Section 3 Calculus Notes

An interior point of the domain of a function f where f' is zero or undefined is a critical point of f

Places where f can possibly have an extreme value (local or global) 1. Critical Points 2. End points of the domain of f

Example

Find absolute max/min of $f(x) = 10x(2 - ln(x))$ on $[1, e^2]$

Step 1: Find Critical Points

$$f'(x) = \frac{d}{dx}(10x)(2 - ln(x)) + 10x\frac{d}{dx}(2 - ln(x))$$
$$f'(x) = 10(2 - ln(x)) + 10x(\frac{-1}{x}) = 0$$
$$10 - 10(ln(x)) = 0 \rightarrow ln(x) = 1 \rightarrow x = e$$

Step 2: Evaluate Function

Critical Points:

$$f(e) = 10e(2 - ln(e)) = 10e$$

End Points:

$$f(1) = 10(1)(2 - ln(1)) = 20$$
$$f(e^2) = 10(e^2)(2 - ln(e^2)) = 0$$

Absolute Max at $x = e$

Absolute Min at $x = e^2$

Example

Find absolute max/min of $f(x) = x^{\frac{2}{3}}$ on $[-2, 3]$

Step 1: Find Critical Points

$$f'(x) = \frac{2}{3}x^{\frac{-1}{3}} \neq 0$$
$$f'(x) = \frac{2}{3x^{\frac{1}{3}}}$$

Undefined at $x = 0$

Step 2: Evaluate Function

Critical Points:

$$f(0) = 0$$

End Points:

$$f(-2) = -2^{2^{\frac{1}{3}}} = \sqrt[3]{4}$$
$$f(3) = 3^{2^{\frac{1}{3}}} = \sqrt[3]{9}$$

Absolute Max at $x = 3$

Absolute Min at $x = 0$

How to find the absolute extrema of a continuous function on a closed interval.

1. Evaluate f at all critical points and endpoints
2. Take the largest and smallest these values

First Derivative Test for Local Extrema

Help us determine if critical point is a local max/min.

Suppose c is a critical point

1. If f' change from negative to positive at c, then f has a local min at $x = c$.
2. If f' changes rom positive to negative at c, then f has a local max at $x = c$.
3. If f' does not change sign at c, then no local extremum at c

Example

$$f(x) = x^{\frac{1}{3}}(x-4)$$
$$f'(x) = \frac{1}{3}x^{\frac{-2}{3}}(x-4) + x^{\frac{1}{3}}(1)$$
$$f'(x) = \frac{x-4}{3x^{\frac{2}{3}}}$$
$$f'(x) = \frac{1}{3}x^{\frac{1}{3}} - \frac{4}{3x^{\frac{2}{3}}} + x^{\frac{1}{3}} \to f'(x)\frac{4}{3}x^{\frac{1}{3}} - \frac{4}{3x^{\frac{2}{3}}}$$

Critical Point $x = 0$

$$x^{\frac{1}{3}} - \frac{1}{x^{\frac{2}{3}}} = 0$$
$$\frac{x^{\frac{1}{3}}}{1} = \frac{1}{x^{\frac{2}{3}}}$$
$$x^{\frac{1}{3}} * x^{\frac{2}{3}} = x = 1$$

Critical Point $x = 1$

Second Derivative tells you how a function bends/turns

The turning or bending behavior defines concavity of the curve.

Decreasing	Increasing
f' decreasing	f' increasing
$f'' < 0$	$f'' > 0$
Concave down	Concave up

The graph of a differentiable function $y = f(x)$ is

1. Concave up on an open interval $I \Leftrightarrow f'' > 0$ on I
2. Concave down on an open interval I if f' is decreasing on $I \Leftrightarrow f'' < 0$ on I
3. Contains an inflection point if the concavity is changing

Example

Characterize the bending of $f(x) = x^4$

$$f'(x) = 4x^3$$

$$f''(x) = 12x^2 > 0 \to x \neq 0$$

$$(-\infty, 0) \cup (0, \infty)$$

At inflection points either $f''(c) = 0$ or $f''(c)$ does not exist

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Example

Sketch a graph of a function with the following conditions:

1. Vertical Asymptote at $x = 0$
2. $f'(x) > 0$ if $x < -2$
3. $f'(x) < 0$ if $x > -2$ $x \neq 0$
4. $f''(x) < 0$ if $x < 0$
5. $f''(x) > 0$ if $x > 0$

Second Derivative Test

Suppose f'' is continuous near c

1. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$
2. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$

Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on (a, b). Then there is at least one point in (a, b) at which $\frac{f(b)-f(a)}{b-a} = f(c)$

Example

A Trucker handed in a ticket at a toll booth showing that in 2 hours she had covered 159 miles on a toll road with a speed limit of $65mph$. The trucker was cited for speeding. Why?

$$\frac{159}{2} = 80mph$$

Show:

If $f'(x) = g'(x)$ for all x in an interval (a, b), then $(f - g)(x)$ is constant on (a, b)

$(f - g)(x) = f(x) - g(x) = k$, where k is constant let $F(x) = f(x) - g(x)$. We need to show $F(x) = k$ for all $x \in (a, b)$

We know $F(x)$ is differentiable on (a, b)

$F'(x) = f'(x) - g'(x) = 0$ for all $x \in (a, b)$

Let x_1 and x_2 be any numbers in (a, b) such that $x_1 < x_2$

We know $F(x)$ is differentiable (x_1, x_2) and continuous on $[x_1, x_2]$. By the **Mean Value Theorem**, we know there exists $c \in (x_1, x_2)$ such that:

$$\frac{F(x_2) - F(x_1)}{x_2 - x_1} = F'(c) = 0$$

Show:

$F(x_1) = F(x_0)$ Because F has the same value at any two numbers x_1, and x_2 in (a, b), $F(x) = k$. Thus $f(x) - g(x) = k$

Linear Approximation

Linear Approximation of f at $x = a$ is

$$f(x) \approx f(a) + f'(x)(x - a)$$

Linearization of f at a:

$$L(x) = f(a) + f'(a)(x - a)$$

When θ is close to 0

$$\sin(\theta) \approx \theta$$
$$f(\theta) = \sin(\theta) \to f'(\theta) = \cos(\theta)$$
$$f(0) = \sin(0) = 0 \to f'(0) = 1$$
$$L(\theta) = f(0) + f'(0)(\theta - 0) = \theta$$

Newton's Method

We are looking for x such that f(x) = 0

Newton's Method helps us approximate x

$f(r) = 0 \rightarrow$ I am Looking for r

As x_3 and x_2 get closer together, the closer you are to r

$$f(x) \approx f(x_1) + f'(x_1)(x - x_1)$$
$$0 = f(x_1) + f'(x_1)(x - x_1)$$
$$x_1 - \frac{f(x_1)}{f'(x_1)}$$

We keep going to generate a sequence of numbers $x_1, x_2, x_3, x_4, ..., x_n$ In most cases, the number x_n becomes closer and closer to the root!

In-Class Problem

$\theta(t)$ be the angle of direct sight of airplane at the time

$L(t)$ is the distance the airplane (mi) is away from the island at the time t

We are looking for $\frac{dL}{dt}$

$$\frac{d\theta}{dt} = \frac{2}{3} * \frac{60sec}{1min} * \frac{60min}{1hr} = \frac{2}{3} * 3600\frac{deg}{hr} * \frac{\pi}{180} = \frac{2400\pi}{180}\frac{rad}{hr}$$

Relationship between variables:

$$\tan(\theta(t)) = \frac{\frac{1200}{528}}{Lt)} = \frac{1200}{528}[L(t)]^{-1}$$

Take Derivative:

$$\sec^2(\theta(t))\frac{d\theta}{dt} = \frac{-1200}{528}[L(t)]^{-2}\frac{dL}{dt}$$
$$(-\sec^2\theta)(\frac{d\theta}{dt})(\frac{528}{1200})L^2 = \frac{dL}{dt}$$

Optimization

Example

An open-top box is to be made by cutting small congruent squares from the corners of a $12in$ by $12in$ sheet of tin and bending up the sides.

How large should the squares cut from the corners be to make the box hold as much as possible?

Let x be the length of the square cutout in inches.

We plan to maximize volume.

$$V(x) = \binom{12-2x}{length}\binom{12-2x}{width}\binom{x}{height} = 12-2x)^2x$$

For $0 <= x <= 6$

Find Critical Points:

$$V'(x) = 2(12-2x)^1(-2)x + (12-2x)^2 * 1$$

$$V'(x) = (12-2x)(-4x+12-2x)$$

$$V'(x) = (12-2x)(-6x+12)$$

$$0 = (12-2x)(-6x+12)$$

$$0 = 12-2x0 = -6x+12$$

Next we check volume at critical points/end Points

$$V(0) = 0$$

$$V(6) = 0$$

$$V(2) = (8)^2 * 2 = 128in^3$$

Volume is maximize $(128in^3)$

Example

A rectangle is to be inscribed in a semicircle of radius 2.

What is the largest area the rectangle can have and what are its dimensions?

Steps

1. Draw a Picture
2. Introduce Variables
   - Let x be the x-coordinate of corner of rectangle
   - $0 \leq x \leq 2$
   - Equation of a circle is $y^2 + x^2 = 4$
   - Top half of circle described by $y = \sqrt{4 - x^2}$
3. Write an equation for unknown quantity

$$A(x) = (2x)(\sqrt{4 - x^2})$$

4. Find Critical points. Evaluate Function at critical points and end points.

$$\frac{dA}{dx} = 2\sqrt{4 - x^2} + 2x(\frac{2x}{2\sqrt{4 - x^2}})$$

$$2\sqrt{4 - x^2} = \frac{2x^2}{\sqrt{4 - x^2}}$$

$$2(4 - x^2) = 2x^2$$

$$4 = 2x^2$$

$$x = \sqrt{2}$$

Critical Point: $x = 2$ and $x = \sqrt{2}$ ##### End Points: $x = 0$ and $x = 2$

$$A(2) = 0$$

$$A(0) = 0$$

$$A(\sqrt{2}) = 2\sqrt{2}\sqrt{4 - \sqrt{2}^2} = 4$$

The area has a maximum of 4 when the rectangle is $sqrt4 - x^2 = \sqrt{2}$ units high and $2x = 2\sqrt{2}$ units long

Sigma Notation

It is convenient to use the Greek letter Σ to indicate sum.

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + ... + a_n$$

Compact way to write a sum with many terms

$$\sum_{k=1}^{n} a_k$$

Index k starts at $k = 1$

Index k ends at $k = n$

Formula for the kth term

k is called the index of summation

$$\sum_{i=0}^{4} 4^{-i} = \sum_{i=0}^{4} \frac{1}{2^{2i}} = \sum_{k=1}^{5} 4^{-(k-1)}$$

Example

Express the sum $8 - \frac{8}{3} + \frac{8}{9} - \frac{8}{27} + \frac{8}{81}$ in sigma notation

$$\frac{8}{3^0} - \frac{8}{3^1} + \frac{8}{3^2} - \frac{8}{3^3} + \frac{8}{3^4}$$

$$= \sum_{k=0}^{4} \frac{8(-1)^k}{3^k} = \sum_{k=0}^{4} \frac{8}{(-3)^k}$$

Algebra Rules for Sums

1. Sum rule

   $$\sum_{k=1}^{n}(a_k+b_k) = (a_1+b_1)+(a_2+b_2)+(a_3+b_3)+...+(a_n+b_n) = \sum_{k=1}^{n} a_k+\sum_{k=1}^{n} b_k$$

2. Difference rule

   $$\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

3. Constant rule

   $$\sum_{k=1}^{n} c(a_k) = c(\sum_{k=1}^{n}(a_k))$$

   Where c is Constant

4. $$\sum_{k=1}^{n} c = n(c)$$