Higher-Order WENO Schemes for Hyperbolic Conservation Laws

A Dissertation Submitted

in Partial Fulfilment of the Requirements

for the Degree of

MASTER OF SCIENCE

in

Mathematics

by

Devika Santhosh
(Roll No. MSC21308)

to

SCHOOL OF MATHEMATICS
INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH
THIRUVANANTHAPURAM - 695 551, INDIA

May 2023

DECLARATION

I, **Devika Santhosh (Roll No: MSC21308)**, hereby declare that, this report entitled **"Higher-Order WENO Schemes for Hyperbolic Conservation Laws"** submitted to Indian Institute of Science Education and Research Thiruvananthapuram towards the partial requirement of **Master of Science** in **Mathematics**, is an original work carried out by me under the supervision of **Dr.Dond Asha Kisan** and has not formed the basis for the award of any degree or diploma, in this or any other institution or university.I have sincerely tried to uphold academic ethics and honesty.Whenever a piece of external information or statement or result is used then, that has been duly acknowledged and cited.

Thiruvananthapuram - 695 551 **Devika Santhosh**

May 2023

CERTIFICATE

This is to certify that the work contained in this project report entitled **"Higher-Order WENO Schemes for Hyperbolic Conservation Laws"** submitted by **Devika Santhosh** (**Roll No: MSC21308**) to Indian Institute of Science Education and Research, Thiruvananthapuram towards the partial requirement of **Master of Science** in **Mathematics** has been carried out by her under my supervision and that it has not been submitted elsewhere for the award of any degree.

Thiruvananthapuram - 695 551	Dr.Dond Asha Kisan
May 2023	Project Supervisor

ACKNOWLEDGEMENT

I want to extend a sincere and heartfelt obligation towards all the personages without whom the completion of the project was not possible.I express my profound gratitude and deep regard to Dr. Dond Asha Kisan, IISER Thiruvananthapuram for her guidance, valuable feedback, and constant encouragement throughout the project.Her valuable suggestions were of immense help.I sincerely acknowledge her constant support and guidance during the project.

I feel a major gratefulness to Rakesh Kumar for supporting me throughout the project work and clearing all my doubts, as well as for helping me with the coding portion. Their constant encouragement and support enabled me to successfully complete the project.I am also grateful to the Indian Institute of Science Education and Research, Thiruvananthapuram, for allowing me to do this project and providing all the required facilities.

Thiruvananthapuram - 695 551 **Devika Santhosh**

May 2023

ABSTRACT

This project focuses on exploring and implementing higher-order numerical schemes for solving hyperbolic conservation laws.The aim is to study and develop more accurate and efficient numerical schemes that minimize oscillations near discontinuities.The Weighted Essentially Non-Oscillatory Schemes (WENO) have been studied and implemented in the finite volume frameworks.The challenges of developing higher-order schemes in the presence of discontinuities and the spurious oscillations that arise in numerical solutions have been examined.In this project, we have developed three new variant of WENO schemes, named as, NWENO-AO(5,3), ENWENO-AO(5,3) and MWENO-AO(5,3).We are making use of MWENO-AO(5,3) for Non-Convex Flux where the other schemes fails.The results of this project can contribute to improving the accuracy and efficiency of numerical solutions for hyperbolic conservation laws.

Keywords: finite difference method, higher order scheme,WENO Schemes

Contents

List of Figures

Chapter 1

Introduction

Consider a domain $\Omega \subset R^n$ and a quantity of interest U, defined for all points $x \epsilon \Omega$. The quantity of interest U may be the temperature of a rod, the pressure of a fluid, the concentration of a chemical or a group of cells or the density of a human population.

The temporal rate of change of U in any fixed sub-domain $\omega \subset \Omega$ is equal to the total amount of U produced or destroyed inside ω and the flux of U across the boundary $\partial\omega$.

The above observation says that the change in U is due to two factors: the source or sink, representing the quantity produced or destroyed, and the flux,representing the amount of U that either goes in or comes out of the sub-domain. This observation is mathematically rendered as

$$\frac{d}{dt}\int_{\omega} U dx = -\int_{\partial\omega} F.\upsilon d\sigma(x) + \int_{\omega} S dx \tag{1.1}$$

where

υ = unit outward normal, $d\sigma(x)$ = surface measure , F and S are the flux and the source respectively.

The minus sign in front of the flux term is for convenience. Note that (1.1) is an integral equation for the evolution of the total amount of U in ω. We simplify (1.1) by using integration by parts (or the Gauss divergence theorem) on the surface integral to obtain

$$\frac{d}{dt}\int_{\omega} U dx + \int_{\partial\omega} div(F) dx = \int_{\omega} S dx \tag{1.2}$$

[The **divergence** of a vector field is a measure of how much the vector field is flowing into or out of a region. It is a scalar quantity that is calculated by taking the dot product of the gradient operator and the vector field.]

The **gradient operator** is a mathematical operator that represents the rate of change of a scalar field in space. It is a vector field that points in the direction of greatest increase of the scalar field and has a magnitude equal to the rate of change of the field in that direction. The divergence of a vector field F is denoted by **div(F)** and is defined as: $div(F) = \nabla.F$

[The **Gauss divergence theorem** states that the flux of a vector field F through a surface S enclosing a volume V is equal to the integral of the divergence of the vector field over the volume V, or:

$$Flux(F, S) = \int V div(F) dV$$

Theorem that relates the flux of a vector field through a surface to the divergence of the vector field within the volume enclosed by the surface. The **flux** of a vector field is a measure of the flow of the vector field through the surface, and the divergence of a vector field is a measure of how much the vector field is flowing into or out of a region.] Since (1.2) holds for all sub-domains ω of Ω, we can use an infinitesimal ω to obtain the following differential equation

$$U_t + div(F) = S \quad \forall \quad (x, t) \in (\Omega, R^+). \tag{1.3}$$

The differential equation (1.3) is often termed as a <u>balance law</u> as it is a statement of the fact that the rate of change in U is a balance of the flux and the source. Frequently, the only change in U is from the fluxes and the source is set to zero. In such cases, (1.3) reduces to (1.4)

$$U_t + div(F) = 0 \quad \forall \quad (x,t) \in (\Omega, R^+). \tag{1.4}$$

Equation (1.4) is termed as a conservation law, as the only change in U comes from the quantity entering or leaving the domain of interest. Conservation law, also called Law of Conservation, in physics, a principle that states that a certain physical property (i.e., a measurable quantity) does not change in the course of time within an isolated physical system(system that cannot exchange either matter or energy with its surroundings).

1.0.1 Examples for conservation laws

Scalar transport equation

Let U denote the *concentration of a chemical*(for example, a pollutant in a river). Let the river flows with a velocity field a(x, t) and we know the velocity field at all points in the river. The pollutant will clearly be transported in the direction of the velocity and so the flux in this case is $F = aU$. Since there is no production or destruction of the pollutant during the flow, the source term is set to zero. Consequently, the conservation law takes the form

$$U_t + div(a(x,t)U) = 0 \forall \quad (x,t) \in (\Omega, R^+). \tag{1.5}$$

This equation is linear. In the simple case of one space dimension and a constant velocity field a(x, t) = a, (1.5) reduces to $\underline{U_t + aU_x = 0.}$

This scalar one-dimensional equation is often referred to as the transport or advection equation.

The heat equation

Let U be the temperature of the material. Diffusion of heat is governed by Fourier's or Fick's law $F(U) = -k\nabla U$. Here, k = *conductivity* tensor for the medium. The minus sign is due to the fact that heat flows from hotter to cooler zones. Substituting Fourier's law into the conservation law (1.4), we obtain the heat equation $U_t - U_{xx} = 0$

Euler equations of gas dynamics.

A gas consists of a large number of molecules. The motion of each molecule can be tracked individually. In a macroscopic model, the key variables of interest are: the density ρ, the velocity field u and the gas pressure p.The relevant conservation laws are

- Conservation of mass
- Conservation of momentum
- Conservation of energy

1.1 Hyperbolic conservation Law

The aim of the project is to develop efficient and accurate fifth order WENO schemes of adaptive order for the hyperbolic conservation laws given by

$$\begin{cases} u_t + f(u)_x = 0 \quad (x,t)\epsilon(a,b) \times (0,t] \\ u(x,0) = u_0(x) \end{cases} \tag{1.6}$$

It is well-known that the classical solution of (1.6) may cease to exist in finite time, even if the initial data is sufficiently smooth. Developing numerical schemes is also not an easy task. The appearance of shocks, contact discontinuities and rarefaction waves in the solution profile makes it difficult to devise stable and high order accurate numerical schemes due to the development of spurious oscillations or even growing numerical instabilities.

WENO schemes are one of the most successful higher order accurate schemes, which computes the solution accurately while maintaining high resolution near the discontinuities in a non-oscillatory fashion. Here we will see WENO schemes in a finite volume frame work. A finite volume scheme approximates the conservation law (1.6) in its integral form. We will see WENO schemes namely the New Weighted Essentially Non Oscillatory scheme of Adaptive order(NWENO-AO(5,3)) and the Efficient New Weighted Essentially Non-Oscillatory scheme of Adaptive order(ENWENO-AO(5,3)).

Despite having excellent approximation properties in computing the solution of hyperbolic conservation with convex flux, WENO reconstruction fails to resolves the compound waves in the case of non-convex hyperbolic conservation laws.In this project, we have developed a modified version of the WENO(MWENO-AO(5,3)) scheme for non-convex hyperbolic conservation laws in the finite-volume framework.

We have detailed the basic formulation of WENO scheme of adaptive order and discussed the construction of NWENO-AO(5,3) ,ENWENO-AO(5,3) and MWENO-AO(5,3) schemes.The analysis of nonlinear weights and polynomial reconstruction has been discussed in Chapter 3.In chapter 5, numerical experiments are performed on a range of test problems. Finally, some conclusions are drawn in chapter 6.

Chapter 2

Finite Volume Schemes

We have the 1D conservation law given by;

$$u_t(x,t) + f_x(u(x,t)) = 0 \tag{2.1}$$

with suitable boundary and initial conditions.We assume that the values of the numerical solution are also available outside the computational domain whenever they are needed. This would be the case for periodic or compactly supported problems.

A finite difference method is based on approximating the point values of the solution of a PDE. This approach is not suitable for conservation laws as the solutions are not continuous and point values may not make sense. Instead, we change the perspective and use the cell averages.. The key idea underlying finite difference schemes is to replace the derivatives in equation with a finite difference. This procedure requires the solutions to be smooth and the equation to be satisfied point-wise. However, the solutions to the conservation law are not necessarily smooth and so the Taylor expansion-essential for replacing derivatives with finite differences-is no longer valid. Hence, the finite difference framework may not suite for approximating conservation laws. Instead, we need to develop a new scheme for designing numerical schemes for scalar conservation laws.

The first step in any numerical approximation is to discretize the computational domain in both space and time.

2.0.1 The Grid

For simplicity, we consider a uniform discretization of the domain $[x_L, x_R]$. The discrete points are denoted as $x_j = x_L + (j + \frac{1}{2})\Delta x$ for j = 0, . . . , N, where $\Delta x = \frac{x_R - x_L}{N+1}$.The computational cells or control volumes is denoted by $C_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$.

2.0.2 Cell Average

The cell average is given by

$$\bar{u_j}^n \approx \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t^n)dx \tag{2.2}$$

The aim of the finite volume method is to update the cell average of the unknown at

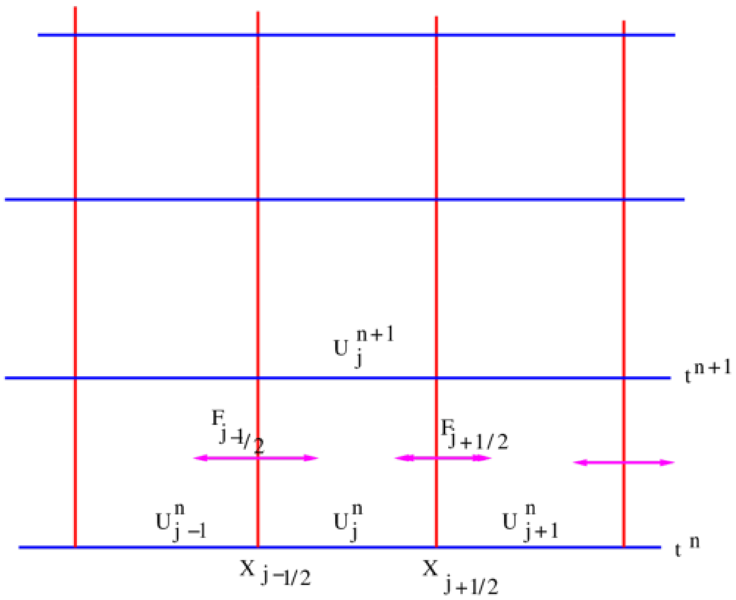


Figure 2.1: A typical finite volume grid displaying cell averages and fluxes

every time step, starting with

$$\bar{u_i}^n \approx \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_0(x)dx \tag{2.3}$$

2.0.3 Integral form of the conservation law.

A finite volume method computes the cell average at the next time level by integrating the conservation law over the domain $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \times [t_n, t_{n+1})$. This gives

$$\int_{t^n}^{t^{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} u_t dxdt + \int_{t^n}^{t^{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} f(u)_x dxdt = 0.$$

Defining

$$\bar{f}^n_{j+1/2} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f\left(u\left(x_{j+1/2}, t\right) dt\right.$$

and using the fundamental theorem of calculus we obtain

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \frac{\Delta t}{\Delta x}(\bar{f}_{j+\frac{1}{2}} - \bar{f}_{j-\frac{1}{2}}) \tag{2.4}$$

We have to make some approximations to estimate the fluxes Suppose we have some method to do this. Then $\bar{f}_{j+\frac{1}{2}}$ is called a numerical flux function. Thus we obtain the semi-discrete finite volume scheme given in equation (2.5).

2.1 Finite volume formulation in the scalar case

For finite volume schemes, or schemes based on cell averages, we do not solve 2.1 directly, but its integrated version. We integrate (2.1) over the interval I_i to obtain

$$\frac{d\bar{u}_i}{dt} = -\frac{1}{\Delta x_i}(f(u_{i+\frac{1}{2}}) - f(u_{i-\frac{1}{2}})), \tag{2.5}$$

where

$$\bar{u}_i = \frac{1}{\Delta x_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,t)dx, \ i = 1, 2, ..N.$$

is the cell average of the solution u(x,t) over the cell I_i. We approximate (2.1) by the following conservative scheme

$$\frac{d\bar{u}(x_i, t)}{dt} = -\frac{1}{\Delta x}(\hat{f}_{i+\frac{1}{2}} - \hat{f}_{i-\frac{1}{2}}), \tag{2.6}$$

where $\bar{u}_i(t)$ is the numerical approximation to the cell average $\bar{u}(x_i, t)$, and the numerical flux $\hat{f}(u_{i+\frac{1}{2}})$ is defined by

$$\hat{f}_{i+\frac{1}{2}} = \hat{f}(u^-_{i+\frac{1}{2}}, u^+_{i+\frac{1}{2}}) \tag{2.7}$$

where $\hat{f}(a, b)$ with the values $u^{\pm}_{i+\frac{1}{2}}$ obtained by the WENO reconstruction. Here we use the Lax–Friedrichs flux given by

$$\hat{f}(a, b) = \frac{1}{2}[f(a) + f(b) - \alpha(b - a)]$$

where $\alpha = \underset{u}{max}|f'(u)|$ is a constant and the maximum is taken over the relevant range of u.

Chapter 3

Polynomial Reconstruction

Consider a general function u represented by its cell averages $\bar{u_j}$ at the grid points x_j. This function may either express the solution or the flux, depending on whether a finite volume or a finite difference scheme is considered. Our objective is to identify an approach that allows the reconstruction of v at the cell interfaces $x_{j+\frac{1}{2}}$ with a given accuracy.

3.1 Reconstruction

Polynomial reconstruction is similar to polynomial interpolation (Process of deriving a simple function from a set of discrete data points so that the function passes through all the given data points (i.e. reproduces the data points exactly) and can be used to estimate data points in-between the given ones) where given a set of averages over a set of intervals, the reconstructed polynomial must have the same averages over the corresponding intervals. We use Legendre polynomials for reconstruction.(this helps in writing the reconstruction polynomials and corresponding smoothness indicators very compactly)[1]. The objective here is to obtain an approximation for $u(x_{i+\frac{1}{2}})$. Here we are approximating polynomials based on 3 sub-stencils and 1 larger stencil. The 3 stencils is given by $\mathbb{S}_0^3 = \{I_{i-1}, I_i, I_{i+1}\}$,$\mathbb{S}_{-1}^3 = \{I_{i-2}, I_{i-1}, I_i\}$ and $\mathbb{S}_1^3 = \{I_i, I_{i+1}, I_{i+2}\}$ and the largest stencil is $\mathbb{S}_0^5 = \{I_{i-2}, I_{i-1}, I_i, I_{i+1}, I_{i+2}\}$. Keeping that in mind the basis elements are chosen as,

$$\begin{aligned}
&L_0(x) = 1, \\
&L_1(x) = \frac{x - x_i}{\Delta x}, \\
&L_2(x) = \left(\frac{x - x_i}{\Delta x}\right)^2 - \frac{1}{12}, \\
&L_3(x) = \left(\frac{x - x_i}{\Delta x}\right)^3 - \frac{3}{20}\left(\frac{x - x_i}{\Delta x}\right), \\
&\quad \dots
\end{aligned} \tag{3.1}$$

Given the location I_i and the order of accuracy k, we first choose a "stencil", based on r cells to the left, s cells to the right, and I_i itself if $r, s > 0$ with $r + s + 1 = k$; $\mathbb{S}(i) = [I_{i-r}, ..., I_{i+s}]$. Now we construct polynomial $\mathbb{P}(x)$.

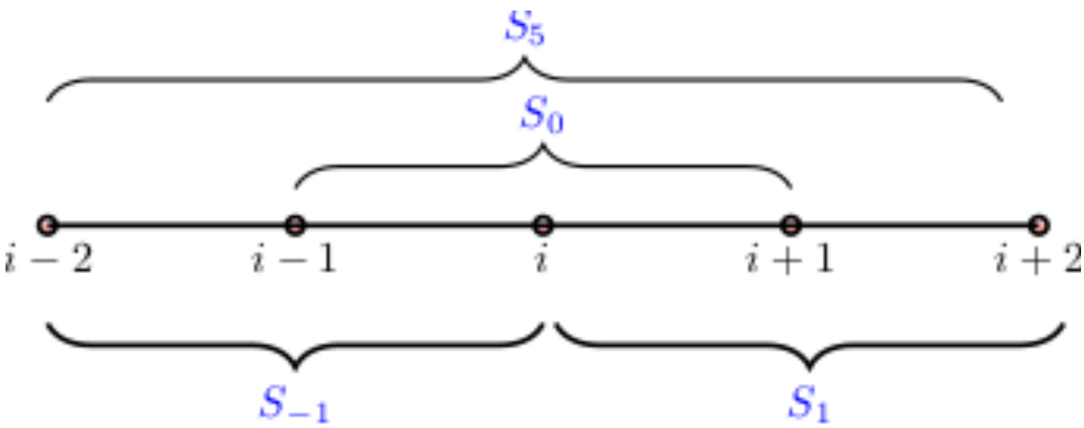


Figure 3.1: Illustration of possible stencils for m = 3 and one biggest possible stencil

Let $\mathbb{P}(x) = a_0 + a_1 L_1(x) + a_2 L_2(x) + ...$ be the polynomial reconstruction for $u(x_{i+\frac{1}{2}})$ corresponding to the stencil S_0. The polynomials are such that they satisfy,

$$\begin{aligned}
&\frac{1}{\Delta x}\int_{x_{j+\frac{1}{2}}}^{x_{j-\frac{1}{2}}} \mathbb{P}_0^5(x)dx = \bar{u}_j, \quad j = i-2, i-1, i, i+1, i+2, \\
&\frac{1}{\Delta x}\int_{x_{j+\frac{1}{2}}}^{x_{j-\frac{1}{2}}} \mathbb{P}_k^3(x)dx = \bar{u}_j, \quad j = i+k-1, i+k, i+k+1, \quad k = -1, 0, 1,
\end{aligned}$$

where $\mathbb{P}_k^m$ denotes the polynomial reconstruction of order m over the stencil $\mathbb{S}_k^m$. Now we consider the map, $\left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right] \to \left[-\frac{1}{2}, \frac{1}{2}\right]$ using the transformation $\xi = \frac{x-x_i}{h}$. Under this transformation, the basis elements becomes,

$$\begin{aligned}
\tilde{L_1}(\xi) &= \xi, \\
\tilde{L_2}(\xi) &= \xi^2 - \frac{1}{12}, \\
\tilde{L_3}(\xi) &= \xi^3 - \frac{3}{20}\xi \\
\tilde{L_4}(\xi) &= \xi^4 - \frac{3}{14}\xi^2 + \frac{3}{560}
\end{aligned}$$

Using the same transformation for rest intervals also and by change of variables, the integrals in can then be rewritten as,

$$\begin{aligned}
&\frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \tilde{\mathbb{P}}_0^5(x)dx = \bar{u}_j, \quad j = -2, -1, 0, 1, 2, \\
&\frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \tilde{\mathbb{P}}_k^3(x)dx = \bar{u}_j, \quad j = k-1, k, k+1, \quad k = -1, 0, 1,
\end{aligned}$$

where, $\tilde{\mathbb{P}}(x) = a_0 + a_1\tilde{L_1}(x) + a_2\tilde{L_2}(x) + ...$

Here we describe polynomial reconstructions of order $n = 3$ and $n = 5$. An nth degree reconstructed polynomial gives $(n+1)th$ order of convergence.

3.1.1 Case $n = 3$ for stencil $\mathbb{S}_0^3 = \{I_{i-1}, I_i, I_{i+1}\}$

Let $\mathbb{P}^3$ be the second degree polynomial approximation of $u(x_{i+\frac{1}{2}})$ based on the stencil $\mathbb{S}_0^3 = \{I_{i-1}, I_i, I_{i+1}\}$. Let $\mathbb{P}^3 = a_0 + a_1\tilde{L}_1(x) + a_2\tilde{L}_2(x)$ which gives a system of equations $Ax = B$, where

$$A = \begin{pmatrix} 1 & \int_{-\frac{3}{2}}^{\frac{1}{2}} \tilde{L}_1(\xi)\, d\xi & \int_{-\frac{3}{2}}^{\frac{1}{2}} \tilde{L}_2(\xi)\, d\xi \\ 1 & \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{L}_1(\xi)\, d\xi & \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{L}_2(\xi)\, d\xi \\ 1 & \int_{\frac{1}{2}}^{\frac{3}{2}} \tilde{L}_1(\xi)\, d\xi & \int_{\frac{1}{2}}^{\frac{3}{2}} \tilde{L}_2(\xi)\, d\xi \end{pmatrix}, x = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

and

$$B = \begin{pmatrix} \bar{u}_{-1} \\ \bar{u}_0 \\ \bar{u}_1 \end{pmatrix}.$$

Upon integration,

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

This can be solved to obtain, $\mathbb{P}^3 = a_0 + a_1 \tilde{L}_1(x) + a_2 \tilde{L}_2(x)$ where

$$\begin{aligned} a_0 &= \bar{u}_1 \\ a_1 &= \frac{\bar{u}_1 - \bar{u}_{-1}}{2} \\ a_2 &= \frac{\bar{u}_1 + \bar{u}_{-1} - 2\bar{u}_0}{2} \end{aligned}$$

3.1.2 Case $n = 3$ for stencil $\mathbb{S}^3_{-1} = \{I_{i-2}, I_{i-1}, I_i\}$

Let $\mathbb{P}^3 = a_0 + a_1 \tilde{L}_1(x) + a_2 \tilde{L}_2(x)$ which gives a system of equations $Ax = B$, where

$$A = \begin{pmatrix} 1 & \int_{-\frac{5}{2}}^{-\frac{3}{2}} \tilde{L}_1(\xi)\, d\xi & \int_{-\frac{5}{2}}^{-\frac{3}{2}} \tilde{L}_2(\xi)\, d\xi \\ 1 & \int_{-\frac{3}{2}}^{-\frac{1}{2}} \tilde{L}_1(\xi)\, d\xi & \int_{-\frac{3}{2}}^{-\frac{1}{2}} \tilde{L}_2(\xi)\, d\xi \\ 1 & \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{L}_1(\xi)\, d\xi & \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{L}_2(\xi)\, d\xi \end{pmatrix}, x = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

and

$$B = \begin{pmatrix} \bar{u}_{-2} \\ \bar{u}_{-1} \\ \bar{u}_0 \end{pmatrix}.$$

Upon integration,

$$A = \begin{pmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

This can be solved to obtain, $\mathbb{P}^3 = a_0 + a_1\tilde{L}_1(x) + a_2\tilde{L}_2(x)$ where

$$\begin{aligned}
a_0 &= \bar{u}_1 \\
a_1 &= \frac{3\bar{u}_0 + \bar{u}_{-2}}{2} - 2\bar{u}_{-1} \\
a_2 &= \frac{\bar{u}_{-2} - 2\bar{u}_{-1} + \bar{u}_0}{2}
\end{aligned}$$

3.1.3 Case $n = 3$ for stencil $\mathbb{S}_1^3 = \{I_i, I_{i+1}, I_{i+2}\}$

Let $\mathbb{P}^3 = a_0 + a_1\tilde{L}_1(x) + a_2\tilde{L}_2(x)$ which gives a system of equations $Ax = B$, where

$$A = \begin{pmatrix} 1 & \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{L}_1(\xi)\, d\xi & \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{L}_2(\xi)\, d\xi \\ 1 & \int_{\frac{1}{2}}^{\frac{3}{2}} \tilde{L}_1(\xi)\, d\xi & \int_{\frac{1}{2}}^{\frac{3}{2}} \tilde{L}_2(\xi)\, d\xi \\ 1 & \int_{\frac{3}{2}}^{\frac{5}{2}} \tilde{L}_1(\xi)\, d\xi & \int_{\frac{3}{2}}^{\frac{5}{2}} \tilde{L}_2(\xi)\, d\xi \end{pmatrix}, x = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

and

$$B = \begin{pmatrix} \bar{u}_0 \\ \bar{u}_1 \\ \bar{u}_2 \end{pmatrix}.$$

Upon integration,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}.$$

This can be solved to obtain, $\mathbb{P}^3 = a_0 + a_1\tilde{L}_1(x) + a_2\tilde{L}_2(x)$ where

$$\begin{aligned}
a_0 &= \bar{u}_1 \\
a_1 &= \frac{-3\bar{u}_0 - 2\bar{u}_2}{2} + 2\bar{u}_1 \\
a_2 &= \frac{\bar{u}_2 - 2\bar{u}_1 + \bar{u}_0}{2}
\end{aligned}$$

3.1.4 Case $n = 5$ for stencil $\mathbb{S}_0^5 = \{I_{i-2}, I_{i-1}, I_i, I_{i+1}, I_{i+2}\}$

$\mathbb{P}^5$ be the second degree polynomial approximation of $u(x_{i+\frac{1}{2}})$ based on the stencil $\mathbb{S}_0^5 = \{I_{i-2}, I_{i-1}, I_i, I_{i+1}, I_{i+2}\}$. Let $\mathbb{P}^5 = a_0 + a_1\tilde{L}_1(x) + a_2\tilde{L}_2(x) + a_3\tilde{L}_3(x) + a_4\tilde{L}_4(x)$ which gives a system of equations $Ax = B$, where

$$A = \begin{pmatrix} 1 & \int_{-\frac{5}{2}}^{-\frac{3}{2}} \tilde{L}_1(\xi)\,d\xi & \int_{-\frac{5}{2}}^{-\frac{3}{2}} \tilde{L}_2(\xi)\,d\xi & \int_{-\frac{5}{2}}^{-\frac{3}{2}} \tilde{L}_3(\xi)\,d\xi & \int_{-\frac{5}{2}}^{-\frac{3}{2}} \tilde{L}_4(\xi)\,d\xi \\ 1 & \int_{-\frac{3}{2}}^{\frac{1}{2}} \tilde{L}_1(\xi)\,d\xi & \int_{-\frac{3}{2}}^{\frac{1}{2}} \tilde{L}_2(\xi)\,d\xi & \int_{-\frac{3}{2}}^{\frac{1}{2}} \tilde{L}_3(\xi)\,d\xi & \int_{-\frac{3}{2}}^{\frac{1}{2}} \tilde{L}_4(\xi)\,d\xi \\ 1 & \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{L}_1(\xi)\,d\xi & \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{L}_2(\xi)\,d\xi & \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{L}_3(\xi)\,d\xi & \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{L}_4(\xi)\,d\xi \\ 1 & \int_{\frac{1}{2}}^{\frac{3}{2}} \tilde{L}_1(\xi)\,d\xi & \int_{\frac{1}{2}}^{\frac{3}{2}} \tilde{L}_2(\xi)\,d\xi & \int_{\frac{1}{2}}^{\frac{3}{2}} \tilde{L}_3(\xi)\,d\xi & \int_{\frac{1}{2}}^{\frac{3}{2}} \tilde{L}_4(\xi)\,d\xi \\ 1 & \int_{\frac{3}{2}}^{\frac{5}{2}} \tilde{L}_1(\xi)\,d\xi & \int_{\frac{3}{2}}^{\frac{5}{2}} \tilde{L}_2(\xi)\,d\xi & \int_{\frac{3}{2}}^{\frac{5}{2}} \tilde{L}_3(\xi)\,d\xi & \int_{\frac{3}{2}}^{\frac{5}{2}} \tilde{L}_4(\xi)\,d\xi \end{pmatrix}, x = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

and

$$B = \begin{pmatrix} \bar{u}_{-2} \\ \bar{u}_{-1} \\ \bar{u}_0 \\ \bar{u}_1 \\ \bar{u}_2 \end{pmatrix}.$$

Upon integration,

$$A = \begin{pmatrix} 1 & -2 & 4 & -\frac{41}{5} & \frac{120}{7} \\ 1 & -1 & 1 & -\frac{11}{10} & \frac{9}{7} \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & \frac{11}{10} & \frac{9}{7} \\ 1 & 2 & 4 & \frac{41}{5} & \frac{120}{7} \end{pmatrix}.$$

This can be solved to obtain, $\mathbb{P}^5 = a_0 + a_1\tilde{L}_1(x) + a_2\tilde{L}_2(x) + a_3\tilde{L}_3(x) + a_4\tilde{L}_4(x)$ where

$$
\begin{aligned}
a_0 &= \bar{u}_1 \\
a_1 &= \frac{82\bar{u}_1 - 82\bar{u}_{-1} + 11\bar{u}_{-2} - 11\bar{u}_2}{120} \\
a_2 &= \frac{40\bar{u}_1 + 40\bar{u}_{-1} - 74\bar{u}_0 - 3\bar{u}_{-2} - 3\bar{u}_2}{56} \\
a_3 &= \frac{-2\bar{u}_1 + 2\bar{u}_{-1} - \bar{u}_{-2} + \bar{u}_2}{12} \\
a_4 &= \frac{-4\bar{u}_1 - 4\bar{u}_{-1} + 6\bar{u}_0 + \bar{u}_{-2} + \bar{u}_2}{24}
\end{aligned}
$$

3.1.5 Polynomial Reconstruction

The first natural question that arises is whether such a reconstruction is possible for a general piece wise smooth function v. The answer is offered by the following result (see [7] for details).

Theorem 3.1.1 *For a piece wise smooth function v, with a finite number of jump discontinuities, there exists an h_0 >0 such that*

$$\pi(x; v) = v(x) + O(h^{m+1}) \quad \forall h \leq h_0,$$

where $\pi(x; v)$ is a polynomial of order m with the property $\pi(x_j; v) = v(x_j)$, $j = \frac{-m}{2}, ...\frac{m}{2}$ at $m + 1$ nodal points.
Furthermore,

$$|\pi(x_j; v)|_{TV} \leq |v|_{TV} + O(h^{m+1})$$

Here the discrete total variation is given by

$$\|U_j^n\| = \sum_j |U_{j+1}^n - U_j{}^n|,$$

We call the scheme is total variation stable, or TV-stable, if $TV_{T(U)} \leq R$ *for some constant* R.

The polynomial reconstruction gives a good approximation for the solution on smooth regions but the numerical solution becomes oscillatory near discontinuities. The WENO (Weighted Essentially Non-Oscillatory) schemes are nonlinear schemes which can achieve high order accuracy in smooth regions and sharp and essentially non-oscillatory discontinuity transitions. Now we will first look into a $5th$ order WENO scheme or WENO-AO(5,3).

Chapter 4

WENO Schemes

This chapter aims to develop WENO schemes for the hyperbolic conservation laws with non-convex flux in a finite difference framework. The hyperbolic conservation laws are given by

$$u_t + f(u)_x = 0 \quad (x,t) \quad \epsilon \quad (a,b) \times (0,t] \tag{4.1}$$

subject to the initial condition $u(x,0) = u_0(x)$. Computing the solutions of (4.1) is a difficult problem, since it involves the formation of compound waves. The compound waves are a sequence of shocks and rarefaction waves, which are difficult to resolve numerically. Many numerical schemes like finite volume WENO and discontinuous Galerkin methods fail to resolve compound waves and converge to the wrong solution. The WENO reconstructions are among the most powerful techniques to provide numerical solutions to the model problems having discontinuities and complicated smooth structures.

WENO-AO(5,3)

Here we describe in detail for the reconstruction procedure of $u^-_{i+\frac{1}{2}}$ to approximate $u(x_{i+\frac{1}{2}},t)$ upto 5^{th} order and the reconstruction procedure of $u^+_{i+\frac{1}{2}}$ is mirror symmetric with respect to $x_{i+\frac{1}{2}}$ of that for $u^-_{i+\frac{1}{2}}$

- Step-1 (Reconstruction of polynomial): Consider the stencil $\mathbb{S}_k^3 = \{i+k-1, i+k, i+k+1\}$, $i = -1, 0, 1$ along with stencil $\mathbb{S}_0^5 = \{i-2, \dots, i+2\}$ with the following conditions

$$
\begin{aligned}
&\frac{1}{\Delta x}\int_{x_{j+\frac{1}{2}}}^{x_{j-\frac{1}{2}}} \mathbb{P}_0^5(x)dx = \bar{u}_j, \quad j = i-2, i-1, i, i+1, i+2,\\
&\frac{1}{\Delta x}\int_{x_{j+\frac{1}{2}}}^{x_{j-\frac{1}{2}}} \mathbb{P}_k^3(x)dx = \bar{u}_j, \quad j = i+k-1, i+k, i+k+1, \quad k = -1, 0, 1,
\end{aligned}
$$

where $\mathbb{P}_k^m$ denotes the polynomial reconstruction of order m over the stencil $\mathbb{S}_k^m$.

- Step-2 (Linear weights): Let γ_k^m denotes the linear weights corresponding to $\mathbb{S}_k^m$. The linear weight are positive linear constant quantity whose sum is equal to one. In WENO-AO(5,3), we have assigned the linear weights as follows(see [1] for more details) :

$$
\gamma_0^5 = \gamma_{\text{Hi}}, \;\; \gamma_0^3 = (1-\gamma_{\text{Hi}})\gamma_{\text{Low}}, \;\; \gamma_{-1}^3 = \gamma_1^3 = \frac{1}{2}(1-\gamma_{\text{Hi}})(1-\gamma_{\text{Low}}).
$$

Here, $\gamma_{\text{Hi}}, \gamma_{\text{Low}} \in (0,1)$ are constant. Rewrite the polynomial $\mathbb{P}_0^5(x)$ as follows:

$$
\begin{aligned}
\mathbb{P}_0^5(x) = \gamma_0^5 \left(\frac{1}{\gamma_0^5}\mathbb{P}_0^5(x) - \frac{\gamma_{-1}^3}{\gamma_0^5}\mathbb{P}_{-1}^3(x) - \frac{\gamma_0^3}{\gamma_0^5}\mathbb{P}_0^3(x) - \frac{\gamma_1^3}{\gamma_0^5}\mathbb{P}_1^3(x) \right) + \gamma_{-1}^3\mathbb{P}_{-1}^3(x)\\
+ \gamma_0^3\mathbb{P}_0^3 + \gamma_1^3\mathbb{P}_1^3.
\end{aligned} \tag{4.2}
$$

Note that $\mathbb{P}_0^5(x)$ is the fifth order approximation to $u(x,t)$

- Step 3 (Smoothness indicator): Here, we construct the smoothness indicator for the $\mathbb{S}_k^m$ stencil. The smoothness indicator for a given stencil is defined as follows (see [8])

$$
\beta_k^m := \sum_{r=1}^{m-1} (\Delta x)^{2r-1} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(\frac{d^r \mathbb{P}_k^m}{dx^r} \right)^2 dx. \tag{4.3}
$$

Balsara etal [1]have demonstrated that the smoothness indicators β_k^m can be written in compact form using the Legendre polynomials. Here, β_k^m corresponds to stencil $\mathbb{S}_k^m$ can be obtained as in compact form (see [1] for more details) given as follow:

$$
\beta_k^m = (a_1^k)^2 + \frac{13}{2}(a_2^k)^2, \quad k = -1, 0, 1
$$

where

$$a_1^{-1} = \frac{1}{2}(\bar{u}_{i-2} - 4\bar{u}_{i-1} + 3\bar{u}_i), \quad a_2^{-1} = \frac{1}{2}(\bar{u}_{i-2} - 2\bar{u}_{i-1} + \bar{u}_i),$$
$$a_1^{0} = \frac{1}{2}(\bar{u}_{i+1} - \bar{u}_{i-1}), \quad a_2^{0} = \frac{1}{2}(\bar{u}_{i-1} - 2\bar{u}_{i} + \bar{u}_{i+1}),$$
$$a_1^{1} = \frac{1}{2}(-3\bar{u}_{i} + 4\bar{u}_{i+1} - \bar{u}_{i+2}), \quad a_2^{1} = \frac{1}{2}(\bar{u}_{i} - 2\bar{u}_{i+1} + \bar{u}_{i+2}).$$

In case of stencil $\mathbb{S}_0^5$, we have

$$\beta_0^5 = \left(a_1^5 + \frac{a_3^5}{10}\right)^2 + \frac{13}{3}\left(a_2^5 + \frac{123}{455}a_4^5\right)^2 + \frac{781}{20}(a_3^5)^2 + \frac{1421461}{2275}(a_4^5)^2,$$

where

$$a_1^5 = \frac{1}{120}(11\bar{u}_{i-2} - 82\bar{u}_{i-1} + 82\bar{u}_{i+1} - 11\bar{u}_{i+2}),$$
$$a_2^5 = \frac{1}{56}(-3\bar{u}_{i-2} + 40\bar{u}_{i-1} - 74\bar{u}_{i} + 40\bar{u}_{i+1} - 3\bar{u}_{i+2}),$$
$$a_3^5 = \frac{1}{12}(-\bar{u}_{i-2} + \bar{u}_{i-1} - 2\bar{u}_{i+1} + \bar{u}_{i+2}),$$
$$a_4^5 = \frac{1}{24}(\bar{u}_{i-2} - 4\bar{u}_{i-1} + 6\bar{u}_{i} - 4\bar{u}_{i+1} + \bar{u}_{i+2}).$$

- Step 4 (Non-linear weights): Using the smoothness indicators discussed in the previous step, we define the unnormalised non-linear weights as follow:

$$\tilde{\omega}_k^m = \gamma_k^m \left(1 + \frac{\tau^2}{(\beta_k^m + \epsilon_w)^2}\right) \quad (m,k) = \{(5,0),(3,-1),(3,0),(3,1)\}$$

where ϵ is a positive number to avoid zero denominator, we take $\epsilon_w = 10^{-12}$ in numerical simulations. The τ is known as global smoothness indicator and it is defined as(see[2])

$$\tau = \frac{1}{3}(|\beta_0^5 - \beta_{-1}^3| + |\beta_0^5 - \beta_0^3| + |\beta_0^5 - \beta_1^3|).$$

The non-linear normalised weights are given as follow:

$$\omega_k^m = \frac{\tilde{\omega}_k^m}{\tilde{\omega}_0^5 + \tilde{\omega}_{-1}^3 + \tilde{\omega}_0^3 + \tilde{\omega}_1^3}, \quad (m,k) = \{(5,0),(3,-1),(3,0),(3,1)\}.$$

- Step 5 (Non-linear reconstruction): We replace the linear weights in 4.2 with the non-linear weights and the final non-linear WENO-AO(5,3) reconstruction is given by

$$\begin{aligned} u^-_{i+\frac{1}{2}} = \omega^5_0 &\left(\frac{1}{\gamma^5_0}\mathbb{P}^5_0(x_{i+\frac{1}{2}}) - \frac{\gamma^3_{-1}}{\gamma^5_0}\mathbb{P}^3_{-1}(x_{i+\frac{1}{2}}) - \frac{\gamma^3_0}{\gamma^5_0}\mathbb{P}^3_0(x_{i+\frac{1}{2}}) - \frac{\gamma^3_1}{\gamma^5_0}\mathbb{P}^3_1(x_{i+\frac{1}{2}}) \right) \\ &+ \omega^3_{-1}\mathbb{P}^3_{-1}(x_{i+\frac{1}{2}}) + \omega^3_0\mathbb{P}^3_0(x_{i+\frac{1}{2}}) + \omega^3_1\mathbb{P}^3_1(x_{i+\frac{1}{2}}). \end{aligned} \tag{4.4}$$

$u^-_{i+\frac{1}{2}}$ is the fifth order approximation to $u(x,t)$ at the point $x_{i+\frac{1}{2}}$

NWENO-AO(5,3) Schemes

- Step-1 (Reconstruction of polynomial): Consider the stencil $\mathbb{S}^3_k = \{i+k-1, i+k, i+k+1\}$, $i = -1, 0, 1$ along with stencil $\mathbb{S}^5_0 = \{i-2, \ldots, i+2\}$ with the following conditions

$$\begin{aligned} &\frac{1}{\Delta x}\int_{x_{j+\frac{1}{2}}}^{x_{j-\frac{1}{2}}} \mathbb{P}^5_0(x)dx = \bar{u}_j, \quad j = i-2, i-1, i, i+1, i+2, \\ &\frac{1}{\Delta x}\int_{x_{j+\frac{1}{2}}}^{x_{j-\frac{1}{2}}} \mathbb{P}^3_k(x)dx = \bar{u}_j, \quad j = i+k-1, i+k, i+k+1, \quad k = -1, 0, 1, \end{aligned}$$

where $\mathbb{P}^m_k$ denotes the polynomial reconstruction of order m over the stencil $\mathbb{S}^m_k$.

- Step-2 (Linear weights): Let γ^m_k denotes the linear weights corresponding to $\mathbb{S}^m_k$. The linear weight are positive linear constant quantity whose sum is equal to one. In NWENO-AO(5,3), we have assigned the linear weights as we did in the WENO-AO(5,3) as follows :

$$\gamma^5_0 = \gamma_{\text{Hi}}, \quad \gamma^3_0 = (1-\gamma_{\text{Hi}})\gamma_{\text{Low}}, \quad \gamma^3_{-1} = \gamma^3_1 = \frac{1}{2}(1-\gamma_{\text{Hi}})(1-\gamma_{\text{Low}}).$$

Rewrite the polynomial $\mathbb{P}^5_0(x)$ as follows:

$$\begin{aligned} \mathbb{P}^5_0(x) = \gamma^5_0 &\left(\frac{1}{\gamma^5_0}\mathbb{P}^5_0(x) - \frac{\gamma^3_{-1}}{\gamma^5_0}\mathbb{P}^3_{-1}(x) - \frac{\gamma^3_0}{\gamma^5_0}\mathbb{P}^3_0(x) - \frac{\gamma^3_1}{\gamma^5_0}\mathbb{P}^3_1(x) \right) + \gamma^3_{-1}\mathbb{P}^3_{-1}(x) \\ &+ \gamma^3_0\mathbb{P}^3_0 + \gamma^3_1\mathbb{P}^3_1. \end{aligned} \tag{4.5}$$

Note that $\mathbb{P}^5_0(x)$ is the fifth order approximation to $u(x,t)$

- Step 3 (Smoothness indicator): Here, we construct the smoothness indicator for the $\mathbb{S}_k^m$ stencil. The smoothness indicator for a given stencil is defined as follows

$$\beta_k^m := \sum_{r=1}^{m-1} (\Delta x)^{2r-1} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(\frac{d^r \mathbb{P}_k^m}{dx^r} \right)^2 dx. \tag{4.6}$$

Balsara etal [1], have demonstrated that the smoothness indicators β_k^m can be written in compact form using the Legendre polynomials. Here, β_k^m corresponds to stencil $\mathbb{S}_k^m$ can be obtained as in compact form (see [1] for more details) given as follow:

$$\beta_k^m = (a_1^k)^2 + \frac{13}{2}(a_2^k)^2, \quad k = -1, 0, 1$$

where

$$\begin{aligned}
a_1^{-1} &= \frac{1}{2}(\bar{u}_{i-2} - 4\bar{u}_{i-1} + 3\bar{u}_i), & a_2^{-1} &= \frac{1}{2}(\bar{u}_{i-2} - 2\bar{u}_{i-1} + \bar{u}_i),\\
a_1^{0} &= \frac{1}{2}(\bar{u}_{i+1} - \bar{u}_{i-1}), & a_2^{0} &= \frac{1}{2}(\bar{u}_{i-1} - 2\bar{u}_{i} + \bar{u}_{i+1}),\\
a_1^{1} &= \frac{1}{2}(-3\bar{u}_{i} + 4\bar{u}_{i+1} - \bar{u}_{i+2}), & a_2^{1} &= \frac{1}{2}(\bar{u}_{i} - 2\bar{u}_{i+1} + \bar{u}_{i+2}).
\end{aligned}$$

In case of stencil $\mathbb{S}_0^5$, we have

$$\beta_0^5 = \left(a_1^5 + \frac{a_3^5}{10} \right)^2 + \frac{13}{3}\left(a_2^5 + \frac{123}{455} a_4^5 \right) + \frac{781}{20}(a_3^5)^2 + \frac{1421461}{2275}(a_4^5)^2,$$

where

$$\begin{aligned}
a_1^5 &= \frac{1}{120}(11\bar{u}_{i-2} - 82\bar{u}_{i-1} + 82\bar{u}_{i+1} - 11\bar{u}_{i+2}),\\
a_2^5 &= \frac{1}{56}(-3\bar{u}_{i-2} + 40\bar{u}_{i-1} - 74\bar{u}_{i} + 40\bar{u}_{i+1} - 3\bar{u}_{i+2}),\\
a_3^5 &= \frac{1}{12}(-\bar{u}_{i-2} + \bar{u}_{i-1} - 2\bar{u}_{i+1} + \bar{u}_{i+2}),\\
a_4^5 &= \frac{1}{24}(\bar{u}_{i-2} - 4\bar{u}_{i-1} + 6\bar{u}_{i} - 4\bar{u}_{i+1} + \bar{u}_{i+2}).
\end{aligned}$$

- Step 4 (Non-linear weights): Using the smoothness indicators discussed in the previous step, we define the unnormalised non-linear weights as follow:

$$\tilde{\omega}_k^m = \gamma_k^m \left(1 + \frac{\tau^2}{(\beta_k^m + \epsilon_w)^2} \right) \quad (m,k) = \{(5,0), (3,-1), (3,0), (3,1)\}$$

where ϵ is a positive number to avoid zero denominator, we take $\epsilon_w = 10^{-12}$ in numerical simulations. The τ is known as global smoothness indicator and it is defined as

$$\tau = \frac{(\beta_0^5)^3}{d+\epsilon}$$

where $d = min\beta_{-1}^3, \beta_0^3, \beta_1^3, \beta_0^5$.

The non-linear normalised weights are given as follow:

$$\omega_k^m = \frac{\tilde{\omega}_k^m}{\tilde{\omega}_0^5 + \tilde{\omega}_{-1}^3 + \tilde{\omega}_0^3 + \tilde{\omega}_1^3}, \quad (m,k) = \{(5,0),(3,-1),(3,0),(3,1)\}.$$

- Step 5 (Non-linear reconstruction): We replace the linear weights in 4.5 with the non-linear weights and the final non-linear NWENO-AO(5,3) reconstruction is given by

$$\begin{aligned} u_{i+\frac{1}{2}}^- = \omega_0^5 \left(\frac{1}{\gamma_0^5}\mathbb{P}_0^5(x_{i+\frac{1}{2}}) - \frac{\gamma_{-1}^3}{\gamma_0^5}\mathbb{P}_{-1}^3(x_{i+\frac{1}{2}}) - \frac{\gamma_0^3}{\gamma_0^5}\mathbb{P}_0^3(x_{i+\frac{1}{2}}) - \frac{\gamma_1^3}{\gamma_0^5}\mathbb{P}_1^3(x_{i+\frac{1}{2}}) \right) \\ + \omega_{-1}^3\mathbb{P}_{-1}^3(x_{i+\frac{1}{2}}) + \omega_0^3\mathbb{P}_0^3(x_{i+\frac{1}{2}}) + \omega_1^3\mathbb{P}_1^3(x_{i+\frac{1}{2}}). \end{aligned} \tag{4.7}$$

$u_{i+\frac{1}{2}}^-$ is the fifth order approximation to $u(x,t)$ at the point $x_{i+\frac{1}{2}}$

ENWENO-AO(5,3) Schemes

- Step-1 (Reconstruction of polynomial): Consider the stencil $\mathbb{S}_k^3 = \{i+k-1, i+k, i+k+1\}$, $i = -1, 0, 1$ along with stencil $\mathbb{S}_0^5 = \{i-2, \ldots, i+2\}$ with the following conditions

$$\frac{1}{\Delta x}\int_{x_{j+\frac{1}{2}}}^{x_{j-\frac{1}{2}}} \mathbb{P}_0^5(x)dx = \bar{u}_j, \quad j = i-2, i-1, i, i+1, i+2,$$
$$\frac{1}{\Delta x}\int_{x_{j+\frac{1}{2}}}^{x_{j-\frac{1}{2}}} \mathbb{P}_k^3(x)dx = \bar{u}_j, \quad j = i+k-1, i+k, i+k+1, \quad k = -1, 0, 1,$$

where $\mathbb{P}_k^m$ denotes the polynomial reconstruction of order m over the stencil $\mathbb{S}_k^m$.

- Step-2 (Linear weights): Let γ_k^m denotes the linear weights corresponding to $\mathbb{S}_k^m$. The linear weight are positive linear constant quantity whose sum is equal to one.

In ENWENO-AO(5,3), we have assigned the linear weights as follows :

$$\gamma_0^5 = \gamma_{\text{Hi}}, \quad \gamma_0^3 = (1-\gamma_{\text{Hi}})\gamma_{\text{Low}}, \quad \gamma_{-1}^3 = \gamma_1^3 = \frac{1}{2}(1-\gamma_{\text{Hi}})(1-\gamma_{\text{Low}}).$$

Rewrite the polynomial $\mathbb{P}_0^5(x)$ as follows:

$$\begin{aligned}\mathbb{P}_0^5(x) = \gamma_0^5 \left(\frac{1}{\gamma_0^5}\mathbb{P}_0^5(x) - \frac{\gamma_{-1}^3}{\gamma_0^5}\mathbb{P}_{-1}^3(x) - \frac{\gamma_0^3}{\gamma_0^5}\mathbb{P}_0^3(x) - \frac{\gamma_1^3}{\gamma_0^5}\mathbb{P}_1^3(x) \right) + \gamma_{-1}^3\mathbb{P}_{-1}^3(x) \\ + \gamma_0^3\mathbb{P}_0^3 + \gamma_1^3\mathbb{P}_1^3.\end{aligned} \tag{4.8}$$

Note that $\mathbb{P}_0^5(x)$ is the fifth order approximation to $u(x,t)$

- Step 3 (Smoothness indicator): Here, we construct the smoothness indicator for the $\mathbb{S}_k^m$ stencil. The smoothness indicator for a given stencil is defined as follows

$$\beta_k^m := \sum_{r=1}^{m-1} (\Delta x)^{2r-1} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(\frac{d^r \mathbb{P}_k^m}{dx^r} \right)^2 dx. \tag{4.9}$$

Balsara etal,[1] have demonstrated that the smoothness indicators β_k^m can be written in compact form using the Legendre polynomials. Here, β_k^m corresponds to stencil $\mathbb{S}_k^m$ can be obtained as in compact form (see [1] for more details) given as follows:

$$\beta_k^m = (a_1^k)^2 + \frac{13}{2}(a_2^k)^2, \quad k = -1, 0, 1$$

where

$$\begin{aligned} a_1^{-1} &= \frac{1}{2}(\bar{u}_{i-2} - 4\bar{u}_{i-1} + 3\bar{u}_i), \quad & a_2^{-1} &= \frac{1}{2}(\bar{u}_{i-2} - 2\bar{u}_{i-1} + \bar{u}_i), \\ a_1^0 &= \frac{1}{2}(\bar{u}_{i+1} - \bar{u}_{i-1}), & a_2^0 &= \frac{1}{2}(\bar{u}_{i-1} - 2\bar{u}_i + \bar{u}_{i+1}), \\ a_1^1 &= \frac{1}{2}(-3\bar{u}_i + 4\bar{u}_{i+1} - \bar{u}_{i+2}), & a_2^1 &= \frac{1}{2}(\bar{u}_i - 2\bar{u}_{i+1} + \bar{u}_{i+2}). \end{aligned}$$

In case of stencil $\mathbb{S}_0^5$, we have

$$\beta_0^5 = \left(a_1^5 + \frac{a_3^5}{10} \right)^2 + \frac{13}{3}\left(a_2^5 + \frac{123}{455}a_4^5 \right) + \frac{781}{20}(a_3^5)^2 + \frac{1421461}{2275}(a_4^5)^2,$$

where

$$
\begin{aligned}
a_1^5 &= \frac{1}{120}(11\bar{u}_{i-2} - 82\bar{u}_{i-1} + 82\bar{u}_{i+1} - 11\bar{u}_{i+2}),\\
a_2^5 &= \frac{1}{56}(-3\bar{u}_{i-2} + 40\bar{u}_{i-1} - 74\bar{u}_i + 40\bar{u}_{i+1} - 3\bar{u}_{i+2}),\\
a_3^5 &= \frac{1}{12}(-\bar{u}_{i-2} + \bar{u}_{i-1} - 2\bar{u}_{i+1} + \bar{u}_{i+2}),\\
a_4^5 &= \frac{1}{24}(\bar{u}_{i-2} - 4\bar{u}_{i-1} + 6\bar{u}_i - 4\bar{u}_{i+1} + \bar{u}_{i+2}).
\end{aligned}
$$

We can observed that β_0^5 involves lots of computation and need to compute in WENO-AO as opposed to classical WENO schemes (which involves only lower order smoothness indicators). To reduce the computational cost, we proposed a new approximate smoothness indicator for the stencil $\mathbb{S}_0^5$ utilising the smoothness indicator of lower order stencil. The approximate smoothness is deonote as $(\beta_0^5)^a$ and it is defined as:

$$
(\beta_0^5)^a = \frac{1}{3}(\beta_{-1}^3 + \beta_0^3 + \beta_1^3),
$$

which involves the smoothness indicator defined over smaller stencils.

- Step 4 (Non-linear weights): Using the smoothness indicators discussed in the previous step, we define the unnormalised non-linear weights as follow:

$$
\tilde{\omega}_k^m = \gamma_k^m \left(1 + \frac{\tau^2}{(\beta_k^m + \epsilon_w)^2}\right) \quad (m,k) = \{(5,0),(3,-1),(3,0),(3,1)\}
$$

where ϵ is a positive number to avoid zero denominator, we take $\epsilon_w = 10^{-12}$ in numerical simulations. The τ is known as global smoothness indicator and it is defined as

$$
\tau = \frac{(\beta_0^5)^3}{d + \epsilon}
$$

where $d = min \beta_{-1}^3, \beta_0^3, \beta_1^3, \beta_0^5$.

The non-linear normalised weights are given as follow:

$$
\omega_k^m = \frac{\tilde{\omega}_k^m}{\tilde{\omega}_0^5 + \tilde{\omega}_{-1}^3 + \tilde{\omega}_0^3 + \tilde{\omega}_1^3}, \quad (m,k) = \{(5,0),(3,-1),(3,0),(3,1)\}.
$$

- Step 5 (Non-linear reconstruction): We replace the linear weights in 4.8 with the non-linear weights and the final non-linear ENWENO-AO(5,3) reconstruction is given by

$$\begin{aligned} u^-_{i+\frac{1}{2}} = \omega^5_0 &\left(\frac{1}{\gamma^5_0}\mathbb{P}^5_0(x_{i+\frac{1}{2}}) - \frac{\gamma^3_{-1}}{\gamma^5_0}\mathbb{P}^3_{-1}(x_{i+\frac{1}{2}}) - \frac{\gamma^3_0}{\gamma^5_0}\mathbb{P}^3_0(x_{i+\frac{1}{2}}) - \frac{\gamma^3_1}{\gamma^5_0}\mathbb{P}^3_1(x_{i+\frac{1}{2}}) \right) \\ &+ \omega^3_{-1}\mathbb{P}^3_{-1}(x_{i+\frac{1}{2}}) + \omega^3_0\mathbb{P}^3_0(x_{i+\frac{1}{2}}) + \omega^3_1\mathbb{P}^3_1(x_{i+\frac{1}{2}}). \end{aligned} \tag{4.10}$$

  $u^-_{i+\frac{1}{2}}$ is the fifth order approximation to $u(x,t)$ at the point $x_{i+\frac{1}{2}}$

The construction of ENWENO-AO(5,3) is same as the NWENO-AO(5,3) scheme except the use of approximate smoothness indicator $(\beta^5_0)^a$, which help us improve the efficiency of the scheme. The closeness of approximate smoothness indicator $(\beta^5_0)^a$ to β^5_0 can be seen in the next section.

Analysis of new smoothness indicator

Theorem 4.0.1 *If $u \in \mathbb{C}^2(\mathbb{S}^5_0)$, then the following estimate holds*

$$|\beta^5_0 - (\beta^5_0)^a| = O(\Delta x^4).$$

Proof: The Taylor series expansion of β^5_0 and $(\beta^5_0)^a$ at point x_i is

$$\beta^5_0 = (u^{'}_i)^2\Delta x^2 - u^{'}_i u^{''}_i \Delta x^3 + \left(\frac{4}{3}(u^{''}_i)^2 + \frac{1}{3}u_i u^{''}_i \right) \Delta x^4 + O(\Delta x^5),$$

and

$$(\beta^5_0)^a = (u^{'}_i)^2\Delta x^2 - u^{'}_i u^{''}_i \Delta x^3 + \left(\frac{4}{3}(u^{''}_i)^2 + \frac{2}{3}u_i u^{''}_i \right) \Delta x^4 + O(\Delta x^5).$$

Using the above estimate, we have the required results.

Now we have to show using new smoothness indicator, new non-linear weights converges to linear weights with required accuracy results.

Theorem 4.0.2 *If u is smooth over the stencil $\mathbb{S}_0^5$, then in WENO-AO(5,3) scheme, we have*

$$\omega_k^m = \gamma_k^m + O(\Delta x^4), \quad (m,k) = \{(5,0),(3,-1),(3,0),(3,1)\}.$$

Proof. The non-linear weight for the WENO-AO(5,3) scheme is given by

$$\omega_k^m = \frac{\gamma_k^m\left(1+\frac{\tau^2}{(\beta_k^m+\epsilon_w)^2}\right)}{\sum_{i=-1}^{1}\gamma_i^3\left(1+\frac{\tau^2}{(\beta_i^3+\epsilon_w)^2}\right)+\gamma_0^5\left(1+\frac{\tau^2}{(\beta_0^5+\epsilon_w)^2}\right)} \tag{4.11}$$

We have The Taylors series expansion of β_0^5,β_1^3,β_0^3,β_1^3

$$\beta_0^5 = (u_i^{'})^2\Delta x^2 - u_i{'}u_i^{''}\Delta x^3 + \left(\frac{4}{3}(u_i^{''})^2+\frac{1}{3}u_iu_i^{''}\right)\Delta x^4 + O(\Delta x^5),$$

$$\beta_{-1}^3 = (u_i^{'})^2\Delta x^2 - u_i{'}u_i^{''}\Delta x^3 + \left(\frac{4}{3}(u_i^{''})^2-\frac{1}{3}u_iu_i^{''}\right)\Delta x^4 + O(\Delta x^5),$$

$$\beta_0^3 = (u_i^{'})^2\Delta x^2 - u_i{'}u_i^{''}\Delta x^3 + \left(\frac{4}{3}(u_i^{''})^2+\frac{2}{3}u_iu_i^{''}\right)\Delta x^4 + O(\Delta x^5),$$

$$\beta_1^3 = (u_i^{'})^2\Delta x^2 - u_i{'}u_i^{''}\Delta x^3 + \left(\frac{4}{3}(u_i^{''})^2-\frac{1}{3}u_iu_i^{''}\right)\Delta x^4 + O(\Delta x^5),$$

From here we can conclude that

$$|\beta_0^5 - \beta_k^3| = O(\Delta x^4) \quad k = -1,0,1$$

Using the above, we can get $\tau = O(\Delta x^4)$. Further, the estimates of smoothness indicators is given as

$$(\beta_k^m + \epsilon_w)^2 = O(\Delta x^4)$$

and

$$\frac{\tau^2}{(\beta_k^m+\epsilon_w)^2} = O(\Delta x^4)$$

Since

$$\gamma_0^5 + \sum_{i=-1}^{1}\gamma_i^3 = 1$$

equation 3.18 can be written as,

$$\omega_k^m = \frac{\gamma_k^m\left(1+n_k^m\right)}{1+\sum_{i=-1}^{1}\gamma_i^3 n_k^m + \gamma_0^5 n_0^5} \tag{4.12}$$

where $n_k^m = \frac{\tau^2}{(\beta_0^5+\epsilon_w)^2}$.Since $n_k^m \to 0$ as $\Delta x \to 0$,we have

$$1 + \sum_{i=-1}^{1} \gamma_i^3 n_k^m + \gamma_0^5 n_0^5 = 1 + O(\Delta x^4)$$

From this we get

$$\frac{1}{1 + \sum_{i=-1}^{1} \gamma_i^3 n_k^m + \gamma_0^5 n_0^5} = \frac{1}{1 + O(\Delta x^4)} = 1 - O(\Delta x^4) \tag{4.13}$$

Finally using the above results we get from 3.13

$$\omega_k^m = \gamma_k^m + O(\Delta x^4), \quad (m,k) = \{(5,0),(3,-1),(3,0),(3,1)\}.$$

The same result follows for N-WENO(5,3) and EN-WENO(5,3). As we have seen for N-WENO the global smoothness indicator is given by

$$\tau = \frac{(\beta_0^5)^3}{d + \epsilon}$$

where $d = min \beta_{-1}^3, \beta_0^3, \beta_1^3, \beta_0^5$.

Using this, we can get $\tau = O(\Delta x^4)$.Further we proceed as above.Similarly,for ENWENO-AO(5,3) from the above theorem we get

$$|\beta_0^5 - (\beta_0^5)^a| = O(\Delta x^4).$$

Proceeding in the similar fashion we can prove the theorem holds for both NWENO-AO(5,3) and ENWENO-AO(5,3) as well (see[9]).

Theorem 4.0.3 *If u is smooth on S_0^5, then NWENO-AO(5,3), ENWENO-AO(5,3) and WENO-AO(5,3) reconstruction are fifth order accurate at the interface*

$$u_{i+\frac{1}{2}}^- - u(x_{i+\frac{1}{2}}, t) = O(\Delta x^5)$$

Proof. In NWENO-AO(5,3) reconstruction we have

$$
\begin{aligned}
u^-_{i+\frac{1}{2}} - u(x_{i+\frac{1}{2}},t) &= \frac{\omega_0^5}{\gamma_0^5}\left(\mathbb{P}_0^5(x_{i+\frac{1}{2}}) - \sum_{k=-1}^{1}\gamma_k^3\mathbb{P}_k^3(x_{i+\frac{1}{2}})\right) + \sum_{k=-1}^{1}\omega_k^3\mathbb{P}_k^3(x_{i+\frac{1}{2}}) - u(x_{i+\frac{1}{2}},t) \\
&= \frac{\omega_0^5}{\gamma_0^5}\left(\mathbb{P}_0^5(x_{i+\frac{1}{2}}) - u(x_{i+\frac{1}{2}},t) + u(x_{i+\frac{1}{2}},t) - \sum_{k=-1}^{1}\gamma_k^3\mathbb{P}_k^3(x_{i+\frac{1}{2}})\right) + \sum_{k=-1}^{1}\omega_k^3\mathbb{P}_k^3(x_{i+\frac{1}{2}}) \\
&\quad - (\omega_0^5 + \sum_{k=-1}^{1}\omega_k^3)u(x_{i+\frac{1}{2}},t) \\
&= \frac{\omega_0^5}{\gamma_0^5}\left(\mathbb{P}_0^5(x_{i+\frac{1}{2}}) - u(x_{i+\frac{1}{2}},t) + (\gamma_0^5 + \sum_{k=-1}^{1}\gamma_k^3)u(x_{i+\frac{1}{2}},t) - \sum_{k=-1}^{1}\gamma_k^3\mathbb{P}_k^3(x_{i+\frac{1}{2}})\right) \\
&\quad + \sum_{k=-1}^{1}\omega_k^3\mathbb{P}_k^3(x_{i+\frac{1}{2}}) - (\omega_0^5 + \sum_{k=-1}^{1}\omega_k^3)u(x_{i+\frac{1}{2}},t) \\
&= \frac{\omega_0^5}{\gamma_0^5}\left((\mathbb{P}_0^5(x_{i+\frac{1}{2}}) - u(x_{i+\frac{1}{2}},t)\right) - \left(\sum_{k=-1}^{1}(\frac{\omega_0^5-\gamma_0^5}{\gamma_0^5}\gamma_k^3 - (\omega_k^3+\gamma_k^3))\right) \\
&\quad (\mathbb{P}_k^3(x_{i+\frac{1}{2}}) - u(x_{i+\frac{1}{2}},t))
\end{aligned}
$$

The sufficient condition to achieve fifth order accurate reconstruction using NWENO-AO(5,3) reconstruction is

$$
\sum_{k=-1}^{1}\left(\frac{\omega_0^5-\gamma_0^5}{\gamma_0^5}\gamma_k^3 - (\omega_k^3+\gamma_k^3)\right) = O(\Delta x^2)
$$

which can be observed from Theorem 4.0.2.
For ENWENO-AO(5,3) we follow the same arguements.Thus we conclude that WENO-AO(5,3),NWENO-AO(5,3) and ENWENO-AO(5,3) reconstruction is of fifth order.

Theorem 4.0.4 *If the solution has critical point ie $u^{'}(x) = 0$,then*

$$
\omega_k^m = \gamma_k^m + O(\Delta x^8), \quad (m,k) = \{(5,0),(3,-1),(3,0),(3,1)\}.
$$

Proof: We have The taylors series expansion of β_0^5,β_{-1}^3,β_0^3,β_1^3

$$
\beta_0^5 = (u_i^{'})^2\Delta x^2 - u_i{'}u_i^{''}\Delta x^3 + \left(\frac{4}{3}(u_i^{''})^2 + \frac{1}{3}u_iu_i^{''}\right)\Delta x^4 + O(\Delta x^5),
$$

$$\beta_{-1}^3 = (u_i^{'})^2\Delta x^2 - u_i{}'u_i^{''}\Delta x^3 + \left(\frac{4}{3}(u_i^{''})^2 - \frac{1}{3}u_iu_i^{''}\right)\Delta x^4 + O(\Delta x^5),$$

$$\beta_0^3 = (u_i^{'})^2\Delta x^2 - u_i{}'u_i^{''}\Delta x^3 + \left(\frac{4}{3}(u_i^{''})^2 + \frac{2}{3}u_iu_i^{''}\right)\Delta x^4 + O(\Delta x^5),$$

$$\beta_1^3 = (u_i^{'})^2\Delta x^2 - u_i{}'u_i^{''}\Delta x^3 + \left(\frac{4}{3}(u_i^{''})^2 - \frac{1}{3}u_iu_i^{''}\right)\Delta x^4 + O(\Delta x^5),$$

Substituting $u_i^{'} = 0$ we get

$$\beta_0^5 = \left(\frac{4}{3}(u_i^{''})^2 + \frac{1}{3}u_iu_i^{''}\right)\Delta x^4 + O(\Delta x^5),,$$

$$\beta_{-1}^3 = \left(\frac{4}{3}(u_i^{''})^2 - \frac{1}{3}u_iu_i^{''}\right)\Delta x^4 + O(\Delta x^5),$$

$$\beta_0^3 = \left(\frac{4}{3}(u_i^{''})^2 + \frac{2}{3}u_iu_i^{''}\right)\Delta x^4 + O(\Delta x^5),$$

$$\beta_1^3 = \left(\frac{4}{3}(u_i^{''})^2 - \frac{1}{3}u_iu_i^{''}\right)\Delta x^4 + O(\Delta x^5),$$

which gives

$$\beta_0^5 = O(\Delta x^4)$$

$$\beta_{-1}^3 = O(\Delta x^4)$$

$$\beta_0^3 = O(\Delta x^4)$$

$$\beta_1^3 = O(\Delta x^4)$$

Proceeding as in Theorem 4.0.2, we get the required result.

4.1 MWENO-AO(5,3) Schemes for Non-Convex Hyperbolic Conservation Law

The Weighted Essentially Non-Oscillatory (WENO) reconstruction provides higher-order accurate solutions to hyperbolic conservation laws for convex flux. But it fails to capture composite structure in the case of non-convex flux and converges to the wrong solution. Redefining a polynomial using the solution obtained by WENO reconstruction (u^{WENO})

and first order scheme (u^{FO}) we get the following solution which ensures the convergence of non-convex flux as,

$$u^M_{i+\frac{1}{2}} = \left(1 - \frac{\omega_0^5}{\gamma_0^5}\right) u^{FO} + \frac{\omega_0^5}{\gamma_0^5} u^{WENO} \tag{4.14}$$

This is the modified WENO Scheme(MWENO-AO(5,3))

Theorem 4.1.1 *If flux function is non-convex on S_0^5, then MWENO-AO(5,3) reconstruction are fifth order accurate at the interface*

$$u^M_{i+\frac{1}{2}} - u(x_{i+\frac{1}{2}}, t) = O(\Delta x^5)$$

Proof. In MWENO-AO(5,3) reconstruction we have

$$\begin{aligned}
u^M_{i+\frac{1}{2}} - u(x_{i+\frac{1}{2}}, t) &= \left(1 - \frac{\omega_0^5}{\gamma_0^5}\right) u^{FO} + \frac{\omega_0^5}{\gamma_0^5} u^{WENO} - u(x_{i+\frac{1}{2}}, t) \\
&= \left(1 - \frac{\omega_0^5}{\gamma_0^5}\right) u^{FO} + \frac{\omega_0^5}{\gamma_0^5} u^{WENO} - \left(1 - \frac{\omega_0^5}{\gamma_0^5} + \frac{\omega_0^5}{\gamma_0^5}\right) u(x_{i+\frac{1}{2}}, t) \\
&= \left(1 - \frac{\omega_0^5}{\gamma_0^5}\right) u^{FO} + \frac{\omega_0^5}{\gamma_0^5} u^{WENO} - \left(1 - \frac{\omega_0^5}{\gamma_0^5}\right)) u(x_{i+\frac{1}{2}}, t) - \left(\frac{\omega_0^5}{\gamma_0^5}\right) u(x_{i+\frac{1}{2}}, t) \\
&= \left(1 - \frac{\omega_0^5}{\gamma_0^5}\right))(u^{FO} - u(x_{i+\frac{1}{2}}, t)) + \frac{\omega_0^5}{\gamma_0^5}(u^{WENO} - u(x_{i+\frac{1}{2}}, t)) \\
&= O(\Delta x^5)
\end{aligned}$$

This completes the proof.

In this chapter, we discussed WENO reconstruction. We have proved theoretically how well the non-linear weights converges to corresponding linear weights in smooth case and whether the reconstruction using all the three WENO schemes gives fifth order accuracy. We also discussed how ENWENO-AO(5,3) become more efficient than NWENO-AO(5,3) by introducing an approximate smoothness indicator. Finally we have proposed a modified WENO scheme for the hyperbolic conservation laws with non-convex flux in the finite-volume framework, which will also give a fifth order approximation to the solution.

Chapter 5

Numerical Results

In this section, we have compared the three new WENO schemes, namely NWENO-AO(5,3),ENWENO-AO(5,3) and MWENO-AO(5,3) are compared in a finite volume framework. For the time integration of the semi-discrete scheme, TVD Runge–Kutta method of order three is used. The semi-discrete scheme for vector conservation laws is written as

$$\frac{du}{dt} = \mathbb{L}(u),$$

Then, the three stage TVD Runge–Kutta method (see [12]) is given by

$$\begin{aligned}
u^{(1)} &= u^{(0)} + \Delta t \mathbb{L}(u^{(0)}) \\
u^{(2)} &= \frac{3}{4}u^{(0)} + \frac{1}{4}(u^{(1)} + \Delta t \mathbb{L}(u^{(1)})) \\
u^{(3)} &= \frac{1}{3}u^{(0)} + \frac{2}{3}(u^{(2)} + \Delta t \mathbb{L}(u^{(2)})) \\
u^{n+1} &= u^{(3)}.
\end{aligned}$$

Here, we have taken CFL = 0.5 in our test cases or mentioned separately. The solution is updated at (n + 1)th level using the information of nth level solution. This is used with time step $\Delta t \approx \Delta x^{5/3}$ to ensure fifth order accuracy for accuracy test. The accuracy of the schemes are measured in L^∞-, L^1- and L^2-error norms which are defined as,

$$||e||_\infty = \max_j |u_j - (u_{\Delta x})_j|,$$
$$||e||_1 = \frac{1}{N+1} \sum_j |u_j - (u_{\Delta x})_j|,$$
$$||e||_2 = \left(\frac{1}{N+1} \sum_j |u_j - (u_{\Delta x})_j|^2 \right)^{\frac{1}{2}},$$

where N denotes the number of subdivisions in the domain while u_j and $(u_{\Delta x})_j$ denotes the exact and approximate solutions (corresponding to $\Delta x = (b-a)/N$) at point x_j respectively over a mesh of width h.

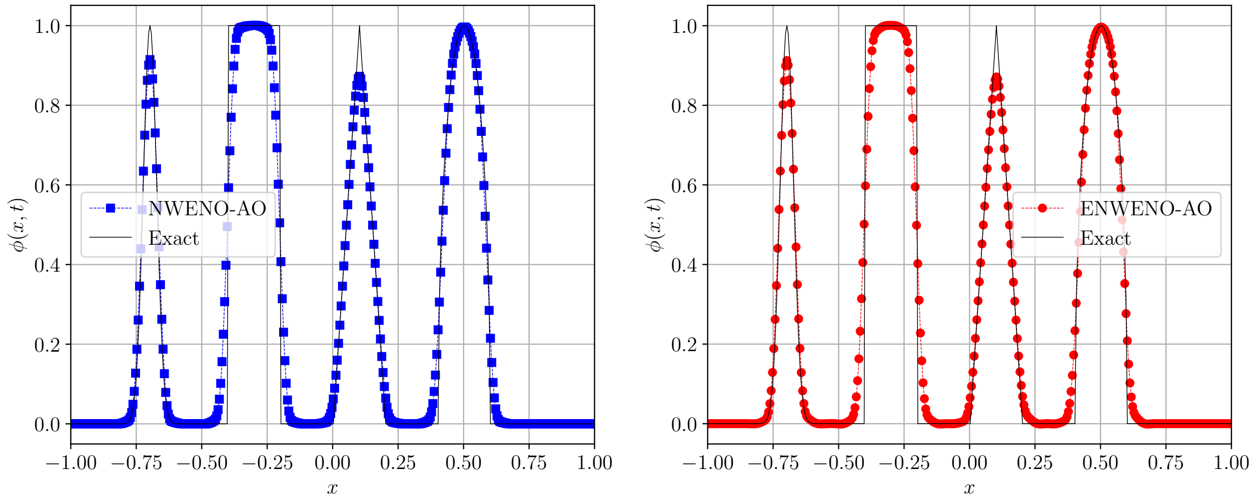
Example 5.0.1 *We consider the linear advection equation*

$$u_t + u_x = 0, \quad \forall \quad (x,t)\epsilon[-1,1] \times (0,T], \tag{5.1}$$

with the initial data $u(x,0) = \sin(\pi x)$*. with a periodic boundary condition. In Table 5.1, we have compared the errors obtained using MWENO-AO(5,3), NWENO-AO(5,3), and ENWENO-AO(5,3) scheme along with their convergence rate. We can observe that accuracy of schemes are comparable and all the scheme converge to exact solution with convergence rate five.*

Table 5.1: Comparison of accuracy and convergence rate of proposed schemes for Example 5.0.1.

N	MWENO-AO(5,3)					
	L^∞-error	Order	L^1-error	Order	L^2-error	Order
20	5.54741e-03	–	1.89023e-03	–	2.52643e-03	–
40	1.22104e-04	5.56	2.57181e-05	5.60	4.09636e-05	5.59
80	4.69096e-07	5.51	2.47569e-07	6.20	2.61557e-07	5.95
160	1.00853e-08	8.02	6.54270e-09	6.70	7.16573e-09	7.29
320	3.14209e-10	5.54	2.00630e-10	5.24	2.22302e-10	5.19
640	6.64102e-12	5.00	4.13649e-12	5.03	4.60809e-12	5.01
N	NWENO-AO(5,3)					
	L^∞-error	Order	L^1-error	Order	L^2-error	Order
20	3.23529e-04	–	2.08340e-04	–	2.30591e-04	–
40	1.02711e-05	5.56	6.55862e-06	5.60	7.27804e-06	5.59
80	3.22380e-07	4.98	2.05390e-07	4.99	2.28077e-07	4.99
160	1.00853e-08	4.99	6.42170e-09	5.00	7.13230e-09	5.00
320	3.14208e-10	5.00	1.99980e-10	5.00	2.22123e-10	5.00
640	6.64857e-12	5.00	4.13551e-12	5.01	4.60684e-12	5.00
N	ENWENO-AO(5,3)					
	L^∞-error	Order	L^1-error	Order	L^2-error	Order
20	3.23529e-04	–	2.08340e-04	–	2.30591e-04	–
40	1.02711e-05	5.56	6.55862e-06	5.60	7.27804e-06	5.59
80	3.22380e-07	4.98	2.05390e-07	4.99	2.28077e-07	4.99
160	1.00853e-08	4.99	6.42170e-09	5.00	7.13230e-09	5.00
320	3.14208e-10	5.00	1.99980e-10	5.00	2.22123e-10	5.00
640	6.64857e-12	5.00	4.13551e-12	5.01	4.60684e-12	5.00



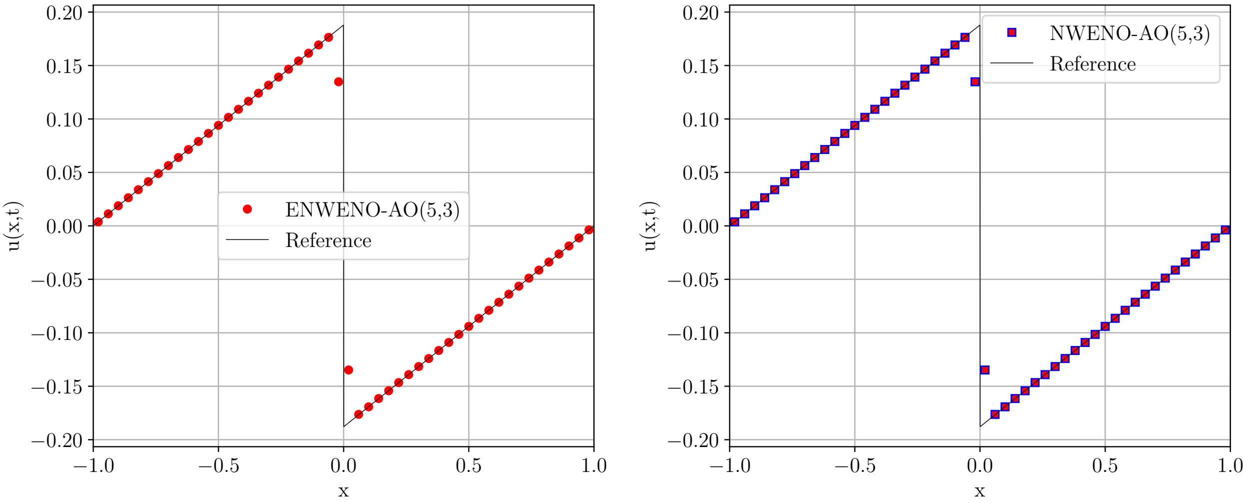
(a) $N = 40$ (b) $N = 80$

Figure 5.1: Comparison of solutions obtained using NWENO-AO(5,3) and EWENO-AO(5,3) for Example 5.0.2.

Example 5.0.2 *The linear equation* (5.1) *with the following initial condition is known as the Zalesak problem. This contains a narrow and smooth combination of Gaussian function, a discontinuous square function, a piece-wise linear triangle function and a semi ellipse function.*

$$\phi(x,0) = \begin{cases} \frac{1}{6}(G(x,\beta,z-\delta)+G(x,\beta,z+\delta)+4G(x,\beta,z)), & -0.8 \le x \le -0.6; \\ 1, & -0.4 \le x \le -0.2; \\ 1-|10(x-0.1)|, & 0 \le x \le 0.2; \\ \frac{1}{6}(F(x,\alpha,a-\delta)+F(x,\alpha,a+\delta)+4G(x,\alpha,a)), & 0.4 \le x \le 0.6; \\ 0, & \textit{otherwise} \end{cases}$$

where $G(x,\beta,z) = \exp^{-\beta(x-z)^2}$ *and* $F(x,\alpha,a) = \sqrt{\max(1-\alpha^2(x-a)^2,0)}$. *The values of the constants are taken as* $a = 0.5$, $z = -0.7$, $\delta = 0.005$, $\alpha = 10$ *and* $\beta = \log(2)/36\delta^2$. *The solutions obtained using the various WENO schemes including EWENO-AO(5,3) for* 400 *grid points at* $T = 20$ *is displayed in Figure 5.1. Both the schemes gives good approximations.*



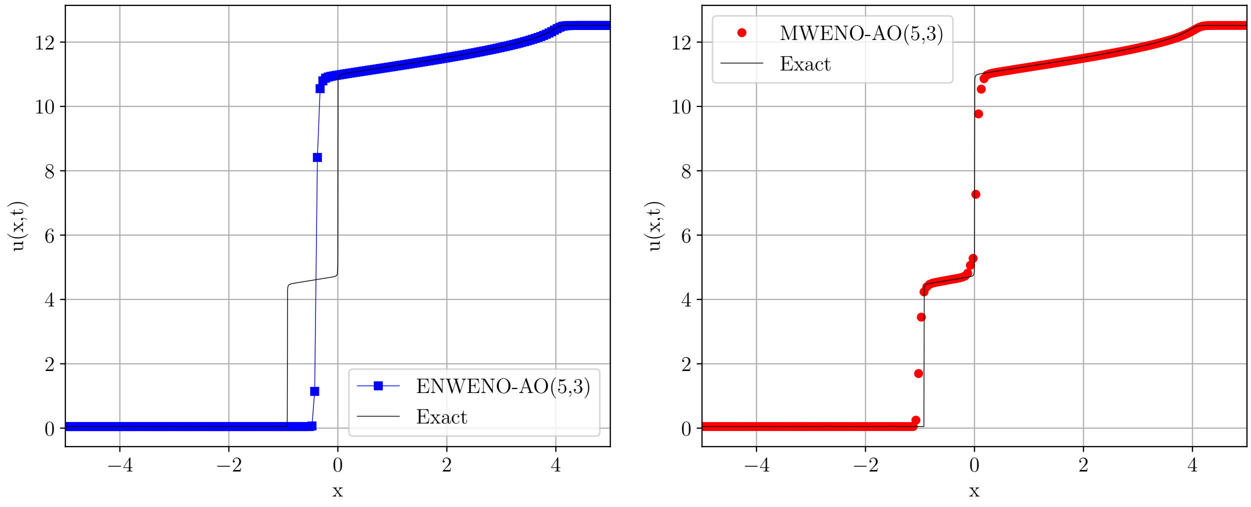
(a) $N = 40$ (b) $N = 80$

Figure 5.2: Comparison of solutions obtained using NWENO-AO(5,3) and EWENO-AO(5,3) for Example 5.0.3 over a grid $N = 40$ at $T = 5$.

Example 5.0.3 *Consider the one-dimensional Burgers' equation*

$$u_t + \left(\frac{u^3}{3}\right)_x = 0$$

with the initial data $u(x, 0) = -\sin(\pi x)$ *over the domain [-1, 1] with a periodic boundary condition. In Figure 5.2, we have compared the solution obtained using NWENO-AO(5,3) and ENWENO-AO(5,3) scheme using 40 grid points at the time* $T = 5$*. Both the scheme are able to capture the shock without oscillations.*



(a) $N = 40$ (b) $N = 80$

Figure 5.3: Comparison of solutions obtained using MWENO-AO(5,3) and EWENO-AO(5,3) for Example 5.0.4 for $N = 40$ and $N = 80$.

Example 5.0.4

$$u_t + (\sin(u))_x = 0 \quad (x,t) \in (-5,5) \times (0,t] \tag{5.2}$$

with initial condition

$$u_0(x) = \begin{cases} \frac{\pi}{64} & \text{if } x < 0 \\ \frac{255\pi}{64} & \text{if } x > 0 \end{cases}$$

The computational domain is $[-5,5]$ with Dirichlet boundary conditions on both sides. It has been reported in literature that for this problem the high order WENO schemes can fail to capture the correct entropy solution by missing the rarefaction between two shocks. In Figure 5.3, we have shown the results obtained using ENWENO-AO(5,3) and MWENO-AO(5,3) schemes over a grid of 200 points at time $T = 4$. We can see ENWENO-AO(5,3) fails to capture correct solution while MWENO-AO(5,3) converge to correct solution.

Chapter 6

Conclusion

- Three WENO Schemes where introduced where ENWENO-AO(5,3), we have proposed a new simple smoothness indicator for the bigger stencil using linear combination of lower order smoothness indicators.
- Theoretically proved that due to the introduction of an approximate smoothness indicator, the ENWENO-AO(5,3) scheme more efficient than NWENO-AO(5,3).
- ENWENO-AO(5,3) helps in saving 20 percentage of the computational time compared to NWENO-AO(5,3).
- MWENO-AO(5,3) will work well for non-convex flux where the rest of the schemes fails.
- WENO schemes is beneficial for solving Hyperbolic Conservation Law numerically.

Bibliography

[1] Dinshaw S. Balsara, Sudip Garain, Vladimir Florinski, and Walter Boscheri. An efficient class of WENO schemes with adaptive order for unstructured meshes. *J. Comput. Phys.*, 404:109062, 32, 2020.

[2] Rafael Borges, Monique Carmona, Bruno Costa, and Wai Sun Don. An improved weighted essentially non-oscillatory scheme for hyperbolic conservation laws. *J. Comput. Phys.*, 227(6):3191–3211, 2008.

[3] Xiaofeng Cai, Jianxian Qiu, and Jingmei Qiu. Finite volume HWENO schemes for nonconvex conservation laws. *J. Sci. Comput.*, 75(1):65–82, 2018.

[4] Marcos Castro, Bruno Costa, and Wai Sun Don. High order weighted essentially non-oscillatory weno-z schemes for hyperbolic conservation laws. *Journal of Computational Physics*, 230(5):1766 – 1792, 2011.

[5] Asha K. Dond and Rakesh Kumar. Finite difference modified WENO schemes for hyperbolic conservation laws with non-convex flux. *Internat. J. Numer. Methods Fluids*, 93(10):3005–3019, 2021.

[6] Sigal Gottlieb, David I Ketcheson, and Chi-Wang Shu. High order strong stability preserving time discretizations. *Journal of Scientific Computing*, 38(3):251–289, 2009.

[7] Jan S. Hesthaven. *The Numerical Methods for Conservation Laws Book*. Society for Industrial and Applied Mathematics, 2018.

[8] Guang-Shan Jiang and Chi-Wang Shu. Efficient implementation of weighted ENO schemes. *J. Comput. Phys.*, 126(1):202–228, 1996.

[9] Rakesh Kumar and Praveen Chandrashekar. Simple smoothness indicator and multi-level adaptive order WENO scheme for hyperbolic conservation laws. *J. Comput. Phys.*, 375:1059–1090, 2018.

[10] Alexander Kurganov, Guergana Petrova, and Bojan Popov. Adaptive semidiscrete central-upwind schemes for nonconvex hyperbolic conservation laws. *SIAM J. Sci. Comput.*, 29(6):2381–2401, 2007.

[11] Jing-Mei Qiu and Chi-Wang Shu. Convergence of high order finite volume weighted essentially nonoscillatory scheme and discontinuous Galerkin method for nonconvex conservation laws. *SIAM J. Sci. Comput.*, 31(1):584–607, 2008.

[12] Chi-Wang Shu and Stanley Osher. Efficient implementation of essentially nonoscillatory shock-capturing schemes. *J. Comput. Phys.*, 77(2):439–471, 1988.

[13] Chi-Wang Shu and Stanley Osher. Efficient implementation of essentially non-oscillatory shock-capturing schemes, ii. *Journal of Computational Physics*, 83(1):32–78, 1989.