STOCHASTIC PROCESS

→ collection of Random variables: $\{X_1, X_2 \cdots X_n\}$
$S(t)$ $\forall$ $t \in [0, \infty)$

~~BROWNIAN MOTION~~:
~~BINOMIAL DISTRIBUTION~~:

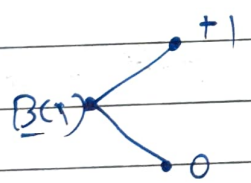
* **Discrete**: A process on descrete intervals t with a parameter 'n' (t: 0, $\frac{1}{n}$, $\frac{2}{n}$ ... $\frac{k}{n}$ ... 1)

~~$W_t(x)$~~
$W_n(t)$ which follows Binomial distribution at any time 't'.

~~1. i.e. $W_n(t)$ can take two values $[-\sqrt{n}, +\sqrt{n}]$ each with probablity $\frac{1}{2}$~~
~~2. $W_n(0) = 0$ i.e. starts at state '0'~~

~~Binomial~~ Binary Random variable:

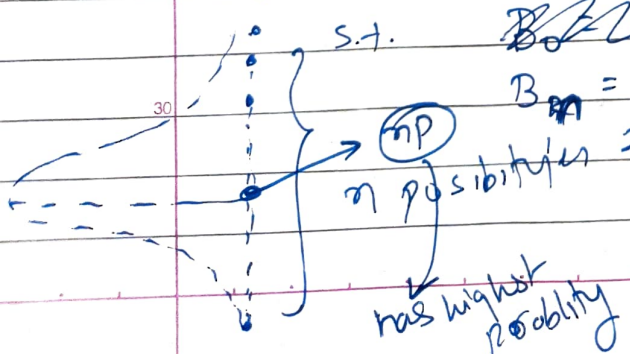
$B(x)$ is B.R.V. iff
i) x can be $[0, 1]$ (discrete)
ii) probality of x taking 1 is p & 0 is $q(1-p)$



Properties:
~~Probality mass function~~:-
mean = $\frac{1}{2}$
variance = $\frac{1}{4}$
$\left(\frac{\frac{1}{4}+\frac{1}{4}}{2} = \frac{\frac{1}{2}}{2}\right)$

Binomial Random ~~Process~~ Variable:

collection of B.R.V: $\{B_0, B_1 \cdots B_k \cdots B_n\}$ defined for some parameter n



s.t. ~~$B_0 = 0$~~
$B_n = [0, 1, 2 \cdots k \cdots n]$ → can take any value

Probablity $(B_n(k)) = {}^nC_k\, p^k q^{n-k}$
mean $(B_n) = np$
variance $(B_n) = npq$.

Each W_n is Binomial Random variable

$$\frac{\sqrt{n}\,B_n(t)+1}{2} \qquad (t \neq 0)$$

$$W_n(t) = \frac{\sqrt{n}\,B_n(t)+1}{2} \longrightarrow (0, t)$$

$W_n(0) = 0$

$$W_n(t) = W_n(t - 1/n) + \frac{B_2}{\sqrt{n}}$$

$$= \frac{\sqrt{n}\,B(t)+1}{2} \longrightarrow (0, t)$$

$$= \frac{\frac{2B(nt) - nt}{\sqrt{n}}}{n} : \left(\frac{-nt}{\sqrt{n}}, \frac{+nt}{\sqrt{n}}\right)$$

$$= \frac{2B(nt) - nt}{\sqrt{n}} = \left(-\sqrt{n}\,t, +\sqrt{n}\,t\right)$$

Where $t = 0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n} \ldots 1 \ldots$

$$P(B_n(x)) = {}^{n}C_x\, p^x q^{n-x}$$

$$P(W_n(t) = x) \Rightarrow P\left(\frac{2B(nt) - nt}{\sqrt{n}} = x\right)$$

$$\Rightarrow P\left(B(nt) = \frac{\sqrt{n}\,x + nt}{2}\right)$$

$$\Rightarrow = {}^{nt}C_{\frac{\sqrt{n}x+nt}{2}} \left(\frac{1}{2}\right)^{nt}$$

$$\Rightarrow \left(\frac{1}{2^{nt}} \cdot {}^{nt}C_{\frac{\sqrt{n}x+nt}{2}}\right)$$

$$\text{mean}(t) = \sum_{x=-\sqrt{n}t}^{+\sqrt{n}t} P(x)\,x = 0 \quad \therefore \text{Variance}(t)$$

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$$\text{mean}(B(n)) = np.$$

(2) $$\text{mean}(W_n(t)) = \text{mean}\left(\frac{2B(nt) - nt}{\sqrt{n}}\right)$$

$$= \frac{2\left(\frac{(nt)}{2}\right) - nt}{\sqrt{n}} = \left(\frac{nt - nt}{\sqrt{2}}\right) = 0$$

$$= 0$$

(3) $$\text{var}(W_n(t)) = \text{var}\left(\frac{2B(nt) - nt}{\sqrt{n}}\right)$$

$$= \frac{4}{n}\,\text{var}(B(nt)) \Rightarrow \frac{4}{n}\left(\frac{nt}{4}\right)$$

$$= t$$

Continuous Brownian Motion

$$W_t = \lim_{n \to \infty} W_n(t)$$

$\therefore$ (1) $W_t(0) = 0$

iv) $f_p(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$

P.d.f of $W_t = X$

v) $f_p(x_1, x_2 \ldots x_n)$ (joint P.d.f)

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi \Delta t_i}} \exp\left(\frac{-\Delta x_i^2}{2\Delta t_i}\right)$$

(2) $$W_t = \lim_{n \to \infty}\left(\frac{2B(nt) - nt}{\sqrt{n}}\right) = N(0, t)$$

(3). $W_{t+s} - W_t \sim N(0, s)$ & is independent to history up until t.

(1) mean $\Rightarrow E[W_t] = 0$

(2) variance $\Rightarrow \text{Var}[W_t] = t$

$$dW_t = \lim_{s \to 0}(W_{t+s} - W_t)$$

(4) $\text{var}[dW_t] = dt$

(1) $$E[dW_t] = \lim_{s \to 0} E[N(0, s)] = 0$$

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Relation Betwen cont & discrete B.M.

$$W_t = \lim_{n \to \infty} \frac{2B(i) - i}{\sqrt{n}}$$ where $i = nt$ (for a very large n)

Properties of B.M.

i) $E[dW_t \mid F_s] = 0$ $\forall\ s < t$

Proof 1. $var(x) = E[x^2] - E[x]^2$

ii) $E[dW_t^2 \mid F_s] = dt$

proof 2 $\int_{-\infty}^{+\infty} x^2 f(x)\,dx$

Proof: $W_t = \lim_{n \to \infty} \frac{2B(i) - i}{\sqrt{n}}$

$$W_{t+dt} = \lim_{n \to \infty} \frac{2(B(i+1) - (i+1))}{\sqrt{n}}$$

$$\therefore dW_t = \lim_{n \to \infty} \frac{2(B(i+1) - B(i)) - 1}{\sqrt{n}}$$

iii) $$\int_0^t (dW_t)^2 = \lim_{n \to \infty} \sum_0^k \left(\frac{2(B(i+1) - B(i)) - 1}{\sqrt{n}} \right)^2$$ (diff of two consecutive Binomial distribution is Binary)

$$= \lim_{n \to \infty} \frac{1}{n} \underbrace{\sum_0^k (2\,Bin - 1)^2}_{\text{Weak law of large numbers}}$$

$Bin \in (0,1)$

$2Bin \in (0,2)$

$2Bin - 1 \in (-1,1)$

$(2Bin - 1)^2 \in [1,1]$

$\therefore E[\ ] = 1$

$$= \lim_{n \to \infty} \frac{1}{n}(k)$$

$k = nt$

$$= \lim_{n \to \infty} \frac{nt}{n} \Rightarrow \underline{t}$$

CALCULUS ON FUNCTION OF BROWNIAN MOTION:

$$Y = f(t, W_t)$$

CHAIN RULE

i) $$dY = \frac{\partial f(t, W_t)}{\partial t} dt + \frac{\partial f}{\partial W_t} dW_t$$

$$+ \frac{1}{2} \frac{\partial^2 f}{\partial t} \cancel{(dt)^2}^{\,0} + \frac{1}{2} \frac{\partial^2 f}{\partial t \partial W} \cancel{dt\,dW}^{\,0}$$

$$+ \frac{1}{2} \frac{\partial^2 f}{\partial W \partial t} \cancel{dW\,dt}^{\,0} + \frac{1}{2} \frac{\partial^2 f}{\partial W_t} (dW)^2$$

$$+ \ldots\ldots$$

$$dY = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial W} dW + \frac{1}{2} \frac{\partial^2 f}{\partial W_t} dt$$

$$dY = \left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial W_t} \right) dt + \frac{\partial f}{\partial W} dW$$

Sp. case

if $Y = f(W_t)$ $\therefore$ $\frac{\partial f}{\partial t} = 0$

$$\therefore dY = \frac{\partial f}{\partial W} dW + \frac{1}{2} \frac{\partial^2 f}{\partial W_t} dt$$

shorcuts.

ii) always try to get a form $Z = f(Y)$ s.t. both Z & Y are stochastic Brownian motions. then apply Ito's:

$$dZ = \frac{\partial f}{\partial Y} dY + \frac{1}{2} \frac{\partial^2 f}{\partial Y^2} (dY)^2$$

& then apply Itos of dY as well & then simplify.

ii) PRODUCT RULE

$$Z = XY = \frac{(X+Y)^2 - X^2 - Y^2}{2}$$

Let $X+Y=K$

$$dZ = \frac{1}{2}\left(\cancel{2(X+Y)} \quad \underline{K^2 - X^2 - Y^2}\right)$$

applying ~~Ito's eqn~~ chain rule on K^2, X^2, Y^2

$$d(K^2) = 2K\,dK + \frac{1}{2}(2)(dK)^2$$

substituting K back & taking 'd' both side

$$dZ = \frac{1}{2}\left(2(X+Y)(dX+dY) + (dX+dY)^2 - \left(2X\,dX + dX^2 + 2Y\,dY + dY^2\right)\right)$$

$$= \frac{1}{2}\left(2X\,dY + 2Y\,dX + 2\,dX\,dY\right)$$

$$\boxed{d(XY) = X\,dY + Y\,dX + dX\,dY}$$ product Rule

EXPECTATION OF A Stoch Process:

Since process is collection of Random Variables, Expectation of process at any time t is nothing but expection of random variable at that time t

$$E[X_t] = \int_{-\infty}^{+\infty} x_t \cdot f_X(x_t)\, dx_t \quad \Bigg| \quad \sum_{x_t=-\infty}^{+\infty} x_t \cdot P(X = x_t)$$

Since $X_t = f(W_t)$ for any t

$\therefore$ Using 'Law of Unconscious Statistican'

$$E[X_t] = \int_{-\infty}^{+\infty} f(w_t) \times \frac{1}{\sqrt{2\pi t^2}} e^{-\frac{1}{2}\left(\frac{w_t}{t}\right)^2} dw_t$$

★ Expectation With Filteration:

Filteration specify history up until time t of a process.

$\therefore$ $E[X_t \mid \mathcal{F}_s]$ = Expectation of process X_t given history up untit time 's' (Conditional Expectation)

Properties

(i) $E[X_t \mid \mathcal{F}_s]$ if $s > t$ $= E[X_t] = X_t$ (Not random anymore)

ii) $E[X_t \mid \mathcal{F}_t] = X_t$

(iii) $E[X_t \mid \mathcal{F}_s]$ if X_t is independent of history upto 's' then

$= E[X_t]$

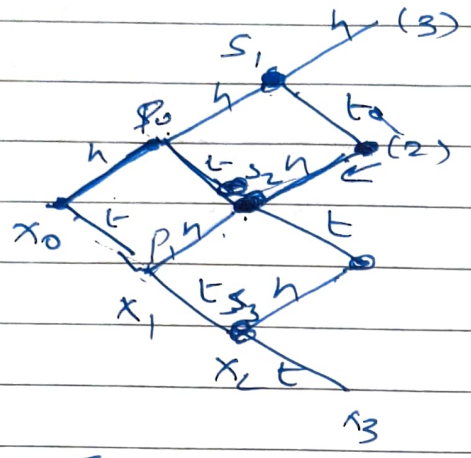
Examples:

1.) X_t : Outcome of cointoss of t $\Rightarrow$ Since outcome of coin toss at t is always independent to any history upto t then

$$E[X_t \mid \mathcal{F}_s] = E[X_t] = \frac{1}{2} \times (1) + \frac{1}{2}(0) = \frac{1}{2}$$

$\forall\ s \leq t$

2. X_t : Number of heads in coin tosse until time t since it includes history of continued time 's' hence X_t is not independent 'S_t', and it will be an another random variable



$$E[X_3 \mid F_2] = \begin{cases} 3\times\frac{1}{2} + 2\times\frac{1}{2} & F_3 = \{P_0\} S_1\} \\ 2\times\frac{1}{2} + 1\times\frac{1}{2} & F_2 = \{P_0\} S_2\} \\ 2\times\frac{1}{2} \times 1\times\frac{1}{2} & F_3 = \{P_1, S_2) \\ 1\times\frac{1}{2} + 0\times\frac{1}{2} & F_4 = \{P_1, S_3\} \end{cases}$$

$\Downarrow$

Its is random variable itself

3. $X_t = W_t$ Random process is Brownian Motion in a measure Q

$E_Q[W_t \mid F_s] \quad \forall s \le t \quad = W_s + W_t - W_s \sim$ ~~$N(0, t-s)$~~

~~random variable following Normal distribution~~

Proof. $E[W_t \mid F_s] = E[W_t - W_s + W_s \mid F_s]$

$= E[\underbrace{W_t - W_s}] F_s] + E[W_s \mid F_s]$

= property of Brownian motion of independent increments

$= E[W_t - W_s] + W_s$

$= 0 + W_s = W_s$

$N(0, t-s)$

4. $X_t = W_t - W_{t-\Delta t}$

$$E[X_t \mid \mathcal{F}_s] = E[\underbrace{W_t - W_{t-\Delta t}}_{} \mid \mathcal{F}_s] = E[N(0,\Delta t)] = 0$$

Since increments are independent up until history $t-\Delta t$ theory must be true for $\forall$ $s<t$

TOWER LAW.

$$E[E[X_t \mid \mathcal{F}_s] \mid \mathcal{F}_r] = E[X_t \mid \mathcal{F}_r]$$

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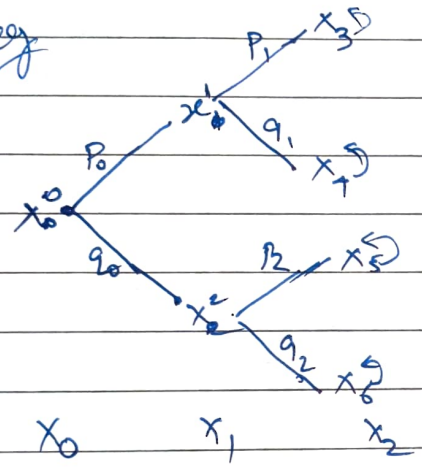


$$E[X_2 \mid \mathcal{F}_0] = E[E[X_2 \mid \mathcal{F}_1] \mid \mathcal{F}_0]$$

$$\Downarrow \qquad\qquad \|$$

$$p_0 p_1 x_3 + p_0 q_1 x_4 + q_0 p_2 x_5 + q_0 q_2 x_6$$

$$\Downarrow$$

$$E\left[\left\{\begin{matrix} x_3 p_1 + x_4 q_1 \\ x_5 p_2 + x_6 q_2 \end{matrix}\right\} \middle| \mathcal{F}_0\right]$$

$$= p_0(x_3 p_1 + x_4 q_1) + q_0(x_5 p_2 + x_6 q_2)$$

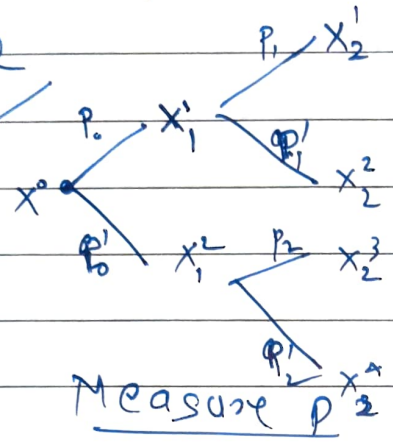
CHANGE OF MEASURE:

(CMG Theorem) Cameron - Martin - Girsanov

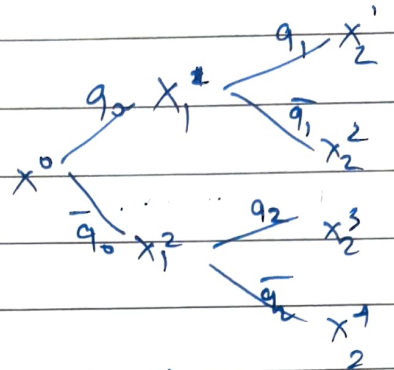
Probability of happening of an event depends upon the measure in which we are, For example real world probability of an event can be very different than one that would ensure no arbitrage.

For example: W_t is a brownian motion only in a specific measure (let's call it Q) but in another measure it might not follow properties of brownian motions. (due to chage in probility distribution).

Discrete



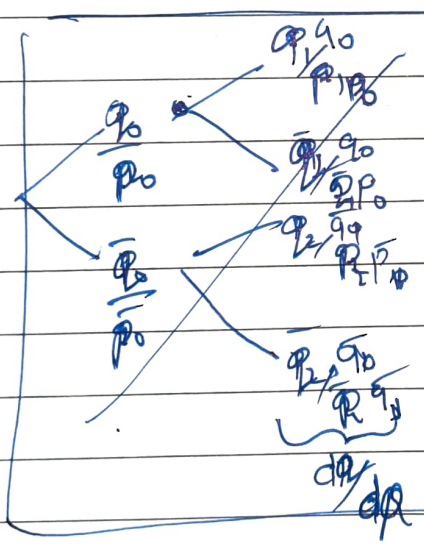
Measure P



Measure Q

$$E_P[X_2|F_1] = \left\{ \begin{matrix} X_2^1 p_1 + X_2^2 \cdot \bar{p}_1 \\ X_2^3 p_2 + X_2^4 \bar{p}_2 \end{matrix} \right\}$$

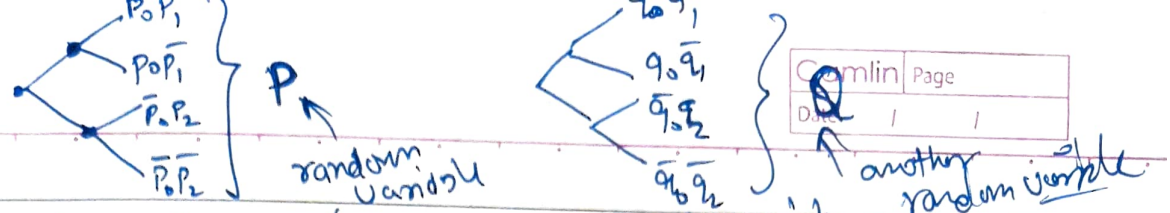
$$E_Q[X_2|F_1] = \left\{ \begin{matrix} X_2^1 q_1 + X_2^2 \bar{q}_1 \\ X_2^3 q_2 + X_2^4 \bar{q}_2 \end{matrix} \right\}$$



$$E\left[\frac{dQ}{dP}\right] =$$

$$E_P\left[\frac{dQ}{dP} \middle| F_1\right] = q_1 + \bar{q}_1$$

Given $E_P[X_t|F_s]$ if we want to infer $E_Q[X_t|F_s]$ we can divide by probbities of P measure for 't' then multiply by probities of Q measure for t but since we want Expection only till filtration s we will need to same process for 's' but in opp. order i.e. divide by P & multiply by Q



To simplify, let define another ~~process~~ random variable which is ~~let~~ ~~call~~ denote is $\left(\frac{dQ}{dP}\right)$ ~~denoted~~ called

Discrete Radon Nikodym ~~Process~~ Derivative which is division of probalities of ξ let's define a process $\xi_t = E\left[\frac{Q}{P} \middle| F_t\right]$.

$$\therefore E_Q[X_t | F_s] = \frac{1}{\xi_s} E_P\left[\xi_t X_t \middle| F_s\right]$$

(P & Q should be equivalent)

such that $\xi_t = E_P\left[\frac{Q}{P} \middle| F_t\right]$ $\begin{cases} \xi_0 = 1 \\ \xi_T = Q/P \end{cases}$

Sp. case $s=0, t=T$

(contin)

i) $\cancel{E_Q[X_t|F_s]} = E_P[\xi_t X_t]$

$$E_Q[X_T | F_s] = \frac{1}{\xi_s} E_P\left[\frac{Q}{P} X_T \middle| F_s\right] \quad ??$$

ii) $s=0$

$$E_Q[X_T] = E_P\left[\frac{Q}{P} X_T\right]$$

Continuous

in Continuous case since ~~a path~~ instead of path probability we have probality density function. Similarly process $\frac{Q}{P}$ is defined rather as $\frac{dQ}{dP}$

which specify ratio of two probity dentisty function for a path from 0 to t or in other term $\omega = \{x_0, x_2 \ldots x_t\}$ taking some specific values all the way for 0 to t

$$\frac{dQ}{dP}(\omega) = \lim_{A \to \omega} \frac{Q(A)}{P(A)}$$

A is t-dimensional Area & ω is one of the path in that t dimension

$$\frac{dQ}{dP}(T) = \lim_{n\to\infty} \frac{f_Q^n(x_1 \cdots x_n)}{f_P^n(x_1 \cdots x_n)}$$: ratio of joint Prob. density function for random variable X_t taking x_i for t_i

(x_n at $t_n = T$)

i) if W_t is brownian motion in P. E_P

$$\frac{dQ}{dP}(T) = \exp\left(-\int_0^T Y_t\, dW_t - \frac{1}{2}\int_0^T Y_t^2\, dt\right)$$

then $\boxed{W_t + \int_0^t Y_s\, ds}$ is brownian motion in Q

Let's call $\tilde{W}_t$; i.e. $\underset{Q\,BM}{\tilde{W}_t} = \underset{P\,BM}{W_t} + \int_0^t Y_s\, ds$

if a process $X_t = f(W_t)$ in P and we want see how process looks in Q measure related by above dQ/dP then

$$X_t = f\left(\tilde{W}_t - \int_0^t Y_s\, ds\right) \text{ in } Q$$ } i.e. we can jump from one measure P to another measure Q just by ~~add~~ subtracting $\int_0^t Y_s\, ds$: wher Y_s is $\mathcal{F}$-previsible process.

~~Proof~~ To prove

$$Y_t = E_P\left[\frac{dQ}{dP} \,\middle|\, \mathcal{F}_t\right] = E_P\left[\exp\left(-\int_0^T Y_t\, dW_t - \frac{1}{2}\int_0^T Y_t^2\, dt\right) \,\middle|\, \mathcal{F}_t\right]$$

$$= E_P\left[\exp\left(\underbrace{-\int_0^t Y_t\, dW_t}_{\text{this should be known by } t} - \int_t^T Y_t\, dW_t - \underbrace{\frac{1}{2}\int_0^T Y_t^2\, dt}_{\text{it has no random ness}}\right) \,\middle|\, \mathcal{F}_t\right]$$

can be taken outside

$$= \underbrace{e^{-\int_0^t Y_t dW_t - \frac{1}{2}\int_0^T Y_t^2 dt}} \cdot E_P\left[\underbrace{\exp\left(-\int_t^T Y_t dW_t\right)} \,\middle|\, \mathcal{F}_t\right]$$

all dW_t are independent to Filteration t hence we can remove Filteration

$$= e^{(\quad '' \quad)} \cdot E_P\left[\exp\left(\int_t^T Y_t dW_t\right)\right]$$

we know that if $X \sim N(\mu, \sigma^2)$ then

$$E[\exp(\theta X)] = \exp\left(\theta\mu + \frac{1}{2}\theta^2\sigma^2\right)$$

here we know $\int_t^T Y_t dW_t \sim N\left(0, \int_t^T Y_t^2 dt\right)$

$$\therefore = e^{(\quad '' \quad)} \cdot \left(\exp\left(\frac{1}{2}\int_t^T Y_t^2 dt\right)\right)$$

$$= e^{\left(-\int_0^t Y_t dW_t - \frac{1}{2}\int_0^t Y_t^2 dt\right)}$$

i.e. $E\left[\frac{dQ}{dP}(T) \,\middle|\, \mathcal{F}_t\right] = \frac{dQ}{dP}(t)$

Proof that $\hat{W}_t$ is Q B.M.

i) $\tilde{W}_0 = W_0 + \int_0^t Y_s ds = 0$ ✓

$E_Q[X_T] = E_P\left(\frac{dQ}{dP} X_T\right)$

$E_Q[X_t | \mathcal{F}_s] = \frac{1}{\xi_s^{-1}} E_P(\xi_t X_t | \mathcal{F}_s)$

ii) $\tilde{W}_t \sim N(0, t) \Rightarrow$ in measure Q $\quad \xi_t = E_P\left[\frac{dQ}{dP} \,\middle|\, \mathcal{F}_t\right]$

to prove

$$E_Q[\exp(\theta \tilde{W}_t)] = \exp\left(\cancel{\theta\mu}^{\,0} + \frac{1}{2}\theta^2 t\right) = \exp\left(\frac{1}{2}\theta^2 t\right)$$

MARTINGALE PROCESS: (Before change of Measure)

A process M_t is called martingale w.r.t a measure P iff

i) $E_P[|M_t|] < \infty$ for $\forall t \in [0, \infty)$

ii) $E_P(M_t | F_s) = M_s$.

Property of M_t: i) $E_P[dM_t | F_s] = 0$.

$\Rightarrow E_P[dM_t] = 0$

Expamples:

i) A constant process, $S_t = C$

ii) P-Brownian Motion; $S_t = W_t \Rightarrow E[W_t | F_s]$

$$= E[W_t - W_s + W_s | F_s]$$
$$= E[W_t - W_s | F_s] + E[W_s | F_s]$$
$$= 0 + W_s = W_s$$

iii) Any process $N_t = E_P[X | F_t]$ is alway P martingale for all P. (From tower law)

iv) Any process $dX = \mu_t dt + \sigma_t dW_t$ is martingale only if $\mu_t = 0$

<u>Proof</u>: if a process X is martingale then

$\Rightarrow$ ~~$E[dX_t | F_t] = 0$~~ $E[dX_t | F_t] = 0$

if this is given then prove that X_t is martingale

$E[X_{t+dt} - X_t | F_t] = 0$

then ~~$E[X_{t+dt} - X_{t-2dt} | F_t] = 0$~~

$E[X_{t+2dt} - X_{t+dt} | F_{t+dt}] = 0$

$E[E[X_{t+2dt} - X_{t+dt} | F_{t+dt}] | F_t]$

$= E[X_{t+2dt} - X_{t+dt} | F_t]$

$= 0$

$\therefore E[dX_s | F_t] = 0$

$\therefore dX_t = d(\mu_t dt) + d(\sigma dW_t)$

For X_t to be martingale

To Prove

$$E[dX_t \mid \mathcal{F}_t] = 0 \Rightarrow E[d(\mu_t t) + d(\sigma dW_t) \mid \mathcal{F}_t] = 0$$

$$\Rightarrow E[d(\mu_t t) \mid \mathcal{F}_t] + E[d(\sigma_t dW_t) \mid \mathcal{F}_t]$$

$$\Rightarrow E[\mu_t dt + t d\mu_t \mid \mathcal{F}_t] + E[\sigma_t dW_t + W_t d\sigma_t + d\sigma_t dW_t \mid \mathcal{F}_t]$$

given $\mathcal{F}_t$ dW_t is not random/stochastic.
(Since $\lim_{\Delta t \to 0} W_{t+\Delta t} - W_t / \Delta t$ is left continuous)

$$\Rightarrow E[\mu_t \mid \mathcal{F}_t] \cdot dt + t E[d\mu_t \mid \mathcal{F}_t] +$$

$$E[\mu dt + \sigma dW_t \mid \mathcal{F}_t] = 0$$

$$E[\mu \mid \mathcal{F}_t] \cdot dt + \sigma_t E[dW_t \mid \mathcal{F}_t] = 0$$

Given filtration t σ_t is not stochastic

σ

$$\therefore E[\mu_t \mid \mathcal{F}_t] = 0 \Rightarrow \mu_t = 0 \text{ for } \forall t$$

$\therefore$ a SDE $dX_t = \mu dt + \sigma dW$ is martingale only if μ is zero $+ E\left[\left(\int_0^T \sigma_s^2 ds\right)^{1/2}\right]$

MARTINGALE REPRESENTATION THEOREM:-

$\Rightarrow dM_t = \phi_t dN_t$. if two processes are martingale (M_t & N_t with volality σ_{1t} & σ_{2t})

$$dM_t = \sigma_t dW_t \Rightarrow dM_t = \frac{\sigma_1}{\sigma_2} dN_t$$
$$dN_t = \sigma_{2t} dW_t$$

such that $\frac{\sigma_{1t}}{\sigma_{2t}} = \phi_t$ as previsible process.

$$\therefore dM_t = \phi_t dN_t$$

CMG. (cont.): To be read after Risk Neutral

Relation between two Measures which are both Risk neutral

CHANGE OF NUMERAIRE

Given both measures are risk neutral, only thing that can differ between the two is Numeraire.

Let's say two measures are Q & P with Numeraires being N_t & M_t;

Since both are risk neural; for an underliy asset S_t,

$S_t N_t^{-1}$: martingale in Q

$S_t M_t^{-1}$: " " P.

	Q	P
Numeraires:	N_t	M_t

Price $(X|\mathcal{F}_0)$: $N_0 E_Q[N_T^{-1} X|\mathcal{F}_0] = M_0 E_P[M_T^{-1} X|\mathcal{F}_0]$

$\downarrow$

(using $E_Q[X] = E_P[\frac{dQ}{dP} X]$)

$\|$

$N_0 E_P\left[\frac{dQ}{dP} N_T^{-1} X|\mathcal{F}_0\right]$

comparing these two

$\frac{dQ}{dP} = \frac{N_T M_0}{M_T N_0}$ (1)

Price $(X|\mathcal{F}_s)$: $N_s E_Q[N_T^{-1} X|\mathcal{F}_s] = M_s E_P[M_T^{-1} X|\mathcal{F}_s]$

$\downarrow$

(using $E_Q[X|\mathcal{F}_s] = \frac{1}{\xi_s} E_P[\xi_T X|\mathcal{F}_s]$)

where $\xi_t = E\left[\frac{dQ}{dP}|\mathcal{F}_t\right]$

$\|$

$\frac{N_s}{\xi_s} E_P\left[\xi_T N_T^{-1} X|\mathcal{F}_s\right]$

comparing two again

$\frac{\xi_T}{\xi_s} = \frac{N_T}{N_s} \times \frac{M_s}{M_T}$

but we know $\xi_T = E_P\left[\frac{dQ}{dP}|\mathcal{F}_T\right] = \frac{dQ}{dP} = \frac{N_T M_0}{M_T N_0}$ (from 1)

$\therefore \xi_s = \frac{N_s M_0}{M_s N_0}$ (2)

$= E_P\left[\frac{dQ}{dP}|\mathcal{F}_s\right] = \frac{N_s M_0}{M_s N_0}$

Process $\frac{N_T M_0}{M_T N_0}$ is martingale in P

$\frac{dQ}{dP}(T)$

if N_t & M_t are in diff currencies, such that N_t is denominated in N_c (currency) & M_t is denominated in M_c & $fx = \cancel{M_t/N_c}\ N_c/M_c$

$$\text{Price}(X|\mathcal{F}_t) = \underset{X}{N_t}\ \underset{Q}{E_Q}[N_T^{-1}X|\mathcal{F}_t] = M_t\ \underset{P}{E_P}[M_T^{-1}X|\mathcal{F}_t]\cdot fx_t$$

in N_c curency

$$\therefore \frac{\xi_T}{\xi_t} N_t N_T^{-1} = M_t fx_t M_T^{-1}$$

$$\therefore \xi_t = \frac{\xi_T}{fx_t} \times \frac{N_t}{N_T} \times \frac{M_T}{M_t} \quad \text{——— (1)}$$

$$\text{Price}(X|\mathcal{F}_0) = N_0\ E_Q[N_T^{-1}X|\mathcal{F}_t] = M_0 fx_0\ E_P[M_T^{-1}X|\mathcal{F}_0]$$

$$\therefore \frac{dQ}{dP} = fx_0 \frac{N_T}{N_0} \cdot \frac{M_0}{M_T} \quad \text{——— (2)}$$

we know $\xi_T = \frac{dQ}{dP}$

$$\therefore \xi_t = \frac{fx_0}{fx_t} \cdot \left(\frac{N_t}{N_0} \times \frac{M_0}{M_t}\right) \text{ martingale in } P.$$

Properties (cont.)

ii) Multidimensional CMG.

Imp Note: these are independent brownian motions.

Brownian Motions

P: $W = (W_t^0, W^1, \ldots W^m)$

Q: $\tilde{W} = (\tilde{W}_t^0, \tilde{W}_t^1, \ldots, \tilde{W}_t^2)$

if $\tilde{W}_t^i = W^i + \int_0^t \gamma_s^i\, ds$. then

$$\frac{dQ}{dP} = \exp\left(-\sum_i\left(\int_0^T \gamma_s^i\, dW_s^i\right) - \frac{1}{2}\sum_i \int_0^T \gamma_i^2\, ds\right)$$

if we represent $\vec{Y}_s = [Y_1 \; Y_2 \; \cdots \; Y_{n-1} \; Y_n]$ (a ~~column~~ row vector)

& $d\vec{W} = [dw_1, dw_2 \cdots dw_{n-1}, dw_n]^T$ (a column ~~row~~ vector)

again $W_1, W_2 \cdots W_n$ are all independent brownian motions

then if Radon Nikodym derivative is

$$\therefore \frac{dQ}{dP} = \exp\left(-\frac{1}{2}\int_0^T \vec{Y}\cdot\vec{Y}^T ds - \int_0^T \vec{Y}\cdot d\vec{W}\right)$$

then $d\vec{\tilde{W}} = [d\tilde{w}_1, d\tilde{w}_2 \cdots d\tilde{w}_{n-1}, d\tilde{w}_n]^T = d\vec{W} + \vec{Y}_t^T dt$

transpose bcz Y_t is row matrix

iii) Multidimension CMG with correlated brownian motion

if we want to move from $d\vec{W}_c$ to $d\vec{\tilde{W}}_c$ such that their independent components are corelated by a correlation matrix C_t, i.e.

$$d\vec{W}_c = L_t d\vec{W} \quad \& \quad d\vec{\tilde{W}}_c = L_t d\vec{\tilde{W}}$$

Correlated b.m. in P. | independent b.m. in measure P. | correlated b.m. in Q | independent b.m. in measure Q

L_t is lower triangular matrix such that (Also called cholesky decomposed matrix)

$L_t L_t^T = C_t$ where C_t is $n \times n$ symmetric correlation matrix, guiding the correlation between various brownian components of $d\vec{W}_c$ and $d\vec{\tilde{W}}_c$

Date: / /

measure : $P \longrightarrow Q$

in case we want to move from W_t to $\tilde{W}_t$ then

$$\frac{dQ}{dP} = \exp\left(-\frac{1}{2}\int_0^T \vec{Y}\cdot C_t \vec{Y}^T ds - \int_0^T \vec{Y}\cdot d\vec{W}_t\right)$$

then $\quad \vec{d\tilde{W}_t} = \vec{dW_t} + C_t Y^T dt$

Proof.

Let's substitute $d\vec{W}_t = L_t \vec{dW}$ in dQ/dP & $C_t = L_t L_t^T$

$$\frac{dQ}{dP} = \exp\left(-\frac{1}{2}\int_0^T \vec{Y}\cdot L_t \cdot L_t^T Y^T ds - \int_0^T \vec{Y}\cdot \vec{L_t}\, d\vec{W}\right)$$

Now here $(\vec{Y}\vec{L})$ is $\vec{Y}$ from CMG of uncorrelated b.m. let's call it $\vec{Y_u}$ ($\vec{Y_u}$ is $1\times n$ hence, $\vec{Y}\vec{L_t}$ is

$$\therefore \frac{dQ}{dP} = \exp\left(-\frac{1}{2}\int_0^T \vec{Y_u}\cdot \vec{Y_u}^T ds - \int_0^T Y_u\, d\vec{W}\right)$$

from that we know

$$\vec{d\tilde{W}} = \vec{dW} + \vec{Y_u}^T dt.$$

Substituting $Y_u^T = (\vec{Y}\vec{L})^T = \vec{L}^T\cdot\vec{Y}^T$

~~also~~ $\cancel{\vec{d\tilde{W}}} =$

$$\vec{d\tilde{W}} = \vec{dW} + \vec{L_t^T}\cdot\vec{Y_t^T}\cdot dt \quad \text{(multiply whole eq}^n \text{ by } L_t\text{)}$$

$$L_t\vec{d\tilde{W}} = L_t\vec{dW} + L_t L_t^T Y_t^T dt$$

$$\tilde{dW_t} = dW_t + CY^T dt \quad \text{hence proved}$$