3) *Non Unique Solutions.*

a) Consider $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{A} = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ -2 & 4 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix}$ and $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

i) Does this system of equations have a solution? Justify your answer.

ii) Is the solution unique? Justify your answer.

iii) Draw the solution(s) in the x_1-x_2 plane using x_1 as the horizontal axis.

b) If the system of linear equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has more than one solution, then there is at least one non zero vector $\boldsymbol{w}$ for which $\boldsymbol{x} + \boldsymbol{w}$ is also a solution. That is, $\boldsymbol{A}(\boldsymbol{x} + \boldsymbol{w}) = \boldsymbol{b}$. Use the definition of linear independence to find a condition on $\text{rank}\{\boldsymbol{A}\}$ that determines whether there is more than one solution.

532 Activity 4 DEVIN BRESSER

1) *Matrix Rank.* Let $\boldsymbol{X} = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

a) What is the rank of $\boldsymbol{X}$?

b) Find a set of linearly independent columns in $\boldsymbol{X}$. Is there more than one set? How many sets of linearly independent columns can you find?

c) A matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & a \\ 1 & 1 & b \\ 0 & 1 & -1 \end{bmatrix}$. Find the relationship between b and a so that rank$\{\boldsymbol{A}\} = 2$. *Hint:* find a, b so that the third column is a weighted sum of the first two columns. Note that there are many choices for a, b that result in rank 2.

a.) $\text{rank}(\underline{X}) = 2$

b.)

$\underline{x}_1$ and $\underline{x}_3$ $\quad$ $\underline{x}_3$ and $\underline{x}_4$

$\underline{x}_1$ and $\underline{x}_4$

$\underline{x}_1$ and $\underline{x}_5$ $\quad$ 5 sets

$\underline{x}_4$ and $\underline{x}_5$

c.) $\underline{A} = \begin{bmatrix} 1 & 0 & a \\ 1 & 1 & b \\ 0 & 1 & -1 \end{bmatrix}$

$$\underbrace{2a_1 - a_2}_{\text{lin. comb}} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \begin{matrix} a \\ b \\ \\ \end{matrix}$$

$$3a_1 - a_2 = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \begin{matrix} a \\ b \\ \\ \end{matrix}$$

$\text{rank}(\underline{A}) = 2 \rightarrow$ $a = b + 1$

2) *Solution Existence.* A system of linear equations is given by $\boldsymbol{Ax} = \boldsymbol{b}$ where $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$.

a) Suppose $\boldsymbol{b} = \begin{bmatrix} 8 \\ 6 \\ -2 \end{bmatrix}$. Does a solution for $\boldsymbol{x}$ exist? If so, find $\boldsymbol{x}$.

b) Suppose $\boldsymbol{b} = \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix}$. Does a solution for $\boldsymbol{x}$ exist? If so, find $\boldsymbol{x}$.

c) Consider the general system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$. This equation says that $\boldsymbol{b}$ is a weighted sum of the columns of $\boldsymbol{A}$. Assume $\boldsymbol{A}$ is full rank. Use the definition of linear independence to find the condition on rank$\{[\ \boldsymbol{A}\ \ \boldsymbol{b}\ ]\}$ that guarantees a solution exists.

a.) $\underline{A}\,\underline{x} = \underline{b}$, $\underline{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$, rank $(\underline{A}) = 2$

3x2

$\underline{b} = \begin{bmatrix} 8 \\ 6 \\ -2 \end{bmatrix}$, $[\underline{A} : \underline{b}] = \begin{bmatrix} 1 & 0 & 8 \\ 1 & 1 & 6 \\ 0 & 1 & -2 \end{bmatrix}$

$[\underline{A} : \underline{b}]_3 = 8[\underline{A} : \underline{b}]_1 - 2[\underline{A} : \underline{b}]_2$

$\Rightarrow$ $[\underline{A} : \underline{b}]_3$ is a linear combination of col. 1&2

$\Rightarrow$ Does not increase rank.

rank $[\underline{A} : \underline{b}] = 2$

$\underline{x}$, unique solution

$\Rightarrow$ Solution $\underline{x}$ exists.

$\Rightarrow$ $\underline{A}\,\underline{x} = \underline{b}$ $\Rightarrow$ $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ -2 \end{bmatrix}$

b) Suppose $\boldsymbol{b} = \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix}$. Does a solution for $\boldsymbol{x}$ exist? If so, find $\boldsymbol{x}$.

c) Consider the general system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$. This equation says that $\boldsymbol{b}$ is a weighted sum of the columns of $\boldsymbol{A}$. Assume $\boldsymbol{A}$ is full rank. Use the definition of linear independence to find the condition on rank$\{[\ \boldsymbol{A}\ \ \boldsymbol{b}\ ]\}$ that guarantees a solution exists.

b.)

$$\underline{b} = \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix}, \quad [\underline{A} : \underline{b}] = \begin{bmatrix} 1 & 0 & 4 \\ 1 & 1 & 6 \\ 0 & 1 & 1 \end{bmatrix}$$

$\Rightarrow$ No linear combination of columns makes $\begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix}$

$\Rightarrow$ rank$([\underline{A} : \underline{b}]) = 3 <$ rank$(\underline{A})$

$\Rightarrow$ No solution

c.) $\underline{A}\ \underline{x} = \underline{b}$

rank$(\underline{A})$ = rank$(\underline{A} : \underline{b})$ $\Rightarrow$ Solution exists

$$\sum_{i=1}^{m} (\underline{A} : \underline{b})_i\, a_i = 0 \iff a_i = 0.$$

linear independence condition for columns of $(A : b)$

In words, this means that $\underline{b}$ must be within the span of $\underline{A}$. or, $\underline{b}$ cannot require $\underline{A}$ to span an additional dimension.

3) *Non Unique Solutions.*

a) Consider $\boldsymbol{Ax} = \boldsymbol{b}$ where $\boldsymbol{A} = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ -2 & 4 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix}$ and $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

i) Does this system of equations have a solution? Justify your answer.

ii) Is the solution unique? Justify your answer.

iii) Draw the solution(s) in the x_1-x_2 plane using x_1 as the horizontal axis.

b) If the system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ has more than one solution, then there is at least one non zero vector $\boldsymbol{w}$ for which $\boldsymbol{x} + \boldsymbol{w}$ is also a solution. That is, $\boldsymbol{A}(\boldsymbol{x} + \boldsymbol{w}) = \boldsymbol{b}$. Use the definition of linear independence to find a condition on rank$\{\boldsymbol{A}\}$ that determines whether there is more than one solution.

a.) i.) $\text{rank}(\underline{A}) = 1$

$\text{rank}(\underline{A} : \underline{b}) = \text{rank}\begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \\ -2 & 4 & -4 \end{bmatrix} = 1$

→ Solution exists because $\text{rank}(\underline{A} : \underline{b}) = \text{rank}(\underline{A})$

ii) $\underset{1}{\text{rank}(\underline{A})} < \underset{2}{\dim(\underline{x})}$

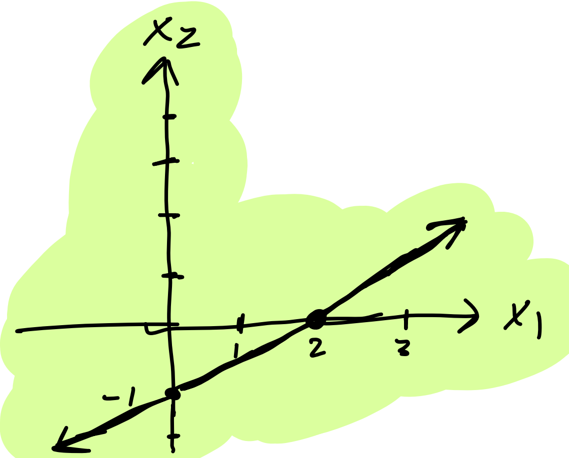
→ Solution is not unique.

iii) $\underline{A}\,\underline{x} = \underline{b}$

$a_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2$

${}_{1\times 2}\begin{bmatrix} 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1} = 1x_1 - 2x_2 = 2 \Rightarrow x_2 = \dfrac{2 - x_1}{-2}$

$\Rightarrow x_2 = \frac{1}{2}x_1 - 1$



b) If the system of linear equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has more than one solution, then there is at least one non zero vector $\boldsymbol{w}$ for which $\boldsymbol{x} + \boldsymbol{w}$ is also a solution. That is, $\boldsymbol{A}(\boldsymbol{x} + \boldsymbol{w}) = \boldsymbol{b}$. Use the definition of linear independence to find a condition on $\text{rank}\{\boldsymbol{A}\}$ that determines whether there is more than one solution.

If $\underline{A}\,\underline{x} = \underline{b}$ has a solution:

$$\text{rank}(\underline{A}) = \text{rank}(\underline{A} \,;\, \underline{b})$$

∞ solutions: $\text{rank}(\underline{A}) < \dim(\underline{x})$

$$\underline{\tilde{x}} = \underline{x} + \underline{w}$$

$$\underline{A}\,\underline{\tilde{x}} = \underline{b}$$

$$\Rightarrow \underline{A}\,\underline{x} + \underline{A}\,\underline{w} = \underline{b}$$

$$\Rightarrow (\underline{A}\,\underline{x} - \underline{b}) + \underline{A}\,\underline{w} = 0$$

$$\Rightarrow \underline{A}\,\underline{w} = 0$$

$$\Rightarrow \sum_{i=1}^{n} \underline{a}_i w_i = 0 \quad \text{for } \underline{w} \neq 0$$

Intuitively, if $\text{rank}(\underline{A}) < \dim(\underline{x})$, it means that there are more variables in $\underline{x}$ than linearly independent equations in $\underline{A}$

AKA, "more unknowns than equations". Thus, ∞ many solutions.

AKA. underdetermined system

Scratch

Spans a 1d line that $\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$ happens to belong to.

rank 1 spans 1d

$$A = \underbrace{\begin{bmatrix} 1 & -2 \\ -1 & 2 \\ -2 & 4 \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{2\times 1} = \underbrace{\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}}_{3\times 1}$$

$$A = \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 2 \\ -2 & 3 \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{2\times 1} = \underbrace{\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}}_{3\times 1}$$

linear indep. columns

rank 2 spans 2d

Spans a particular 2d plane that $\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$ does not belong to.

$\underline{A}\,\underline{x} = \underline{b}$ Scratch

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$\text{rank}(\underline{A}) = 2$

$\underline{A}$ spans all of 2d-space

$\text{rank}(\underline{A} \,;\, \underline{b}) = \text{rank}\left(\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \end{bmatrix}\right) = 2$

A solution exists.

$\left.\begin{array}{l} \text{rank}(\underline{A}) = 2 \\ \dim(\underline{x}) = 2 \end{array}\right\} \Rightarrow$ 2 independent equations, 2 variables

$\Rightarrow$ determined system, exactly 1 solution

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$\text{rank}(\underline{A}) = 1$

$\underline{A}$ spans only the vectors along line $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\text{rank}(\underline{A} : \underline{b}) = \text{rank}\left(\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 5 \end{bmatrix}\right) = 2$

No solution exists. $\underline{b}$ requires $\underline{A}$ to span an additional dimension.

a) Consider $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{A} = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ -2 & 4 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix}$ and $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Scratch

i) Does this system of equations have a solution? Justify your answer.
ii) Is the solution unique? Justify your answer.
iii) Draw the solution(s) in the x_1-x_2 plane using x_1 as the horizontal axis.

$$\begin{bmatrix} 1 & -2 \\ -1 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix}$$

$\text{rank}(\underline{A}) = 1$. only spans the vectors a scalar of $\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

$\text{rank}(\underline{A} \vdots \underline{b}) = \text{rank}\left(\begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \\ -2 & 4 & -4 \end{bmatrix}\right) = 1$

only spans the same 1d space. $\underline{b}$ happens to be a scalar along that line.

$\text{rank}(\underline{A}) = 1$, $\dim(\underline{x}) = 2$

→ 1 linearly independent equation, 2 variables

→ underdetermined system w/ ∞ many solutions.