

## Intro to Sets

[Discrete Mathematics]

Set Theory

## Set Theory

The Trevtutor

A set is a collection of objects called elements

$$\{1, 2, 3\} = A$$

Visual

List notation

Sets can be finite or infinite

Sets defined  
use uppercase #'s

$$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$\mathbb{Z}^+ = \{1, 2, 3, 4, \dots\}$$

"positive integers"      ↗ Applied Pattern forever!

## additional Points

- Repeated elements are listed once

$$\{a, b, a, c; b, a\} = \{a, b, c\}$$

- There is no order in a set.

$$\{3, 2, 1\} = \{1, 2, 3\}$$

$$\cdot \{2, 1, 3\}$$

## Common Sets

### Natural Numbers

$$\mathbb{N} = \{0, 1, 2, 3, 4, \dots\} \text{ or } \{1, 2, 3, \dots\}$$

### Integers

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

### Rational #'s

$$\mathbb{Q} = \{\dots, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \dots\}$$

~ # you can write as a fraction.

**DEFINITION 1**

A *set* is an unordered collection of objects, called *elements* or *members* of the set. A set is said to *contain* its elements. We write  $a \in A$  to denote that  $a$  is an element of the set  $A$ . The notation  $a \notin A$  denotes that  $a$  is not an element of the set  $A$ .

These sets, each denoted using a boldface letter, play an important role in discrete mathematics:

- $\mathbf{N} = \{0, 1, 2, 3, \dots\}$ , the set of **natural numbers**
- $\mathbf{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ , the set of **integers**
- $\mathbf{Z}^+ = \{1, 2, 3, \dots\}$ , the set of **positive integers**
- $\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, \text{ and } q \neq 0\}$ , the set of **rational numbers**
- $\mathbf{R}$ , the set of **real numbers**
- $\mathbf{R}^+$ , the set of **positive real numbers**
- $\mathbf{C}$ , the set of **complex numbers**.

**DEFINITION 2**

Two sets are *equal* if and only if they have the same elements. Therefore, if  $A$  and  $B$  are sets, then  $A$  and  $B$  are equal if and only if  $\forall x(x \in A \leftrightarrow x \in B)$ . We write  $A = B$  if  $A$  and  $B$  are equal sets.

**DEFINITION 5**

A set is said to be *infinite* if it is not finite.

Theorem 1 shows that every nonempty set  $S$  is guaranteed to have at least two subsets, the empty set and the set  $S$  itself, that is,  $\emptyset \subseteq S$  and  $S \subseteq S$ .

**THEOREM 1**

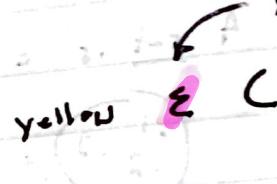
For every set  $S$ , (i)  $\emptyset \subseteq S$  and (ii)  $S \subseteq S$ .

Size

### Elements and Cardinality

Let  $C = \{\text{yellow, blue, red}\}$

"yellow is an element of C"



"Green is not an element of C"



"The cardinality of C is 3"

(size)

3 different elements  
in C?

$$|C| = 3$$

absolute value

### The Empty Set (or null set)

- Special set with no elements.

Symbol  $\emptyset = \{\}$

$$\{\emptyset\}$$

$$|\emptyset| = 0$$

$$\rightarrow |\{\emptyset\}| = 1$$

$$|\{\emptyset\}| = 0$$

empty set

\* set has an empty set as an element

### Set-Builder Notation

- How to build a set

$$\cdot Q = \{\dots, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \dots\}$$

↳ Better way to express the set?

$$= \left\{ \frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0 \right\}$$

here to be integers

and n don't be zero?

Another Example: Even integers

$$2\mathbb{Z} = \{\dots, -4, -2, 0, 2, 4, \dots\}$$

$$= \{2n \mid n \in \mathbb{Z}\}$$

n is an integer

Such that  
Read as ↗

## More on the Empty Set:

$\{\emptyset\}$  has one more element than  $\emptyset$ .

**THE EMPTY SET** There is a special set that has no elements. This set is called the **empty set**, or **null set**, and is denoted by  $\emptyset$ . The empty set can also be denoted by  $\{\}$  (that is, we represent the empty set with a pair of braces that encloses all the elements in this set). Often, a set of elements with certain properties turns out to be the null set. For instance, the set of all positive integers that are greater than their squares is the null set.

A set with one element is called a **singleton set**. A common error is to confuse the empty set  $\emptyset$  with the set  $\{\emptyset\}$ , which is a singleton set. The single element of the set  $\{\emptyset\}$  is the **empty set itself!** A useful analogy for remembering this difference is to think of **folders** in a computer file system. The empty set can be thought of as an **empty folder** and the set consisting of just the empty set can be thought of as a folder with exactly one folder inside, namely, the empty folder.

A Set is a collection of objects



What about a set with NO objects?



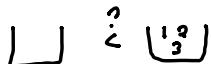
Notation:  $\{\}$  or  $\emptyset$

What is  $\{\emptyset\}$ ?

- Set that contains the empty set  
~ another set inside it



Is  $\emptyset \subset \{1, 2, 3\}$ ?



Recall:  $A \subset B$  means:  
if  $x \in A$ , then  $x \in B$

This is vacuous Truth.

## \* Real-World Example \*

Desk =  $\{ \text{drink, laptop, microphone} \}$

=  $\{ x \mid x \text{ is on my desk} \}$   
 Such that

### Exercises

- 1) List the elements of D:  $D = \{ x \in \mathbb{Z}^+ \mid x < 6 \}$   
 Such that  
 $1, 2, 3, 4, 5$

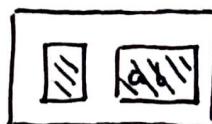
- 2.) What is the cardinality of D?

$$|D| = 5$$

{ "elements" in set }

- 3.) What is the cardinality of  $|\{\emptyset, \{a\}, \{b\}\}| = 2$

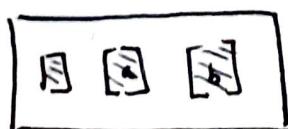
like looking into a box



Note: look at commas !

How does it differ from this set here?  $\rightarrow |\{\emptyset, \{a\}, \{b\}\}| = 3$

$$|\{\emptyset, a, \{b\}\}| = 3$$



or 3 elements in our box



## Cartesian Products and Order Pairs

### Cartesian Products

- An ordered pair  $(a, b)$  is a set

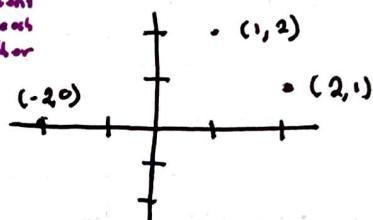
$$\{ \{a\}, \{a, b\} \}$$

- You've seen ordered pairs before on graph coordinates

$$(1, 2) = \{ \{1\}, \{1, 2\} \}$$

$$(2, 1) = \{ \{2\}, \{2, 1\} \}$$

$$(-2, 0)$$



The Cartesian Product,  $A \times B$ , is the Set cross Product

$$\{ (a, b) \mid a \in A \text{ and } b \in B \}$$

Given  $X = \{0, 1, 2\}$  and  $Y = \{0, 1\}$

$$X \times Y = \{ (0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1) \}$$

Cross Product  
↓

X	Y
0	0
1	1
2	

$$Y \times X = \{ (0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2) \}$$

1st      2nd

**DEFINITION 7**

The *ordered n-tuple*  $(a_1, a_2, \dots, a_n)$  is the ordered collection that has  $a_1$  as its first element,  $a_2$  as its second element, ..., and  $a_n$  as its  $n$ th element.

**DEFINITION 8**

Let  $A$  and  $B$  be sets. The *Cartesian product* of  $A$  and  $B$ , denoted by  $A \times B$ , is the set of all ordered pairs  $(a, b)$ , where  $a \in A$  and  $b \in B$ . Hence,

$$A \times B = \{ (a, b) \mid a \in A \wedge b \in B \}.$$

What is the Cardinality of  $A \times B$ ?

If  $|A| = m$  and  $|B| = n$  then

$$|A \times B| = mn$$

$$|X| = 3$$

$$|Y| = 2$$

$$|X \times Y| = 3 \cdot 2$$

$$= 6$$

Cartesian Products can generalize to n-tuples

3-tuples

$$A \times B \times C = \{(a, b, c) \mid a \in A, b \in B, c \in C\}$$

n-tuples

$$A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \dots, a_n \in A_n\}$$

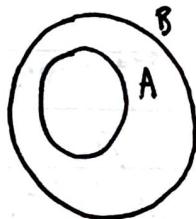
#### DEFINITION 9

The *Cartesian product* of the sets  $A_1, A_2, \dots, A_n$ , denoted by  $A_1 \times A_2 \times \dots \times A_n$ , is the set of ordered  $n$ -tuples  $(a_1, a_2, \dots, a_n)$ , where  $a_i$  belongs to  $A_i$  for  $i = 1, 2, \dots, n$ . In other words,

$$A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \dots, n\}.$$

## Subsets and Powersets

If  $A$  is a subset of  $B$ , then every element in  $A$  must also be in  $B$ .



$$A \subseteq B$$

↑  
subset of  
con equal  
equivalent

$$A \subset B$$

↑  
 $A$  must strictly  
be smaller than  $B$



if  $A$  and  $B$  are the same size (the little bar at the bottom)

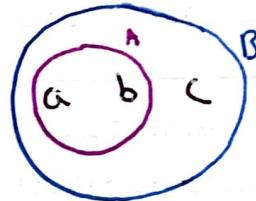
### Questions:

True or False? Exemplar

$$\{a, b, a, c\} \subseteq \{a, b, c\}$$

repeated elements do not matter

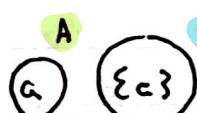
$$\{a, b\} \subseteq \{a, b, c\} \quad \text{True}$$



$$\{c, d\} \subseteq \{c, d\} \quad \text{True}$$

A      B

$$\{a\} \not\subseteq \{\{a\}\} \quad \text{False}$$



list out all the elements

We don't see  $a$  in  $B$ ,  
We just see the set containing  $a$

line thru it means note a subset

$$\emptyset \subseteq \{x, y, z\} \quad \text{True}$$

on empty set is a subset of every set!

### DEFINITION 3

The set  $A$  is a *subset* of  $B$  if and only if every element of  $A$  is also an element of  $B$ . We use the notation  $A \subseteq B$  to indicate that  $A$  is a subset of the set  $B$ .

## Power Sets

A power set of A,  $P(A)$ , is the set containing all possible subsets of A.

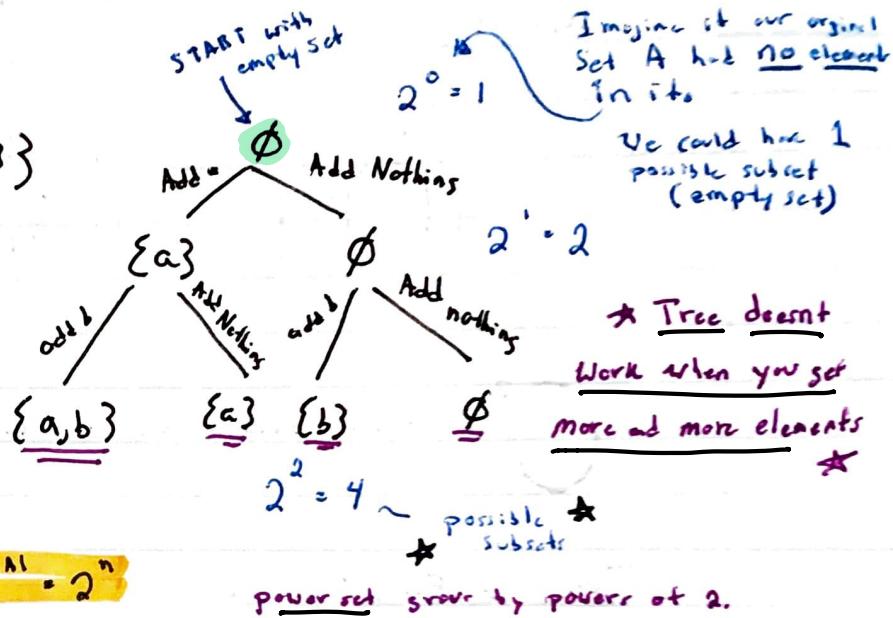
$$A = \{a, b\}$$

$$P(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$



Power Sets, contain all possible subsets of A.

= possible subset of set containing a and b



## Size of Power Set

\* If  $|A| = n$ , then  $|P(A)| = 2^{|A|} = 2^n$

- for each element, we can add it to a subset, or we cannot add it to a subset.

→ 2 Choices per element ~ add or do not add

$$|A|=6 \quad |P(A)| = 2^{|A|} = 2^6 = 64 \sim \text{possible subsets}$$

The empty set is an element of every single power set

## Tricky Questions

powerset of the empty set

$$P(\emptyset) = \{\emptyset\} = \{\{\emptyset\}\}$$

\* Set contains nothing in it.

{size of the empty set}

$$|\emptyset| = 0$$

$$|P(\emptyset)| = 2^0 = 1$$

powerset of the  $\emptyset$   $P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$

Set contains nothing

"set containing the empty set"

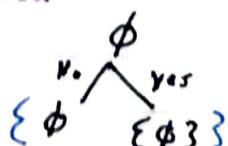
So the power set of the empty set is just the set containing the empty set

$$|\{\emptyset\}| = 1 \quad |P(\{\emptyset\})| = 2^1 = 2$$

? size of the set contains nothing

Should have 2 elements in it.

Can draw tree for this example



↑ ↑

cc The Powerset is just going to be these two possible subsets in a set

$$P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$$

A  
 ↑  
 we see A  
 is an element

No      Yes  
 ↗      ↘  
 { }      { }  
 ↓  
 A

Is  $A \subseteq P(A)$  for any  $A$ ? No  
 Is  $A \subset$  of the powerset of  $A$  for any  $A$ ?

Is  $A \in P(A)$  for any  $A$ ? Yes

Element Notation

### Exercises

1.) Let  $|C| = k$  and  $|D| = j$ , what is  $|P(C \times D)|$ ?

$$|C \times D| = |C| \times |D| = k \cdot j$$

$$|P(C \times D)| = 2^{|C \times D|} = 2^{kj}$$

2.) List the elements of  $P(P(\emptyset))$

$$P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$$

3.) if  $|A|=m$ , what is  $|P(P(P(A)))|$ ?  $= 2^{|P(P(A))|}$

$$\begin{aligned}
 & \uparrow \\
 & \text{but indicated size} \\
 & \text{is what is the size of the} \\
 & \text{powerset of the powerset} \\
 & \text{of the Powerset of } A? \quad = 2^{2^{|P(A)|}} \\
 & \quad = 2^{2^{2^{|A|}}} \\
 & \quad = 2^{2^{2^m}}
 \end{aligned}$$

### DEFINITION 6

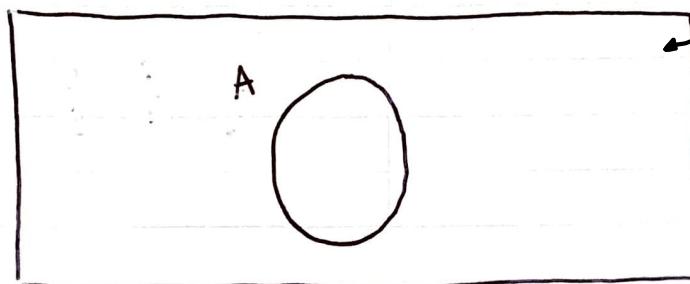
Given a set  $S$ , the power set of  $S$  is the set of all subsets of the set  $S$ . The power set of  $S$  is denoted by  $\mathcal{P}(S)$ .

## Set Operations

Set Operations and Venn Diagrams ↗ used to show relationships b/w sets.

Every set  $A$  exists within some universe  $U$ .

Box represents  
 $U$ , some universe



$$\begin{aligned} U &= \mathbb{Z} \\ &= \mathbb{IR} \\ &= \mathbb{Z}^+ \end{aligned}$$

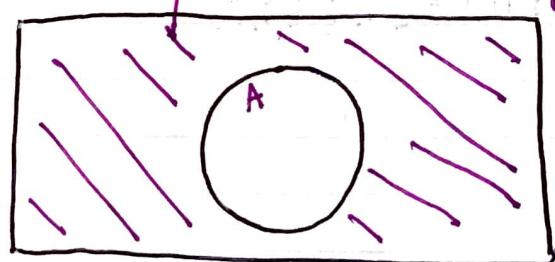
The Complement of  $A$  is written as  $\bar{A}$ .

$$\bar{A} = \{a \in U \mid a \notin A\}$$

Can be written as  $A'$  or  $A^c$  as well.

in order to define a complement you need a universe to talk about.

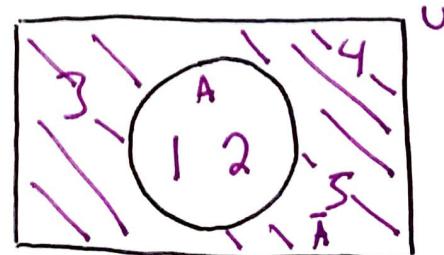
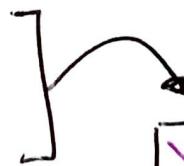
The Complement of  $A$  is everything outside of the set  $A$  that is in the universe



Example:  $A = \{1, 2\}$

$$U = \{1, 2, 3, 4, 5\}$$

$$\bar{A} = \{3, 4, 5\}$$



### DEFINITION 5

Let  $U$  be the universal set. The *complement* of the set  $A$ , denoted by  $\bar{A}$ , is the complement of  $A$  with respect to  $U$ . Therefore, the complement of the set  $A$  is  $U - A$ .

## Intersection

Given Sets A and B, the intersection of A and B is:

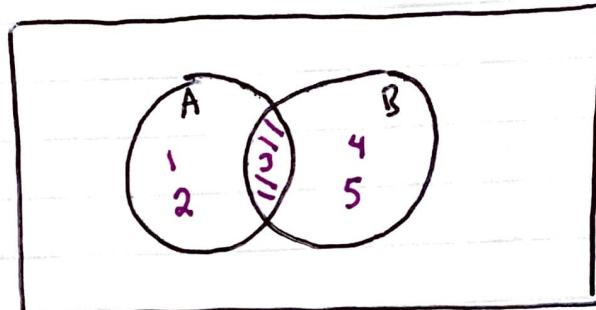
$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

$$A = \{1, 2, 3\}$$

\* upside  
down U

$$B = \{3, 4, 5\}$$

$$A \cap B = \{3\}$$



## Union

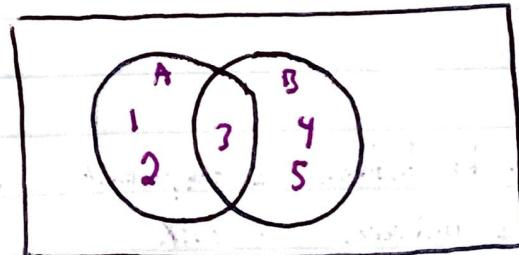
Given Sets A and B, the union of A and B is:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Union is whether the element is in A or B, can be in both or just one

$$A \cup B = \{1, 2, 3, 4, 5\}$$

↑  
take all elements and put them  
in a set!



### DEFINITION 1

Let A and B be sets. The *union* of the sets A and B, denoted by  $A \cup B$ , is the set that contains those elements that are either in A or in B, or in both.

### DEFINITION 2

Let A and B be sets. The *intersection* of the sets A and B, denoted by  $A \cap B$ , is the set containing those elements in both A and B.

### DEFINITION 3

Two sets are called *disjoint* if their intersection is the empty set.

## Difference

Given the sets  $A$  and  $B$ , the difference,  $A - B$ , is defined as

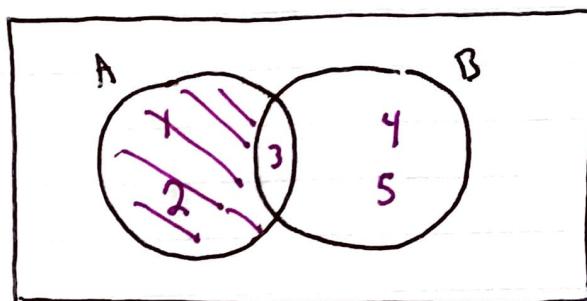
[Set Operations Video](#)

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

or  $A \setminus B$   
or  $A \cap \bar{B}$

$$A - B = \{1, 2\}$$

You take everything in A and subtract whatever is in B.



## Exercises

$$A = \{1, 3, 5, 7, 9\} \quad B = \{4, 8, 12, 16\} \quad C = \{1, 4, 9, 16\}$$

$$A \cup B = \{1, 3, 5, 7, 9, 4, 8, 12, 16\}$$

$$C \cap B = \{4, 16\}$$

$$C - B = \{1, \cancel{4}, 9, \cancel{16}\} = \{1, 9\} \quad \leftarrow \text{difference } B$$

$$\emptyset \cap B = \emptyset$$

$\leftarrow$  b/c there's no elements in the empty set  $\emptyset$

### DEFINITION 4

Let  $A$  and  $B$  be sets. The *difference* of  $A$  and  $B$ , denoted by  $A - B$ , is the set containing those elements that are in  $A$  but not in  $B$ . The difference of  $A$  and  $B$  is also called the *complement of  $B$  with respect to  $A$* .

## Useful definitions

**Union**  $A \cup B$ : All elements that are either in  $A$  OR in  $B$

**Intersection**  $A \cap B$ : All elements that are both in  $A$  AND in  $B$ .

**Difference**  $A - B$ : All elements in  $A$  that are NOT in  $B$  (complement of  $B$  with respect to  $A$ ).

**Complement**  $\bar{A}$ : All elements in the universal set  $U$  NOT in  $A$ .

**Negation**  $\neg p$ : not  $p$

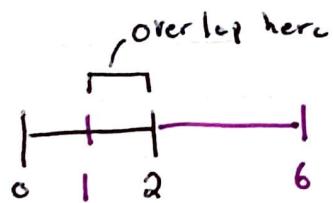
**Disjunction**  $p \vee q$ :  $p$  or  $q$

**Conjunction**  $p \wedge q$ :  $p$  and  $q$

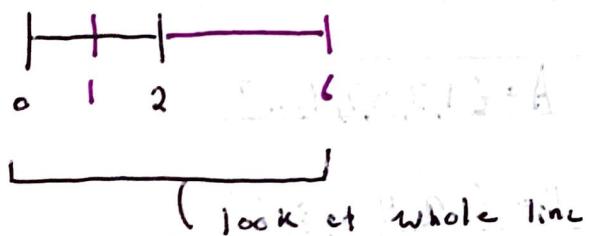
## Set Operations Examples

If  $A = [0, 2]$  and  $B = [1, 6]$ , determine the following where  $\mathbb{U} = \mathbb{R}$

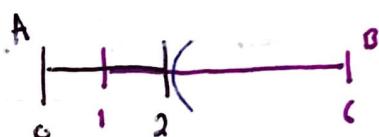
- $A \cap B = [1, 2]$



- $A \cup B = \text{[0, 6]}$



- $B - A = (2, 6)$  ↗ open interval

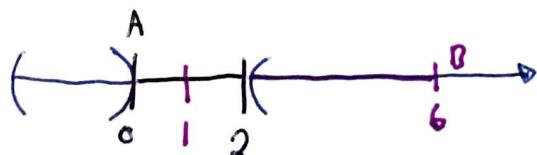


- $\bar{A} = (-\infty, 0) \cup (2, \infty)$

{

everything in the universe - A

here to use  
union to denote  
this set.



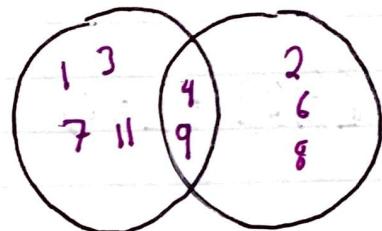
Find sets A and B where:

$$A - B = \{1, 3, 7, 11\}$$

$$B - A = \{2, 6, 8\}$$

$$B \cap A = \{4, 9\}$$

A      B



$$A = \{1, 3, 4, 7, 9, 11\}$$

$$B = \{2, 4, 6, 8, 9\}$$

Prov: If  $A \subseteq B$  and  $C \subseteq D$ , then:

(i)  $A \cap C \subseteq B \cap D$

Let  $x \in A \cap C$   
↳  $x \in A$  and  $x \in C$   
     $x \in B$  and  $x \in D$   
     $x \in B \cap D$

(ii)  $A \cup C \subseteq B \cup D$

Let  $x \in A \cup C$   
↳  $x \in A$  or  $x \in C$   
     $x \in B$  or  $x \in D$   
     $x \in B \cup D$

Exercise : Let  $A = \{a, b\}$  and  $B = \{c, d\}$

$$A \times B = \{(a, c), (a, d), (b, c), (b, d)\}$$

$$B^2 = B \times B = \{(c, c), (c, d), (d, c), (d, d)\}$$

$$\underset{\text{empty set}}{\emptyset} \times A = \emptyset \quad \leftarrow \text{you get the empty set back}$$

$$|\emptyset \times A| = |\emptyset| \cdot |A| = 0 \cdot 2 = 0$$

more Questions:

if  $|B|=m$  and  $|A|=n$  then find... What is the size of these

$$|A \times B| = n \cdot m$$

$$|A^2| = n \cdot n = n^2$$

$$|B^{32} \times A^{19}| = m^{32} n^{19}$$

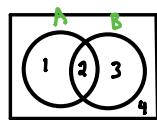
## Set Identities

## Section 2.2

TABLE 1 Set Identities.

Identity	Name
$A \cap U = A$	Identity laws
$A \cup \emptyset = A$	
$A \cup U = U$	Domination laws
$A \cap \emptyset = \emptyset$	
$A \cup A = A$	Idempotent laws
$A \cap A = A$	
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws Can take in either order
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

## De Morgan's Law Examples:



$$\begin{aligned} A &= \{1, 2\} \\ B &= \{2, 3\} \\ U &= \{1, 2, 3, 4\} \end{aligned}$$

$$A \cup B = \{1, 2, 3\}$$

- We want the Complement, which is anything not included: which is 4

$$\begin{aligned} \overline{A} &= \{3, 4\} \\ \overline{B} &= \{1, 4\} \end{aligned}$$

$$\overline{A \cup B} = \overline{\overline{A} \cap \overline{B}}$$

= 4

Union

## Proving Set Identities

### Section 2.2

#### Methods of Identity Proof

- 1) Prove each set in the identity is a subset of the other
- 2) Use propositional logic ~ 2 column proof
- 3) Use a membership table showing the same combinations of sets do or don't belong to the identity.

if  $A = B$   
 $A \subseteq B$   
 $B \subseteq B$

① Prove  $\overline{A \cap B} = \overline{A} \cup \overline{B}$  by showing  $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$  and  $\overline{A} \cup \overline{B} \subseteq A \cap B$

↙ Prove this

<i>re-arrange it differently</i>	$x \in \overline{A \cap B}$	By assumption ~ like our premise, we need to show
	$x \notin A \cap B$	Definition of Complement if element is in both
	$\neg((x \in A) \wedge (x \in B))$	Definition of intersection
	$\neg(x \in A) \vee \neg(x \in B)$	De Morgan's for Prop. Logic
	$x \notin A \vee x \notin B$	Definition of Negation
	$x \in \overline{A} \vee x \in \overline{B}$	Definition of Complement
<i>Proved</i> ↗	$x \in \overline{A} \cup \overline{B}$	Definition of Union
		↙ Proving this
	$x \in \overline{A} \cup \overline{B}$	By Assumption ~ $x$ belongs, by assumption
	$(x \in \overline{A}) \vee (x \in \overline{B})$	Def of Union
	$x \notin A \vee x \notin B$	Def of Complement
	$\neg(x \in A) \vee \neg(x \in B)$	Def of Negation
	$\neg((x \in A) \wedge (x \in B))$	De Morgan's Law Prop. Logic
	$\neg(x \in A \cap B)$	Def of Intersection
	$x \in \overline{A \cap B}$	Def of Complement

∴  $\overline{A \cap B} = \overline{A} \cup \overline{B}$  ~ We proved both directions

Prove  $A \cap B = \bar{A} \cup \bar{B}$  using set Builder notation and propositional logic.

$$\begin{aligned}
 \overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{Def. of Complement} \\
 &= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{Def. of } \notin \\
 &= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{Def. of intersection} \\
 &= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{De Morgan Prop Logic} \\
 &= \{x \mid x \notin A \vee x \notin B\} && \text{Definition of } \notin \\
 &= \{x \mid x \in \bar{A} \vee x \in \bar{B}\} && \text{Definition of Complement} \\
 &= \{x \mid x \in \bar{A} \cup \bar{B}\} && \text{Definition of a Union} \\
 \\ 
 &= \bar{A} \cup \bar{B} && \text{Proven its True}
 \end{aligned}$$

Prove  $A \cap B = \bar{A} \cup \bar{B}$  using a membership table.

A	B	$A \cap B$	$\overline{A \cap B}$	$\bar{A}$	$\bar{B}$	$\bar{A} \cup \bar{B}$
1	1	1	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	0	1	1	1	1

### Generalized Union and Intersection

Union of several items  $\rightarrow \bigcup_{i=1}^n A_i = A_1 \cup A_2 \cup \dots \cup A_n$

$$\bigcap_{i=1}^n A_i = A_1 \cap A_2 \cap \dots \cap A_n$$

## Section 2.2

### Indexed Sets and Well Ordering Principle

### SET THEORY

Can we shorten  $A_0 \cap (A_1 \cap (A_2 \cap (A_3 \cap (A_4 \cap A_5))))$ ? ~ Yes!  
What about Unions?

Intersection notation →  $\bigcap_{i=0}^n A_i = A_0 \cap A_1 \cap A_2 \cap \dots \cap A_n$   
intersection from  $i=0$  to  $n$

Union notation →  $\bigcup_{i=0}^n A_i = A_0 \cup A_1 \cup \dots \cup A_n$

Similar to  $\sum_{i=0}^n a_i = a_1 + a_2 + \dots + a_n$

Example:  $\bigcap_{i=0}^{n+1} A_i = \left( \bigcap_{i=0}^n A_i \right) \cap A_{n+1}$  Brackets

### Well ordering Principle

Any non empty subset of  $\mathbb{N}$  has a least element.  
Used in the Division Algorithm which will be shown  
when we get to number Theory.

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

$\leftarrow k < j < i$   $k$  is the smallest element.

$$A = \{i, j, k\} \subseteq \mathbb{N}$$

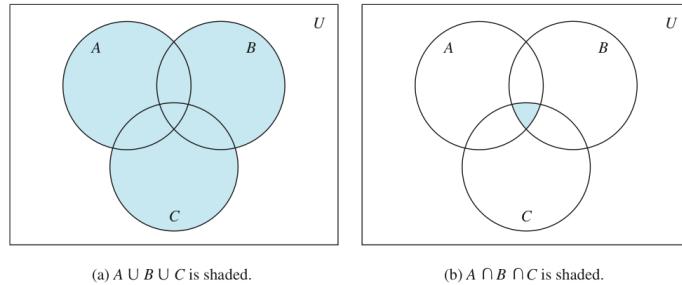
However: ✓ Set of full integers  
 $\mathbb{Z} = \dots -3, -2, -1, 0, 1, 2, 3, \dots$ , we can have negative numbers  
 \* We can't say there's a small element  
 However if you have  $\mathbb{Z}^+ \checkmark$  + is positive integers ✓

## Section 2.2

$$\bigcap_{i=0}^{n+m} A_i = \left( \bigcap_{k=0}^n A_k \right) \cap \left( \bigcap_{j=n+1}^{n+m} A_j \right)$$
$$(A_0 \cap \dots \cap A_n)_n (A_{n+1} \cap \dots \cap A_{n+m})$$

## Generalized Unions and Intersections

Because unions and intersections of sets satisfy associative laws, the sets  $A \cup B \cup C$  and  $A \cap B \cap C$  are well defined; that is, the meaning of this notation is unambiguous when  $A$ ,  $B$ , and  $C$  are sets. That is, we do not have to use parentheses to indicate which operation comes first because  $A \cup (B \cup C) = (A \cup B) \cup C$  and  $A \cap (B \cap C) = (A \cap B) \cap C$ . Note that  $A \cup B \cup C$  contains those elements that are in at least one of the sets  $A$ ,  $B$ , and  $C$ , and that  $A \cap B \cap C$  contains those elements that are in all of  $A$ ,  $B$ , and  $C$ . These combinations of the three sets,  $A$ ,  $B$ , and  $C$ , are shown in Figure 5.



**FIGURE 5** The Union and Intersection of  $A$ ,  $B$ , and  $C$ .

**EXAMPLE 15** Let  $A = \{0, 2, 4, 6, 8\}$ ,  $B = \{0, 1, 2, 3, 4\}$ , and  $C = \{0, 3, 6, 9\}$ . What are  $A \cup B \cup C$  and  $A \cap B \cap C$ ?

~ Contain elements in at least one of  $A$ ,  $B$ , and  $C$ .

$$A \cup B \cup C = \{0, 1, 2, 3, 4, 6, 8, 9\}$$

$$A \cap B \cap C = \{0\}$$

~ Element contained in all 3 sets.

### DEFINITION 6

The *union* of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

We use the notation

$$A_1 \cup A_2 \cup \dots \cup A_n = \bigcup_{i=1}^n A_i$$

to denote the union of the sets  $A_1, A_2, \dots, A_n$ .

**DEFINITION 7**

The *intersection* of a collection of sets is the set that contains those elements that are members of all the sets in the collection.

We use the notation

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^n A_i$$

**EXAMPLE 16** For  $i = 1, 2, \dots$ , let  $A_i = \{i, i + 1, i + 2, \dots\}$ . Then,

$$\bigcup_{i=1}^n A_i = \bigcup_{i=1}^n \{i, i + 1, i + 2, \dots\} = \{1, 2, 3, \dots\},$$

and

$$\bigcap_{i=1}^n A_i = \bigcap_{i=1}^n \{i, i + 1, i + 2, \dots\} = \{n, n + 1, n + 2, \dots\} = A_n.$$

We can extend the notation we have introduced for unions and intersections to other families of sets. In particular, we use the notation

$$A_1 \cup A_2 \cup \cdots \cup A_n \cup \cdots = \bigcup_{i=1}^{\infty} A_i$$

to denote the union of the sets  $A_1, A_2, \dots, A_n, \dots$ . Similarly, the intersection of these sets is denoted by

$$A_1 \cap A_2 \cap \cdots \cap A_n \cap \cdots = \bigcap_{i=1}^{\infty} A_i.$$

More generally, when  $I$  is a set, the notations  $\bigcap_{i \in I} A_i$  and  $\bigcup_{i \in I} A_i$  are used to denote the intersection and union of the sets  $A_i$  for  $i \in I$ , respectively. Note that we have  $\bigcap_{i \in I} A_i = \{x \mid \forall i \in I (x \in A_i)\}$  and  $\bigcup_{i \in I} A_i = \{x \mid \exists i \in I (x \in A_i)\}$ .

**EXAMPLE 17** Suppose that  $A_i = \{1, 2, 3, \dots, i\}$  for  $i = 1, 2, 3, \dots$ . Then,

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \{1, 2, 3, \dots, i\} = \{1, 2, 3, \dots\} = \mathbf{Z}^+$$

and

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} \{1, 2, 3, \dots, i\} = \{1\}.$$

To see that the union of these sets is the set of positive integers, note that every positive integer  $n$  is in at least one of the sets, because it belongs to  $A_n = \{1, 2, \dots, n\}$ , and every element of the sets in the union is a positive integer. To see that the intersection of these sets is the set  $\{1\}$ , note that the only element that belongs to all the sets  $A_1, A_2, \dots$  is 1. To see this note that  $A_1 = \{1\}$  and  $1 \in A_i$  for  $i = 1, 2, \dots$ .

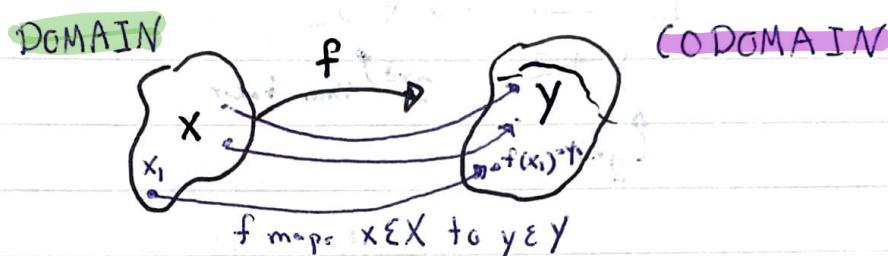
## Section 2.3

### Functions Notes

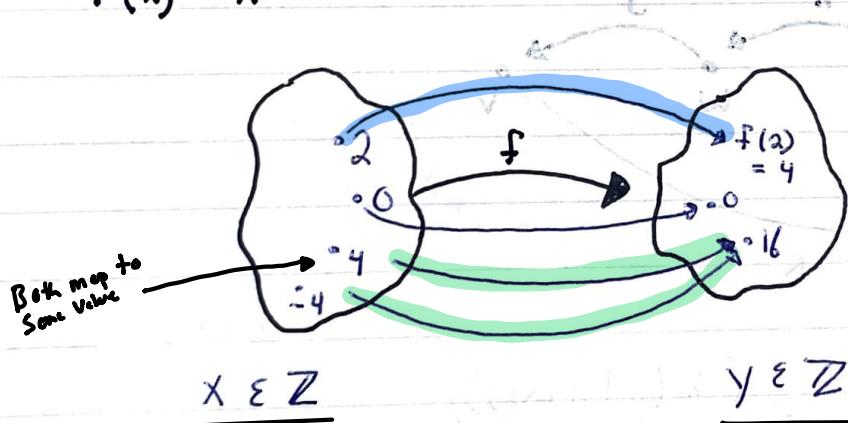
$f : \text{domain} \rightarrow \text{codomain}$   
 $x \in X \xrightarrow{\text{maps}} y \in Y$   
 $x \text{ maps to } y \text{ if there exists } y \in Y \text{ such that } f(x) = y$   
 - range  
 - image

$x \in X \quad y \in Y$   
 $f(x) = y$   
 range  
 image

### Big Picture



$$f(x) = x^2 \quad \mathbb{Z}$$



$f : \mathbb{Z} \rightarrow \mathbb{Z}^+$   
 maps integer into a positive integer  
 codomain:  $\mathbb{Z}^+$  ~ Since  $x^2$  will always return a + #  
 Range:  $\{x^2 \mid \sqrt{x} \text{ is an integer}\}$

### Characteristic Function

$$f(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}$$

$$A = \{a, b, c, \dots\}$$

$$f(1) = 1 \quad f : \mathbb{Z} \rightarrow \{0, 1\}$$

$$f(2) = 0 \quad \text{checks if # is odd}$$

$$f(3) = 1$$

$$f(4) = 0$$

Kind of like truth tables

$P(x)$	$X$
1	1
0	2
1	3
0	4
1	5

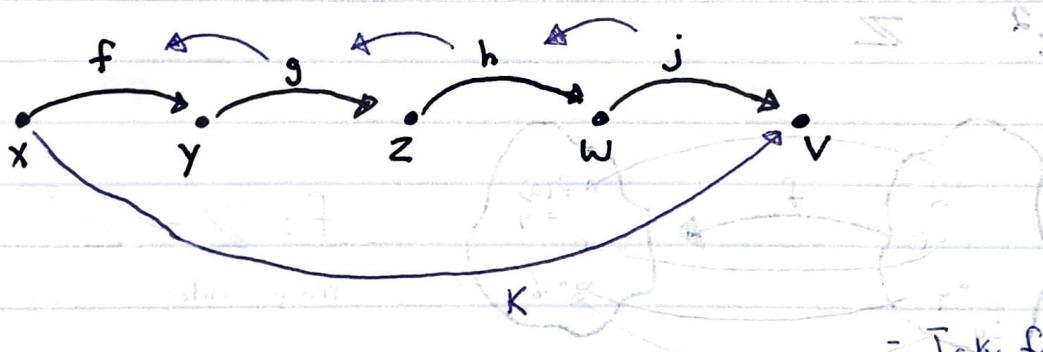
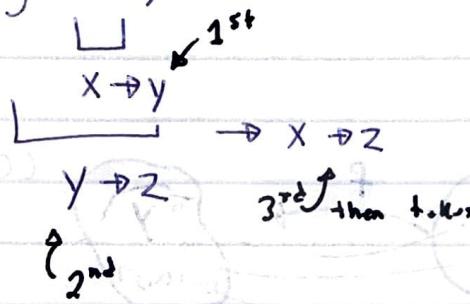
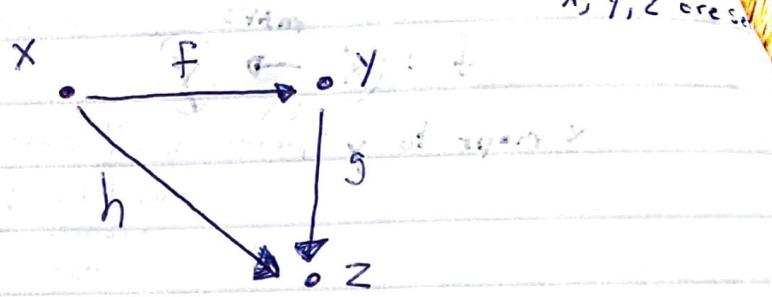
## Composite functions

$$f: X \rightarrow Y$$

$$g: Y \rightarrow Z$$

$$h: X \rightarrow Z$$

$$h: ? = g \circ f = g(f(x))$$



- Take functions in reverse

$$K = j \circ (h \circ (g \circ f)) = j(h(g(f(x))))$$

$$= j(h(g(y)))$$

$$= j(h(z))$$

$$j(w) = j(w)$$

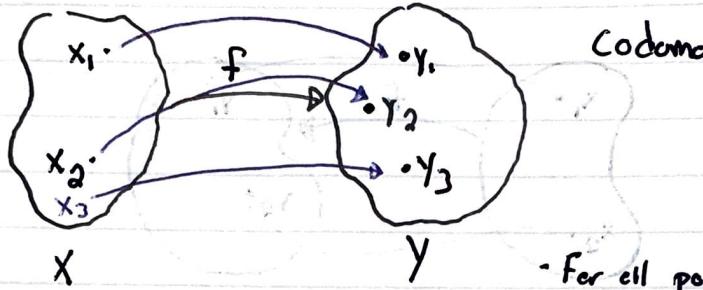
$$v = v$$

$$\begin{array}{c} x \\ | \\ y \\ | \\ z \\ | \\ w \\ | \\ v \end{array}$$

## Surjective (onto)

Let  $f: X \rightarrow Y$

$f$  is surjective iff  $\forall y \in Y, \exists x \in X$  such that  $f(x) = y$



Codomain = Range

- For all possible values of  $y$  there  
is some  $x$  that gives you  $y$

Show  $f(x) = 5x + 2$  is surjective for  $\forall x \in \mathbb{R}$ . What about  $\forall x \in \mathbb{Z}$ ?  
↑ real numbers

$$y = f(x)$$

if we pick  $y=0 \quad x = -2/5$

$$y=1 \quad x = \frac{-1}{5}$$

$$f: \mathbb{Z} \rightarrow \mathbb{Z}$$

$$y=7 \rightarrow x=1$$

$$y=\underline{5} \rightarrow x=\underline{3/5}$$

$\mathbb{Z}$  is not in the integers

Not in the range

Let's say we have  $f: \mathbb{R} \rightarrow \mathbb{Z}$

$x$  any real #'s

$y$  only integers

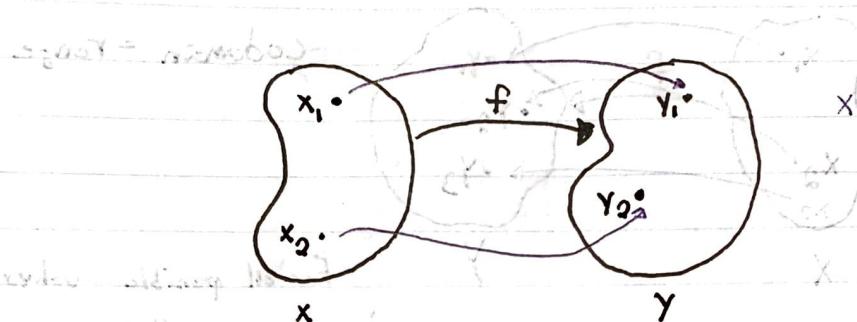
Yes so  $\mathbb{Z}$  surjective?

# Injective, Surjective, Bijective Functions

## Injective (one-to-one)

if  $x_1 \neq x_2$  then  $f(x_1) \neq f(x_2)$

function from  $X \rightarrow Y$   
 $f: X \rightarrow Y$



Prove Equation is Injective:

Show  $f(x) = 3x - 2$  is injective

Implies

$$\begin{aligned} x_1 \neq x_2 &\Rightarrow f(x_1) \neq f(x_2) \\ f(x_1) = f(x_2) &\Rightarrow x_1 = x_2 \end{aligned}$$

$$f(x_1) = f(x_2) \sim \text{Assume}$$

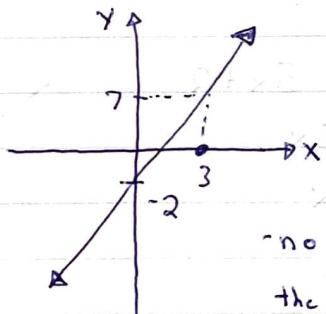
$$3x_1 - 2 = 3x_2 - 2$$

$$3x_1 = 3x_2$$

$$x_1 = x_2$$

proven it's injective  
or one-to-one

Injective



is easier to prove  
the contrapositive

no other x that gives  
the same y

Is  $f(x) = x^2$  injective?

$$f(x_1) = f(x_2)$$

take the  
square root

$$\pm x_1 = \pm x_2$$

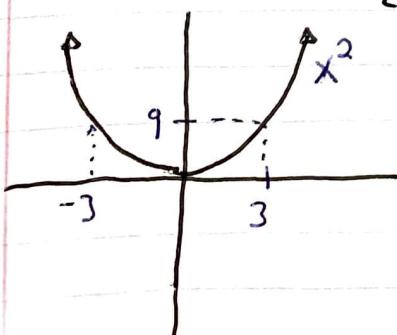
does it equal? No!

Why? b/c of the  $\pm$

$$+ x_1 = -x_2$$

Example  $3 \cancel{x} - 3$

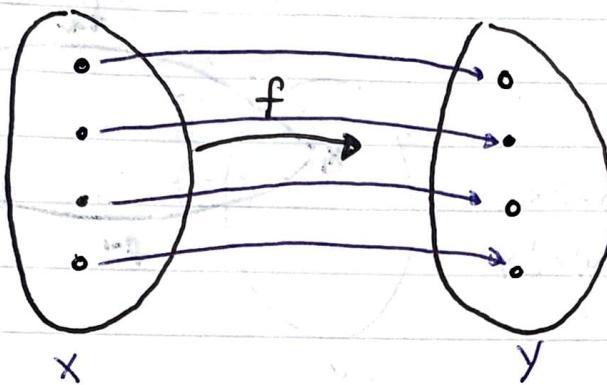
not injective!



- no other x-value should give you the y-value  
- but -3 and 3 both give the y-value of 9

**Bijective = Injective AND Surjective**

for  $f: X \rightarrow Y$ , Each  $x \in X$  maps to exactly one unique  $y \in Y$



What does this say about  $|X|$  and  $|Y|$ ?

$$|X| = |Y|$$

Size of domain is equal to the codomain

### Inverse

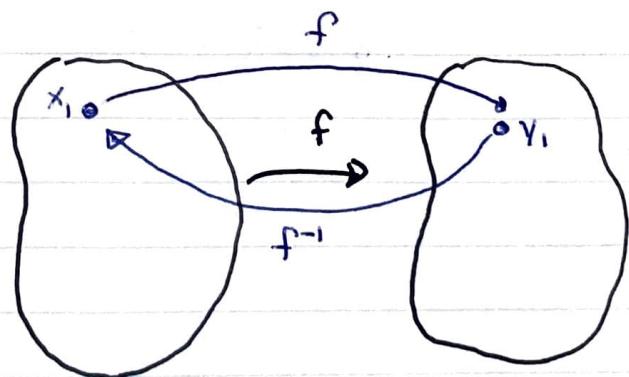
Given  $f: X \rightarrow Y$ , we define

Inverse the inverse of

$$f^{-1}: Y \rightarrow X$$

$$f(x) = y$$

$$f^{-1}(y) = x$$



Find the inverse of  $f(x) = 5x + 2$

↓  
implies

it's

bijective

→ prove

injective  
+  
surjective

When you prove a function is surjective, you always set the inverse

Quation

Wiederholung: Invertible Funktionen

$$f(x) = y \quad f(f^{-1}(y)) = f(x) = y \quad \text{für } x \in \text{dom } f \quad \text{und } y \in \text{ran } f$$

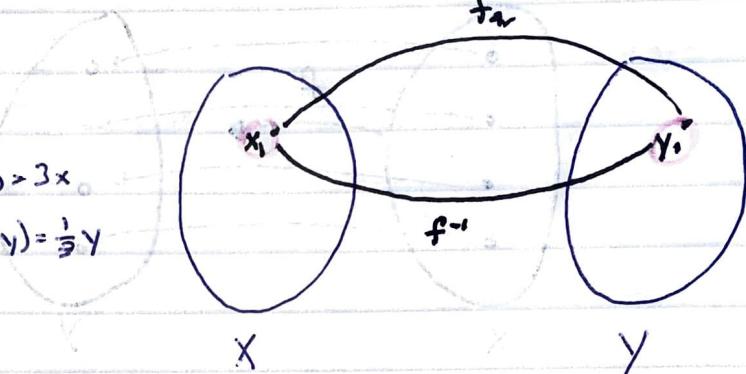
$f'(y) = x$        $\frac{\text{Interval}}{\text{"cycle"}}$

Kinda like:

$$f \begin{matrix} \nearrow \\ \uparrow \\ \searrow \end{matrix} 3 \cdot \frac{1}{3} x = x$$

$$f(x) = 3x$$

$$f^{-1}(y) = \frac{1}{3}y$$



1<sup>st</sup> take  $y_1$  and map it to  $x_1$ ,  
2<sup>nd</sup> take  $f^{-1}$  back and get to  $y_1$

numbered after the image of numbers in set B

discrete

continuous  $y = x + b$  mixed

no arrows but discrete

$y = x + b$

discrete

$y = \sin x$  function has a range of  $y$

open

closed

(1)

## Function Examples

(A)  $\{f: A \rightarrow A\}$  where  $A \subseteq \mathbb{R}$ . Is  $f: A \rightarrow A$  injective? What is the range?

a.)  $f: \mathbb{R} \rightarrow \mathbb{R}$  where  $f(x) = e^x + 8$

$$\{e^x\} = \{f(x)\}$$

$$e^x = e^y$$

$$\ln e^x = \ln e^y \quad \text{YES!}$$

$$x = y$$

Therefore  $f(x) = f(y) \Rightarrow x = y$

$$\{e^x\} = \{f(x)\}$$

$\uparrow$  Prove

range of function  $\{e^x\} = \mathbb{R}$

$$(0, +\infty)$$

$$\{e^x\} = \{f(x)\}$$

$$\{e^x\} = \{f(x)\}$$

could be factored

b.)  $g: \mathbb{Z} \rightarrow \mathbb{Z}$  where  $g(x) = x^3 - x$

$$\begin{cases} g(0) = 0^3 - 0 = 0 \\ g(1) = 1^3 - 1 = 0 \end{cases}$$

We have two elements in the domain mapping to each other, therefore not injective

Range?

$$\{n^3 - n \mid n \in \mathbb{Z}\}$$

(Give an Example of a function  $f: A \rightarrow B$  and  $A_1, A_2 \subseteq A$  where  $f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2)$ )

Example set given by Srinivasan notes part 2 i  
intersection

$$A = \{1, 2\}$$

$$B = \{3\}$$

$$A_1 = \{1\}$$

$$f(1) = 3$$

$$A_2 = \{2\}$$

$$f(2) = 3$$

$$f(A_1) = \{3\}$$

$$f(A_2) = \{3\}$$

$$f(A_1 \cap A_2) = f(\emptyset) = \emptyset$$

(  
empty set

When you put the empty set  
into a function you get the  
empty set back.

Not equal to

$$f(A_1) \cap f(A_2) = \{3\} \cap \{3\} = \{3\}$$

To prove work with simple sets

## Surjective Function Examples

Show that  $f: \mathbb{R} \rightarrow \mathbb{R}$  where  $f(x) = x^3$  is surjective

$$y = x^3$$

$$\sqrt[3]{y} = x$$

all  $y$ 's are going to be produced from values of  $x$

$$\sqrt[3]{y} \in \mathbb{R}$$

is the cube root of  $y$  in the real H's?

Yes, Surjective

Yes it is, we can put in any  $y$  in and get an  $x$  back

Let  $|A| = 4$  and  $|B| = 6$

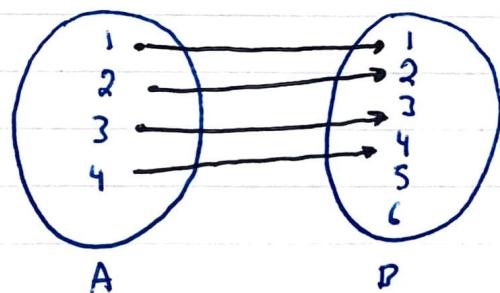
i) How many functions  $f: A \rightarrow B$  are there?

$$6 \times 6 \times 6 \times 6 = 6^4$$

$$\stackrel{\text{# of options if}}{\Rightarrow} \text{can map to.} = |B|^{|A|}$$

ii) How many are injective?

$$4! \binom{6}{4} = \frac{6!}{2! 4!} = \frac{6!}{2!}$$



Everything in our domain must map to something in our codomain.

iii) How many are surjective?

0

$\hookrightarrow$  means the codomain is equal to the range, we can't do that we can only map to 4 things.

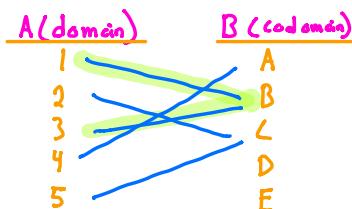
## Introduction to functions

Section 23.1

A function  $f$  from  $A$  to  $B$ , denoted  $f: A \rightarrow B$

assigns each element of  $A$  to exactly one element of  $B$ .

Functions may also be called mappings or transformations.



$f(1) = B$  then we can say

Image:  $B$  is the image of  $1$  under  $f$   
 $\{A, B, C\}$  image of  $f$

$\{A, B, C\}$  image of  $f$

Preimage:  $\{1, 3\}$  the preimage of  $B$  under  $f$

range:  $\{A, B, C\}$

↑ All values mapped in codomain.

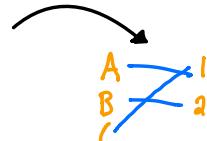
## Representing Functions

1. Explicit Statement  $f(A) = 1, f(B) = 2, f(C) = 1$

2. Formula  $f(x) = x^2 + 1$

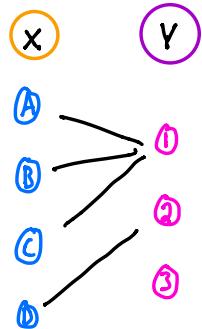
3. Computer Program

4. A relation  $A \times B$

$$\begin{array}{ll} (1, 2) & (1, 7) \\ (2, 3) & (2, 6) \\ (4, 5) & \end{array}$$


Answer the following for  $f: X \rightarrow Y$

- domain  $X$
- Co-domain  $Y$
- range  $\{1, 2, 3\}$
- Preimage of 2  $\{A, D\}$
- Images of A  $\{1, 3\}$
- $f(D) = 2$



## One-One and Onto Functions

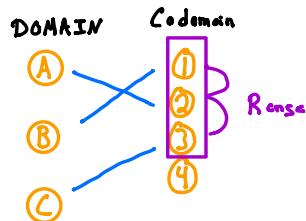
Section 23.1

### Injective (one-to-one) Functions

- Each Value in the range corresponding to exactly one element in the domain.

$$\boxed{\forall a \forall b ((a \neq b) \rightarrow (f(a) \neq f(b)))} \quad \text{OR} \quad \boxed{\forall a \forall b ((f(a) = f(b)) \rightarrow (a = b))}$$

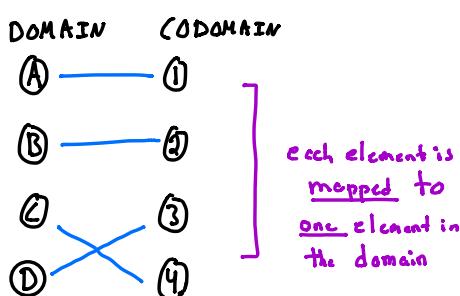
Contrapositive



### Surjective (onto) Functions

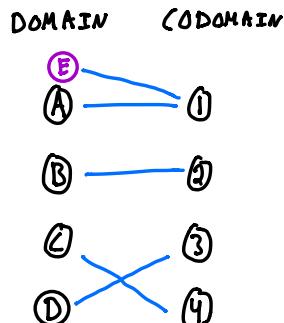
Every element in the codomain maps to at least one element in the domain

$$\forall y \exists x (f(x) = y)$$



Question:

What if we add another Domain Value (E) that is mapped to a same codomain?



It's still Surjective but not One-to-One anymore.

### Bijective (one-to-one correspondence) Functions

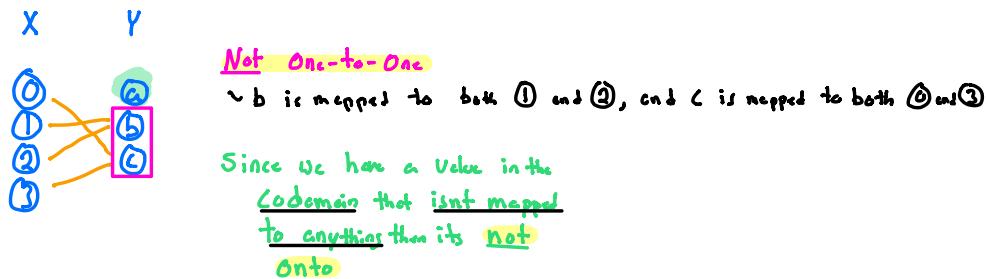
Functions that are both one-to-one and onto, or both surjective and injective.



### Practice:

Let  $f$  be a function from  $X = \{0, 1, 2, 3\}$  to  $y = \{a, b, c\}$  defined by  $f(0)=c$ ,  $f(1)=b$ ,  $f(2)=b$  and  $f(3)=c$ . Is  $f: X \rightarrow Y$  either one-to-one or onto?

mapping:



Practice:

Is the function  $f(x) = x^2$  from the set of integers to the set of integers either one-to-one or onto?

$$f: \mathbb{Z} \rightarrow \mathbb{Z}$$

One-to-one

1 way to show one-to-one if  $f(a) = f(b)$  then  $a = b$

$$f(a) = 4 \quad f(b) = 4 \quad a = 2 \quad b = -2$$

Not one-to-one

$$f(a) = f(b) \quad \text{but} \quad a \neq b$$

Onto

Is the codomain ever going to be negative?

$$f(a) \neq -1$$

~ Won't be in the range

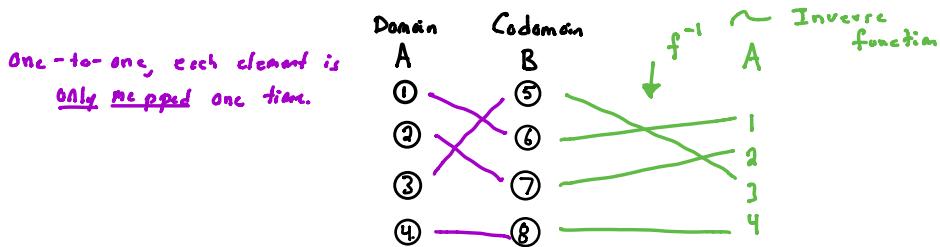
Not onto

## Inverse Functions and Composition Functions

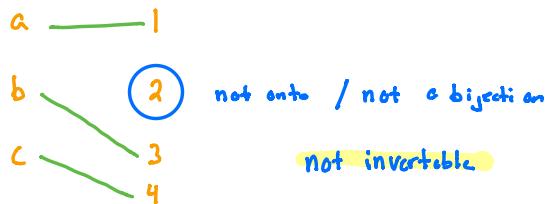
Section 23.1

### Inverse Functions

Let  $f$  be a bijection (both one-to-one and onto) from set  $A$  to  $B$ . The inverse of  $f$ ,  $f^{-1}$ , is the function from  $B$  to  $A$  defined as  $f^{-1}(y) = x$  iff  $f(x) = y$ .



If  $f$  is the function from  $\{a, b, c\}$  to  $\{1, 2, 3, 4\}$  such that  $f(a)=1, f(b)=3$  and  $f(c)=4$ , is  $f$  invertible? If so, what is the inverse?



Let  $f: \mathbb{Z} \rightarrow \mathbb{Z}$  such that  $f(x) = x+3$ . Is  $f$  invertible? If so, what is the inverse?

One-to-one?  $f(a) = f(b)$   $\frac{a}{7} = \frac{b}{4}$   $\sim$  same value

$$\begin{array}{ll} f(a) = 7 & f(b) = 7 \\ a+3=7 & b+3=7 \\ a=4 & b=4 \end{array} \quad \text{Yes}$$

Invertible

Onto?

$$\begin{array}{ccc} & \downarrow \text{Codomain} & \\ 4 & \rightarrow & 7 \\ & & \text{Yes} \\ 5 & \rightarrow & 8 \end{array} \quad \text{Everything is going to get mapped to}$$

Inverse:  $f(x) = x + 3$

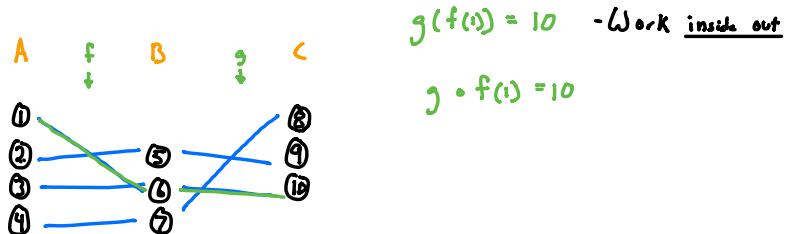
$$\begin{array}{c} \downarrow \\ x = f^{-1}(x) + 3 \\ x - 3 = f^{-1}(x) \end{array}$$

$f^{-1}(x) = x - 3$

$\sim$  inverse has to undo

### Composition of Functions

Let  $f: A \rightarrow B$  and  $g: B \rightarrow C$ . The composition of  $g$  with  $f$ ,  $g \circ f$ , is the function from  $A$  to  $C$  defined by  $g \circ f(x) = g(f(x))$ .



### Examples:

If  $f(x) = x + 3$  and  $g(x) = x^2 - 2$  find:

*Other way to solve*

$$f \circ g(1) = f(g(1)) = f(-1) = -1 + 3 = 2$$

$$f \circ g(x) = f(g(x)) = (x^2 - 2) + 3 = x^2 + 1$$

$$f \circ g(1) = 1^2 + 1 = 1 + 1 = 2 \quad \checkmark \quad \sim \text{checked}$$

$$g \circ f(1) = g(1+3) = g(4) = 4^2 - 2 = 14$$

$$g \circ f(x) = g(f(x)) = (x+3)^2 - 2 = x^2 + 6x + 9 - 2 = x^2 + 6x + 7$$

$$g \circ f(1) = 1^2 + 6(1) + 7 = 14 \quad \checkmark \quad \sim \text{checked}$$

## Useful Functions to Know

Section 23.1

Floor Function  $f(x) = \lfloor x \rfloor$

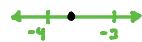
Largest integer less than or equal to  $x$ .

Ceiling Function  $f(x) = \lceil x \rceil$

Smallest integer greater than or equal to  $x$ .

$$\lceil 2.2 \rceil = 3$$

$$\lceil -3.7 \rceil = -3$$



$$\lfloor 2.2 \rfloor = 2$$

$$\lfloor -3.7 \rfloor = -4$$

## Factorial Function

$f: \mathbb{N} \rightarrow \mathbb{Z}^*$  denoted by  $f(n) = n!$  is the product of the first  $n$  positive integers when  $n$  is a non-negative integer.

$$f(n) = n \cdot (n-1) \cdot (n-2) \cdots \cdots 3 \cdot 2 \cdot 1$$

$$f(4) = 4 \cdot 3 \cdot 2 \cdot 1$$

Stirling formula:  $n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$

Question:

Let  $f(x) = \lfloor \frac{x^3}{2} \rfloor$ . Find  $f(s)$  if  $S = \{0, 1, 2, 3\}$ .

$$f(0) = \lfloor \frac{0^3}{2} \rfloor = \lfloor 0 \rfloor = 0$$

$$f(1) = \lfloor \frac{1^3}{2} \rfloor = \lfloor \frac{1}{2} \rfloor = 0$$

$$f(2) = \lfloor \frac{2^3}{2} \rfloor = \lfloor 2 \rfloor = 2$$

$$f(3) = \lfloor \frac{3^3}{2} \rfloor = \lfloor \frac{27}{2} \rfloor = 4$$

## Introduction to Sequences

### Section 2.4

#### Sequences

An ordered list of elements created by a function mapping the integers to a set  $S$ .

Notation:  $a_n$  represents the  $n^{\text{th}}$  term

Ex. If  $a_n = 2n$ , find  $\{a_n\} = \{0, 2, 4, 6, \dots\}$

$$\begin{aligned}a_0 &= 2(0) = 0 \\a_1 &= 2(1) = 2 \\a_2 &= 2(2) = 4 \\a_3 &= 2(3) = 6\end{aligned}$$

Pattern would continue.

Arithmetic Sequence  $\rightarrow a_n = a + dn$  \* ~ How you write an Arithmetic Sequence Explicitly

A sequence formed by adding the initial term,  $a$ , and the product of the common difference,  $d$ , and the term number,  $n$ .

$$\{a_n\} = [a] + a + d, a + 2d, a + 3d, \dots a + nd$$

#### Questions:

1. Let  $a = 5$  and  $d = 2$ . Find the first 5 terms of  $\{a_n\}$

Written Explicitly  $\rightarrow a_n = 5 + 2n$

$\{5, 7, 9, 11, 13, \dots\}$   $\xrightarrow{+2 \quad +2}$  Written Recursively  
 $a_0, a_1, a_2, a_3, a_4$   $a_4 = 5 + 2(4)$   
 $a_4 = 13 \checkmark$

2. Find  $a$  and  $d$  for the sequence  $\{7, 4, 1, -2, \dots\}$

$\{7, 4, 1, -2, \dots\}$   
 $a_0, a_1, a_2, a_3$

$a_n = 7 + dn \rightarrow a_n = 7 - 3n$

must find the common difference, which we can see is  $-3$ .

Let's check:  
 $a_3 = 7 - 3(3)$   
 $a_3 = -2 \checkmark$

### Geometric Sequence

$$a_n = ar^n \quad \leftarrow \text{Geometric Sequence Explicitly}$$

A sequence formed by multiplying the initial term  $a$ , by the common ratio  $r$  to the  $n^{\text{th}}$  power,  $r^n$ .

$$\{a_n\} = \underbrace{a, \underbrace{ar, \underbrace{ar^2, \underbrace{ar^3, \dots, ar^n}}_r}_r}_r$$

### Questions:

1. Let  $a=4$  and  $r=3$ . Find the first 5 terms of the geometric progression.

$$a_n = 4(3)^n$$

$$\text{One way to do problem} \Rightarrow \left\{ \begin{array}{c} \overbrace{4}^3, \overbrace{12}^2, \overbrace{36}^2, \overbrace{108}^3, \overbrace{324}^4, \dots \\ a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \end{array} \right\}$$

$$a_3 = 4(3)^3 = 4(27)$$

$$a_3 = 108 \checkmark$$

2. Find  $a$  and  $r$  for  $\{a_n\} = \{3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \dots\}$

$$a_n = 3r^n$$

$$\{a_n\} = \left\{ 3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \dots \right\}$$

$$a_n = 3\left(\frac{1}{2}\right)^n$$

$$\begin{array}{l} a=3 \\ r=\frac{1}{2} \end{array}$$

$$\frac{\frac{3}{2}}{3} = \frac{3}{2} \cdot \frac{1}{2} = \frac{1}{2} = r$$

~ divide by previous term

## Recurrence Relations

### Recurrence Relation

An equation that expresses  $a_n$  in terms of one or more of the previous terms of the sequence. Initial conditions are required to specify terms that precede the first term where the relation takes effect.

Ex. ①  $a_n = a_{n-1} + 2a_{n-2}$  for  $n \geq 2$  where  $\begin{array}{l} a_0 = 2 \\ a_1 = 5 \end{array}$

$$\{2, 5, 9, 19, 37\}$$

When do we start

$$a_2 = a_1 + 2a_0 = 5 + 2(2) = 5 + 4 = 9$$

$$a_3 = a_2 + 2a_1 = 9 + 2(5) = 9 + 10 = 19$$

$$a_4 = a_3 + 2(a_2) = 19 + 2(9) = 19 + 18 = 37$$

## Questions

Let  $a_n$  be the sequence that satisfies the recurrence relation  $a_n = a_{n-1} + 6$  for  $n \geq 1$

a.) If  $a_0 = 3$  Find  $a_1, a_2, a_3$ .

$$\begin{aligned} a_1 &= a_0 + 6 = 3 + 6 = 9 \\ a_2 &= a_1 + 6 = 9 + 6 = 15 \\ a_3 &= a_2 + 6 = 15 + 6 = 21 \end{aligned}$$

$$\begin{aligned} a_n &= a + dn \\ a_n &= 3 + 6n \end{aligned}$$

If you wanted to find  $a_{100}$   
 $a_{100} = 3 + 6(100) = 3 + 600$

$$a_{100} = 603$$

b.) If  $a_0 = -7$  Find  $a_1, a_2, a_3$ .

$$\begin{aligned} a_1 &= a_0 + 6 = -7 + 6 = -1 \\ a_2 &= a_1 + 6 = -1 + 6 = 5 \\ a_3 &= a_2 + 6 = 5 + 6 = 11 \end{aligned}$$

$$a_n = -7 + 6n$$

## Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ..

Expressed as a recurrence relation

$$\textcircled{2} \quad \begin{cases} f_0 = 0 \\ f_1 = 1 \end{cases}$$

$$\textcircled{1} \quad f_n = f_{n-1} + f_{n-2} \quad \text{for } n \geq 2$$

## Solving Recurrence Relations

a<sub>100</sub>

Finding a non-recursive formula to calculate  $a_n$  is called solving the recurrence relation.  
The solution is called a closed formula.

One method for solving is called iteration, involving substitution.

Let  $a_n$  be the sequence that satisfies the recurrence relation  $a_n = a_{n-1} + 3$  for  $n \geq 1$  with  $a_0 = 2$ .

$$a_0 = 2$$

$$a_1 = 2 + 3$$

$$a_2 = (2+3) + 3 = 2 + 2(3)$$

$$a_3 = (2+3+3) + 3 = 2 + 3(3)$$

$$a_4 = 2 + 4(3)$$

$$\hookrightarrow a_n = 2 + 3n$$

iteration

$$a_n = 2 + 3n$$

2<sup>nd</sup> Way

Let  $a_n$  be the sequence that satisfies the recurrence relation  $a_n = a_{n-1} + 3$  for  $n \geq 1$  with

$$a_0 = 2.$$

$$\begin{aligned} a_n &= [a_{n-1}] + 3 \quad \stackrel{\text{rewrite}}{\sim} \quad \stackrel{\text{using}}{\curvearrowright} \text{this EQ} \\ &= (a_{n-2} + 3) + 3 = a_{n-2} + 2(3) \\ &= (a_{n-3} + 3) + 2(3) = a_{n-3} + 3(3) \\ &\vdots \\ &= a_{n-n} + n(3) \quad \text{3rd Way} \end{aligned}$$

$$a_n = a_0 + 3n$$

## Summations and Sigma Notation

Section 2.4.3

### Sigma Notation

To express the sum of the terms of the sequence  $a_n = \{a_m, a_{m+1}, \dots, a_n\}$ , we write

$$\sum_{\substack{i=m \\ \text{Lower limit}}}^n a_i = \sum_{i=m}^n a_i = \sum_{m \leq i \leq n} a_i$$

Limits      index

which represents  $a_m + a_{m+1} + a_{m+2} + \dots + a_n$

$$\sum_{i=m}^n a_i = \sum_{i=m}^n a_i = \sum_{m \leq i \leq n} a_i$$

Use sigma notation to express the sum of the first 100 terms of the sequence  $a_i$  where  $a_i = \frac{1}{i}$  for  $i=1, 2, \dots$

$$a_1 = \frac{1}{1}$$

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{100}$$

$$a_2 = \frac{1}{2}$$

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{100}$$

$$a_3 = \frac{1}{3}$$

$$\sum_{i=1}^{100} \frac{1}{i}$$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} +$$

Questions

What is the value of  $\sum_{i=5}^9 i^2$ ?

$$\sum_{i=5}^9 i^2 = 5^2 + 6^2 + 7^2 + 8^2 + 9^2 = 25 + 36 + 49 + 64 + 81 = \boxed{255}$$

What is the value of  $\sum_{i=7}^{10} (-1)^i$ ?

$$\sum_{i=7}^{10} (-1)^i = (-1)^7 + (-1)^8 + (-1)^9 + (-1)^{10} = -1 + 1 - 1 + 1 = \boxed{0}$$

## Summation Properties and Formula

### Section 2.4.4

#### Basic Properties

$$\sum_{k=1}^n c a_k = c \sum_{k=1}^n a_k$$

↙ Constant      ↗ move to outside

$$* \sum_{k=1}^n c = \underbrace{c + c + \dots + c}_n \sim \text{if you have just a constant}$$

$$\sum_{k=1}^n (a_k \pm b_k) = \sum_{k=1}^n a_k \pm \sum_{k=1}^n b_k$$

$\underbrace{\quad}_{\text{Two different summation}}$

$$\sum_{k=1}^n a_k = \sum_{k=1}^n a_k + \sum_{k=j+1}^n a_k \text{ for } i \stackrel{\text{lower}}{=} j < j+1 < n \text{ and } j \in \mathbb{N} \quad \stackrel{\text{upper}}{\curvearrowleft}$$

\*  $\sum_{k=1}^n a_k = \sum_{k=1+m}^{n+m} a_{k-m}$  for all  $r \in \mathbb{W}$

$\underbrace{\quad}_{c}$  might have to "+ " or "- "

~ Complicated

#### Summation Formula:

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

Practice:

$$\sum_{i=1}^7 i = 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$$

### Summation Formula:

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

### Summation Formula:

$$\sum_{k=0}^n ar^k = \frac{ar^{n+1} - a}{r-1}$$

~ Geometric Progression / Sequence

$$a_n = 0$$

### Example:

$$\begin{aligned} \sum_{k=0}^7 (2 \cdot 3^k + 5 \cdot 2^k) &= \sum_{k=0}^7 2 \cdot 3^k + \sum_{k=0}^7 5 \cdot 2^k \\ &= \frac{2(3)^8 - 2}{2} + \frac{5(2)^8 - 5}{1} \\ &= \frac{13122 - 2}{2} + 1275 \\ &= 6560 + 1275 = \boxed{7835} \end{aligned}$$

### Useful Summation Formulae:

$$\sum_{k=0}^n ar^k = \frac{ar^{n+1} - a}{r-1}$$

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}$$

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

$$\sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4}$$

## Matrices and Matrix Operations

### Section 2.6

#### Matrices

A rectangular array of numbers. A matrix with  $m$  rows and  $n$  columns is called an  $m \times n$  matrix.

$$M_{3 \times 2} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{32} \end{bmatrix}$$

- 3 rows  
- 2 Columns

element address  $m_{11}$  1<sup>st</sup> row 1<sup>st</sup> column

3<sup>rd</sup> row 2<sup>nd</sup> column

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 4 & 5 \end{bmatrix}$$

dimensions  $2 \times 3$   
 $n_{23} = 5$

#### Matrix Addition (and Subtraction)

Matrices of the same size can be added by finding the sum of elements of the same address.

$$A_{2 \times 2} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$B_{2 \times 2} = \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}$$

~~$C_{2 \times 2} = \begin{bmatrix} 2 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$~~

- Can't add/subtract

$$A + B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix} = \boxed{\begin{bmatrix} 5 & 7 \\ 9 & 11 \end{bmatrix}}$$

$$A - B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix}$$

## Matrix Multiplication

Matrices M and N can be multiplied if M is a  $m \times k$  matrix and N is a  $k \times n$  matrix. The resulting product matrix will be an  $m \times n$  matrix.

$$A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 3 & -2 \\ 5 & -1 & 0 \end{bmatrix} \quad \begin{matrix} \text{- Can't always} \\ \text{multiply} \end{matrix}$$

$$AB = \begin{matrix} 2 \times 2 & 2 \times 3 \end{matrix} \begin{bmatrix} -1 & 3 \\ 4 & -2 \end{bmatrix} \cdot \begin{bmatrix} 4 & 3 & -2 \\ 5 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -4+15 & -3+3 & 2+0 \\ 16+10 & 12+2 & -8+0 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 2 \\ 6 & 14 & -8 \end{bmatrix}$$

$$\boxed{BA} = \text{Not Possible!}$$

## Examples:

$$\text{Let } A = \begin{bmatrix} 9 & 1 & 7 \\ 4 & -7 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 & 6 \\ 7 & 6 \end{bmatrix}$$

$$A - B = \text{NP} \quad \text{dimensions are not possible}$$

$$\boxed{AB} = \text{NP}$$

$$\boxed{BA} = \begin{bmatrix} -2 & 6 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 9 & 1 & 7 \\ 4 & -7 & 3 \end{bmatrix} = \begin{bmatrix} 6+32 & -2+54 & 2+24 \\ 0+21 & 7+42 & -7+18 \end{bmatrix} = \begin{bmatrix} 32 & 54 & 26 \\ 24 & -25 & 11 \end{bmatrix}$$

$$\boxed{A^2 = NP} \quad \boxed{B^2 = \begin{bmatrix} -2 & 6 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} -2 & 6 \\ 7 & 6 \end{bmatrix} = \begin{bmatrix} 9+36 & 16+36 \\ 49+42 & 56+36 \end{bmatrix} = \begin{bmatrix} 60 & 52 \\ 91 & 92 \end{bmatrix}}$$

$$2B = 2 \begin{bmatrix} -2 & 6 \\ 7 & 6 \end{bmatrix} = \begin{bmatrix} -4 & 12 \\ 14 & 12 \end{bmatrix}$$

## Identity Matrix $A \cdot I = A$

The identity matrix is a square matrix  $I_n = \delta_{ij}$  where  $\delta_{ij} = 1$  if  $i=j$  and  $\delta_{ij} = 0$  if  $i \neq j$ . (First diagonal is 1's and all other elements are 0).

Example: Find  $AI$  where  $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -2 & 7 \\ 0 & 1 & -1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & -2 & 7 \\ 0 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -2 & 7 \\ 0 & 1 & -1 \end{bmatrix}$$

Exact same matrix as we had before

## Transpose of Matrices

The transpose of  $A_{m \times n}$  denoted  $A^t$  is the  $n \times m$  matrix obtained by interchanging the rows and columns of  $A$ . If  $A = A^t$  the  $A$  is symmetric.

$$A_{2 \times 3} = \begin{bmatrix} 4 & 5 & 7 \\ 0 & 1 & -1 \end{bmatrix} \quad A^t = \begin{bmatrix} 4 & 0 \\ 5 & -1 \end{bmatrix} \quad \text{not symmetric}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad A^t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \sim \text{symmetric}$$