Introduction to Relations Chapter 9.1-9.5

Relations

Let A and B be sets. A binary relation from A to B is a subset of A x B.

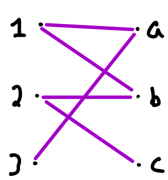
Essentially, a binary relation is a set R of ordered pairs where the first element of each ordered pair comes from A, and the second element comes from B.

Let A = { 1, 2, 3 } and B = { a, b, c }.

Then R = { (1,a), (1,b), (2,b), (2,c), (3,a) } is a relation from A to B.

Recall that a function is a relation where each element of B is mapped to by only one element of A.

Different ways to draw relations:



R	a	b	c
1	x	x	
2		x	x
3	x		

$1 \; R \; a$ ~ One is in a relation with a

$1 \; \cancel{R} \; c$ ~ doesnt map 1 to c

If Adam takes Discrete Math, Programming and Nutrition, and Kevin takes Discrete Math and Composition, give the relation from A to B, if A = { Adam, Kevin } and B = {Discrete Math, Programming, Nutrition, Composition }

Just Give the Order pairs

- R would be the set containing all the ordered pairs

R = { (Adam, Discrete Mathematics)

(Adam, Programming)

(Adam, Nutrition)

(Kevin, Discrete Mathematics)

(Kevin, Composition) }

Binary Relation on a Set

A binary relation on a set A is subset of A x A, or a relation from set A onto itself.

If A = {1,2,3,4}, then list the ordered pairs in set R if $R = \{ (a,b) \mid a \text{ divides } b \}$

$\frac{b}{a} = c \in \mathbb{Z}$

has to be an integer

Using 1 as A:

$\frac{1}{1} = 1$ $\frac{2}{1} = 2$ $\frac{3}{1} = 3$ $\frac{4}{1} = 4$

$$\boxed{R = \{ (1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4) \}}$$

listing of all the elements in the relation

Using 2 as A:

$\frac{1}{2} \neq$ $\frac{2}{2} = 1$ $\frac{3}{2} \neq$ $\frac{4}{2} = 2$

Using 3 as A:

$\frac{1}{3} \neq$ $\frac{2}{3} \neq$ $\frac{3}{3} = 1$ $\frac{4}{3} \neq$

Using 4 as A:

$\frac{1}{4} \neq$ $\frac{2}{4} \neq$ $\frac{3}{4} \neq$ $\frac{4}{4} = 1$

Finding Integers

How many relations are there from set A to itself?

Assuming $|A| = n$

cardinality

$|A \times A| = n^2$

$\boxed{2^{n^2} \text{ or } 2^{|A|^2}}$

m elements has 2^m subsets

Consider the relations below. Which contain (1,1), (1,3), (2,4) and (2,1) ?

a b

$R_1 = \{(a,b) \mid a = b\}$ $(1,1)$

$R_2 = \{(a,b) \mid a \leq b\}$ $(1,1)$, $(1,3)$, $(2,4)$

$R_3 = \{(a,b) \mid a > b\}$ $(2,1)$

$R_4 = \{(a,b) \mid a+b \leq 3\}$ $(1,1)$, $(2,1)$

Properties of Relations

Reflexive Relations

A relation R is reflexive iff (a, a) $\in$ R for every a $\in$ A.

Tell whether the following relations are reflexive.

$R_1 = \{(a,b) \mid a = b\}$ (a,a)

$R_2 = \{(a,b) \mid a \leq b\}$ (a,a) $a \leq a$

$R_3 = \{(a,b) \mid a > b\}$ $a \not> a$

$R_4 = \{(a,b) \mid a+b \leq 3\}$ $a + a \not\leq 3$

has to work for all a's

Symmetric Relations

A relation R is symmetric iff (b,a) $\in$ R whenever (a,b) $\in$ R.

Tell whether the following relations are symmetric.

$R_1 = \{(a,b) \mid a = b\}$ ~ Yes (2,2) if you switch the order, would it be ~~symmetric~~ symmetric?

$\cancel{R_2} = \{(a,b) \mid a \leq b\}$ (3,4) (4,3) **NO** $a \not\leq b$

$\cancel{R_3} = \{(a,b) \mid a > b\}$ (4,3) (3,4)

$R_4 = \{(a,b) \mid a+b \leq 3\}$ True (0,0), (0,1), (3,0)
(0,0), (1,0), (0,3)

Anti-Symmetric Relations

A relation R on set A is anti-symmetric if $\forall$a, b $\in$ A if (a,b) R and (b, a)$\in$ R, then a = b

$$\forall a \forall b (\underline{((a,b) \in R \wedge (b,a) \in R)} \to (\underline{a = b}))$$

Implication

Tell whether the following relations are anti-symmetric.

$R_1 = \{ (a,b) \mid a = b \}$ $(3,3), (3,3)$ $(5,5)(5,5)$ TRUE

$R_2 = \{ (a,b) \mid a \leq b \}$ switch order $(2,2)$ $(2,2)$ ✓ TRUE Counter Example: $\cancel{(3,5)}\cancel{(5,3)}$

$R_3 = \{ (a,b) \mid a > b \}$ $\cancel{(4,2)}\cancel{(2,4)}$ * Were concered about if $\underline{a = b}$, so check if pair is in the relation

Vacuously TRUE

$a > b$ $b > a$

$\cancel{R_4} = \{ (a,b) \mid a+b \leq 3 \}$ $\cancel{(2,1)}\cancel{(1,2)}$

Transitive Relations

A relation R on set A is transitive if whenever $(a,b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ $\forall$ a, b, c $\in$ A.

Tell whether the following relations are transitive.

$R_1 = \{(a,b) \mid a = b\}$ $(2,2)\ (2,2) \longrightarrow (2,2)$ ✓ TRUE

implies

$R_2 = \{(a,b) \mid a \leq b\}$ $(2,5)\ (5,8) \longrightarrow (2,8)$ ✓ TRUE

$2 \leq 5$ $5 \leq 8$

$R_3 = \{(a,b) \mid a > b\}$ $(4,3)\ (3,0) \longrightarrow (4,0)$ ✓ TRUE

$R_4 = \{(a,b) \mid a+b \leq 3\}$ $(3,0)\ (0,2) \longrightarrow (3,2)$ ✗

Counter-Example

Combining Relations

Combining Relations

We can combine relations in the same manner we combine any other sets.

Let $A = \{1, 2, 3, 4\}$ $\quad R_1 = \{(1,0), (1,2), (1,1), (2,2)\}$

$B = \{0, 1, 2\}$ $\quad R_2 = \{(1,1), (3,2), (4,2)\}$

Find:

Union $(R_1 \cup R_2) = \{(1,0), (1,1), (1,2), (2,2), (3,2), (4,2)\}$

Intersection $(R_1 \cap R_2) = \{(1,1)\}$

Difference $(R_1 - R_2) = \{(1,0), (1,2), (2,2)\}$

Difference $(R_2 - R_1) = \{(3,2), (4,2)\}$

$(R_1 \oplus R_2) = \{(1,0), (1,2), (2,2), (3,2), (4,2)\}$

Composing Relations

$(f \circ g)x = f(g(x))$

Let R, be a relation from set A to set B, and Let R2 be a relation from set B to set C. The composition R2 o R1, consists of the ordered pairs (a, c), such that a $\in$ A, b $\in$ B, c $\in$ C, (a, b) $\in$ R, and (b, c) $\in$ R2.

What is R2 o R1 if R1 is a relation from {1, 2, 3} to {1, 2, 3, 4 }

with R1 = { (1,1), (1,3), (1,4), (2,3), (3,1), (3,4) } and

R2 is a relation from {1, 2, 3, 4} to {0, 1, 2 } where R2 =. { (1,0), (2,0), (3,1), (3,2), (4,1) }

$$\boxed{R_2 \circ R_1 = \{(1,0),(1,1),(1,2),(2,1),(2,2),(3,0),(3,1)\}}$$

Note: You don't have to include duplicates pairs

Composing Relations with Themselves

(a, c) $\in$ R o R if (a, b) $\in$ R and (b, c) $\in$ R. That is, if a is a grandparent of c.

Find R o R if R is a relation from { 0, 1, 2, 3 } to {1, 2, 3 } where R = { (1,1), (2,1), (3,2), (4,3) }

2 tells us to map to 1

$$\boxed{R \circ R = \{ (1,1), (2,1), (3,1), (4,2) \}}$$

Powers of Relations

If R is a relation from { 0, 1, 2, 3 } to { 0, 2, 3 } where R = { (1,1), (2,1), (3,2), (4,3) }, find R^n for n = 2, 3, 4, 5.

$R^2 = R \circ R$ $\quad$ $R^2 = R \circ R = \{(1,1), (2,1), (3,1), (4,2)\}$ $\quad$ What they map to!

$R^2 = \{(1,1), (2,1), (3,1), (4,2)\}$

$R^3 = R^2 \circ R$ $\quad$ $R^3 = R^2 \circ R = \{(1,1), (2,1), (3,1), (4,1)\}$

$R^3 = \{(1,1), (2,1), (3,1), (4,1)\}$

$R^4 = R^3 \circ R$ $\quad$ $R^4 = R^3 \circ R = \{(1,1), (2,1), (3,1), (4,1)\}$

$R^4 = \{(1,1), (2,1), (3,1), (4,1)\}$

$R^5 = R^4 \circ R$ $\quad$ $R^5 = R^4 \circ R = \{(1,1), (2,1), (3,1), (4,1)\}$

$R^5 = \{(1,1), (2,1), (3,1), (4,1)\}$

9.3 Matrix Representations of Relations and Properties

Representing Relations Using Matrices

A relation between two finite sets can be represented using a zero – one matrix. If R is a relation between A = $\{a_1, a_2, a_3, \ldots a_m\}$ and B = $\{b_1, b_2, \ldots b_n\}$

Then $M_R = [m_{ij}]$ where $m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R \\ 0 \text{ if } (a_i, b_j) \notin R \end{cases}$

Suppose $\overset{=3}{A} = \{0, 1, 2\}$ and $\overset{=4}{B} = \{1, 2, 3, 4\}$, with R a relation between A and B

with R= $\{(0,1), (0,3), (1,1), (1,2), (1,4), (2,2)\}$

Give the matrix that represents R.

- Going to be a 3×4 matrix

$\overset{A\,B}{(0,1)}$

B Values

Is (0,1) in the relation? ~ Yes so put a 1

$$M_R = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

A Values

$$\boxed{M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}}$$

Let A = $\{a_1, a_2, a_3\}$ and B = $\{b_1, b_2, b_3, b_4, b_5\}$ Which ordered pairs are in the matrix represented by:

$$M_R = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccccc} b_1 & b_2 & b_3 & b_4 & b_5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

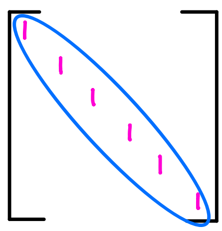
$R = \{(a_1, b_3), (a_1, b_4), (a_1, b_5), (a_2, b_1), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}$

Properties from Matrices – Reflexive

The Matrix of a relation on a set, which is a square matrix, can determine if the relation has a certain property.

Reflexive - if $(a,a) \in R$

In a Matrix, this means $m_{ii} = 1$ for $i = 1, 2, \ldots n$.



Properties from Matrices – Irreflexive

The Matrix of a relation on a set, which is a square matrix, can determine if the relation has a certain property.

Irreflexive = if (a, a) R

In a Matrix, this means for

Properties from Matrices – Symmetric

The Matrix of a relation on a set, which is a square matrix, can determine if the relation has a certain property.

Symmetric - if (a, a) R implies (b, a) R.

In a Matrix, this means for

~~Properties from Matrices – Anti-Symmetric~~

The Matrix of a relation on a set, which is a square matrix, can determine if the relation has a certain property.

Anti-Symmetric - if (a, a) R implies (b, a) R then a = b

In a Matrix, this means or , or both.

Properties from Matrices – Asymmetric

The Matrix of a relation on a set, which is a square matrix, can determine if the relation has a certain property.

Asymmetric - if (a, a) R and (b, a) R then a = b AND (a, a) R. So this means or , or both, but

Operations on Matrices

Properties From Matrices – Transitive

The matrix of a relation on a set, which is a square matrix, can determine if the relation has a certain properties.

Transitive if (a, b) R , then a, b   R.

Is R, represented in a matrix below, reflexive, irreflexive, symmetric, anti-symmetric, and/or transitive?

Reflexive
Irreflexive
Symmetric
Anti-symmetric
Transitive

Representing Relations Using Digraphs

Operations on Matrices

Digraphs – Directed Graphs

A digraph is a pictoral representation of a relation. It consist of a set V of vertices (or nodes _ together with a set E of ordered pairs elements of V, called edges (or arcs). The vertex a is called the initial vertex of edge (a, b), with b as the terminating vertex.

Draw a diagraph for the relation R on { 1, 2, 3 } where R = { (1,1), (1,2), (2,2) ,(2,3), (3,1), (3,2), (3,3) }

Is this relation

Reflexive ?

Symmetric ?

Antisymmetric ?

Transitive ?

9.4 Closures of Relations

9.5. Equivalence Relations

A relation on a set A is called an equivalence relation if it is reflexive, symmetric and transitive. Two elements related by an equivalence relation are said to be equivalent, denoted a b.

Let R be the relation on the set of real numbers such that   iff a -b is an integer. Is this an equivalence relation?

Reflexive?

Symmetric?
Transitive?

Suppose R is the relation on the set of strings of English letters where a R b iff ,
where is the length of string x. Is this an equivalence relation ?

Is R, a relation on the positive integers where a R b iff a | b an equivalence relation ?

Partitions

A partition of a set S is a collection of disjointed non-empty subsets that have S as their union.

If x, y   S, then:

What are the sets in the partition of integers formed by congruence mod 4?

Which are partitions of $A = \{ 1, 2, 3, 4, 5, 6, 7 \}$?