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MATH 331: Geometry

Independent Project

Geometric Manifolds and the Shape of Space

Geometry can be found at the root of many problems in the real world, one being the potential shape and structure of the Universe. There are many theories and speculations regarding the shape of space, and since this idea is so open and complex, each theory is based upon a series of assumptions that allow it to blossom. Given the assumption that the Universe is a Geometric 3-Manifold, scientists can attempt to determine the possible shape of space using the geometric ideas along with gathered data that describes patterns in the Cosmic Microwave Background (CMB). Discussing these concepts in further detail requires exposure to the fundamentals behind geometric manifolds as well as an introduction to Cosmic Background Microwaves and the Last Scattering Sphere. This paper will provide an introduction to these topics, before bringing them all together and applying them to the quest for the shape of space.

The most basic version of a geometric manifold is a 2-manifold, which is a connected surface that is locally isometric to the Euclidean plane, hyperbolic plane, or sphere [2]. This means that any neighborhood on the smooth surface of a 2-manifold has the same geometric properties as one of these three surfaces. Throughout the semester, we have been unknowingly working with 2-manifolds because cones and cylinders are two common examples of flat 2-manifolds, meaning that their smooth surfaces are locally isometric to the Euclidean plane. A common technique for understanding these 2-manifolds is to decompose them into flat coverings, where the boundaries contain the same set of points in a specific orientation. The way that these two sets of similar points line up is known as gluing, because the two boundaries are essentially "glued" together in such a way that their points match correctly. As a reminder, figure 1 below depicts the flat coverings of a cylinder and 90° cone, along with the appropriate gluing.

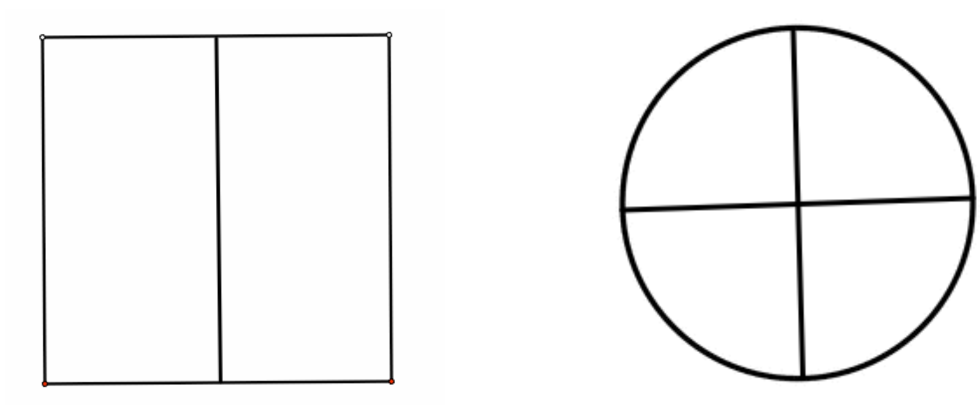


Figure 1: Flat coverings of the cylinder and 90° cone

Another example of a flat 2-manifold is a torus, which is constructed using the gluing in the left-hand image of figure 2 below. The covering of a torus is similar to a single covering of a cylinder, except the additional pair of opposite sides of the covering is glued together in a direct way. The right-hand image of figure 2 below shows how the torus is physically constructed from the flat covering first being folded into a cylinder, then folded a second time so the two extending ends meet, creating a finite, flat 2-manifold [1].

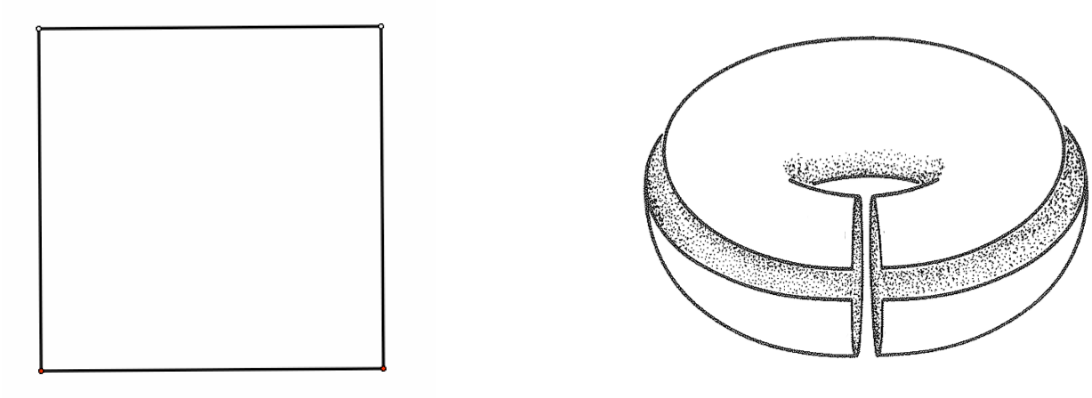


Figure 2: Gluing of a Torus (left) and its physical construction (right).

Looking at the construction of the torus in the figure above, each vertex has a 90° interior angle, and all four vertices of the covering meet at the same exact point on the surface of the torus. This supports the claim that the torus is locally isometric to the Euclidean plane, because on this surface, the angles meeting at any given vertex must add up to 360°.

If the vertices meet at a point and the angles add up to a measure greater than or less than 360°, then the 2-manifold is locally isometric to the hyperbolic plane or sphere, respectively. This because the angles greater than 360° would overlap on the plane, and angles less than 360° would leave gaps. By studying triangles on these surfaces, we have discovered that increasing the area of a triangle on the hyperbolic plane will make the interior angles smaller, while increasing the area of a triangle on the sphere will make the interior angles larger. So, if greater than 360° worth of angles meet at a vertex on the hyperbolic plane, then expanding the area will cause the angles to shrink until eventually their sum is 360° and they fit together perfectly. Similarly, if there is less than 360°, then expanding the area on the sphere will cause the angles to increase until they fit together.

An example of this situation occurs with the octagonal gluing in figure 3 below, which results in each of the eight vertices being mapped to the same point [2]. Since each vertex has an interior angle of 135° degrees, then there would be a total of 1080° around a single point, which would cause overlap on the plane. Using the technique mentioned above, if the gluing described a hyperbolic 2-manifold, in this case it is actually a two-holed torus, then the area of the octagon could be increased, decreasing the angles until they fit together in a perfect 360°.

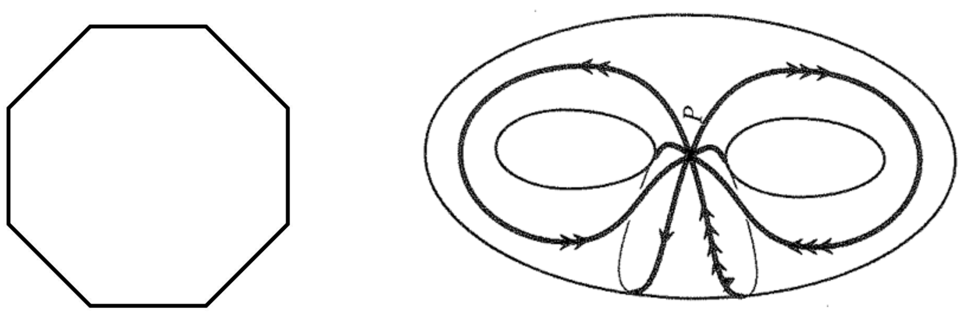
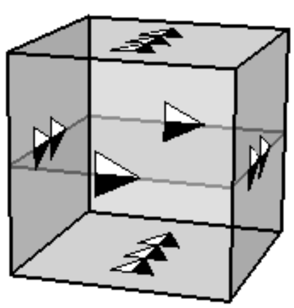


Figure 3: Gluing of a two-holed torus (left) and its construction (right).

The basic ideas are easier to experiment with and understand using 2-manifolds, but when it comes to the shape of space, the real applications stem from geometric 3-manifolds, which are spaces in which the region surrounding any point is isometric to Euclidean 3-space, hyperbolic 3-space, or the 3-sphere. The concepts are similar to those regarding 2-manifolds, except with a three-dimensional space rather than a two dimensional surface. Euclidean 3-space is the standard of three-dimensional space, where a third dimension is added to the plane. Hyperbolic 3-space is an expansion of the hyperbolic plane into a third dimension, and the 3-sphere is the set of points in four-dimensional space that are equidistant from a center point (an expansion of the standard 2-sphere, which is the set of all points in 3-space equidistant from a center) [1]. The same concept of gluing also applies to 3-

manifolds, except instead of gluing sides of a polygon; the faces of a polyhedron are glued together. These polyhedra with their specific gluing will tile one of the three given spaces perfectly, indicating which classification of 3-manifolds in which it belongs.

Since 3-space can be thought of as an infinite stack of Euclidean planes, and squares completely tile the plane since each vertex is surrounded by 360°, it implies that the cube can tile 3-space perfectly since each face is a square. The left-hand image of figure 4 below depicts the gluing of a 3-torus, which is constructed by gluing the opposite sides of a cube with a direct orientation, meaning that the two points directly across from each other on opposite faces are the same point. Various flat 3-manifolds (isometric to Euclidean 3-space) are formed depending on the orientation of the gluing, for example, where the top and bottom faces of a cube are glued with a ¼ or ½ rotation, producing the quarter- turn and half-turn manifolds, respectively [2]. In the half-turn manifold depicted in the right-hand side of figure 4 below, the top and bottom faces contain the same set of points, but in a reverse orientation as a result of the ½ turn gluing.



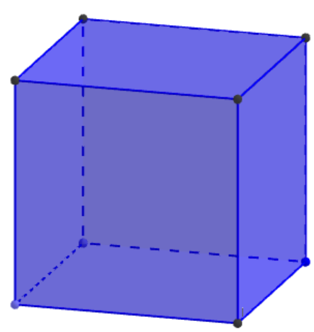
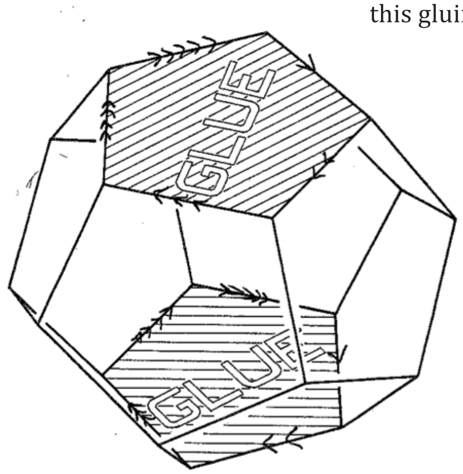
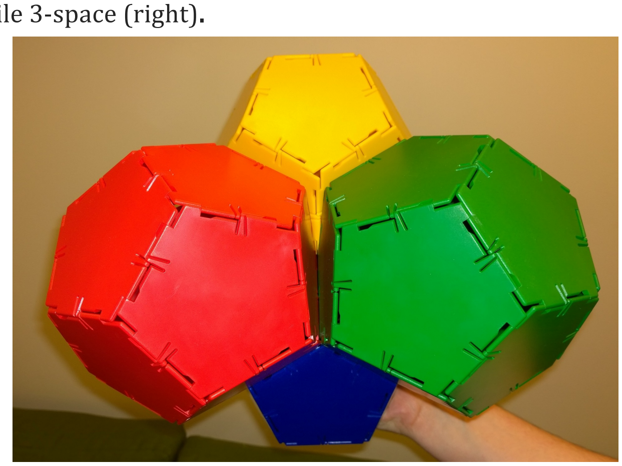


Figure 4: 3-Torus gluing (left) and half-manifold gluing (right).

Instead of using a cube as the polyhedron in the 3-manifold, another possibility is a regular dodecahedron. The Poincaré Dodecahedral Space is formed by gluing opposite sides of a dodecahedron with a 1/10 clockwise rotation, which can be seen in the left-hand image of figure 5 below [1]. Since opposite faces do not line up in a direct way, rather than similar points being directly across from each other, they are rotated 1/10 around the opposite face. It turns out that with the orientation of this gluing, the vertices of the dodecahedron map to a similar point in groups of four, meaning four dodecahedrons meet at the same vertex. Because of the angles of the pentagonal faces along with the 1/10 gluing, dodecahedrons won't completely tile 3-space, as seen in the right-hand image of figure 5 below, because there will be gaps in between the polehedra. Since in this case the angles must be larger in order for the four dodecahedrons to fit together, Poincaré Dodecahedral Space is isometric to the 3-sphere, because if this 3-Manifold shares the same properties as said space, we can use a similar technique that was used to increase the angles in a spherical 2-manifold. By increasing the volume of the dodecahedrons in the 3-sphere, the angles will expand until they fit together at a single point, eliminating the gaps.

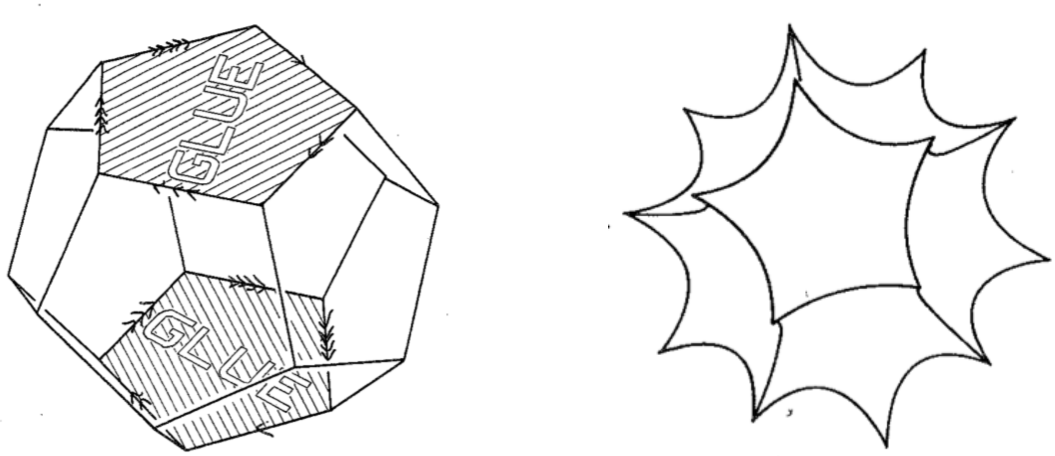
Figure 5: The gluing for the Poincaré Dodecahedral Space (left) and an image visualizing how this gluing doesn't tile 3-space (right).





Another 3-manifold produced by dodecahedrons is the Seifert-Weber Space, which is formed by gluing opposite sides of a dodecahedron with a 3/10 clockwise rotation, which can be seen in the left-hand image of figure 6 below [1]. Much like how the gluing for a 3-Torus maps all eight of the vertices to the same point in space, this 3/10 gluing maps all twenty of the vertices to the same point. With twenty dodecahedrons meeting at the same vertex, it is clear that there will be a lot of overlap because there simply is not enough space around a single point to fit all of the polyhedra. In order for these twenty dodecahedrons to fit together without overlap, the angles of their faces must be smaller, implying that the Seifert-Weber Space is isometric to hyperbolic 3-space. Using the properties of hyperbolic 3-space, increasing the volume of the dodecahedrons will cause the face angles to decrease and eventually fit together perfectly without any overlap. The right-hand side of figure 6 below is an example of how the angles of a dodecahedron decrease as the volume increases [1]. It is important to note that the faces only *appear* to be caving in on the polyhedron as a consequence of attempting to represent the three-dimensional hyperbolic figure on the Euclidean plane. In actual hyperbolic 3-space, for a dodecahedron with the same angles as in the picture below, it would not be a concave polyhedron.

Figure 6: The gluing for the Seifert-Weber Space (left) and a dodecahedron in hyperbolic 3-space after expanding its volume (right).



Using these examples, assuming the Universe is some 3-manifold will result in a situation where the Universe has a finite volume, but has no boundaries because of the infinite tiling of the same reoccurring image. This assumption also leads to the idea that opposite "sides" of the Universe contain the same set of "points" arranged according to some gluing pattern. If these sets of similar points on opposite "sides" can be analyzed in a way that distinguishes the orientation of the 3-manifold's gluing, then the geometric structure of the Universe can potentially be determined. In this high level explanation, the Last Scattering Sphere can be used to determine the set of similar points on opposite "sides" of the Universe, and Cosmic Microwave Background within those similar points can hint at the orientation of the gluing.

Immediately following the Big Bang, radiation waves scattered throughout the Universe, eventually cooling to microwaves over time. These microwaves were first observed by Arno Penzias and Robert Wilson of Bell Labs in 1965 [1]. Evenly distributed throughout space, these microwaves all began scattering at the same exact time, and all travel at the same speed as the scatter in all directions. CMB acts as a type of blueprint for the Universe at the time of the Big Bang, because the temperature of the microwaves reflects the density of the region in space in which they originated. Although the overall temperature of CMB waves is nearly uniform, small patterns can be distinguished due to the temperature fluctuations as a result of these differences in density.

Since all of these microwaves started at the same time and travel at the same speed, then it implies that all of the microwaves that are observed from the Earth at any given time all have travelled the same distance. In other words, the Earth is equidistant to the starting points of each CMB wave being observed at the same time, which is reminiscent to the definition of a sphere. If we imagine the Earth at the center of a very large sphere, then the origin of each microwave being observed at a given time would exist along the surface of this Last Scattering Sphere (LSS) as depicted in figure 7 below [1].

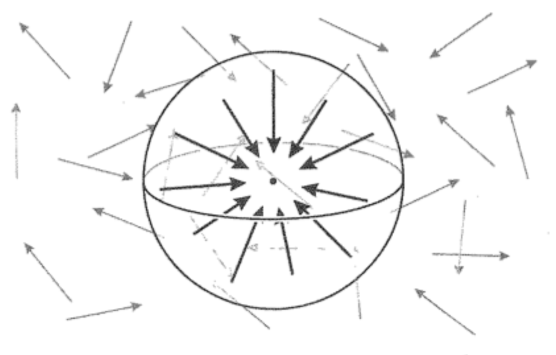
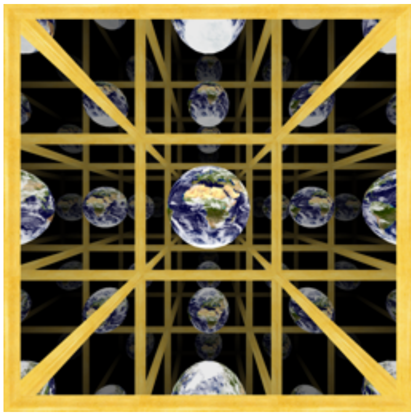


Figure 7: Last Scattering Sphere with the Earth at its relative center point.

In order to connect the important idea of the Last Scattering Surface with the potential shape of space, let's assume the Universe is a flat 3-manifold, the 3-Torus, that it is constructed by gluing opposite sides of a cube in a direct orientation. Considering the 3-Torus as a whole, it is an infinite tiling of 3-space that consists of identical cubic images of the Universe, and for reference, let the Earth be at the center of each image as shown in the left-hand picture of figure 8 below. Since the Earth is at the center of the LSS as well as each image of the Universe, there will eventually be a LSS that reaches beyond the boundaries of a single cubic image. This makes sense because all of the microwaves observed at a particular time have travelled the same distance from the surface of some LSS, and all of the microwaves began scattering at the same time, so the set of microwaves observed at a later point in time would have travelled a farther distance and therefore originated on an LSS that has a greater radius. So at some point, people on Earth will be able to observe microwaves from an LSS that is larger than the image of the Universe itself, as depicted in the right-hand picture of figure 8 below.



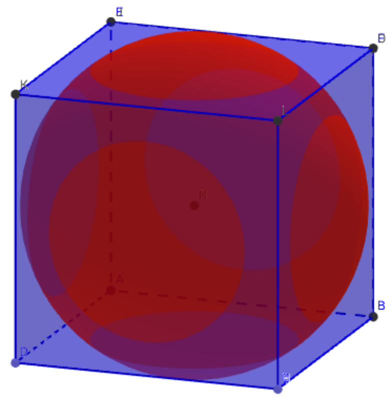


Figure 8: The Universe as a 3-Torus (left) and an LSS that extends beyond the boundaries of a cubic image of the Universe (right).

If the LSS extends beyond the boundary of the cube, and each image of the cube Universe has an instance of the LSS, then the LSS will intersect itself on every side of the cube as depicted in the left-hand picture of figure 9 below. The self-intersection of the Last Scattering Sphere produces a circle along opposite faces of the cube Universe, and this circle is comprised of the starting positions of the same set of CMB microwaves. To visualize this, we can reduce the situation from a 3-Torus with a Last Scattering Sphere to a 2-manifold torus with a "last scattering circle" as shown in the right-hand picture of figure 9 below. Within this diagram, each instance of point A is the origin of some microwave, and since the microwaves travel in all directions, two images of the same microwave will approach the Earth from different images of point A.

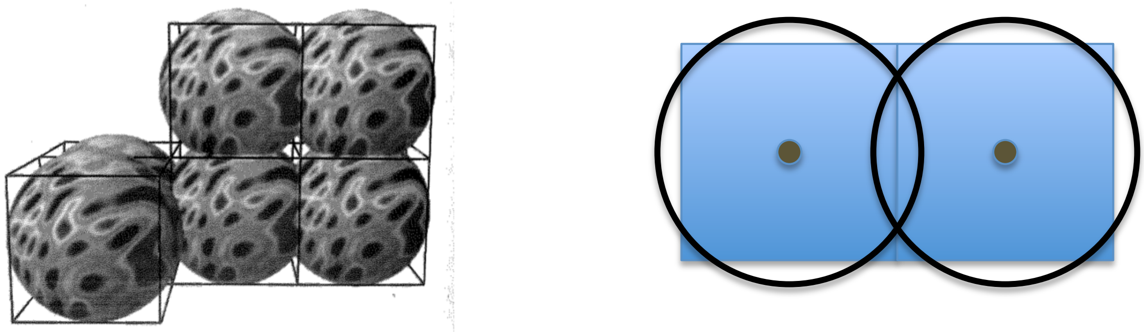


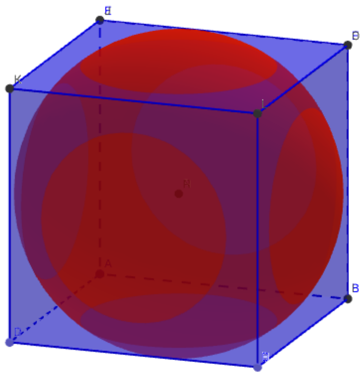
Figure 9: Self-Intersections of a large Last Scattering Sphere (left) and the 2-manifold version of one self-intersection.

Since opposite faces of the cube Universe contain a circle with the same set of CMB waves, we can observe them from the Earth and examine the different patterns within the two circles. These patterns are a result of the slight temperature fluctuations amongst the waves, which can be used to determine how the patterns match and ultimately the gluing of the 3-manifold. The orientation of the gluing is a hint towards the classification of 3-manifold to which the space belongs, so analyzing these "circles in the sky" can lead to determining the geometric structure of the 3-manifold Universe [2]. For example, if opposite pairs of circles with similar CMB waves match in a direct mapping like the gluing of the 3-Torus, then it implies that the geometric structure of the Universe is isometric to Euclidean 3-space. Other flat 3-manifolds include the hexagonal 3-torus, the quarter-turn manifold, the half-turn manifold, the one-sixth-turn manifold, and the one-third-turn manifold, which would also indicate that the Universe shares geometric properties with Euclidean 3-space. Figure 10 below shows various 3-manifolds with the self-intersection circles of the Last Scattering Sphere, along with how the corresponding gluings affects the patterns of CMB microwaves within these circles (Problem 24.5 (b.) of *Experiencing Geometry*, page 360) [2].

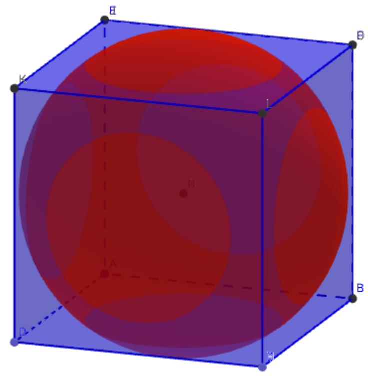
Figure 10: Self-intersections of the Last Scattering Sphere in various 3-Manifolds.

Flat 3-Manifolds

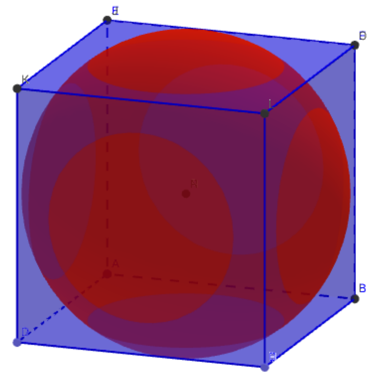
3-Torus



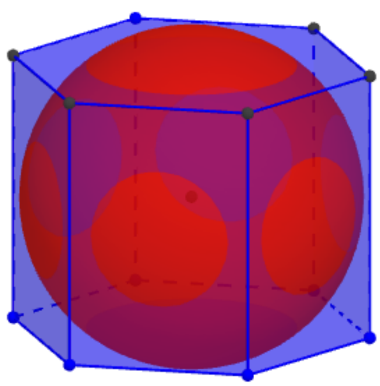
Quarter-Turn Manifold



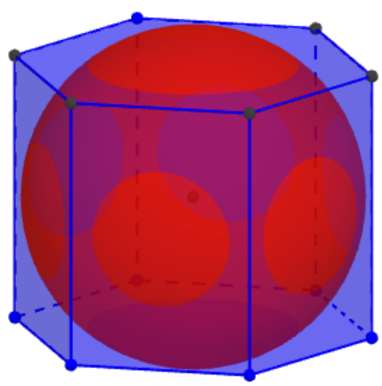
Half-Turn Manifold



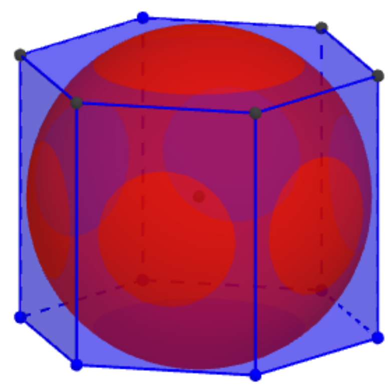
Hexagonal 3-Torus



One-Sixth-Turn Manifold

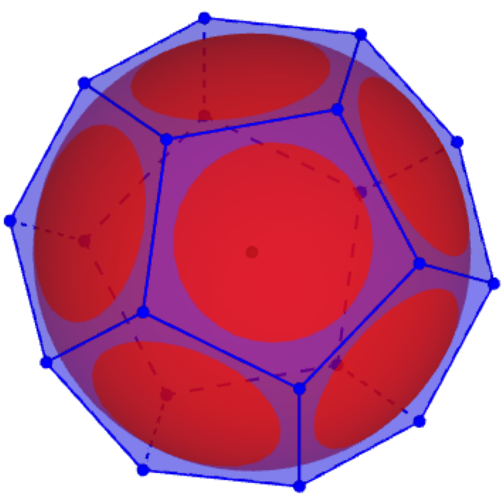


One-Third-Turn Manifold



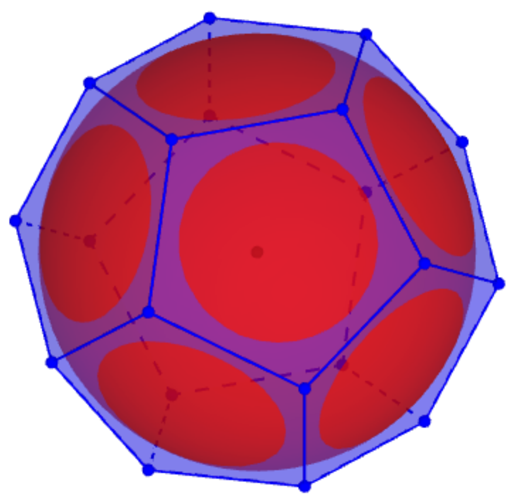
Hyperbolic 3-Manifolds

Seifert-Weber Dodecahedral Space



Spherical 3-Manifolds

Poincaré Dodecahedral Space



In attempt to identify these “circles in the sky”, NASA launched the Wilkinson Microwave Anisotropy Probe (WMAP) in June 2001 with a goal to measure various aspects of space, including mapping out the Cosmic Microwave Background of the entire sky [4]. After nine years of gathering data and two years of analysis, the discoveries and results of the mission were released in 2012, including the image in figure 11 below displaying the temperature fluctuations in the Cosmic Microwave Background. In addition to coming up with evidence that supports the Big Bang theory regarding our Universe, WMAP also produced results that indicate that the Universe *should* obey the rules of Euclidean Geometry [4]. This means that if the assumption that the Universe is a 3-manifold were indeed accurate, then it would likely be a flat 3-manifold isometric to Euclidean 3-space.

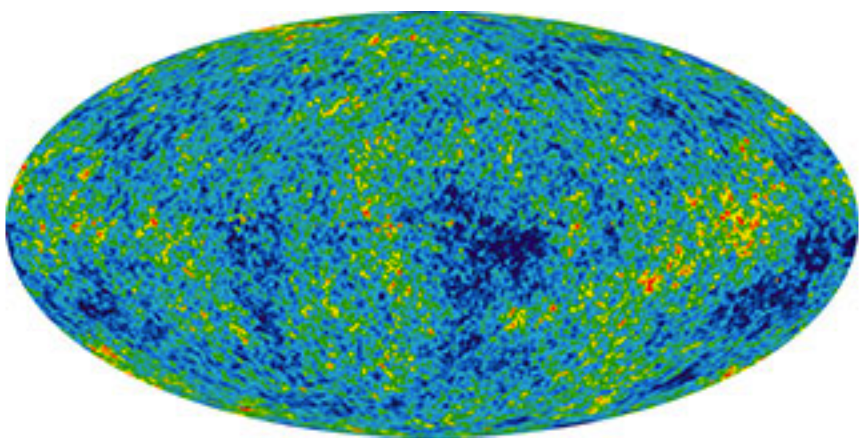


Figure 11: The map of Cosmic Microwave Background using the data gathered from NASA’s WMAP mission.

The Universe is such a vast, unknown topic that some assumptions must be made in order to formulate any educated theories and ideas regarding it, including the shape of space. By assuming that the Universe is a geometric 3-manifold, we are able to apply known ideas such as Cosmic Microwave Background and the Last Scattering Sphere to the geometric properties of 3-manifolds in order to make a reasonable attempt at discovering the shape of space. These speculations, later supported by the data gathered from NASA's WMAP exploration mission, form a strong case that the Universe is indeed a geometric 3-manifold, specifically one that holds properties similar to Euclidean 3-space.

Bibliography

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2. Henderson, David W., and Daina Taimiņa. *Experiencing Geometry: Euclidean and Non-Euclidean with History*. 3rd ed. Upper Saddle River, NJ: Pearson Prentice Hall, 2005. Print.
3. Luminet, Jean-Pierre. *The Wraparound Universe*. Wellesley, MA: A.K. Peters, 2008. Print.
4. Griswold, Britt. "Wilkinson Microwave Anisotropy Probe." *(WMAP)*. National Aeronautics and Space Administration (NASA), 1 July 2013. Web. 08 Dec. 2014. <http://map.gsfc.nasa.gov/>.

Images

Figure 2 (right): The Shape of Space [1], page 117, figure 7.14

Figure 3: (right): Experiencing Geometry [2], page 259, figure 18.15

Figure 4 (left): http://euler.slu.edu/escher/index.php/The_Three_Geometries

Figure 5 (left): The Shape of Space [1], page 222, figure 16.3

Figure 6 (left): The Shape of Space [1], page 220, figure 16.1

Figure 6 (right): The Shape of Space [1], page 221, figure 16.2

Figure 7: The Shape of Space [1], page 301, figure 22.6

Figure 8 (left):
http://web.math.princeton.edu/conference/Thurston60th/lectures.html

Figure 9 (left): Experiencing Geometry [2], page 304, figure 22.8

Figure 11: http://map.gsfc.nasa.gov/media/121238/index.html