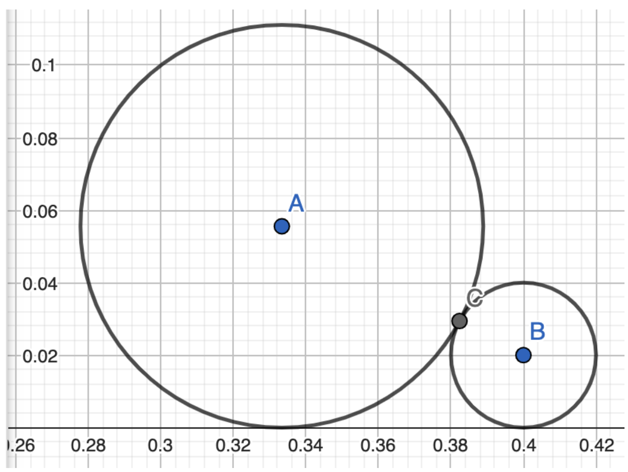


Figure 84: Ford Circles $C(1, 3)$ and $C(2, 5)$

**Exercise 102**

Next, choose two fractions $\frac{a}{b}$ and $\frac{c}{d}$ that are not adjacent in $F_6$, and clearly graph $C(a,b)$ and $C(c,d)$ on the same set of axes. What do you observe about the two circles?

**Solution**

I have graphed the Ford circles corresponding to the two fractions $\frac{2}{5}$ and $\frac{3}{5}$, $C(2,5)$ and $C(3,5)$. $C(2,5)$ is the ford circle with center $(\frac{2}{5}, \frac{1}{50})$ and radius $\frac{1}{50}$. $C(3,5)$ is the ford circle with center $(\frac{3}{5}, \frac{1}{50})$ and radius $\frac{1}{50}$. Unlike the previous two exercises, these two Ford circles are not tangent to each other. Because their corresponding fractions have the same denominator, they have the same radius.

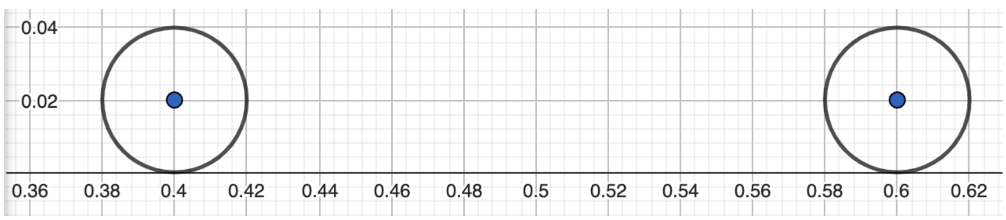


Figure 85: Ford Circles $C(2,5)$ and $C(3,5)$

### Exercise 103

Repeat Exercises 101 and 102 for several additional pairs of fractions in $F_6$ keeping track of what you observe about the circles in the cases where the fractions are adjacent and are not adjacent.

### Solution

I have graphed the Ford circles corresponding to all the fractions in $F_6$. The x coordinate of the circles are the fractions themselves, so the circles are in the same order as the fractions in the sequence $F_6$. We can see the Ford circles corresponding to two adjacent fractions of the Farey sequence are always tangent to each other. However, the Ford circles corresponding to two non-adjacent fractions of the Farey sequence may or may not be tangent to each other.

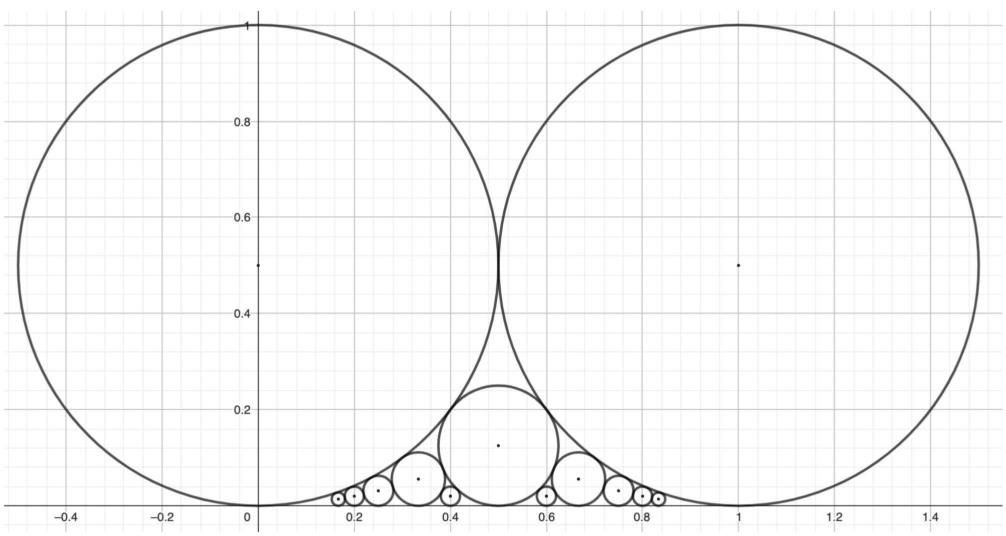


Figure 86: Ford Circles of fractions in Farey Sequence $F_6$

**Exercise 104**

Prove that the representative Ford circles of two distinct fractions are either tangent at one point or wholly external.

**Solution**

We are attempting to show that the representative Ford circles of two distinct fractions in a Farey Sequence intersect at most one point. This means the distance between their centers is equal to or greater than the sum of their radii. Consider two Ford circles $C(a,b)$ and $C(c,d)$. $C(a,b)$ has center $(\frac{a}{b}, \frac{1}{2b^2})$ and radius $\frac{1}{2b^2}$ and $C(c,d)$ has center $(\frac{c}{d}, \frac{1}{2d^2})$ and radius $\frac{1}{2d^2}$. Using the Pythagorean Theorem, the distance between their radii is $\sqrt{(\frac{c}{d} - \frac{a}{b})^2 + (\frac{1}{2d^2} - \frac{1}{2b^2})^2}$ and the sum of their radii is $\frac{1}{2b^2} + \frac{1}{2d^2}$.

$$\begin{aligned}
\sqrt{\left(\frac{c}{d} - \frac{a}{b}\right)^2 + \left(\frac{1}{2d^2} - \frac{1}{2b^2}\right)^2} &\geq \frac{1}{2b^2} + \frac{1}{2d^2} \\
\left(\frac{c}{d} - \frac{a}{b}\right)^2 + \left(\frac{1}{2d^2} - \frac{1}{2b^2}\right)^2 &\geq \left(\frac{1}{2b^2} + \frac{1}{2d^2}\right)^2 \\
\frac{c^2}{d^2} - \frac{2ac}{bd} + \frac{a^2}{b^2} + \frac{1}{4d^4} - \frac{1}{2b^2d^2} + \frac{1}{4b^4} &\geq \frac{1}{4d^4} + \frac{1}{2b^2d^2} + \frac{1}{4b^4} \\
\frac{c^2}{d^2} - \frac{2ac}{bd} + \frac{a^2}{b^2} &\geq \frac{1}{b^2d^2} \\
b^2c^2 - 2abcd + a^2d^2 &\geq 1 \\
(bc - ad)^2 &\geq 1
\end{aligned}$$

Because $a, b, c, d \in \mathbb{Z}$, $(ad - bc)^2 > 0$. $ad - bc \neq 0$ because otherwise we get $\frac{a}{b} = \frac{c}{d}$, which is not true because these are two distinct fractions in a Farey sequence. Therefore, we get $(bc - ad)^2 \geq 1$, which means the distance between the centers of the two ford circles is greater than or equal to the sum of the two radii.

**Exercise 105**

Show that the representative Ford circles of two distinct fractions are tangent at one point precisely when the fractions are adjacent in some Farey sequence $F_n$.

**Solution**

In our proof of Exercise 104, we showed that the distance between the centers of two the circles is equal to the sum of their radii if and only if $ad - bc = 1$. In Exercise 91, we proved that for two successive terms of $F_n$ $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$, $p_2q_1 - p_1q_2 = 1$. Therefore, if $ad - bc = 1$, $\frac{a}{b}$ and $\frac{c}{d}$ are successive terms of a Farey sequence. This completes the proof.

**Exercise 106**

Suppose that $C(a, b)$ and $C(c, d)$ are tangent Ford circles. Prove that $C(a + c, b + d)$ is the unique circle tangent to the real line and to both of the circles $C(a, b)$ and $C(c, d)$. In other words, the circle associated with the mediant fraction, $C(a + c, b + d)$ is the largest circle between $C(a, b)$ and $C(c, d)$.

**Solution**

In Exercise 105, we showed that the Ford circles corresponding to two adjacent fractions, $f_1$ and $f_2$, in the Farey sequence are tangent. We also know that the mediant fraction is adjacent to both fractions $f_1$ and $f_2$. Therefore, the Ford circle corresponding to the mediant fraction is tangent to both circles $C(a, b)$ and $C(c, d)$. Moreover, because its radius is equal to the y-coordinate of its center, it is adjacent to the x-axis. Therefore, it is the largest circle between $C(a, b)$ and $C(c, d)$ when bounded by the x-axis.

Shown below are the Ford Circles $C(1,3)$ and $C(2,5)$ and the Ford Circle corresponding to their mediant fraction $C(3,8)$.

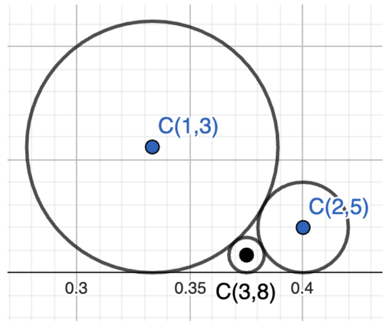


Figure 87: Ford Circles $C(1,3)$, $C(3,8)$, and $C(2,5)$

# 9 Lattice Points In and On a Circle

### Exercise 107

Prove that no two lattice points are the same distance from the point $\left(\sqrt{2}, \frac{1}{3}\right)$.

### Solution

Suppose there are two lattice points with coordinates $(x_1, y_1)$ and $(x_2, y_2)$ such that the coordinates $x_1, x_2, y_1, y_2 \in \mathbb{Z}$. Let us show that they cannot be the same distance from the point $\left(\sqrt{2}, \frac{1}{3}\right)$. We can write the distances using the Pythagorean Theorem $\sqrt{(\sqrt{2} - x_1)^2 + (\frac{1}{3} - y_1)^2}$ and $\sqrt{(\sqrt{2} - x_2)^2 + (\frac{1}{3} - y_2)^2}$. Equating them and squaring both sides, we get the following.

$$
\begin{aligned}
(\sqrt{2} - x_1)^2 + (\frac{1}{3} - y_1)^2 &= (\sqrt{2} - x_2)^2 + (\frac{1}{3} - y_2)^2 \\
2 - 2\sqrt{2}x_1 + x_1^2 + \frac{1}{9} - \frac{2}{3}y_1 + y_1^2 &= 2 - 2\sqrt{2}x_2 + x_2^2 + \frac{1}{9} - \frac{2}{3}y_2 + y_2^2 \\
-2\sqrt{2}x_1 + x_1^2 - \frac{2}{3}y_1 + y_1^2 &= -2\sqrt{2}x_2 + x_2^2 - \frac{2}{3}y_2 + y_2^2
\end{aligned}
$$

Rearranging and solving for $\sqrt{2}$, we arrive at the expression $\sqrt{2} = \frac{-x_1^2+\frac{2}{3}y_1-y_1^2+x_2^2-\frac{2}{3}y_2+y_2^2}{-2x_1+2x_2}$, when $x_1 \neq x_2$. Because $x_1, x_2, y_1, y_2 \in \mathbb{Z}$, the expression on the right hand side is rational. Therefore, $\sqrt{2} \in \mathbb{Q}$, which is a contradiction.

If $x_2 = x_1$, the expression simplifies to $0 = \frac{2}{3}y_1 - y_1^2 - \frac{2}{3}y_2 + y_2^2$.

$\frac{2}{3}(y_2 - y_1) = y_2^2 - y_1^2 = (y_2 - y_1)(y_2 + y_1)$.

If $y_2 \neq y_1$, $\frac{2}{3} = y_2 + y_1$. This also results in a contradiction because $y_1$ and $y_2$ are integers.

If $y_2 = y_1$, because $x_1 = x_2$, the two points are the same point. Thus, there do not exist two lattice points that are the same distance from the point $\left(\sqrt{2}, \frac{1}{3}\right)$.

**Exercise 108**

Prove that for every natural number n, there exists in the plane a circle with exactly n lattice points in its interior.

**Solution**

In Exercise 107, we proved that there do not exist two lattice points with equal distance from the point $Q = \left(\sqrt{2}, \frac{1}{3}\right)$. This means each lattice point has a difference distance from Q. Let us order the lattice points according to their distance from Q and label them as $P_1, P_2, ..., P_n, ...$ where $P_1$ is the lattice point that is closest to Q. For each n, we can choose a radius that is greater than the distance from point Q to $P_n$ but smaller than the distance from point Q to $P_{n+1}$. One possible radius is $\frac{|QP_n|+|QP_{n+1}|}{2}$, where $|QP_n|$ and $|QP_{n+1}|$ are the distances between Q and $P_n$ and Q and $P_{n+1}$ respectively. Because we have ordered the points, the circle with radius $\frac{|QP_n|+|QP_{n+1}|}{2}$ will contain the n lattice points $P_1$ to $P_n$.

**Exercise 109**

Show that the result of Exercise 107 holds if the point $(\sqrt{2}, \frac{1}{3})$ is replaced with any point of the form $(\sqrt{e}, \frac{1}{f})$, where e and f are positive integers with $e > 1$ and square-free and $f > 2$.

**Solution**

Similar to our approach in Exercise 107, let us consider two lattice points with coordinates $(x_1, y_1)$ and $(x_2, y_2)$ such that the coordinates $x_1, x_2, y_1, y_2 \in \mathbb{Z}$. Let us show that they cannot be the same distance from the point $\left(\sqrt{e}, \frac{1}{f}\right)$, where e and f are positive integers with $e > 1$ and square-free and $f > 2$. We can write the distances using the Pythagorean Theorem $\sqrt{(\sqrt{e} - x_1)^2 + (\frac{1}{f} - y_1)^2}$ and $\sqrt{(\sqrt{e} - x_2)^2 + (\frac{1}{f} - y_2)^2}$. Equating them and squaring both sides, we get the following.

$$\begin{aligned}
(\sqrt{e} - x_1)^2 + (\frac{1}{f} - y_1)^2 &= (\sqrt{e} - x_2)^2 + (\frac{1}{f} - y_2)^2 \\
e - 2\sqrt{e}x_1 + x_1^2 + \frac{1}{f^2} - \frac{2}{f}y_1 + y_1^2 &= e - 2\sqrt{e}x_2 + x_2^2 + \frac{1}{f^2} - \frac{2}{f}y_2 + y_2^2 \\
-2\sqrt{e}x_1 + x_1^2 - \frac{2}{f}y_1 + y_1^2 &= -2\sqrt{e}x_2 + x_2^2 - \frac{2}{f}y_2 + y_2^2
\end{aligned}$$

Rearranging and solving for $\sqrt{e}$, we arrive at the expression $\sqrt{e} = \frac{-x_1^2+\frac{2}{f}y_1-y_1^2+x_2^2-\frac{2}{f}y_2+y_2^2}{-2x_1+2x_2}$, when $x_1 \neq x_2$. Because $x_1, x_2, y_1, y_2 \in \mathbb{Z}$, the expression on the right hand side is rational. Therefore, $\sqrt{e} \in \mathbb{Q}$. However, this is a contradiction because e is a square free number.

If $x_2 = x_1$, the expression simplifies to $0 = \frac{2}{f}y_1 - y_1^2 - \frac{2}{f}y_2 + y_2^2$.

$\frac{2}{f}(y_2 - y_1) = y_2^2 - y_1^2 = (y_2 - y_1)(y_2 + y_1)$.

If $y_2 \neq y_1$, $\frac{2}{f} = y_2 + y_1$. This also results in a contradiction because $y_1$ and $y_2$ are integers and f is greater than 2.

If $y_2 = y_1$, because $x_1 = x_2$, the two points are the same point. Thus, there do not exist two lattice points that are the same distance from the point $\left(\sqrt{e}, \frac{1}{f}\right)$, where e and f are positive integers with $e > 1$ and square-free and $f > 2$.

**Definition 39**

Let $C(\sqrt{n})$ denote the circle with center $(0, 0)$ and radius $\sqrt{n}$.

**Definition 40**

Let $L(n)$ be the number of lattice points in the interior and on the boundary of the circle $C(\sqrt{n})$.

**Exercise 110**

Find $L(5)$, $L(7)$, and $L(10)$.

**Solution**

The circle with its center at the origin and radius $\sqrt{5}$ has a total of 21 points in its interior and on its boundary as shown in Figure 86 below. The circle with its center at the origin and radius $\sqrt{7}$ also has a total of 21 points in its interior and on its boundary as shown in Figure 87 below. However, this time, all the points are in its interior.

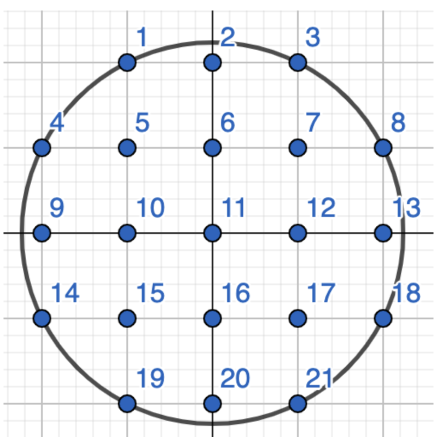


Figure 88: Lattice Points $C(\sqrt{5})$

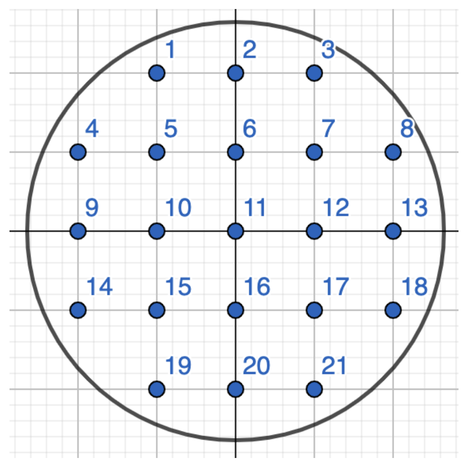


Figure 89: Lattice Points $C(\sqrt{7})$

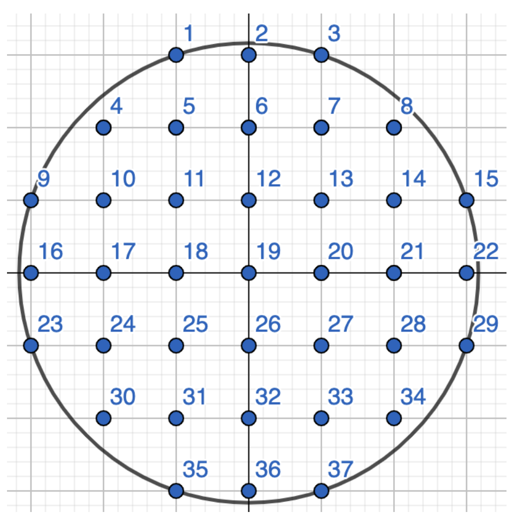


Figure 90: Lattice Points $C(\sqrt{7})$

**Exercise 111**

Let $A(n)$ denote the area of all of the unit lattice squares with horizontal and vertical sides that are cut by the boundary of the circle $C(\sqrt{n})$. Show that $\left|\frac{L(n)}{n} - \pi\right| \leq \frac{A(n)}{n}$

**Solution**

$L(n)$ is the number of lattice points inside the circle. Let us construct a unit lattice square corresponding to each lattice point such that the lower left corner of the square is the lattice point. A construction of $L(n)$ for a circle with radius 4 is shown in Figure 89 below.

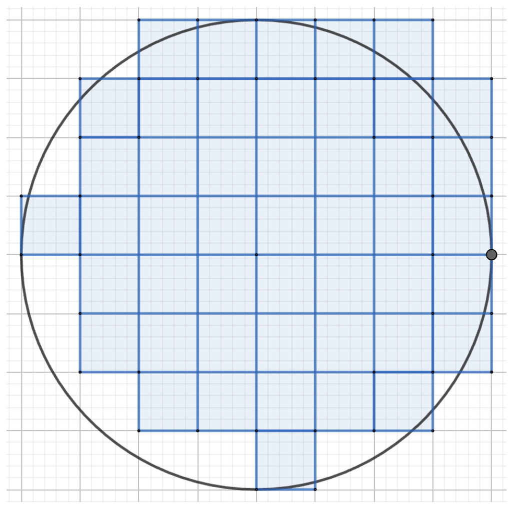

Figure 91: $L(n)$ for $C(4)$

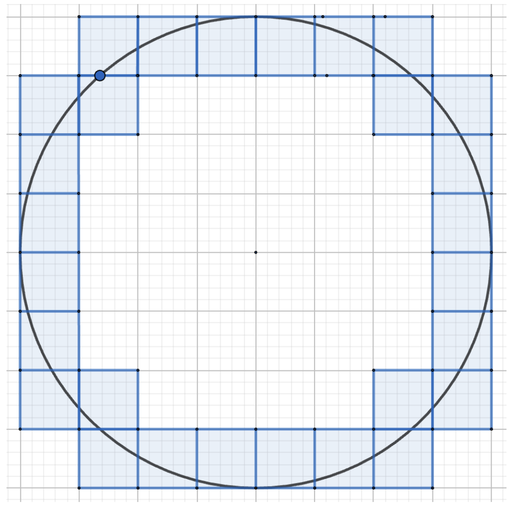

Figure 92: $A(n)$ for $C(4)$

$A(n)$ is the area of the unit lattice squares that intersect with the boundary of the circle. A construction of $A(n)$ for a circle with radius 4 is shown in Figure 90.

The expression $L(n) - A(n)$ is the area of the region of the circle that is comprised of unit squares that are fully within its borders. Because the boundary of the circle has a nonzero curvature, there are some unit squares that intersect the boundary. Thus, this area is less than the area of the circle. In other words, $L(n) - A(n) \geq n\pi$. Now, let us consider the area $L(n) + A(n)$. This can be achieved by including each unit square that is intersecting the boundary of the circle in addition to the unit squares contained within the circle's boundary. This includes regions outside the circle's boundaries, so $L(n) + A(n) \leq n\pi$. Combining the two expressions, we get $L(n) - A(n) \geq n\pi \geq L(n) + A(n)$. Subtracting $L(n)$ from all sides, we can express the inequality as $-A(n) \geq n\pi - L(n) \geq A(n)$. This means $|L(n) - n\pi| \leq A(n)$. Dividing both sides by n, we get $\left|\frac{L(n)}{n} - \pi\right| \leq \frac{A(n)}{n}$.

**Exercise 112**

All of the squares that are cut by the boundary of the circle are contained in an annulus of width $2\sqrt{2}$. Show that the area $R(n)$ of this annulus is $R(n) = 4\pi\sqrt{2n}$.

**Solution**

The largest distance between the boundary of the circle and the edge of the annulus is $\sqrt{2}$. Therefore, we can construct two circles, one with radius $n+\sqrt{2}$ and the other with radius $n-\sqrt{2}$ to completely contain the annulus. The area of the region in the larger circle outside the smaller circle is $\pi((n+\sqrt{2})^2-(n-\sqrt{2})^2) = \pi(n^2+2\sqrt{2n}+2-(n^2-2n\sqrt{2}+2)) = 4\pi\sqrt{2n}$.

**Exercise 113**

Show that $\left|\frac{L(n)}{n}-\pi\right| \leq \frac{4\sqrt{2}\pi}{\sqrt{n}}$

**Solution**

In Exercise 111, we showed that $\left|\frac{L(n)}{n}-\pi\right| \leq \frac{A(n)}{n}$. Here, $A(n)$ is the area of the unit lattice squares that intersect the boundary of the circle. Because not all unit squares in the annulus intersect the circle's boundary, $A(n) < R(n)$. Therefore, $\left|\frac{L(n)}{n}-\pi\right| \leq \frac{A(n)}{n} \leq \frac{R(n)}{n}$, which we can write as $\left|\frac{L(n)}{n}-\pi\right| \leq \frac{4\pi\sqrt{2n}}{n} = \frac{4\pi\sqrt{2}}{\sqrt{n}}$.

**Exercise 114**

Show that $\lim_{n\to\infty} \frac{L(n)}{n} = \pi$.

**Solution**

In Exercise 113, we showed that $\left|\frac{L(n)}{n}-\pi\right| \leq \frac{4\pi\sqrt{2}}{\sqrt{n}}$. Let us take the limit of the expression on the right. $\lim_{n\to\infty}\left(\frac{4\pi\sqrt{2}}{\sqrt{n}}\right) = 0$. Therefore, the limit of $\frac{L(n)}{n}-\pi$, which is smaller than the expression $\frac{4\pi\sqrt{2}}{\sqrt{n}}$, as n approaches infinity is 0. Therefore, $\lim_{n\to\infty} \frac{L(n)}{n} = \pi$.

# 10 Minkowski's Theorem

**Exercise 115**

This problem is an introduction to how Pick's Theorem generalizes in higher dimensions. First, we'll rewrite Pick's Theorem as follows. Let P be a lattice polygon, and let $L(P)$ denote the total number of lattice points in the interior and on the sides of P, so $L(P) = B(P) + I(P)$. Then Pick's Theorem can be restated as follows: $L(P) = A(P) + \frac{1}{2}B(P) + 1$. This generalization of Pick's Theorem describes how $L(P)$ changes as the polygon undergoes dilation by a positive integer. For each positive integer n, we define the lattice polygon $nP$ as $nP = \{nx|x \in P\}$. Prove that $L(nP) = A(P)n^2 + \frac{1}{2}B(P)n + 1$.

**Solution**

To prove this statement, we need to show that the lattice polygon $nP$ has $n$ times the number of boundary points and $n^2$ times the area of the lattice polygon $P$.

Let us first show the former statement. In Exercise 31, we showed that there are exactly $gcd(a, b) - 1$ lattice points on the line segment $L$ between the origin and point $(a, b)$, not including endpoints. Including the endpoint that is not the origin, there are $gcd(a, b)$ points. Now, suppose we take the line segment $nL = nx|x \in L$. The endpoint of this line is $(na, nb)$, which means there are $gcd(na, nb)$ points on the line including $(na, nb)$. If $gcd(a, b) = d$, $d|a$ and $d|b$. If we multiply both a and b by n, $gcd(na, nb) = nd = n \times gcd(a, b)$.

In Exercise 32, we showed that for a lattice n-gon with vertices $p_1 = (a_1, b_1)$, $p_2 = (a_2, b_2)$, ..., $p_n = (a_n, b_n)$, the lattice points on its boundary is given by

$$B(P) = \sum_{i=1}^{n} d_i$$

where $d_i = gcd(a_{i+1} - a_i, b_{i+1} - b_i)$ for i = 1, 2, . . . , $n - 1$ and $d_n = gcd(a_1 - a_n, b_1 - b_n)$. Because each $d_n$ increases by a multiple of n, the total number of boundary points increase by a factor of n.

$$B(nP) = \sum_{i=1}^{n} nd_i = n\sum_{i=1}^{n} d_i = nB(P)$$

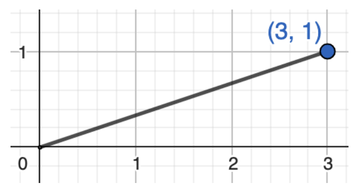


Figure 93: L with endpoint $(3, 1)$

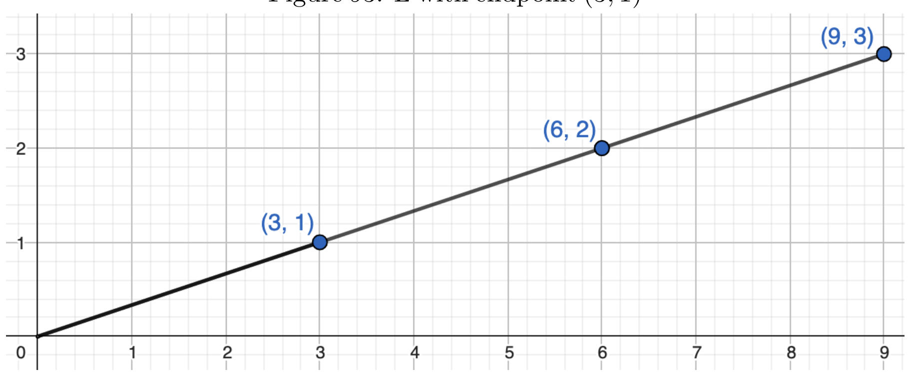


Figure 94: 3L with endpoint $(9, 3)$

In Figures 91 and 92, is an example of a line segment $L$ with endpoints at the origin and point $(3, 1)$ and a line segment $3L$ with endpoints at the origin and point $(9, 3)$. Without including the origin, line segment $L$ has 1 point and line segment $3L$ has 3 points, as expected.

In Exercise 62, we showed that each lattice polygon can be dissected into primitive lattice triangles. Let us dissect the lattice polygon P into primitive lattice triangles. When we enlarge $P$ into $nP$, we enlarge each primitive lattice triangle $T$ into a lattice triangle $nT$. Let us calculate the area of each $nT$ with base b and height h. $A(T) = \frac{bh}{2}$. We showed that the number of lattice points on a side increases by a factor of n. Therefore, for triangle $nT$, $A(nT) = \frac{(nb)(nh)}{2} = \frac{n^2bh}{2} = n^2A(T)$. This means the area of a lattice polygon increases by a factor of $n^2$.

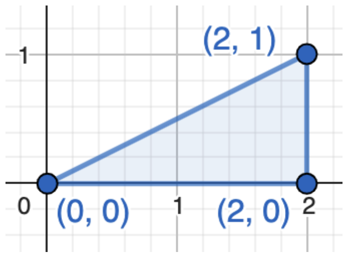

Figure 95: T with vertices $(2, 0)$ and $(2, 1)$

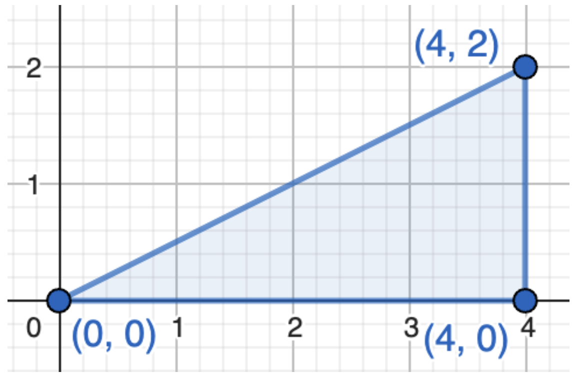

Figure 96: 2T with vertices $(4, 0)$ and $(4, 2)$

In Figure 93 is a triangle $T$ with vertices at the origin and two points $(2, 0)$ and $(2, 1)$. $A(T) = \frac{2\times1}{2} = 1$. In Figure 94 is a triangle $2T$ with vertices at the origin and two points $(4, 0)$ and $(4, 2)$. $A(2T) = \frac{4\times2}{2} = 4 = 2^2A(T)$, as expected.

Therefore, $L(nP) = A(nP) + \frac{1}{2}B(nP) + 1 = A(P)n^2 + \frac{1}{2}B(P)n + 1$

**Exercise 116**

Let P be the triangle with vertices $(0, 0)$, $(3, 1)$, and $(1, 4)$.

(a) Sketch $P$ and $5P$

(b) Compute $L(5P)$ directly using your sketch from part (a).

(c) Compute $L(5P)$ using the result in Exercise 115, and verify that you obtain the same result as in part (b).

**Solution**

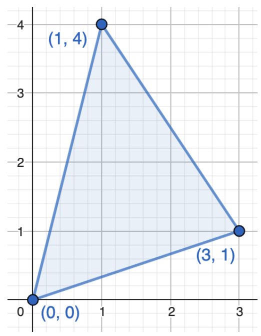


Figure 97: Triangle $P$

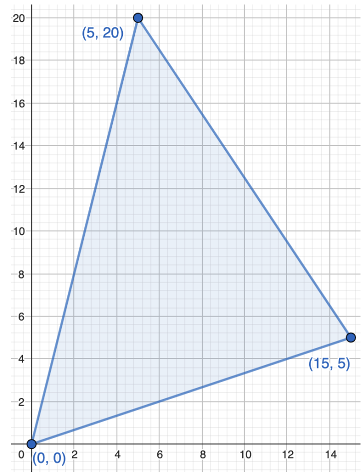


Figure 98: Triangle $5P$

(a)

(b) $L(5P) = A(5P) + \frac{1}{2}B(5P) + 1 = 137.5 + \frac{1}{2}(15) + 1 = 146.$

(c) $L(5P) = A(5P) + \frac{1}{2}B(5P) + 1 = A(P)5^2 + \frac{1}{2}B(P)5 + 1 = (5.5)5^2 + \frac{5\times 3}{2} + 1 = 146$, which is the same result we got in (b).

**Definition 41**

Let $R \subseteq \mathbb{R}^n$. R is convex if for all points x and y in R, the line segment joining x and y is contained in R.

**Definition 42**

Let $R \subseteq \mathbb{R}^n$. The convex hull of R is the intersection of all of the convex sets that contain R. Alternatively, the convex hull of R is the smallest convex set that contains R.

To prove Minkowski's Theorem, we will need the following result, which we will state but not prove.

**Theorem 7**

Let R be a bounded, closed, convex set in $\mathbb{R}^2$ that contains three non-collinear lattice points. Then the convex hull of the set of all lattice points in R is a lattice polygon P that contains the same number of lattice points as R. Moreover, $A(P) \leq A(R)$ and $p(P) \leq p(R)$, where $p(X)$ denotes the perimeter of X.

**Exercise 117**

Illustrate Theorem 7 for the following regions:

(a) A square with sides of length 3 whose center is the point $(0, 0)$.

(b) An equilateral triangle with vertices $(0, 3)$, $(-2.5, 3 - \frac{5\sqrt{3}}{2})$, and $(2.5, 3 - \frac{5\sqrt{3}}{2})$. Note: this is an equilateral triangle with base 5.

(c) A circle with center $(0, 0)$ and radius $\frac{5}{4}$.

(d) A circle with center $(0, 0)$ and radius $\frac{7}{8}$.

### Solution

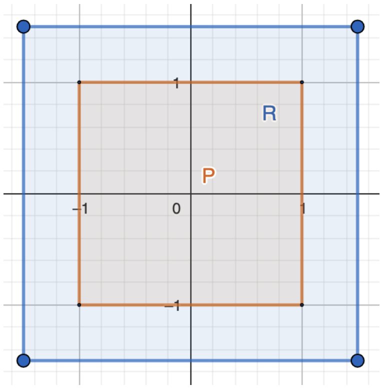


Figure 99: (a) Square with side lengths 3 and R

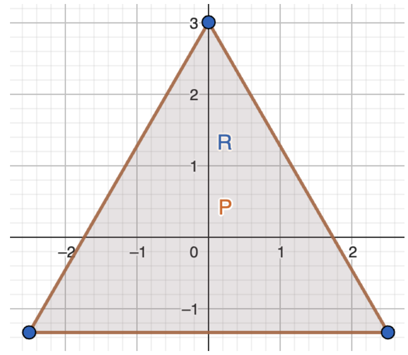


Figure 100: (b) Triangle

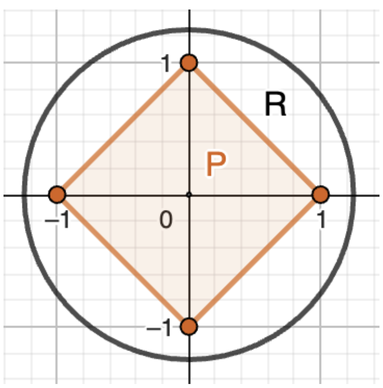


Figure 101: (c) Circle with radius $\frac{5}{4}$

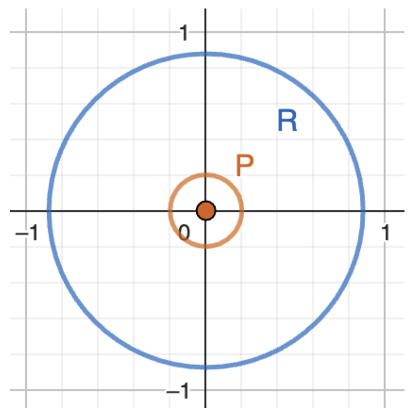


Figure 102: (d) Circle with radius $\frac{7}{8}$

The next exercise, a theorem of Ehrhart, generalizes Pick's Theorem to bounded, convex regions in the plane.

**Exercise 118**

Let R be a bounded, convex region in $\mathbb{R}^2$. Let $L(R)$ denote the total number of lattice points in the interior and boundary of R. $L(R) = B(R) + I(R)$. Use Pick's Theorem to prove that $L(R) \leq A(R) + \frac{1}{2}p(R) + 1$.

**Solution**

According to Theorem 7, there is a polygon with the same number of lattice points in its interior and on its boundary as the region R. Let us name this polygon P. The region R can have a larger area and a larger perimeter than P. In Exercise 115, we determined $L(P) = A(P) + \frac{1}{2}B(P) + 1$. Because $A(R) \geq A(P)$ and $p(R) \geq p(P)$, $L(P) = A(P) + \frac{1}{2}p(A) + 1 \geq A(R) + \frac{1}{2}p(R) + 1$. However, the region R has no more lattice points in or on its boundary. Therefore, $L(P) = L(R)$. Therefore, $L(R) \leq A(R) + \frac{1}{2}p(R) + 1$.

**Exercise 119**

Illustrate Exercise 118 for the same regions as in Exercise 117.

**Solution**

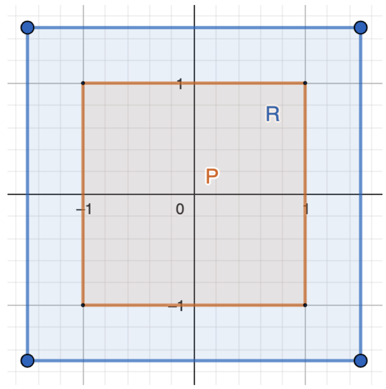


Figure 103: (a) Square with side lengths 3 and R

$A(P) = 9$ but $A(R) > A(P) = 9$. Moreover, $p(P) = 12$ but $p(R) > p(P) = 12$. Therefore, $L(R) \leq A(R) + \frac{1}{2}p(R) + 1$.

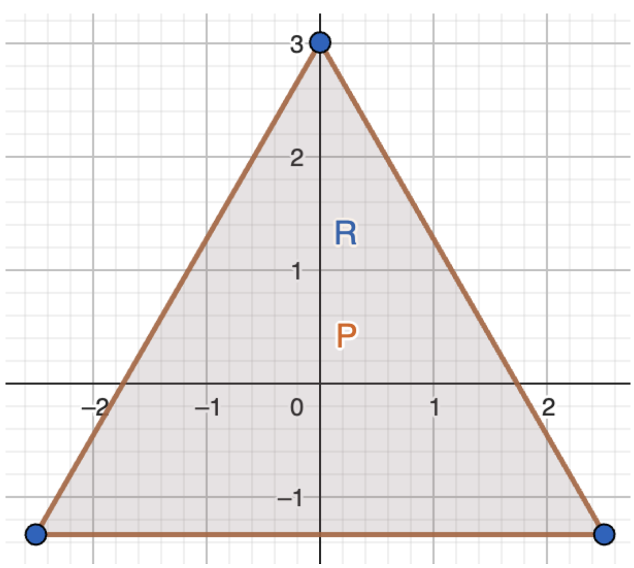


Figure 104: (b) Triangle

$h_P = \frac{5\sqrt{3}}{2}$ and $b_P = 5$. Therefore, $A(P) = \frac{5 \cdot 5\sqrt{3}}{2 \cdot 2} = \frac{25\sqrt{3}}{4}$. However, the area of the region R is larger than the area of the triangle. $A(R) > A(P) = 9$. Moreover, $p(R) > p(P) = 12$. Therefore, $L(R) \leq A(R) + \frac{1}{2}p(R) + 1$.

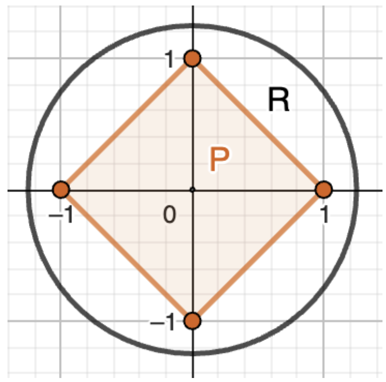


Figure 105: (c) Circle with radius $\frac{5}{4}$

$A(R) > A(P)$ and $p(R) > p(P)$. Therefore, $L(R) \leq A(R) + \frac{1}{2}p(R) + 1$.

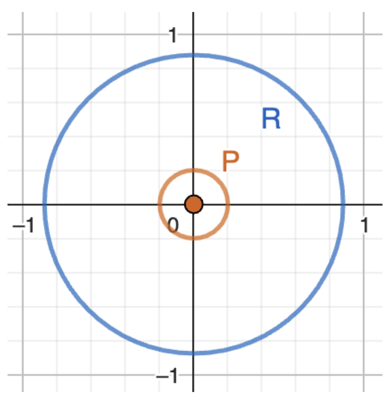


Figure 106: (d) Circle with radius $\frac{7}{8}$

$A(R) > A(P)$ and $p(R) > p(P)$. Therefore, $L(R) \leq A(R) + \frac{1}{2}p(R) + 1$.

**Theorem 8 Blichfeldt's Theorem**

Let R be a bounded set in $\mathbb{R}^2$ with area greater than 1. Then R must contain two distinct points $(x_1, y_1)$ and $(x_2, y_2)$ such that the point $(x_2 - x_1, y_2 - y_1)$ is an integer point (not necessarily in R).

To prove Blichfeldt's Theorem, let us introduce the following notation:

- Let S denote the unit square, i.e. $S = \{(x, y)$ such that $0 \leq x < 1$ and $0 \leq y < 1\} = [0, 1) \times [0, 1)$.
- For integers i and j, let $I_{i,j} = [i, i+1) \times [j, j+1)$.
- Let $R_{i,j} = I_{i,j} \cap R$, i.e. $R_{i,j}$ is the portion of R that lies in the unit interval $[i, i+1) \times [j, j+1)$
- Let $T_{i,j} = R_{i,j} - (i, j)$. This translates each $R_{i,j}$ to the unit square S.

**Exercise 120**

Show that there must exist i,j and m,n such that $T_{i,j} \cap T_{m,n} \neq \emptyset$ and $i \neq m$ or $j \neq n$.

**Solution**

R is a bounded set in $\mathbb{R}^2$ that has an area greater than 1. Because it is bounded, there are a finite number of unit squares $I_{i,j}$ that it intersects. Moreover, each $R_{i,j} = I_{i,j} \cap R$ is distinct and do not intersect. Let us translate each $R_{i,j}$ to the unit square S and label it as $T_{i,j}$. Because area and shape are preserved under translation, the area spanned by $R_{i,j}$ is equal to the area spanned by $T_{i,j}$.

Let us write an expression equal to the area of R. $A(R) = \sum_{i,j} A(R_{i,j}) = A(\cup_{i,j} R_{i,j}) > 1$. However, because $\sum_{i,j} A(T_{i,j}) = \sum_{i,j} A(R_{i,j})$, $\sum_{i,j} A(T_{i,j}) > 1$. But because each $T_{i,j}$ is contained in the unit square S with area equal to 1, there must be a region that is contained in two different regions $T_{i,j}$ and $T_{m,n}$ such that $i \neq m$ and $j \neq n$. Therefore, for $i \neq m$ and $j \neq n$, $T_{i,j} \cap T_{m,n} \neq \emptyset$.

**Exercise 121**

Complete the proof of Blichfeldt's Theorem.

**Solution**

From Exercise 120, we know there is a region in S that is contained in both $T_{i,j}$ and $T_{m,n}$, where $i \neq m$ and $j \neq n$. $\exists (\alpha, \beta) \in T_{i,j} \cap T_{m,n}$, where $\alpha$ and $\beta$ do not need to be integers. This means there exist a point in $R_{i,j}$ with coordinates $T_{i,j} + (i,j)$ and a point in $R_{m,n}$ with coordinates $T_{m,n} + (m,n)$. Let us call the new coordinates $(\alpha_1, \beta_1) = T_{i,j} + (i,j)$ and $(\alpha_2, \beta_2) = T_{m,n} + (m,n)$. Because $i, j, m, n \in \mathbb{Z}$, we are translating the points horizontally and vertically by integer values. This means the decimal part of $\alpha_1$ and $\alpha_2$ are equal and the decimal part of $\beta_1$ and $\beta_2$ are equal. $(\alpha_1, \beta_1) - (\alpha_2, \beta_2) = ((\alpha, \beta) + (i,j)) - ((\alpha, \beta) + (m,n)) = (i,j) - (m,n) = (i-m, j-n)$. $i - m \in \mathbb{Z}$ and $j - n \in \mathbb{Z}$. Therefore, there exist two distinct points in R such that their x and y coordinates have the same decimal part, completing the proof.

**Definition 43**

A set $R$ in $\mathbb{R}^n$ is symmetric about the origin if whenever the point $(x_1, x_2, ..., x_n)$ is in R, the point $(-x_1, -x_2, ..., -x_n)$ is also in R.

**Theorem 9 Minkowski's Theorem**

Let R be a bounded, convex region in $\mathbb{R}^2$ that is symmetric about the origin and has area greater than 4. Then R contains a lattice point other than the origin.

**Exercise 122**

Illustrate Minkowski's Theorem for the following regions:

(a) A circle with center $(0, 0)$ and radius $\frac{5}{4}$

(b) A circle with center $(0, 0)$ and radius $\frac{7}{8}$

**Solution**

(a) The circle is shown in Figure 105 below. Each point in the first quadrant have positive x and y coordinates. Their counterparts, all points with negative x and y coordinates are contained in the sector of the circle in the third quadrant. Similarly, each point in the second quadrant has negative x coordinates and positive y coordinates. Their counterparts, with positive x coordinates and negative y coordinates are in the fourth quadrant.

(b) The circle is shown in Figure 106 below. Each point in the first quadrant have positive x and y coordinates. Their counterparts, all points with negative x and y coordinates are contained in the sector of the circle in the third quadrant. Similarly, each point in the second quadrant has negative x coordinates and positive y coordinates. Their counterparts, with positive x coordinates and negative y coordinates are in the fourth quadrant.

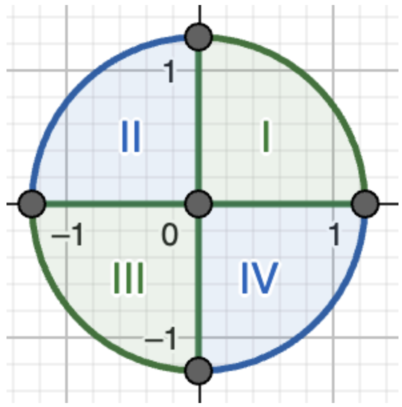


Figure 107: Circle with center $(0, 0)$ and radius $\frac{5}{4}$

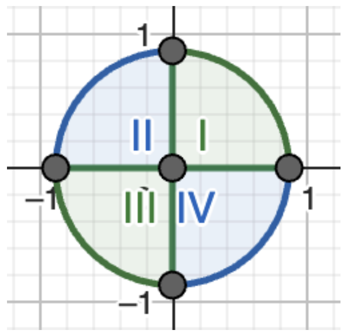


Figure 108: Circle with center $(0, 0)$ and radius $\frac{7}{8}$

**Exercise 123**

Let R be a bounded, convex region in $\mathbb{R}^2$ that is symmetric about the origin and has area greater than 4. Consider the region $R' = \left\{\frac{1}{2}x \text{ such that } x \in R\right\}$. Since $R'$ is just a smaller version of R, it is clear that $R'$ is convex and symmetric about the origin. Show that there are points $x'$ and $y'$ in $R'$ such that $x' - y'$ is a nonzero lattice point.

**Solution**

In Exercise 121, we proved Blichfeldt's Theorem, which states that for a bounded set $S$ in $\mathbb{R}^2$, there are two points $(x_1, y_1)$ and $(x_2, y_2)$ in $S$ such that $(x_2 - x_1, y_2 - y_1)$ is an integer. Here, the bounded set is $R'$. For every point in $R$, $R'$ contains a point with half its value. In Exercise 115, we showed that scaling all side lengths of a polygon by n causes its area to scale by a factor of $n^2$. Values in $R'$ are scaled by a factor of a half, so the area is scaled by a factor of a quarter. Therefore, because the area of $R$ is greater than 4, the area of $R'$ is greater than 1. Therefore, by Blichfeldt's Theorem, there are two points $x'$ and $y'$ such that $x' - y'$ is a lattice point. Because $x'$ and $y'$ are two different points, the difference of their components is nonzero.

**Exercise 124**

Let $x'$ and $y'$ be as in Exercise 123. Show that $x' - y'$ is in R.

**Solution**

R is symmetric across the origin, which means because $X = (x_1, y_1)$ and $Y = (x_2, y_2)$ are in R, $-X = (-x_1, -y_1)$ and $-Y = (-x_2, -y_2)$ are also in R. Let us write a parametric equation of a line between X and -Y.

$$X = x_1 + t(-x_2 - x_1)Y = y_1 + t(-y_2 - y_1)$$

$X' = (\frac{1}{2}x_1, \frac{1}{2}y_1)$ and $Y' = (\frac{1}{2}x_2, \frac{1}{2}y_2)$. Therefore, $X' - Y' = \left(\frac{1}{2}x_1 - \frac{1}{2}x_2, \frac{1}{2}y_1 - \frac{1}{2}y_2\right)$. Let us look at the point on the line from X to -Y when $t = \frac{1}{2}$.

$$X = x_1 + \frac{1}{2}(-x_2 - x_1) = \frac{1}{2}x_1 - \frac{1}{2}x_2Y = y_1 + \frac{1}{2}(-y_2 - y_1) = \frac{1}{2}y_1 - \frac{1}{2}y_2$$

This means the point $X' - Y'$ is on the line between X and -Y. Moreover, because R is convex, this means $X' - Y'$ must be within the region R.

**Definition 44 General Lattice**

Let $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$ be a set of linearly independent vectors in $\mathbb{R}^n$. The lattice with basis $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ is defined as $V = V(\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}) = \{\mathbf{x_1 v_1} + \mathbf{x_2 v_2} + ... + \mathbf{x_n v_n} | \mathbf{x_i} \in \mathbb{Z} \text{ for } \mathbf{i = 1, 2, ...n}\}$.

**Definition 45**

The **determinant of the lattice $\mathbf{\Lambda}$** is the determinant of the $n \times n$ matrix A with columns $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$.
Geometrically, det $\Lambda$ is the volume of the parallelepiped $\{\alpha_1 \mathbf{v_1} + \alpha_\mathbf{2} \mathbf{v_2} + ... + \alpha_\mathbf{n} \mathbf{v_n} | \alpha_\mathbf{i} \in [\mathbf{0}, \mathbf{1}]\}$

**Theorem 10 Minkowski's Theorem for a General Lattice**

Let $V$ be a lattice in $\mathbb{R}^n$, and let $C \subseteq \mathbb{R}^n$ be a symmetric convex set with vol$(C) > 2^n \det\Lambda$.
Then C contains a point of V other than 0.

# 11 Appendix A

Let us calculate the area of a triangle with three vertices $P = (P_x, P_y)$, $Q = (Q_x, Q_y)$, and $R = (R_x, R_y)$. Let us construct two vertices $\vec{PQ}$ and $\vec{PR}$. The area of the triangle is equal to half of the determinant of the matrix whose columns are the vectors $\vec{PQ} = \begin{bmatrix} Q_x - P_x \\ Q_y - P_y \end{bmatrix}$ and $\vec{PR} = \begin{bmatrix} R_x - P_x \\ R_y - P_y \end{bmatrix}$. The determinant of the matrix is $\begin{vmatrix} Q_x - P_x & R_x - P_x \\ Q_y - P_y & R_y - P_y \end{vmatrix} = R_xQ_y - R_xP_y - P_xQ_y + P_xP_y - (R_yQ_x - R_yP_x - P_yQ_x + P_xP_y) = P_x(R_y - Q_y) + Q_x(P_y - R_y) + R_x(Q_y - P_y)$. The area of the triangle is non-negative. Therefore, we need to take the absolute value of the determinant and divide it by 2. Therefore, the area is $\left|\frac{P_x(R_y - Q_y) + Q_x(P_y - R_y) + R_x(Q_y - P_y)}{2}\right|$.

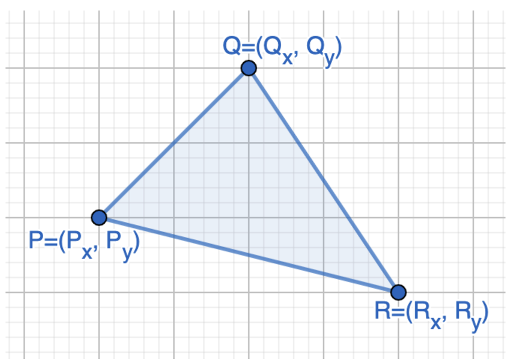


Figure 109: Triangle with vertices $P$, $Q$, and $R$.

# 12 Appendix B

Let us calculate $\sin(108°)$. Using $x = 18°$, we can write $\sin(108°) = \sin(6x) = 2\sin(3x)\cos(3x)$ and expand this expression using the double angle formulas as follows.

$$\begin{aligned} \sin(3x) = \sin(2x + x) &= \sin(2x)\cos(x) + \sin(x)\cos(2x) \\ &= 2\sin(x)\cos^2(x) + \sin(x)(\cos^2(x) - \sin^2(x)) \\ &= 3\sin(x)\cos^2(x) - \sin^3(x) \end{aligned}$$

$$
\begin{aligned}
\cos(3x) = \cos(2x + x) &= \cos(2x)\cos(x) - \sin(2x)\sin(x) \\
&= (\cos^2(x) - \sin^2(x))\cos(x) - 2\sin^2(x)\cos(x) \\
&= \cos^3(x) - 3\sin^2(x)\cos(x)
\end{aligned}
$$

Using these two expressions, we get $\sin(6x) = 6\sin(x)\cos^5(x) - 20\sin^3(x)\cos^3(x) + 6\sin^5(x)\cos(x)$. Because $\sin(90^\circ) = 1 = \sin(18^\circ \times 5)$, let us expand the expression $\sin(5x) = \sin(3x + 2x)$ to calculate the value of $\sin(18^\circ)$.

$$
\begin{aligned}
\sin(3x + 2x) &= \sin(3x)\cos(2x) + \sin(2x)\cos(3x) \\
&= (3\sin(x)\cos^2(x) - \sin^3(x))(\cos^2(x) - \sin^2(x)) + (2\sin(x)\cos(x))(\cos^3(x) - 3\sin^2(x)\cos(x)) \\
&= 5\sin(x)\cos^4(x) - 10\sin^3(x)\cos^2(x) + \sin^5(x)
\end{aligned}
$$

$$
\begin{aligned}
\cos(3x + 2x) &= \cos(3x)\cos(2x) - \sin(3x)\sin(2x) \\
&= (\cos^3(x) - 3\sin^2(x)\cos(x))(\cos^2(x) - \sin^2(x)) - (3\sin(x)\cos^2(x) - \sin^3(x))(2\sin(x)\cos(x)) \\
&= \cos^5(x) - 10\sin^2(x)\cos^3(x) + 5\sin^4(x)\cos(x)
\end{aligned}
$$

Plugging in $x$ for $\sin(x)$ and $y$ for $\cos(x)$, we get $\sin(5x) = 1 = 5xy^4 - 10x^3y^2 + x^5$ and $\cos(5x) = 0 = y^5 - 10x^2y^3 + 5x^4y$. Solving the system of equations, $\sin(18^\circ) = \frac{\sqrt{5}-1}{4}$ and $\cos(18^\circ) = \frac{\sqrt{10+2\sqrt{5}}}{4}$. We can plug these values into the equation we found for $\sin(108^\circ) = \sin(6x)$

$$
\begin{aligned}
&= 6\sin(x)\cos^5(x) - 20\sin^3(x)\cos^3(x) + 6\sin^5(x)\cos(x) \\
&= 6\left(\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right)^5 - 20\left(\frac{\sqrt{5}-1}{4}\right)^3\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right)^3 + 6\left(\frac{\sqrt{5}-1}{4}\right)^5\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right)
\end{aligned}
$$

From the equation above, we can see that $\sin(108^\circ)$ is not rational because $\sqrt{5}$ raised to an

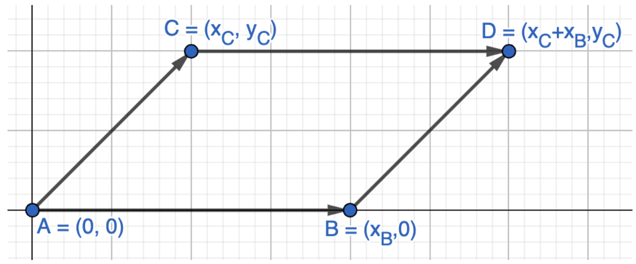


Figure 110: Parallelogram

odd power is irrational.

# 13 Appendix C

The area of a parallelogram is given by the height times the base length. Translation, rotation, and reflection do not change the area of a parallelogram. Therefore, it is possible to construct a coordinate system such that one of the vertices of the parallelogram is at the origin and one of the sides lies on the positive x-axis with the parallelogram contained within the first quadrant by translating, rotating, and reflecting the parallelogram as needed. Let us label the vertices A, B, C, and D counterclockwise starting with the vertex at the origin as shown below. We can construct vectors $\vec{AB}$ and $\vec{AC}$ with an angle $\theta$ between them. The height of the parallelogram is $||\vec{AC}|| \sin(\theta)$ and the base length is $||\vec{AB}||$, so the area of the parallelogram can be written as $||\vec{AC}|| ||\vec{AB}|| \sin(\theta)$. This is equal to the magnitude of the cross product. Therefore, we can calculate the area of a parallelogram by taking the magnitude of the cross product between two vectors with the same tail. Because our vectors are in the $\mathbb{Z}^2$ basis, we can take the third component to be 0 to calculate the cross product. This first two terms of the cross product will be 0, so this is equivalent to calculating the determinant.