

CS/ECE/ME532 Activity 7

Estimated Time: 15 min for each problem

1. Let $\mathbf{X} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

- a) Use the Gram-Schmidt orthogonalization procedure and hand calculation to find an orthonormal basis for the space spanned by the columns of \mathbf{X} . What geometric object is described by the span of these bases?

b) Now interchange the columns of \mathbf{X} , that is, define $\tilde{\mathbf{X}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

- Do the columns of \mathbf{X} span the same space as the columns of $\tilde{\mathbf{X}}$?
- Use the Gram-Schmidt orthogonalization procedure to find an orthonormal basis for the space spanned by the columns of $\tilde{\mathbf{X}}$. How does the geometric object described by the span of this set of orthonormal bases compare to the one in Part a?
- Are the bases vectors you found for \mathbf{X} and $\tilde{\mathbf{X}}$ the same? Does the space spanned by the columns of a matrix depend on the order of the columns?

2. Let $\mathbf{X} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ as in the previous problem.

- Place the orthonormal bases you found as columns of a matrix \mathbf{U} .
- Find $\mathbf{U}^T \mathbf{U}$.
- Since \mathbf{U} contains a basis for space spanned by the columns of \mathbf{X} you decide to write each column of \mathbf{X} as a linear combination of the columns of \mathbf{U} : $\mathbf{X} = \mathbf{U} \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$. What is the dimension of \mathbf{a}_1 ? Briefly describe the meaning of \mathbf{a}_1 and \mathbf{a}_2 .
- Let $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$ so that $\mathbf{X} = \mathbf{U} \mathbf{A}$. Multiply both sides of this equation by \mathbf{U}^T and solve for \mathbf{A} .

3. Let the columns of an n -by- p ($n > p$) matrix \mathbf{X} be linearly independent and \mathbf{U} be an orthonormal basis for the p -dimensional space spanned by the columns of \mathbf{X} .
- a) It can be shown that $\mathbf{X} = \mathbf{U}\mathbf{T}$ where \mathbf{T} is a p -by- p invertible matrix. Briefly explain why \mathbf{T} should be invertible without resorting to a mathematical proof. That is, explain why this result is intuitively reasonable.
 - b) Use the result in the previous item to show that the projection onto the space spanned by \mathbf{X} is identical to that onto the space spanned by \mathbf{U} . That is, show $\mathbf{P}_x = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T = \mathbf{P}_U = \mathbf{U}(\mathbf{U}^T\mathbf{U})^{-1}\mathbf{U}^T$. *Hint:* Recall that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
 - c) Express \mathbf{P}_U without a matrix inverse.

4. Consider the matrix and vector

$$\mathbf{X} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Note that \mathbf{X} is defined identically in the preceding problems.

- a) Make a sketch of the orthonormal bases \mathbf{U} and the columns of \mathbf{X} in three dimensions.
 - b) Use \mathbf{U} and the result of the previous problem to compute the LS estimate $\hat{\mathbf{b}} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{b}$.
5. Let $\mathbf{z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and define $\mathbf{Q} = \mathbf{z}\mathbf{z}^T$.
- a) Sketch the surface $y = \mathbf{x}^T\mathbf{Q}\mathbf{x}$ where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. If you find 3-D sketching too difficult, you may draw a contour map with labeled contours.
 - b) Let $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Sketch the subspace spanned by \mathbf{z} and the subspace spanned by \mathbf{w} on your sketch of the surface $y = \mathbf{x}^T\mathbf{Q}\mathbf{x}$.
 - c) Does the problem $\min_{\mathbf{x}} \mathbf{x}^T\mathbf{Q}\mathbf{x}$ have a unique solution?
 - d) Is $\mathbf{Q} \succ 0$? Is $\mathbf{Q} \succeq 0$?