1. Consider the following matrix and vector:

$$\underset{3\times 2}{\boldsymbol{A}} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} [2\times 1] \quad \boldsymbol{d} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

$\operatorname{rank}(\underline{A}) < \operatorname{rank}(\underline{A} \vdots \underline{d})$

No solution!

a) Find the solution $\widehat{\boldsymbol{w}}$ to $\min_{\boldsymbol{w}} \|\boldsymbol{d} - \boldsymbol{A}\boldsymbol{w}\|_2$. 3×1

Use approximation within span(A)

b) Make a sketch of the geometry of this particular problem in $\mathbb{R}^3$, showing the columns of $\boldsymbol{A}$, the plane they span, the target vector $\boldsymbol{d}$, the residual vector and the solution $\widehat{\boldsymbol{d}} = \boldsymbol{A}\widehat{\boldsymbol{w}}$.

a) from orthogonality condition:

$\hat{\underline{d}}$: closest solution within span(A)

$\underline{d}$: desired solution $\dim(\underline{d}) > \operatorname{rank}(A)$

cannot be obtained by LC of $\underline{A}$.

min($\underline{w}$) occurs when:

$$\underline{A}^T(\underline{d} - \hat{\underline{d}}) = \underline{0}$$

$$\rightarrow \underline{A}^T(\underline{d} - \underline{A}\hat{\underline{w}}) = \underline{0}$$

$$\rightarrow \underline{A}^T\underline{d} - \underline{A}^T\underline{A}\,\hat{\underline{w}} = \underline{0}$$

$\underline{A}^T\underline{A}$

$$\rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\rightarrow \begin{aligned} 2\hat{w}_1 - \hat{w}_2 &= 1 \\ -\hat{w}_1 + 2\hat{w}_2 &= -1 \end{aligned}$$

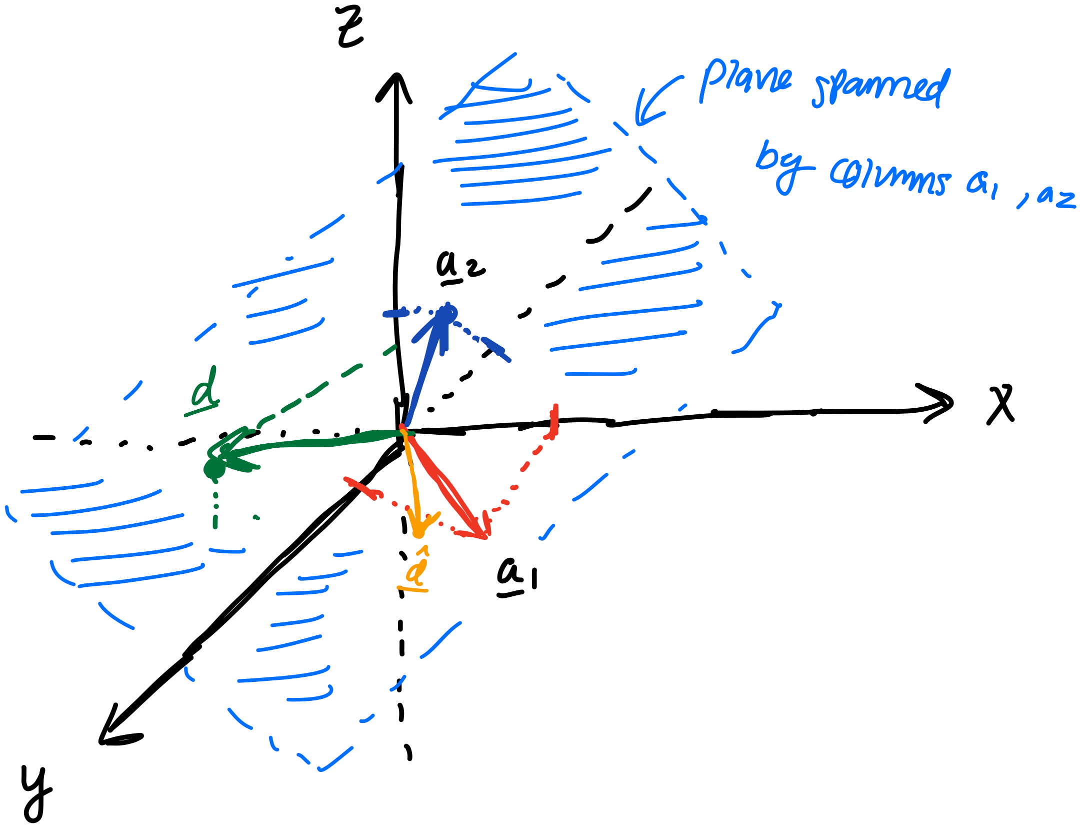
$$\rightarrow 3\hat{w}_2 = -1$$

$$\rightarrow w_2 = -1/3$$

$$\rightarrow \hat{w}_1 = 1 + 2\hat{w}_2 = 1 + (-2/3) = 1/3$$

$$\rightarrow \hat{\underline{w}} = \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix}, \quad \hat{\underline{d}} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ -1/3 \end{bmatrix}$$

b.)



In [18]:
```
import numpy as np
A = np.array([[25,0,1],[20,1,2],[40,1,6]])
b = np.array([[110],[110],[210]])

# To see rank, use:
# np.linalg.matrix_rank(A)

# To invert a matrix, use:
# np.linalg.inv(A)

# To multiply matrices in Python 3, use:
# A@B
```

In [19]:
```
# Problem 2a.)

np.linalg.inv(A)@b

# The array does not agree with the known values.
```

Out[19]:
```
array([[ 4.25],
       [17.5 ],
       [ 3.75]])
```

In [54]:
```
import numpy as np
A = np.array([[25,15,10,0,1],[20,12,8,1,2],[40,30,10,1,6], [30,15,15,0,3], [35,20,15,2,4
b = np.array([[104],[97],[193],[132],[174]])

print(np.linalg.matrix_rank(A))

# Note: you can use np.hstack() to concatinate vectors, for example np.hstack((A,b))

# Note: you can select all the columns, except the first of a matrix A as:  A[:,1:]
```

```
4
```

In [56]:
```
# Problem 2b.)

Awithb = np.hstack((A,b))
print(f"Rank of a with b appended: {np.linalg.matrix_rank(Awithb)}")

# i. rank(A) = 4
#    rank(A | b) = 4
#    because rank(A) = rank(A | b), b does lie within the span of A.
#    So at least one exact solution does exist.

# ii. rank(A) = 4
#     dim(x) = 5
#     More items in x than linearly independent equations.
#     Underdetermined system, infinitely many solutions.

# iii.

Awithout1stcol = A[:,1:]
print(f"new rank without 1st col: {np.linalg.matrix_rank(Awithout1stcol)}")
# So this new matrix without the 1st column is rank 4
# but the x-vector is dimension 4 now.

np.linalg.matrix_rank(np.hstack((Awithout1stcol,b)))
# Before: 5x5, 5x1 = 5x1
# After removing 1st col: 5x4, 4x1 = 5x1

# So there is now a single unique solution now
# Cannot simply multiply A^-1 to both sides since Awithout1stcol is nonsquare
# Use transpose method
```

```python
Awithout1stcol_transpose = np.matrix.transpose(Awithout1stcol)
inverse = np.linalg.inv(Awithout1stcol_transpose @ Awithout1stcol)
x =  inverse @ Awithout1stcol_transpose @ b
print(x)

# The squared error will be zero since b is within span(A).
print(b-Awithout1stcol@x)

# A bunch of floating point errors if I had to guess
```

```
Rank of a with b appended: 4
new rank without 1st col: 4
[[4.]
 [4.]
 [9.]
 [4.]]
[[-1.70530257e-13]
 [-1.56319402e-13]
 [ 0.00000000e+00]
 [ 8.52651283e-14]
 [-2.27373675e-13]]
```

In [ ]:

In [ ]:

3. Suppose the four-by-three matrix $\boldsymbol{A} = \boldsymbol{T}\boldsymbol{W}^T$ where $\boldsymbol{T} = \begin{bmatrix} \boldsymbol{t}_1 & \boldsymbol{t}_2 \end{bmatrix}$ and $\boldsymbol{W}^T = \begin{bmatrix} \boldsymbol{w}_1^T \\ \boldsymbol{w}_2^T \end{bmatrix}$. Further, let $\boldsymbol{t}_1 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$, $\boldsymbol{t}_2 = \begin{bmatrix} 0.5 \\ -0.5 \\ -0.5 \\ 0.5 \end{bmatrix}$, and $\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{w}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

a) What is the rank of $\boldsymbol{A}$?

b) What is the dimension of the subspace spanned by the columns of $\boldsymbol{T}$?

a.) $\underline{A} = \underline{T}\,\underline{W}^T = \begin{bmatrix} \underline{t_1} & \underline{t_2} \end{bmatrix} \begin{bmatrix} \underline{w_1}^T \\ \underline{w_2}^T \end{bmatrix}$

$$= \underset{4\times 2}{\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \\ 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}} = \underset{4\times 3}{\begin{bmatrix} 1 & -0.5 & 1 \\ 0 & 1.5 & 0 \\ 0 & 1.5 & 0 \\ 1 & -0.5 & 1 \end{bmatrix}}$$

$\underline{a}_1 = \underline{a}_3$

So $\underline{a_1}$ and $\underline{a_3}$ are linearly dependent.

$\rightarrow$ rank $\underline{A} = 2.$

b.) $\underline{T} = \underset{4\times 2}{\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \\ 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}}$,

column rank : 2

dimension of subspace : 4

$\rightarrow$ 2d plane in $\mathbb{R}^4$.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$\uparrow$

$\underline{x}$ is some linear comb. of cols of $\underline{W}$

$\underline{x} = \underset{3\times 2}{\underline{W}}\ \underset{2\times 1}{\underline{\tilde{x}}}$ $\quad \begin{bmatrix} \vdots & \vdots \end{bmatrix} \begin{bmatrix} \vdots \end{bmatrix}$ $\quad$ ex: $\underline{x} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

c) Is $\boldsymbol{Q} = \boldsymbol{A}^T\boldsymbol{A} \succ 0$?

d) Suppose $\boldsymbol{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$. Does the least squares problem $\min_{\boldsymbol{x}} \|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|_2^2$ have a unique solution?

e) Suppose we force $\boldsymbol{x}$ to lie in the subspace spanned by $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$, that is, we constrain $\boldsymbol{x} = \boldsymbol{W}\tilde{\boldsymbol{x}}$ where $\tilde{\boldsymbol{x}}$ is a two-by-one vector. Does the least squares problem $\min_{\boldsymbol{x}} \|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|_2^2$ have a unique solution for $\tilde{\boldsymbol{x}}$? Find at least one solution. Note that the numbers are chosen in this problem so you can easily do the calculations on paper.

c.) $\underline{Q} = \underline{A}^T \underline{A} \succ 0$? (positive definite)

$$\underline{Q} = \underset{3\times4}{\begin{bmatrix} 1 & 0 & 0 & 1 \\ -0.5 & 1.5 & 1.5 & -0.5 \\ 1 & 0 & 0 & 1 \end{bmatrix}} \underset{4\times3}{\begin{bmatrix} 1 & -0.5 & 1 \\ 0 & 1.5 & 0 \\ 0 & 1.5 & 0 \\ 1 & -0.5 & 1 \end{bmatrix}} = \underset{3\times3}{\begin{bmatrix} 2 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 2 \end{bmatrix}}$$

$\underline{Q}$ is not full rank so it is not invertible.

Therefore $\underline{Q}$ is not positive definite.

d.) $\underline{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$, $\min_{x} \|\underline{b} - \underline{A}\,\underline{x}\|_2^2$ have a unique solution?

$\underline{A}\,\underline{x} = \underline{b}$

$$\underset{4\times3}{\begin{bmatrix} 1 & -0.5 & 1 \\ 0 & 1.5 & 0 \\ 0 & 1.5 & 0 \\ 1 & -0.5 & 1 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{4\times1}{\begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}}$$

$\text{rank}(\underline{A}) = 2$

$\text{rank}(\underline{A} : \underline{b}) = 2$ (computer)

$\dim(\underline{x}) = 3$

$\rightarrow \infty$ many solutions

more unknowns than linear indep. equations.

Desired solution $\rightarrow$ $\underline{b} - \underline{A}\,\underline{x}$ $\leftarrow$ Closest solution in col. space of $\underline{A}$.

e) Suppose we force $\boldsymbol{x}$ to lie in the subspace spanned by $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$, that is, we constrain $\boldsymbol{x} = \boldsymbol{W}\tilde{\boldsymbol{x}}$ where $\tilde{\boldsymbol{x}}$ is a two-by-one vector. Does the least squares problem $\min_{\boldsymbol{x}} \|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|_2^2$ have a unique solution for $\tilde{\boldsymbol{x}}$? Find at least one solution. Note that the numbers are chosen in this problem so you can easily do the calculations on paper.

$$\underline{A}\,\underline{x} = \underline{b}$$

$$\begin{bmatrix} 1 & -0.5 & 1 \\ 0 & 1.5 & 0 \\ 0 & 1.5 & 0 \\ 1 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\mathrm{rank}(\underline{A}) = 2$$

$$\mathrm{rank}(\underline{A} : \underline{b}) = 2$$

$$1 + \left(-\tfrac{1}{3}\right) = 1 + 0.5 \cdot \tfrac{1}{3} \cdot -2$$

$$\dim(\underline{x}) = 3$$

$\rightarrow \infty$ many solutions

$$\tilde{x}_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \tilde{x}_2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} \cdot \\ 2/3 \\ \cdot \end{bmatrix}$$

$$x_1 - 0.5x_2 + x_3 = 2 \qquad \tilde{x}_1 = 1, \quad \tilde{x}_2 = 1/6 \quad \rightarrow \begin{bmatrix} 7/6 \\ 2/3 \\ 7/6 \end{bmatrix}$$

$$1.5x_2 = 1 \rightarrow x_2 = 2/3$$

$$1.5x_2 = 1$$

$$x_1 - 0.5x_2 + x_3 = 2$$

$$x_1 - 1/3 + x_3 = 2 \Rightarrow x_1 + x_3 = 7/3$$

$$x_1 - 1/3 + x_3 = 2 \rightarrow x_1 + x_3 = 7/3$$