INTEREST RATE

Interest rate represents time value of money.

- Tradables
  - Instruments
    - Cash
    - Cash Bonds.
  - Claims
    - Vanilla
      - Discount bonds / zero coupon bonds
      - Forwards
      - Coupon bonds
      - Bond options.
      - Swap
      - Caplets / Floors
      - Swaptions
    - Exotics
      - Callable swaps / Comcallable swaps

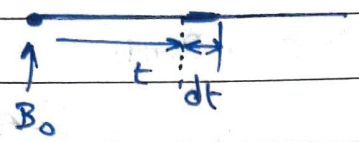
INSTRUMENTS :

i) CASH :
- → Notes issued by country's central / Federal bank
- → Value of cash stays same throughout time, i.e. zero interest rate risk
- → Most liquid.
- → Inflation might affect the buying power of that cash. For example if a basket of goods worth $100 a year ago costs $110 now then buying capacity of that cash has gone down.

ii) CASH BOND / BANK ACCOUNT

Qualitative : If we deposit some cash in a comercial / Investment bank's saving account then it grows with some interest rate.

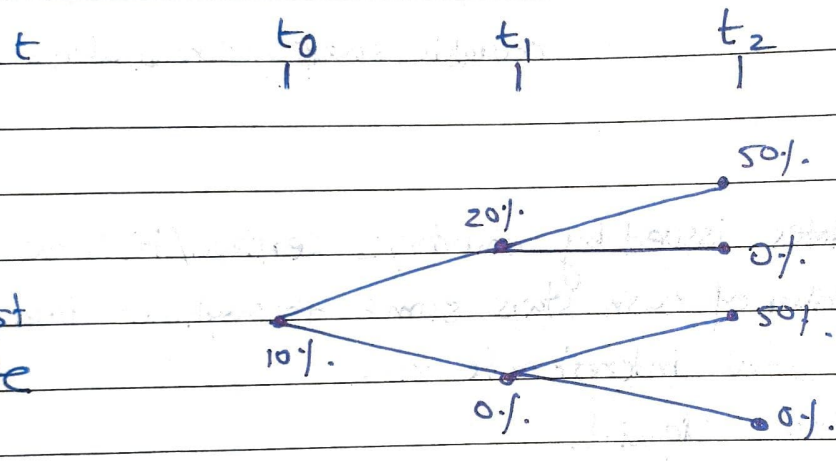
Quantitative: Type of interest rate can quantitately determine valuation of cash bond. For sake of consistency we assume continuously compounded interest rate

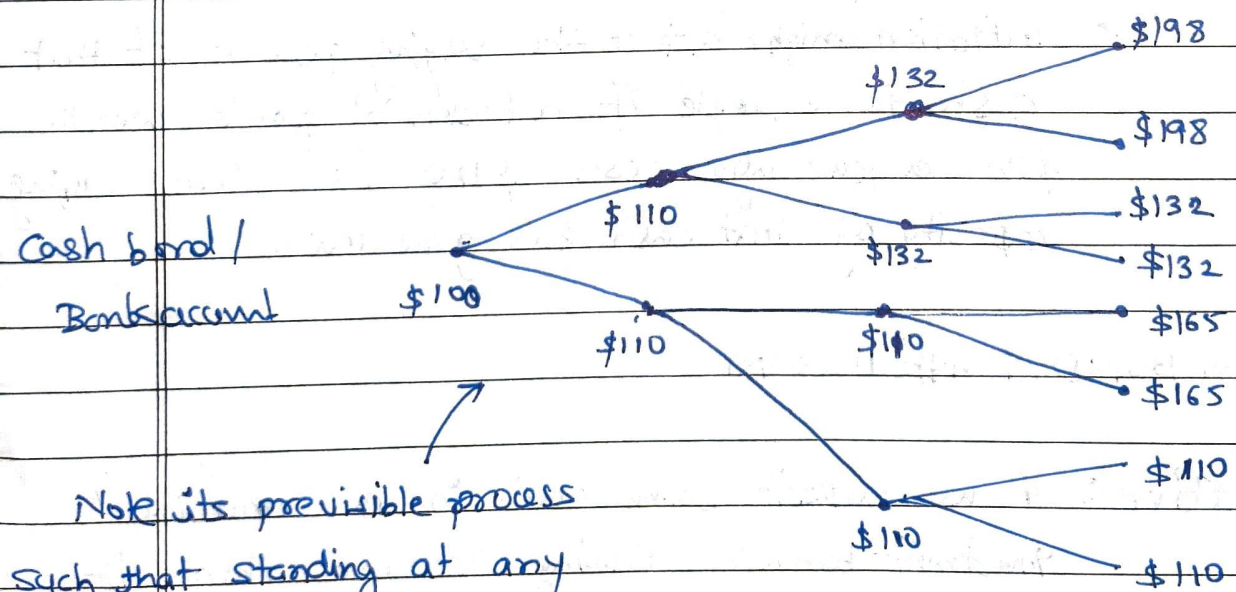


$$dB_t = r_t \cdot B_t \cdot dt$$

r_t is interest rate fixed at time t for duration of dt from t to $t+dt$

$$\therefore \quad B_t = B_0 e^{\int_0^t r_s ds}$$





Note its previsible process such that standing at any node t_i we know exactly what would be bond value at t_{i+1} since we know interest rate at t_i already

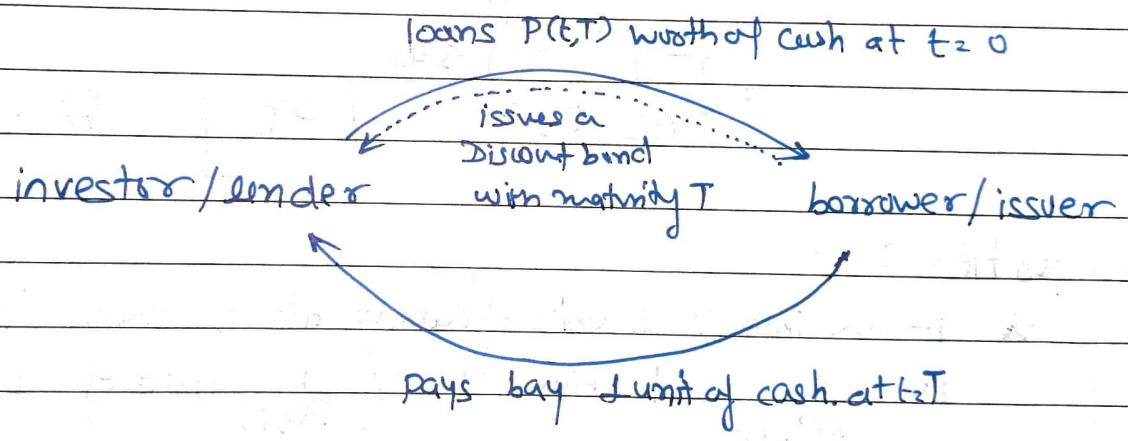
CLAIM > VANILLAS

DISCOUNT BOND / ZERO COUPON BOND:

Qualitative: Claim bonds are fixed income trables that allow borrower to take loan at some 'issuance price' & issue a bond and after maturity (T) pay back the face value of bond (issuance price + interest) to lender/investor i.e.

Claim of one unity of currency at T is called ZCB.

Its value/worth/price at any time t is denoted by P(t,T)



TYPES OF Discount Bonds.

Depending upon issuer ther can be various types of bonds

1. GOVENMENT / SOVERIGN / TRESURY
   a. issued by govt to raise funding for road/shools/dams/infra etc
   b. Most security, almost zero credit risk.

2. MUNICIPAL BONS:
   a. Issued by state/local municipal to raise funding for their respective projects
   b. May offer some tax benefits.

3. CORPORATE BONDS..
   a. Have credit Risk b. Issued by companies to grow business

QUANTITATIVE:

Assuming Numeraire of market to be ~~as~~ cash bonds / bank account (i.e. $N_t = B_t$), using risk neutral pricing worth of a claim $X = 1$ paid at time T is given by.

$$\text{Price} = P(t,T) = B_t \cdot E_Q\left[B_T^{-1} X \mid \mathcal{F}_t\right]$$

where Q is measure which make $B_t \cdot B_t^{-1}$ martingale. since $B_t \cdot B_t^{-1} = 1$ is alway martingale, Q can be any measure.

$$\therefore \quad P(t,T) = B_t E_Q\left[B_T^{-1} \mid \mathcal{F}_t\right] = E_Q\left[\frac{B_t}{B_T} \mid \mathcal{F}_t\right]$$

$$= E_Q\left[e^{-\int_t^T r_s ds} \mid \mathcal{F}_t\right]$$

for any Q (if there are no other assets in market otherwise Q^R: i.e. risk neutral measure)

Properties:

★ if we invest 1 unit of money at t then it would grow to $\frac{1}{P(t,T)}$ at time T ($\frac{1}{P(t,T)}$ units of 1 ZCB) at value at t

Note: \$1 at t can grow to many different function of instrument depending upon market & instrument., For example a claim: $Y = B_T / B_t$ will have price at $t = 1$ as well but that does not mean that $P(t,T) = B_T / B_t = f(\text{other instruments})$ for all $t \le T$.

★ $P(t,t) = 1$ i.e. worth of \$1 paid at t, valued at t is same \$1

★ ZCB is used as numeraire in T-forward measure, specifically

$$N_t = \frac{P(t,T)}{P(0,T)}$$ here T is constant coming from T of T-forward measure.

$$N_0 = 1. \qquad N_T = \frac{P(T,T)}{P(0,T)} = \frac{1}{P(0,T)}$$

This is quite useful in cases when we want to estimate fair value of a claim by just taking its expectation. For example, if we have a claim X paid at T then it value at any time t before T is

$$\text{Price}_t(X) = N_t \cdot E_{Q^T}\left[N_T^{-1} \cdot X \mid \mathcal{F}_t\right]$$

Substituting N_t from above

$$= \frac{P(t,T)}{P(0,T)} \cdot E_{Q^T}\left[\frac{P(0,T)}{P(T,T)} \cdot X \mid \mathcal{F}_t\right]$$

$P(T,T) = 1.$

$$= P(t,T) \cdot E_{Q^T}\left[X \mid \mathcal{F}_t\right]$$

Notice how there is no other term inside expectation other than X

★ $T_1 \quad T_2 \quad T_3$.

$$E\left[P(T_2,T_3)\,|\,\mathcal{F}_{T_1}\right] = \frac{P(T_1,T_3)}{P(T_1,T_2)}$$

Proof

$$E\left[E\left[e^{-\int_{T_2}^{T_3} r_s ds}\,|\,\mathcal{F}_2\right]\,|\,\mathcal{F}_{T_1}\right]$$

$$= E\left[e^{-\int_{T_1}^{T_3} r_s ds}\,|\,\mathcal{F}_1\right]$$

multiply & divide by $E\left[e^{-\int_{T_1}^{T_2} r_s ds}\,|\,\mathcal{F}_1\right]$

$$= E\left[e^{-\int_{T_1}^{T_3} r_s ds}\,|\,\mathcal{F}_1\right] / E\left[e^{-\int_{T_1}^{T_2} r_s ds}\,|\,\mathcal{F}_1\right]$$

★ ZCB is also used as Numerair for Libor forward measure.

i.e. if we assume $T_1\, T_2 \cdots T_n$ to importaint claim dates for a particallar deal starting at T_0 than Numeraire for i^{th} date is defined as

$$N_i = \frac{1}{1 \prod_{j=0}^{i-1} P(T_j, T_{j+1})}$$

$$N_0 = 1 \;;\; N_1 = \frac{1}{P(T_0,T_1)} \;,\; N_2 = \frac{1}{P(T_0,T_1)\,P(T_1,T_2)} \cdots$$

$$\cdots N_{T_n} = \frac{1}{P(T_0,T_1)\cdots\cdots P(T_{n-1},T_n)}$$

Example:

$T_0 \quad T_1 \quad T_2 \quad T_3 \quad T_4$ ↑ \$1.

What's worth of \$1 paid a T_4 as of T_1.

Risk Neutral Measure.

$$\cdots = B_{T_1} \cdot E_Q\left[B_{T_4}^{-1} \cdot 1\,|\,\mathcal{F}_1\right] = E_Q\left[e^{-\int_{T_1}^{T_4} r_s ds}\,|\,\mathcal{F}_1\right]$$

T4-forward Measure

$$= \frac{P(T_1,T_4)}{P(0,T_4)}\, E_Q\left[\left(\frac{P(T_4,T_4)}{P(0,T_4)}\right)^{-1} 1\,|\,\mathcal{F}_1\right]$$

$$= P(T_1,T_4) \cdot E_Q\left[1\,|\,\mathcal{F}_1\right] = P(T_1,T_4)$$

$$= E_Q\left[e^{-\int_{T_1}^{T_4} r_s ds}\,|\,\mathcal{F}_1\right]$$

Libor forward Measure.

$$= \frac{1}{P(T_0,T_1)} E_Q\left[P(T_0,T_1)\cdot P(T_1,T_2)P(T_2T_3)P(T_3T_4) \,\middle|\, \mathcal{F}_1 \right]$$

$$= E_Q\left[E_Q\left[P(T_1,T_2)P(T_2T_3)P(T_3T_4) \,\middle|\, \mathcal{F}_2 \right] \,\middle|\, \mathcal{F}_1 \right]$$

(using Tower law

$$= E_Q\left[P(T_1T_2)P(T_2T_3)\cdot E_Q\left[P(T_3T_4) \,\middle|\, \mathcal{F}_2 \right] \,\middle|\, \mathcal{F}_1 \right]$$

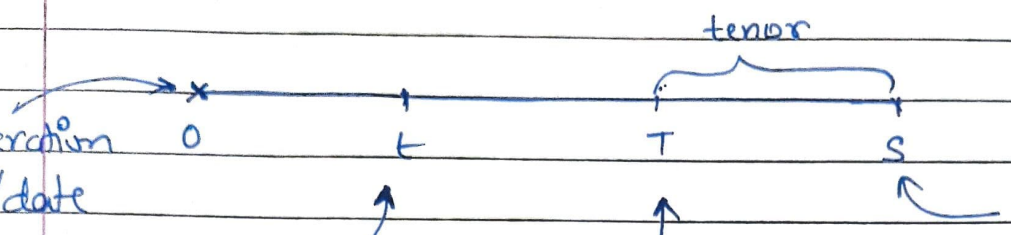
(using property above

$$= E_Q\left[\cancel{P(T_1T_2)P(T_2T_3)}\cdot \frac{P(T_2T_4)}{\cancel{P(T_2T_3)}} \,\middle|\, \mathcal{F}_1 \right]$$

$$= E_Q\left[P(T_1,T_2)\cdot P(T_2T_4) \,\middle|\, \mathcal{F}_1 \right]$$

$$= E_Q\left[P(T_1,T_4) \,\middle|\, \mathcal{F}_1 \right] = P(T_1,T_4)$$

TYPES OF INTEREST RATES



	Simple Compounded (SC)	Continuous Compounded (C.C)
FORWARD (t<T)	$\frac{1}{P(t,S)} = \frac{1}{P(t,T)}\left(1 + S.C.F.(S-T)\right)$	$\frac{1}{P(t,S)} = \frac{1}{P(t,T)} e^{C.C.F(S-T)}$
	$\therefore SCF = \frac{1}{(S-T)}\left(\frac{P(t,T)}{P(t,S)} - 1\right)$ (Forward Rate agreement)	$\therefore CCF = \frac{\log\left(\frac{P(t,T)}{P(t,S)}\right)}{(S-T)}$ (forward)
SPOT (t=T) (t=t)	$SC.S = \frac{1}{(S-t)}\left(\frac{1 - P(t,S)}{P(t,S)}\right)$	$SC.S = \frac{-\log(P(t,S))}{(S-t)}$ (yield)

INSTANTANEOUS FORWARD SC. IF =

~~T=t~~

$S = T + dT$

$$CC.I.F = \lim_{\Delta T \to 0} \frac{-\left(\log(t,T+\Delta T) - \log(t,T)\right)}{\Delta T}$$

~~CFO~~ Instantaneous forward

$$(f(t,T) = -\frac{\partial \log(t,T)}{\partial T}$$

INSTANTANEOUS S.C. IS SPOT.

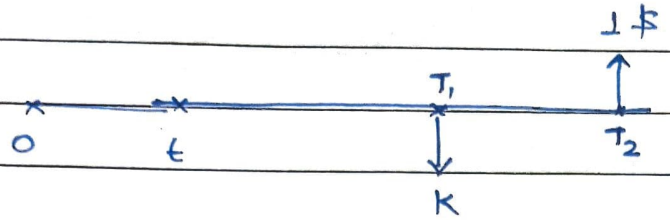
$S = t + \Delta T$

$T = t$

$$C.C.IS = \lim_{\Delta t \to 0} \frac{-\log P(t,t+\Delta t)}{\Delta t}$$

$$= -\frac{\partial}{\partial t} \log(P(t,t+dt))$$

FORWARD CONTRACT



Pay K dollar at T_1 & receive 1$ at T_2

$$\text{Price}(t) = B_t E_Q\left[-K \cdot B_{T_1}^{-1} + 1 \cdot B_{T_2}^{-1} \,\Big|\, \mathcal{F}_t\right]$$

$$\text{using} = \left(E_Q\left(\frac{B_i}{B_j} \,\Big|\, \mathcal{F}_i\right) = P(t_i, t_j) \right)$$

$$\text{Price}(t) = -K_t\left(P(t,T_1) + P(t,T_2)\right)$$

at t wto make ~~fair~~ fair deal, Price (t) = 0

$$\therefore \quad K_t = P(t,T_2) / P(t,T_1) \quad \text{i.e.} \quad K_0 = \frac{P(0,T_2)}{P(0,T_1)}$$

BOND OPTION:

ZERO BOND.

CALL: Bond option gives right to buy a bond at future date (ZBC) for a given price. An option on a T maturity bond with strike K exercised at $t_{decision}$ ~~given by~~ has worth

$$\text{Price}(t) = B_t E_Q\left(B_{t_{decision}}^{-1} \left(P(t_{decision}, T) - K\right)^+ \Big| \mathcal{F}_t\right)$$

where $t < t_{decision}$

$$ZBC(t, t_{decision}, T, K) = P(t, t_{decision}) \cdot E_{Q^T}\left(\left(P(t_{decision}, T) - K\right)^+ \Big| \mathcal{F}_t\right)$$

ZERO BOND PUT (ZBP): Put option gives owner to sell T-maturity ZCB at $t_{decision}$ against strike K

$$ZBP(t, t_{decision}, T, K) =$$

$$P(t, t_{decision}) \, E_{Q^T}\left[\left(K - P(T_{decision}, T) \right)^+ \middle| \mathcal{F}_t \right]$$

Note: if we have a bond with coupons ~~then~~ (i.e. $P(T_{desim}, T) + \sum_i P(T, T_{i+1}) \cdot K \cdot \delta_i$ is underlying then above expression is call "coupon bond option"

CAPLETS / CAPS.

~~1 unit currency. Simply compounded with~~

→ A caplet claim, caps the spot forward rate with some strike. This allows an Institute/ buyer to get protected from unexpect jump in spot forward rate.

from T to S

$$X = \left(\text{Spot Rate}(T, S) - K \right)^+ \cdot N \cdot (S - T)$$

Period / day count fractio

Notional

$$\text{Spot Rate}(T, S) = \frac{1}{(S-T)} \left(\frac{1}{P(T,S)} - 1 \right)$$

(simply compounded)

$$X = N(S-T) \left(\frac{1}{(S-T)} \left(\frac{1}{P(T,S)} - 1 \right) - K \right)^+$$

$$= \frac{N}{P(T,S)} \left(1 - P(T,S) - K P(T,S)(S-T) \right)^+$$

$$= \frac{N}{P(T,S)} \left(1 + K(S-T) \right) \left(\frac{1}{1 + K(S-T)} - P(T,S) \right)^+$$

Date: / /

$$\text{Price}(t) = P(t,S)\, E_{Q^L}\left[\frac{N}{P(T,S)}(1+K(S-T))\left(\frac{1}{1+K(S-T)} - P(T,S)\right)^+ \Big| \mathcal{F}_t\right]$$

$$= N\, P(t,T)\,(1+K(S-T))\, \underbrace{E_{Q^T}\left(\left(K' - P(T,S)\right)^+ \Big| \mathcal{F}_t\right)}_{ZBP(t,T,S,K')}$$

$$\left(P(t,S) = P(t,T)\,P(T,S) \right)$$

$$= N(1+K(S-T)) \times ZBP(t,T,S,K')$$

with

where $K' = \left(\frac{1}{1+K(S-T)}\right)$

day count fraction

CAPS:

if we have time periods $T_0\ T_1, \ldots T_{n-1}, T_n$ and we want to protect against spot forward between T_i to T_{i+1} then we execute series of Caplets each with $T = T_i$ & $S = T_{i+1}$

$$\text{Price}(t) = \sum_i P(\text{Caplets}_i(t))$$

$$= N \sum_i (1+K_i S_i)\, ZBP(t, T_i, T_{i+1}, K_i')$$

$S_i = (T_{i+1} - T_i)$ & $K_i' = \left(1/(1+K_i S_i)\right)$

FLOORLETS / FLOORS.

FLOOLETS floors the spot forward rate from T to S with some strike, this allows buyers to protect selling price.

$$X = \left(K - \text{SpotRate}(T,S)\right)^+ N.(S-T)$$

$$\text{Price}(t) = N(1+K(S-T))\, ZBC(t,T,S,K')$$ where $K' = \frac{1}{(1+K(S-T))}$

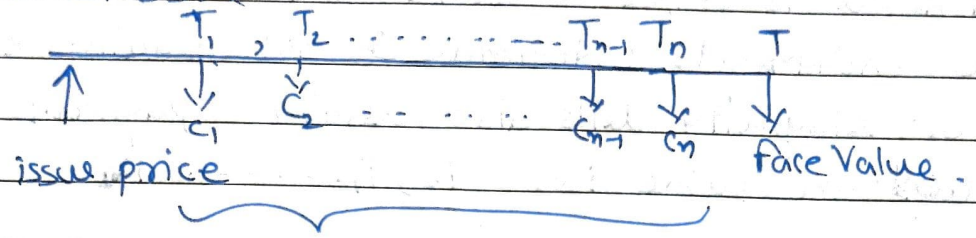
Floors : Series of floorlets for $T_0, T_1 \cdots T_{n-1}, T_n$

$$\text{Floors} = \sum_i \text{Floorlets}(T_i, T_{i+1})$$

$$\text{Price}(t) = N \cdot \sum_i (1 + K_i \delta_i)\, ZBC(t, T_i, T_{i+1}, K_i')$$

$$\delta_i = T_{i+1} - T_i \quad \& \quad K_i' = \frac{1}{1 + K\delta_i}$$

COUPON BONDS. (should be after ZCB & before forward contract)

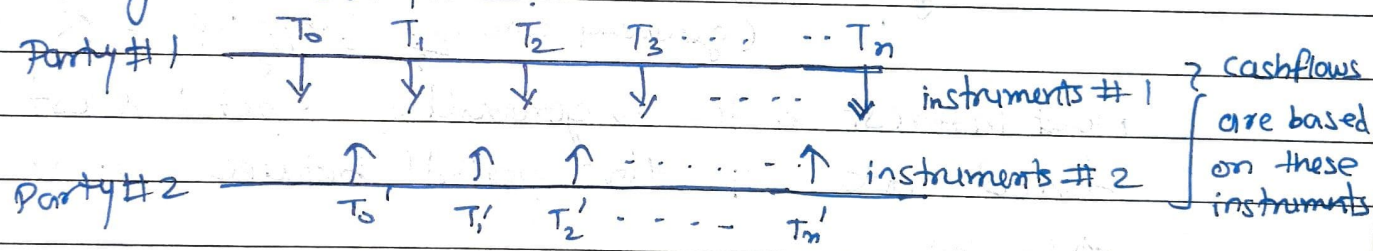


also pay coupons with some rate (on face value) at t_i for $i \in [0, n]$

$$\text{issue Price} = \sum C_i P(t_0, T_i) + F.V. P(t_0, T)$$

SWAP.

A swap is derivative contract through which two parties exchange the cash flows.



Swaps are not traded on exchanges. These are OTC (Over the counter) derivatives not engaged by retail investor but financial institutions.

Types Of SWAP :

INTEREST RATE SWAP :

Instrument # 1 : Fixed interest rate based on a Notional, called Fixed leg.

Instrument # 2 : Float interest rate based on Same Notional called Flout leg.

→ Notional (N) is fixed amount on which interest payments are exchanged. (Notional is not usually exchanged).
→ For simplicity let's also assume payments on both legs are same i.e. $T_0, T_1, \ldots, T_N$.

Quantitative:

this N is Number of periods

$$\text{Claim} = \text{Float leg} - \text{Fixed Leg}$$

$$= \sum_{i=0}^{N-1} N \cdot dcf_i \, L_i - \sum_{i=0}^{N-1} N \cdot dcf_i \, K_i$$

this is Notional.

where N : Notional
dcf_i : Day Count fraction = No. of day between T_i to T_{i+1}
L_i : Float interest rate between T_i to T_{i+1}
K_i : Fixed " " " = K
(generally does not vary)

Float interest rate is generally spot interest rate fixed at T_i for period between T_i to T_{i+1} given by

$$L_i = \frac{1}{dcf_i}\left(\frac{1}{P(T_i, T_{i+1})} - 1\right)$$

$$\therefore \text{claim}_{swap} = N \cdot \sum_{i=0}^{N_T-1}\left(\frac{1}{P(T_i, T_{i+1})} - 1 - dcf_i K\right)$$

Price

Fair value of interest rate swap at any time t is given by

$$= N_t \, E_Q\left[\text{claim}_{swap} \cdot N_T^{-1} \mid \mathcal{F}_t\right]$$

For making analytics easy, we will use libor forward measure Q with numeraire N_t,

$$N_t = \frac{1}{\left(\prod_{i=0}^{k-1} P(T_i, T_{i+1})\right) \cdot P(t_k, t)}$$

assuming t to be betwen (T_k, T_{k+1})

$$N_{T_{i+1}} = \frac{1}{\prod_{j=0}^{i} P(T_j, T_{j+1})}$$

$$\text{Price} = E_Q\left[\underbrace{\frac{1}{\left(\prod_{j=0}^{k-1} P(T_j, T_{j+1})\right) P(T_k, t)}}_{N_t} \times \underbrace{\sum_{i=0}^{N-1}\left(\frac{1}{P(T_i, T_{i+1})} - 1 - d(t_i, k)\right)}_{\times} \times \underbrace{\prod_{j=0}^{i} P(T_j, T_{j+1})}_{(N_{T_{i+1}})^{-1}} \,\middle|\, \mathcal{F}_t\right]$$

taking sumation over i outside.

$$= \sum_{i=0}^{N-1} E_Q\left[\frac{\prod_{j=k}^{i} P(T_j, T_{j+1})}{P(T_k, t)} \times \left(\frac{1}{P(T_i, T_{i+1})} - 1 - d(t_i, k)\right) \,\middle|\, \mathcal{F}_t\right]$$

Using property

$$\prod_{j=k}^{i} P(T_j, T_{j+1}) = \cancel{P(T_k, T_j)}\, P(T_k, t) \cdot P(t, T_i)$$

$$\& \prod_{j=1}^{i-1} P(T_j, T_{j+1}) = P(T_k, t) \cdot P(t, T_{i-1})$$

Date: / /

$$= \sum_{i=0}^{N-1} E_Q\left[P(t, T_i) - P(t, T_{i+1})(1 + dcf_i \cdot K) \,\middle|\, \mathcal{F}_t \right]$$

we know $E_Q\left[P(t, T) \,|\, \mathcal{F}_t \right] = P(t, T)$

$$= \underbrace{\sum_{i=K}^{N-1} \left(P(t, T_i) - P(t, T_{i+1}) \right)}_{\text{these summation terms will cancell out till } k+1} - \sum_{i=K}^{N-1} P(t, T_{i+1})\, dcf_i\, K$$

$$= \left(P(t, T_{k+1}) - P(t, T_N) \right) - K \sum_{i=K}^{N-1} P(t, T_{i+1})\, dcf_i$$

$$= \underbrace{\left(\sum_{i=0}^{N-1} P(t, T_{i+1}) \cdot dcf_i \right)}_{\text{Annuity}} \left(\underbrace{\frac{P(t, T_{k+1}) - P(t, T_N)}{\sum_{i=0}^{N-1} P(t, T_{i+1}) \cdot dcf_i}}_{\text{swap rate}} - \underbrace{K}_{\text{strike /fixed rate.}} \right)$$

Note here k is index of Time periods such that $t \in (T_k, T_{k+1})$

(More ~~Accurately~~)
Some times its called forward rate at we are looking at price of swap at $T_k < t < T_{k+1}$ it can be some fixed $t = T_k$

SWAPTION (SWAP OPTION)

Option to enter into a swap (underlying swap could be interestrate swap, credit default swap, commodity swap, cross currency swap etc.)

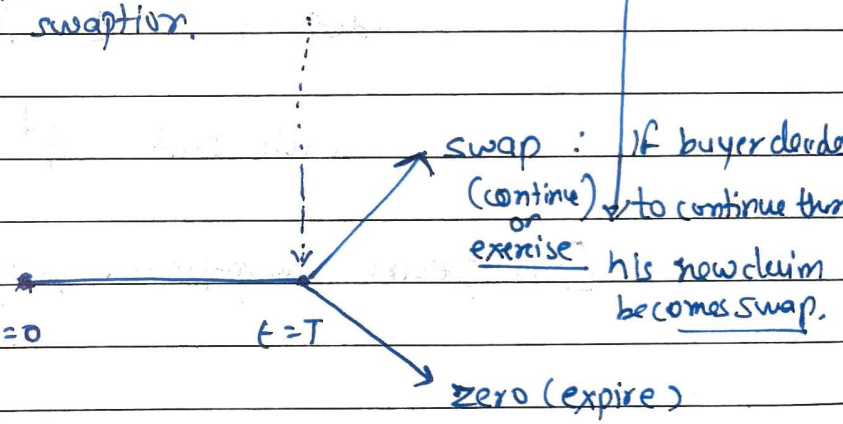
OTC (Not standardised i.e. Not traded on exchange)

Pays Price (swaption) also called premium

Seller → Buyer

in exchange of recieving swaption that gives buyer right to continue with swap (exercise) or expire at or before some time T, also called expiration of swaption.

* Earns the premium
* If buyer decides to continue, seller pays and receives either floating or fixed leg.

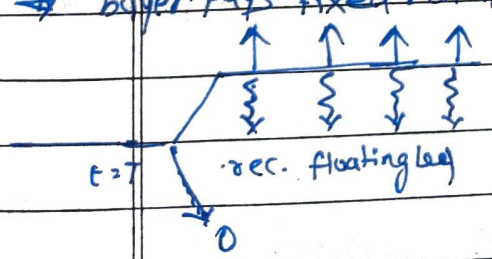


TYPES OF SWAPTIONS.

1. Based on direction of underlying Swap. with respect to purchaser (buyer)

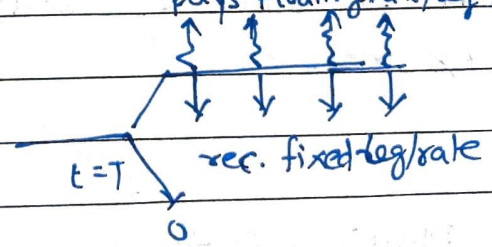
Payer Swaption

* buyer pays fixed rate/leg



Receiver ~~Buyer~~ Swaption

pays floating rate/leg.



2. Based on number of decision dates.

American Swaption : Buyer can exercise his option (i.e. Enter into swap) between any date between start of deal till expiration of swaption.

$$T_{decision} \in [0, T]$$

Bermudan swaption : Buyer can exercise at predermined dates between origin of deal until expiry

$$T_{decision} \in \{ T_1^d, T_2^d, T_3^d \cdots T_n^d \}$$

such that $T_i^d \in [0, T]$

European Swaption : Buyer can exercises or expire only at expiry of swaption

$$T_{decision} = T$$

EUROPEAN SWAPTION

There are further types of european swaption based on ~~what~~ is nature of swap two parties enter into after an exercise.

Physical settled

Cash settled.

Definition. if buyer choose to exercise, both enter into ~~new~~ contract, swap i.e. after T they will follow fixed/float leg schedule of underlying swap.

if buyer choose to exercise, then expection of swap / price of swap at T is settled in cash.

Claim: if(price(swap)@T > 0) swap else 0

here we are entering into another series of cashflows dictated by swap. (so actually deal will continue until swap's last cashflow)

= if (price(swap)@T > 0) Price(swap)@T else 0

$:= \text{Max}(\text{price}(\text{swap})@T, 0)$

Note: here we are directly settling cash equal to Price(swap)@T at T

CASH SETTLED EURO SWAPTION:

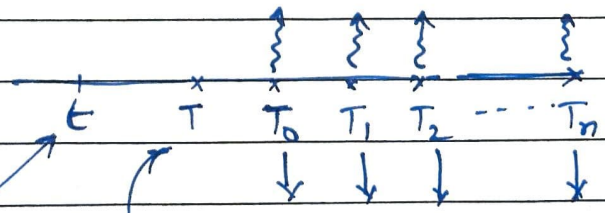
$$\text{Claim}(X) = \text{Max}(\text{price}(\text{swap})@T, 0)$$

where $\text{price}(\text{swap})@T = N_T E_Q[\text{swap} \cdot N_{T+Tenor}^{-1} \mid \mathcal{F}_T]$

N_t is numerair & Q is measure s.t. $N_t^{-1} \cdot B_t$ is martingale.

from swap section, Price(swap)@t is

swap:



any time t before T_0

Expiration of swaption

$$\text{Price}(\text{swap})@t = (P(t,T_0) - P(t,T_n)) - K\sum_{i=0}^{n-1} P(t,T_{i+1})\,dcf_i$$

where $dcf_i = (T_{i+1} - T_i)$

Assuming first swap payment date (T_0) is same swaption expiration.

and we want Price of this swap at $t = T \; (= T_0)$

$$\therefore \text{Price(swap)@}T = \left(1 - P(T, T_n) - K \sum_{i=0}^{n-1} P(T, T_{i+1}) \, dcf_i\right)$$

Now going back to swaption:

$$\therefore \text{Price (swaption)@}t = N_t \cdot E_Q\left[\text{claim}_{\text{swaption}} \times N_T^{-1} \mid \mathcal{F}_t\right]$$

$$= N_t \, E_Q\left[\max\left(\text{price(swap)@}T, 0\right) \cdot N_T^{-1} \mid \mathcal{F}_t\right]$$

we can use $Q = Q_{\text{Libor fwd}}$ $\therefore N_t = \dfrac{1}{P(0,t)}$

$$= \frac{1}{P(0,t)} E_{Q_{\text{Libor}}}\left[\max\left(1 - \left(P(T,T_n) + K\sum_{i=0}^{n-1} P(T,T_{i+1})\delta_i\right), 0\right) \times P(0,T) \mid \mathcal{F}_t\right]$$

Note, if we call $B_{\text{coupon}}(T_1, T_2 \cdots T_n) = \underbrace{P(T,T_n)}_{\text{Bond}} + \underbrace{K\sum_{i=0}^{n-1} P(T,T_{i+1})\delta_i}_{\text{coupons}}$

then above expression is nothing but 'coupon bond' option with strike '1'. (this is alternative definition of swaption)