DISTRIBUTIONS

When we have a variable 'x' which can take any value from a specific domain 'D' and a particular value it takes from this domain is assigned with a probability then creates a distribution (Graph of 'D' on x axis & Problity (D) on y axis).

Each distr. is characterized by:
(i) mean (μ)
(ii) variance (σ^2)
(iii) PMF/PDF
iv) CDF
All 4 parameters of uniquely identified by Moment Generating Function (Read that first)

DISCRETE DISTRIBUTIONS:

When domain discrete then distribution is called Discrete distribution. Probablity is defined for any value of $x \in D$ and is often called Probability Mass function ($P_{MF}(x)$)

$$\text{mean}(X) = \sum_{x \in D} x \cdot P_{MF}(x) = E[X] = \mu$$

$$\text{variance}(X) = E[(x-\mu)^2] = \sum_{x \in D} (x-\mu)^2 P_{MF}(x)$$
$$= E[X^2] - (E[X])^2$$

$$CDF(X) = \sum_{x=-\infty}^{x} P_{MF}(x)$$

over discrete increments from D

$$MGF_X(t) = E[e^{tX}] = \sum e^{tx} \cdot P_{MF}(x)\big|_{x \in D}$$

BERNOULLI DISTRIBUTION: Ber(p)

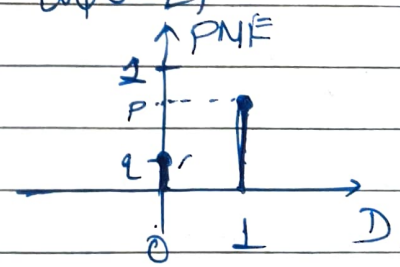
Domain = $\{0, 1\}$

PMF(x) = $P(X=0) = 1-p = q$; $P(X=1) = p$

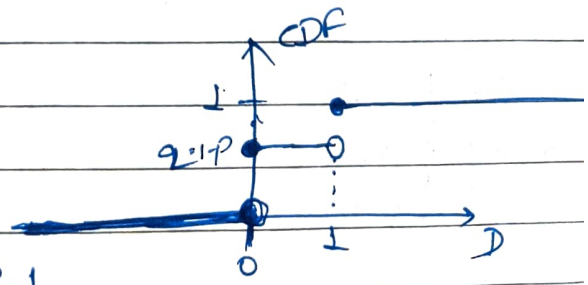
Mean = p

Variance = $p(1-p) = p \cdot q$

$$CDF(x) = \begin{cases} 0 & x < 0 \\ 1-p & 0 \le x < 1 \\ 1 & 1 \le x \end{cases}$$



All CDF(X) are follow Uniform distribution, i.e. Each of value of CDF(x) is equiprobable (Note: Domain now is CDF(x); i.e. 0, 1-p, 1 not x)

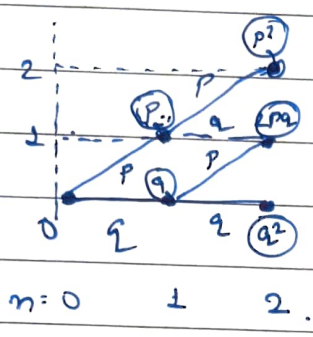


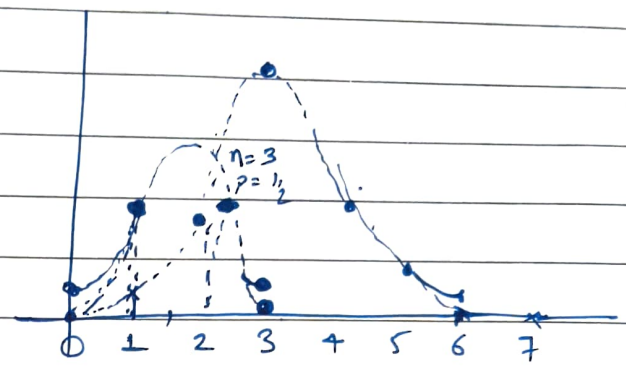
$$MGF_X(t) = \sum e^{tx(0)} \cdot (1-p) + e^{t \cdot 1}(p)) = p(e^t - 1) + 1 = (pe^t + q)$$

BINOMIAL DISTRIBUTION: $Bin(n,p)$ = Collection of n Bernoulli(p) distributions

Domain: $\{0, 1, 2 \dots n\}$

PMF(x): ${}^nC_x \cdot p^x \cdot (1-p)^{n-x}$ → Problity follow Binomial distribution





Mean = np.

Variance = $np.q$

$MGF(t) = (pe^t + q)^n$

$$CDF(x) = \sum_{x=0}^{x} {}^nC_x p^x q^{n-x}$$

Imp. Properties.

1. $Bin(1,p) = Ber(p)$
2. if $X \sim Bin(n,p)$ & $Y \sim Bin(m,p)$
   - then $X+Y \sim Bin(m+n, p)$

POISSON DISTRIBUTION: $Pois(\lambda)$ = λ is rate, probality of 'x' number of certain event occuring give constant mean rate of occurance 'λ' since last event.

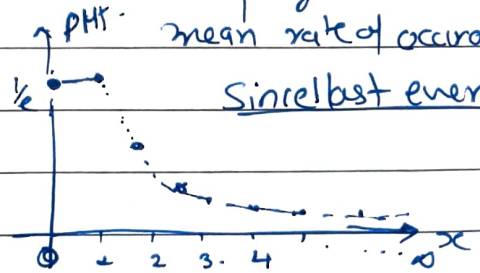
λ: avg no of events in that interval (in which we want to find probality many of that event happing x time)

Domain: All natural numbers $(0,1,2 \dots \infty)$

$$PMF(X) = \frac{\lambda^x e^{-\lambda}}{x!}$$

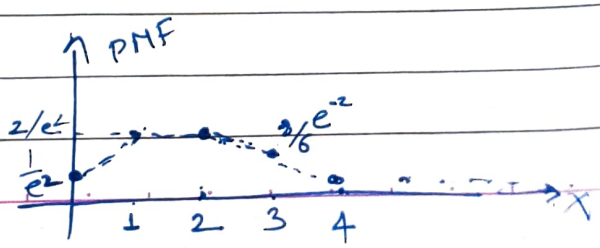
sp. cases: $\lambda = 1$.

$$PMF(x) = \frac{e^{-1}}{x!}$$



$\lambda = 2$

$$PMF(X) = \frac{2^x e^{-2}}{x!}$$



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mean $= \lambda$. $\quad MGF_X(t) = e^{\lambda(e^t - 1)}$ for $-\infty < t < \infty$

variance $= \lambda$

$$PDF(x) = \sum_{0}^{x} \frac{\lambda^x e^{-\lambda}}{x!}$$

eg. No. of average calls : 3/min $(= \lambda)$

then most likely no. of calls to occure = mean $= \lambda = 3$

$$Pr.(\text{no. of calls} = 1) = \frac{3^1 \cdot e^{-3}}{1!} = 3e^{-3} < \left(Pr = \frac{3^3 \cdot e^{-3}}{3!} \approx 4.5 e^{-3}\right)$$

$$Pr(\text{" " } = 4) = \frac{3^4 \cdot e^{-3}}{4!} = \frac{81}{24} e^{-3} \approx 3.4 e^{-3}$$

Generalization :

if instead no. of avg event in same period, we have rate of event happening in Δt interval then $\left(\gamma = \frac{d\lambda}{dt}\right)$

$$PMF(x) = \frac{(\gamma t)^x e^{-\gamma t}}{x!}$$

Derivation : Let's assuming γ_t is the rate with which an event occurs then probablity that event happen in Δt duration is

$$Pr(E) = \gamma_t \Delta_t \quad (\gamma_t \Delta_t)$$

Note : 1) Occurance of event in two Δt_1 & Δt_2 are independent

2) Probality of exactly 2 or more events in short Δt period is zero.

Probality of that event happening x times in an interval ΔT interval

$p \; p \cdots \cdots \overset{x \text{ time}}{} \; q \cdots q \rightarrow \left(\frac{\Delta T}{\Delta t} - x\right)$

$\Delta_t \; \Delta_t \cdots \cdots \Delta_t$

$\leftarrow \Delta T \longrightarrow$

$\therefore n = \Delta T / \Delta t$

Using binomial prob theorem

$$Pr(x \text{ times}) = {}^nC_x (p)^x (q)^{n-x}$$

$\Delta t \to 0$

$$= \lim_{\Delta t \to 0} \frac{n!}{(n-x)!\,x!} (\gamma_t \Delta t)^x (1 - \gamma_t \Delta t)^{n-x} = \frac{(\gamma_t \Delta T)^x \cdot e^{-\gamma_t \Delta T}}{x!}$$

Let $\lambda = \gamma_t \Delta T$ = avg no of event in ΔT period

$$= \frac{\lambda^x e^{-\lambda}}{x!}$$

$$cdf(Y) = \Phi\left(\frac{\ln y - \mu}{\sigma}\right)$$

$$\text{mean} = e^{\mu + \sigma^2/2}$$

Proof

$$= \int_{-\infty}^{+\infty} e^{x} f_X(x)\,dx = \cancel{\int_{-\infty}^{+\infty} e^{x}\left(\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right)dx}$$

$$= E[e^{x}] = e^{\mu + \frac{1}{2}\sigma^2}$$ (2nd property of Normal distribution)

$$\text{variance} = e^{2\mu + \sigma^2}\left(e^{\sigma^2} - 1\right)$$

Proof

$$\text{Variance} = E[X^2] - (E[X])^2$$

$$= E[e^{2x}] - E[e^{x}]^2$$

$$= e^{2\mu + \frac{4\sigma^2}{2}} - e^{\left(\mu + \frac{1}{2}\sigma^2\right)2}$$

$$= e^{2\mu + \sigma^2}\left(e^{\sigma^2} - 1\right)$$

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Moment Generating Function (mgf) : It is parametrized by a variable t & For a distribution $f_X(x)$ on random variable x is given by

$$M_X(t) = E_X[e^{tX}]$$

Discrete

$$M_X(t) = \sum_{x \in D} e^{tx} \cdot P.M.f(x)$$

Continuous

$$M_X(t) = \int_{x \in D} e^{tx} f_X(x) dx$$

Moments from MGF :

i) Mean $= E[X] = \left.\frac{\partial M_X(t)}{\partial t}\right|_{t=0} = \left.E[e^{tx} \cdot x]\right|_{t=0}$

$= E[x]$

$\therefore \quad = M'(0)$

ii) Variance $= E[x^2] - E[x]^2 =$

$$\left.\frac{\partial^2 M(t)}{\partial t^2}\right|_{t=0} = \left.E\left[x^2 e^{tx} + 0 \cdot e^{tx}\right]\right|_{t=0} = E[x^2]$$

$$= M''(0) - M'(0)^2$$

iii) Skew $= E\left[\left(\frac{x-\mu}{\sigma}\right)^3\right] = \frac{E[x^3] - 3\mu\sigma^2 - \mu^3}{\sigma^3}$

$$\left.\frac{\partial^3 M(t)}{\partial t^3}\right|_{t=0} = \left.\frac{\partial}{\partial t}\left(x^2 e^{tx}\right)\right|_{t=0} = \left.x^2 e^{tx}(x) + 0\right|_{t=0} = E[x^3]$$

$$\therefore \quad \frac{M'''(0) - 3\mu\sigma^2 - \mu^3}{\sigma^3}$$

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Properties.

i) $CDF(+\infty) = 1 \Rightarrow \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1$

$$\Rightarrow \int_{-\infty}^{+\infty} e^{-x^2/2} dx = \sqrt{2\pi}$$

ii) if $X \sim N(\mu, \sigma)$ then. (identifying Normals)

$$E[e^{\theta X}] = e^{\theta\mu + \frac{1}{2}\theta^2\sigma^2}$$

Proof:

$$E[e^{\theta X}] = \int_{-\infty}^{+\infty} e^{\theta x} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\frac{x^2 - 2x\mu + \mu^2 - 2\sigma^2\theta x}{\sigma^2}\right)} dx$$

$$= \text{''} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\frac{(x + (\theta\sigma^2 - \mu))^2 - (\theta\sigma^2 - \mu)^2 + \mu^2}{\sigma^2}\right)} dx$$

$$= \text{''} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\frac{(\mu + \theta\sigma^2 + \mu)(\mu - \theta\sigma^2 - \mu)}{\sigma^2}\right)} e^{-\frac{1}{2}\left((x + (\theta\sigma^2 - \mu)\right)} d$$

$$= \text{''} \cdot e^{-\frac{1}{2}\left(\frac{(2\mu + \theta\sigma^2)(-\theta\sigma^2)}{\sigma^2}\right)} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\underbrace{\frac{x + \theta\sigma^2 - \mu}{\sigma}}_{Y.}\right)^2} dx$$

$$= \text{''} \quad e^{+\frac{1}{2}\theta(2\mu + \theta\sigma^2)(\cdot} \cdot \sigma \cdot \int_{-\infty}^{+\infty} e^{-\frac{1}{2}Y^2} dY$$

$$= \frac{1}{\cancel{\sqrt{2\pi\sigma^2}}} e^{+\theta\mu + \frac{1}{2}\theta^2\sigma^2} \cdot \cancel{\sigma} \cdot \sqrt{2\pi}$$

iii) $\phi(-x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{+\infty} e^{\frac{-x^2}{2}} dx = 1 - \phi(x)$

iv) if $X_1 \sim N(\mu_1, \sigma_1^2)$ & $X_2 \sim N(\mu_2, \sigma_2^2)$ then $Y = X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

LOG-NORMAL DISTRIBUTION: $LogN(\mu, \sigma^2)$: μ & σ^2 are variance of Underlying Normal distribution.., $Y = e^X$ where X is $\sim N(\mu, \sigma^2)$

Domain : $(-\infty, +\infty)$

Pdf(x) : $\frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right)$

Proof: $pdf(X) = \frac{d}{dx} Pr(X \le x) = \frac{d}{dx} Pr(\ln X < \ln x)$

$= \frac{d}{dy} Pr(X < \ln y) = \frac{d}{dy} \phi\left(\frac{\ln y - \mu}{\sigma}\right)$

Let $t = \frac{\ln y - \mu}{\sigma} \Rightarrow dt = \frac{1}{y} dy \left(\frac{1}{\sigma}\right)$

$= \frac{d}{dy} \phi(t) = \frac{d\phi(t)}{dt} \times \frac{dt}{dy} = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \cdot \left(\frac{1}{y\sigma}\right)$

$= \frac{1}{y\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{\ln y - \mu}{\sigma}\right)^2}$

CONTINUOUS DISTRIBUTION:

In continuous distribution 'D' is continuous & instead of PMF we define Pdf, prob. Density function; i.e. prob of x taking value between $(x, x+dx)$

Uniform Distribution: $U(a,b)$

a is lower bound & b is upper bound.

Domain: $(-\infty, +\infty)$ or $[a, b]$

$$Pdf(x) = \begin{cases} 1/(b-a) & \text{if } x \in [a,b] \\ 0 & \text{else} \end{cases}$$

$$cdf(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x < b \\ 1 & b < x \end{cases}$$

mean $= a+b/2$

variance $= \frac{1}{12}(b-a)^2$

$$MGF = \begin{cases} \frac{e^{tb} - e^{ta}}{t(b-a)} & \text{for } t \neq 0 \\ 1 & t = 0 \end{cases}$$

Normal Distribution: $N(\mu, \sigma^2)$

μ is mean & σ^2 is variance

Domain: $(-\infty, +\infty)$

$$Pdf(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

$$cdf(x) = \phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

sp. case, standard Normal distribution

$\mu = 0$ & $\sigma = 1$

$$cdf = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \phi(x) \rightarrow \text{its called } \underline{\phi(x)}$$

mean $= \mu$

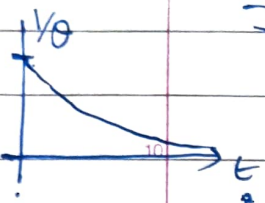
variance $= \sigma^2$

$\therefore CDF(X) = \phi\left(\frac{x-\mu}{\sigma}\right)$

$\mu t + \frac{1}{2}\sigma^2 t^2$

EXPONENTIAL DISTRIBUTION (θ)

Assuming an event occurs with rate r_t, Exponential distribution assign probality to each time $t \in R^+$ which represent waiting time unit 1st that event occurs. It is parameterized with θ_t which represent mean waiting time ~~before~~ until first event occurs.



Domain : $[0, \infty)$

Pdf(t) $\frac{1}{\theta} e^{-t/\theta}$

$\therefore$ it is the probality that waiting time before 1st that event occurs is dt.

$r_t \Delta T$ = Avg no of events ΔT

$\therefore$ Avg Time taken per event $= \frac{\Delta T}{r_t \Delta T}$

Since that's the avg time until first event occurs

$\therefore \theta_t = 1/r_t$ —— (1)

$$cdf = 1 - e^{-t/\theta}$$
$$= 1 - e^{-t/\theta}.$$

mean $= \theta$

variance θ^2

$mgf(t) = \left(\frac{1}{1-\theta t}\right)$ → this t is diff from above variable t.

Derivation.

Let's start with cdf, = Pr(waiting time is less than t)

$= 1 -$ Pr(" " greater than t)

$= 1 -$ Pr(0 event occurs until t)

$$= 1 - \left(\frac{(r_t t)^x e^{-r_t t}}{x!}\right)\Big|_{x=0}$$

$$= 1 - e^{-r_t T} \quad (\text{from 1, } \theta = 1/r_t)$$

$$= 1 - e^{-t/\theta}$$

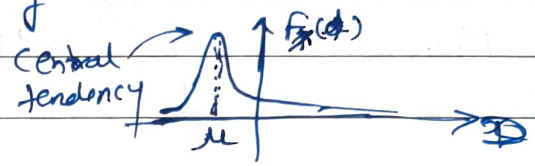
MOMENTS & MOMENTS GENERATING FUNCTION

Moments of used to describe the shape of distribution

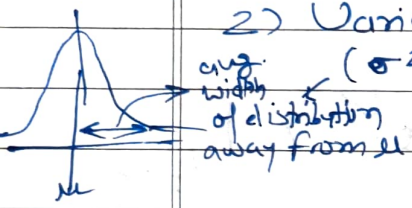
Four major moments:

1) Mean: Indicates central tendency of distribution. It is given by expection of random variable following that distribution, i.e.
(μ)

$$\text{Mean} = E[X]$$

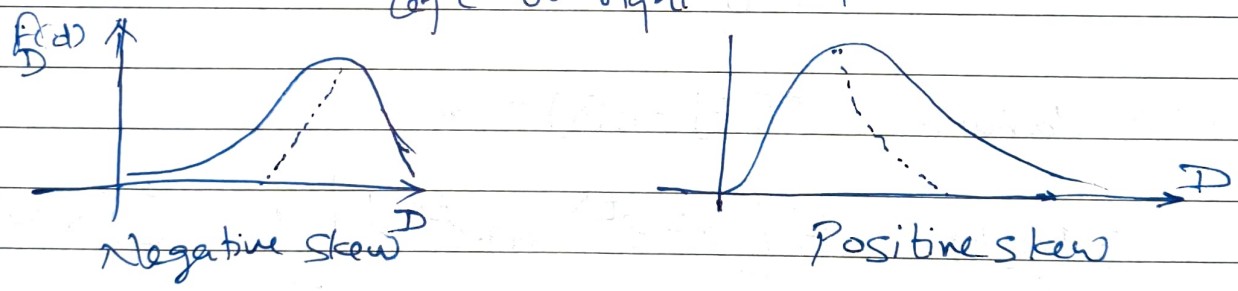


2) Variance: Indicates width of deviation
(σ^2)



$$\text{Variance} = E[(X-\mu)] = E[X^2] - E[X]^2$$
$$= E[X^2] - \mu^2$$

3. Skewness: Asymmetric leaning of distribution to either left or right
μ_3



$$\text{skewness} = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{E[X^3] - 3\mu\sigma^2 - \mu^3}{\sigma^3}$$

(Pearson's Moment coefficient of skewness)

4. Kurtosis: Degree of fatness of outer tails.

$$\text{kurtosis} = E\left[\left(\frac{X-\mu}{\sigma}\right)^4\right]$$