SMFD.

1. N letters.
N separate envelopes. | at least 1 letter in correct.

$$\Downarrow$$

$$P = 1 - \left(\text{no item correct}\right) \rightarrow \frac{D_N}{N!}$$

all wrong $\Rightarrow$ ~~(N-1)(N~~

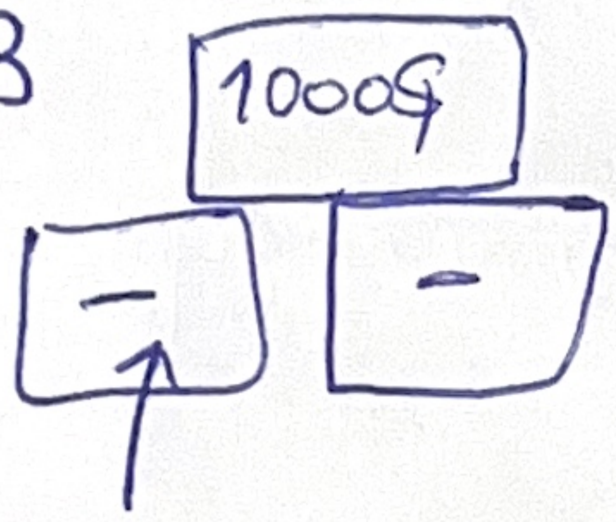
$$\hookrightarrow D_N = N!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \cdot \frac{(-1)^N 1}{N!}\right)$$

So $P = 1 - \frac{1}{2} + \frac{1}{3} + (-1)^{N+1}\frac{1}{N!}$

$N \to \infty$

$$\Rightarrow 1 - \frac{1}{e} \approx 0.6321$$

2. $N=3$



	P1	P2	P3
	0	0	2000 $\times\left(\frac{1}{3}\right)$
Chosen P_1	–	✓ host	Sw ✓
Chosen P_2	✓ host	–	Sw ✓
Chosen P_3	✓ host		– don't switch

win $= \frac{1}{2}$

win $= \frac{1}{2}$

$\frac{2}{3}$ cases switch

exp win $\Rightarrow$ 667.7

3. $P(B \cap C) > 0$. A B C D

a) $P(A \cap B|C) = P(A|B \cap C)P(B|C)$

$$P(A \cap B \cap C) = P(A|B \cap C) \cdot P(B \cap C)$$

$$\Rightarrow P(A \cap B|C) = \frac{P(A \cap B \cap C)}{P(C)}$$

3.

$$P(A\cap B|C) = \frac{P((A\cap B)\cap C)}{P(C)}$$

$$P(A|B\cap C) = \frac{P(A\cap B\cap C)}{P(B\cap C)}$$

$$P(B|C) = \frac{P(B\cap C)}{P(C)}$$

a) $P(A\cap B|C) = \frac{P(A|B\cap C)\, P(B\cap C)}{P(C)}$

$= P(A|B\cap C)\, P(B|C)$ A ✓

b) $P(A\cap B|C) = P(A|C)\, P(B|C)$ for indep A, B.

$P(A\cap B) = P(A)P(B)$

$P(A|C)P(B|C)$

Cond indep of A & B; given C.

B ✗.

c)

$P(A|D\cap B^c) > P(A|D\cap B)$

&

$P(A|D^c\cap B^c) > P(A|D^c\cap B)$

Compare $P(A/B) < P(A/B^c)$

C ✗

$P(A/B)$

$$P(A|D\cap B^c) = \frac{P(A\cap D\cap B^c)}{P(D\cap B^c)} > \frac{P(A\cap B\cap D)}{P(D\cap B)}$$

$$\frac{P(A\cap D^c\cap B^c)}{P(D^c\cap B^c)} > \frac{P(A\cap D^c\cap B)}{P(D^c\cap B)}$$

$P(A|B) = \cancel{P(A|D\cap B)\, P(D|B)}$

$$\frac{P(A\cap B)}{P(B)} \sim P(A|\cancel{B}\cap B) + P(A|D^c\cap B)$$

$$P(A|B^c) \sim P(A|D\cap B^c) + P(A|D^c\cap B^c)$$

$\Rightarrow P(A|B^c) > P(A|B)$

c) false

4.

a) X $\quad E(x) = \text{finite.}$

discrete $\cdot\, E(X^2) = \infty. \quad \Rightarrow \dfrac{c}{k^3}$

b) $X_{\text{cont.}}$

c) $E(x) = 1$

$E(e^{-x}) < 1/3$

5. $1, 2, \ldots N$ with rep. $\hookrightarrow$ n tickets.

$E(M)$

$\hookrightarrow$ money obtained = ?

probability mass function M

$$P(M=k) = P(M \leq k) - P(M \leq k-1)$$

$\hookrightarrow$ large N; Riemann Integral;

$$U = \frac{k}{N} \;;\; \text{as } N \to \infty;\; \frac{1}{N^n} \sum_{k=0}^{N-1} k^n = \frac{1}{N} \sum_{k=0}^{N-1} \left(\frac{k}{N}\right)^n \sim \int_0^1 u^n du = \frac{1}{n+1}$$

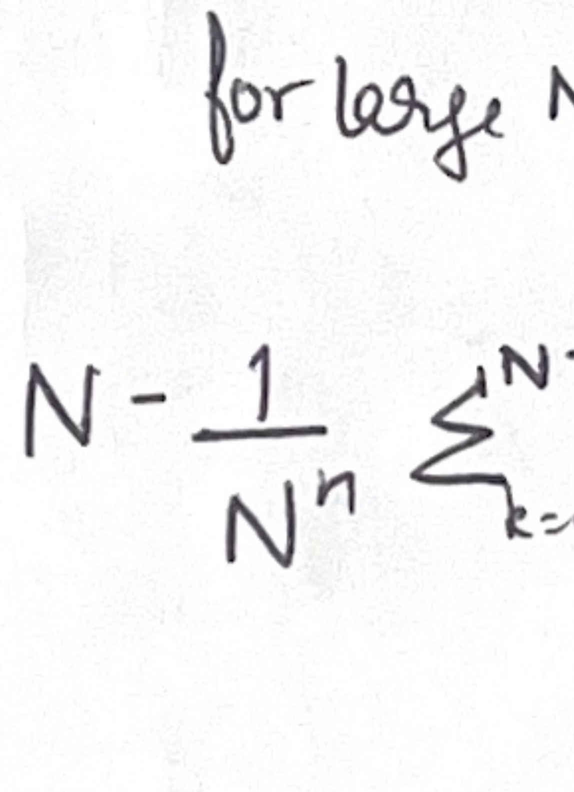
for large N; $E(M) \sim N - \dfrac{N}{n+1} \simeq \dfrac{nN}{n+1}$

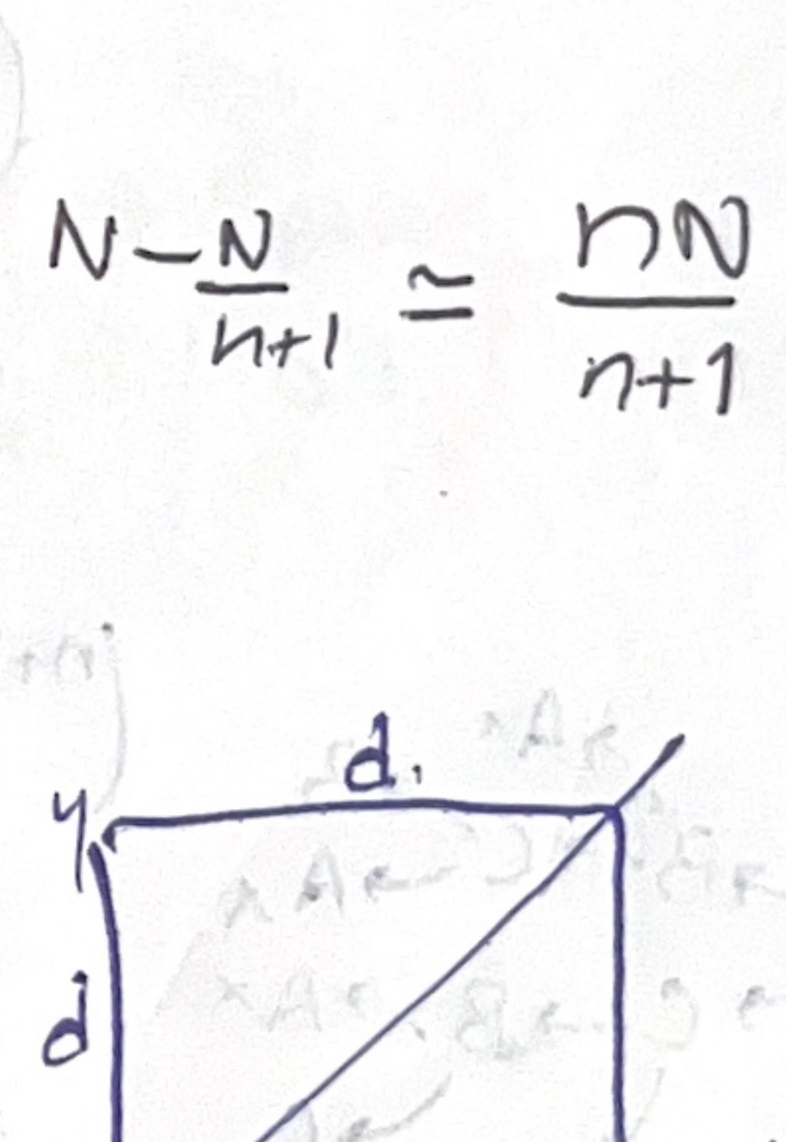
$$E(M) = N - \frac{1}{N^n} \sum_{k=0}^{N-1} k^n$$

6.

d.

$\overline{AB} < \dfrac{d}{3}$





$\hookrightarrow |a-b| < \dfrac{d}{3} \qquad -\dfrac{d}{3} < a-b < \dfrac{d}{3} \Rightarrow 1 - \left(\dfrac{2d}{3}\right)^2 \Rightarrow \dfrac{5d^2}{9}$

total $\Rightarrow d^2$

$\Rightarrow 5/9$

7. $(n+1)$

$n \quad n-1 \quad n-1$

1 2 3

a) $P(r)$

3 ppl.

$A \to B \to C \ \frac{1}{2}$
$\quad\ \ \frac{1}{2} \to A \ \frac{1}{2}$

$A \to C \to B \ \frac{1}{2}$
$\quad\quad\ \to A$

$\left(\frac{1}{2}\right)$ $P(\text{not to } A \text{ no rep})$ $= \dfrac{\frac{1}{2}\left(\frac{1}{2}\right)}{\frac{1}{2}} = 1$

$\dfrac{n(n-1)^{r-1}}{n^r} = \left(\dfrac{n-1}{n}\right)^{r-1}$

→ can tell back to person who told

for one set

$P(\text{not originator}) = \dfrac{n}{n+1}$

$P(\text{no repeat in } r \text{ steps}) = \left(\dfrac{n}{n+1}\right)^r$

total sets,

$P(\text{originator not in gap}) = \left(\dfrac{n}{n+1}\right)^N$

$P(\text{no return in } r \text{ steps}) = \left(\dfrac{n}{n+1}\right)^{Nr}$

b) $P(\text{w/o repetition}) \Rightarrow \cancel{\dfrac{n!}{(n-r)!\,(n+1)^r}}$

1. $\xrightarrow{n}$ 2 $\xrightarrow{n-1}$ 3 $\xrightarrow{n-2}$ $\times r$

r rows

$\dfrac{\cancel{n!}}{n^r}\ {}^{n}C_r$

$A \to B \to A \times$
$\quad\quad B \to C \to B$
$\quad\quad\quad\quad\ \to A \times$
$A \to C \to B \to A \times$
$\quad\quad C \to A$
$\quad\quad\quad\quad\ \to C$

$({}^{n+1}C_{r+1})$

r times

3 times

$n+1$

$\left[\dfrac{{}^{n}C_r \times r!}{n^r}\right] = \dfrac{n!\,\cancel{r!}}{\cancel{r!}(n-r)!} \times \dfrac{1}{n^r}$

$= \dfrac{n!}{(n-r)!\,n^r}$

8. $P(\bigcap_{i=1}^{n} A_i^c) \leq e^{-P(A_1) - P(A_2) - \cdots - P(A_n)}$

$\hookrightarrow A_i \Rightarrow$ indepevents

$\prod_{i=1}^{n} P(A_i^c) = \prod_{i=1}^{n} \underbrace{(1 - P(A_i))}_{\leq e^{-P(A_i)}}$

$\boxed{1+x \leq e^x}$

so

$\prod_{i=1}^{n} (1 - P(A_i)) \leq \prod_{i=1}^{n} e^{-P(A_i)}$

$\leq e^{-\sum P(A_i)}$

H.P

Alt Proof

$\ln(P(\bigcap_{i=1}^{n} A_i^c)) = \sum_{i=1}^{n} \ln(1 - P(A_i))$

$\ln(1+x) \leq x$ for $x \geq -1$

$\Rightarrow \ln(1 - P(A_i)) \leq -P(A_i)$

9. $F, G \Rightarrow$ 2 distribution functions F & G. $\rightarrow$ nondecreasing right continuous.

convolution $(F*G)$:

$$(F*G)(z) = \int_{-\infty}^{\infty} F(z-t)\, dG(t)$$

$\int_{x}^{-\infty} H(x) = 0$ & $\int_{x}^{\infty} H(x) = 1$

$$= \int_{-\infty}^{\infty} G(z-t)\, dF(t)$$

$\rightarrow$ F, G: dist func.

- $F(z_1 - t) \leq F(z_2 - t)$

$dG(t)$ = non-ve

so $\int F(z_1 - t)\, dG(t) \leq \int F(z_2 - t)\, dG(t)$

$\Rightarrow$ monotonicity ✓

$\rightarrow \lim_{h \to 0^+} H(z+h) = \lim_{h \to 0^+} \int_{-\infty}^{\infty} F(z+h-t)\, dG(t) = \int_{-\infty}^{\infty} F(z-t)\, dG(t) = H(z).$

right continuity ✓

$\rightarrow$ lower $\int_{z}^{\infty}$ $\Rightarrow \lim_{z\to-\infty} H(z) = \lim_{z\to-\infty} \int_{-\infty}^{\infty} F(z-t)\, dG(t)$

@ fixed t $\xrightarrow{\ }$ $= 0$. $\Big| \Rightarrow \lim_{z\to-\infty} H(z) = \int_{-\infty}^{\infty} 0 \, dG(t) = 0$

upper. $\int_{z}^{\infty}$ $\lim_{z\to\infty} \underbrace{\int_{-\infty}^{\infty} F(z-t) G(t)}_{=1}$

$\lim_{z\to\infty} H(z) \Rightarrow \int_{-\infty}^{\infty} 1 dG(t) = G(\infty) - G(-\infty)$

$= 1 - 0$

$= \underline{\underline{1}}$

11. $u \in \mathbb{R}$

convex function $\varphi(x) = e^{ux}$ $x \in \mathbb{R}$

$\mu = EX$ ↳ normal random variable

$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$\sigma = [E(X-\mu)^2]^{1/2}$,

i) $Ee^{uX} = e^{u\mu + \frac{1}{2}u^2\sigma^2}$