

Homework #5 : sections 2.3-2.5

2.3

12. Determine whether each of these functions from \mathbb{Z} to \mathbb{Z} is one-to-one.

- a) $f(n) = n - 1$
- b) $f(n) = n^2 + 1$
- c) $f(n) = n^3$
- d) $f(n) = \lceil n/2 \rceil$

a) $f(n) = n - 1$

if $f(a) = f(b)$ then $a = b$ ← To prove one-to-one problems
 $f(c) = 3$ $f(d) = 3$
 $a = 4$ $b = 4$

Yes one-to-one

b) $f(n) = n^2 + 1$

if $f(a) = f(b)$ then $a = b$
 $f(c) = 5$ $f(d) = 5$
 $a = 2$ $b = -2$ ✗
 $f(c) = f(d)$ but $c \neq d$

Not one-to-one

c) $f(n) = n^3$

if $f(a) = f(b)$ then $a = b$

One-to-one

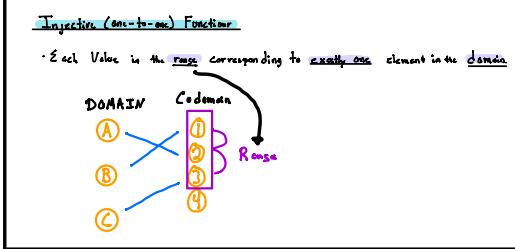
$f(e) = 8$ $f(f) = 8$ ✓
 $a = 2$ $b = -2$ ✓

d) $f(n) = \lceil n/2 \rceil$

if $f(a) = f(b)$ then $a = b$

Not one-to-one $f(a) = 2$ $f(b) = 2$

$\lceil \quad \rceil$ ← Will round #'s



2. Determine whether f is a function from \mathbf{Z} to \mathbf{R} if

- a) $f(n) = \pm n.$
- b) $f(n) = \sqrt{n^2 + 1}.$
- c) $f(n) = 1/(n^2 - 4).$

a) $f(n) = \pm n.$

No

Because each # is mapped to two integers

b) $f(n) = \sqrt{n^2 + 1}.$

Yes

Because each # is mapped to one number

c) $f(n) = 1/(n^2 - 4).$

No

Because the function is undefined at -2 and 2

8. Find these values.

a) $\lfloor 1.1 \rfloor$
c) $\lfloor -0.1 \rfloor$
e) $\lceil 2.99 \rceil$
g) $\lfloor \frac{1}{2} + \lceil \frac{1}{2} \rceil \rfloor$

b) $\lceil 1.1 \rceil$
d) $\lceil -0.1 \rceil$
f) $\lceil -2.99 \rceil$
h) $\lceil \lfloor \frac{1}{2} \rfloor + \lceil \frac{1}{2} \rceil + \frac{1}{2} \rceil$

Floor Function $f(x) = \lfloor x \rfloor$

Largest integer less than or equal to x .

a) $\lfloor 1.1 \rfloor$

b) $\lceil 1.1 \rceil$

1

2

Ceiling Function $f(x) = \lceil x \rceil$

Smallest integer greater than or equal to x .

c) $\lfloor -0.1 \rfloor$



-1

d) $\lceil -0.1 \rceil$



0

e) $\lceil 2.99 \rceil$

3

f) $\lceil -2.99 \rceil$

-2



g) $\lfloor \frac{1}{2} + \lceil \frac{1}{2} \rceil \rfloor$

$\lfloor \frac{1}{2} + \lceil \frac{1}{2} \rceil \rfloor$

$\lceil \frac{1}{2} \rceil = 1$

$\lfloor \frac{1}{2} + 1 \rfloor$

$\lfloor 1.5 \rfloor$

1

h) $\lceil \lfloor \frac{1}{2} \rfloor + \lceil \frac{1}{2} \rceil + \frac{1}{2} \rceil$

$\lfloor \frac{1}{2} \rfloor = 0$

$\lceil 0 + 1 + \frac{1}{2} \rceil$

$\lceil 1.5 \rceil$

2

13. Which functions in Exercise 12 are onto?

Surjection (onto) Functions

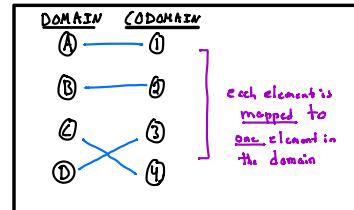
Every element in the codomain maps to at least one element in the domain.

a) $f(n) = n - 1$

onto

- because every integer is & less than some integer

b) $f(n) = n^2 + 1$



Not onto

- Because everything is positive (n^2), so the range cannot include any negative integer

c) $f(n) = n^3$

Not onto

$2^{\frac{1}{3}} \neq \text{Integer}$

d) $f(n) = \lceil n/2 \rceil$

onto

22. Determine whether each of these functions is a bijection from \mathbf{R} to \mathbf{R} .

- $f(x) = -3x + 4$
- $f(x) = -3x^2 + 7$
- $f(x) = (x+1)/(x+2)$
- $f(x) = x^5 + 1$

Bijective (one-to-one correspondence) Functions

Functions that are both one-to-one and onto or both surjective and injective.

a) $f(x) = -3x + 4$

One-to-one ✓
onto ✓

Yes, Bijection

$$\begin{aligned} f(a) &= f(b) \quad a = b \\ -3a + 4 &= -3b + 4 \\ -3(a) + 4 &= -3(b) + 4 \\ -3 &= -3 \quad \checkmark \end{aligned}$$

$a = b \checkmark$

onto:
 $f(x) = -3x + 4$

$$f\left(\frac{4-x}{3}\right) = x \checkmark$$

b) $f(x) = -3x^2 + 7$

$$\begin{aligned} \text{if } f(a) &= f(b) \text{ then } a = b \\ f(-a) &\neq f(b) \times \end{aligned}$$

Not Bijection

c) $f(x) = (x+1)/(x+2)$

$$f(x) = \frac{(x+0)}{(x+2)} = 1 ? \quad \text{There is no real # for } x \text{ to equal 1. So not onto}$$

Not Bijection

Plus when $x = -2$ it is undefined, so it doesn't have a image.
just another observation.

d) $f(x) = x^5 + 1$

Yes, Bijection

~ Increasing function

26. a) Prove that a strictly increasing function from \mathbf{R} to itself is one-to-one.
 b) Give an example of an increasing function from \mathbf{R} to itself that is not one-to-one.

- a) Prove that a strictly increasing function from \mathbf{R} to itself is one-to-one.

Have to prove $f(x)$ is one-to-one

If f is "strictly" increasing then it implies $x < y$, then $f(x) < f(y)$

Assume $f(a) = f(b)$

If $a < b$, then by definition of strictly increasing $f(a) < f(b)$, then it's not possible that $a < b$ when $f(a) = f(b)$

If $b < a$, then by definition of strictly increasing $f(b) < f(a)$, then it's not possible that $b < a$ when $f(a) = f(b)$

Since $a < b$ is not true and since $b < a$ is not true, a and b then have to be the same:

$$a = b$$

Then by the definition of one-to-one we've shown that f is one-to-one

- b) Give an example of an increasing function from \mathbf{R} to itself that is not one-to-one.

$f(x) = \lceil x \rceil$, is an increasing function, but not one-to-one, because it's possible for some values to have the same image

$$\text{eg. } f(1) = \lceil 1 \rceil = 1$$

$$f(0.4) = \lceil 0.4 \rceil = 1$$

30. Let $S = \{-1, 0, 2, 4, 7\}$. Find $f(S)$ if

- a) $f(x) = 1$. b) $f(x) = 2x + 1$.
 c) $f(x) = \lceil x/5 \rceil$. d) $f(x) = \lfloor (x^2 + 1)/3 \rfloor$.

DEFINITION 4

Let f be a function from A to B and let S be a subset of A . The *image* of S under the function f is the subset of B that consists of the images of the elements of S . We denote the image of S by $f(S)$, so

$$f(S) = \{t \mid \exists s \in S \ (t = f(s))\}.$$

We also use the shorthand $\{f(s) \mid s \in S\}$ to denote this set.

a) $f(x) = 1$.

$$S = \{-1, 0, 2, 4, 7\}$$

Remark: The notation $f(S)$ for the image of the set S under the function f is potentially ambiguous. Here, $f(S)$ denotes a set, and not the value of the function f for the set S .

Determine the image of every element in the set S .

$$f(-1) = 1$$

$$f(0) = 1$$

$$f(2) = 1$$

$$f(4) = 1$$

$$f(7) = 1$$

$f(S)$ is the set of all the images of
every element in S .

$$f(S) = \{1\}$$

b) $f(x) = 2x + 1$.

$$f(-1) = 2(-1) + 1 = -1$$

$$f(0) = 2(0) + 1 = 1$$

$$f(2) = 2(2) + 1 = 5$$

$$f(4) = 2(4) + 1 = 9$$

$$f(7) = 2(7) + 1 = 15$$

$$f(S) = \{-1, 1, 5, 9, 15\}$$

c) $f(x) = \lceil x/5 \rceil$.

$$f(-1) = \lceil \frac{-1}{5} \rceil = \lceil -0.2 \rceil = 0$$

$$f(0) = \lceil \frac{0}{5} \rceil = \lceil 0 \rceil = 0$$

$$f(2) = \lceil \frac{2}{5} \rceil = \lceil 0.4 \rceil = 1$$

$$f(4) = \lceil \frac{4}{5} \rceil = \lceil 0.8 \rceil = 1$$

$$f(7) = \lceil \frac{7}{5} \rceil = \lceil 1.4 \rceil = 2$$

$$f(S) = \{0, 1, 2\}$$

d) $f(x) = \lfloor (x^2 + 1)/3 \rfloor$.

$$f(-1) = \lfloor \frac{(-1)^2 + 1}{3} \rfloor = \lfloor \frac{2}{3} \rfloor = 0$$

$$f(0) = \lfloor \frac{(0)^2 + 1}{3} \rfloor = \lfloor \frac{1}{3} \rfloor = 0$$

$$f(2) = \lfloor \frac{(2)^2 + 1}{3} \rfloor = \lfloor \frac{5}{3} \rfloor = 1$$

$$f(4) = \lfloor \frac{(4)^2 + 1}{3} \rfloor = \lfloor \frac{17}{3} \rfloor = 5$$

$$f(7) = \lfloor \frac{(7)^2 + 1}{3} \rfloor = \lfloor \frac{50}{3} \rfloor = 16$$

$$f(S) = \{0, 1, 5, 16\}$$

$$\frac{17}{3} = 5.67$$

$$\frac{50}{3} = 16.67$$

33. Suppose that g is a function from A to B and f is a function from B to C .
- Show that if both f and g are one-to-one functions, then $f \circ g$ is also one-to-one.
 - Show that if both f and g are onto functions, then $f \circ g$ is also onto.

One-to-One:

Each value in the range corresponding to exactly one element in the domain.

$$\forall a \forall b ((a \neq b) \rightarrow (f(a) \neq f(b)))$$

- a) Show that if both f and g are one-to-one functions, then $f \circ g$ is also one-to-one.

Using definitions.

Assume that both f and g are One-to-one

- We need to show that x and y are two distinct elements of A , $f(g(x)) \neq f(g(y))$

$$g(x) \neq g(y)$$

Since $g(x)$ and $g(y)$ are distinct elements of B , then we can say:

$$f(g(x)) \neq f(g(y))$$

- b) Show that if both f and g are onto functions, then $f \circ g$ is also onto.

Assume both are onto, we need to show $f \circ g$ is onto

Need to show that if z is any element of C , then there exists some element $x \in A$, such that $f(g(x)) = z$

Since f is onto, we can say that there is an element $y \in B$ such that $f(y) = z$

Then since g is onto, and $y \in B$, we can conclude that there exists some element $x \in A$ such that $g(x) = y$

$$z = f(y) = f(g(x))$$

***35.** If f and $f \circ g$ are onto, does it follow that g is onto?
Justify your answer.

43. Let $g(x) = \lfloor x \rfloor$. Find

- a) $g^{-1}(\{0\})$.
- b) $g^{-1}(\{-1, 0, 1\})$.
- c) $g^{-1}(\{x \mid 0 < x < 1\})$.

- Must find all numbers whose floor is 0

a) $g^{-1}(\{0\})$.

$$g^{-1}(\{0\}) = \left\{ x \mid 0 \leq x < 1 \right\} = [0, 1)$$

such that

b) $g^{-1}(\{-1, 0, 1\})$.

$$g^{-1}(\{-1, 0, 1\}) = \{x \mid -1 \leq x < 2\} = [-1, 2)$$

c) $g^{-1}(\{x \mid 0 < x < 1\})$.

- Since x is an integer there is no values such that $g(x)$ is strictly b/w 0 and 1.

TABLE 1 Useful Properties of the Floor and Ceiling Functions.

(n is an integer, x is a real number)

(1a) $\lfloor x \rfloor = n$ if and only if $n \leq x < n + 1$

(1b) $\lceil x \rceil = n$ if and only if $n - 1 < x \leq n$

(1c) $\lfloor x \rfloor = n$ if and only if $x - 1 < n \leq x$

(1d) $\lceil x \rceil = n$ if and only if $x \leq n < x + 1$

(2) $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$

(3a) $\lfloor -x \rfloor = -\lceil x \rceil$

(3b) $\lceil -x \rceil = -\lfloor x \rfloor$

(4a) $\lfloor x + n \rfloor = \lfloor x \rfloor + n$

(4b) $\lceil x + n \rceil = \lceil x \rceil + n$

Section 2.4

4. What are the terms a_0, a_1, a_2 , and a_3 of the sequence $\{a_n\}$, where a_n equals
- a) $(-2)^n$?
 - b) 3^n ?
 - c) $7 + 4^n$?
 - d) $2^n + (-2)^n$?

a) $(-2)^n$?

$$a_0 = (-2)^0 \quad a_1 = (-2)^1 \quad a_2 = (-2)^2 \quad a_3 = (-2)^3$$

$$a_0 = 1 \quad a_1 = -2 \quad a_2 = 4 \quad a_3 = -8$$

b) 3^n ?

$$a_0 = a_1 = a_2 = a_3 = 3$$

c) $7 + 4^n$?

$$\begin{aligned} a_0 &= 7 + 4^0 & a_1 &= 7 + 4^1 & a_2 &= 7 + 4^2 & a_3 &= 7 + 4^3 \\ a_0 &= 7 + 1 & & & a_2 &= 7 + 16 & & \\ a_0 &= 8 & a_1 &= 11 & a_2 &= 23 & a_3 &= 71 \end{aligned}$$

d) $2^n + (-2)^n$?

$$\begin{aligned} a_0 &= 2^0 + (-2)^0 & a_1 &= 2^1 + (-2)^1 & a_2 &= 2^2 + (-2)^2 & a_3 &= 2^3 + (-2)^3 \\ a_0 &= 1 + 1 & a_1 &= 2 + -2 & a_2 &= 4 + 4 & a_3 &= 8 + -8 \\ a_0 &= 2 & a_1 &= 0 & a_2 &= 8 & a_3 &= 0 \end{aligned}$$

9. Find the first five terms of the sequence defined by each of these recurrence relations and initial conditions.

- a) $a_n = 6a_{n-1}, a_0 = 2$
- b) $a_n = a_{n-1}^2, a_1 = 2$
- c) $a_n = a_{n-1} + 3a_{n-2}, a_0 = 1, a_1 = 2$
- d) $a_n = na_{n-1} + n^2a_{n-2}, a_0 = 1, a_1 = 1$
- e) $a_n = a_{n-1} + a_{n-3}, a_0 = 1, a_1 = 2, a_2 = 0$

a) $a_n = 6a_{n-1}, a_0 = 2$

$$\begin{aligned} a_0 &= 2 \\ a_1 &= 6(2) = 12 \end{aligned}$$

$$a_2 = 6(12) = 72$$

$$a_3 = 6(72) = 432$$

$$2, 12, 72, 432, 2592$$

b) $a_n = a_{n-1}^2, a_1 = 2$

$$\begin{aligned} a_1 &= a_0^2 \\ a_2 &= a_1^2 \\ a_3 &= a_2^2 \\ a_4 &= a_3^2 \\ a_5 &= a_4^2 \end{aligned}$$

$$a_1 = a_0^2 = 4$$

$$a_2 = 4^2 = 16$$

$$a_3 = 16^2 = 256$$

$$a_4 = 256^2 = 65536$$

$$2, 4, 16, 256, 65536$$

c) $a_n = a_{n-1} + 3a_{n-2}, a_0 = 1, a_1 = 2$

$$a_0 = a_{n-1} + 3a_{n-2} \quad a_0 = 1, a_1 = 2$$

$$a_2 = a_1 + 3a_0 = 2 + 3(1) = 5$$

$$1, 2, 5, 11, 26$$

$$a_3 = a_2 + 3a_1 = 5 + 3(2) = 11$$

$$a_4 = a_3 + 3a_2 = 11 + 3(5) = 26$$

d) $a_n = na_{n-1} + n^2a_{n-2}, a_0 = 1, a_1 = 1$

$$\begin{aligned} a_2 &= 2a_1 + (2)^2 \cdot a_0 = 2(1) + (2)^2 \cdot 1 = 6 \\ a_3 &= 3a_2 + (3)^2 \cdot a_1 = 3(6) + (3)^2 \cdot 1 = 18 + 9 = 27 \\ a_4 &= 4a_3 + (4)^2 \cdot a_2 = 4(27) + 4^2 \cdot 6 = 204 \end{aligned}$$

$$1, 1, 6, 27, 204$$

e) $a_n = a_{n-1} + a_{n-3}, a_0 = 1, a_1 = 2, a_2 = 0$

$$a_3 = a_2 + a_0 = 0 + 1 = 1$$

$$1, 2, 0, 1, 3$$

$$a_4 = a_3 + a_1 = 1 + 2 = 3$$

12. Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = -3a_{n-1} + 4a_{n-2}$ if
- $a_n = 0.$
 - $a_n = 1.$
 - $a_n = (-4)^n.$
 - $a_n = 2(-4)^n + 3.$

a) $a_n = 0.$

$$a_n = -3a_{n-1} + 4a_{n-2}$$

$$\begin{aligned} 0 &= -3(0) + 4(0) \\ 0 &= 0 \end{aligned} \quad \checkmark$$

b) $a_n = 1.$

$$a_n = -3a_{n-1} + 4a_{n-2}$$

$$\begin{aligned} 1 &= -3(1) + 4(1) \\ 1 &= 1 \end{aligned} \quad \checkmark$$

c) $a_n = (-4)^n.$

$$a_n = -3a_{n-1} + 4a_{n-2}$$

$$(-4)^n = -3((-4)^n) + 4((-4)^n)$$

$$(-4)^n = (-4)^n$$

d) $a_n = 2(-4)^n + 3.$

19. Suppose that the number of bacteria in a colony triples every hour.

- a) Set up a recurrence relation for the number of bacteria after n hours have elapsed.
- b) If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?

- a) Set up a recurrence relation for the number of bacteria after n hours have elapsed.

$$b_n = \text{number of bacteria after } n \text{ hours}$$

$$b_n = 3b_{n-1}$$

- b) If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?

$$b_0 = 100 \quad n = 10$$

$$b_n = 3b_{n-1} = 3^2 b_{n-2} = \underbrace{3^n b_0}_{\text{arrow}}$$

$$b_{10} = 3^{10}(100) =$$

$$b_{10} = 5904900$$

29. What are the values of these sums?

a) $\sum_{k=1}^5 (k+1)$

b) $\sum_{j=0}^4 (-2)^j$

c) $\sum_{i=1}^{10} 3$

d) $\sum_{j=0}^8 (2^{j+1} - 2^j)$

$$\text{a) } \sum_{k=1}^5 (k+1) = (1+1) + (2+1) + (3+1) + (4+1) + (5+1) = 2 + 3 + 4 + 5 + 6 = 20$$

$$\text{b) } \sum_{j=0}^4 (-2)^j = (-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + (-2)^4 = 1 + -2 + 4 + -8 + 16 = 11$$

$$\text{c) } \sum_{i=1}^{10} 3 = 3 \cdot 10 = 30$$

$$\text{d) } \sum_{j=0}^8 (2^{j+1} - 2^j) = (2^0 + 1) - 2^0$$

