Intro to Sets [Discrete Mathmatics] Set Theory

Thetrevtutor

Set Theory

A set is a collection of objects called elements

 $= \{1, 2, 3\} = A$

visual list notation

sets can be finite or infinite

$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$\mathbb{Z}^+ = \{1, 2, 3, 4 \ldots\}$

"positive integers" Applied Pattern forever?

additional Points

- Repeated elements are listed once

$\{a, b, a, c, b, a\} = \{a, b, c\}$

- There is no order in a set.

$\{3, 2, 1\} = \{1, 2, 3\}$
$= \{2, 1, 3\}$

Common Sets

- Natural Numbers

$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$ or $\{1, 2, 3, \ldots\} \to \mathbb{Z}^+$

- Integers

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Rational #s

$\mathbb{Q} = \{\ldots, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \ldots\}$ ~ # you can write as a fraction.

Elements and Cardinality

Let $C = \{ \text{yellow}, \text{blue}, \text{red} \}$

yellow is in the set

"yellow is an element of C" $\quad$ $\text{yellow} \in C$

"Green is not an element of C" $\quad$ $\text{Green} \notin C$

"The cardinality of C is 3" $\quad$ $|C| = 3$
(size)
3 different Elements in C !

absolute Value

The Empty Set

symbol $\rightarrow \emptyset = \{ \}$

$\{ \emptyset \}$
element

$|\emptyset| = 0$

$|\{ \}| = 0$
empty set

$|\{\emptyset\}| = 1$
set has an empty set as an element

Set - Builder Notation

- $\mathbb{Q} = \{ .., 1/1, 1/2, 1/3, 2/3, .. \}$

Better way to express the set?

and n can't be zero!

$= \left\{ \frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0 \right\}$

have to be integers

Another Example: even integers

$2\mathbb{Z} = \{ \ldots, -4, -2, 0, 2, 4, \ldots \}$

$= \{ 2n \mid n \in \mathbb{Z} \}$

Such that $\quad$ n is an integer

★ Real-World Example ★

Desk = { drink, laptop, microphone }

= { x | x is on my desk }

<u>Exercises</u>

1) List the Elements of D $\quad D = \{ x \in \mathbb{Z}^+ \mid x < 6 \}$

such that

1, 2, 3, 4, 5

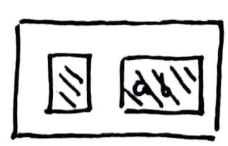
2.) What is the cardinality of D?

$|D| = 5$

elements in set

3.) What is the cardinality of $|\{ \phi, \{a, b\} \}| = 2$

like looking into a box



- look @ comma's !

How does in differ from this set here? $|\{ \phi, \{a\}, \{b\} \}| = 3$

3 elements in our box

$|\{ \phi, a, \{b\} \}| = 3$

Cartesian Products and Order Pairs

Cartesian Products

· An ordered pair (a,b) is a set

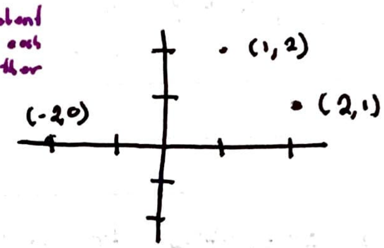
$$\{\{a\}, \{a,b\}\}$$

· You've seen ordered pairs before as graph coordinates

$(1,2) = \{\{1\}, \{1,2\}\}$ equivalent to each other

$(2,1) = \{\{2\}, \{2,1\}\}$

$(-2,0)$



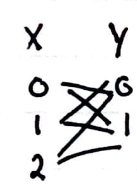
The Cartesian Product, $A \times B$, is the Set   Cross Product

$$\{(a,b) \mid a \in A \text{ and } b \in B\}$$

Given $X = \{0,1,2\}$ and $Y = \{0,1\}$

$$X \times Y = \{(0,0), (0,1), (1,0), (1,1), (2,0), (2,1)\}$$

Cross Product



$$Y \times X = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\}$$

1st 2nd

What is the Cardinality of $A \times B$?

if $|A| = m$ and $|B| = n$ then

$|A \times B| = mn$

$|X| = 3$ $\qquad$ $|X \times Y| = 3 \cdot 2$

$|Y| = 2$ $\qquad\qquad\qquad = 6$

Cartesian Products can generalize to n-tuples

3-tuple

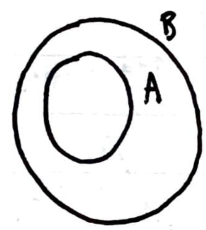
$A \times B \times C = \{ (a, b, c) \mid a \in A, b \in B, c \in C \}$

n-tuple

$A_1 \times A_2 \times \ldots \times A_n = \{ (a_1, a_2, \ldots a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n \}$

Subsets and Powersets

If A is a subset of B, then every element in A must also be in B.



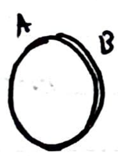
subset of

$A \subseteq B$

— can equal
equivalent

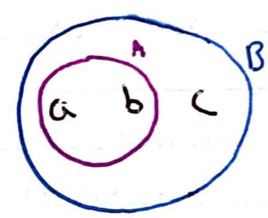
$A \subset B$

A must strictly be smaller than B



• if A and B are the same size (the little bar _ at the bottom)

True or False? Examples



$\{a, b, a, a\} \subseteq \{a, b, c\}$

-repeated elements do not matter

$\overset{A}{\{a, b\}} \subseteq \overset{B}{\{a, b, c\}}$ True

$\{c, d\} \subseteq \{c, d\}$ True

list out all the elements

$\overset{A}{\{a\}} \not\subseteq \overset{B}{\{\{a\}\}}$ False

A: a B: {a}

We dont see a in B,
We just see the set containing a

line thru it means not a subset

$\emptyset \subseteq \{x, y, z\}$ True

^ an empty set is a subset of every set !

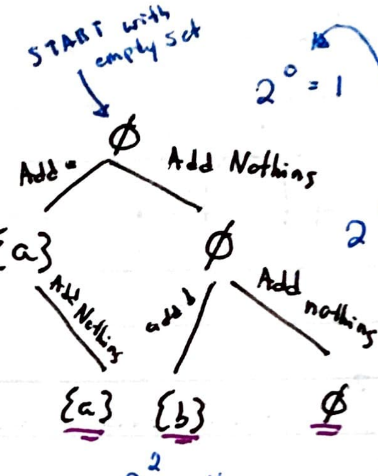
Power Sets

A **power set** of A, $p(A)$, is the set containing all possible subsets of A.

$A = \{a, b\}$

$p(A) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$

Power Set, contains all possible subsets of A.



$2^0 = 1$ — Imagine if our original Set A had no element in it. We could have 1 possible subset (empty set)

$2^1 = 2$

★ Tree doesn't work when you get more and more elements ★

= possible subset of set containing a and b

$2^2 = 4$ ~ possible subsets ★

power set grows by powers of 2.

Size of Power Set

if $|A| = n$, then $|p(A)| = 2^{|A|} = 2^n$

- for each element, we can add it to a subset, or we cannot add it to a subset.
  - → 2 Choices per element ~ add or do not add

$|A| = 6 \qquad |p(A)| = 2^{|A|} = 2^6 = 64$ ~ possible subsets

The Empty set is an element of every single Power set

Tricky Questions

★ Set contains nothing in it. ★

powerset of the empty set → $p(\emptyset) = \{\emptyset\} = \{\{\}\}$

So the power set of the empty set is just the set containing the empty set

Size of the Empty set
$|\emptyset| = 0$
$|p(\emptyset)| = 2^0 = 1$

powerset of the set containing nothing ↗ $p(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$

"set containing" the empty set

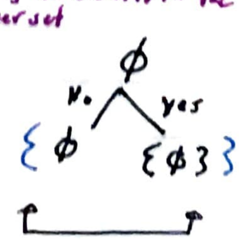
So there are a set inside containing no element in the powerset

$|\{\emptyset\}| = 1 \qquad |p(\{\emptyset\})| = 2^1 = 2$

size of the set containing nothing

Should have 2 elements in it.

Can draw tree for this example



∴ The Powerset is just going to be these two possible subsets in a set

$p(\{\emptyset\}) = \{\emptyset, \underbrace{\{\emptyset\}}_{A}\}$

We see A is an element

$\emptyset$

No Yes

$\{\emptyset \quad \underbrace{\{\emptyset\}}_{A}\}$

Is $A \subseteq p(A)$ for any A? No

is A a subset of the power set of A for any A?

Is $A \in p(A)$ for any A? Yes

Element Notation

Exercises

1.) Let $|C| = K$ and $|D| = j$, what is $|p(C \times D)|$?

$|C \times D| = |C| \times |D|$

$= Kj$

$|p(C \times D)| = 2^{|C \times D|} = 2^{Kj}$

2.) List the elements of $p(p(\emptyset))$

$p(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$

3.) if $|A| = m$, what is $|p(p(p(A)))|$?

bar indicated size

i.e. what is the size of the powerset of the powerset of the powerset of A?

$= 2^{|p(p(A))|}$

$= 2^{2^{|p(A)|}}$

$= 2^{2^{2^{|A|}}}$

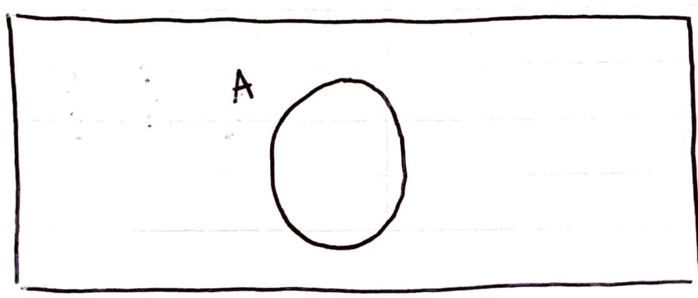
$= 2^{2^{2^m}}$

Set Operations

Set Operations and Venn Diagrams

Every Set A exists within some universe U.

Box represents U, some universe



$\sim U = \mathbb{Z}$
$= \mathbb{R}$
$= \mathbb{Z}^+$

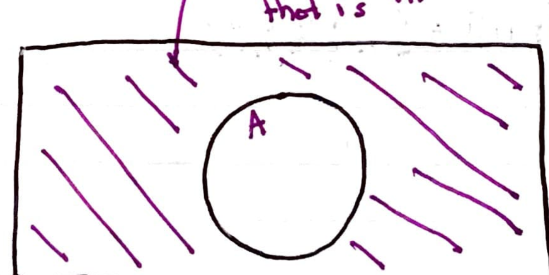
The Complement of A is written as $\bar{A}$.

$\bar{A} = \{ a \in U \mid a \notin A \}$

Can be written as A^c or A' as well.

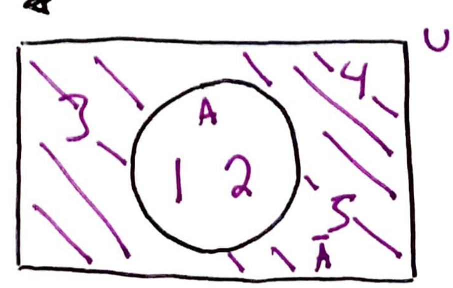
· in order to define a complement you need a universe to talk about.

The Complement of A, is everything outside of the set A that is in the universe



Example: $A = \{1,2\}$
$U = \{1,2,3,4,5\}$

$\bar{A} = \{3,4,5\}$

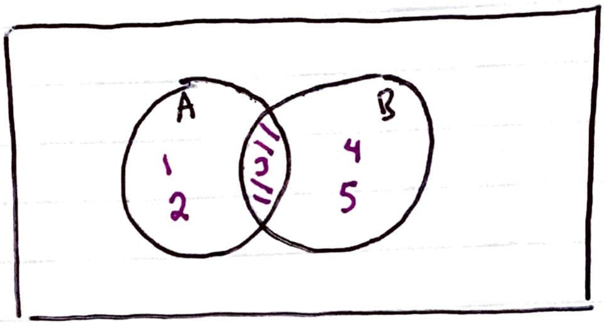


Given Sets A and B, the intersection of A and B is:

$$A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$$

$A = \{1,2,3\}$

* upside down U



$B = \{3,4,5\}$

$A \cap B = \{3\}$

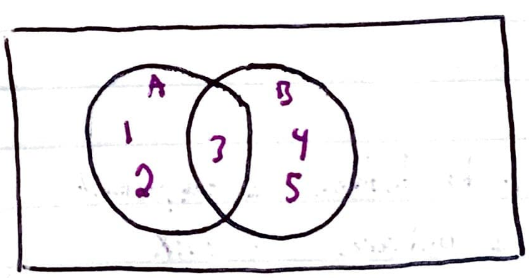
Given Sets A and B, the union of A and B is

$$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$$

Union is wether the element is in A or B, can be in both or just one

$A \cup B = \{1,2,3,4,5\}$

take all elements and put them in a set!



Set Operations Video

Given the sets A and B, the difference, $A-B$, is defined as

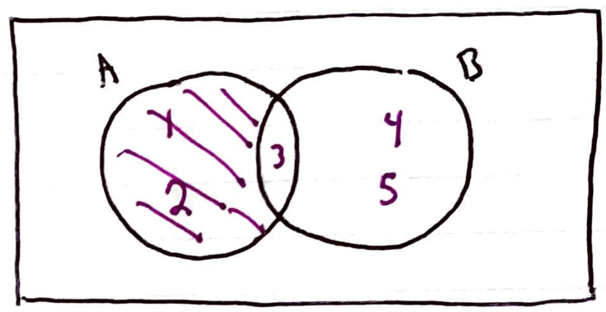
$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

or $A \setminus B$

or $A \cap \overline{B}$

$A - B = \{1, 2\}$

You take everything in A and subtract whatever is in B



Exercises

$A = \{1,3,5,7,9\}$ $B = \{4,8,12,16\}$ $C = \{1,4,9,16\}$

$A \cup B = \{1,3,5,7,9,4,8,12,16\}$

$C \cap B = \{4,16\}$

$C - B = \{1, \cancel{4}, 9, \cancel{16}\} = \{1,9\}$ C difference B

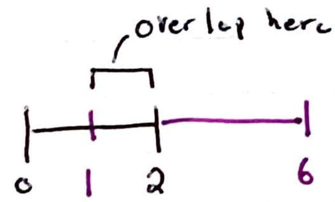
$\emptyset \cap B = \{\}$ $\emptyset$

b/c there's no elements in the empty set!

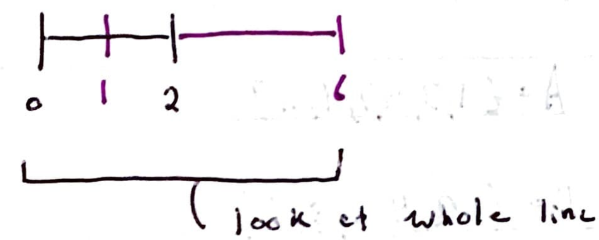
Set Operations Examples

If $A=[0,2]$ and $B=[1,6]$, determine the following where $\mathcal{U}=\mathbb{R}$

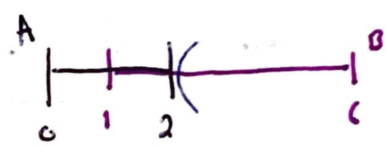
- $A \cap B = [1,2]$



- $A \cup B = = [0,6)$



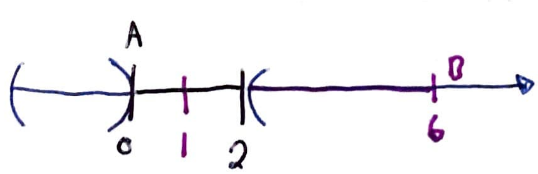
- $B - A = (2,6)$ open interval



- $\bar{A} = (-\infty, 0) \cup (2, \infty)$

have to use a union to describe this set.

everything in the universe − A

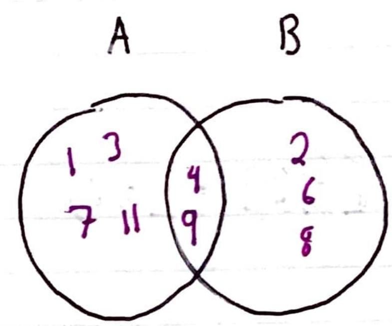


Find Sets A and B where:

$A - B = \{1, 3, 7, 11\}$

$B - A = \{2, 6, 8\}$

$B \cap A = \{4, 9\}$



$A = \{1, 3, 4, 7, 9, 11\}$

$B = \{2, 4, 6, 8, 9\}$

Prove: If $A \subseteq B$ and $C \subseteq D$, then:

(i) $A \cap C \subseteq B \cap D$

Let $x \in A \cap C$

$x \in A$ and $x \in C$

$x \in B$ and $x \in D$

$x \in B \cap D$

(ii) $A \cup C \subseteq B \cup D$

Let $x \in A \cup C$

$x \in A$ or $x \in C$

$x \in B$ or $x \in D$

$x \in B \cup D$

Exercise : Let $A = \{a,b\}$ and $B = \{c,d\}$

$A \times B = \{(a,c), (a,d), (b,c), (b,d)\}$

$B^2 = B \times B = \{(c,c), (c,d), (d,c), (d,d)\}$

$\emptyset \times A = \emptyset$ ← you get the empty set back

empty set

$|\emptyset \times A| = |\emptyset| \cdot |A| = 0 \cdot 2 = 0$

more Questions:

if $|B| = m$ and $|A| = n$ then find... What is the size of these

$|A \times B| = n \cdot m$

$|A^2| = n \cdot n = n^2$

$|B^{32} \times A^{19}| = m^{32} n^{19}$

Indexed Sets and Well Ordering Principle

SET THEORY

Can we shorten $A_1 \cap (A_2 \cap (A_3 \cap (A_4 \cap A_5)))$? ~ Yes!
What about Unions?

Intersection notation → $\bigcap_{i=0}^{n} A_i = A_0 \cap A_1 \cap A_2 \cap \ldots \cap A_n$

intersection from i=0 to n

Similar to $\sum_{i=0}^{n} a_i = a_1 + a_2 + \ldots a_n$

Union Notation → $\bigcup_{i=0}^{n} A_i = A_0 \cup A_1 \cup \ldots \cup A_n$

Example:

Brackets

$$\bigcap_{i=0}^{n+1} A_i = \left(\bigcap_{i=0}^{n} A_i\right) \cap A_{n+1}$$

Well ording Principle

↳ Any non empty subset of $\mathbb{N}$ has a least element.
Used in the Divison Algorithm which will be shown when we get to number Theory.

$\mathbb{N} = \{1, 2, 3, \ldots\}$

←

$k < j < i$ k is the smallest element.

$A = \{i, j, k\} \subseteq \mathbb{N}$

→ However: set of all integer

$\mathbb{Z} = \ldots -\infty$, we can have negative ∞

* We can't say there's a small element

However if you have $\mathbb{Z}^+$ ✓ + is positive integers ✓

$\{\ldots, 1, 2, 3, 3, \ldots\}$

$$\bigcap_{i=0}^{n+m} A_i = \underbrace{\left(\bigcap_{k=0}^{n} A_k\right)}_{(A_0 \cap \ldots \cap A_n)} \cap \underbrace{\left(\bigcap_{j=n+1}^{n+m} A_j\right)}_{(A_{n+1} \cap \ldots \cap A_{n+m})}$$