Student ID: ______________________________

Collaborators: ______________________________

CS181 Winter 2018 – Problem Set 5

Due Wednesday, March 7, 11:59 PM

- Please write your student ID **and the names of anyone you collaborated with** in the spaces provided and attach this sheet to the front of your solutions. **Please do not include your name anywhere since the homework will be blind graded.**
- An extra credit of **5%** will be granted to solutions written using LaTeX. Here is one place where you can create LaTeXdocuments for free: `https://www.sharelatex.com/`. The link also has tutorials to get you started. There are several other editors you can use.
- If you are writing solutions by hand, please write your answers in a neat and readable hand-writing.
- Always explain your answers. When a proof is requested, you should provide a rigorous proof.
- If you don't know the answer, write "I don't know" along with a clear explanation of what you tried. For example: "I couldn't figure this out. I think the following is a start, that is correct, but I couldn't figure out what to do next. [[Write down a start to the answer that you are sure makes sense.]] Also, I had the following vague idea, but I couldn't figure out how to make it work. [[Write down vague ideas.]]" At least 20% will be given for such an answer.

  Note that if you write things that do not make any sense, no points will be given.
- The homework is expected to take anywhere between 6 to 10 hours. You are advised to start early.
- Submit your homework online on the course webpage on CCLE. You can also hand it in at the end of any class before the deadline.

1.

Theorem 1. $\text{SUBSET}_{\text{TM}}$ is undecidable.

Proof. The idea is to prove the theorem by contradiction. Let us assume that $\text{SUBSET}_{\text{TM}}$ is decidable. That means there exists a decider H for $\text{SUBSET}_{\text{TM}}$. We then let Z = $< M >$ via the recursion theorem, where M is a TM. We then run our decider H with input Z and another input $< N >$. N is a TM which accepts a language L(N) = $\{1^p|$ p is a prime number$\}$ (Don't worry about where this language comes from; it is a completely arbitrary example language). We now have all the components of our decider. Our decider is as follows:

$$H(Z, < N >) \tag{1}$$

There are two cases. If H accepts, that means H believes that L(M) is a subset of L(N). If H rejects, that means H believes that L(M) is not a subset of L(N). To get a contradiction, we simply introduce a language L(M) such that it contradicts the result of H.

For the first case, we simply let L(M) = $\{1^n|$ n is a natural number$\}$. Despite the fact that H accepts, L(M) is clearly not a subset of L(N). This is a contradiction.

For the second case, we simply let L(M) = $\emptyset$. Despite the fact that H rejects, L(M) is clearly a subset of L(N). This is a contradiction.

Now that we have contradictions for all cases, we have proven by contradiction that $\text{SUBSET}_{\text{TM}}$ is undecidable. ☐

2a.
We can construct $\text{L}_2{}^{\text{DIAG}}$ by simply applying diagonalization to an enumeration that includes $\text{L}_1{}^{\text{DIAG}}$. Therefore, the first step is to attach $\text{L}_1{}^{\text{DIAG}}$ to the enumeration (L_1, L_2, L_3, ...). Then, we take the resulting enumeration and apply Cantor's Diagonalization to it, which gives us our $\text{L}_2{}^{\text{DIAG}}$. Our $\text{L}_2{}^{\text{DIAG}}$ cannot equal any of the languages in the enumeration (L_1, L_2, L_3, ...) or $\text{L}_1{}^{\text{DIAG}}$ because they are both in the new enumeration. Thus, we have successfully constructed $\text{L}_2{}^{\text{DIAG}}$.

2b.
The construction is simple. We can combine our original enumeration (L_1, L_2, L_3, ...) with our set of diagonal languages. Then, we apply diagonalization to create a new diagonal language. We can keep repeating this process and thus create an infinite set of distinct diagonal languages.

Base Step: We have the original enumeration of (L_1, L_2, L_3, ...). Thus our first diagonal language $\text{L}_1{}^{\text{DIAG}}$ must be different from any language in the enumeration by Cantor's Diagonalization.

Induction Step: Assume that we have an m number of diagonal languages ($\text{L}_1{}^{\text{DIAG}}$, $\text{L}_2{}^{\text{DIAG}}$, ..., $\text{L}_\text{m}{}^{\text{DIAG}}$). We also assume that each diagonal language is distinct from other diagonal languages and distinct from the languages in the original enumeration (L_1, L_2, L_3, ...). We then attach our diagonal languages to the original enumeration to form a new enumeration. Using this newly created enumeration, we apply Cantor's Diagonalization to create a new diagonal language ${L_{m+1}}^{DIAG}$. We know that ${L_{m+1}}^{DIAG}$ is not equal to any of the languages in our enumeration by diagonalization. Thus, our proof is complete.

2c.
To construct our $L^{SUPERDIAG}$, we simply change the order of languages in our enumeration. Then we can apply Cantor's Diagonalization to our new enumeration to get our $L^{SUPERDIAG}$.

To change the order of languages in our enumeration, simply move the set of diagonal languages. If the enumeration was originally (L_1, L_2, L_3, ..., ${L_1}^{DIAG}$, ..., ${L_m}^{DIAG}$), then simply move the set of diagonal languages to the front like so: (${L_1}^{DIAG}$, ..., ${L_m}^{DIAG}$, ..., L_1, L_2, L_3, ...). If the set of diagonal languages was originally at the front, then move them to the back.

Now that we have our new enumeration, we can apply diagonalization to it to construct $L^{SUPERDIAG}$. $L^{SUPERDIAG}$ is different from any of the languages in the enumeration by diagonalization. Thus, it satisfies the requirements that $L^{SUPERDIAG}$ be distinct from any ${L_j}^{DIAG}$ and L_j.

2d.
To construct ${L_2}^{SUPERDIAG}$, we simply take our constructed enumeration in part (c) and attach $L^{SUPERDIAG}$ to it. Then, we apply diagonalization to get ${L_2}^{SUPERDIAG}$.

To recap, our constructed enumeration in part (c) consisted of (L_1, L_2, L_3, ...) as well as the set of diagonal languages (${L_1}^{DIAG}$, ..., ${L_m}^{DIAG}$). We then attach $L^{SUPERDIAG}$ to this enumeration. Then, we apply diagonalization on the resulting enumeration to get ${L_2}^{SUPERDIAG}$. We know ${L_2}^{SUPERDIAG}$ is distinct by diagonalization. Because $L^{SUPERDIAG}$ and the set of diagonal languages are in the enumeration as well as (L_1, L_2, L_3, ...), we can conclude that ${L_2}^{SUPERDIAG}$ satisfies all the requirements.