Homework #8

Section 4.4

6. Find an inverse of a modulo m for each of these pairs of relatively prime integers using the method followed in Example 2.

a) $a = 2, m = 17$
b) $a = 34, m = 89$

c) $a = 144, m = 233$
d) $a = 200, m = 1001$

12. Solve each of these congruences using the modular inverses found in parts (b), (c), and (d) of Exercise 6.

a) $34x \equiv 77 \pmod{89}$

b) $144x \equiv 4 \pmod{233}$

c) $200x \equiv 13 \pmod{1001}$

34. Use Fermat's little theorem to find 23^{1002} **mod** 41.

Section 5.1

4. Let $P(n)$ be the statement that $1^3 + 2^3 + \cdots + n^3 = (n(n+1)/2)^2$ for the positive integer n.

a) What is the statement $P(1)$?

b) Show that $P(1)$ is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step, identifying where you use the inductive hypothesis.

f) Explain why these steps show that this formula is true whenever n is a positive integer.

Let $P(n) = 1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

a) What is the statement $P(1)$?

b) Show that $P(1)$ is true, completing the basis step of the proof.

$$\underbrace{P(1) = (1)^3 = \left(\frac{1(1+1)}{2}\right)^2}_{\text{Statement for } P(1)} = 1 = \left(\frac{2}{2}\right)^2 = 1 = (1)^2 = 1 = 1 \checkmark$$

P(1) is True

c) What is the inductive hypothesis? $P(K) = P(K+1)$

Inductive Hypothesis: $1^3 + 2^3 + \ldots + K^3 = \left(\frac{K(K+1)}{2}\right)^2$

What we want!

$\star$ Must Show: $1^3 + 2^3 + \ldots + (K+1)^3 = \left(\frac{(K+1)(K+2)}{2}\right)^2$ - Extra Step

$1^3 + 2^3 + \ldots + (K+1)^3 = \left(\frac{K(K+1)}{2}\right)^2 + (K+1)^3$ - Add $(K+1)^3$ to both sides

$$= \frac{K^2(K+1)^2}{2^2} + \frac{4(K+1)^3}{4}$$

d) What do you need to prove in the inductive step?

If IH P(K) is True, then P(K+1) is also true

$$= \frac{K^2(K^2+2K+1)}{4} + \frac{4(K+1)^3}{4}$$

$$= \frac{K^4 + 2K^3 + K^2}{4} + \frac{4(K+1)^3}{4}$$

$$= \frac{(K^4 + 2K^3 + K^2 + (4(K+1)^3)}{4}$$

e) Complete the inductive step, identifying where you use the inductive hypothesis.

$$= \frac{K^4 + 6K^3 + 13K^2 + 12K + 4}{4}$$

$$= \left(\frac{(K^2 + 3K + 2)}{2}\right)^2$$

$$= \left(\frac{(K+1)(K+2)}{2} \right)^2$$

$$= \left(\frac{(K+1)((K+1)+1)}{2} \right)^2$$

$$\left(\frac{(K+1)((K+1)+1)}{2} \right)^2 = \left(\frac{(K+1)(K+2)}{2} \right)^2$$ ~ Equal to the inductive Hypothesis

f) Explain why these steps show that this formula is true whenever n is a positive integer.

If IH P(K) is True, then P(K+1) is also true

$$\boxed{\therefore P(n) \text{ is true } \forall\ n \in \mathbb{Z},\ n > 0}$$

6. Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer.

$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$$

Let $P(n): 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$

① Basis: Prove $P(1)$ is true

$$P(1) = (1) \cdot (1!) = (1+1)! - 1$$

$1 \cdot 1! = (2)! - 1$ $\qquad 2! = 2 \cdot 1 = 2$

$1 = 1$ ✓

② Inductive step: $P(K) \rightarrow P(K+1)$

Inductive Hypothesis: $1 \cdot 1! + 2 \cdot 2! + \cdots + K \cdot K! = ((K+1)+1)! - 1$

10. a) Find a formula for

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}$$

by examining the values of this expression for small values of n.

b) Prove the formula you conjectured in part (a).

<u>Summation Problem</u>

16. Prove that for every positive integer n,

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2)$$
$$= n(n+1)(n+2)(n+3)/4.$$

Let $P(n) = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots n(n+1)(n+2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$

① Basis: Prove $P(1)$ is true

For $P(1)$: $1(1+1)(1+2) = \dfrac{1(1+1)(1+2)(1+3)}{4}$

$6 = 6$ ✓

② Inductive step: $P(k) \rightarrow P(k+1)$

Inductive Hypothesis: $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots k(k+1)(k+2) = \dfrac{k(k+1)(k+2)(k+3)}{4}$

* Must Show: $(k+1)((k+1)+1)((k+1)+2) = \dfrac{(k+1)((k+1)+1)((k+1)+2)((k+1)+3)}{4}$

18. Let $P(n)$ be the statement that $n! < n^n$, where n is an integer greater than 1.

a) What is the statement $P(2)$?

b) Show that $P(2)$ is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step.

f) Explain why these steps show that this inequality is true whenever n is an integer greater than 1.

* Inequlity Problem *

a) What is the statement $P(2)$?

$n! < n^n$

b) Show that $P(2)$ is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

Section 5.2

4. Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for $n \geq 18$.

   a) Show statements $P(18)$, $P(19)$, $P(20)$, and $P(21)$ are true, completing the basis step of the proof.

   b) What is the inductive hypothesis of the proof?

   c) What do you need to prove in the inductive step?

   d) Complete the inductive step for $k \geq 21$.

   e) Explain why these steps show that this statement is true whenever $n \geq 18$.

12. Use strong induction to show that every positive integer n can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, and so on. [*Hint:* For the inductive step, separately consider the case where $k + 1$ is even and where it is odd. When it is even, note that $(k + 1)/2$ is an integer.]

Section 5.3

3. Find $f(2)$, $f(3)$, $f(4)$, and $f(5)$ if f is defined recursively by $f(0) = -1$, $f(1) = 2$, and for $n = 1, 2, \ldots$

a) $f(n+1) = f(n) + 3f(n-1)$.
b) $f(n+1) = f(n)^2 f(n-1)$.
c) $f(n+1) = 3f(n)^2 - 4f(n-1)^2$.
d) $f(n+1) = f(n-1)/f(n)$.

8. Give a recursive definition of the sequence $\{a_n\}$, $n = 1, 2, 3, \ldots$ if

a) $a_n = 4n - 2$.
b) $a_n = 1 + (-1)^n$.
c) $a_n = n(n + 1)$.
d) $a_n = n^2$.

24. Give a recursive definition of

a) the set of odd positive integers.

b) the set of positive integer powers of 3.

c) the set of polynomials with integer coefficients.

37. Give a recursive definition of w^i, where w is a string and i is a nonnegative integer. (Here w^i represents the concatenation of i copies of the string w.)