Relations

A relation R on a set X is a subset of $X \times X$.
If $(a,b) \in R$, we write xRy
"x is related to y"

$R(x,y)$ $\quad Rxy$ ← Predicate logic

Example:

xGy : x is greater than y. $\quad (x,y) \in \mathbb{Z}$

$(4,3)$ $\quad 4G3$ $\quad G(4,3)$ $\quad$ yes true

$(6,2)$ $\quad 6G2$ $\quad$ yes

$(1,7)$ $\quad 1G7$ $\quad$ × No, False

$(2.1,1)$ $\quad G(2.1,1)$ $\quad$ don't make a claim
↑ not an integer

Reflexive ~ equality is just the relation b/w two things

for all notation → $\forall x \quad xRx$

~ a diagram

x is going to be related to itself

Reflexive used with the equal sign

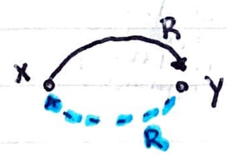
Example: = equal relationship $\quad 4 \in X \quad (4,4) \in R \quad 4=4$

4 in our subset X so 4,4 will always be in R since 4 = 4.

Symmetric

for all elements x and y → $\forall x \forall y, \; xRy \to yRx$

(if x is related to y then y is related to x)



$\neq$ $\quad 4 \not> 3 \to 3 \not> 4$

$xRy \to yRx$

Transitive

$\forall x \forall y \forall z \quad xRy \wedge yRz \to xRz$

↑ and

↓ less than sign

$1<2 \quad 2<3 \to 1<3$

Note: These do not hold for everything, these are properties used to describe certain relationships, they don't enforce relationships.

Examples

$\leq$? yes Refl? $1 \leq 1$ yes $X \leq X$

no Symm? $3 \leq 4 \to 4 \leq 3$? ~ not true

yes Trans? $1 \leq 2, 2 \leq 3 \to 1 \leq 3$? $a \leq b, b \leq c \to a \leq c$?

$X - y \neq 0$? xRy $x - y \neq 0$

No Refl? $xRx \to X - X \neq 0$ $\therefore$ But we know any # minus itself is always zero so this property can't be true

Yes Symm? $4-3 \neq 0$, $3-4 \neq 0$? ~seems like this is the case

But lets try something else:

What if x and y are the same number?

$4-4 \neq 0$, $\to 4-4 \neq 0$?

if the first part is false, symm still holds

Trans?

$X - y \neq 0$ $y - z \neq 0 \to X - z \neq 0$?

What about $X = y$?

Whats so special about the equality?

Creates an Equivalence Class

- Sym
- reflexive
- Transitive

Anything that satisfies these 3 properties is considered an equivalence Class

Refl. $X = X$

Symm $X = y \to y = X$

Trans $X = y \wedge y = z \to X = Z$

Example $X \neq y$

Ref? no $X \neq X$

Sym? yes $X \neq y \to y \neq X$

Trans? no $X \neq y \wedge y \neq z \to x \neq z$?

example $X = 2, y = 1, z = 2$ $2 \neq 1, 1 \neq 2 \to 2 \neq 2$

Claim isnt True

Relations Examples

example 1

Prove: if R_1 and R_2 are reflexive, then $R_1 \cap R_2$ is reflexive.

Intersection

1st step Some arbitrary $(x,x) \in R_1, R_2$

that's in both R1 and R2

2nd step then we know that $(x,x) \in R_1 \cap R_2$

show it's in the intersection

(x,x) is going to be in the intersection of r1 and r2 b/c it's in both and $R_1 \cap R_2$ is reflexive

3rd step: b/c its true for any arbitrary pair we picked the intersection is reflexive

Example 2:

If $\forall x, (x,x) \notin R$ then R is irreflexive ~ X is never related to itself

(i) Give an example where R is irreflexive and transitive, but not symmetric. < or >

$x < y, \; x > y$

Does it satisfy these conditions? ✓

~ x is never going to be less than itself ✓

~ Transitive? $x < y$ and $y < z \to x < z$ yes ✓

~ not symmetric b/c $x < y$ and y can't be less than x ✓

(ii) Show that if R is transitive and symmetric, it cannot be irreflexive.

$xRy \wedge yRz \to xRz$

$xRy \to yRx$

Assume xRy then by symm $yRx \xrightarrow{\text{by trans}} xRx$

if R is transitive and symmetric then its going to be reflexive.

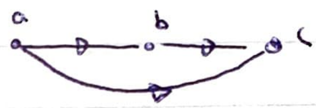
Partial Orders

Equivalence Relations

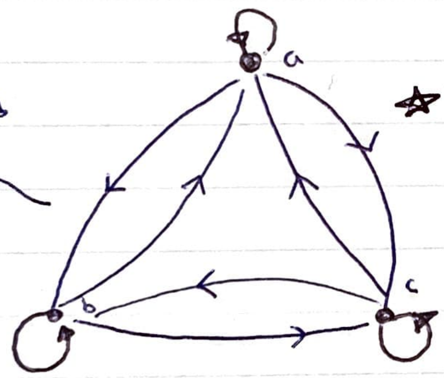
1.) Reflexive $a, \quad aRa$

2.) Symmetric $a,b \quad aRb \rightarrow bRa$

3.) Transitive $aRb \wedge bRc \rightarrow aRc$



a related to b



~ all related to itself

b related to c

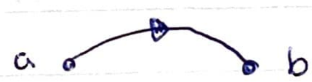
★ $a = b = c$

$| = | = |$ ★

Question

Suppose R is an equivalence relation on some set X.

for each $x \in X$, let $[x] = \{y \mid xRy\}$



Then $[x] = [y]$

or $[x] \cap [y] = \emptyset$

Functions

domain

$f : X \xrightarrow{\text{maps}} Y$ codomain

$x \in X$

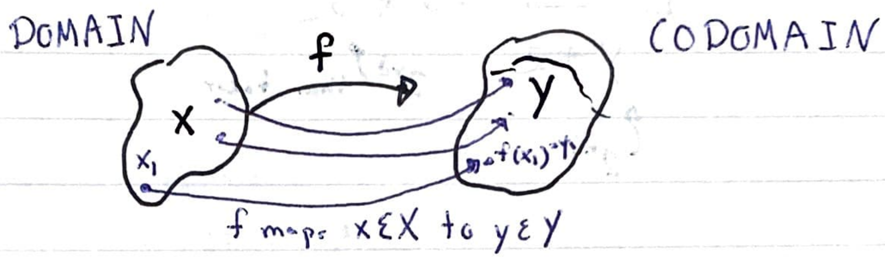
x maps to y • preimage

$y \in Y$ such that $f(x) = y$
- range
- image

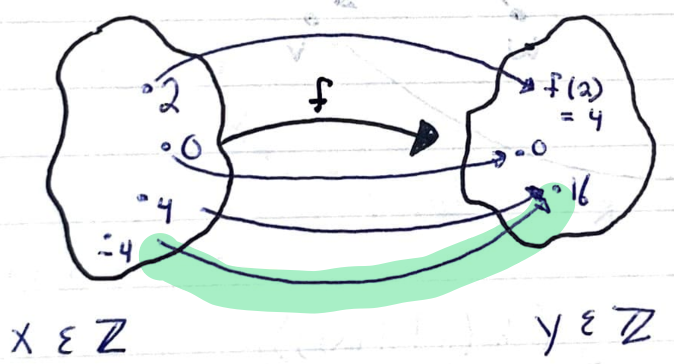
$x \in X \quad y \in Y$

$f(x) = y$ range image

Big Picture



$f(x) = x^2 \quad \mathbb{Z}$



$f : \mathbb{Z} \to \mathbb{Z}^+$

maps integer into a positive integer

Codomain: $\mathbb{Z}^+$

Range: $\{x^2 \mid \sqrt{x} \text{ is an integer}\}$

Characteristic Function

$A = \{a, b, c, \ldots\}$

$$f(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}$$

$f(1) = 1$

$f(2) = 0$

$f(3) = 1$

$f(4) = 0$

$f : \mathbb{Z} \to \{0, 1\}$

Checks if # is odd

Kind of like truth tables!

P(x)	X
1	1
0	2
1	3
0	4
1	5

x, y, z are se

Composite functions

$f: x \to y$

$g: y \to z$

$h: x \to z$

$h: ? = g \circ f = g(f(x))$

$x \to y$ 1st

$\to x \to z$

$y \to z$ 3rd then take

2nd

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[dr, "h"'] & y \arrow[d, "g"] \\ & z \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[rrrr, bend right, "K"'] & y \arrow[r, bend left, "g"] & z \arrow[r, bend left, "h"] & w \arrow[r, bend left, "j"] & v \end{tikzcd}$$

- Take functions in reverse

$K = j \circ (h \circ (g \circ f)) = j(h(g(f(x))))$

$x \to v$

$= j(h(g(y)))$

$= j(h(z))$

$= j(w)$

$= v$

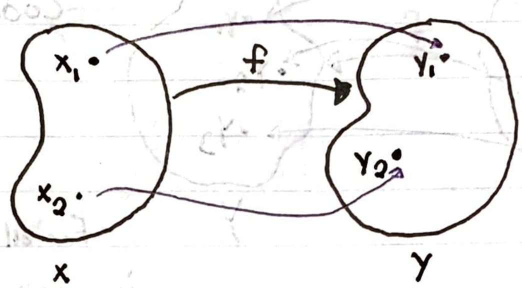
Injective, Surjective, Bijective Functions

Injective (one-to-one)

function from x to y

$f: x \to y$

if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$



x's going to different y values then x_2 value

Prove Equation is Injective:

Implies

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

$\boxed{f(x_1) = f(x_2) \Rightarrow x_1 = x_2}$ easier to prove the contrapositive

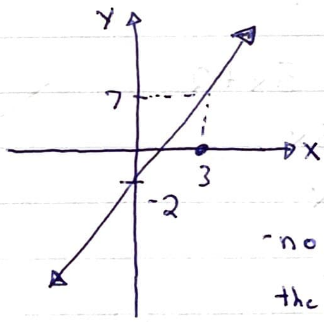
Show $f(x) = 3x - 2$ is injective.

Proof

$$f(x_1) = f(x_2) \quad \sim \text{Assume}$$
$$3x_1 - 2 = 3x_2 - 2$$
$$3x_1 = 3x_2$$
$$x_1 = x_2$$

proven it's injective or one-to-one

Injective



Graphed

- no other x that gives the same y

Is $f(x) = x^2$ injective?

$$f(x_1) = f(x_2)$$

take the square root

$$x_1^2 = x_2^2$$
$$\pm x_1 = \pm x_2$$

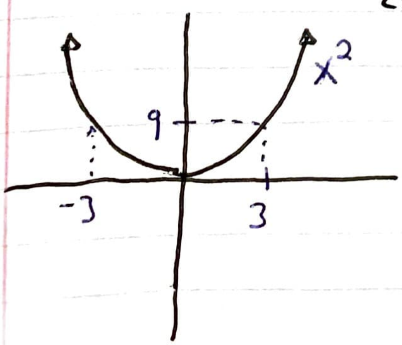
does it equal? No!

Why? b/c of the $\pm$

$$+x_1 = -x_2$$

Example $3 \cancel{=} -3$

not injective!

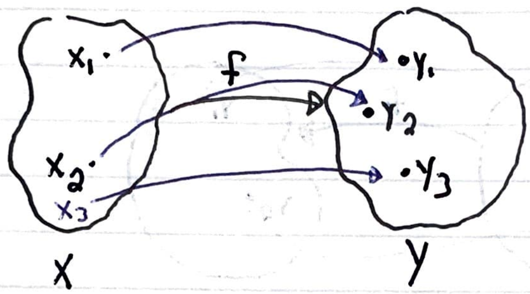


- no other x-value should give you the y-value
- but -3 and 3 both give the y-value of 9

Surjective (onto)

Let $f: X \to Y$

f is surjective iff $\forall y \in Y, \exists x \in X$ such that $f(x) = y$



Codomain = range

- For all possible values of Y there is some X that gives you Y

Show $f(x) = 5x+2$ is surjective for $\forall x \in \mathbb{R}$. What about $\forall x \in \mathbb{Z}$?

real numbers

$y = f(x)$

[Get EQ in terms of X]

$y = 5x+2$

$y-2 = 5x$

$\frac{y-2}{5} = x$

if we pick $y=0$ $x = -2/5$

$y=1$ $x = \frac{-1}{5}$

$f: \mathbb{Z} \to \mathbb{Z}$

$y = 7 \to x = 1$

$y = 5 \to x = 3/5 \notin \mathbb{Z}$ not in the integers

Not in the range

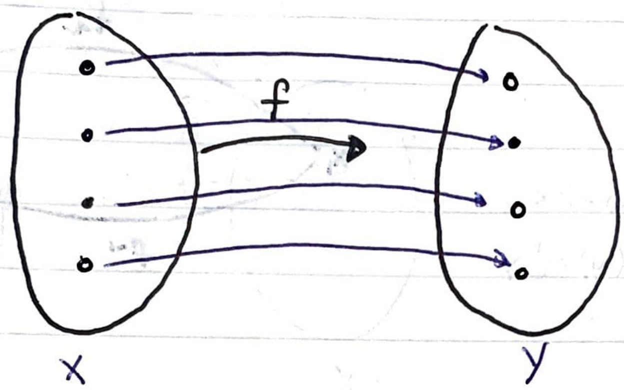
Lets say we have $f: \mathbb{R} \to \mathbb{Z}$

x any real #'s y only integers

Yes so b surjective!

Bijective: Injective AND Surjective

for $f: X \to Y$, Each $x \in X$ maps to exactly one unique $y \in Y$



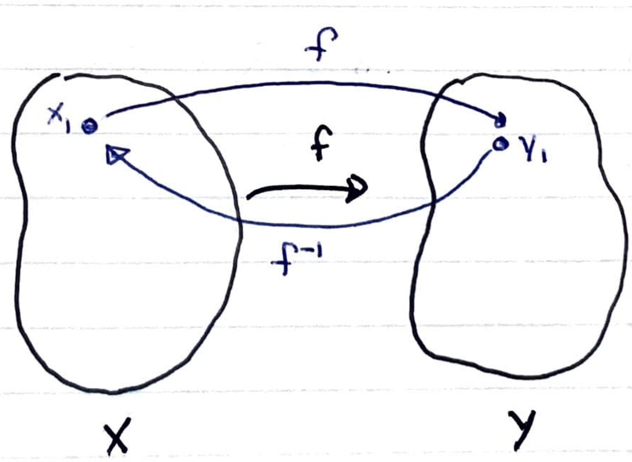
What does this say about $|X|$ and $|Y|$?
size size

$|X| = |Y|$

Size of domain is equal to the codomain

Inverse

Given $f: X \to Y$, we define
Inverse the inverse of
$f^{-1}: Y \to X$



$f(x) = y$
$f^{-1}(y) = x$

Find the inverse of $f(x) = 5x + 2$

implies it's bijective → prove injective + Surjective

When you prove a function is surjective, you always get the inverse

Quartin

$f(x) = y$
$f^{-1}(y) = x$

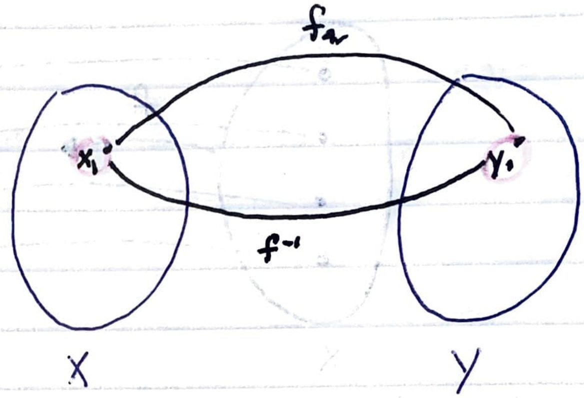
$$\underbrace{f(f^{-1}(y))}_{\text{Internal "Cycle"}} = f(x) = y$$

Kinda like:

$$3 \cdot \tfrac{1}{3} x = X$$

f $\quad$ f^{-1}

$f(x) = 3x$
$f^{-1}(y) = \frac{1}{3} y$



1st take y_1 and map it to x_1
2nd take f^{*} back and set to y

Function Examples

is the function injective? What is the range?

a.) $f: \mathbb{R} \to \mathbb{R}$ where $f(x) = e^x$

$f(x) = f(y) \Rightarrow x = y$

Prove

$e^x = e^y$

$\ln e^x = \ln e^y$ YES !

$x = y$

range of function

$(0, +\infty)$

could be factored

b.) $g: \mathbb{Z} \to \mathbb{Z}$ where $g(x) = x^3 - x$ $\quad x((x-1)(x+1))$

0 1 -1

$\boxed{g(0)} = 0^3 - 0 = 0$

$\boxed{g(1)} = 1^3 - 1 = 0$

same element in co domain

We have two elements in the domain mapping to each other, therefore not injective

Range?

$\{ n^3 - n \mid n \in \mathbb{Z} \}$

Give an Example of a function $f: A \to B$ and $A_1 A_2 \subseteq A$ where $f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2)$

intersection

$A = \{1,2\}$ $B = \{0\}$

$A_1 = \{1\}$

$A_2 = \{2\}$

$f(1) = 3$

$f(2) = 3$

$f(A_1) = \{3\}$

$f(A_2) = \{3\}$

$f(A_1 \cap A_2) = f(\phi) = \phi$

empty set

When you put the empty set into a function you get the empty set back.

Not Equal to

$f(A_1) \cap f(A_2) = \{3\} \cap \{3\} = \{3\}$

- To prove work with simple sets

Surjective Function Examples

Show that $f: \mathbb{R} \to \mathbb{R}$ where $f(x) = x^3$ is surjective

$$y = x^3$$

$$\boxed{\sqrt[3]{y} = x}$$

all y's are going to be produce from values of x

$\sqrt[3]{y} \in \mathbb{R}$

is the cube root of y in the real #'s?

Yes, Surjective

Yes it is, we can put in any y in and get an x back

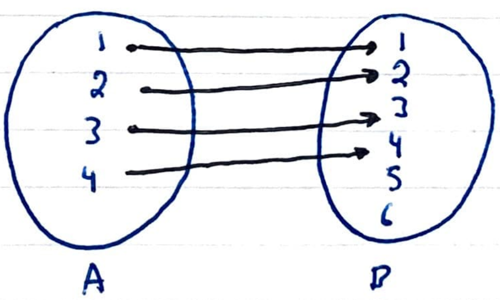
Let $|A| = 4$ and $|B| = 6$

i) How many functions $f: A \to B$ are there?

$6 \times 6 \times 6 \times 6 = 6^4$

\# of options it can map to.

$= |B|^{|A|}$



ii) How many are injective?

$$4!\binom{6}{4} = \frac{6!\,\cancel{4!}}{2!\,\cancel{4!}} = \frac{6!}{2!}$$

Everything in our domain must map to something in our codomain.

iii) How many are surjective?

0

↳ means the codomain is equal to the range, we can't do that we can only map to 4 things.