Mathematical Induction Chapter 5.1-5.4

5.1

Proof Using Mathematical Induction - Summation Formulas

Mathematical Induction

To prove P(x) is true for $x \in \mathbb{Z}^+$, where P(x) is a propositional function, we complete two steps:

1) Basis Step - Verify $\boxed{P(1)}$ is true

2) Inductive step - Verify if P(K) is true, then P(K+1) is true $\forall K \in \mathbb{Z}^+$

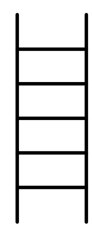
* Inductive Hypothesis : P(K) is true

* Must Show: $P(K) \rightarrow P(K+1)$

Conclusion : $\boxed{P(x) \text{ is true } \forall\ K \in \mathbb{Z}^+}$

Other instructors might not write this

like a ladder with ∞ # of runs



Proving a Summation Formula

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Show $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

Let $P(n)$: $\boxed{1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}}$

① Basis: Prove $P(1)$ is true

$P(1) = 1 = \frac{1(1+1)}{2}$ $\quad 1 = \frac{1(2)}{2}$ $\quad 1 = 1$ ✓

~ Prove Basis Step

~ easy its just math

② Inductive step: $P(K) \rightarrow P(K+1)$

Inductive Hypothesis: $1 + 2 + 3 + \ldots + K = \frac{K(K+1)}{2}$

★ Must Show: $1 + 2 + 3 \ldots + K + (K+1) = \frac{(K+1)((K+1+1))}{2} = \boxed{\frac{(K+1)(K+2)}{2}}$

Exactly What We Wanted

$1 + 2 + 3 + \ldots + K + (K+1) = \frac{K(K+1)}{2} + (K+1)$

1st Step: Added $(K+1)$ to both sides

Write as fraction

$= \frac{K(K+1)}{2} + \frac{2(K+1)}{2}$

2nd Step: simplfy solution

$$\boxed{= \frac{(K+1)(K+2)}{2}}$$

$\therefore$ $P(n)$ is true $\forall n \in \mathbb{Z}$

Problem 2:

Use mathematical induction to show that for all non-negative integer n, $1+2+2^2+\ldots+2^n=2^{n+1}-1$

$n \geq 0$

$$1+2+2^2+\ldots+2^n=2^{n+1}-1$$

Let $P(n)$: $1+2+2^2+\ldots+2^n=2^{n+1}-1$

① Basis: Prove $P(0)$ is true

$$2^0=2^{0+1}-1$$
$$1=1\ \checkmark$$

② Inductive step: $P(k)\rightarrow P(k+1)$

Inductive Hypothesis: $1+2+2^2+\ldots+2^k=2^{k+1}-1$

★ Must Show: $1+2+2^2+\ldots+2^k+2^{k+1}=2^{(k+1)+1}-1=\boxed{2^{k+2}-1}$ ← We want to get to this

$$1+2+2^2+\ldots+2^k+2^{k+1}=2^{k+1}-1+2^{k+1}$$ ~ Add to both sides

$$=2(2^{k+1})-1$$

$$\boxed{=2^{k+2}-1}$$

can add exponent

$$3^2\cdot 3^4=3^6$$

∴ $P(n)$ is True for all $n\geq 0$, $n\in\mathbb{Z}$

Problem 3

Conjective and prove a summation formula for the sum of the first n positive odd integers.

$n = 1$	$1 = 1$	1^2	looking for a Pattern
$n = 2$	$1 + 3 = 4$	2^2	
$n = 3$	$1 + 3 + 5 = 9$	3^2	
$n = 4$	$1 + 3 + 5 + 7 = 16$	4^2	
$\vdots$		$\vdots$	
$n = n$	$1 + 3 + 5 + \ldots + (2n-1) =$	n^2	~ $P(n)$

Prove $1 + 3 + 5 + \ldots + (2n-1) = n^2$

Let $P(n)$: the proposition that $1 + 3 + 5 + \ldots + (2n-1) = n^2$

① Basis: Prove $P(1)$ is true

$$1 = (1)^2$$
$$1 = 1 \checkmark$$

② Inductive step: $P(k) \rightarrow P(k+1)$

Inductive Hypothesis: $1 + 3 + 5 + \ldots + (2k+1) = k^2$

Show: $1 + 3 + 5 + \ldots + (2k-1) + (2k+1) = (k+1)^2$

if we added the next odd integer, which is $+2$.

$$1 + 3 + 5 + \ldots + (2k-1) + (2k+1) = k^2 + (2k+1)$$

$$1 + 3 + 5 + \ldots + (2k-1) + (2k+1) = (k+1)(k+1)$$ - Factor

$$1 + 3 + 5 + \ldots + (2k-1) + (2k+1) = (k+1)^2$$

$\therefore P(n)$ is true

Proof Using Mathematical Induction - Inequalities

Proving an Inequality:

Prove $n < 2^n \quad \forall n \in \mathbb{Z}^+$ using mathematical induction.

Let $P(n)$: $n < 2^n \quad \forall n \in \mathbb{Z}^+$

① Basis: $P(1)$

$P(1)$: $1 < 2^1$

$1 < 2^1$ ✓

② Inductive $P(k) \rightarrow \underline{P(k+1)}$

IH: $k < 2^k$

Show: $k+1 < 2^{k+1}$

$k+1 < 2^k + 1$

~ Since its an inequality that don't have same rules as an equals

~ We just have to make sure the <u>inequality</u> <u>still holds</u>

We know that: $1 < 2^k \quad \forall k \in \mathbb{Z}^+$

~ Proved this in Basis Step ①

$k+1 < 2^k + 1 < 2^k + 2^k = 2(2^k)$

- How can we do this?

B/c $1 < 2^k$

- it's ok to do this b/c were not breaking any <u>mathematical</u> <u>law</u>.

- Replacing, inequality still works!

$$\boxed{k+1 < 2^{k+1}}$$

Example 2:

Prove $2^n < n! \quad \forall n \in \mathbb{Z}^+$ and $n \geq 4$.

Let $P(n): 2^n < n!$

① Basis $P(4)$

$2^4 < 4! \rightarrow 4 \cdot 3 \cdot 2 \cdot 1$

$16 < 24$ ✓

② Inductive $P(k) \rightarrow P(k+1)$

IH: $2^k < k!$

SHOW: $2^{k+1} < \underline{(k+1)!}$

Trying to get here!

$2^k \cdot 2^1 < k! \cdot 2$

$k+1 > 2$

- Just here to make sure the <u>inequality</u> still holds

$2^{k+1} < 2k!$

$2^{k+1} < (k+1)k!$

$(k+1)(k)(k-1)\ldots(2)(1) = (k+1)!$

$$\boxed{2^{k+1} < (k+1)!}$$

Proof Using Mathematical Induction - Divisibility

Use mathematical induction to prove $7^{n+2} + 8^{2n+1}$ is divisible by 57 for all non-negative integers n.

Let P(n): $7^{n+2} + 8^{2n+1}$ is divisible by 57

① Basis: P(0)

$$P(0): \quad 7^{0+2} + 8^{2(0)+1} = 7^2 + 8 = 49 + 8 = \underline{\underline{57}} \checkmark$$

~ is divisible by 57

② Inductive:

IH : $7^{k+2} + 8^{2k+1}$ is divisible by 57

Show: $7^{k+3} + 8^{2k+3}$ is divisible by 57

$$7 \cdot 7^{k+2} + 8^{2k+1} \cdot 8^2 = 7^{k+3} + 8^{2k+3}$$

~ Shown we can turn the IH into what were suppose to Show.

$$7 \cdot 8^{2k+1} + 57 \cdot 8^{2k+1}$$

$$= 7 \cdot 7^{k+2} + \boxed{(7+57)\, 8^{2k+1}}$$

If you multiply out

- lets re write the right-hand side.

$$= 7(7^{k+2} + 8^{2k+1}) + 57 \cdot 8^{2k+1}$$

$$= 7(7^{k+2} + 8^{2k+1}) + 57 \cdot 8^{2k+1}$$

- By Inductive Hypothesis this is divisible by 57

- Then this portion is also divisible by 57

Since $7^{k+2} + 8^{2k+1}$ is divisible by 57 and $57 \cdot 8^{2k+1}$ is divisible by 57,

then $7^{k+3} + 8^{2k+3}$ is divisible by 57.

$$\boxed{\therefore\ 7^{n+2} + 8^{2n+1} \text{ is divisible by } 57 \text{ for all } n \geq 0,\ n \in \mathbb{Z}}$$

The Well-Ordering Principle and Strong Induction

5.2

The Well-Ordering Principle

Every non-empty set of non-negative integers has a least element

$\mathbb{Z}^+$ $\qquad$ $n = 1$

$n \in \mathbb{Z}^+$ $\quad$ $n \geq 4$ $\qquad$ $n = 4$

non-negative int $\qquad$ $n = 0$

Prove that every amount of postage of \$ 0.12 or more can be formed using \$ 0.04 and \$ 0.05 stamps.

$P(n)$ is the statement that postage of n-cents can be formed using 4-cent and 5-cent stamps if $n \geq 12$.

Basis: show postage of 12-cents can be made

Inductive: show if $P(k)$ is true, then $P(k+1)$ is true for $k \geq 12$

Revisiting Recursive Definitions

Fibonacci Numbers

Recall the set of Fibonacci numbers:

0, 1, 1, 2, 3, 5, 8, 13, 21, ...

How each number found?

add 2 previous terms

① Initial conditions

$f_0 = 0$

$f_1 = 1$

② Function

$f_n = f_{n-1} + f_{n-2}, \quad n \geq 2$

Recursive Defined Functions

Basis Step: Specifies the value of the function for the first term(s)

Recursive Step: Gives a rule for finding subsequent values using a previous value(s) beginning at those defined in the basis step.

If f is defined recursively by $f(0) = 2$ and $f(n+1) = 3f(n) - 1$, find $f(1)$, $f(2)$, $f(3)$, and $f(4)$.

Give a recursive definition for a^n for $a \in \mathbb{R}$ and non-negative and $n \in \mathbb{Z}^+$

Give a recursive definition of the sequence $\{a_n\}$, $n = 1, 2, 3 \ldots$ if $a_n = 2n + 1$

Recursively Defined Sets and structures

Basis step: Specifies an inital collection of elements

Recursive step: Gives rules for forming new elements from those already in the set.

Recursively Defined Set

Let S be a subset of the integers defined

recursively by:

Basis step:

Recursive Step:

List the elements of S produced by the first 5 application of the recursive definition.

Recursively Defined structure

A set of rooted trees, where a rooted tree consists of a set of vertices

Example of Rooted Trees Defn:

Basis Step:

Recursive Step 1:

Step 2:

Structural Induction

Recursive Algorithms

An algorithm is called recursive if it solves a problem by reducing it to an instance of the same problem with a smaller input.