Hare Krishna

Permutation & Combination

CHEAT SHEET

RAGHAVENDRA BAZARI

Contents

Operations

Name	Formula	Properties
Factorial, $n!$	$n! = 1 \times 2 \times 3 \times \ldots \times (n-1) \times n$	
Combination, ${}^{n}_{i}C$	${}^{n}_{i}C = \frac{n!}{(n-i)!\,i!}$	• ${}^{n}_{i}C = {}^{n}_{n-i}C$ • $if\ {}^{n}_{i}C == {}^{n}_{j}C$ $then\ either\ i == j\ or\ i + j == n$ • ${}^{n}_{i}C + {}^{n}_{i+1}C = {}^{n+1}_{i+1}C$ • ${}^{n}_{i}C = \frac{n}{i} \times {}^{n-1}_{i-1}C$
Permutation, ${}^{n}_{i}P$	${}^{n}_{i}P = {}^{n}_{i}C \times i! = \frac{n!}{(n-i)!}$	

Representing Number of ways: #,

Arrangements

Each distinct object with one occurrence

n objects each distinct i.e. $\{o_1, o_2, \dots, o_{n-1}, o_n\}$

Places		Number of ways to arrange n objects on r places
r distinct places each with one occurrence	$\{p_1, p_2, \dots, p_{r-1}, p_r\}$	${}^{n}_{r}C \times r!$ if n >=r ${}^{r}_{n}C \times n!$ otherwise

Properties

- This is also called arrangement without replacement.
- When a particular object is to be included each time: ${}^{n-1}_{r-1}C \times r!$
- When a particular object is never included: ${}^{n-1}_{r}C \times r! = {}^{n-1}_{r}P$
- When two specified objects always occur & occur together: ${}^{n-2}_{r-2}C \times (r-1)! \times 2$
- Arranging n objects on n places such that no two objects from a subset m objects (out of n) occur together:

$$(n-m)!\,{}^{n-m+1}_{m}C \times m!$$

- If $r = n$ and there is one specific way in which each object can occupy its place correctly (let's say i^{th} object occupying i^{th} place is correct) then:

$$n! = \#\, no\ object\ occpy\ its\ place\ correctly + \#\, atleast\ one\ obect\ occupy\ its\ place\ correctly$$

Let's defining A_h as # in which h^{th} object occupies the its place ($h^{th} place$) correctly then

$$\#\, atleast\ one\ object\ occupy\ its\ place\ correctly = \bigcup_{h=1}^{r} A_h$$

1. **# at-least one place is occupied correctly**

Notice union does not included repeated counts, if |.| represents cardinality of set then

$$\left|\bigcup_{h=1}^{r} A_h\right| = \sum_{i=1}^{r} |A_i| - \sum_{i=1}^{r}\sum_{j=i+1}^{r} |A_i \cap A_j| + \sum_{i=1}^{r}\sum_{j=i+1}^{r}\sum_{k=j+1}^{r} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1}|A_1 \cap A_2 \cap \ldots \cap A_n|$$

In words
+ (# 1 place is selected from r) times (# at-least that place is occupied correctly with repetitions)
- (# 2 place is selected from r) times (#at-least those 2 places are occupied correctly with repetitions)
+(# 3 place is selected from r) times (#at-least those 3 places are occupied correctly with repetitions)

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(+/-) (# r place is selected from r) times (#at-least those r places are occupied correctly with repetitions)

2. # at-least k places are occupied correctly

$$\left|\bigcup A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap \ldots \cap A_{i_k}\right| = \sum_{i=k}^{n} (-1)^{i+k} \times {}^{i-1}_{k-1}C \times S_i$$

Where S_i is sum of all possible combinations with i objects in correct places.

3. # exactly k places are occupied correctly

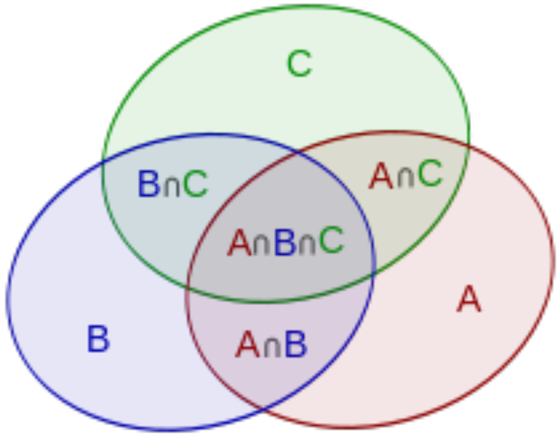
$$= \#\, atleast\ k\ places\ are\ occupied\ correctly - \#\, atleast\ k+1\ places\ are\ occupied\ correctly$$

$$= \sum_{i=k}^{n} (-1)^{i+k} \times {}^{i-1}_{k-1}C \times S_i - \sum_{i=k+1}^{n} (-1)^{i+k+1} \times {}^{i-1}_{k}C \times S_i$$

$$= S_k + \sum_{i=k-1}^{n} (-1)^{i+k} \times \left({}^{i-1}_{k-1}C + {}^{i-1}_{k}C\right) \times S_i$$

$$= S_k + \sum_{i=k+1}^{n} (-1)^{i+k} \times {}^{i}_{k}C \times S_i$$

$$= \sum_{i=k}^{n} (-1)^{i+k} \times {}^{i}_{k}C \times S_i$$

Example	Suppose there is a deck of *n* cards numbered from 1 to n. Suppose a card numbered *m* is in the correct position if it is the m^{th} card in the deck. How many ways, *W, then* # for at-least **specific** k cards in correct position: $$(n-k)!$$ # for at least k cards in correct position: $$\neq\ {}^{n}_{k}C(n-k)!$$ Because we will end up repeating all the # for at-least $k+1, k+2, \ldots so\ on$ cards to be in correct postion in ${}^{n}_{k}C$ possible combinations of choosing k cards.  For example with three sets $\{A, B, C\}$ # in atleast any two $\neq \|A \cap B\| + \|A \cap C\| + \|B \cap C\|$ as they all three include $\|A \cap B \cap C\|$ # at-least one card is in correct position

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$$
\begin{aligned}
&= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{n!}{k!\,(n-k)!} (n-k)! \\
&= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{n!}{k!} \\
&= n! \times \left(1 - \sum_{i=0}^{\infty} \frac{-1^{i}}{(i)!}\right) \\
&= n! \left(1 - \frac{1}{e}\right)
\end{aligned}
$$

no card in correct position

$$
\begin{aligned}
&= n! - \text{ at} - \text{least one card is in correct position} \\
&= \mathrm{n}! - \mathrm{n}! \left(1 - \frac{1}{e}\right) \\
&= \frac{1}{e}
\end{aligned}
$$

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r distinct places each with finite possible occurrences

x out of r places identical + y places identical + *y* object identical too and rest (r-3) all are distinct

In other word, first place allows x objects to be placed second one allows *y* objects and so on

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$\{\underbrace{p_1, \dots, p_1}_{\text{x times}}, \underbrace{p_2, \dots, p_2}_{\text{y times}}, \underbrace{p_3, \dots, p_3}_{\text{y times}}, \underbrace{\dots, p_{r-1}, p_r}_{\text{1-time: such (r-3) places}}\}$

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$$\frac{n!}{x! * y! * y! * 1! \dots 1!} \times \left(\frac{\big(x + y + y + (r-3)\big)!}{\underbrace{2! * (r-3)!}}\right)$$

Division by number of times those places allowing same objects, for example y was allowed by 2 places and 1 was allowed by r-3 times.
(this whole term including numerator is only needed if we want to arrange r places)

This assumes that all sum of all finite occurrences of r objects eventually sum to n, if that's not the case then add a dummy place and dump remaining $n - \big(x + y + y + (r-3)\big)$ objects in that place and then use above formula.

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<tr>
<td rowspan="8">Special Case</td>
<td colspan="2">1 distinct place with r (finite) occurrences</td>
<td>$\underbrace{\{p_1, p_1, \dots, p_1, p_1\}}_{\text{r times}}$</td>
<td>${}^{n}_{r}C$ if n >=r
${}^{r}_{n}C$ otherwise</td>
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<tr>
<td rowspan="7">Examples</td>
<td colspan="3">Given n points, m being collinear:
• Number of lines formed: To form a line we need 2 points, where ordering does not matter, hence ${}^{n}_{2}C - {}^{m}_{2}C + 1$
• Number of triangles formed: ${}^{n}_{3}C - {}^{m}_{3}C$

Number of Quadrilateral formed: ${}^{n}_{4}C - {}^{m}_{3}C \times {}^{n-m}_{1}C - {}^{m}_{4}C$</td>
</tr>
<tr>
<td colspan="3">Number of diagonals of n sided polygon = ${}^{n}_{2}C - n$</td>
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<tr>
<td colspan="3">Maximum number of regions n non-parallels lines divide a plane into: $\frac{n(n+1)}{2} + 1$</td>
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<tr>
<td colspan="3">Number of parallelograms in two set of parallel lines m and n respectively: ${}^{n}_{2}C \times {}^{m}_{2}C$</td>
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<tr>
<td colspan="3">In a rectangle of size $w \times l$ with $w \geq l$:
• Number of lines: $(w+1)+(l+1)$
• Number of smaller rectangles: ${}^{w+1}_{2}C \times {}^{l+1}_{2}C$
• Number of smaller squares: $w \times l + (w-1) \times (l-1) + \cdots + (w-(l-1) \times (l-(l-1))$</td>
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<tr>
<td colspan="3">Maximum number of points of intersection of $m\ lines\ and$ n circles: ${}^{m}_{2}C + 2{}^{n}_{2}C + 2{}^{m}_{1}C{}^{n}_{1}C$</td>
</tr>
<tr>
<td colspan="3">Number of ways of arranging n objects on r places such that places are arranged circularly, then actually we can only arrange $(r-1)$ places, hence:
${}^{n}_{r}C \times (r-1)!$</td>
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<tr>
<td colspan="3">r distinct places each with infinite possible occurrences

In other word each of r places can get any number of objects (including no objects).</td>
<td>$(p_1, p_1, \dots, \infty, p_2, p_2, \dots, \infty, \dots, p_r, p_r, \dots, \infty)$</td>
<td>r^n</td>
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</table>

Properties	This is also called arrangement with replacement. This includes all the possible combinations, i.e. if we define a set A_h which represents h^{th} place occupied for sure (at least by one object) then $$r^n = \bigcup_{h=1}^{r} A_h$$ Notice union does not included repeated counts, if $\|.\|$ represents cardinality of set then **1. # at least one place is occupied:** $$\left\|\bigcup_{h=1}^{r} A_h\right\| = \sum_{i=1}^{r} \|A_i\| - \sum_{i=1}^{r} \sum_{j=i+1}^{r} \|A_i \cap A_j\| + \sum_{i=1}^{r} \sum_{j=i+1}^{r} \sum_{k=j+1}^{r} \|A_i \cap A_j \cap A_k\| - \cdots + (-1)^{n+1} \|A_1 \cap A_2 \cap \ldots \cap A_n\|$$ In words: + (# 1 place is selected from r) times (#at-least that place is included for sure, including repetition) - (# 2 place is selected from r) times (#at-least those 2 place is included for sure including repetition) +(# 3 place is selected from r) times (#at-least those 3 place is included for sure including repetition) . . . (+/-) (# r place is selected from r) times (#at-least those r place is included for sure including repetition) **2. # at least any k places are occupied:**

$$\left|\bigcup_{1\leq i_1<i_2\ldots<i_k<n} A_{i_1}\cap A_{i_2}\cap\ldots\cap A_{i_k}\right|$$

$$= \sum_{i_1=1}^{r}\sum_{i_2=i_1+1}^{r}\ldots\sum_{i_k=i_{k-1}+1}^{r}\left|A_{i_1}\cap A_{i_2}\cap\ldots\cap A_{i_k}\right| - \cdots (1)^{j+1}\,{}^{k+j-2}_{j-1}C\sum_{i_1=1}^{r}\sum_{i_2=i_1+1}^{r}\ldots\sum_{i_{k+j}=i_{k+j-1}+1}^{r}\left|A_{i_1}\cap A_{i_2}\cap\ldots\cap A_{i_{k+j}}\right|\ldots$$
$$+ (-1)^{n+1}|A_1\cap A_2\cap\ldots\cap A_n|$$

at-least k places are occupied

- (# k place is selected from r) times (# selecting 0 from k-1) times (#at-least those k place is included for sure including repetition)
+ (# k+1 place is selected from r) times (# selecting 1 from k) times (#at-least those k+1 place is included for sure including repetition)
-(# k+2 place is selected from r) times (# selecting 2 from k+1) times (#at-least those k+2 place is included for sure including repetition)
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.
(+/-) (# r place is selected from r) times (# selecting r-k from r) times (#at-least those r place is included for sure including repetition)

exactly k places are occupied:

$$\#\ atleast\ k\ places\ are\ occupied - \#atleast\ k+1\ places\ are\ occupied$$

<table>
<tr>
<td>Example</td>
<td colspan="2">
Given two groups of n men and m women, Find number of ways where each men can like any women from 2nd group and similarly each women can like any man from 1st group:

$$m^n n^m$$

Bifurcation for like back:

$$m^n n^m = \#\, no\ one\ is\ liked\ back\ + \#\, atleast\ one\ is\ liked\ back$$

at-least specific k men are liked back:

$$= {}^{m}_{k}C k!\, m^{n-k} n^{m-k}$$

at least 1 is liked back

$$= \sum_{i=1}^{\infty} (-1)^{i-1}\, {}^{n}_{i}C \times {}^{m}_{i}C \times i! \times m^{n-i} n^{m-i}$$

at least k is liked back

$$= \sum_{i=k}^{\infty} (-1)^{i-k}\, {}^{i-1}_{k-1}C \times {}^{n}_{i}C \times {}^{m}_{i}C \times i! \times m^{n-i} n^{m-i}$$

exactly k are liked back

$$= \sum_{i=k}^{\infty} (-1)^{i-k}\, {}^{i}_{k}C \times {}^{n}_{i}C \times {}^{m}_{i}C \times i! \times m^{n-i} n^{m-i}$$
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<td>r distinct places each with infinite possible occurrences
but each distinct place should be used at least once.

In other word each of r places can get any number of objects (excluding no objects).</td>
<td>$(p_1, p_1, \dots, \infty, p_2, p_2, \dots, \infty, \dots, p_r, p_r, \dots, \infty)$</td>
<td>$r^n - {}^{r}_{1}C(r-1)^n + {}^{r}_{2}C(r-2)^n - {}^{r}_{3}C(r-3)^n + {}^{r}_{4}C(r-4)^n + \cdots + \infty$</td>
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</table>

Example	Number of ways such that exactly k are liked back: $N(k) = N_{atleast}(k) - N_{atleast}(k+1) - N_{atleast}(k+2) + N_{atleast}(k+3) + \cdots + \infty$ $N(k) = \sum_{r=0}^{\infty} (-1)^r \, {}^{n}_{k+r}C \, {}^{m}_{k+r}C \, (k+r)! \, m^{n-k-r} \, n^{m-k-r}$ Type equation here. $m^n n^m$

Each distinct object with finite many occurrences

a out of n objects identical of one type + *b* places identical of another type + *b* object identical too and rest (n-3) all are distinct

$$\{\underbrace{o_1, \dots, o_1}_{a\text{ times}}, \underbrace{o_2, \dots, o_2}_{b\text{ times}}, \underbrace{o_3, \dots, o_3}_{b\text{ times}}, \underbrace{\dots, o_{n-1}, o_n}_{\text{1-time: such (n-3) objects}}\}$$

		Number of ways to arrange n objects on r places
r distinct places each with one occurrence	$\{p_1, p_2, \dots, p_{r-1}, p_r\}$	Coefficient of x^r in $\left(\frac{x^0}{0!} + \cdots + \frac{x^a}{a!}\right) \times \left(\frac{x^0}{0!} + \cdots + \frac{x^b}{b!}\right) \times \left(\frac{x^0}{0!} + \cdots + \frac{x^b}{b!}\right) \times \left(\frac{x^0}{0!} + \frac{x^1}{1!}\right)^{n-3} \times r!$ Special case, r = sum of all finite occurences then $\frac{(a+b+b+n-3)!}{a!\,b!\,b!\,1!\dots.1!} = \frac{r!}{a!\,b!\,b!\,1!\dots.1!}$
Examples	Number of ways to arrange {1, 2, 3, 4, 5} such that 2 always comes before 3 and 3 always comes before 5. On first look it might seem like there are 5 distinct objects and 5 places but due to constraint above, actually {2, 3, 5} are now identical as we can't arrange them, hence answer is $\frac{5!}{3!}$	
r distinct places each with finite possible occurrences x out of r places identical + y places identical + *y* object identical too and rest (r-3) all are distinct	$\{\underbrace{p_1, \dots, p_1}_{\text{x times}}, \underbrace{p_2, \dots, p_2}_{\text{y times}}, \underbrace{p_3, \dots, p_3}_{\text{y times}}, \underbrace{\dots, p_{r-1}, p_r}_{\text{1-time: such (r-3) places}}\}$	NA

<table>
<tr><td colspan="3">In other word, first place allows x objects to be placed second one allows *y* objects and so on</td><td></td><td></td></tr>
<tr><td rowspan="3">Special Case</td><td colspan="2">1 distinct place with r (finite) occurrences</td><td>$\underbrace{\{p_1, p_1, \dots, p_1, p_1\}}_{\text{r times}}$</td><td>Coefficient of x^r in
$$(x^0 + \cdots + x^a) \times (x^0 + \cdots + x^b) \times (x^0 + \cdots + x^b) \times (x^0 + x^1)^{n-3}$$</td></tr>
<tr><td rowspan="2">Examples</td><td colspan="3">Number of Integral solutions of $x_1 + x_2 + \cdots + x_r = n \; s.t. \; 0 \leq x_i \leq n$

Answer: Here we can assume n being finite occurrences of 1 distinct place on which we are trying to place $\{0, 1, 2, \dots, n-1, n\}$ units from x_1 (basically n objects of 1 type) and n objects of another type (x_2)and so on. So basically:

Coefficient of x^n in $(x^0 + \cdots + x^n) \times (x^0 + \cdots + x^n) \times \dots \times (x^0 + \cdots + x^n) = (x^0 + \cdots + x^n)^r$

which is ${}^{n+r-1}_{r-1}C$</td></tr>
<tr><td colspan="3">Number of Integral solutions of $k_1x_1 + k_2x_2 + \cdots + k_rx_r = n \; s.t. \; 0 \leq x_i \leq n$

Answer: Here we can assume n being finite occurrences of 1 distinct place on which we are trying to place $\{0, k_1, 2k_1, \dots, \}$ units from x_1 and $\{0, k_2, 2k_2, \dots, \}$ objects of another type (x_2)and so on. So basically:

Coefficient of x^n in $(x^0 + x^{k_1} + x^{2k_1} + \cdots) \times (x^0 + x^{k_2} + x^{2k_2} + \cdots) \times \dots \times (x^0 + x^{k_r} + x^{2k_r} + \cdots)$</td></tr>
<tr><td colspan="3">r distinct places each with infinite possible occurrences</td><td>$(p_1, p_1, \dots, \infty, p_2, p_2, \dots, \infty, \dots, p_r, p_r, \dots, \infty)$</td><td>NA</td></tr>
</table>

In other word each of r places can get any number of objects (including no objects).		
r distinct places each with infinite possible occurrences but each distinct place should be used at least once. In other word each of r places can get any number of objects (excluding no objects).	$(p_1, p_1, \dots, \infty, p_2, p_2, \dots, \infty, \dots, p_r, p_r, \dots, \infty)$	NA

One distinct object with finite many occurrences (sp. case)

only *1 object with* n occurrences:

$\underbrace{\{o_1, \dots, o_1\}}_{\text{n times}}$

	Places	Number of ways to arrange n objects on r places
r distinct places each with one occurrence	$\{p_1, p_2, \dots, p_{r-1}, p_r\}$	1
r distinct places each with finite possible occurrences x out of r places identical + y places identical + *y* object identical too and rest (r-3) all are distinct In other word, first place allows x objects to be placed second one allows *y* objects and so on	$\{\underbrace{p_1, \dots, p_1}_{\text{x times}}, \underbrace{p_2, \dots, p_2}_{\text{y times}}, \underbrace{p_3, \dots, p_3}_{\text{y times}}, \underbrace{\dots, p_{r-1}, p_r}_{\text{1-time: such (r-3) places}}\}$	1
r distinct places each with infinite possible occurrences In other word each of r places can get any number of objects (including no objects).	$(p_1, p_1, \dots, \infty, p_2, p_2, \dots, \infty, \dots,, p_r, p_r, \dots, \infty)$	${}^{n+r-1}_{r-1}C$
Examples	Number of Integral solutions of $x_1 + x_2 + \cdots + x_r = n \; s.t. \; 0 \leq \; x_i \leq n$ Answer: ${}^{n+r-1}_{r-1}C$	

	Number of Integral solutions of $x_1 + x_2 + \cdots + x_r \le n, s.t. 0 \le x_i \le n$ Answer: Add a dummy variable $x_{r+1}, s.t. 0 \le x_{r+1} \le n$ Then we can re-write equation as $x_1 + x_2 + \cdots + x_r + x_{r+1} = n, s.t. 0 \le x_i \le n$ now use above solution with $r+1$ places.		
r distinct places each with infinite possible occurrences but each distinct place should be used at least once. In other word each of r places can get any number of objects (excluding no objects).	$(p_1, p_1, \ldots, \infty, p_2, p_2, \ldots, \infty, \ldots, p_r, p_r, \ldots, \infty)$	${}^{n-1}_{r-1}C$	
Examples	Number of Integral solutions of $x_1 + x_2 + \cdots + x_r = n \; s.t. 1 \le x_i \le n$ Answer: ${}^{n-1}_{r-1}C$		

Each distinct Object each with infinite many occurrences

n distinct and infinite occurrence for each object:
$(o_1, o_1, \dots, \infty, o_2, o_2, \dots, \infty, \dots, o_n, o_n, \dots, \infty)$

Places		Number of ways to arrange n objects on r places
r distinct places each with one occurrence	$\{p_1, p_2, \dots, p_{r-1}, p_r\}$	n^r
r distinct places each with finite possible occurrences x out of r places identical + y places identical + *y* object identical too and rest (r-3) all are distinct In other word, first place allows x objects to be placed second one allows *y* objects and so on	$\{\underbrace{p_1, \dots, p_1}_{\text{x times}}, \underbrace{p_2, \dots, p_2}_{\text{y times}}, \underbrace{p_3, \dots, p_3}_{\text{y times}}, \underbrace{\dots, p_{r-1}, p_r}_{\text{1-time: such (r-3) places}}\}$	NA
r distinct places each with infinite possible occurrences In other word each of r places can get any number of objects (including no objects).	$(p_1, p_1, \dots, \infty, p_2, p_2, \dots, \infty, \dots, p_r, p_r, \dots, \infty)$	NA
r distinct places each with infinite possible occurrences but each distinct place should be used at least once.	$(p_1, p_1, \dots, \infty, p_2, p_2, \dots, \infty, \dots, p_r, p_r, \dots, \infty)$	NA

In other word each of r places can get any number of objects (excluding no objects).		