CS/ECE/ME532 Activity 7

Estimated Time: 15 min for each problem

1. Let $\boldsymbol{X} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

a) Use the Gram-Schmidt orthogonalization procedure and hand calculation to find an orthonormal basis for the space spanned by the columns of $\boldsymbol{X}$. What geometric object is described by the span of these bases?

b) Now interchange the columns of $\boldsymbol{X}$, that is, define $\tilde{\boldsymbol{X}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

i. Do the columns of $\boldsymbol{X}$ span the same space as the columns of $\tilde{\boldsymbol{X}}$?

ii. Use the Gram-Schmidt orthogonalization procedure to find an orthonormal basis for the space spanned by the columns of $\tilde{\boldsymbol{X}}$. How does the geometric object described by the span of this set of orthonormal bases compare to the one in Part a?

iii. Are the bases vectors you found for $\boldsymbol{X}$ and $\tilde{\boldsymbol{X}}$ the same? Does the space spanned by the columns of a matrix depend on the order of the columns?

2. Let $\boldsymbol{X} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ as in the previous problem.

a) Place the orthonormal bases you found as columns of a matrix $\boldsymbol{U}$.

b) Find $\boldsymbol{U}^T\boldsymbol{U}$.

c) Since $\boldsymbol{U}$ contains a basis for space spanned by the columns of $\boldsymbol{X}$ you decide to write each column of $\boldsymbol{X}$ as a linear combination of the columns of $\boldsymbol{U}$: $\boldsymbol{X} = \boldsymbol{U}\begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{bmatrix}$. What is the dimension of $\boldsymbol{a}_1$? Briefly describe the meaning of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$.

d) Let $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{bmatrix}$ so that $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{A}$. Multiply both sides of this equation by $\boldsymbol{U}^T$ and solve for $\boldsymbol{A}$.

3. Let the columns of an n-by-p ($n > p$) matrix $\boldsymbol{X}$ be linearly independent and $\boldsymbol{U}$ be an orthonormal basis for the p-dimensional space spanned by the columns of $\boldsymbol{X}$.

a) It can be shown that $\boldsymbol{X} = \boldsymbol{UT}$ where $\boldsymbol{T}$ is a p-by-p invertible matrix. Briefly explain why $\boldsymbol{T}$ should be invertible without resorting to a mathematical proof. That is, explain why this result is intuitively reasonable.

b) Use the result in the previous item to show that the projection onto the space spanned by $\boldsymbol{X}$ is identical to that onto the space spanned by $\boldsymbol{U}$. That is, show $\boldsymbol{P}_x = \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T = \boldsymbol{P}_U = \boldsymbol{U}(\boldsymbol{U}^T\boldsymbol{U})^{-1}\boldsymbol{U}^T$. *Hint:* Recall that $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.

c) Express $\boldsymbol{P}_U$ without a matrix inverse.

4. Consider the matrix and vector

$$\boldsymbol{X} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Note that $\boldsymbol{X}$ is defined identically in the preceding problems.

a) Make a sketch of the orthonormal bases $\boldsymbol{U}$ and the columns of $\boldsymbol{X}$ in three dimensions.

b) Use $\boldsymbol{U}$ and the result of the previous problem to compute the LS estimate $\hat{\boldsymbol{b}} = \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{b}$.

5. Let $\boldsymbol{z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and define $\boldsymbol{Q} = \boldsymbol{z}\boldsymbol{z}^T$.

a) Sketch the surface $y = \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}$ where $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. If you find 3-D sketching too difficult, you may draw a contour map with labeled contours.

b) Let $\boldsymbol{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Sketch the subspace spanned by $\boldsymbol{z}$ and the subspace spanned by $\boldsymbol{w}$ on your sketch of the surface $y = \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}$.

c) Does the problem $\min_{\boldsymbol{x}} \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}$ have a unique solution?

d) Is $\boldsymbol{Q} \succ 0$? Is $\boldsymbol{Q} \succeq 0$?

$P_A \underline{d} \Rightarrow$ projection of $\underline{d}$ onto subspace spanned by $\underline{A}$

$A\underline{w}_0 - \underline{d} \Rightarrow$ error

$\underline{P}_{A\perp} \Rightarrow$ every column is $\perp$ to $\underline{A}$ matrix.

$\underline{P}_A$ is symmetric

$\underline{P}_A \underline{A} = \underline{A}$ $\quad \underline{P}_A^T = \underline{P}_A$

$\underline{P}_{A\perp} \underline{A} = 0$ $\quad \underline{P}_{A\perp}^T = \underline{P}_{A\perp}$

ECE 532 Activity 7 - DEVIN BRESSER

1. Let $\boldsymbol{X} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

a) Use the Gram-Schmidt orthogonalization procedure and hand calculation to find an orthonormal basis for the space spanned by the columns of $\boldsymbol{X}$. What geometric object is described by the span of these bases?

b) Now interchange the columns of $\boldsymbol{X}$, that is, define $\tilde{\boldsymbol{X}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

i. Do the columns of $\boldsymbol{X}$ span the same space as the columns of $\tilde{\boldsymbol{X}}$?

ii. Use the Gram-Schmidt orthogonalization procedure to find an orthonormal basis for the space spanned by the columns of $\tilde{\boldsymbol{X}}$. How does the geometric object described by the span of this set of orthonormal bases compare to the one in Part a?

iii. Are the bases vectors you found for $\boldsymbol{X}$ and $\tilde{\boldsymbol{X}}$ the same? Does the space spanned by the columns of a matrix depend on the order of the columns?

a.) $$\underline{X} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{u}_1 = \underline{a}_1 / \|\underline{a}_1\|_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix}$$

$$c_2 = (\underline{I} - \underline{u}_1 \underline{u}_1^T)\, \underline{a}_2$$

$$\underline{u}_1 \underline{u}_1^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{c}_2 = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \underline{a}_2 = \underset{3\times3}{\begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}$$

$$\underline{c}_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} \rightarrow \underline{U}_2 = \frac{1}{\sqrt{3/2}} \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{6}/6 \\ -\sqrt{6}/6 \\ \sqrt{6}/3 \end{bmatrix}$$

a.) $\underline{U} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{6}/6 \\ \sqrt{2}/2 & -\sqrt{6}/6 \\ 0 & \sqrt{6}/3 \end{bmatrix}$, 2D-plane in $\mathbb{R}^3$

b) Now interchange the columns of $\boldsymbol{X}$, that is, define $\tilde{\boldsymbol{X}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

i. Do the columns of $\boldsymbol{X}$ span the same space as the columns of $\tilde{\boldsymbol{X}}$?

ii. Use the Gram-Schmidt orthogonalization procedure to find an orthonormal basis for the space spanned by the columns of $\tilde{\boldsymbol{X}}$. How does the geometric object described by the span of this set of orthonormal bases compare to the one in Part a?

iii. Are the bases vectors you found for $\boldsymbol{X}$ and $\tilde{\boldsymbol{X}}$ the same? Does the space spanned by the columns of a matrix depend on the order of the columns?

b.) i.) Yes, $\text{span}(\underline{X}) = \text{span}(\tilde{\underline{X}})$

$\text{span}(\underline{X}) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$ for all a, b, $\text{span}(\tilde{\underline{X}}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$ for all a, b

ii) $\underline{u}_1 = \underline{a}_1 / \|\underline{a}_1\|_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}$

$c_2 = (\underline{I} - \underline{u}_1 \underline{u}_1^T)\, \underline{a}_2$

$\underline{u}_1 \underline{u}_1^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$

$\underline{c}_2 = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} \right) \underline{a}_2 = \underset{3\times3}{\begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & -1 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}$

$\underline{c}_2 = \begin{bmatrix} 1/2 \\ -1 \\ -1/2 \end{bmatrix} \rightarrow \underline{u}_2 = \frac{1}{\sqrt{3/2}}\begin{bmatrix} 1/2 \\ -1 \\ -1/2 \end{bmatrix} = \begin{bmatrix} \sqrt{6}/6 \\ -\sqrt{6}/3 \\ -\sqrt{6}/6 \end{bmatrix}$

b.) $\underline{U} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{6}/6 \\ 0 & -\sqrt{6}/3 \\ \sqrt{2}/2 & -\sqrt{6}/6 \end{bmatrix}$ 2D plane in $\mathbb{R}^3$

ii) Same geometric object (2D plane in $\mathbb{R}^3$)

iii) Basis vectors are <u>not</u> the same

space spanned <u>is</u> the same.

Order of columns does not matter for column span.

2. Let $\boldsymbol{X} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ as in the previous problem.

a) Place the orthonormal bases you found as columns of a matrix $\boldsymbol{U}$.

b) Find $\boldsymbol{U}^T\boldsymbol{U}$.

c) Since $\boldsymbol{U}$ contains a basis for space spanned by the columns of $\boldsymbol{X}$ you decide to write each column of $\boldsymbol{X}$ as a linear combination of the columns of $\boldsymbol{U}$: $\boldsymbol{X} = \boldsymbol{U}\begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{bmatrix}$. What is the dimension of $\boldsymbol{a}_1$? Briefly describe the meaning of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$.

d) Let $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{bmatrix}$ so that $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{A}$. Multiply both sides of this equation by $\boldsymbol{U}^T$ and solve for $\boldsymbol{A}$.

a.) $$\underline{U} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{6}/6 \\ \sqrt{2}/2 & -\sqrt{6}/6 \\ 0 & \sqrt{6}/3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3\sqrt{2} & \sqrt{6} \\ 3\sqrt{2} & -\sqrt{6} \\ 0 & 2\sqrt{6} \end{bmatrix}$$

b.) $$\underline{U}^T\underline{U} = \frac{1}{36}\underset{2\times 3}{\begin{bmatrix} 3\sqrt{2} & 3\sqrt{2} & 0 \\ \sqrt{6} & -\sqrt{6} & 2\sqrt{6} \end{bmatrix}}\underset{3\times 2}{\begin{bmatrix} 3\sqrt{2} & \sqrt{6} \\ 3\sqrt{2} & -\sqrt{6} \\ 0 & 2\sqrt{6} \end{bmatrix}}$$

$$= \frac{1}{36}\begin{bmatrix} 18+18+0 & 0 \\ 0 & 6+6+24 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

c) Since $\boldsymbol{U}$ contains a basis for space spanned by the columns of $\boldsymbol{X}$ you decide to write each column of $\boldsymbol{X}$ as a linear combination of the columns of $\boldsymbol{U}$: $\boldsymbol{X} = \boldsymbol{U}\left[\begin{array}{cc}\boldsymbol{a}_1 & \boldsymbol{a}_2\end{array}\right]$. What is the dimension of $\boldsymbol{a}_1$? Briefly describe the meaning of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$.

d) Let $\boldsymbol{A} = \left[\begin{array}{cc}\boldsymbol{a}_1 & \boldsymbol{a}_2\end{array}\right]$ so that $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{A}$. Multiply both sides of this equation by $\boldsymbol{U}^T$ and solve for $\boldsymbol{A}$.

c.) $$\underset{3\times 2}{\underline{X}} = \underset{3\times 2}{\underline{U}} \underset{2\times 2}{[\underline{a_1}\ \underline{a_2}]}$$

$$\overset{\underline{X}}{\begin{bmatrix}\cdot&\cdot\\ \cdot&\cdot\\ \cdot&\cdot\end{bmatrix}} = \overset{\underline{U}}{\begin{bmatrix}\cdot&\cdot\\ \cdot&\cdot\\ \cdot&\cdot\end{bmatrix}} \overset{\underline{A}}{\begin{bmatrix}\cdot&\cdot\\ \cdot&\cdot\end{bmatrix}}$$

$\rightarrow$ $\underline{a_1}$, $\underline{a_2}$ are 2-dimensional.

The matrix $\underline{A}$ represents the linear transformation required to transform $\underline{U}$ into $\underline{X}$

d.) $$\underline{X}_{3\times 2} = \underline{U}_{3\times 2}\, \underline{A}_{2\times 2}$$

$$\rightarrow \underline{U}^T_{2\times 3}\, \underline{X}_{3\times 2} = \underline{U}^T_{2\times 3}\, \underline{U}_{3\times 2}\, \underline{A}_{2\times 2}$$

$$\rightarrow \underline{U}^T\underline{X}_{2\times 2} = \underline{I}_{2\times 2}\, \underline{A}_{2\times 2}$$

$$\rightarrow \underline{U}^T\underline{X}_{2\times 2} = \underline{A}_{2\times 2}\ .$$

3. Let the columns of an n-by-p ($n > p$) matrix $\boldsymbol{X}$ be linearly independent and $\boldsymbol{U}$ be an orthonormal basis for the p-dimensional space spanned by the columns of $\boldsymbol{X}$.

a) It can be shown that $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{T}$ where $\boldsymbol{T}$ is a p-by-p invertible matrix. Briefly explain why $\boldsymbol{T}$ should be invertible without resorting to a mathematical proof. That is, explain why this result is intuitively reasonable.

b) Use the result in the previous item to show that the projection onto the space spanned by $\boldsymbol{X}$ is identical to that onto the space spanned by $\boldsymbol{U}$. That is, show $\boldsymbol{P}_x = \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T = \boldsymbol{P}_U = \boldsymbol{U}(\boldsymbol{U}^T\boldsymbol{U})^{-1}\boldsymbol{U}^T$. *Hint:* Recall that $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.

c) Express $\boldsymbol{P}_U$ without a matrix inverse.

a.) $\underline{X}_{n \times p}$ rank p

$\underline{U}$ is orthonormal basis for p-dim space spanned by cols ($\underline{X}$

$$\underline{X}_{n\times p} = \underline{U}_{n \times p}\, \underline{T}_{p \times p}$$

→ $\underline{T}$ must be invertible because it is the linear transformation that converts the columns of $\underline{U}$ into the columns of $\underline{X}$. $\underline{U}$ and $\underline{X}$ have the same span, so $\underline{T}$ cannot have a determinant of 0.

If $\det(\underline{T}) = 0$, it would shrink $\underline{U}$ into a lower dimension and the span would not be equal to span $(\underline{X})$.

b) Use the result in the previous item to show that the projection onto the space spanned by $\boldsymbol{X}$ is identical to that onto the space spanned by $\boldsymbol{U}$. That is, show $\boldsymbol{P}_x = \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T = \boldsymbol{P}_U = \boldsymbol{U}(\boldsymbol{U}^T\boldsymbol{U})^{-1}\boldsymbol{U}^T$. *Hint:* Recall that $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.

c) Express $\boldsymbol{P}_U$ without a matrix inverse.

b.) $\underline{X} = \underline{U}\,\underline{T}$

simplifies to $\underline{I}$ because $\underline{U}$ is orthonormal basis

$$\underbrace{\underline{X}(\underline{X}^T\underline{X})^{-1}\underline{X}^T}_{} = \underline{U}\overbrace{(\underline{U}^T\underline{U})}^{}{}^{-1}\underline{U}^T = \underline{U}\,\underline{U}^T$$

$$(\underline{U}\,\underline{T})(\underline{T}^T\cancelto{I}{\underline{U}^T\underline{U}}\,\underline{T})^{-1}(\underline{U}\,\underline{T})^T$$

$$= (\underline{U}\,\underline{T})(\underline{T}^T\underline{T})^{-1}(\underline{T}^T\underline{U}^T)$$

$$= \underline{U}\,\cancelto{I}{\underline{T}\,\underline{T}^{-1}}\,\cancelto{I}{\underline{T}^{T^{-1}}\underline{T}^T}\,\underline{U}^T$$

$$= \underline{U}\,\underline{U}^T$$

So $P_X = P_U$

c.) $P_U = \underline{U}(\cancelto{I}{\underline{U}^T\underline{U}})^{-1}\underline{U}^T = \underline{U}\,\underline{U}^T$

4. Consider the matrix and vector

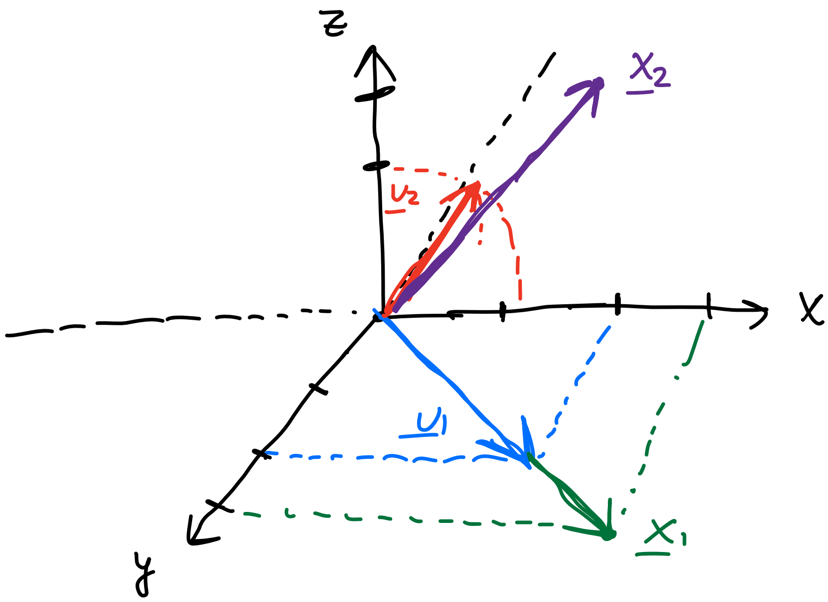
$$\boldsymbol{X} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Note that $\boldsymbol{X}$ is defined identically in the preceding problems.

a) Make a sketch of the orthonormal bases $\boldsymbol{U}$ and the columns of $\boldsymbol{X}$ in three dimensions.

b) Use $\boldsymbol{U}$ and the result of the previous problem to compute the LS estimate $\hat{\boldsymbol{b}} = \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{b}$.

a.) from previous, $\underline{U} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{6}/6 \\ \sqrt{2}/2 & -\sqrt{6}/6 \\ 0 & \sqrt{6}/3 \end{bmatrix}$



b.) $\text{proj}_{\underline{x}}\, \hat{\underline{b}} = \text{proj}_{\underline{U}}\, \hat{b} = \underline{U}\,\underline{U}^T\, \hat{\underline{b}}$

$$= \begin{bmatrix} 2/3 & 1/3 & 1/3 \\ 1/3 & 2/3 & -1/3 \\ 1/3 & -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 4/3 \\ 1/3 \end{bmatrix}$$

5. Let $\boldsymbol{z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and define $\boldsymbol{Q} = \boldsymbol{z}\boldsymbol{z}^T$.

a) Sketch the surface $y = \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x}$ where $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. If you find 3-D sketching too difficult, you may draw a contour map with labeled contours.

b) Let $\boldsymbol{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Sketch the subspace spanned by $\boldsymbol{z}$ and the subspace spanned by $\boldsymbol{w}$ on your sketch of the surface $y = \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x}$.

c) Does the problem $\min_{\boldsymbol{x}} \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x}$ have a unique solution?

d) Is $\boldsymbol{Q} \succ 0$? Is $\boldsymbol{Q} \succeq 0$?

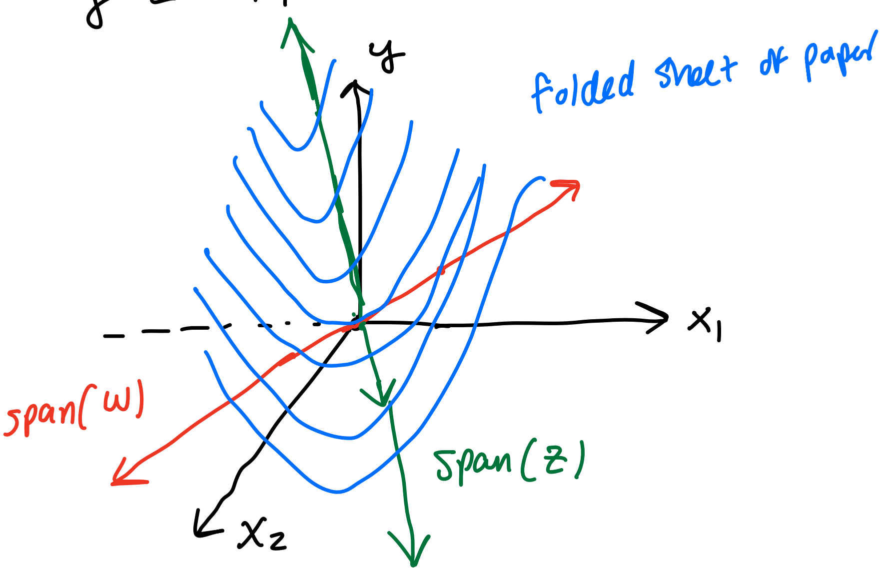
a.) $$\underline{Q} = \underline{z}\,\underline{z}^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$y = \underline{x}^T \underline{Q}\, \underline{x} = \underbrace{\begin{bmatrix} x_1 & x_2 \end{bmatrix}}_{1\times 2} \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{2\times 1}$$

$$= \underbrace{\begin{bmatrix} x_1 + x_2 & x_1 + x_2 \end{bmatrix}}_{1\times 2} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{2\times 1}$$

$$y = x_1^2 + x_2^2 + 2 x_1 x_2$$

b.)



c.) $\min_{x} \underline{x}^T \underline{Q} \underline{x}$ has solution $x_1 = -x_2$, ∞ solns.

d.) Eigenvalues $\lambda_1 = 0$, $\lambda_2 = 2$

positive semidefinite.