INTEREST RATE

Interest rate represents time value of money.

- Tradables
  - Instruments
    - Cash
    - Cash Bonds.
  - Claims
    - Vanilla
      - Discount bonds / zero coupon bonds
      - Forwards
      - Coupon bonds
      - Bond options.
      - Swap
      - Caplets / Floors
      - Swaptions
    - Exotics
      - Callable swaps / Comcallable swaps

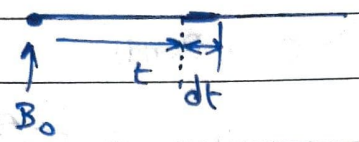
INSTRUMENTS :

i) CASH :
- → Notes issued by country's central / Federal bank
- → Value of cash stays same throughout time, i.e. zero interest rate risk
- → Most liquid.
- → Inflation might affect the buying power of that cash. For example if a basket of goods worth $100 a year ago costs $110 now then buying capacity of that cash has gone down.

ii) CASH BOND / BANK ACCOUNT

Qualitative : If we deposit some cash in a comercial / Investment bank's saving account then it grows with some interest rate.

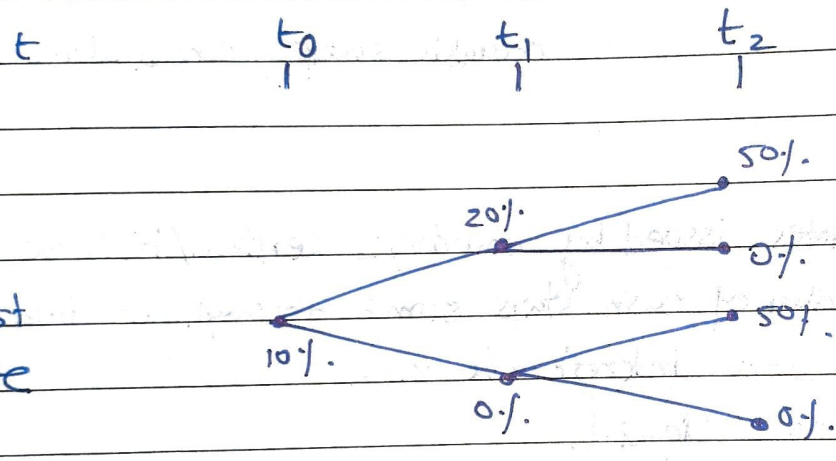
Quantitative: Type of interest rate can quantitately determine valuation of cash bond. For sake of consistency we assume continuously compounded interest rate

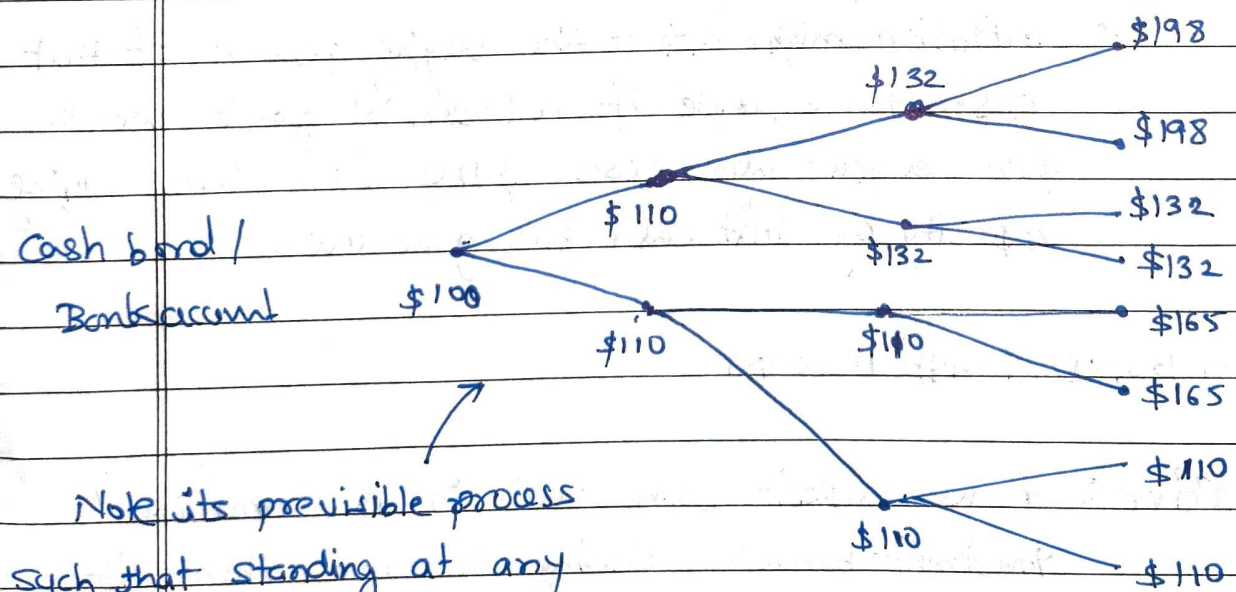


$$dB_t = r_t \cdot B_t \cdot dt$$

r_t is interest rate fixed at time t for duration of dt from t to $t+dt$

$$\therefore \quad B_t = B_0 e^{\int_0^t r_s ds}$$





Note its previsible process such that standing at any node t_i we know exactly what would be bond value at t_{i+1} since we know interest rate at t_i already

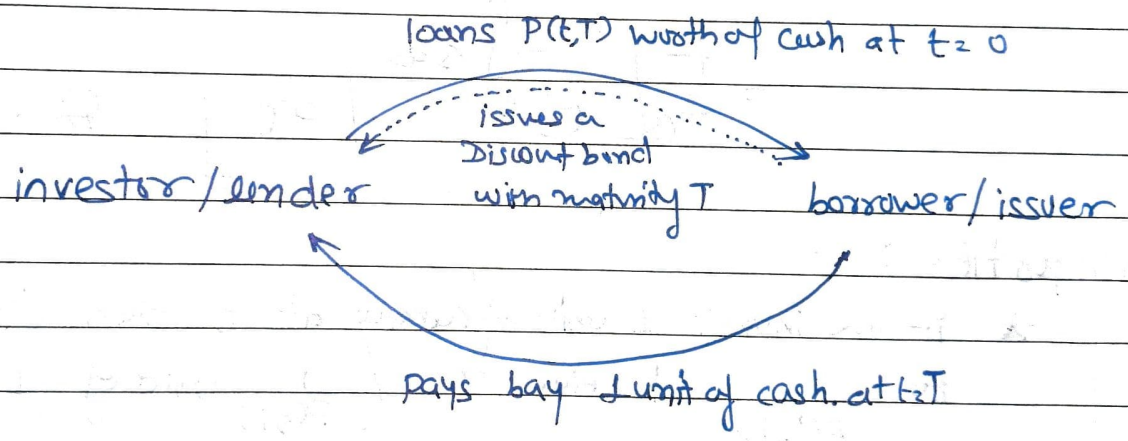
CLAIM > VANILLAS

DISCOUNT BOND / ZERO COUPON BOND:

Qualitative: Claim bonds are fixed income trables that allow borrower to take loan at some 'issuance price' & issue a bond and after maturity (T) pay back the face value of bond (issuance price + interest) to lender/investor i.e.

Claim of one unity of currency at T is called ZCB.

Its value/worth/price at any time t is denoted by P(t,T)



TYPES OF Discount Bonds.

Depending upon issuer ther can be various types of bonds

1. GOVENMENT / SOVERIGN / TRESURY
   a. issued by govt to raise funding for road/shools/dams/infra etc
   b. Most security, almost zero credit risk.

2. MUNICIPAL BONS:
   a. Issued by state/local municipal to raise funding for their respective projects
   b. May offer some tax benefits.

3. CORPORATE BONDS..
   a. Have credit Risk b. Issued by companies to grow business

QUANTITATIVE:

Assuming Numeraire of market to be ~~as~~ cash bonds / bank account (i.e. $N_t = B_t$), using risk neutral pricing worth of a claim $X = 1$ paid at time T is given by.

$$\text{Price} = P(t,T) = B_t \cdot E_Q\left[B_T^{-1} X \mid \mathcal{F}_t\right]$$

where Q is measure which make $B_t \cdot B_t^{-1}$ martingale. since $B_t \cdot B_t^{-1} = 1$ is alway martingale, Q can be any measure.

$$\therefore \quad P(t,T) = B_t E_Q\left[B_T^{-1} \mid \mathcal{F}_t\right] = E_Q\left[\frac{B_t}{B_T} \mid \mathcal{F}_t\right]$$

$$= E_Q\left[e^{-\int_t^T r_s ds} \mid \mathcal{F}_t\right]$$

for any Q (if there are no other assets in market otherwise Q^R: i.e. risk neutral measure)

Properties:

★ if we invest 1 unit of money at t then it would grow to $\frac{1}{P(t,T)}$ at time T ($\frac{1}{P(t,T)}$ units of 1 ZCB) at value at t

Note: \$1 at t can grow to many different function of instrument depending upon market & instrument., For example a claim: $Y = B_T / B_t$ will have price at $t = 1$ as well but that does not mean that $P(t,T) = B_T / B_t = f(\text{other instruments})$ for all $t \le T$.

★ $P(t,t) = 1$ i.e. worth of \$1 paid at t, valued at t is same \$1

★ ZCB is used as numeraire in T-forward measure, specifically

$$N_t = \frac{P(t,T)}{P(0,T)}$$ here T is constant coming from T of T-forward measure.

$$N_0 = 1. \qquad N_T = \frac{P(T,T)}{P(0,T)} = \frac{1}{P(0,T)}$$

This is quite useful in cases when we want to estimate fair value of a claim by just taking its expectation. For example, if we have a claim X paid at T then it value at any time t before T is

$$\text{Price}_t(X) = N_t \cdot E_{Q^T}\left[N_T^{-1} \cdot X \mid \mathcal{F}_t\right]$$

Substituting N_t from above

$$= \frac{P(t,T)}{P(0,T)} \cdot E_{Q^T}\left[\frac{P(0,T)}{P(T,T)} \cdot X \mid \mathcal{F}_t\right]$$

$P(T,T) = 1.$

$$= P(t,T) \cdot E_{Q^T}\left[X \mid \mathcal{F}_t\right]$$

Notice how there is no other term inside expectation other than X

★ $T_1 \quad T_2 \quad T_3$.

$$E\left[P(T_2,T_3)\,|\,\mathcal{F}_{T_1}\right] = \frac{P(T_1,T_3)}{P(T_1,T_2)}$$

Proof

$$E\left[E\left[e^{-\int_{T_2}^{T_3} r_s ds}\,|\,\mathcal{F}_2\right]\,|\,\mathcal{F}_{T_1}\right]$$

$$= E\left[e^{-\int_{T_1}^{T_3} r_s ds}\,|\,\mathcal{F}_1\right]$$

multiply & divide by $E\left[e^{-\int_{T_1}^{T_2} r_s ds}\,|\,\mathcal{F}_1\right]$

$$= E\left[e^{-\int_{T_1}^{T_3} r_s ds}\,|\,\mathcal{F}_1\right] / E\left[e^{-\int_{T_1}^{T_2} r_s ds}\,|\,\mathcal{F}_1\right]$$

★ ZCB is also used as Numerair for Libor forward measure.

i.e. if we assume $T_1\, T_2 \cdots T_n$ to importaint claim dates for a particallar deal starting at T_0 than Numeraire for i^{th} date is defined as

$$N_i = \frac{1}{1 \prod_{j=0}^{i-1} P(T_j, T_{j+1})}$$

$$N_0 = 1 \;;\; N_1 = \frac{1}{P(T_0,T_1)} \;,\; N_2 = \frac{1}{P(T_0,T_1)\,P(T_1,T_2)} \cdots$$

$$\cdots N_{T_n} = \frac{1}{P(T_0,T_1)\cdots\cdots P(T_{n-1},T_n)}$$

Example:

$T_0 \quad T_1 \quad T_2 \quad T_3 \quad T_4$ ↑ \$1.

What's worth of \$1 paid a T_4 as of T_1.

Risk Neutral Measure.

$$\cdots = B_{T_1} \cdot E_Q\left[B_{T_4}^{-1} \cdot 1\,|\,\mathcal{F}_1\right] = E_Q\left[e^{-\int_{T_1}^{T_4} r_s ds}\,|\,\mathcal{F}_1\right]$$

T4-forward Measure

$$= \frac{P(T_1,T_4)}{P(0,T_4)}\, E_Q\left[\left(\frac{P(T_4,T_4)}{P(0,T_4)}\right)^{-1} 1\,|\,\mathcal{F}_1\right]$$

$$= P(T_1,T_4) \cdot E_Q\left[1\,|\,\mathcal{F}_1\right] = P(T_1,T_4)$$

$$= E_Q\left[e^{-\int_{T_1}^{T_4} r_s ds}\,|\,\mathcal{F}_1\right]$$

Libor forward Measure.

$$= \frac{1}{P(T_0,T_1)} E_Q\left[P(T_0,T_1)\cdot P(T_1,T_2)P(T_2T_3)P(T_3T_4) \,\middle|\, \mathcal{F}_1 \right]$$

$$= E_Q\left[E_Q\left[P(T_1,T_2)P(T_2T_3)P(T_3T_4) \,\middle|\, \mathcal{F}_2 \right] \,\middle|\, \mathcal{F}_1 \right]$$

(using Tower law

$$= E_Q\left[P(T_1T_2)P(T_2T_3)\cdot E_Q\left[P(T_3T_4) \,\middle|\, \mathcal{F}_2 \right] \,\middle|\, \mathcal{F}_1 \right]$$

(using property above

$$= E_Q\left[\cancel{P(T_1T_2)P(T_2T_3)}\cdot \frac{P(T_2T_4)}{\cancel{P(T_2T_3)}} \,\middle|\, \mathcal{F}_1 \right]$$

$$= E_Q\left[P(T_1,T_2)\cdot P(T_2T_4) \,\middle|\, \mathcal{F}_1 \right]$$

$$= E_Q\left[P(T_1,T_4) \,\middle|\, \mathcal{F}_1 \right] = P(T_1,T_4)$$

RELATION BETWEEN Q^R, Q^T & Q^L

★ ~~Relation between~~ CONVERSION from Q^R to Q^T

(1) if we have claim at T (let's call it X_T) then its expectation under Q^R can be translated to Q^T using

$$E_{Q^R}[X_T] = E_{Q^T}\left[\frac{dQ^T}{dQ^R} \cdot X_T\right]$$

we know $\frac{dQ^T}{dQ^R} = \left(\frac{N^{Q^T}(T)}{N^{Q^T}(0)}\right) \Big/ \left(\frac{N^{Q^R}(T)}{N^{Q^R}(0)}\right)$ $\left\{ \begin{array}{l} N^{Q^T} = \text{Numeraire in } Q^T \\ N^{Q^R} = \text{Numeraire in } Q^R \end{array} \right.$

$$N^{Q^T}(t) = \frac{P(t,T)}{P(0,T)} \quad \text{and} \quad N^{Q^R}(t) = B(t)$$

$$\therefore \frac{dQ^T}{dQ^R} = \frac{P(T,T)}{P(0,T)} \Big/ \frac{B(T)}{B(0)} = \frac{B(0)}{B(T)P(0,T)}$$

$$\therefore E_{Q^R}[X_T] = E_{Q^T}\left[\frac{B(0)}{B(T)P(0,T)} \cdot X_T\right]$$

(2) if we have process X_T then it its conditional expectation at t can be translated to

$$E_{Q^R}[X_T | \mathcal{F}_t] = \xi_t E_{Q^T}[X_T \cdot \xi_T^{-1} | \mathcal{F}_t]$$

where $\xi_t = E_{Q^R}\left[\frac{dQ^T(t)}{dQ^R} \Big| \mathcal{F}_t\right]$, since $\frac{dQ^T}{dQ^R}$ is martingale

$$\xi_t = \frac{dQ^T(t)}{dQ^R} = \frac{P(t,T)}{P(0,T)} \times \frac{B(0)}{B(t)}$$

$$E_{Q^R}[X_T | \mathcal{F}_t] = \frac{P(t,T)}{B(t)} E_{Q^T}[X_T \times B(T) | \mathcal{F}_t]$$

Conversion from Q^T to Q^L

$T_0 \quad T_i \quad T_{i+1}$

* if we have process X_t at $t = T_{i+1}$ & its conditional expectation at T_i can be converted from ~~$Q^{T_{i+1}}$ to~~ $Q^{T_{i+1}}$ to Q^L

$$E_{Q^{T_{i+1}}}[X_{T_{i+1}} \mid \mathcal{F}_{T_i}] = \xi_{T_i} E_{Q^L}[\xi_{T_{i+1}} X_{T_{i+1}} \mid \mathcal{F}_{T_i}]$$

where $\xi_t = \dfrac{dQ^L}{dQ^{T_{i+1}}}(t) = \left(\dfrac{N^{Q^L}(t)}{N^{Q^L}(0)}\right) \Big/ \left(\dfrac{N^{Q^{T_{i+1}}}(t)}{N^{Q^{T_{i+1}}}(0)}\right)$

$$N^{Q^L}(t) = \frac{1}{\prod_{j=0}^{k-1} P(T_j, T_{j+1})} \qquad T_k < t < T_{k+1}$$

$$N^{T_{i+1}}(t) = \frac{P(t, T_{i+1})}{P(0, T_{i+1})}$$

$$\therefore \xi_t = \frac{1}{\prod_{j=0}^{k-1} P(T_j, T_{j+1})} \times \left(\frac{P(0, T_{i+1})}{P(t, T_{i+1})}\right)$$

$$\therefore \xi_{T_i} = \frac{P(0, T_{i+1})}{\prod_{j=0}^{i} P(T_j, T_{j+1})} \qquad (k = i)$$

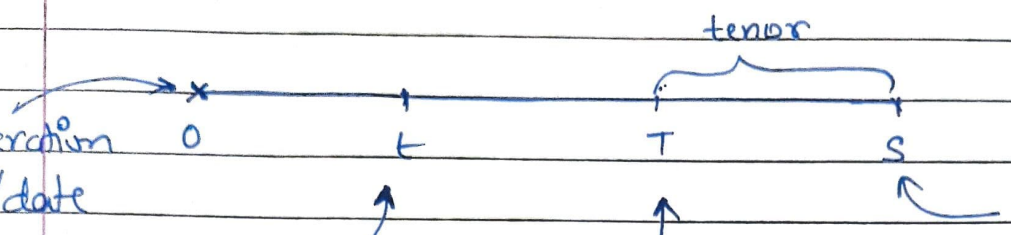
$$\xi_{T_{i+1}} = \frac{P(0, T_{i+1})}{\prod_{j=0}^{i} P(T_j, T_{j+1})\, P(T_{i+1}, T_{i+2})} \qquad (k = i+1)$$

$$(\xi_{T_i} = \xi_{T_{i+1}})$$

$$E_{Q^{T_{i+1}}}[X_{T_{i+1}} \mid \mathcal{F}_{T_i}] = E_{Q^L}[X_{T_{i+1}} \mid \mathcal{F}_{T_i}]$$

i.e. Conditional distribution of process at T_{i+1} conditioned at T_i is same between $Q^{T_{i+1}}$ forward measure & Q^L

TYPES OF INTEREST RATES



	Simple Compounded (SC)	Continuous Compounded (C.C)
FORWARD (t<T)	$\frac{1}{P(t,S)} = \frac{1}{P(t,T)}\left(1 + S.C.F.(S-T)\right)$	$\frac{1}{P(t,S)} = \frac{1}{P(t,T)} e^{C.C.F(S-T)}$
	$\therefore SCF = \frac{1}{(S-T)}\left(\frac{P(t,T)}{P(t,S)} - 1\right)$ (Forward Rate agreement)	$\therefore CCF = \frac{\log\left(\frac{P(t,T)}{P(t,S)}\right)}{(S-T)}$ (forward)
SPOT (t=T) (t=t)	$SC.S = \frac{1}{(S-t)}\left(\frac{1 - P(t,S)}{P(t,S)}\right)$	$SC.S = \frac{-\log(P(t,S))}{(S-t)}$ (yield)

INSTANTANEOUS FORWARD SC. IF =

~~T=t~~

$S = T + dT$

$$CC.I.F = \lim_{\Delta T \to 0} \frac{-\left(\log(t,T+\Delta T) - \log(t,T)\right)}{\Delta T}$$

~~CFO~~ Instantaneous forward

$$(f(t,T) = -\frac{\partial \log(t,T)}{\partial T}$$

INSTANTANEOUS S.C. IS SPOT.

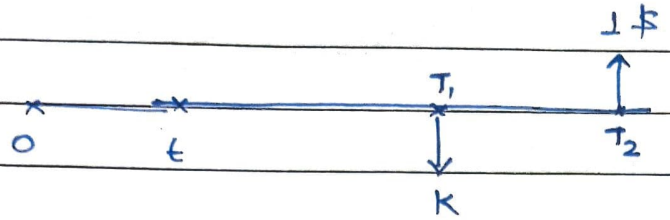
$S = t + \Delta T$

$T = t$

$$C.C.IS = \lim_{\Delta t \to 0} \frac{-\log P(t,t+\Delta t)}{\Delta t}$$

$$= -\frac{\partial}{\partial t} \log(P(t,t+dt))$$

FORWARD CONTRACT



Pay K dollar at T_1 & receive 1$ at T_2

$$\text{Price}(t) = B_t E_Q\left[-K \cdot B_{T_1}^{-1} + 1 \cdot B_{T_2}^{-1} \,\Big|\, \mathcal{F}_t\right]$$

$$\text{using} = \left(E_Q\left(\frac{B_i}{B_j} \,\Big|\, \mathcal{F}_i\right) = P(t_i, t_j) \right)$$

$$\text{Price}(t) = -K_t\left(P(t,T_1) + P(t,T_2)\right)$$

at t wto make ~~fair~~ fair deal, Price (t) = 0

$$\therefore \quad K_t = P(t,T_2) / P(t,T_1) \quad \text{i.e.} \quad K_0 = \frac{P(0,T_2)}{P(0,T_1)}$$

BOND OPTION:

ZERO BOND.

CALL: Bond option gives right to buy a bond at future date (ZBC) for a given price. An option on a T maturity bond with strike K exercised at $t_{decision}$ ~~given by~~ has worth

$$\text{Price}(t) = B_t E_Q\left(B_{t_{decision}}^{-1} \left(P(t_{decision}, T) - K\right)^+ \Big| \mathcal{F}_t\right)$$

where $t < t_{decision}$

$$ZBC(t, t_{decision}, T, K) = P(t, t_{decision}) \cdot E_{Q^T}\left(\left(P(t_{decision}, T) - K\right)^+ \Big| \mathcal{F}_t\right)$$

ZERO BOND PUT (ZBP): Put option gives owner to sell T-maturity ZCB at $t_{decision}$ against strike K

$$ZBP(t, t_{decision}, T, K) =$$

$$P(t, t_{decision}) \, E_{Q^T}\left[\left(K - P(T_{decision}, T) \right)^+ \middle| \mathcal{F}_t \right]$$

Note: if we have a bond with coupons ~~then~~ (i.e. $P(T_{desim}, T) + \sum_i P(T, T_{i+1}) \cdot K \cdot \delta_i$ is underlying then above expression is call "coupon bond option"

CAPLETS / CAPS.

~~1 unit currency. Simply compounded with~~

→ A caplet claim, caps the spot forward rate with some strike. This allows an Institute/ buyer to get protected from unexpect jump in spot forward rate.

from T to S

$$X = \left(\text{Spot Rate}(T, S) - K \right)^+ \cdot N \cdot (S - T)$$

Period / day count fractio

Notional

$$\text{Spot Rate}(T, S) = \frac{1}{(S-T)} \left(\frac{1}{P(T,S)} - 1 \right)$$

(simply compounded)

$$X = N(S-T) \left(\frac{1}{(S-T)} \left(\frac{1}{P(T,S)} - 1 \right) - K \right)^+$$

$$= \frac{N}{P(T,S)} \left(1 - P(T,S) - K P(T,S)(S-T) \right)^+$$

$$= \frac{N}{P(T,S)} \left(1 + K(S-T) \right) \left(\frac{1}{1 + K(S-T)} - P(T,S) \right)^+$$

Date: / /

$$\text{Price}(t) = P(t,S)\, E_{Q^L}\left[\frac{N}{P(T,S)}(1+K(S-T))\left(\frac{1}{1+K(S-T)} - P(T,S)\right)^+ \Big| \mathcal{F}_t\right]$$

$$= N\, P(t,T)\,(1+K(S-T))\, \underbrace{E_{Q^T}\left(\left(K' - P(T,S)\right)^+ \Big| \mathcal{F}_t\right)}_{ZBP(t,T,S,K')}$$

$$\left(P(t,S) = P(t,T)\,P(T,S) \right)$$

$$= N(1+K(S-T)) \times ZBP(t,T,S,K')$$

with

where $K' = \left(\frac{1}{1+K(S-T)}\right)$

day count fraction

CAPS:

if we have time periods $T_0\ T_1, \ldots T_{n-1}, T_n$ and we want to protect against spot forward between T_i to T_{i+1} then we execute series of Caplets each with $T = T_i$ & $S = T_{i+1}$

$$\text{Price}(t) = \sum_i P(\text{Caplets}_i(t))$$

$$= N \sum_i (1+K_i S_i)\, ZBP(t, T_i, T_{i+1}, K_i')$$

$S_i = (T_{i+1} - T_i)$ & $K_i' = \left(1/(1+K_i S_i)\right)$

FLOORLETS / FLOORS.

FLOOLETS floors the spot forward rate from T to S with some strike, this allows buyers to protect selling price.

$$X = \left(K - \text{SpotRate}(T,S)\right)^+ N.(S-T)$$

$$\text{Price}(t) = N(1+K(S-T))\, ZBC(t,T,S,K')$$ where $K' = \frac{1}{(1+K(S-T))}$

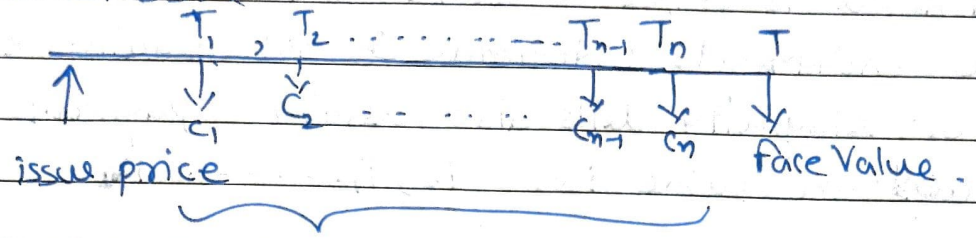
Floors : Series of floorlets for $T_0, T_1 \cdots T_{n-1}, T_n$

$$\text{Floors} = \sum_i \text{Floorlets}(T_i, T_{i+1})$$

$$\text{Price}(t) = N \cdot \sum_i (1 + K_i \delta_i)\, ZBC(t, T_i, T_{i+1}, K_i')$$

$$\delta_i = T_{i+1} - T_i \quad \& \quad K_i' = \frac{1}{1 + K\delta_i}$$

COUPON BONDS. (should be after ZCB & before forward contract)

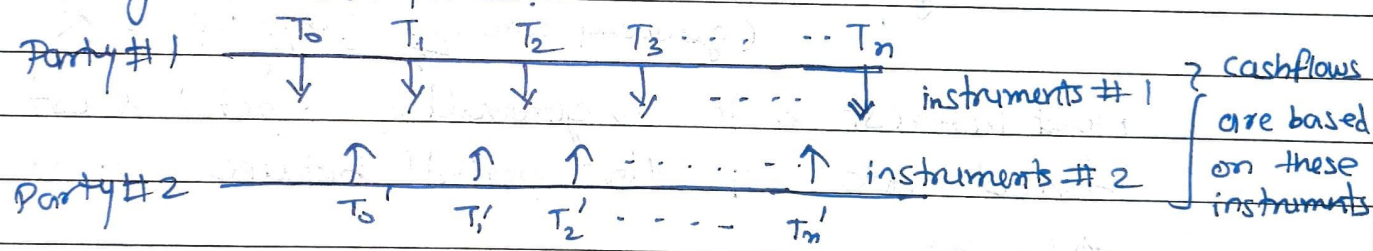


also pay coupons with some rate (on face value) at t_i for $i \in [0, n]$

$$\text{issue Price} = \sum C_i P(t_0, T_i) + F.V. P(t_0, T)$$

SWAP.

A swap is derivative contract through which two parties exchange the cash flows.



Swaps are not traded on exchanges. These are OTC (Over the counter) derivatives not engaged by retail investor but financial institutions.

Types Of SWAP :

INTEREST RATE SWAP :

Instrument # 1 : Fixed interest rate based on a Notional, called Fixed leg.

Instrument # 2 : Float interest rate based on Same Notional called Flout leg.

→ Notional (N) is fixed amount on which interest payments are exchanged. (Notional is not usually exchanged).
→ For simplicity let's also assume payments on both legs are same i.e. $T_0, T_1, \ldots, T_N$.

Quantitative:

this N is Number of periods

$$\text{Claim} = \text{Float leg} - \text{Fixed Leg}$$

$$= \sum_{i=0}^{N-1} N \cdot dcf_i \, L_i - \sum_{i=0}^{N-1} N \cdot dcf_i \, K_i$$

this is Notional.

where N : Notional
dcf_i : Day Count fraction = No. of day between T_i to T_{i+1}
L_i : Float interest rate between T_i to T_{i+1}
K_i : Fixed " " " = K
(generally does not vary)

Float interest rate is generally spot interest rate fixed at T_i for period between T_i to T_{i+1} given by

$$L_i = \frac{1}{dcf_i}\left(\frac{1}{P(T_i, T_{i+1})} - 1\right)$$

$$\therefore \text{claim}_{swap} = N \cdot \sum_{i=0}^{N_T-1}\left(\frac{1}{P(T_i, T_{i+1})} - 1 - dcf_i K\right)$$

Price

Fair value of interest rate swap at any time t is given by

$$= N_t \, E_Q\left[\text{claim}_{swap} \cdot N_T^{-1} \mid \mathcal{F}_t\right]$$

For making analytics easy, we will use libor forward measure Q with numeraire N_t,

$$N_t = \frac{1}{\left(\prod_{i=0}^{k-1} P(T_i, T_{i+1})\right) \cdot P(t_k, t)}$$

assuming t to be betwen (T_k, T_{k+1})

$$N_{T_{i+1}} = \frac{1}{\prod_{j=0}^{i} P(T_j, T_{j+1})}$$

$$\text{Price} = E_Q\left[\underbrace{\frac{1}{\left(\prod_{j=0}^{k-1} P(T_j, T_{j+1})\right) P(T_k, t)}}_{N_t} \times \underbrace{\sum_{i=0}^{N-1}\left(\frac{1}{P(T_i, T_{i+1})} - 1 - d(t_i, k)\right)}_{\times} \times \underbrace{\prod_{j=0}^{i} P(T_j, T_{j+1})}_{(N_{T_{i+1}})^{-1}} \,\middle|\, \mathcal{F}_t\right]$$

taking sumation over i outside.

$$= \sum_{i=0}^{N-1} E_Q\left[\frac{\prod_{j=k}^{i} P(T_j, T_{j+1})}{P(T_k, t)} \times \left(\frac{1}{P(T_i, T_{i+1})} - 1 - d(t_i, k)\right) \,\middle|\, \mathcal{F}_t\right]$$

Using property

$$\prod_{j=k}^{i} P(T_j, T_{j+1}) = \cancel{P(T_k, T_j)}\, P(T_k, t) \cdot P(t, T_i)$$

$$\& \prod_{j=1}^{i-1} P(T_j, T_{j+1}) = P(T_k, t) \cdot P(t, T_{i-1})$$

$$= \sum_{i=0}^{N-1} E_Q\left[P(t, T_i) - P(t, T_{i+1})(1 + dcf_i \cdot K) \,\middle|\, \mathcal{F}_t \right]$$

we know $E_Q\left[P(t,T) \mid \mathcal{F}_t \right] = P(t,T)$

$$= \underbrace{\sum_{i=k}^{N-1} \left(P(t, T_i) - P(t, T_{i+1}) \right)}_{\text{these summation terms will cancell out till } k+1} - \sum_{i=k}^{N-1} P(t, T_{i+1})\, dcf_i K$$

$$= \left(P(t, T_{k+1}) - P(t, T_N) \right) - K \sum_{i=k}^{N-1} P(t, T_{i+1}) dcf_i$$

$$= \underbrace{\left(\sum_{i=0}^{N-1} P(t, T_{i+1}) \cdot dcf_i \right)}_{\text{Annuity}} \left(\underbrace{\frac{P(t, T_{k+1}) - P(t, T_N)}{\sum_{i=0}^{N-1} P(t, T_{i+1}) \cdot dcf_i}}_{\text{swap rate}} - \underbrace{K}_{\text{strike /fixed rate.}} \right)$$

Note here k is index of Time periods such that $t \in (T_k, T_{k+1})$

(More Accurately)
Some times its called forward rate at we are looking at price of swap at $T_k < t < T_{k+1}$ it can be some fixed $t = T_k$

SWAPTION (SWAP OPTION)

Option to enter into a swap (underlying swap could be interestrate swap, credit default swap, commodity swap, cross currency swap etc.)

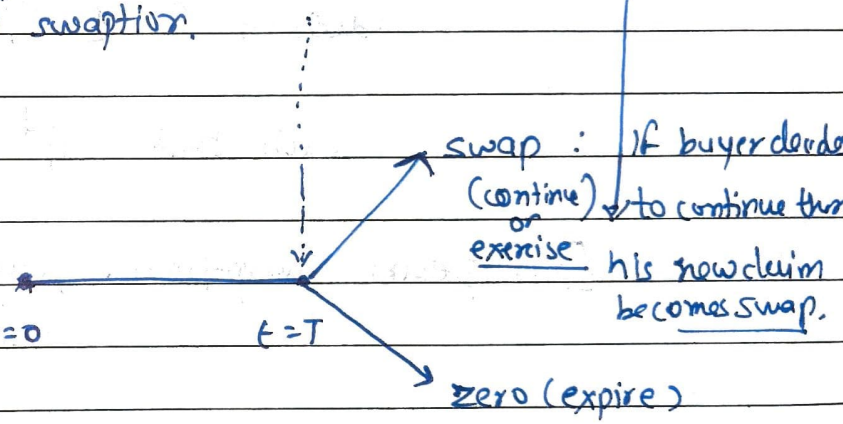
OTC (Not standardised i.e. Not traded on exchange)

Pays Price (swaption) also called premium

Seller → Buyer

in exchange of recieving swaption that gives buyer right to continue with swap (exercise) or expire at or before some time T, also called expiration of swaption.

* Earns the premium
* If buyer decides to continue, seller pays and receives either floating or fixed leg.

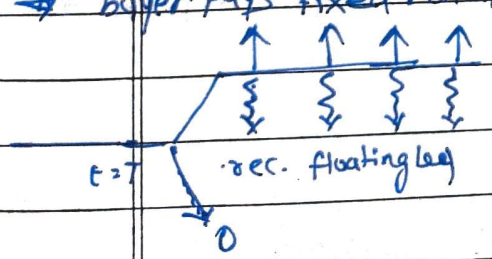


TYPES OF SWAPTIONS.

1. Based on direction of underlying Swap. with respect to purchaser (buyer)

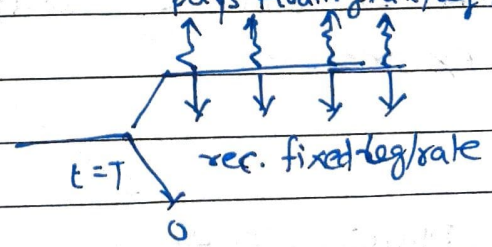
Payer Swaption

* buyer pays fixed rate/leg



Receiver ~~Buyer~~ Swaption

pays floating rate/leg.



2. Based on number of decision dates.

American Swaption : Buyer can exercise his option (i.e. Enter into swap) between any date between start of deal till expiration of swaption.

$$T_{decision} \in [0, T]$$

Bermudan swaption : Buyer can exercise at predermined dates between origin of deal until expiry

$$T_{decision} \in \{ T_1^d, T_2^d, T_3^d \cdots T_n^d \}$$

such that $T_i^d \in [0, T]$

European Swaption : Buyer can exercises or expire only at expiry of swaption

$$T_{decision} = T$$

EUROPEAN SWAPTION

There are further types of european swaption based on ~~what~~ is nature of swap two parties enter into after an exercise.

Physical settled

Cash settled.

Definition. if buyer choose to exercise, both enter into ~~new~~ contract, swap i.e. after T they will follow fixed/float leg schedule of underlying swap.

if buyer choose to exercise, then expection of swap / price of swap at T is settled in cash.

Claim: if(price(swap)@T > 0) swap else 0

here we are entering into another series of cashflows dictated by swap. (so actually deal will continue until swap's last cashflow)

= if (price(swap)@T > 0) Price(swap)@T else 0

$:= \text{Max}(\text{price}(\text{swap})@T, 0)$

Note: here we are directly settling cash equal to Price(swap)@T at T

CASH SETTLED EURO SWAPTION:

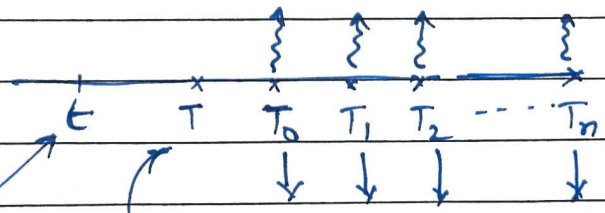
$$\text{Claim}(X) = \text{Max}(\text{price}(\text{swap})@T, 0)$$

where $\text{price}(\text{swap})@T = N_T E_Q[\text{swap} \cdot N_{T+Tenor}^{-1} \mid \mathcal{F}_T]$

N_t is numerair & Q is measure s.t. $N_t^{-1} \cdot B_t$ is martingale.

from swap section, Price(swap)@t is

swap:



any time t before T_0

Expiration of swaption

$$\text{Price}(\text{swap})@t = (P(t,T_0) - P(t,T_n)) - K\sum_{i=0}^{n-1} P(t,T_{i+1})\,dcf_i$$

where $dcf_i = (T_{i+1} - T_i)$

Assuming first swap payment date (T_0) is same swaption expiration.

and we want Price of this swap at $t = T \; (= T_0)$

$$\therefore \text{Price(swap)@}T = \left(1 - P(T, T_n) - K \sum_{i=0}^{n-1} P(T, T_{i+1}) \, dcf_i\right)$$

Now going back to swaption:

$$\therefore \text{Price (swaption)@}t = N_t \cdot E_Q\left[\text{claim}_{\text{swaption}} \times N_T^{-1} \mid \mathcal{F}_t\right]$$

$$= N_t \, E_Q\left[\max\left(\text{price(swap)@}T, 0\right) \cdot N_T^{-1} \mid \mathcal{F}_t\right]$$

we can use $Q = Q_{\text{Libor fwd}}$ $\therefore N_t = \dfrac{1}{P(0,t)}$

$$= \frac{1}{P(0,t)} E_{Q_{\text{Libor}}}\left[\max\left(1 - \left(P(T,T_n) + K\sum_{i=0}^{n-1} P(T,T_{i+1})\delta_i\right), 0\right) \times P(0,T) \mid \mathcal{F}_t\right]$$

Note, if we call $B_{\text{coupon}}(T_1, T_2 \cdots T_n) = \underbrace{P(T,T_n)}_{\text{Bond}} + \underbrace{K\sum_{i=0}^{n-1} P(T,T_{i+1})\delta_i}_{\text{coupons}}$

then above expression is nothing but 'coupon bond' option with strike '1'. (this is alternative definition of swaption)

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RATE EXOTICS

~~Value of a unit~~ Interest rate products which cannot have closed form solution for fair price generally falls into this category. Some of the features for a claim which makes it exotic are

- ★ Callability
- ★ barriers
- ★ Conditional Expectations, for modelling credit risky cashflows
- ★ joint probability of distribution of two or more rates (with spread)

INTEREST RATE MODELS:

For the purpose of pricing these exotics/derivatives we make assumptions on real world dynamics of underlying instrument such that those dynamics

i) Satisfy real world behaviour of that instrument, i.e. for example, $r(t)$ or $f(t,T)$ should be > 0 for most currencies.

ii) Con reproduce the fair values/market prices of all the ~~for~~ some of liquid tradables (for example swap caps, floors, swaptions, Bond options etc.). Sometimes we would want to match the nth moments of dynamics those vanilla products employ, for example implied volatities or implied skew

Some of the attractive qualities of models are

(a) Analytical tractability, i.e. to be able to price vanilla/liquid tradable using some formula rather heavy monte carlo.

CLASSIFICATION:

Broadly interest rate models are classified into two categories based on which quantities is fundamentally modelled

Short Rate Models

these models shortrate / instantanous spot rate r_t. Some of the famous dynamics are as follows.

1. Ho & Lee

$$dr_t = \sigma dW_t + \theta dt$$

i) rate can go negative (X)

ii) volatility of $f(t,T)$ is $\sigma^2(T-t)$, i.e. not dependency on T, ~~hence~~ so can't price vanills (X)
(even $\sigma \to \sigma_t$ & $\theta \to \theta_t$ does not allow capturing T)

a). Analytically tractable (✓)

2. Vasicek / OU process.

$$dr_t = \sigma dW_t + (\theta - \alpha r_t) dt$$
$$= \sigma dW_t + \alpha(\underbrace{\theta/\alpha}_{\text{long term mean (LTM)}} - r_t) dt$$

speed of reversion

As r_t go above LTM, drift become negative & tries to bring r_t back to LTM with speed α; likewise in opp dir.

TERM STRUCTURE MODELS.

These models instantanous forward rate $f(t,T)$ directly. There are two broad classes of models

1. Heath-Jarrow Morton framework

$$df(t,T) = \alpha(t,T)\, dt + \sum_{i=1}^{n} \sigma_i(t,T)\, dW_i$$

n: Number of factors in Model.

i) $f(t,T)$ can still go negative but with proper σ_i, it can be prevented

ii) depending upon choice of n & σ_i, variable numbers of instruments can be matched

a) Analytically tractable under few assumptions.

2. Libor Market Model.

↳ Log Normal forward Model (LFM)

↳ Log Normal swap Model (LSM)

i) rate & forward rate can still go negative (×)

ii) $\sigma_f(t,T) = \sigma \exp(-\alpha(T-t))$ (✓)

it does allow matching more volatilities but since its one-factor with constant σ; not possible to match all instrument + $f(t,T)$ & $f(t,S)$ are perfecting correlated.

c) Analytically tractable. (✓)

3. Hulls-White Extension

Same as vasicek with α & θ being time dependent

4 Cox Ingersoll Ross (CIR)

$$dr_t = \sigma_t \sqrt{r_t}\, dW_t + (\theta_t - \alpha_t r_t)\, dt$$

as with OU process drift tries to move r_t toward θ_t/α_t in addition vol term also gets smaller at r_t tends to zero which allows drift to dominate & prevent r_t to go below 0

i) rate > 0 (✓)

ii) still $f(t,T)$ & $f(t,S)$ are perfectly correleated. (~)

& cannot match all tenors

a) Not analytically tractable.

LFM.

$$\frac{df(t,T)}{f(t,T)} = \sigma dW$$

(i) $f(t,T)$ are positive

(ii) These dynamic are very consistent with Black formulas used for pricing Caps/Caplets hence very much coherent with market (but can be either swap or caps, bcz if forward rate is Log normal & then swap rate cannot be LogNormal & vice versa).

5. Black-Karasinski Model.

$$dX_t = \sigma_t \, dW_t + (\theta_t - \alpha_t X_t) dt$$
$$r_t = \exp(X_t)$$

i) $r_t \geq 0$ (✓)

ii) $f(t,T)$ & $f(t,S)$ are still perfectly correlated & parameters still can't match all liquid points.

a) Analytically tractable (✓)

HEATH JARROW MORTON (HJM) FRAMEWORK.

SDE for forward rate $f(t,T)$ in general HJM framework is given by (in ~~risk neutral~~ real world Measure)

$$df(t,T) = \alpha(t,T)\,dt + \sum_{i=1}^{n} \sigma_i(t,T)\,dW_t^i$$

n : is number of factors involved in determining $f(t,T)$

$\alpha(t,T)$: Single drift in real world P.

$\sigma_i(t,T)$: volality for each W_i brownian motion

W_t^i : i^{th} B.M. in ~~risk neutral measure~~ ~~Q~~ real world measure P

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)\,ds + \sum_{i=1}^{n} \int_0^t \sigma_i(s,T)\,dW_s^i$$

PROPERTIES:

i) Real world / risk neutral,

$$\text{Variance}(f(t,T)) = \sqrt{\left(\sum \sigma_i(t,T)\right)}$$

$$\text{covariance}(f(t,T), f(t,S)) = \sum_{i=1}^{n} \sigma_i(t,T)\,\sigma_i(t,S)$$

this allows more freedom in specifying stronger/weaker correlation ~~with~~ between two IFRs of different tenor.

RISK NEUTRAL MEASURE (Q)

Let's try to determine dynamics of $f(t,T)$ (hence for $P(t,T)$ & $r(t)$) under risk neutral measure.

Numeraire $(N_t) = B_t = B_0 e^{\int_0^t r_u du}$

tradable $(S_t) = P(t,T) = \cancel{e^{-\int_t^T f(t,s)ds}}\; e^{-\int_t^T f(t,u)du}$

Dynamics of $S_t N_t^{-1}$ should be martingale in Risk netural world i.e. it should be driftless.

$\cancel{B_t = B\, r_s = f(s,s) = f(0,s) + \int_0^s \alpha(}$

Let's look at process for r_u, B_t & $P(t,T)$

$$r_u = f(u,u) = f(0,u) + \int_0^u \alpha(s,u)\,ds + \sum_{i=1}^{n} \int_0^u \sigma_i(s,u)\,dW_s$$

$$f(t,u) = f(0,u) + \int_0^t \alpha(s,u)\,ds + \sum_{i=1}^{n} \int_0^t \sigma_i(s,u)\,dW_s$$

$$\therefore B_t = B_0 e^{\int_0^t r_u du} = B_0\, e^{\int_0^t f(0,u)du + \int_0^t\int_0^u \alpha(s,u)\,ds\,du + \sum_{i=1}^{n}\int_0^t\int_0^u \sigma_i(s,u)\,dW_s\,du}$$

$$\left(\int_0^t\int_0^u \alpha(s,u)\,ds\,du = \right.$$

$$= \int_0^t\int_s^t \alpha(s,u)\,du\,ds =$$

$$\left.\right)$$

$$= B_0 \exp\left(\int_0^t f(0,u)\,du + \int_0^t\int_s^t \alpha(s,u)\,du\,ds + \sum_{i=1}^{n} \int_0^t\int_s^t \sigma_i(s,u)\,du\,dW_s \right.$$

$$P(t,T) = \exp\left(-\left(\int_t^T f(0,u)\,du + \int_t^T\int_0^t \alpha(s,u)\,ds\,du + \sum_{i\geq 1}^{n}\int_t^T\int_0^t \sigma_i(s,u)\,dW_s\,du\right)\right)$$

again using change of integral

$$= \exp\left(-\left(\int_t^T f(0,u)\,du + \int_0^t\int_t^T \alpha(s,u)\,du\,ds + \sum_{i=0}^{n}\int_0^t\int_t^T \sigma_i(s,u)\,du\,dW_s^i\right)\right)$$

Now computing $Z_t = S_t N_t^{-1} = P(t,T).\,B_t(t)^{-1}$

$$Z_t = B_0 \exp\left(-\left(\int_0^T f(0,u)\,du + \int_0^t\int_s^T \alpha(s,u)\,du\,ds + \sum_{i=0}^{N}\int_0^t\int_s^T \sigma_i(s,u)\,du\,dW_s^i\right)\right)$$

$dZ_t = \cancel{Z_t \cdot d}$

Now to make Z_t process martingale, drift of ~~SED~~ SDE of Z_t should be zero, let's compute SDE of above process

assuming $Y_t = -\Big(\underbrace{\int_0^T f(0,u)\,du}_{a_t} + \underbrace{\int_0^t\int_s^T \alpha(s,u)\,du\,ds}_{b_t = f^2(t,T)} + \underbrace{\sum_{i=0}^{N}\int_0^t\int_s^T \sigma_i(s,u)\,du\,dW}_{c_t = f^3(t,W_t,T)}\Big)$

$\therefore\ da_t = d\left(\int_0^T f(0,u)\,du\right) = 0$ (No t dependence)

$db_t = \frac{\partial b_t}{\partial t}dt + \frac{\partial b_t}{\partial T}dT$ (Now behave assumed a fixed T, i.e. it not variable atm $\therefore dT = 0$)

$= \frac{\partial}{\partial t}\left(\int_0^t\int_s^T \alpha(s,u)\,du\,ds\right)dt$ (Apply Lebniz's rule)

$= \int_t^T \alpha(t,u)\,du$

$$dc_t = \frac{\partial c_t}{\partial t} dt + \frac{\partial c_t}{\partial W_t} dW_t + \frac{\partial c_t}{\partial T} dT \quad \left(\begin{array}{l} \frac{\partial c_t}{\partial t} = 0 \\ \text{and } dT = 0 \end{array} \right)$$

$$= \frac{\partial c_t}{\partial W_t} \cdot dW_t = \frac{\partial}{\partial W_t} \left(\int_0^t \int_s^T \sigma_i(s,u)\, du\, dW_s \right)$$

Note limits for W_s for 0 to W_t)

$$= \left(\int_t^T \sigma_i(t,u)\, du \right) dW_t$$

$$\therefore dY_t = -(da_t + db_t + dc_t) = -\left(\int_t^T \alpha(t,u)\, du \right) dt - \sum_{i=1}^{n} \left(\int_t^T \sigma_i(t,u)\, du \right) dW_t^i$$

$$\therefore (dY_t)^2 = \left(\int_t^T \sigma_i(t,u)\, du \right)^2 \cdot dW_t^2 = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\Sigma_i^2(t,T) \times \Sigma_j^2(t,T) \right) \rho_{ij} \right) dt$$

correlation between W_i & W_j

Let's call, $-\int_t^T \sigma_i(t,u)\, du = \Sigma_i(t,T)$

Coming back to Z_t $(= e^{Y_t})$

$$\therefore dZ_t = e^{Y_t} \cdot dY_t + \frac{1}{2} e^{Y_t} \cdot (dY_t)^2 \quad \text{(using Ito's)}$$

$$= e^{Y_t} \left(dY_t + \frac{1}{2} (dY_t)^2 \right)$$

$$= Z_t \cdot \left(\left(-\left(\int_t^T \alpha(t,u)\, du \right) dt + \sum_{i=1}^{n} \Sigma_i(t,T)\, dW_t \right) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\Sigma_i^2(t,T) \Sigma_j(t,T) \right) \rho_{ij}\, dt \right)$$

$$\frac{dZ_t}{Z_t} = \underbrace{\left(\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2} \Sigma_i^2(t,T) \cdot \Sigma_j(t,T) \rho_{ij} - \int_t^T \alpha(t,u)\, du \right)}_{\text{drift}} dt + \sum_{i=1}^{n} \Sigma_i(t,T)\, dW_t^i$$

To make drift term of Z_t zero, let's use multidimensional CMG.

$$\tilde{W}_i(t) = W_i(t) + \int_0^t \gamma_i(s)\, ds$$

$\tilde{W}_i$ = n dimension brownian motion in Risk Neutral Measure Q

i.e $d\tilde{W}_i(t) = dW_i(t) + \gamma_i(t)\,dt$, substituting $dW_i(t)$

$$\frac{dZ_t}{Z_t} = \left(\sum_i \sum_j \rho_{ij} \frac{\Sigma_i \Sigma_j}{2} - \int_t^T \alpha(t,u)\,du - \sum_i \gamma_i(t) \cdot \Sigma_i \right) dt + \sum_i \Sigma_i\, d\tilde{W}_i$$

For simplicity
$\Sigma_i = \Sigma_i(t,T)$
$\sigma_i = \sigma_i(t,T)$

To make this zero, we get

$$\int_t^T \alpha(t,u)\,du = \sum_i \left(\sum_j \frac{\rho_{ij} \Sigma_i \Sigma_j}{2} - \Sigma_i \gamma_i(t) \right)$$

DRIFT IN REAL WORLD ($\alpha(t,T)$)

Now, above should hold for all maturity which mean γ_i in above should be 'T' independent. (otherwise not all the $P(t,T)$ will be martingale in measure Q causing arbitrage). Diff above w.r.t T.

$$\alpha(t,T) = \sum_i \left(\sum_j \frac{\rho_{ij}}{2} \frac{\partial}{\partial T}(\Sigma_i \Sigma_j) - \frac{\partial}{\partial T}(\Sigma_i \gamma_i) \right)$$

$$\Sigma_i = -\int_t^T \sigma_i(t,u)\,du \qquad \frac{\partial \Sigma_i \Sigma_j}{\partial T} = -\Sigma_i\left(\sigma_j(t,T) - \Sigma_j \sigma_i(t,T)\right)$$

$$\frac{\partial \Sigma_i}{\partial T} = -\sigma_i(t,T) \quad \therefore \quad \frac{\partial \Sigma_i \gamma_i}{\partial T} = \gamma_i\left(-\sigma_i(t,T)\right)$$

$$\alpha(t,T) = \sum_i \left(\frac{1}{2} \sum_j \rho_{ij} \left(-\Sigma_i \sigma_j - \Sigma_j \sigma_i \right) + \gamma_i \sigma_i \right)$$

$\downarrow$

Basically α in real world needs to follow above equation
* (it can't be just any arbitrary function of t & T)

SDE OF ZCB

Now back to determining SDE of $P(t,T)$ in Risk Neutral Measure $\mathbb{Q}_t$ =

$$\frac{dZ_t}{Z_t} = \sum_i \Sigma_i \, d\tilde{W}_i$$

(SDE for Z_t in $\mathbb{Q}$)
$\tilde{W}_i$ is brownian motion in $\mathbb{Q}$

we know $Z_t = \dfrac{P(t,T)}{B(t)}$

$$\therefore \; dZ_t = P(t,T)\left(-\frac{1}{B^2(t)} dB(t) \right) + \frac{1}{B(t)} dP(t,T)$$

$$\frac{dZ_t}{Z_t} = \frac{-dB_t}{B_t} + \frac{dP(t,T)}{P(t,T)}$$

Substituting dZ_t/Z_t from above & $\dfrac{dB}{B_t} = r_t \, dt$

$$\star \quad \frac{dP(t,T)}{P(t,T)} = r_t \, dt + \sum_i \Sigma_i(t,T) \, d\tilde{W}_i^t$$

$\therefore$ $\Sigma_i(t,T)$ are volality of $P(t,T)$ in $\mathbb{Q}$

SDE OF IFR ($f(t,T)$)

We know in real world

$$d f(t,T) = \alpha(t,T)\, dt + \sum_i \sigma_i(t,T)\, dW_i$$

real world

substituting $\alpha(t,T)$ from above & $dW_i = d\tilde{W}_i - \gamma_i \, dt$

$$df = \left(-\frac{1}{2} \sum_i \sum_j \rho_{ij} \left(\Sigma_i \sigma_j + \Sigma_j \sigma_i \right) \right) dt + \sum_i \sigma_i(t,T)\, d\tilde{W}_i$$

($\sum \gamma_i \, dt$ gets cancelled)

Further simplifying the drift

$$df(t,T) = -\sum_i \sum_j \rho_{ij} \Sigma_i \sigma_j \, dt + \sum_i \sigma_i \, dW_t$$

$$\left(\Sigma_i = -\int_t^T \sigma_i(t,u)\,du \right)$$

$\therefore$ $\sigma_i(t,T)$ (σ_i) are i^{th} volatility of $f(t,T)$ in Q

SDE For $r(t)$

$r(t) = f(t,t)$ where

$$f(t,T) = f(0,T) - \sum_i \sum_j \rho_{ij} \int_0^t \Sigma_i \sigma_j \, ds + \sum_j \int_0^t \sigma_i \, dW_s$$

here

$$\Sigma_i = \Sigma_i(s,T) = -\int_s^T \sigma_i(s,u)\,du \quad (i^{th} \text{ volatility of } P(s,T))$$

$\sigma_j = \sigma_j(s,T)$ and ρ_{ij} is constant correlation between i^{th} & j^{th} brownian motions in Q, i.e. $\tilde{W}_i$ & $\tilde{W}_j$

(i^{th} volatility of $f(s,T)$)

$$\therefore \quad r(t) = f(0,t) - \sum_i \sum_j \rho_{ij} \int_0^t \Sigma_i(s,t)\,\sigma_j(s,t)\,ds + \sum_i \int_0^t \sigma_i(s,t)\,dW_s$$

To simplify differentiating, collecting/grouping all $\sigma_i(s,t)$ terms as one of the rate factor $r_i(t)$

i.e. $r(t) = f(0,t) + \sum_i r_i(t)$

where

$$\cancel{r_i(t) = -\sum_i \sum_j}$$

$$r_i(t) = -\int_0^t \sigma_i(s,t) \sum_j \rho_{ij} \Sigma_j(s,t)\, ds + \int_0^t \sigma_i(s,t)\, dW_s^i$$

(interchange i & j in drift term as all terms are in + or × which is associative)

$$\therefore r_i(t) = -\sum_j \underbrace{\int_0^t \rho_{ij}\, \sigma_i(s,t)\, \Sigma_j(s,t)\, ds}_{a_{t(ij)}} + \underbrace{\int_0^t \sigma_i(s,t)\, dW_s^i}_{b_t^i}$$

Taking derivatives,

$$dq_t^{ij} = d\left(\int_0^t \rho_{ij}\, \sigma_i(s,t)\, \Sigma_j(s,t)\, ds\right) = \frac{\partial a_t}{\partial t} dt \quad \text{(using Libnitz rule)}$$

$$= \left(\int_0^t \rho_{ij} \frac{\partial}{\partial t}\left(\sigma_i(s,t)\Sigma_j(s,t)\right) ds + \rho_{ij}\, \sigma_i(t,t)\, \Sigma_j(t,t)\right) dt$$

(using product rule) $\qquad \Sigma_j(t,t) = -\int_t^t \sigma_j(t,t)\, dt = 0$

$$= \left(\rho_{ij} \int_0^t \left(\Sigma_j(s,t) \frac{\partial}{\partial t} \sigma_i(s,t) + \sigma_i(s,t)\left(-\sigma_j(s,t)\right)\right) ds\right) dt$$

$$= \rho_{ij}\left(\int_0^t \Sigma_j(s,t) \frac{\partial \sigma_i(s,t)}{\partial t} ds\right) dt - \rho_{ij}\left(\int_0^t \sigma_i(s,t)\, \sigma_j(s,t)\, ds\right)$$

Note the limit are for W_s from 0 to W_t

$$db_t^i = \frac{\partial b_t}{\partial t}\, dt + \frac{\partial b_t}{\partial W_t}\, dW_t^i \qquad \left(\text{where } b_t^i = \int_0^t \sigma_i(s,t)\, dW_s^i\right)$$

$$= \left(\sigma_i(t,t)\frac{\partial W_t^i}{\partial t} + \int_0^t \frac{\partial}{\partial t}\sigma_i(s,t)\, dW_s^i\right) dt +$$

partial derivative of W_t^i w.r.t t is zero

$$\left(\sigma_i(t,t)\frac{\partial W_t^i}{\partial W_t^i} + \int_0^t \frac{\partial}{\partial W_t}\sigma_i(s,t)\, dW_s^i\right) dW_t^i$$

Note σ_i is not stochastic w.r.t maturity t hence this will be zero

$$= \left(\int_0^t \frac{\partial}{\partial t}\sigma_i(s,t)\, dW_s^i\right) dt + \left(\cancel{\int_0^t \frac{\partial}{\partial W_t}\sigma_i(s,t)\, dW_s^i}\right) dW_t$$

$$+ \sigma_i(t,t)\, dW_t^i$$

Now combining all,

$$dr_i(t) = -\sum_j da_t^{ij} + db_t^i$$

this is outside summation of j

$$= \left(-\sum_j \left(\rho_{ij}\overbrace{\left(\int_0^t \Sigma_j(s,t)\frac{\partial}{\partial t}\sigma_i(s,t)\, ds\right)}^{x} - \rho_{ij}\left(\int_0^t \sigma_i(s,t)\sigma_j(s,t)\, ds\right)\right) + \underbrace{\int_0^t \frac{\partial}{\partial t}\sigma_i(s,t)\, dW_s^i}_{y}\right) dt$$

$$+ \sigma_i(t,t)\, dW_t^i$$

Notice how term x & y are almost resembling $r_i(t)$ except for the $\frac{\partial}{\partial t}$. if we somehow can remove it, the can replace entire expression by r_t

Above expression of dr_t (SDE) also say something about process r_t, that is its non-markov in nature, and we know tractability is one of desired property from model (Interest Rate Model required/ desired properties on first page of Exotics)

So to make r_t process markov, ~~if~~ we can represent $\sigma_i(s,t)$ as separable function of s & t; for example

$$\sigma_i(s,t) = g_i(s) \cdot f_i(t)$$

~~here~~ $\sigma_i(s) \neq \sigma_i(s,s) \neq \sigma_{r_i}(s)$ ~~(Volatility of $r(t)$)~~

we have ~~taken~~

$$\therefore \frac{\partial \sigma_i(s,t)}{\partial t} = g_i(s) \frac{\partial f(t)}{\partial t}$$

$$= \sigma_i(s,t)\left(\frac{1}{f(t)} \times \frac{\partial f(t)}{\partial t}\right)$$

Now this term is s independent $\therefore$ can be taken out from integration w.r.t s in both x & y term

$$dr_i(t) = +\left(\sum_j \rho_{ij} \int_0^t \sigma_i(s,t)\, \sigma_j(s,t)\, ds\right) dt$$

$$+ \frac{1}{f(t)} \frac{\partial f(t)}{\partial t}\left(-\sum_j \rho_{ij} \int_0^t \Sigma_j(s,t)\, \sigma_i(s,t)\, ds + \int_0^t \sigma_i(s,t)\, dW_s^i\right) dt$$

this is nothing by $r_i(t)$

$$+ \sigma_i(t,t)\, dW_t^i$$

defining $\Phi_{ij} = \int_0^t \sigma_i(s,t)\, \sigma_j(s,t)\, ds$ & $\sigma_{i\gamma}(t) = \sigma_i(t,t)$

$$dr_i(t) = \left(\left(\sum_j \rho_{ij} \Phi_{ij} \right) + \frac{1}{f^i(t)} \frac{\partial f^i(t)}{\partial t} \cdot r_i(t) \right) dt$$

$$+ \cancel{\sigma_i(t,t)\, dW} \; \sigma_{i\gamma}(t)\, dW_t$$

this is eventually mean reverting process in W_t^i dimension.

$$\therefore \quad dr(t) = \sum_i dr_i(t)$$

and $r(t)$ is summation of all the r_i factors following mean reverting process.

* So now we have dynamics/SDE for all three r_t, $f(t,T)$ & $P(t,T)$ in risk neutral measure $\mathbb{Q}$
* All the dynamics of SDE have been generall (i.e. with no assumptions) except for r_t where for tractability we made an assumption forward rate volality.

GENERAL HJM

Gaussian HJM
$\sigma_i(t,T) = \sigma$ (constant)
(cheyette Model)

LMM — Non Markovian-HJM
(~~short rates~~)

Markovian HJM
$(\sigma_i(t,T) = g(t) f(T))$

Merton & Moraleda Model

$\sigma_i(t,T) = \sigma(\gamma(T-t)+1) e^{-\frac{\lambda}{2}(T-t)}$
$\sigma, \gamma, \lambda \geq 0$

(LGM) Ritchken & Sankarasubramanian

$g(t) = \sigma_r(t)\, e^{\int_0^t \lambda_u du}$

$f(T) = e^{-\int_0^T \lambda_u du}$

SDE for ϕ_{ij}

We assumed in short rate SDE; $\phi_{ij} = \int_0^t \sigma_i(s,t)\sigma_j(s,t)\,ds = f(t)$ (some function of t)

$$d\phi_{ij} = df(t) = \frac{\partial f(t)}{\partial t}dt$$

$$= \left(\frac{\partial}{\partial t}\int_0^t \sigma_i(s,t)\sigma_j(s,t)\,ds\right)dt$$

$$= \left(\sigma_i(t,t)\,\sigma_j(t,t) + \int_0^t \frac{\partial}{\partial t}\left(\sigma_i(s,t)\sigma_j(s,t)\right)ds\right)dt$$

$$\frac{\partial \sigma_i(s,t)}{\partial t} = \frac{\partial}{\partial t}\left(f(s)\cdot g^i(t)\right) = \frac{f(s)\cdot g^i(t)}{g^i(t)} \times \frac{\partial g^i(t)}{\partial t} = \sigma_i(s,t) \times \left(\frac{1}{g_i(t)}\frac{\partial g^i(t)}{\partial t}\right)$$

(Markovain HJM assumption
$\sigma_i(s,t) = f(s)\,g^i(t)$)

$$= \left(\sigma_i^{\gamma}(t)\sigma_j^{\gamma}(t) + \left(\frac{1}{g_i(t)}\frac{\partial g_i(t)}{\partial t} + \frac{1}{g_j(t)}\frac{\partial g_j(t)}{\partial t}\right)\underbrace{\int_0^t \sigma_i(s,t)\sigma_j(s,t)\,ds}_{\phi_{ij}}\right)dt$$

$$\therefore\ d\phi_{ij} = \sigma_i^{\gamma}(t)\cdot\sigma_j^{\gamma}(t) + \left(\frac{1}{g_i(t)}\frac{\partial g_i(t)}{\partial t} + \frac{1}{g_j(t)}\frac{\partial g_j(t)}{\partial t}\right)\phi_{ij}\,dt$$

T-forward Measure: (Q^T)

So far we have derived all HJM processes ($f(t,T)$, $P(t,T)$, $r(t)$, $\Phi_{ij}(t)$) in Risk Neutral measure, let's derive them in T-forward measure where Numeraire (M_t) is given by:

$$M_t\,(\text{T-forward}) = \frac{P(t,T)}{P(0,T)} \quad (\text{here } T \text{ is predefined})$$

We will try to use CMG to directly change the measure (Q^T) & hence transform SDE's into T forward measure from Risk neutral measure (Q^R). (Although we can derive them from scratch in T-forward measure, its just more convienient to use CMG).

$$\frac{dQ^T}{dQ^R} = \frac{M_t}{M_0} \times \frac{N_0}{N_t}$$

M_t is numeraire in $Q^T = \dfrac{P(t,T)}{P(0,T)}$

N_t is numeraire in $Q = B(t)$

$$= \frac{\left(\dfrac{P(t,T)}{P(0,T)}\right)}{\left(\dfrac{P(0,T)}{P(0,T)}\right)} \times \frac{B(0)}{B(t)} = \frac{P(t,T)}{P(0,T)} \times \frac{B(0)}{B(t)}$$

Let's ~~write~~ derive the process for $P(t,T)$ in Q^R, we know SDE =

$$\frac{dP(t,T)}{P(t,T)} = r_t\,dt + \sum_i \Sigma_i(t,T)\,dW_i^t$$

W_t^i is ith Brownian Motion in Q^R

To integrate, let's use Ito's

$$d(\ln P(t,T)) = \frac{1}{P(t,T)}\,dP(t,T) + \frac{1}{2}\left(-\frac{1}{P(t,T)^2}\right)(dP(t,T))^2$$

Date: / /

$$d(\ln P(t,T)) = \left(r_t\,dt + \sum_i \Sigma_i(t,T)\,dW_{it}\right) - \frac{1}{2}\left(\sum_i \Sigma_i(t,T)\,dW_{it}\right)^2$$

using

$$\left(\left(\sum_i x_i\right)^2 = \sum_i \sum_j x_i x_j\right)$$

$$d(\ln(P(t,T)) = \left(r_t - \frac{1}{2}\sum_i \sum_j \Sigma_i \Sigma_j \rho_{ij}\right) dt + \sum_i \Sigma_i(t,T)\,dW_t^i$$

integrating both side from 0, to t (and changing all t to s)

$$\ln(P(S,T))\Big]_0^t = \int_0^t \left(r_s - \frac{1}{2}\sum_i \sum_j \Sigma_i \Sigma_j \rho_{ij}\right) ds + \sum_i \int_0^t \Sigma_i(s,T)\,dW_s^i$$

$$P(t,T) = P(0,T)\cdot \exp\left(\quad\right)$$

& we already know

$$B(t) = B(0)\exp\left(\int_0^t r_s\,ds\right)$$

Substituting both $P(t,T)$ & $B(t)$, we get

$$\frac{dQ^T}{dQ^R} = \exp\left(-\frac{1}{2}\sum_i \int_0^t \Sigma_i(s,T) \sum_j \rho_{ij}\, \Sigma_j(s,T)\, ds + \sum_i \int_0^t \Sigma_i(s,T)\, dW_s^i \right)$$

representing

$$\vec{\Sigma} = [\Sigma_1\ \Sigma_2 \cdots \Sigma_n] \quad \& \quad d\vec{W} = \begin{bmatrix} dw_1 \\ \vdots \\ dw_n \end{bmatrix} \quad \& \quad C = \begin{bmatrix} \rho_{11} & \rho_{12} & \cdots \\ & \rho_{22} & \\ & & \ddots \end{bmatrix}$$

$$\frac{dQ^T}{dQ^R} = \exp\left(-\frac{1}{2}\int_0^T \vec{\Sigma}\cdot C \cdot \vec{\Sigma}^T ds - \int_0^T \vec{\Sigma}\, d\vec{W} \right)$$

using multidimensional CMG for correlated b.m.

$$d\vec{Z} = d\vec{W} - C\Sigma^T dt$$

where $\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$ are correlated b.m in Q^T

in other words,

$$dZ_i = dW_i - \sum_j \rho_{ij}\Sigma_j\, dt$$

SDE for $P(t,T)$ in T-forward measure.

$$\frac{dP(t,T)}{P(t,T)} = r_t dt + \vec{\Sigma}_i d\vec{W} \qquad \text{(in risk neutral measure)}$$

$$\frac{dP(t,T)}{P(t,T)} = r_t\, dt + \vec{\Sigma}_i\left(d\vec{Z} + C\vec{\Sigma}^T dt\right)$$

$$\frac{dP(t,T)}{P(t,T)} = \left(r_t + \vec{\Sigma}_i C \vec{\Sigma}^T\right) dt + \vec{\Sigma}_i d\vec{Z} \qquad \text{(in fwd measure)}$$

SDE of $f(t,T)$:

$$df(t,T) = -\sum_i \sum_j \rho_{ij} \Sigma_{ij} \sigma_i \, dt + \sum_i \sigma_i \, dW_t^i$$

Assuming $\vec{\sigma} = (\sigma_1, \sigma_2 \cdots \sigma_n)$

$$df(t,T) = -\vec{\sigma} \cdot C \cdot \Sigma^T dt + \vec{\sigma} \, d\vec{W} \qquad \text{(risk neutral)}$$

$$= -\vec{\sigma} \, C \, \Sigma^T dt + \vec{\sigma} \left(d\vec{Z} + C \Sigma^T dt \right)$$

$$df(t,T) = \vec{\sigma} \, d\vec{Z} \qquad \text{(t-forward measure)}$$

Imp: $f(t,T)$ is martingale in T-forward measure.

SDE of $r(t)$: (Markovian HJM)

$\sigma_i(t,T) =$

$$dr(t) = \sum_i dr_i(t)$$

$$dr_i(t) = \left(\sum_j \rho_{ij} \phi_{ij} + \frac{1}{f^i} \frac{\partial f}{\partial t} r_i(t) \right) dt + \sigma_i^r \, dW_t^i \qquad \text{(in risk neutral measure)}$$

$$= \left(\qquad \right) dt + \sigma_i^r \left(dZ_t^i + \sum_j \rho_{ij} \Sigma_j \, dt \right)$$

$$= \left(\sum_j \rho_{ij} (\phi_{ij} + \Sigma_j) + \frac{1}{f^i} \frac{\partial f^i}{\partial t} r_i(t) \right) dt + \sigma_i^r \, dZ_t^i$$

Markovian HJM.

SDE from general HJM remains valid in markovian HJM, In this section we will start deriving their processes.

RITCHKEN & SANKARASUBRAMNIAN (LGM)/R&S)

Out of many possible markovians, we will specificall focus on R&S version, where

$$\sigma_i(t,T) = g_i(t).f_i(T)$$

$$g_i(t) = \sigma_i(t,t)e^{\int^t \lambda_i dv} = \sigma_i^r(t)e^{\int^t \lambda_i(v)dv} \text{ (volatility of short rate)}$$

& $$f_i(T) = \exp\left(-\int_0^T \lambda_i(v)\,dv\right)$$

here λ_i is i^{th} variable of 't' which is piecewise linear function. Let's write down SDE's for $P(t,T)$, $f(t,T)$ & $r(t)$ in R&S. for n-dimensional ~~Z~~ (T-forward measure)

* $$\frac{dP(t,T)}{P(t,T)} = (r_t + \vec{\Sigma}_i C \vec{\Sigma}^T)dt + \vec{\Sigma}_{(t,T)} d\vec{Z}_t$$ [C is $n \times n$ correlation matrix

* $$df(t,T) = \vec{\sigma}_{(t,T)} d\vec{Z}_t$$

Note
1. $f(t,T)$ is martingale in T-forward measure
2. it is normally distributed for deterministic $\vec{\sigma}$ on $\vec{Z}$

* $$dr(t) = \sum_i dr_i(t)$$ where

* $$dr_i(t) = \left(\sum_j \rho_{ij}(\Phi_{ij} + \overset{\sigma_i^r}{\Sigma_{ij}}) + \frac{1}{f_i}\frac{\partial f^i}{\partial t} r_i(t)\right)dt + \sigma_i^r dz_t^i$$

for R&S,

$$\frac{1}{f}\frac{\partial f(t)}{\partial t} = -\lambda_i(t) \Rightarrow dr_i(t) = \left(\sum_j \rho_{ij}(\Phi_{ij} + \overset{\sigma_i^r}{\Sigma_j}) - \lambda_i r_i(t)\right)dt + \sigma_i^r dz_t^i$$

MODEL PARAMETERS:

Befor we move forward with derivation for their process, let's formalize all the model parameters. We will assume model descritization for $\{T_0, T_1, T_2 \cdots T_m\}$ & λ will be vary betwee T_i to T_{i+1} to be λ_i.

i) FParam. $(F_i(t)) = \exp\left(-\int_0^t \lambda_i(v)\,dv\right)$

assuming $T_k < t < T_{k+1}$, we can simplify this integration as

$$F_i(t) = F_i(T_k) \cdot \exp\left(-\lambda_k (t - T_k)\right)$$

ii) PSi Param $(\psi_i(t))$

$$\psi_i(t) = \int_0^t \exp\left(-\int_0^u \lambda_s\,ds\right) du$$

descritization: $(T_k < t < T_{k+1})$

$$\psi_i(t) = \psi_i(T_k) + \int_{T_k}^t \exp\left(-\int_0^u \lambda_s\,ds\right) du$$

$$= \psi_i(T_k) + \int_{T_k}^t F_i(u)\,du$$

$$= \psi_i(T_k) + \int_{T_k}^t F_i(T_k)\exp\left(-\lambda_k(u - T_k)\right) du$$

$$= \psi_i(T_k) + F_i(T_k)\exp(\lambda_k T_k)\int_{T_k}^t \exp(-\lambda_k u)\,du$$

$$= \psi_i(T_k) + F_i(T_k)\exp(\lambda_k T_k)\left(\frac{\exp(-\lambda_k u)(-\lambda_k)}{-\lambda_k}\right]_{T_k}^t$$

$$= \psi_i(T_k) + F_i(T_k)\exp(\lambda_k T_k)\left(\frac{\exp(-\lambda_k t) - \exp(-\lambda_k T_k)}{-\lambda_k}\right)$$

$$= \psi_i(T_k) + F_i(T_k)\left(\frac{1 - \exp(-\lambda_k(t-T_k))}{\lambda_k}\right)$$

iii) GParam ($G_{ij}(t)$)

$$G_{ii}(t) = \int_0^t \left(\frac{\sigma_i^r(s)}{F_i(s)}\right)^2 ds.$$

$$G_{ij}(t) = \int_0^t \frac{\sigma_i^r(s)\,\sigma_j^r(s)}{F_i(s)\,F_j(s)}\, ds.$$

Discretization.

★ Assumption, $\sigma_i(s)$ is constant between T_k & T_{k+1} & is equals to ~~$\sigma_k(T_k)$~~ $\sigma_i(T_k)$ (since $T_k < t < T_{k+1}$, $\sigma^r(t) = \sigma^r(T_k)$)

$$G_{ij}(t) = G_{ij}(T_k) + \int_{T_k}^t \frac{\sigma_i^r(T_k)\,\sigma_j^r(T_k)}{F_i(T_k)\,F_j(T_k)}\left(\exp(+\lambda_i(T_k)(s-T_k) + \lambda_j(T_k)(s-T_k))\right) ds.$$

$$= G_{ij}(T_k) + \frac{\sigma_i^r(T_k)\,\sigma_j^r(T_k)}{F_i(T_k)\,F_j(T_k)} \int_{T_k}^t \exp\left((\lambda_i(T_k) \overset{\text{plus}}{+} \lambda_j(T_k))(s-T_k)\right) ds$$

$(\lambda_i(T_k) = \lambda_i^k)$

$$= \quad '' \quad + \quad '' \quad \left(\frac{\exp((\lambda_i^k + \lambda_j^k)(s-T_k))}{(\lambda_i + \lambda_j)\cancel{(s-T_k)}}\right]_{T_k}^{t}$$

$$= G_{ij}(T_k) + \frac{\sigma_i^r(T_k)\,\sigma_j^r(T_k)}{F_i(T_k)\,F_j(T_k)}\left(\frac{\exp((\lambda_i^k + \lambda_j^k)(t-T_k)) - 1}{\lambda_i^k + \lambda_j^k}\right)$$

Note: this derivation is not valid for cases when $\sigma^r(t)$ is not piecewise constant function rather modelled stochastically.

iv) Capital Lambda ($\Lambda_i(t,T)$)

$$\Lambda_i(t,T) = \int_t^T \exp\left(-\int_t^u \lambda_s ds\right) du$$

it can be written in terms of PsiParam & FParam as

$$\Lambda_i(t,T) = \frac{\psi_i(T) - \psi_i(t)}{F_i(t)}$$

substituting descritized versions of ψ_i & F_i, we get descritized version of $\Lambda_i(t,T)$

v) PhiParam ($\phi_{ij}(t)$)

Introduced in derivation of SDE of $r_i(t)$, we know

$$\phi_{ij}(t) = \int_0^t \sigma_i^*(s,t)\, \sigma_j^*(s,t)\, ds$$

(σ_i & σ_j are volatity of forward rate

in R&S markovian HJM,

$$\sigma_i(s,t) = \sigma_i^r(s) e^{-\int_s^t \lambda_u du}$$

$$= \sigma_i^r(s) \cdot \frac{F_i(t)}{F_i(s)}$$

$$\therefore \phi_{ij}(t) = F_i(t) F_j(t) \underbrace{\int_0^t \frac{\sigma_i^r(s)\, \sigma_j^r(s)}{F_i(s)\, F_j(s)} ds}$$

$$= F_i(t) F_j(t)\, G_{ij}(t)$$

PROCESS DERIVATION (T-forward measure)

Short Rate.

We know shortrate dynamics in risk neutral measure Q^R is given by

$$r(t) = f(0,t) + \sum_i r_i(t)$$

where

$$r_i(t) = -\sum_j \int_0^t \rho_{ij}\, \sigma_i(s,t)\, \Sigma_j(s,t)\, ds + \int_0^t \sigma_i(s,t)\, dW_s^i$$

also from change of measure we know

this T is constant

$$dW_s^i = dZ_s^i + \sum_j \rho_{ij}\, \Sigma_j(s,T)\, ds$$

$\therefore r_i(t)$ in T-forward measure

$$\cancel{r_i(t)} = r_i(t) = \sum_j \int_0^t \rho_{ij}\, \sigma_i(s,t) \left(\Sigma_j(s,T) - \Sigma_j(s,t) \right) ds + \int_0^t \sigma_i(s,t)\, dZ_s^i$$

in R&S approx we know

$$\Sigma_j(s,T) = -\int_s^T \sigma_j(s,u)\, du = -\int_s^t \sigma_j(s,u)\, du - \int_t^T \sigma_j(s,u)\, du$$

$$= \Sigma_j(s,t) - \int_t^T \sigma_j(s,t)\, e^{-\int_t^u \lambda_v^j dv}\, du$$

$$= \Sigma_j(s,t) - \sigma_j(s,t) \int_t^T e^{-\int_t^u \lambda_v^j dv}\, du$$

$$\Sigma_j(s,T) - \Sigma_j(s,t) = -\sigma_j(s,t)\Lambda_j(t,T)$$

$$\therefore \quad r_i(t) = -\sum_j \int_0^t \rho_{ij}\, \sigma_i(s,t)\sigma_j(s,t)\Lambda_j(t,T)\,ds + \int_0^t \sigma_i(s,t)\,dZ_s^i$$

$$= -\sum_j \rho_{ij}\Lambda_j(t,T)\phi_{ij}(t) + \int_0^t \sigma_i(s,t)\,dZ_s^i$$

FORWARD RATES ($f(t,T)$)

We know $f(t,T)$ is martingale in T-forward measure

$$df(t,T) = \vec{\sigma}_i(t,T)\,d\vec{Z}_t$$

(imp 'T' is a constant from T-forward measure, if we do change of variable (T) then forward measure will change too

$$\int_0^t df(s,T) = \int_0^t \vec{\sigma}_i(s,T)\,d\vec{Z}_s$$

$$f(t,T) = f(0,T) + \int_0^t \vec{\sigma}_i(s,T)\,d\vec{Z}_s$$

$$\vec{\sigma}(s,T) = [\sigma_1(s,T), \sigma_2(sT) \cdots \sigma_n(sT)]$$

$$d\vec{Z}_s = [dZ_1(s), dZ_2(s) \cdots dZ_n(s)]$$

n is number of factors.

What about $f(t,u)$ in T-forward measure (Note this is not martingale)

$$df(t,u) = +\sum_i\sum_j \rho_{ij}\,\sigma_i(t,u)\left(\Sigma_j(t,T) - \Sigma_j(t,u)\right)dt + \sum_i \sigma_i(t,u)\,dW_t^i$$

(changing t to u & integrating both side)

$$f(t,u) = f(0,u) + \sum_i\sum_j \rho_{ij}\int_0^t \sigma_i(s,u)\left(\Sigma_j(s,T) - \Sigma_j(s,u)\right)ds + \sum_i \int_0^t \sigma_i(s,u)\,dW_s^i$$

ZERO COUPON BOND ($P(t,T)$)

$$P(t,T) = \exp\left(-\int_t^T f(t,u)\,du\right)$$

substituting $f(t,u)$ from previous section

~~where~~

$$p(t;T) = \exp\Bigg(-\Bigg(\underbrace{\int_t^T f(0,u)\,du}_{a} + \underbrace{\sum_i\sum_j \rho_{ij}\int_t^T\int_0^t \sigma_i(s,u)\left(\Sigma_j(s,T)-\Sigma_j(s,u)\right)ds\,du}_{b}$$

$$+ \underbrace{\sum_i \int_t^T\int_0^t \sigma_i(s,u)\,dW_s^i\,du}_{c}\Bigg)\Bigg)$$

Let's handle a, b, & c terms separately.

a

$$\exp\left(-\int_t^T f(0,u)\,du\right) = \frac{\exp\left(-\int_0^T f(0,u)\,du\right)}{\exp\left(-\int_0^t f(0,u)\,du\right)} = \frac{P(0,T)}{P(0,t)}$$

b

Let's replace $\sigma_i(s,u) = \sigma_i(s,t)\,e^{-\int_t^u \lambda_v^i\,dv}$ and

$$\Sigma_j(s,u) = \Sigma_j(s,t) - \sigma_j(s,t)\,\Lambda_j(t,u)$$

$$= \sum_i\sum_j \rho_{ij}\int_t^T e^{-\int_t^u \lambda_v^i\,dv}\left(\underbrace{\int_0^t \sigma_i(s,t)\left(\Sigma_j(s,T)-\Sigma_j(s,t)\right)ds}\right)du$$

this expression is free from u

$$+ \rho_{ij}\int_t^T e^{-\int_t^u \lambda_v^i\,dv}\int_0^t \sigma_i(s,t)\,\sigma_j(s,t)\,\Lambda_j(t,u)\,ds\,du$$

first term here is drift of $r(t)$ in T forward measure

$$= \sum_i \underbrace{\text{drift}(r_i(t))}_{\times \Lambda_i(t,T)} + \underbrace{\sum_i \sum_j \rho_{ij}\, \phi_{ij}(t) \int_t^T e^{\int_t^u \lambda_i dv} \cdot \Lambda_j(t,u)\, du}_{d}$$

putting $e^{\int_t^u \lambda_i dv} = \dfrac{\partial \Lambda_i(t,u)}{\partial u}$

solving d

$$\sum_i \sum_j \rho_{ij}\phi_{ij} \int_t^T \frac{\partial \Lambda_i(t,u)}{\partial u} \Lambda_j(t,u)\, du \quad ; \text{ we know } \rho_{ij}\phi_{ij} = \rho_{ji}\phi_{ji}$$

$$\therefore = \sum_{i=1}^{n} \sum_{j=i}^{n} \rho_{ij}\phi_{ij} \left(\int_t^T \left(\frac{\partial \Lambda_i}{\partial u} \Lambda_j + \Lambda_i \frac{\partial \Lambda_j}{\partial u} \right) du \right)$$

$$d(\Lambda_i \Lambda_j) = \Lambda_i \frac{\partial \Lambda_j}{\partial u} du + \Lambda_j \frac{\partial \Lambda_i}{\partial u} du + \frac{\partial \Lambda_i}{\partial u} \frac{\partial \Lambda_j}{\partial u} \cancel{(du)^2} \to 0$$

integrating both side from t to T ($\Lambda_j(t,t) = 0$)

$$= \sum_{i=1}^{n} \sum_{j=i}^{n} \rho_{ij}\phi_{ij} \Lambda_i(t,T) \Lambda_j(t,T)$$

to keep summations on both side i & j from 1 to n

$$= \sum_i \sum_j \rho_{ij}\phi_{ij} \frac{\Lambda_i(t,T)\Lambda_j(t,T)}{2}$$

hence b

$$= \sum_i \underbrace{\text{drift}(r_i(t))}_{\times \Lambda_i(t,T)} + \frac{1}{2} \sum_i \sum_j \rho_{ij}\phi(t)_{ij} \Lambda_i(t,T)\Lambda_j(t,T)$$

Now handling C

$$\sum_i \int_t^T \int_0^t \sigma_i(s,u)\, dW_s^i\, du \quad - \quad \left(\text{using } \sigma_i(s,u) := \sigma_i(s,t)\, e^{-\int_t^u \lambda_v^i dv} \right)$$

$$= \sum_i \left(\int_t^T e^{-\int_t^u \lambda_v^i dv} \int_0^t \sigma_i(s,t)\, dW_s^i\, du \right)$$

$$= \sum_i \int_t^T e^{-\int_t^u \lambda_v^i dv}\, du \underbrace{\int_0^t \sigma_i(s,t)\, dW_s^i}_{\text{vol term of } r_i(t)}$$

$$= \sum_i \Lambda_i(t,T) \cdot \text{Volterm}(r_i(t))$$

Now putting a & b & c together

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \cdot \exp\left(- \left(\sum_i \Lambda_i(t,T) \left(r_i(t) + \frac{1}{2} \sum_j \rho_{ij} \phi_{ij}(t) \Lambda_j \right) \right) \right)$$

$$= \frac{P(0,T)}{P(0,t)} \exp\left(- \sum_i \Lambda_i(t,T) \left(r_i(t) + \frac{1}{2} \sum_j \rho_{ij} \phi_{ij}(t) \Lambda_j(t,T) \right) \right)$$