CS/ECE/ME532 Activity 4

Estimated Time: 15 min for P1, 10 min for P2, 15 min for P3

1) *Matrix Rank.* Let $\boldsymbol{X} = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   a) What is the rank of $\boldsymbol{X}$?

   b) Find a set of linearly independent columns in $\boldsymbol{X}$. Is there more than one set? How many sets of linearly independent columns can you find?

   c) A matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & a \\ 1 & 1 & b \\ 0 & 1 & -1 \end{bmatrix}$. Find the relationship between b and a so that rank$\{\boldsymbol{A}\} = 2$. *Hint:* find a, b so that the third column is a weighted sum of the first two columns. Note that there are many choices for a, b that result in rank 2.

2) *Solution Existence.* A system of linear equations is given by $\boldsymbol{Ax} = \boldsymbol{b}$ where $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$.

   a) Suppose $\boldsymbol{b} = \begin{bmatrix} 8 \\ 6 \\ -2 \end{bmatrix}$. Does a solution for $\boldsymbol{x}$ exist? If so, find $\boldsymbol{x}$.

   b) Suppose $\boldsymbol{b} = \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix}$. Does a solution for $\boldsymbol{x}$ exist? If so, find $\boldsymbol{x}$.

   c) Consider the general system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$. This equation says that $\boldsymbol{b}$ is a weighted sum of the columns of $\boldsymbol{A}$. Assume $\boldsymbol{A}$ is full rank. Use the definition of linear independence to find the condition on rank$\left\{\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \end{bmatrix}\right\}$ that guarantees a solution exists.

3) *Non Unique Solutions.*

a) Consider $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{A} = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ -2 & 4 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix}$ and $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

i) Does this system of equations have a solution? Justify your answer.

ii) Is the solution unique? Justify your answer.

iii) Draw the solution(s) in the x_1-x_2 plane using x_1 as the horizontal axis.

b) If the system of linear equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has more than one solution, then there is at least one non zero vector $\boldsymbol{w}$ for which $\boldsymbol{x} + \boldsymbol{w}$ is also a solution. That is, $\boldsymbol{A}(\boldsymbol{x} + \boldsymbol{w}) = \boldsymbol{b}$. Use the definition of linear independence to find a condition on $\text{rank}\{\boldsymbol{A}\}$ that determines whether there is more than one solution.