MPCS 50103 Discrete Mathematics—Autumn 2019

Homework 9. This problem set is due Monday December 2 at 11:59 pm.

Reading: Rosen 7e, chapter 8, sections 8.1–8.3.

Written assignment:

- Solve the following "DO" exercises and homework problems "HW".
- "DO" exercises are strongly recommended to check your understanding of the concepts. **Do not submit them**.
- **Problems labeled "HW" must be submitted**.
- You are responsible for the material covered in **both** "DO" exercises and HW problems.

"Do" Exercises (*not* to be submitted):

1. **"DO"** Rosen 7e, section 8.1, exercises 3, 7, 13, and 27, on pages 510–512.

2. **"DO"** Rosen 7e, section 8.2, exercises 5 and 9, on pages 524–525.

3. **" DO"** Rosen 7e, section 8.3, exercises 14 and 17, on page 535.

Homework Problems (to be submitted Monday December 2 at 11:59 pm):

- **Collaboration policy: If you work with others, indicate their names as part of your submission. You must answer each question by yourself without assistance. It is a violation of this policy to submit a solution that you cannot explain orally to the instructor/TAs.**
- **Looking for solutions to problems on the internet, even when acknowledged, is STRONGLY DISCOURAGED.**
- Write out your work for every problem. If you just write your answer without showing your work, you will not receive credit.

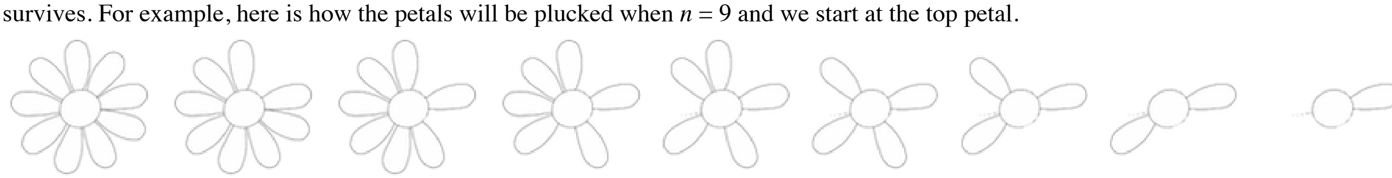
1. **HW**
   - Give a recurrence for the number of strictly increasing sequences of positive integers that have 1 as their first term and n as their last term, where n is a positive integer. That is, sequences $a_1, a_2, \ldots, a_k$, where $a_1 = 1$, $a_k = n$, and $a_j < a_{j+1}$ for $j = 1, 2, \ldots, k-1$. What are the initial condition(s)? Explain why and how your recurrence models the problem. (3 points)
   - Solve this recurrence: how many sequences of the type described are there when n is an integer with $n \geq 2$? Show all your work. (2 points)
   - What method did you use to solve the recurrence? (1 point)

2. **HW** In the Tower of Hanoi problem, suppose your goal is to transfer all n disks from the left peg 1 to the right peg 3 but you cannot move a disk directly between pegs 1 and 3: i.e., *each move must be to or from the middle peg 2*. As usual, a larger disk must never be placed on top of a smaller one, and you can only move one disk at a time.
   - Give a recurrence for the minimum number of moves required to solve the Tower of Hanoi problem for n disks with this added restriction. Include the initial condition. Explain why and how your recurrence models the problem. (3 points)
   - Solve this recurrence to find a closed-form expression (i.e., an explicit formula in terms of n) for the minimum number of moves required to solve the problem for n disks. Use the "guess and verify" method, and use induction to verify your guess. (3 points)
   - How many different arrangements are there of the n disks on three pegs so that no disks is on top of a smaller one? Explain your answer. (2 points)
   - Show that, in the process of transferring a tower under the above restrictions, you will actually encounter every properly stacked arrangement of n disks on three pegs. (2 points)

3. **HW** What is the maximum number L_n of regions defined by n lines in the plane?

   Example. The plane with no lines has one region, the plane with 1 line has 2 regions, and the plane with 2 lines has 4 regions:
   - Give a recurrence for L_n. Include the initial condition(s). Assume no two of the lines are parallel and no three of the lines go through the same point. Explain why and how your recurrence models the problem. (3 points)
   - Solve this recurrence to find a closed form expression for L_n. Show all your work. (2 points)
   - What method did you use to solve the recurrence? (1 point)

4. **HW** Consider a sunflower with n petals ($n \geq 1$), numbered 1 to n. We go through the petals clockwise, pluck every alternate petal starting with the second petal, until just one remains, going around the sunflower as many times as possible. Let $S(n)$ be the number of the petal that survives. For example, here is how the petals will be plucked when $n = 9$ and we start at the top petal.



   Numbering the petals from 1 to n clockwise, with the first petal plucked labeled 2, we have $S(9) = 3$.
   - Determine the value of $S(n)$ for each integer n for $n = 1, 2, \ldots, 17$. (2 points)
   - Guess a closed formula for $S(n)$ based on the values found in the previous part. *Hint:* Write $n = 2^m + k$, where $m \geq 0$ and $0 \leq k \leq 2^m$, i.e., 2^m is the largest power of 2 less than or equal to n. (2 points)
   - Show that $S(n)$ satisfies the recurrence $S(2n) = 2S(n) - 1$ and $S(2n+1) = 2S(n) + 1$ for $n \geq 1$, and $S(1) = 1$. (1 point)

- Use mathematical induction to prove your guess for $S(n)$, making use of the recurrence. (5 points)

5. **HW** Students in a Star Trek fan club meet every week to initiate new members. Students who have been in the club for two or more weeks initiate 3 new members each week and students who have been in the club for only one week initiate one new member each. Let $T(n)$ be the number of members in the club on week n.
   - Suppose $T(0) = 0$ and $T(1) = 1$. Determine the value of $T(n)$ for each integer n for $n = 1, 2, \ldots, 7$. (2 points)
   - Write a recurrence for $T(n)$, based on the values found in the previous part. (4 points)
   - Now we modify the problem a bit. Members who have been members of the club for i weeks add i members each week. Now find the recurrence for $T(n)$. Explain why and how your recurrence models the problem. (3 points)

Gerry Brady
Monday November 25 16:12:16 CST 2019