ECE 532 Activity 5 DEVIN BRESSER

1. Let $\boldsymbol{z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

a) Sketch the subspace spanned by $\boldsymbol{z}$ in $\mathbb{R}^2$.

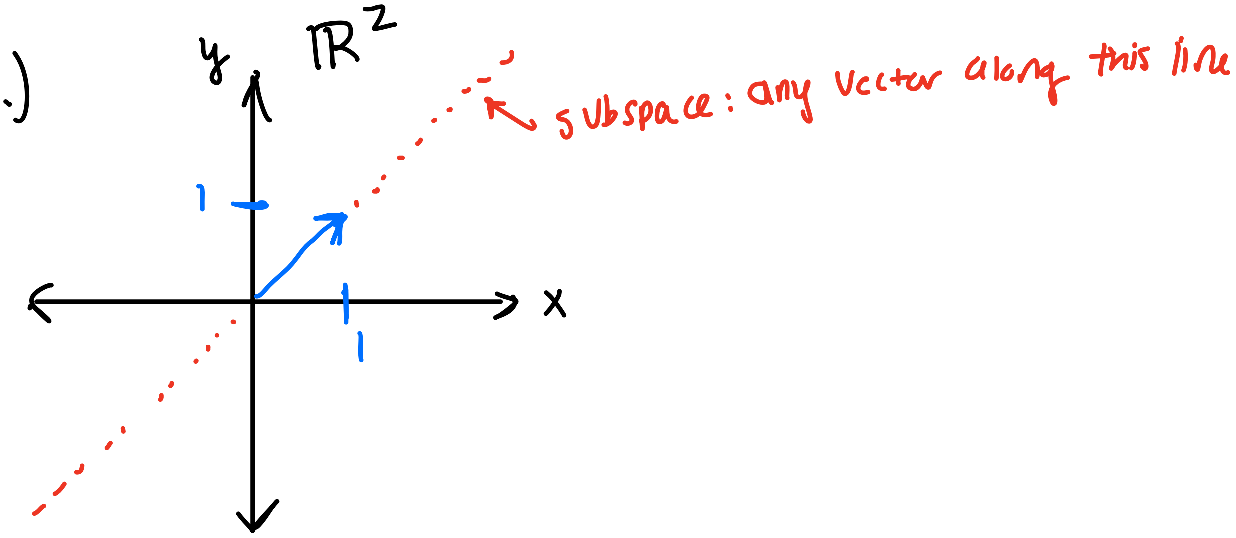
b) Sketch the subspace spanned by $\boldsymbol{w}$ in $\mathbb{R}^2$.

c) Sketch span $\{\boldsymbol{z}, \boldsymbol{w}\}$ in $\mathbb{R}^2$.

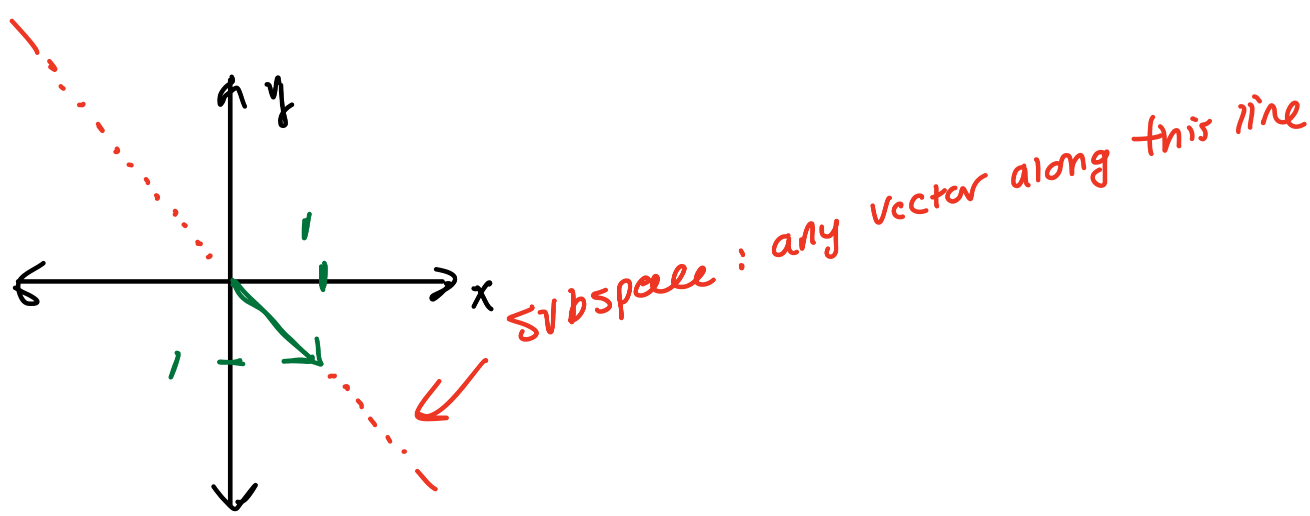
d) Are $\boldsymbol{z}$ and $\boldsymbol{w}$ orthogonal? Why or why not?

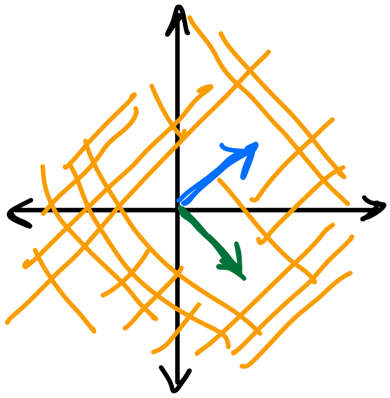
e) Do $\{\boldsymbol{z}, \boldsymbol{w}\}$ form an orthonormal basis? Why or why not? If not, can you modify $\boldsymbol{z}$ and $\boldsymbol{w}$ to form an orthonormal basis?

a.)



b.)



c.)  $\text{span}\{\underline{z}, \underline{w}\} = \mathbb{R}^2$

d.) orthogonal if $\underline{z}^T \underline{w} = 0$

$\rightarrow \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 + -1 = 0 \checkmark$

$\rightarrow$ $\underline{z}$ and $\underline{w}$ are orthogonal.

e.) $\{\underline{z}, \underline{w}\}$ do not form an orthonormal basis.

because $\|\underline{z}\| = \sqrt{2} \neq 1$, $\|w\| = \sqrt{2} \neq 1.$

If $\underline{z}^* = \frac{1}{\|z\|} \underline{z}$ and $\underline{w}^* = \frac{1}{\|w\|} \underline{w}$

$\underline{z}^* = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\underline{w}^* = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

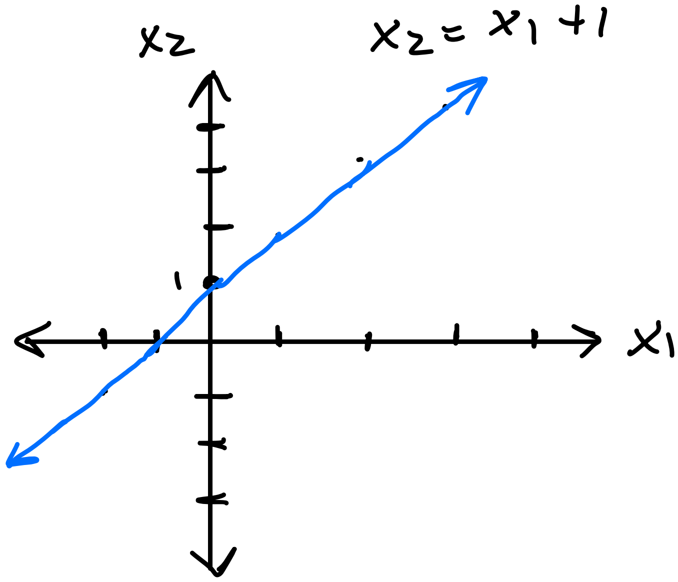
Then $\underline{z}^*$ and $\underline{w}^*$ form an orthonormal basis.

2. Consider the line in $\mathbb{R}^2$ defined by the equation $x_2 = x_1 + 1$.

a) Sketch the line in $\mathbb{R}^2$.

b) Does this line define a subspace of $\mathbb{R}^2$? Why or why not?

a.)



b.) Does not define a subspace of $\mathbb{R}^2$
Because it does not include $(0,0)$.

3. You collect ratings of three space-related science fiction movies and two romance movies from seven friends on a scale of 1-10.

Movie	Jake	Jennifer	Jada	Theo	Ioan	Bo	Juanita
Star Trek	4	7	2	8	7	4	2
Pride and Prejudice	9	3	5	6	10	5	5
The Martian	4	8	3	7	6	4	1
Sense and Sensibility	9	2	6	5	9	5	4
Star Wars: Empire Strikes	4	9	2	8	7	4	1

You put this data into a matrix $\boldsymbol{X}$ (available in the file `movie.mat`) and decide to model (approximate) as the product of a rank-r taste matrix with orthonormal columns and a weight matrix. That is, $\boldsymbol{X} \approx \boldsymbol{T}\boldsymbol{W}$.

a) What is the rank of $\boldsymbol{X}$? Relevant Python commands are `numpy.linalg.matrix_rank()`.

b) What are the dimensions of $\boldsymbol{T}$ and $\boldsymbol{W}$ (in terms of r)?

3a.) per the python code, rank $(\underline{X}) = 5$

3b.)

$$\underset{5\times 7}{\underline{X}} \approx \underset{5\times r}{\underline{T}}\ \underset{r\times 7}{\underline{W}}$$

r affinity vectors

7 users

$$[5\times 7] \approx [5\times r][r\times 7]$$

r taste vectors

5 movies

c) You know that each user's ratings have an average value that is greater than zero because the scale is 1-10. And you suspect the baseline (average) rating may differ from user to user. To account for this you decide your first basis vector in the taste matrix should be

$$\boldsymbol{t}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}$$

Choose w_{1j} so that each element of the vector $\boldsymbol{t}_1 w_{1j}$ equals the average value j^{th} column of $\boldsymbol{X}$, denoted as $\boldsymbol{X}_{:,j}$. Find an expression for w_{1j} that depends on $\boldsymbol{t}_1$ and $\boldsymbol{X}_{:,j}$.

d) Define $\boldsymbol{w}_1^T = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{17}\end{bmatrix}$ and find the rank-1 approximation to $\boldsymbol{X}$ that reflects the baseline ratings of each friend, $\boldsymbol{t}_1\boldsymbol{w}_1^T$.

e) Which friend has the highest baseline rating? Which friend has the lowest baseline rating?

f) Find the residual not modeled by $\boldsymbol{t}_1\boldsymbol{w}_1^T$, that is, $\boldsymbol{X} - \boldsymbol{t}_1\boldsymbol{w}_1^T$. Do you see any patterns in the residual? Briefly describe them qualitatively.

This problem is continued in a homework assignment.

Movie	Jake	Jennifer	Jada	Theo	Ioan	Bo	Juanita	i
ar Trek	4	7	2	8	7	4	2	1
rejudice	9	3	5	6	10	5	5	2
Martian	4	8	3	7	6	4	1	⋮
nsibility	9	2	6	5	9	5	4	
Strikes	4	9	2	8	7	4	1	
j	1	2	3	.	.	.		

3c.) $\underline{t}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$, $\underline{w}_1^T = \begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} & w_{15} & w_{17}\end{bmatrix}$

mean is $\frac{1}{5}$ (sum down column)

each w_{1j} needs to be the sum down that user's column in $\underline{X}$.

so, $w_{1j} = \frac{1}{\sqrt{5}} \sum_{i=1}^{5} \underline{X}_{i1}$

Movie	Jake	Jennifer	Jada	Theo	Ioan	Bo	Juanita	
ar Trek	4	7	2	8	7	4	2	1
rejudice	9	3	5	6	10	5	5	2
Martian	4	8	3	7	6	4	1	⋮
nsibility	9	2	6	5	9	5	4	
Strikes	4	9	2	8	7	4	1	

d) Define $\boldsymbol{w}_1^T = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{17} \end{bmatrix}$ and find the rank-1 approximation to $\boldsymbol{X}$ that reflects the baseline ratings of each friend, $\boldsymbol{t}_1\boldsymbol{w}_1^T$.

e) Which friend has the highest baseline rating? Which friend has the lowest baseline rating?

f) Find the residual not modeled by $\boldsymbol{t}_1\boldsymbol{w}_1^T$, that is, $\boldsymbol{X} - \boldsymbol{t}_1\boldsymbol{w}_1^T$. Do you see any patterns in the residual? Briefly describe them qualitatively.

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3d.) $\underline{w}_1^T = \frac{1}{\sqrt{5}}\left[30, 29, 18, 34, 39, 22, 13\right]$

$$\underline{\tilde{X}} = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}\frac{1}{\sqrt{5}}\left[30, 29, 18, 34, 39, 22, 13\right]$$

$$= \frac{1}{5}\begin{bmatrix}30 & 29 & 18 & 34 & 39 & 22 & 13\\30 & 29 & 18 & 34 & 39 & 22 & 13\\30 & 29 & 18 & 34 & 39 & 22 & 13\\30 & 29 & 18 & 34 & 39 & 22 & 13\\30 & 29 & 18 & 34 & 39 & 22 & 13\end{bmatrix} = \begin{bmatrix}6 & 5.8 & 3.6 & 6.8 & 7.8 & 4.4 & 2.6\\6 & 5.8 & 3.6 & 6.8 & 7.8 & 4.4 & 2.6\\6 & 5.8 & 3.6 & 6.8 & 7.8 & 4.4 & 2.6\\6 & 5.8 & 3.6 & 6.8 & 7.8 & 4.4 & 2.6\\6 & 5.8 & 3.6 & 6.8 & 7.8 & 4.4 & 2.6\end{bmatrix}$$

3e.) Highest: Ioan 7.8

Lowest: Juanita 2.6

f) Find the residual not modeled by $\boldsymbol{t}_1\boldsymbol{w}_1^T$, that is, $\boldsymbol{X} - \boldsymbol{t}_1\boldsymbol{w}_1^T$. Do you see any patterns in the residual? Briefly describe them qualitatively.

$$\underline{X} - \underline{t}_1 \underline{w}^T =$$

$$\underline{X} - \begin{bmatrix} 6 & 5.8 & 3.6 & 6.8 & 7.8 & 4.4 & 2.6 \\ 6 & 5.8 & 3.6 & 6.8 & 7.8 & 4.4 & 2.6 \\ 6 & 5.8 & 3.6 & 6.8 & 7.8 & 4.4 & 2.6 \\ 6 & 5.8 & 3.6 & 6.8 & 7.8 & 4.4 & 2.6 \\ 6 & 5.8 & 3.6 & 6.8 & 7.8 & 4.4 & 2.6 \end{bmatrix}$$

MatrixForm=

$$= \begin{pmatrix} -2 & 1.2 & -1.6 & 1.2 & -0.8 & -0.4 & -0.6 \\ 3 & -2.8 & 1.4 & -0.8 & 2.2 & 0.6 & 2.4 \\ -2 & 2.2 & -0.6 & 0.2 & -1.8 & -0.4 & -1.6 \\ 3 & -3.8 & 2.4 & -1.8 & 1.2 & 0.6 & 1.4 \\ -2 & 3.2 & -1.6 & 1.2 & -0.8 & -0.4 & -1.6 \end{pmatrix}$$

→ The lower the values of the residual matrix provide a measure of how good the mean is as an approximation.