CS/ECE/ME532 Activity 13

Estimated time: 25 minutes for Q1 and 35 minutes for Q2.

1. We've previously considered several low rank approximations for matrices based on the SVD. Let an n-by-p matrix $\boldsymbol{X}$ with $n \leq p$ be expressed as

$$\boldsymbol{X} = \sum_{i=1}^{n} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T$$

where σ_i is the ith singular value with left singular vector $\boldsymbol{u}_i$ and right singular vector $\boldsymbol{v}_i$. The rank-r approximation is

$$\boldsymbol{X}_r = \sum_{i=1}^{r} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T$$

where $r \leq n$. Define the error between $\boldsymbol{X}$ and the rank-r approximation as $\boldsymbol{E}_r = \boldsymbol{X} - \boldsymbol{X}_r$.

a) Find the SVD of $\boldsymbol{E}_r$ in terms of the σ_i, $\boldsymbol{u}_i$, and $\boldsymbol{v}_i$.

b) Suppose $\boldsymbol{X}$ is full rank. What is the rank of $\boldsymbol{E}_r$?

c) Find the operator norm (which is also called the 2-norm of a matrix) of the error matrix $||\boldsymbol{E}_r||_{op}$ in terms of the SVD parameters for $\boldsymbol{X}$.

d) Explain the conditions under which $\boldsymbol{X}_r$ will be a "good" approximation to $\boldsymbol{X}$.

2. Image compression. A digital image can be represented with a matrix, where each element of the matrix represents a pixel in the image. A low-rank approximation to the matrix is one way to compress the image, as explored in this problem. A data file contains a matrix $\boldsymbol{A} \in \mathbb{R}^{600\times 400}$ of grayscale values scaled to lie between 0 and 1. A helper script loads the data and displays the corresponding image. There are three lines of code that require completion before you can run the code: one in the section labeled "Bucky's Singular Values" and two in the section labeled "Low-Rank Approximation".

a) Take the SVD of $\boldsymbol{A}$ by completing the code. Inspect the singular value spectrum. What do you conclude about the approximate rank of $\boldsymbol{A}$? Why is it useful to plot the logarithm of the singular values?

b) Approximate $\boldsymbol{A}$ as a rank r matrix $\boldsymbol{A}_r$ by only keeping the r largest singular values and making the rest zero. Try this for $r \in \{10, 20, 50, 100\}$ and plot the corresponding low-rank images. Also find the fractional squared error

$$e = \frac{||\boldsymbol{A} - \boldsymbol{A}_r||_F^2}{||\boldsymbol{A}||_F^2}$$

Comment on the how the quality of the approximation changes as r increases.

c) Compare the space required to store the full $\boldsymbol{A}$ matrix with the space required to store the rank r SVD approximation of $\boldsymbol{A}$; how many times smaller is the storage requirement for $r \in \{10, 20, 50, 100\}$? You may assume that storage space requirements are proportional to the number of numbers that must be stored. e.g. a 10×10 matrix contains 100 numbers.

d) Use the last section of the code to find the rank of the low-rank approximation that minimizes the sum of the bias squared and variance for a noisy version of Bucky. Note that since the "Bias-Variance Tradeoff in Low-Rank Approximations" lecture assumes an N-by-M matrix with $N < M$, we work with the transpose of $\boldsymbol{A}$ so that $M = 600$ and $N = 400$.

i. Assume the variance of each row $\sigma_g^2 = \sum_{j=1}^{M} g_{ij}^2$ in the "Bias-Variance Tradeoff in Low-Rank Approximations" lecture is $\sigma_g^2 = 10$.

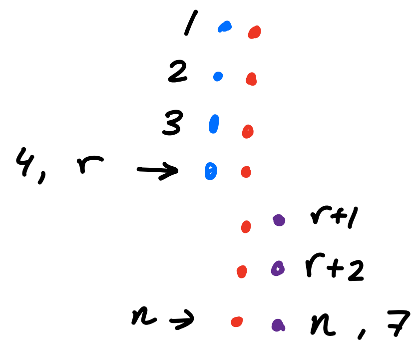
ii. Assume the variance of each row $\sigma_g^2 = \sum_{j=1}^{M} g_{ij}^2$ in the "Bias-Variance Tradeoff in Low-Rank Approximations" lecture is $\sigma_g^2 = 50$.

e) Optional. Simulate the noisy case by performing low-rank approximations to a noisy version of $\boldsymbol{A}$. You may create this using the command `Anoise = A + np.sqrt(sigma2/600)* np.random.randn(np.shape(A)[0], np.shape(A)[1])` in Python, where `sigma2` corresponds to σ_g^2. Note that the division by $M = 600$ is necessary because `randn` creates random matrices where each element (not the row) has unit variance.

Activity 13 DEVIN BRESSER

1.) $\underline{E}_r = \underline{X} - \underline{X}_r = \sum_{i=1}^{n} \sigma_i \underline{u}_i \underline{v}_i^T - \sum_{i=1}^{r} \sigma_i \underline{u}_i \underline{v}_i^T$

$$\underline{E}_r = \sum_{r+1}^{n} \sigma_i \underline{u}_i \underline{v}_i^T$$



b.) If $\underline{X}$ is full rank (n), $\underline{E}_r$ is rank $n-r$

c.)

Operator Norm ("2 norm") of a Matrix:

largest singular value.

$$\|\underline{A}\|_2 = \|A\|_{op} := \max_{\underline{x} \neq 0} \frac{\|\underline{A}\underline{x}\|_2}{\|\underline{x}\|_2} = \sigma_1$$

$$\rightarrow \|\underline{E}_r\|_{op} = \sigma_{r+1}$$

d.) $\underline{X}_r$ will be a good approximation to $\underline{X}$ when the singular values $\sigma_{r+1}, \dots \sigma_n$ are small relative to $\sigma_1, \dots, \sigma_r$

2.) a.) The approximate rank of A is 400 as that is the quantity of nonnegative σ's.

It is useful to plot the log of the σ's to determine which ones contribute the most variance to the image & determine a good rank for an approximation.

b.)

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In [ ]:
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In [17]: # Find and display low-rank approximations

r_vals = np.array([10, 20, 50, 100 ])
err_fro = np.zeros(len(r_vals))

# display images of various rank approximations
for i, r in enumerate(r_vals):

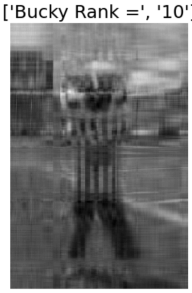
    # Complete and uncomment two lines below
    Ar = np.sum([s[i] * np.outer(U[:, i], VT[i, :]) for i in range(r)], axis=0)
    Er = A - Ar
    err_fro[i] = np.linalg.norm(Er,ord='fro')

    fig = plt.figure()
    ax = fig.add_subplot(111)
    ax.imshow(Ar,cmap='gray',interpolation='none')
    ax.set_axis_off()
    ax.set_title(['Bucky Rank =', str(r_vals[i])], fontsize=18)
    plt.show()

# plot normalized error versus rank
norm_err = err_fro/np.linalg.norm(A,ord='fro')

fig = plt.figure()
ax = fig.add_subplot(111)
ax.stem(r_vals,norm_err)
ax.set_xlabel('Rank', fontsize=16)
ax.set_ylabel('Normalized error', fontsize=16)
plt.show()
```

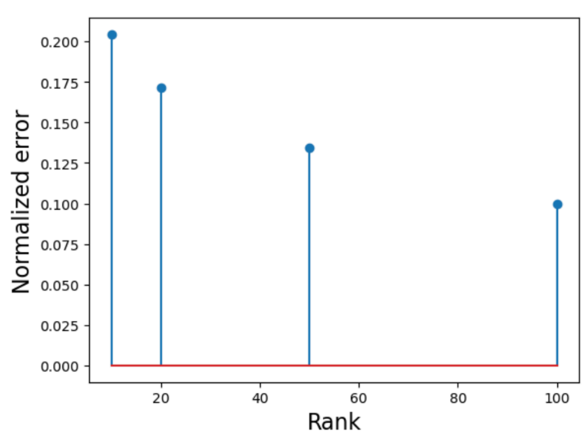
['Bucky Rank =', '10']



['Bucky Rank =', '50']

['Bucky Rank =', '20']

['Bucky Rank =', '100']



Comment: By rank 100, the image visually has the same effect as the original. As rank increases, image fidelity increases.

c.) $\underline{A}$: $600 \times 400 = 240,000$

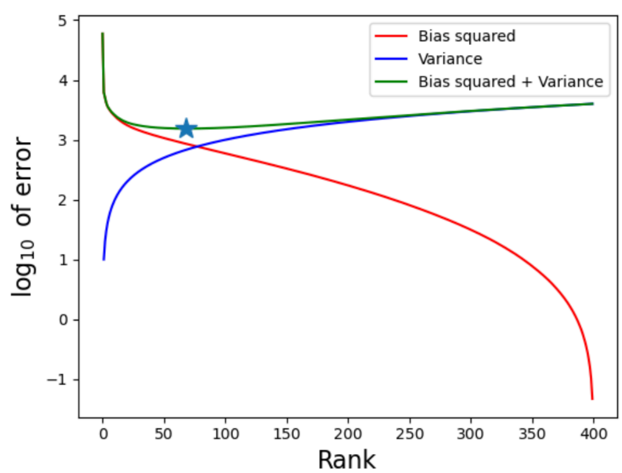
In general, for rank r approx. :

$\underline{U}_r$ is $600 \times r$, $\underline{V}_r^T$ is $r \times 400$, δ_r is 1 scalar

$\rightarrow$ $1001 \cdot r$ numbers to store.

$\rightarrow$ $\{10, 20, 50, 100\}$ r $\rightarrow$ $\{10{,}010, 20{,}020, 50{,}050, 100{,}100\}$ numbers.

d.) $\delta_g^2 = 10$:



$\delta_g^2 = 50$: