CS/ECE/ME532 Assignment 2

1. Answer the following questions. Justify your answers.

a) Are the columns of the following matrix linearly independent?

$$\boldsymbol{A} = \begin{bmatrix} +0.92 & +0.92 \\ -0.92 & +0.92 \\ +0.92 & -0.92 \\ -0.92 & -0.92 \end{bmatrix}$$

b) Are the columns of the following matrix linearly independent?

$$\boldsymbol{A} = \begin{bmatrix} +1 & +1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & -1 \end{bmatrix}$$

c) Are the columns of the following matrix linearly independent?

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 8 \end{bmatrix}$$

d) What is the rank of the following matrix?

$$\boldsymbol{A} = \begin{bmatrix} +5 & +2 \\ -5 & +2 \\ +5 & -2 \end{bmatrix}$$

e) Suppose the matrix in part d is used in to solve the system of linear equations $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{w} = \boldsymbol{d}$. Does a unique solution exist? Explain why.

2. *Norm additivity.* Suppose $\|\cdot\|_a$ and $\|\cdot\|_b$ are norms on $\mathbb{R}^n$.

a) Prove that $f(\boldsymbol{x}) = \|\boldsymbol{x}\|_a + \|\boldsymbol{x}\|_b$ is also a norm on $\mathbb{R}^n$.

b) The "norm ball" is defined as the set of $\boldsymbol{x}$ for which an (arbitrary) norm $f(\boldsymbol{x}) = 1$. Sketch the norm ball in $\mathbb{R}^2$ for the norm $f(\boldsymbol{x}) = \|\boldsymbol{x}\|_1 + \|\boldsymbol{x}\|_\infty$.

ECE 532 Assignment 2 - DEVIN BRESSER

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b) Are the columns of the following matrix linearly independent?

$$A = \begin{bmatrix} +1 & +1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & -1 \end{bmatrix}$$

c) Are the columns of the following matrix linearly independent?

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 8 \end{bmatrix}$$

d) What is the rank of the following matrix?

$$A = \begin{bmatrix} +5 & +2 \\ -5 & +2 \\ +5 & -2 \end{bmatrix}$$

a.) $$\underline{A} = \begin{array}{cc} c_1 & c_2 \\ \end{array}$$
$$\underline{A} = \begin{bmatrix} 0.92 & 0.92 \\ -0.92 & 0.92 \\ 0.92 & -0.92 \\ -0.92 & -0.92 \end{bmatrix}$$

if linearly dependent, $c_1 = \alpha c_2$

$c_{11} = 0.92 = 1 \cdot c_{21}$

$c_{12} = -0.92 = -1 \cdot c_{21}$

Thus no linear combination exists. Linear independent.

b.) $$\underline{A} = \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \end{array}$$

Looking at top row:
only way to make one column a linear comb. of the others is to set:

$c_{11} = \alpha c_{21} + (1-\alpha) c_{31}$ for all α

Then try on middle row:

$c_{12} \stackrel{?}{=} \alpha c_{22} + (1-\alpha) c_{32}$, $\alpha = 1$

$\Rightarrow -1 \stackrel{?}{=} 1 \cdot 1 + (1-1) -1$

$\Rightarrow -1 \stackrel{?}{=} 1$. Therefore no linear combination works for both the 1st and second rows,

all all 3 columns are linearly independent.

c) Are the columns of the following matrix linearly independent?

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 8 \end{bmatrix}$$

1.) c.)

No. $C_1 = C_3 - 0.5C_2$

d) What is the rank of the following matrix?

$$A = \begin{bmatrix} +5 & +2 \\ -5 & +2 \\ +5 & -2 \end{bmatrix}$$

d.) Rank $\begin{array}{cc} C_1 & C_2 \end{array}$
$$\begin{bmatrix} 5 & 2 \\ -5 & 2 \\ 5 & -2 \end{bmatrix}$$

Columns are only linearly independent if: $C_{1i} = 2.5C_{2i}$ for all i.

$C_{12} = -5 \neq 2.5\, C_{22}$

Thus columns are linearly independent and rank $= 2$.

e) Suppose the matrix in part d is used in to solve the system of linear equations $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{w} = \boldsymbol{d}$. Does a unique solution exist? Explain why.

$$\boldsymbol{A} = \begin{bmatrix} +5 & +2 \\ -5 & +2 \\ +5 & -2 \end{bmatrix}$$

3×2

$$\underline{A}^T\,\underline{A} = \underset{2\times3}{\begin{bmatrix} 5 & -5 & 5 \\ 2 & 2 & -2 \end{bmatrix}} \underset{3\times2}{\begin{bmatrix} 5 & 2 \\ -5 & 2 \\ 5 & -2 \end{bmatrix}} = \underset{2\times2}{\begin{bmatrix} 75 & -10 \\ -10 & 12 \end{bmatrix}}$$

rank $\underline{A}^T\underline{A} = 2$ by inspection.

So, if $\underline{A}^T\underline{A}\,\underline{w} = \underline{d}$

$$\rightarrow [2\times2][2\times1] = [2\times1]$$

Because $\text{rank}\,\underset{2\times2}{(\underline{A}^T\underline{A})}$ must be equal to $\text{rank}\,\underset{2\times3}{(\underline{A}^T\underline{A} \,\vdots\, \underline{d})}$

and $\text{rank}(\underline{A}^T\underline{A}) = \dim(\underline{w}) = 2$

There is exactly 1 unique solution.

2. *Norm additivity.* Suppose $\|\cdot\|_a$ and $\|\cdot\|_b$ are norms on $\mathbb{R}^n$.

a) Prove that $f(\boldsymbol{x}) = \|\boldsymbol{x}\|_a + \|\boldsymbol{x}\|_b$ is also a norm on $\mathbb{R}^n$.

b) The "norm ball" is defined as the set of $\boldsymbol{x}$ for which an (arbitrary) norm $f(\boldsymbol{x}) = 1$. Sketch the norm ball in $\mathbb{R}^2$ for the norm $f(\boldsymbol{x}) = \|\boldsymbol{x}\|_1 + \|\boldsymbol{x}\|_\infty$.

Norm properties:

1.) $\|\underline{x}\| \geq 0$ for all $\underline{x}$.

2.) $\|x\| = 0$ if and only if $\underline{x} = \underline{0}$.

3.) $\|b\,\underline{x}\| = |b|\,\|\underline{x}\|$ for all $b \in \mathbb{R}$, $\underline{x} \in \mathbb{R}^N$.

4.) $\|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$.

a.) If $\|\cdot\|_a$ and $\|\cdot\|_b$ are norms on $\mathbb{R}^N$ then they both satisfy 1.) – 4.) above.

$f(\underline{x}) = \|\underline{x}\|_a + \|\underline{x}\|_b$ must also satisfy 1.) – 4.)

1.) $f(\underline{x}) \geq 0$ for all $\underline{x}$.

$f(\underline{x})$ is the sum of $\|\underline{x}\|_a$ and $\|\underline{x}\|_b$ which are both ≥ 0 since they are norms. ✓

2.) $f(\underline{x}) = 0$ iff $\underline{x} = \underline{0}$.

If $f(\underline{x}) = 0$, then $\|\underline{x}\|_a + \|\underline{x}\|_b = 0$

Since $\|\underline{x}\|_a$ and $\|\underline{x}\|_b$ are both norms, then they can only be 0 if $\underline{x}$ is the zero vector.

And since $\|\underline{x}\|_a$ and $\|x\|_b$ are norms, they must be non-negative.

Thus the only conditions when $\|\underline{x}\|_a + \|\underline{x}\|_b = 0$ are when $\underline{x} = \underline{0}$.

Thus, $f(\underline{x}) = 0$ only when $\underline{x} = \underline{0}$. ✓

3.) $f(c\,\underline{x}) = |c|\,f(\underline{x})$ if $f(\underline{x})$ is a norm.

→ $f(c\,\underline{x}) = \|c\,\underline{x}\|_a + \|c\,\underline{x}\|_b = |c|\,\|\underline{x}\|_a + |c|\,\|\underline{x}\|_b$

$= |c|\left(\|\underline{x}\|_a + \|\underline{x}\|_b\right) = |c|\,f(\underline{x})$. ✓

4.) $f(\underline{x}+\underline{y}) \leq f(\underline{x}) + f(\underline{y})$ if $f(\underline{x})$ is a norm.

$\rightarrow$ $f(\underline{x}+\underline{y}) = \|\underline{x}+\underline{y}\|_a + \|\underline{x}+\underline{y}\|_b$

$f(\underline{x}) + f(\underline{y}) = \|\underline{x}\|_a + \|\underline{y}\|_a + \|\underline{x}\|_b + \|\underline{y}\|_b$.

$\rightarrow$ Check: $\|\underline{x}+\underline{y}\|_a + \|\underline{x}+\underline{y}\|_b \leq \|\underline{x}\|_a + \|\underline{y}\|_a + \|\underline{x}\|_b + \|\underline{y}\|_b$?

Since $\|\underline{x}\|_a$ and $\|\underline{x}\|_b$ are norms, $\|\underline{x}+\underline{y}\|_a \leq \|\underline{x}\|_a + \|\underline{y}\|_a$

and $\|\underline{x}+\underline{y}\|_b \leq \|\underline{x}\|_b + \|\underline{y}\|_b$.

Then the sum of the inequalities must also be true.

Then $\|\underline{x}+\underline{y}\|_a + \|\underline{x}+\underline{y}\|_b \leq \|\underline{x}\|_a + \|\underline{y}\|_a + \|\underline{x}\|_b + \|\underline{y}\|_b$

and $f(\underline{x}+\underline{y}) \leq f(\underline{x}) + f(\underline{y})$. ✓

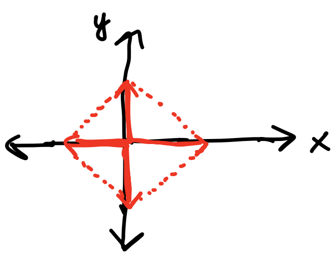
Since all 4 properties are satisfied, $f(\underline{x})$ is also a norm on $\mathbb{R}^N$.

b) The "norm ball" is defined as the set of $\boldsymbol{x}$ for which an (arbitrary) norm $f(\boldsymbol{x}) = 1$. Sketch the norm ball in $\mathbb{R}^2$ for the norm $f(\boldsymbol{x}) = \|\boldsymbol{x}\|_1 + \|\boldsymbol{x}\|_\infty$.

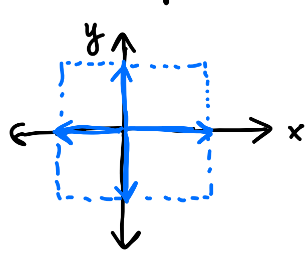
$f(\underline{x}) = \|\underline{x}\|_1 + \|\underline{x}\|_\infty$

manhattan norm

longest dimension only



$\|\underline{x}\|_1 = 1$



$\|\underline{x}\|_\infty = 1$

See next page for graph

$\|\underline{x}\|_1 + \|\underline{x}\|_\infty = 1$

$(1/3, 1/3) \rightarrow 1/3 + 1/3 + 1/3 = 1$ ✓

$(0, 0.5) \rightarrow 0 + 0.5 + 0.5 = 1$ ✓

$(0.2, 0.4) \rightarrow 0.2 + 0.4 + 0.4 = 1$ ✓

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$$\|\underline{x}\|_1 + \|\underline{x}\|_\infty = 1$$