Activity: PageRank and networks

Eigen decomposition

$$\boldsymbol{B}\boldsymbol{e}_i = \lambda_i \boldsymbol{e}_i$$

eigenvector eigenvalue

$\boldsymbol{B}$ (square) symmetric matrix:

$$\boldsymbol{B} = \boldsymbol{E}\boldsymbol{\Lambda}\boldsymbol{E}^T$$

diagonal

orthonormal rows, cols

Connection between eigenvecs and SVD

$$\boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T\boldsymbol{V}\boldsymbol{\Sigma}^T\boldsymbol{U}^T = \boldsymbol{U}\boldsymbol{\Sigma}^2\boldsymbol{U}^T$$

→ Eigenvectors of $\boldsymbol{A}\boldsymbol{A}^T$ are left singular vectors of $\boldsymbol{A}$

$$\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Sigma}^T\boldsymbol{U}^T\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T = \boldsymbol{V}\boldsymbol{\Sigma}^2\boldsymbol{V}^T$$

→ Eigenvectors of $\boldsymbol{A}^T\boldsymbol{A}$ are right singular vectors of $\boldsymbol{A}$

Eigenvalues: $\lambda_i = \sigma_i^2$

Power iteration (main idea)

$$(\boldsymbol{A}\boldsymbol{A}^T)^k = \boldsymbol{U}\Sigma^2\boldsymbol{U}^T\boldsymbol{U}\Sigma^2\boldsymbol{U}^T \dots \boldsymbol{U}\Sigma^2\boldsymbol{U}^T$$
$$= \boldsymbol{U}\Sigma^{2k}\boldsymbol{U}^T$$

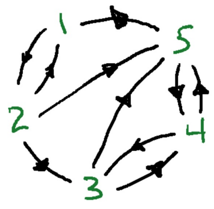
→ imagine that $\sigma_1 = 1$, others less than 1

$$\lim_{k\to\infty} \boldsymbol{\Sigma}^{2k} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 0 \end{bmatrix}$$

$$= \boldsymbol{U}\Sigma^{2k}\boldsymbol{U}^T \to \boldsymbol{u}_1\boldsymbol{u}_1^T$$

Adjacency matrix and PageRank

- Graph: nodes with edges between them
- Adjacency matrix: non-zero entry $\tilde{\boldsymbol{A}}_{ij}$ if edge from j to i
- *Transition probability matrix: normalize columns of* $\tilde{\boldsymbol{A}}$ *to 1*



Graph

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \rightarrow \boldsymbol{Q} = \begin{bmatrix} 0 & 1/3 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1 \\ 1/2 & 1/3 & 1/2 & 1/2 & 0 \end{bmatrix}$$

$\lambda_1 = 1$

Adjacency matrix

Transition probability matrix (Markov Chain)

$\boldsymbol{Q}^k\boldsymbol{b} \rightarrow$ direction of first eigenvector of $\boldsymbol{Q}$.

The first eigenvector is the steady-state probability distribution