Risk Neutral / Arbitrage Pricing

Expectation Vs. Arbitrage :-

Contract : What should be worth of a contract today that pays a stock in exchange of K denom of stock ~~in exc~~ at time T $\Rightarrow$ this shold be zero at $t=0$

i.e. $\text{Price}((S_T - K) @ T) = 0$

$= \text{Price}((S_T - K)e^{-rT}) = 0$ continuous compounded

Using Expectation :-

is S_T denote value of stock at T, let assume it take a process such that

$$S_T = S_0 e^{x} \quad \text{where } x \text{ is } N(\mu, \sigma)$$

$$P((S_T - K)e^{-rT}) := E[(S_T - K)e^{-rT}] = 0 \quad (\text{it should be zero at } T)$$

$$\therefore \quad K = E[S_T e^{-rT}] = E[S_0 e^{x - rT}]$$

$$= S_0 e^{-rT} E[e^{x}]$$

~~Using~~

~~Using law of conconcious statistician~~

$$\cancel{= S_0 e^{-rT} \cdot \left(\int_{-\infty}^{+\infty} e^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \right)}$$

~~Using Identify Normal~~

$$\cancel{= \frac{S_0 e^{-rT}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{x - \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx}$$

$$K = S_0 e^{-rT} \left(e^{\mu + \frac{1}{2}\sigma^2} \right)$$

Using Identifying Normals (Moment generating func)

using ARBITRAGE:

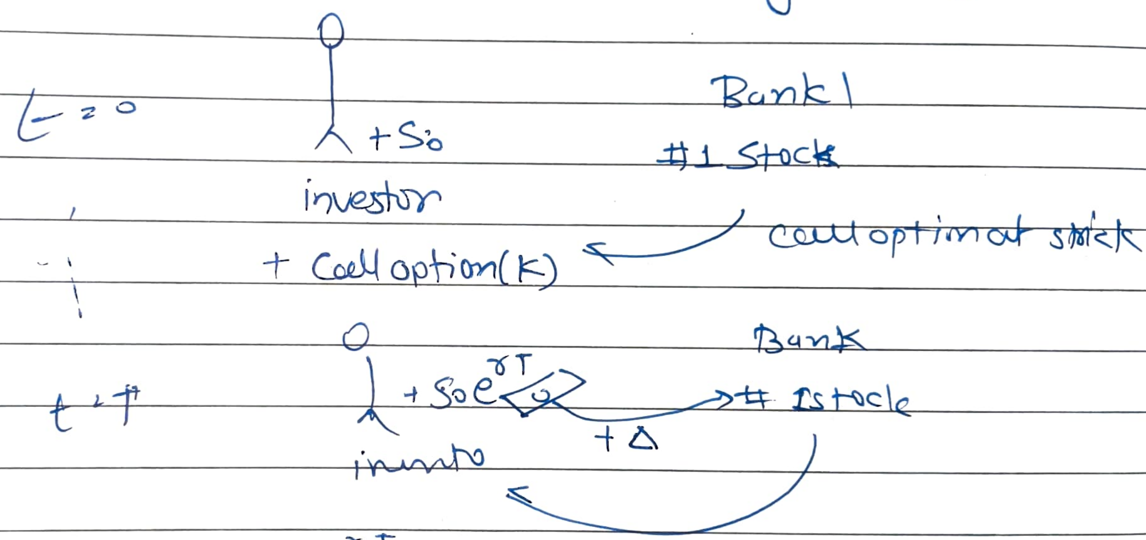
at $t = 0$; ~~buy~~ sell 1 unit of stock ~~with~~ worth S_0 ~~money borrowed~~ & obtain S_0 money from bank.

$t = 0 \Rightarrow$ -1 unit of stock (lend) $+ S_0$ (cash)

at $t = T \Rightarrow$ -1 unit of stock $+ S_0 e^{rT}$

For avoiding arbitrage; worth of 1 unit of stock at T, should be

(1) $S_0 e^{rT}$; if ~~may~~ strike is greater than $S_0 e^{rT}$ then investor can buy that ~~much~~ stock at $K > S_0 e^{rT}$ & sell ~~curr stock~~ then he will be at loss of $K - S_0 e^{rT}$



$\Delta = K - S_0 e^{rT}$

(2) if strike $K < S_0 e^{rT}$ then Bank is at loss.

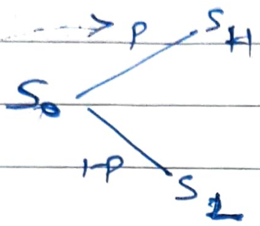
So arbitrage free price of forward is $S_0 e^{rT}$

i.e. Expection in real world measure of stock (where we assumed it follows exp. Brownian motion) will not give fair value estimate of contract (any contract)

ARBITRAGE FREE Pricing

① DISCRETE STOCK MODEL World

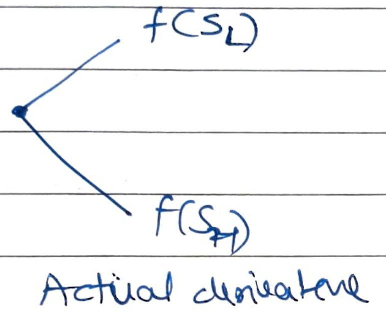
real world probabilities

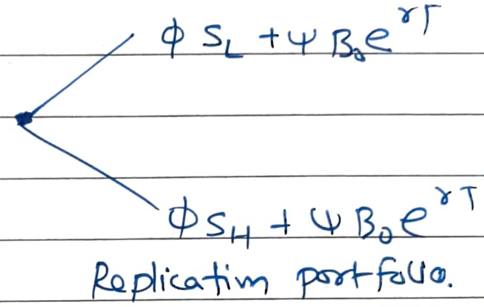


$B_0 \longrightarrow B_0 e^{rT}$

Let assume a claim $X = f(S_T, B_T)$

depending upon two states of stock; let take ϕ stock & ψ Bonds





We know, for actual derivative to be arbit free

$$f(S_L) = \phi S_L + \psi B_0 e^{rT}$$
$$f(S_H) = \phi S_H + \psi B_0 e^{rT}$$

$$\therefore \quad \phi = \frac{f(S_H) - f(S_L)}{S_H - S_L}$$

$$\psi = \frac{f(S_L) - (\phi) S_L}{B_0 e^{rT}}$$

i.e. at $t=0$ we know ~~what~~ how many stocks / Bonds we need to replicate f at $t=T$ (one tick later)

$\therefore$ Price Equation at $t=0 \Rightarrow \phi S_0 + \psi B_0$

$$= \phi S_0 + e^{-rT}\left(f(S_L) - \phi S_L\right)$$
$$= \phi\left(S_0 - S_L e^{-rT}\right) + e^{-rT} f(S_L)$$
$$= f(S_H) \times \left(\frac{S_0 - S_L e^{-rT}}{S_H - S_L}\right) + f(S_L)\left(e^{-rT} - \frac{S_0 - S_L e^{-rT}}{S_H - S_L}\right.$$

$$= e^{-rT}\left(f(S_H) \times \left(\frac{S_0 e^{rT} - S_L}{S_H - S_L} \right) + f(S_L) \times \left(1 - \left(\quad \right)\right)\right)$$

if we assume $p = \dfrac{S_0 e^{rT} - S_L}{S_H - S_L}$

$$= e^{-rT}\left(f(S_H) \times p + f(S_L)(1-p) \right)$$

$$= e^{-rT} \cdot E_Q[f(T)]$$

in a measure which is called risk neutral measure

Note how probulity

CONTINOUS WORLD.

Given a numeraire Z_t, ~~for~~ Price of derivate X_T is given by

$$\text{Price}(X@t)\ \cancel{B_t} = Z_t E_Q[Z_T^{-1} X_T \mid \mathcal{F}_t]$$

Q is measure where $Z_t^{-1} S_t$ is martingale where S_t is ~~stock process~~ underlying asset process. (it should be tradable in market)

Replication;

Let say we want to replicate a claim X at T assuming we have previsible process ϕ_t & ψ_t (i.e. value of these at $t+dt$ is known at t)

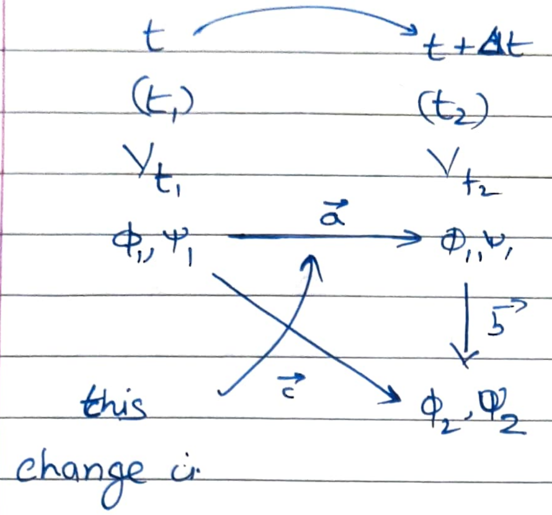
	Derivative		Replicating strategy
at $t=T$	X_T	$=$	$\phi_T S_T + \psi_T B_T$

Final goal

Assump $Z_t = B_t$

REPLICATION STRATEGY

DISCRETE WORLD



Let's say

$\vec{a}$: change in Portfolio price from t_1 to t_2, keeping ϕ_{t_1} & $\psi_{t_1} = \phi_1 \Delta S_1 + \psi_2 \Delta B$

$\vec{c} = \vec{a} + \vec{b}$

$\vec{b}$: cost of changing ϕ_{t_1} to ϕ_{t_2} & ψ_{t_1} to ψ_{t_2} at t_2 $= S_2 \Delta\phi + B_2 \Delta\psi_2$

$\vec{c}$: overall change in portfolio from t_1 to t_2 : $V_2 - V_1$.

(1) Self Financing Strategy

- For a replication strategy to be self financing we only put/invest at $t=0$; at all other times; any change in portfolio should be at zero cost.

i.e. $\vec{b} = 0$

i.e $S_2 \Delta\phi + B_2 \Delta\psi = 0$

(2) It should produce the claim at time T.

i.e.

$$\phi_T S_T + \psi_T B_T = X_T$$

Both are necessary conditions for a strategy to perfectly hedge

Proof: To prove that Price@t $= B_t E_Q[B_T^{-1} X \mid \mathcal{F}_t]$ will be able to give a replication strategy

Let $E_t = E_Q[B_T^{-1} X \mid \mathcal{F}_t]$; Since its in an expection, it is always martingale in Q

& we already know $B_t^{-1} S_t$ $(= Z_t)$ is also martingale in this measure.

Let $\phi_t = dE_t / dZ_t$ (from M.R.T.)

$\psi_t = E_t - \phi_t Z_t$ (as per our assumption $B_t E_t = \phi_t S_t + \psi_t B_t$ from here we make expression for ψ_t)

1. To prove #1 (self financing strategy)

$$\Delta\phi \cdot S_2 + \Delta\psi B_2 = 0.$$

we need to prove; $\Delta V(V_2 - V_1) = \phi_1 \Delta S + \psi \Delta B$

$(dV_t = \phi_t dS_t + \psi_t dB_t$ // in cont world$)$

$V_t = B_t E_t$ // from our defination of price

$dV_t = dB_t E_t + B_t dE_t$ // $dE_t = \phi_t dZ_t$

$= dB_t \cdot E_t + B_t \phi_t dZ_t$ // $E_t = \phi_t Z_t + \psi_t$

$= dB_t \psi_t + \phi_t Z_t dB_t + B_t \phi_t dZ_t$

$= \psi_t dB_t + \phi_t (Z_t dB_t + B_t dZ_t)$

$= \psi_t dB_t + \phi_t d(Z_t B_t)$ // $Z_t = B_t^{-1} S_t$

$= \psi_t dB + \phi_t dS_t$

2. To prove #2

at $t = T$

$V_T = B_T E_T = B_T E_Q[B_T^{-1} X | F_T]$

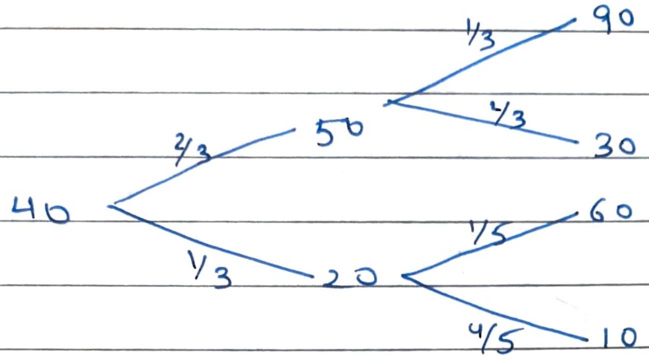
$= B_T B_T^{-1} X = X$ ///

EXAMPLE:

Let's assume following Binomial model on particular spot

$t =$ 0 1 2

S_t:



Attaching risk neutral probalities, probablity (H) $= \dfrac{S_0 e^{+r\Delta t} - S_L}{S_H - S_L}$

also considering zero interest rate $(r = 0)$ market

B_t : 1 1 1.

in this market, let's say we have a claim (call option)
$= \text{Max}(X - 40, 0)$ at time 2.

X :

50

0

20

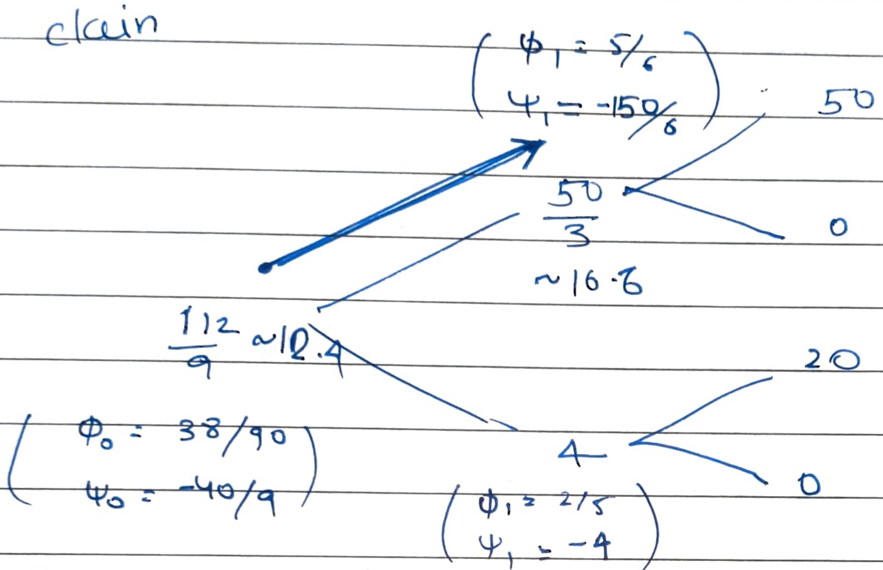
0

Date / /

Following
$$P_t = B_t^{-1} E_Q[X_T B_T^{-1} | \mathcal{F}_t]$$

$$B_t = 1 \ \forall t \ \therefore \ P_t = E_Q[X_T | \mathcal{F}_t]$$

Q: Risk neutral measure; Since probabilities are already calculated in Q measure.; let's build pricing tree from claim



Let's assume spot moves from $t=0$ to $t=1$ (from 40 to 50)

$$\vec{c} = \Delta V = V_2 - V_1 = \frac{50}{3} - \frac{112}{9} = \frac{150-112}{9} = \frac{38}{9}$$

$$\vec{a} : \phi_0 \cdot (S_2 - S_1) + \psi_1 (B_2 - B_1) = \frac{38}{90} \times (50-40) = \frac{38}{9}$$

$$\vec{b} \quad S_2 \cdot (\phi_1 - \phi_0) + B_2(\psi_2 - \psi_1) = 50 \times \left(\frac{5}{6} - \frac{38}{90}\right) + 1 \cdot \left(\frac{-150}{6} - \left(\frac{-40}{9}\right)\right)$$

$$= 50 \times \left(\frac{75-38}{90}\right) + \left(\frac{40}{9} - \frac{150}{6}\right)$$

$$= \frac{50 \times 37}{90} + \frac{400 - 2250}{90} = \frac{1850 - 1850}{90} = 0$$