Hare Krishna

Permutation & Combination

CHEAT SHEET

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Contents

Operations

Name	Formula	Properties
Factorial, $n!$	$n! = 1 \times 2 \times 3 \times \ldots \times (n-1) \times n$	
Combination, ${}^{n}_{i}C$	${}^{n}_{i}C = \frac{n!}{(n-i)!\,i!}$	• ${}^{n}_{i}C = {}^{\;\;n}_{n-i}C$ • $if\ {}^{n}_{i}C == {}^{n}_{j}C$ $then\ either\ i == j\ or\ i + j == n$ • ${}^{n}_{i}C + {}^{\;\;n}_{i+1}C = {}^{n+1}_{i+1}C$ • ${}^{n}_{i}C = \frac{n}{i} \times {}^{n-1}_{i-1}C$
Permutation, ${}^{n}_{i}P$	${}^{n}_{i}P = {}^{n}_{i}C \times i! = \frac{n!}{(n-i)!}$	

Arrangements

Each distinct object with one occurrence

n objects each distinct i.e. $\{o_1, o_2, \dots, o_{n-1}, o_n\}$

Places		Number of ways to arrange n objects on r places
r distinct places each with one occurrence	$\{p_1, p_2, \dots, p_{r-1}, p_r\}$	${}^n_rC \times r!$ if n >=r ${}^r_nC \times n!$ otherwise Special cases: • When a particular object is to be included each time: ${}^{n-1}_{r-1}C \times r!$ • When a particular object is never included: ${}^{n-1}_{r}C \times r! = {}^{n-1}_{r}P$ • When two specified objects always occur & occur together: ${}^{n-2}_{r-2}C \times (r-1)! \times 2$ • Arranging n objects on n places such that no two objects from a subset m objects (out of n) occur together: $(n-m)!\, {}^{n-m+1}_{m}C \times m!$
r distinct places each with finite possible occurrences x out of r places identical + y places identical + *y* object identical too and rest (r-3) all are distinct In other word, first place allows x objects to be placed second one allows *y* objects and so on	$\{\underbrace{p_1, \dots, p_1}_{\text{x times}}, \underbrace{p_2, \dots, p_2}_{\text{y times}}, \underbrace{p_3, \dots, p_3}_{\text{y times}}, \underbrace{\dots, p_{r-1}, p_r}_{\text{1-time: such (r-3) places}}\}$	$\frac{n!}{x! * y! * y! * 1! \dots 1!} \times \left(\frac{\left(x+y+y+(r-3)\right)!}{\underbrace{2! * (r-3)!}}\right)$ Division by number of times those places allowing same objects, for example y was allowed by 2 places and 1 was allowed by r-3 times. (this whole term including numerator is only needed if we want to arrange r places) This assumes that all sum of all finite occurrences of r objects eventually sum to n, if that's not the case then add a dummy place and dump remaining $n - \left(x+y+y+(r-3)\right)$ objects in that place and then use above formula.

<table>
<tr>
<td rowspan="8">Special Case</td>
<td colspan="2">1 distinct place with r (finite) occurrences</td>
<td>$\underbrace{\{p_1, p_1, \dots, p_1, p_1\}}_{r\ \text{times}}$</td>
<td>${}^{n}_{r}C$ if n >=r
${}^{r}_{n}C$ otherwise</td>
</tr>
<tr>
<td rowspan="7">Examples</td>
<td colspan="3">Given n points, m being collinear:
• Number of lines formed: To form a line we need 2 points, where ordering does not matter, hence ${}^{n}_{2}C - {}^{m}_{2}C + 1$
• Number of triangles formed: ${}^{n}_{3}C - {}^{m}_{3}C$
Number of Quadrilateral formed: ${}^{n}_{4}C - {}^{m}_{3}C \times {}^{n-m}_{1}C - {}^{m}_{4}C$</td>
</tr>
<tr>
<td colspan="3">Number of diagonals of n sided polygon = ${}^{n}_{2}C - n$</td>
</tr>
<tr>
<td colspan="3">Maximum number of regions n non-parallels lines divide a plane into: $\frac{n(n+1)}{2} + 1$</td>
</tr>
<tr>
<td colspan="3">Number of parallelograms in two set of parallel lines m and n respectively: ${}^{n}_{2}C \times {}^{m}_{2}C$</td>
</tr>
<tr>
<td colspan="3">In a rectangle of size $w \times l$ with $w \geq l$:
• Number of lines: $(w + 1) + (l + 1)$
• Number of smaller rectangles: ${}^{w+1}_{2}C \times {}^{l+1}_{2}C$
• Number of smaller squares: $w \times l + (w-1) \times (l-1) + \cdots + (w - (l-1) \times (l - (l-1))$</td>
</tr>
<tr>
<td colspan="3">Maximum number of points of intersection of $m\ lines\ and$ n circles: ${}^{m}_{2}C + 2{}^{n}_{2}C + 2{}^{m}_{1}C{}^{n}_{1}C$</td>
</tr>
<tr>
<td colspan="3">Number of ways of arranging n objects on r places such that places are arranged circularly, then actually we can only arrange $(r-1)$ places, hence:
${}^{n}_{r}C \times (r-1)!$</td>
</tr>
<tr>
<td colspan="3">r distinct places each with infinite possible occurrences

In other word each of r places can get any number of objects (including no objects).</td>
<td>$(p_1, p_1, \dots, \infty, p_2, p_2, \dots, \infty, \dots, p_r, p_r, \dots, \infty)$</td>
<td>r^n</td>
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but each distinct place should be used at least once.

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<td>$(p_1, p_1, \dots, \infty, p_2, p_2, \dots, \infty, \dots, p_r, p_r, \dots, \infty)$</td>
<td>$r^n - {}^{r}_{1}C(r-1)^n + {}^{r}_{2}C(r-2)^n - {}^{r}_{3}C(r-3)^n + {}^{r}_{4}C(r-4)^n + \cdots + \infty$</td>
</tr>
</table>

Each distinct object with finite many occurrences

a out of n objects identical of one type + *b* places identical of another type + *b* object identical too and rest (n-3) all are distinct

$$\{\underbrace{o_1, \dots, o_1}_{a\text{ times}}, \underbrace{o_2, \dots, o_2}_{b\text{ times}}, \underbrace{o_3, \dots, o_3}_{b\text{ times}}, \underbrace{\dots, o_{n-1}, o_n}_{\text{1-time: such (n-3) objects}}\}$$

Places		Number of ways to arrange n objects on r places
r distinct places each with one occurrence	$\{p_1, p_2, \dots, p_{r-1}, p_r\}$	Coefficient of x^r in $\left(\frac{x^0}{0!} + \cdots + \frac{x^a}{a!}\right) \times \left(\frac{x^0}{0!} + \cdots + \frac{x^b}{b!}\right) \times \left(\frac{x^0}{0!} + \cdots + \frac{x^b}{b!}\right) \times \left(\frac{x^0}{0!} + \frac{x^1}{1!}\right)^{n-3} \times r!$ Special case, r = sum of all finite occurences then $\frac{(a+b+b+n-3)!}{a!\,b!\,b!\,1!\dots.1!} = \frac{r!}{a!\,b!\,b!\,1!\dots.1!}$
Examples	Number of ways to arrange {1, 2, 3, 4, 5} such that 2 always comes before 3 and 3 always comes before 5. On first look it might seem like there are 5 distinct objects and 5 places but due to constraint above, actually {2, 3, 5} are now identical as we can't arrange them, hence answer is $\frac{5!}{3!}$	
r distinct places each with finite possible occurrences x out of r places identical + y places identical + *y* object identical too and rest (r-3) all are distinct	$\{\underbrace{p_1, \dots, p_1}_{\text{x times}}, \underbrace{p_2, \dots, p_2}_{\text{y times}}, \underbrace{p_3, \dots, p_3}_{\text{y times}}, \underbrace{\dots, p_{r-1}, p_r}_{\text{1-time: such (r-3) places}}\}$	NA

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<tr><td colspan="3">In other word, first place allows x objects to be placed second one allows y objects and so on</td><td></td><td></td></tr>
<tr><td rowspan="3">Special Case</td><td colspan="2">1 distinct place with r (finite) occurrences</td><td>$\underbrace{\{p_1, p_1, \dots, p_1, p_1\}}_{\text{r times}}$</td><td>Coefficient of x^r in
$(x^0 + \cdots + x^a) \times \left(x^0 + \cdots + x^b\right) \times \left(x^0 + \cdots + x^b\right) \times (x^0 + x^1)^{n-3}$</td></tr>
<tr><td rowspan="2">Examples</td><td colspan="3">Number of Integral solutions of $x_1 + x_2 + \cdots + x_r = n \; s.t. \; 0 \leq \; x_i \leq n$

Answer: Here we can assume n being finite occurrences of 1 distinct place on which we are trying to place $\{0\,, 1, 2, \dots, n-1, n\}$ units from x_1 (basically n objects of 1 type) and n objects of another type $(x_2)$and so on. So basically:

Coefficient of x^n in $(x^0 + \cdots + x^n) \times (x^0 + \cdots + x^n) \times \dots \times (x^0 + \cdots + x^n) = \; (x^0 + \cdots + x^n)^r$

which is ${}^{n+r-1}_{\;\;r-1}C$</td></tr>
<tr><td colspan="3">Number of Integral solutions of $k_1x_1 + k_2x_2 + \cdots + k_rx_r = n \; s.t. \; 0 \leq \; x_i \leq n$

Answer: Here we can assume n being finite occurrences of 1 distinct place on which we are trying to place $\{0, k_1, 2k_1, \dots, \}$ units from x_1 and $\{0, k_2, 2k_2, \dots, \}$ objects of another type $(x_2)$and so on. So basically:

Coefficient of x^n in $\left(x^0 + x^{k_1} + x^{2k_1} + \cdots\right) \times \left(x^0 + x^{k_2} + x^{2k_2} + \cdots\right) \times \dots \times \left(x^0 + x^{k_r} + x^{2k_r} + \cdots\right)$</td></tr>
<tr><td colspan="3">r distinct places each with infinite possible occurrences

In other word each of r places can get any number of objects (including no objects).</td><td>$(p_1, p_1, \dots, \infty, p_2, p_2, \dots, \infty, \dots, p_r, p_r, \dots, \infty)$</td><td>NA</td></tr>
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r distinct places each with infinite possible occurrences but each distinct place should be used at least once. In other word each of r places can get any number of objects (excluding no objects).	$(p_1, p_1, \dots, \infty, p_2, p_2, \dots, \infty, \dots, p_r, p_r, \dots, \infty)$	NA

One distinct object with finite many occurrences (sp. case)

only *1 object with* n occurrences:

$\underbrace{\{o_1, \dots, o_1\}}_{\text{n times}}$

<table>
<tr><th colspan="2">Places</th><th>Number of ways to arrange n objects on r places</th></tr>
<tr><td>r distinct places each with one occurrence</td><td>$\{p_1, p_2, \dots, p_{r-1}, p_r\}$</td><td>1</td></tr>
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x out of r places identical + y places identical + y object identical too and rest (r-3) all are distinct

In other word, first place allows x objects to be placed second one allows y objects and so on</td><td>$\{\underbrace{p_1, \dots, p_1}_{\text{x times}}, \underbrace{p_2, \dots, p_2}_{\text{y times}}, \underbrace{p_3, \dots, p_3}_{\text{y times}}, \underbrace{\dots, p_{r-1}, p_r}_{\text{1-time: such (r-3) places}}\}$</td><td>1</td></tr>
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<tr><td>Examples</td><td colspan="2">Number of Integral solutions of $x_1 + x_2 + \dots + x_r = n \; s.t. \; 0 \le \; x_i \le n$

Answer: ${}^{n+r-1}_{r-1}C$</td></tr>
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<tr><td></td><td colspan="3">Number of Integral solutions of $x_1 + x_2 + \cdots + x_r \leq n, s.t. 0 \leq x_i \leq n$

Answer: Add a dummy variable $x_{r+1}, s.t. 0 \leq x_{r+1} \leq n$
Then we can re-write equation as $x_1 + x_2 + \cdots + x_r + x_{r+1} = n, s.t. 0 \leq x_i \leq n$ now use above solution with $r+1$ places.</td></tr>
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<tr><td>Examples</td><td colspan="3">Number of Integral solutions of $x_1 + x_2 + \cdots + x_r = n \; s.t. 1 \leq x_i \leq n$

Answer: ${}^{n-1}_{r-1}C$</td></tr>
</table>

Each distinct Object each with infinite many occurrences

n distinct and infinite occurrence for each object:

$(o_1, o_1, \dots, \infty, o_2, o_2, \dots, \infty, \dots, o_n, o_n, \dots, \infty)$

Places		Number of ways to arrange n objects on r places
r distinct places each with one occurrence	$\{p_1, p_2, \dots, p_{r-1}, p_r\}$	n^r
r distinct places each with finite possible occurrences x out of r places identical + y places identical + *y* object identical too and rest (r-3) all are distinct In other word, first place allows x objects to be placed second one allows *y* objects and so on	$\{\underbrace{p_1, \dots, p_1}_{\text{x times}}, \underbrace{p_2, \dots, p_2}_{\text{y times}}, \underbrace{p_3, \dots, p_3}_{\text{y times}}, \dots, \underbrace{p_{r-1}, p_r}_{\text{1-time: such (r-3) places}}\}$	NA
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r distinct places each with infinite possible occurrences but each distinct place should be used at least once. In other word each of r places can get any number of objects (excluding no objects).	$(p_1, p_1, \dots, \infty, p_2, p_2, \dots, \infty, \dots, p_r, p_r, \dots, \infty)$	NA