Maximizing National Welfare Under Price Controls: A Tariff Optimization Model

MAT 4800: Introduction to Nonlinear Optimization

Devin Khun
Dept. of Mathematics & Statistics
Cal Poly Pomona
Pomona, United States
khun@cpp.edu

Luc Frost-Neto
Dept. of Mathematics & Statistics
Cal Poly Pomona
Pomona, United States
lmfrostneto@cpp.edu

James Leckie
Dept. of Mathematics & Statistics
Cal Poly Pomona
Pomona, United States
jlleckie@cpp.edu

***Abstract*—A tariff is a tax imposed on imported goods, often used as a tool for political leverage or to support domestic industries. However, determining the optimal tariff rate, that is, one that maximizes a nation's welfare poses a complex challenge. This paper frames the problem as a nonlinear optimization task aimed at identifying the tariff rate that yields the greatest benefit to the importing country. Three scenarios are considered: (1) the importing country maximizes its own welfare without regard for its trade partner, (2) both countries' welfare are jointly considered, and (3) the curvature of the demand function is varied to analyze the effects of price responsiveness. Through this framework, we explore how different assumptions impact the optimal policy choice.**

I. INTRODUCTION

Tariffs are a tax imposed on imported goods from another country. Today, within the global trade landscape, the issue of tariffs and taxes on imported goods has been a thoroughly debated topic. There has been a lot of conversation as to if tariffs can benefit the average citizen, what a mutually beneficial tariff rate looks like, or if tariffs are necessary at all. The fundamental idea behind a tariff is that the importing country who imposes the tax stands to benefit, while leading to the exporting country's detriment. However, it's illogical to max out a tariff rate on an imported good, since at a certain point, the dead-weight loss outweighs the gains of the tariff rate. At its heart, this is an optimization problem. Which leads to the following research questions: *Is there a tariff rate on an imported good that maximizes the gains on the importing country, while also minimizing the dead-weight loss? What is the rate that maximizes the gains for both the importing and exporting countries?*

II. RELATED WORK

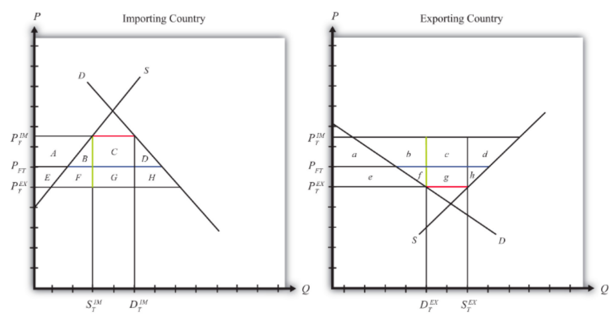

Fig. 1. Welfare Effects of a Tariff [1]

In order to properly assess the impact of a tariff rate, a simple way to do so is to look at the National Welfare. The following economic principles must be considered under the assumption that the market is perfectly competitive. For an importing country, the national welfare is the consumer surplus (represented by $-(A+B+C+D)$ in 1), combined with the producer surplus (represented by A in 1) and the government revenue (represented by C and G in 1). In total, this comes out to be the positive terms of trade effect (G) minus the dead-weight loss (B and D). To answer the research questions, the optimization problem involves a multivariable function that outputs the area of this region, in terms of the tariff rate imposed, and the price floor set by the importing country. Similarly, it's worthwhile to assess the national welfare loss of the exporting country. Since the exporting government does not gain any revenue, the national welfare of the exporting country is entirely defined by the consumer and producer surplus (represented by $-(f+g+h)$ in 1). To analyze the negative impacts on the exporting country, subtract the area of this region from the objective function.

III. Methodology

Finding the area of the described region in the previous section, results in the following welfare function for the importing country:

$$\begin{aligned} W(t, p_i) = (p_f - p_e)(D(p_i) - S(p_i)) && \text{(G)} \\ - \frac{1}{2}(p_i - p_f)(S(p_i) - S(p_f)) && \text{(B)} \\ - \frac{1}{2}(p_i - p_f)(D(p_f) - D(p_i)) && \text{(D)} \end{aligned}$$

The national welfare is a function of the imposed tariff rate t, and the price of the good in the importing country p_i (which in our example is a price floor set by the importing country). The following variables are constants, specific to the good being analyzed:

- t: Tariff Rate
- p_i: Price Floor Set in Importing Country
- p_e: Price in the Exporting Country
- p_f: Free Trade Equilibrium Price
- $S : P \to Q$ is the supply function mapping quantity supplied to price
- $D : P \to Q$ is the demand function mapping quantity demanded to price

Some constraints that we considered incorporating into the model are grounded in both economic logic and mathematical feasibility:

- **Non-negative Tariff Rate:** $t \geq 0$. This ensures that the tariff rate remains economically realistic. In practice, a negative tariff would imply a subsidy, which is a different policy instrument altogether. By restricting t to non-negative values, we focus the analysis solely on protective trade measures.
- **Price Relationships:** $p_e \leq p_f \leq p_i$. This captures the typical structure of prices in a tariff-affected market. Here, p_e represents the external (world) price, p_f the domestic price after the tariff, and p_i the price that the importing country ultimately pays. The inequalities reflect that tariffs usually raise domestic prices above the world price, and the consumer price p_i should not fall below either.
- **Market Feasibility:** $D(p_i) - S(p_i) \geq 0$. This constraint ensures that demand is at least as large as supply at the consumer price p_i, preventing a situation where excess supply would destabilize the market. It reflects a basic feasibility condition for the equilibrium in the domestic market under the modeled pricing structure.

To obtain the price in the exporting country, consider the tariff rate:

$$t = \frac{p_i}{p_e} - 1 \quad \text{where } t \geq 0,\ p_i \geq p_f \tag{1}$$

Plugging in equation (1), the welfare equation becomes:

$$\begin{aligned} W(t, p_i) = (p_f - \frac{p_i}{t+1})(D(p_i) - S(p_i)) \\ - \frac{1}{2}(p_f - \frac{p_i}{t+1})(S(p_i) - S(\frac{p_i}{t+1})) \\ - \frac{1}{2}(p_f - \frac{p_i}{t+1})(D(\frac{p_i}{t+1}) - D(p_i)) \end{aligned} \tag{2}$$

To optimize the function, first, it is necessary to come up with general expressions for the supply and demand functions, so that the inverses can be computed. Consider the generic supply and demand functions where m_d and m_s represent the slope of the demand and supply functions respectively, and b_d and b_s represent the y-intercepts of the demand and supply functions.

$$\begin{aligned} D(p) = b_d - m_d \cdot p \\ S(p) = b_s + m_s \cdot p \end{aligned}$$

IV. Results

Now we can apply this optimization problem setup into a real world example of steel prices. Using data from UN Comtrade [4], we can see that the free trade price of steel is approximately \$833 per metric ton.

Period	Reporter	Partner	Commodity Code	Trade Value (US\$)	Net Weight (kg)	Free Trade Price
2024	USA	World	7208 (steel)	\$2,431,079,947	2,753,484,814	\$883 per metric ton

Fig. 2. U.S. Steel Import Data (HS Code 7208) from UN Comtrade [4]

Also, using price and quantity data from Focus Economics and steel.org we can estimate the supply and demand functions as:

$$\begin{aligned} D(p) = 8 - 0.001716 \cdot p \\ S(p) = 1 + 0.000189 \cdot p \end{aligned}$$

Now we will need to estimate the price of the floors and ceilings of the tariff rate. The tariff floor and ceiling we will use the values of 0% and 50%. Thus giving the total list of these parameters as:

$$\begin{aligned} p_f = 883 \\ t_{floor} = 0 \\ t_{ceiling} = 0.5 \end{aligned}$$

Using MATLAB's nonlinear optimization solver `fmincon`, we can solve the optimization problem under these parameters and constraints. The optimal values for p_i and t in order to maximize the welfare function are:

- Optimal Tariff: $t = 0.50$
- Optimal Price: $p_i = \$883$
- Actual Welfare: \$1482 Million USD

V. Extension

The two ways that we will look at extending the framework of this problem will be by adding another condition to the objective function and looking at changes in the supply or demand functions. First by extending the objective function, we can add a condition that takes into account the exporting country's welfare loss. This loss comes from subtracting areas f, g, and h from the exporting country graph in 1. This value is represented as:

$$-\frac{1}{2}(p_f - p_e)\left(D(p_f) - S(p_f) + D(p_e) - S(p_e)\right)$$

Adding this condition into the objective function will now consider the overall impact of the tariff on both countries instead of just the one country benefiting from it. The new objective function will look like:

$$\begin{aligned} W_2(t, p_i) = {} & (p_f - \frac{p_i}{t+1})(D(p_i) - S(p_i)) \\ & - \frac{1}{2}(p_f - \frac{p_i}{t+1})(S^{-1}(p_i) - S^{-1}(\frac{p_i}{t+1})) \\ & - \frac{1}{2}(p_f - \frac{p_i}{t+1})(D(\frac{p_i}{t+1}) - (p_i)) \\ & - \frac{1}{2}(p_f - p_e)\left(D(p_f) - S(p_f) + D(p_e) - S(p_e)\right) \end{aligned} \tag{3}$$

Using the same parameters and supply and demand functions to optimize this new welfare equation gives the results:

- Optimal tariff t: 0.00
- Optimal price p_i: \$883
- Actual welfare: \$0 million USD

The two ways that we will look at extending the framework of this problem will be by adding another condition to the objective function and looking at changes in the supply or demand functions. First by extending the objective function, we can add a condition that takes into account the exporting country's welfare loss. This loss can be represented as:

$$-\frac{1}{2}(p_f - p_e)\left(D(p_f) - S(p_f) + D(p_e) - S(p_e)\right)$$

The second extension of the project that we examined was changing the curvature of the demand function. For the original optimization problem without the exporter welfare taken into consideration, we introduced a new parameter a_d. This parameter will control the curvature of price in the demand function. The new demand function will look like:

$$D(p) = b_d - m_d \cdot p^{a_d}$$

Modifying the demand function will make customers more sensitive to increases in prices, meaning the demand will now decrease much quicker when a_d increases. When optimizing equation (3) with the modified demand equation, the tables and graphs below show the resulting optimal t and the welfare values for the changing a_d:

a_d	**Welfare in (\$USD Million)**
1.000	1482
1.025	1384
1.050	1268
1.075	1130
1.100	996
1.125	771
1.150	539
1.175	264
1.200	55
1.225	0

TABLE I
WELFARE LEVELS UNDER VARYING DEMAND CURVATURE PARAMETERS (a_d)

a_d	**Optimal Tariff** t
1.160	0.5000
1.170	0.5000
1.180	0.3969
1.190	0.2584
1.200	0.1549
1.210	0.0748
1.220	0.0114
1.230	0.0000

TABLE II
OPTIMAL TARIFF t AT VARYING DEMAND CURVATURE VALUES a_d

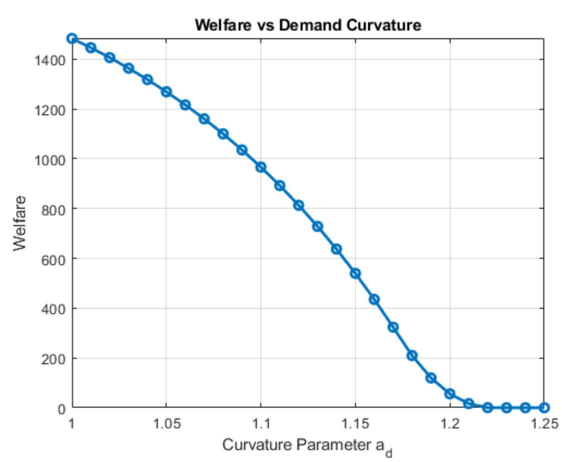

Fig. 3. Welfare vs a_d

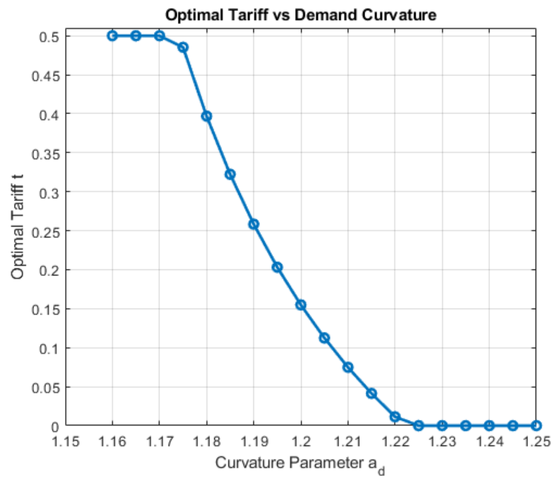

Fig. 4. Tariff rate vs a_d

In addition, we also changed the parameters for a supply-constrained market and a more elastic demand market. Regarding the supply-constrained example, supply was less responsive, but the optimal tariff generates more welfare. This is possible due to imports remaining high as the inelastic supply barely increases, which boosts government revenue. As for the elastic demand scenario, demand is highly responsive, so even a small price increase causes demand to fall, and we see a slight decrease in welfare. Even when the optimal tariff converges to a bound, the realized welfare gains are highly dependent on market responsiveness. In supply-constrained scenarios, welfare gains can be significantly higher despite limited price flexibility, while elastic demand sharply reduces importer advantage.

- **Supply-Constrained Example**
  - Optimal Price: $p_i = \$826$
  - Welfare: $2452.43 Million USD
- **Elastic Demand Example**
  - Optimal Price: $p_i = \$847$
  - Welfare: $1654.47 Million USD

VI. Discussion

Our model of the national welfare as an indicator of the impact on a tariff rate can be used as a tool to determine the tariff rate on an imported good that would maximize the advantage of a country in an international trade war. Additionally, we provide a model that factors in a country's relationship with their trade partner. Our second model shares the tariff rate that maximizes the difference between an importing country's welfare increase, and an exporting country's national welfare decrease.

In the first model, if a country is only interested in their gains, the government may decide to impose a high tax on imported goods to maximize their welfare. However, if the same country is interested in protecting their relationship with a trade partner, they may not want to introduce a tariff, as that will result in a decrease of welfare for the other country.

In the examples provided, the optimal solution to the tariff war is either to not introduce a tax, or have the highest possible tax allowed under the parameters.

Moving forward, a natural extension of our work would be to introduce new constraints or a new supply/demand/national welfare model that more accurately resembles the modern day market. In our model, we must assume perfect market conditions while also largely ignoring much of the politics happening in the background. Another opportunity for further research, would be to look at the same optimization problem, but measure the GDP change as a result of a tariff rate. While significantly harder to calculate (as it includes the national consumption, investment, spending, and net exports), the GDP yields a more accurate measure on the impacts of a trade tax. Finally, at the intersection of mathematics, economics, and international affairs, one may resolve to investigate current tariffs imposed to determine if such a tax is rooted in economic theory, or if the rate was arbitrarily selected.

VII. Conclusion

We enjoyed working on this project and how we were able to use nonlinear optimization techniques to model and solve scenarios involving determining optimal tariff rates. By exploring the welfare effects of tariff rates and market responsiveness, we showed how mathematical modeling can inform certain trade strategies. A direction of future work could incorporate more complex market dynamics, political factors, or economic indicators like GDP to build a more realistic framework.

References

[1] "7.5: Import Tariffs – Large Country Welfare Effects." *Social Sci LibreTexts*, 19 Jan. 2020. https://socialsci.libretexts.org/Bookshelves/Economics/International_Trade_-_Theory_and_Policy/07. Accessed 6 May 2025.

[2] "Industry Data." *American Iron and Steel Institute*. https://www.steel.org/industry-data/. Accessed 6 May 2025.

[3] "Steel (USA)." *FocusEconomics*. https://www.focus-economics.com/commodities/base-metals/steel-usa/. Accessed 6 May 2025.

[4] UN Comtrade. https://comtradeplus.un.org/. Accessed 6 May 2025.

Appendix

Code Repository

The full source code (MATLAB and Python) used in this report can be found at the following GitHub repository: https://github.com/devingineer/AcademiaMath