Counting Rules Chapter 6.1-6.6

6.1 The Basics of Counting

Product Rule

When a procedure can be broken into a sequence of tasks where the number of outcomes of each task is expressed as n1, n2, n3….. etc, then there are n1 * n2 * n3 *…*nk different ways to perform the procedure.

Example:

If I have 4 different t-shirts and 3 different pairs of shorts, how many different outfits do I have?

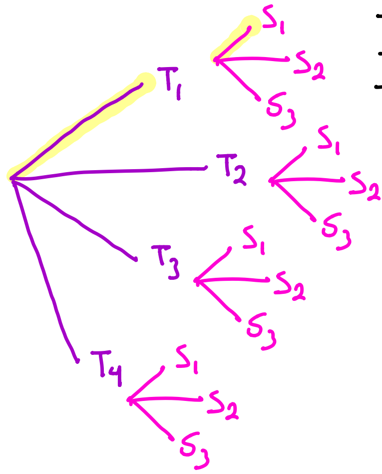
$$4 \cdot 3 = \boxed{12}$$

Tree Diagrams

A visual strategy we can use to represent counting problems.

Example:

If I have 4 different t-shirts and 3 different pairs of shorts how many different outfits do I have?



$T_1 S_1$
$T_1 S_2$
$T_1 S_3$

$4 \cdot 3 = 12$

12 different options

Sum Rule

$$\boxed{|A_1 \cup A_2 \cup A_3 \dots \cup A_n| = |A_1| + |A_2| + |A_3| + \dots + |A_n|}$$

If a task can be performed in one of n, ways or one of n2 ways then there are n1 + n2 ways to perform the task.

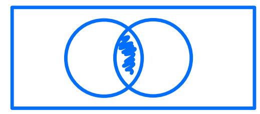
Example:

I want to take a trip to the beach. I can travel to one of 37 international beaches or one of 14 domestic beaches. How many beach vacations choices do I have?

$$37 + 14 = \boxed{51}$$

Subtraction Rules

If a task can be done in either one of n, ways or one of n2 ways, then the total number of ways to do the task is n1 + n2 minus the number of ways that are common to the two different ways



Principle of Inclusion- Exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

How many bitstrings of length 7 either start with 1 bit or end with the 3 bit 000 ?

- # options

that start with 1 $\quad 1\ \overbrace{\_\ \_\ \_\ \_\ \_\ \_}^{2^6}$

that end with 000 $\quad \underbrace{\_\ \_\ \_\ \_}_{2^4}\ 0\ 0\ 0$

Can be either 1 or 0

that start with 1 and end in 000 $\quad 1\ \underbrace{\_\ \_\ \_}_{2^3}\ 0\ 0\ 0$

~ # Ways they're Common

$$= 2^6 + 2^4 - 2^3$$

$$= 64 + 16 - 8$$

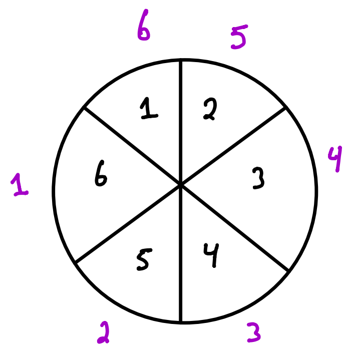
$$\boxed{= 72}$$

Division Rule

There are n/d ways to do a task if It can be done using a procedure that can be carried out in n ways, where there are d corresponding outcomes per group.

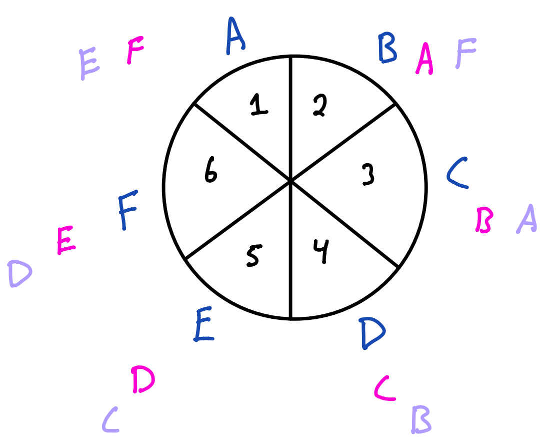
Example:

How many ways can I sit 6 people around a circular table where two seatings are considered the same when each person has the same left and right neighbor?



$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \frac{6!}{6}$$

6 different positions that are actually the same seating (same order)



~ moved everyone clockwise

~ These two seatings are considered the same

~ Shifted one more

6.2 The Pigeonhole Principle

The Pigeonhole Principle

Pigeonhole Principle: which states that if there are more pigeons than pigeonholes then there must be at least one pigeonhole with at least two pigeons in it.

If k is positive integer and k + 1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects

- Pigeonhole Principle uses proof by Contraposition

A function f from a set with k + 1 or more elements to a set with k elements is not one-to-one.

Proof:

Suppose that for each element y in the codomain of f we have a box that contains all elements x of the domain of f such that f (x) = y. Because the domain contains k + 1 or more elements and the codomain contains only k elements, the pigeonhole principle tells us that one of these boxes contains two or more elements x of the domain. This means that f cannot be one-to-one.

Example 1

Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

Example 2

In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.

Example 3

How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

Theres 101 possible scores on the final

So the pigeonhole principle says there must be at least 102 students to

have 2 students with the same score

The Generalized Pigeonhole Principle

If N objects are placed into k boxes, then there is at least one box containing at least objects

Example 5

Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

Example 6

What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

$\left\lceil \frac{N}{5} \right\rceil = 6$ $\quad N = 5 \cdot 5 + 1 = \underline{26}$

Permutations and Combinations

6.3

Permutation

An Ordered arrangement of distinct objects. An r-permutation is the arrangement of r-elements of a set?

Let $S = \{A, B, C\}$. Find all 2-permutations.

$P(n,r)$
$P(n,K)$
$_nP_r$
$_nP_K$
Can be written

AB
AC
BA
BC
CA
CB

6 options

$$\underset{n}{\underline{3}} \cdot \underset{n-1}{\underline{2}} = 6$$

$$P(3,2) = \frac{3!}{(3-2)!} = 3 \cdot 2 = 6$$

$$P(n,k) = \frac{n!}{(n-k)!} = \frac{n(n-1)(n-2)\ldots(n-(k-1))\cancel{(n-k)}\cancel{(n-k+1)}\ldots\cancel{(2)}\cancel{(1)}}{\cancel{(n-k)}\cancel{(n-(k+1))}\cdots\cancel{(2)}\cancel{(1)}}$$

Combinations:

An unordered arrangement of elements of a set. An r-combination is a subset of the set with r elements.

Let $\boxed{S = \{A, B, C\}}$ Find all 2-combinations. Relate this to the number of 2-permutations

AB	BA
AC	CA
BC	CB

~ Are really the same set if order didn't matter

Perm = 6 $\xrightarrow{\div 2}$

Combination = 3

$$\boxed{C(n,r) = \binom{n}{r} = \frac{n!}{(n-r)! \cdot r!}}$$

$$P(n,r) = \frac{n!}{(n-r)!}$$

$$C(n,r) = \frac{n!}{(n-r)! \cdot r!}$$

Take out any redudencies

$$C = (3,2) = \frac{3!}{(3-2)!\,2!} = \frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 1} = 3$$

How many poker hands of 5 cards can be dealt from a standard deck of 52 cards?

Do we care about the order?

- no, so its combination problem

$C(52,5)$

$$C(n,k) = \frac{n!}{(n-k)!\,k!}$$

$$C(52,5) = \frac{52!}{(52-5)!\,5!}$$

$$= \frac{52\cdot 51\cdot 50\cdot 49\cdot 48\cdot \cancel{47!}}{\cancel{47!}\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$\boxed{C(52,5) = 2,598,960}$$

In how many ways can 100 marathon runners place in 1st, 2nd, and 3rd ?

$$P(n,k) = \frac{n!}{(n-k)!}$$

$$P(100,3) = \frac{100!}{(100-3)!} = \frac{100\cdot 99\cdot 98\cdot \cancel{97!}}{\cancel{97!}}$$

$$\boxed{P(100,3) = 97020}$$

Counting Rules Practice

How many bit strings of length 6 are there?

$$\underset{01}{\underline{2}} \quad \underset{01}{\underline{2}} \quad \underset{01}{\underline{2}} \quad \underset{01}{\underline{2}} \quad \underset{01}{\underline{2}} \quad \underset{01}{\underline{2}} \quad = 2^6 = 64$$

How many bit strings of length 6 start with 1 or end with 1?

Addition

or comes in

$$|A| + |B| - |A \cap B|$$

here to subtract duplicates

$|A|$ Starts with 1 $\quad \underline{1} \quad \underline{2} \cdot \underline{2} \cdot \underline{2} \cdot \underline{2} \cdot \underline{2} = 2^5$

$|B|$ End in 1 $\quad \underline{2} \quad \underline{2} \quad \underline{2} \quad \underline{2} \quad \underline{2} \quad \underline{1} = 2^5$

$|A \cap B|$ Starts and end in 1 $\quad \underline{1} \quad \underline{2} \quad \underline{2} \quad \underline{2} \quad \underline{2} \quad \underline{1} = 2^4$

$$|A| + |B| - |A \cap B| = 2^5 + 2^5 - 2^4 = 48$$

How many bit strings of length n are there, not counting the empty string?

String with all zeros.

$$\boxed{2^n - 1}$$

Subtracting one empty string

Residents of Douglas County, NE have license plate beginning with 3 letters and ending with 3 digits. How many unique license plates are there in Douglas County?

$\underline{26} \cdot \underline{26} \cdot \underline{26} \cdot \underline{10} \cdot \underline{10} \cdot \underline{10} = \boxed{17,576,000}$

$26^3 \cdot 10^3$

If zeros are no longer allowed as the first of the three digits, how many unique license plates are there?

$\underline{26} \cdot \underline{26} \cdot \underline{26} \cdot \underline{9} \cdot \underline{10} \cdot \underline{10} = \boxed{15,818,400}$

only 9 options for 1st digit

$26^3 \cdot 10^2 \cdot 9$

Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$.

· Proof needed

$S = \{0, 1\}$ $|S| = 2$ $2^2 = 4$

# of different subsets:

$|\{\},\{0\},\{1\},\{0,1\}| = 4$

If S has 3 elements

$S = \{0,1,2\}$ $|S| = 3$ $2^3 = 8$

$\{\},\{0\},\{1\},\{2\}$

$\{0,1\},\{0,2\},\{1,2\},\{0,1,2\}$

$$\underbrace{\underline{2}\cdot\underline{2}\cdot\underline{2}\ \dots\ \underline{2}\cdot\underline{2}\cdot\underline{2}}_{S \text{ elements}}$$

$2^{|S|}$ - each element is either in the subset

If $|A_1| = 7$ and $|A_2| = 4$, how many elements are in the Cartesian product $A_1 \times A_2$?

$A_1 = \{1,2,3,4,5,6,7\}$

$A_2 = \{0,1,2,3,7\}$

(1,0)	(2,0)
(1,1)	(2,1)
(1,3)	(2,3)
(1,7)	(2,7)

$7 \cdot 4 = \boxed{28}$

· just a product rule example

A computer password can be 5-6 characters. The password must contain at least 1 digit. All the other characters must either be digits or lower-case letters. How many unique passwords exist?

5 characters

$$P_5 = 36^5 - 26^5$$

10 digit + 26 Letters

Contain at least one digit

$$P_5 = 48{,}584{,}800$$

$$P_6 = 36^6 - \underline{26}^6 = 1{,}867{,}866{,}560$$

$$P_6 = 1{,}867{,}866{,}560$$

Use Addition Rule to find the total # of options

$$P_5 + P_6$$

$$48{,}584{,}800 + 1{,}867{,}866{,}560$$

$$\boxed{= 1{,}916{,}451{,}360}$$

In how many ways can a photographer at a wedding arrange the bride, groom, and all 6 members of the wedding party if the bride and groom must be next to one another?

8 people → 7 group

since the bride and groom must sit together

1 2 3 4 5 6 $\overset{2}{\widehat{B|g}}$

$$7! = \underline{7} \cdot \underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1}$$

If didn't specify what side the bride and groom should be on. Bride could be on the right or left

$$\boxed{2 \cdot 7! = 10{,}080}$$

How many arrangements if each wedding party couple, those that walked together, must be next to one another, and the bride and groom must stand together with the bride to the left of the groom

1 | 2	3 | 4	5 | 6	7 | 8
2	2	2	

$$\boxed{4! \cdot 2^3 = 192}$$

How many 4-permutations of the positive integers not exceeding 100 contain 3 consecutive integers in the correct order in consecutive positions in the permutation.

$$\left.\begin{array}{c} 1,2,3 \\ 2,3,4 \\ 3,4,5 \\ \vdots \\ 98,99,100 \end{array}\right\} \text{98 ways}$$

After using 1 of the 98 Digits

$$2 \quad 98 \cdot 97 - 97$$

Counted 97 two times

$\underbrace{\_\ \_\ \_}\ \underline{97}$

$\underline{97}\ \underbrace{\_\ \_\ \_}$

$\underline{1}\ \underline{2}\ \underline{3}\ \underline{4}$

$$\boxed{= 18{,}915}$$

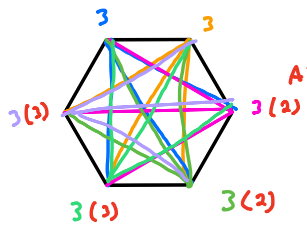
If the math faculty of a large University consists of 5 men and 6 women. How many committees of size 4 can be formed if the committee must contain at least one member of each gender?

Combination

$$C(11,4) - \overbrace{C(6,4)}^{\text{Consists of only women}} - \overbrace{C(5,4)}^{\text{only men}}$$

$$\boxed{= 310}$$

How many diagonals does a convex hexagon have?



Already Been Counted

$3+3+3+3+3+3-(1+2+3+3)$

$= 9$

$\frac{6(6-3)}{2} = \frac{18}{2} = 9$

How many diagonals does a convex $\boxed{\text{n-gon}}$ have?

$$\boxed{\frac{n(n-3)}{2}}$$

The Binomial Theorem

Binomial Expansion

Find the binomial expansion of $(3x+2)^3$.

$= (3x+2)(3x+2)(3x+2)$

$= (9x^2 + 12x + 4)(3x+2)$

$= 27x^3 + 18x^2 + 36x^2 + 24x + 12x + 8$

$\boxed{= 27x^3 + 54x^2 + 36x + 8}$

What if I asked for the coefficient of the term containing x^{62} of $(3x+2)^{100}$?

Binomial Theorem:

Let x and y be variables, and let n be nonnegative integer. Then:

$$\boxed{(\underline{x}+\underline{y})^n} = \underbrace{\sum_{j=0}^{n}}_{\text{Summation}} \binom{n}{j} \underline{x}^{n-j} \underline{y}^{j} = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \dots \binom{n}{n-1}x\cdot y^{n-1} + \binom{n}{n}y^n$$

What is the binomial expansion of $(x+y)^4$?

$$(x+y)^4 = \underset{\substack{4-0 \\ 0}}{\overset{1}{\cancel{\binom{4}{0}}}} x^4y^0 + \underset{4-3=1}{\overset{4}{\cancel{\binom{4}{1}}}} x^3y^1 + \overset{6}{\cancel{\binom{4}{2}}} x^2y^2 + \overset{4}{\cancel{\binom{4}{3}}} x^1y^3 + \overset{1}{\cancel{\binom{4}{4}}} x^0y^4$$

$$\binom{4}{0} = {}_4C_0 = C(4,0) = \frac{4!}{0!(4-0)!} = \frac{4!}{4!} = 1$$

Answer to Expansion:

$$(x+y)^4 = x^4 + 4x^3y^1 + 6x^2y^2 + 4xy^3 + y^4$$

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \dots \binom{n}{n-1} x \cdot y^{n-1} + \binom{n}{n} y^n$$

Binomial Expansion

Find the binomial expansion of $(3x+2)^3$.

$$\binom{3}{1} = \frac{3!}{2!\,1!} = \frac{3\cdot\cancel{2\cdot 1}}{\cancel{2\cdot 1}\cdot 1} = 3$$

$$\frac{3!}{3!\,0!} = 1$$

$$\sum_{j=0}^{3} \binom{3}{j} (3x)^{3-j} (2)^j$$

$$\binom{3}{0}(3x)^{3-0}(2)^0 + \binom{3}{1}(3x)^{3-1}(2)^1 + \binom{3}{2}(3x)^{3-2}(2)^2 + \binom{3}{3}(3x)^{3-3}(2)^3$$

$$(1)(27x^3)(1) + 3(9x^2)(2) + 3(3x)(4) + 1(1)(x^0)(8)$$

$$\boxed{27x^3 + 54x^2 + 36x + 8}$$

What if I asked for the coefficient of the term containing x^{62} of $(3x+2)^{100}$?

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \dots \binom{n}{n-1} x \cdot y^{n-1} + \binom{n}{n} y^n$$

$$(3x+2)^{100} = \underline{\sum_{j=0}^{100} \binom{100}{j} (3x)^{\boxed{100-j}} (2)^j}$$

$100 - j = 62$

$j = \underline{38}$

What is our j going to be?

$$= \binom{100}{38} (3x)^{62} (2)^{38}$$

$$\boxed{= \underline{\left(\frac{100!}{38!\,62!} 3^{62}\, 2^{38}\right)} x^{62}}$$

This would be our coefficent

Pascals Identity

Let n, k be positive integers with n >= k. Then

$$\boxed{\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}}$$

Proof: Let T be a set where $|T| = n+1$, $a \in T$ and $S = T - \{a\}$. there are

$\binom{n+1}{k}$ Subsets of T containing k elements. Each of these subsets either:

Contains a with k-1 other elements or $\binom{n}{k-1}$

Contains k elements of S and not a $\binom{n}{k}$

Pascals Triangle

- Based on that identity

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{1} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\
 & & & & \vdots & & & &
\end{array}
$$

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

$3+1=4$

$$(x+y)^4 = \boxed{x^4 + 4x^3y + 6x^2y^2 + 4xy + y^4}$$

$$(x+y)^5 = x^5 + 5xy^4 + 10x^3y^3 + 10x^2y^3 + 5xy^4 + y^5$$

Generalized Permutations and Combinations 6.5

Permutations with Repetition

Example 1: How many strings of length r can be formed from the uppercase letters of the English alphabet?

By the product Rule

26^r

Theorem 1: The number of r-permutations of a set of n objects with repetition allowed is n .

Proof:

There are n ways to select an element of the set for each of the r positions in r-permutation

Combinations with Repetition

Example 2:

How many ways are there to select four pieces of fruit from a bowl containing apples, oranges, and pears if the order in which the pieces are selected does not matter, only the type of fruit and not the individual piece matters, and there are at least four pieces of each type of fruit in the bowl?

Solution:

Theorem 2:

There are $C(n + r - 1, r) = C(n + r - 1, n - 1)$ r-combinations from a set with n elements when repetition of elements is allowed.

Proof:

Example 4

Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen? Assume that only the type of cookie, and not the individual cookies or the order in which they are chosen, matters.

From Theorem 2

$$C(4+6-1,6) = C(9,6)$$

$$C(9,6) = C(9,3) = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84$$

84 different ways to choose 6 cookies

Example 5

How many solutions does the equation have, where x1, x2, and x3 are nonnegative integers?

$$X_1 + X_2 + X_3 = 11$$

Combinations and Permutations with and without Repetition

Type	Repetition Allowed?	Formula
r-permutations	no	$\frac{n!}{(n-r)!}$
r-combinations	no	$\frac{n!}{r!(n-r)!}$
r-permutations	yes	n^r
r-combinations	yes	$\frac{(n+r-1)!}{r!(n-1)!}$

Permutation with Indistinguishable Objects

How many different strings can be made by reordering the letters of the word SUCCESS?

Theorem 3:

The number of different permutations of n objects, where there are n1 indistinguishable objects of type 1, n2 indistinguishable objects of type 2, . . . , and nk indistinguishable objects of type k, is:

$$\frac{n!}{n_1! \cdot n_2! \cdots n_k!}$$

Proof :

Distributing Objects into Boxes

How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?

$$C(52,5)C(47,5)C(42,5)C(37,5) = \frac{52!}{47!\,5!} \cdot \frac{47!}{42!\cdot 5!} \cdot \frac{42!}{37!\,5!} \cdot \frac{37!}{32!\cdot 5!}$$

$$\boxed{= \frac{52!}{5!\,5!\,5!\,5!\,32!}}$$

Theorem 4

The number of ways to distribute n distinguishable objects into k distinguishable boxes so that ni objects are placed into box i, i = 1,2,...,k, equals

$$\frac{n!}{n_1!\cdot n_2!\cdots n_k!}$$