The Foundations: Logic and Proofs Section 1.1 – 1.7

1.1 Propositions, Negations, Conjunctions and Disjunctions

A Proposition is a Declarative Statement that is either True or False.

a. The sky is blue

b. The Moon is made of cheese

c. Luke, I am your Father

d. Sit Down

e. $X + 1 = 2$

Constructing Compound Propositions

A compound proposition is comprised of Propositions and one or more of the following connectives:

Negation $\neg$ "Not" $\neg p$

Conjunction $\wedge$ "And"

Disjunction $\vee$ "Or"

Implication $\rightarrow$ "If, then "

Biconditional $\leftrightarrow$ "If and Only if"

Each Proposition is Represented by a Propositional Variable (p, q, r, s, ….)

Example: $\boxed{p \rightarrow q}$

Negation

The Negation of the Propositional p is $\neg p$ (not p).

Example:

If p denotes " the grass is green," then $\neg p$ denotes "It is not the case that grass is green ", or more simply, " The Grass is not Green."

A. My Dog is the cutest dog.

p $\neg p$: My dog is not the cutest dog

B. The door is not open $\neg p$ The door is open

p

C. Are ~~we~~ there yet?

Truth Tables

Each row of a truth table gives us one possibility for the truth values of our proposition(s). Since each proposition has two possible truth values, True or False, we will have 2 rows for each proposition (or 2^n Rows were n = number of propositions)

Truth Table for $\neg$ p

Combinations	Connective
p	$\neg p$
T	F
F	T

p My dog is the cutest dog.

$\neg p$ My dog is not the cutest dog

Conjunction

The Conjunction of Propositions p and q is denoted $p \wedge q$ and read p "and " q.

For a conjunction to be True, BOTH Propositions must be True.

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

$p \wedge q = \min(p, q)$

- Only true when both p and q are true.

Disjunction

The disjunction of Propositions p and q is denoted p V q and read p "OR" q.

For a disjunction to be True, EITHER Proposition must be True.

p	q	p V q
1	1	1
1	0	1
0	1	1
0	0	0

$p \vee q = \max(p, q)$

The Connective "OR" in English " X OR"

Inclusive "OR" $p \vee q$

The prerequisite for MA420 is either MA315 OR MA335

Exclusive "OR" $p \oplus q$

You get soup or salad with your entrée.

Truth Table:

p	q	$p \vee q$	$p \oplus q$
T	T	T	F
T	F	T	T
F	T	T	T
F	F	F	F

Implications Converse, Inverse, Contrapositive and Biconditionals

Constructing Compound Propositions

A compound proposition is comprised of propositions and one or more of the following connectives:

Negation $\neg$ "NOT"

Conjunctions $\land$ "AND"

Disjunction $\lor$ "OR"

Implication $\rightarrow$ "If, then"

Biconditional $\leftrightarrow$ "If and Only if"

Each Proposition is represented by a propositional variable (p, q, r, s …)

Example: $p \rightarrow q$

Implication (Conditional Statement)

The implication of propositions p and q is denoted p $\rightarrow$ q and read “If p then q” or “p implies q”.

When the hypothesis is True, the conclusion must be True for the implication to be True.
When the Hypothesis is False, then conclusion is True.

p denotes “it is a holiday”

q denotes “The store is closed “

p	q	$p \rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

Converse, Inverse, Contrapositive

From our implication, $p \to q$, we can construct 3 new conditional statements

1. Converse $q \to p$ Switch Order
2. Inverse $\neg p \to \neg q$ Negate
3. Contrapositive $\neg q \to \neg p$ Switch and negate

Example:

It is raining is a sufficient condition for me not going to town

$\boxed{\text{If}}$ $\underbrace{\text{it is raining,}}_{p}$ $\boxed{\text{then}}$ $\underline{\text{I won't go to town}}$

$p \to q$

Practice

Give the converse, inverse and contrapositive or the implication:

Prof. B is happy when you complete your homework

q p

If you complete your HW, Then Prof B. is happy

Converse

$q \to p$

If Prof. B is happy, then you completed your HW.

Inverse

$\neg p \to \neg q$

If You didn't complete your HW, then Prof. B is not happy.

Contrapositive

$\neg q \to \neg p$

If Prof B is not happy, then you did not complete your HW

Biconditional

The biconditional of propositions p and q is denoted $p \leftrightarrow q$ and read " p IF AND ONLY IF q"

For a biconditional to be true, both propositions must share the same truth value

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

A Preview…

Our Biconditional $p \leftrightarrow q$ can also be written as a compound proposition.

$$(p \leftrightarrow q) \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	1	0	0	0
0	0	1	1	1	1

Constructing a Truth Table for Compound Propositions

Compound Proposition Truth Table Walk-Thru

Row

- Need a row for every possible combination of values for the compound propositions

Columns

- Need a column for each propositional variable
- Need a column for the truth value of each expression that occurs in the compound proposition as it is built up

- Need a column for the compound proposition (usually at far right)

Order of Operations

Precedence	Operator
1	$\neg$
2	$\wedge$
3	$\vee$
4	$\rightarrow$
5	$\leftrightarrow$

Compound Proposition Truth Table Walk-Thru

Construct a truth table for p V q $\rightarrow$ ¬r

1. First, construct columns for each proposition; p, q, r (there may be referred to as atomic propositions)

p	q	r

2. Next Create a column for each compound proposition; p V q, ¬r

p	q	r	p V q	¬r

3. Lastly, create a column for the final compound proposition

p	q	r	p V q	¬r	pVq $\rightarrow$ ¬r

4. Now create as many rows as necessary for all possible combinations of expressions that occur in your compound proposition. Because we have 3 propositions, we need $2^3 = 8$ rows.

5. Fill in the Truth Table for your proposition first. Now complete the table

p	q	r	p V q	¬r	pVq $\rightarrow$ ¬r
1	1	1	1	0	0
1	1	0	1	1	1
1	0	1	1	0	0
1	0	0	1	1	1
0	1	1	1	0	0
0	1	0	1	1	1
0	0	1	0	0	1
0	0	0	0	1	1

Practice

Create a truth table for $(p \vee \neg q) \rightarrow (p \wedge q)$

"or"

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
1	1	0	1	1	1
1	0	1	1	0	0
0	1	0	0	0	1
0	0	1	1	0	0

1.2 Translating Propositional Logic Statements

Translating English Sentences

1. Identify Atomic Propositions

2. Determine appropriate logical connective

p q

If I go to the store or the movies, I won't do my homework

r

p : I go to the store

q : I go to the movie

r : I do my Homework

$$\boxed{(p \vee q) \longrightarrow \neg r}$$

Practice

$H \to C$

1. You can get a free sandwich on Thursday if you buy a sandwich or a cups of soup.

C H

p: I buy a sandwich
q: I buy a cup of soup
r: I get a free sandwich on Thursday

$\boxed{(p \vee q) \to r}$

2. You can get a free sandwich on Thursday only if you buy a sandwich or 4 cups of soup.

$\boxed{r \to (p \vee q)}$

if

3. The automated reply can't be sent when the system is full.

p: The system is full

q: The automated reply can be sent

$\boxed{p \to \neg q}$

Translating Propositions

q: You can ride the rollercoaster

r: You are under 4 feet tall

s: you are older than 16 years old

Translate: $(r \vee \neg s) \rightarrow \neg q$

$\boxed{\text{if r or not s, then not q}}$

Not older

If you are under 4 ft tall or younger than 16 years old then you can not ride the rollercoaster.

1.3 Proving Logical Equivalences with Truth Tables

Some Terminology

"or" "and"

p	$-p$	$p \vee \neg q$	$p \wedge \neg p$
1	0	1	0
0	1	1	0

A tautology is a proposition which is always true. Example: $p \vee \neg p$

A contradiction is a proposition which is always false. Example: $p \wedge \neg p$

A contingency is a proposition which is neither a tautology nor a contradiction.
Example: p

Logical Equivalences

"iff"

Two compound propositions p and q are **logically equivalent** if $p \leftrightarrow q$ is a **tautology**, meaning they have the same truth value in all possible cases.

- We write this as $p \equiv q$ where p and q are compound propositions
- Let's use this truth table to show that: $\boxed{\neg p \vee q \equiv p \rightarrow q}$

"if p then q"

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
1	1	0	1	1
1	0	0	0	0
0	1	1	1	1
0	0	1	1	1

Proof that $\neg p \vee q \equiv p \rightarrow q$

Practice

Determine if $\neg p \wedge q \equiv \neg p \vee \neg q$

"or"

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
1	1	1	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	0	1	1	1	1

Proof that $\neg p \wedge q \equiv \neg p \vee \neg q$

Practice

Determine if $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Proof that $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

Key Logical Equivalences including De Morgan's Law

Logical Equivalence

Identity Laws

$p \wedge T \equiv p$

$p \vee F \equiv p$

Double Negation Law

$\neg(\neg p) \equiv p$

T· tautology

F· Contradiction

Domination Laws

$p \vee T \equiv T$

$p \wedge F \equiv F$

Absorption Laws

$p \vee (p \wedge q) \equiv p$

$p \wedge (p \vee q) \equiv p$

Idempotent Laws

$p \vee p \equiv p$

$p \wedge p \equiv p$

Negation Laws

$p \vee \neg p \equiv T$

$p \wedge \neg p \equiv F$

Commutative Laws - Order

$p \vee q \equiv q \vee p$

$p \wedge q \equiv q \wedge p$

Distribution Laws

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Associative Laws ~ Grouping

$(p \vee q) \vee r \equiv p \vee (q \vee r)$

$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

De Morgan's Laws

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

A few More Equivalences

$p \to q \equiv \neg p \vee q$

$p \to q \equiv \neg q \to \neg p$

$p \vee q \equiv \neg p \to q$

$p \wedge q \equiv \neg(p \to \neg q)$

$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$

$(p \to r) \wedge (q \to r) \equiv (p \wedge q) \to r$

$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$

$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Constructing New Logical Equivalences

Reminder – Showing equivalence with a truth table

Show $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

p	q	$\neg p$	$\neg q$	$\neg p \wedge q$	$p \vee (\neg p \wedge q)$	$\neg(p \vee (\neg p \wedge q))$	$\neg p \wedge \neg q$
1	1	0	0	0	1	0	0
1	0	0	1	0	1	0	0
0	1	1	0	1	1	0	0
0	0	1	1	0	0	1	1

Constructing New Logical Equivalences

Show $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$ by developing a series of logical equivalences.

$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$ By 2nd De Morgans Law

$\equiv \neg p \wedge (\neg\neg p \vee \neg q)$ By 1st De Morgans Law

$\equiv \neg p \wedge (p \vee \neg q)$ By Double Negation Law

$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$ By 2nd Dist. Law

$\equiv F \vee (\neg p \wedge \neg q)$ By 2nd Negation law

$\equiv (\neg p \wedge \neg q) \vee F$ By Comm. Law for Disjunction

$\equiv \neg p \wedge \neg q$ By identity Law

Show $(p \wedge q) \to (p \vee q) \equiv T$ is a tautology by developing a series of logical equivalences.

$$
\begin{aligned}
(p \wedge q) \to (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && p \to q \equiv \neg p \vee q \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && 1^{st} \text{ Demorgans Law} \\
&\equiv (p \vee \neg p) \vee (q \vee \neg p) && \text{Commutative, associative laws for Disjunction, } 2^{nd} \text{ Negation Law} \\
&\equiv T \vee T \\
&\equiv T
\end{aligned}
$$

Practice

Use logical equivalences to show

$$\neg(\neg p \vee q) \equiv \neg q \wedge p$$

$$
\begin{aligned}
\neg(\neg p \vee q) &\equiv \neg\neg p \wedge \neg q && 1^{st} \text{ DeMorgan Law} \\
&\equiv p \wedge \neg q && \text{Double Negation Law} \\
&\equiv \neg q \wedge p && \text{Comm. Law}
\end{aligned}
$$

$$\boxed{\therefore \neg(\neg p \vee q) \equiv \neg q \wedge p}$$

1.4 Predicate Logic

When Propositional Logic Fails

If I say,

- All candy is made with chocolate is delicious
- M&M's are made with chocolate

Does it follow that M&M's are delicious?

We can't model this relationship with propositions ~ Statements that are True or False

This is where we need predicate logic which includes:

Variables: x, y, z, there are the subjects of the statement(s)

Predicates: A property the variable can have. (example "is greater than 3") > 3

Quantifiers: Covered in the next video

Predicates

- Statements involving variables, such as “x < 2 “ and “ x + y = z “ are often found in mathematical assertions, in computer programs and in system specifications. The statements are neither true nor false when the values of the variables aren’t specified.

- The statement “ x is less than 2 “ has two parts. First, the variable x is the subject of the statement. The second part, the predicate, “is less than 2 “ refers to the property that the subject of our statement can have. The predicate, “ is less than 2 “ can be denoted by P (x), where P denotes the predicate and x the variable.

Predicate logic involves variables

Variables : x, y, z are the subject of these statements

$x < 2$

x is the subject of the statement
The predicate is " less than 2"

$P(x)$

Propositional Functions

Propositional functions become propositions (and have truth values) when variables are each replaced by a value from the domain (or bound by a quantifier, we will see later).

The statement P (x) is said to be the value of the propositional function P at x.

For example, let P (x) denote “x > 0 “ and the domain be the integers.

Then:

P (-3) is false $-3 > 0$ False

P (0) is false $0 > 0$ False

P (3) is true $3 > 0$ True

Often the domain is denoted by U. So in this example U is the integer

Examples of Propositional Functions

Variables

Domain

Let “ x + y = z “ be denoted by R (x,y,z) and U (for all three variables) be the integers.
Find these truth values

Right now this is considered a propositional function.

R(2,-1,5) $2 + -1 = 5$ False

R(3,4,7) $3 + 4 = 7$ True

R(x,3,z) Not a Proposition

2 variables not yet defined

Now that we've Given x,y,z Values its now a Proposition (Has Truth Value of True or False)

Compound Expressions

Connectives from propositional logic carry over to predicate logic

If P(x) denotes “ x > 0”, find these truth values:

P(3) V P (-1)	T V F	Solution: T
P(3) P(-1)	T F	Solution: F
P(3) P(-1)	T F	Solution: F
P(3) P(-1)	T T	Solution: T

Expressions with variables are not propositions and therefore do not have truth values. For example:

$$P(3) \land P(y)$$
$$P(x) \rightarrow P(y)$$

When used with quantifiers (to be introduced next), these expressions (propositional functions) become propositions

Quantifiers

Quantifiers

As we know, a propositional function P(x) is not a proposition until it has a Truth Value.
Up to this point, we could only do this by assigning a value to our variable.

Example:

If P(x) represents " $x > 0$ ", Find the truth value for P (4).

$4 > 0$ TRUE

Now we will turn a Propositional function into a proposition using a quantifier.

Let's Focus on the two most widely used Quantifiers:

$\forall_{\underline{\underline{x}}} P(x)$

" For All "	$\forall$	Universal Quantifier
" There Exists "	$\exists$	Existential Quantifier

The Universal Quantifier " $\forall$ "

The statement, $\forall x\, P(x)$ tells us that the proposition P(x) must be True for ALL values of x in the domain of discourse/universe.

" For every x in the domain "

Example:

Let P(x) represent " x > 0 ". Find each Truth value for $\forall x\, P(x)$

a. U is $\mathbb{Z}$ ~ Integers $-3 \in \mathbb{Z}$ $-3 > 0$ False " Counter Example "

b. U is $\mathbb{Z}^+$ ~ Positive Integers TRUE

The Existential Quantifier “ $\exists$ “

The statement, $\exists x\, P(x)$ tells us that the proposition P(x) is True for SOME value(s) of x in the domain of discourse/universe.

Example:

Let P(x) represent “ x > 0 “. Find each Truth value for $\exists x\, P(x)$

a. U is $\mathbb{Z}$ $\quad 7 \in \mathbb{Z} \quad 7 > 0$ TRUE

b. U is $\mathbb{Z}^-$ $\quad$ FALSE

~ All negative integers are < 0, so false

The Quantifiers

Universal	Existential
$\forall$	$\exists$
"For All"	"There Exists"
When True? **When P (x) is true for every x in the domain.**	**When True?** **There is an x in the domain for which P(x) is True**
When False? **There is an x in the domain for which P(x) is False**	**When False?** **When P(x) is false for every x in the domain**
$\forall x P(x) \equiv P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$	$\exists x P(x) \equiv P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$

Practice

Let P(x) be " $x^2 > 0$ " if the domain consists of all integers. Find the Truth values of:

$\forall x P(x)$ $\quad 0 \in \mathbb{Z} \quad 0^2 > 0$ FALSE

$\therefore \forall x\, P(x)$ is FALSE

$\exists x P(x)$ $\quad 1 \in \mathbb{Z} \quad 1^2 > 0$ TRUE $\quad \therefore \exists x\, P(x)$ is TRUE

← There only needs to be one

Let P(x) be " $x^2 < 0$ ". If the domain consist of all integers, find the truth values of:

$\forall x P(x)$ FALSE $\quad (-1)^2 < 0 \quad 1 < 0$, False $\quad \therefore \forall x\, P(x)$ is False

$\exists x P(x)$ FALSE $\quad \therefore \exists x\, P(x)$ is False

Let P(x) be " $x + 1 = 2x$ " if the domain consists of all integers. Find the Truth values of:

$\forall x P(x)$ $\quad 0 \in \mathbb{Z},\ 0+1 = 2(0)$: FALSE $\quad \therefore \forall x\, P(x)$ is False

$\exists x P(x)$ $\quad 1 \in \mathbb{Z},\ 1+1 = 2(1)$: TRUE $\quad \therefore \exists x\, P(x)$ is True

Let P(x) be " $x^2 < 16$ ". If the domain consists of all integers, find the truth values of:

$\forall x P(x)$ FALSE

$\exists x P(x)$ TRUE

The Uniqueness Quantifier $\exists !$

The statement, $\exists !x\ P(x)$, tells us that the proposition P(x) is true for exactly ONE value of x in the domain of discourse/universe.

Give the truth value for $\exists !x\ P(x)$ for each proposition in the domain of all integers.

a. P(x) represents " 2x = 4 " $2x = 4$ $x = 2$ TRUE

b. P(x) represents " 2x > 4 " $2x > 4$ $x > 2$ FALSE $x = 3, 4$ more than one solution

c. P(x) represents " 2x = 3" $2x = 3$ $x = 1.5$ FALSE

Negating the Translating with Quantifiers

Quantifiers we've learned

Let P(x) be the statement " x has taken a course in programming " for the domain of students in your class.

$\forall x P(x)$ Every student in my class has taken a course in programming.

$\exists x P(x)$ There is a student in my class who has taken a course in programming.

Negating Quantifiers

Let P(x) be the statement " x has taken a course in programming " for the domain of students in your class.

$\neg \forall x P(x)$ There is a student in my class who hasnt taken a course in programming.

$\xleftrightarrow{\text{Equivilent}}$ $\exists x \neg P(x)$

$\neg \exists x P(x)$ All students in my class have not taken a course in programming.

$\xleftrightarrow{\text{Equivilent}}$ $\forall x \neg P(x)$

De Morgan's Laws for Quantifiers

$$\boxed{\neg \exists x P(x) \equiv \forall x \neg P(x)}$$

True when P(x) is False for every x
False when there is an x for which P(x) is True

$$\boxed{\neg \forall x P(x) \equiv \exists x \neg P(x)}$$

True when there is an x for which P(x) is False
False when P(x) is True for every x

Translating and Negating

Negate the statements:

- "There is an honest politician"

- All Americans eat cheeseburgers

There is an honest politician

H(x) Represents "x is Honest" For domain of all Politicans

$\exists x\, H(x)$

"Not" operator

Translate Negation

$$\boxed{\neg \exists x\, H(x) \equiv \forall x \neg H(x)}$$

Every politician is dishonest

DM Law

All Americans eat cheeseburgers

C(x) Represent "x Eats Cheeseburgers" For domain of all Americans.

$\forall x\, C(x)$

$$\boxed{\neg \forall x\, C(x) \equiv \exists x \neg C(x)}$$

Not every American eats cheeseburgers.

Translating

Some student in this class has visited Mexico.

Domain: Students in class

M(x) Rep: "x has visited Mexico"

There exist a student

$$\boxed{\exists x\, M(x)}$$

Domain: All people

We have to look at everyone.

Two Propositional Function

M(x) Rep: "x has visited Mexico"

C(x) Rep: "x is a student in this class"

student in the class

* Parenthesis are important!

$$\boxed{\exists x\, (M(x) \land C(x))}$$

has to have both of these conditions

More Practice

Every student in this class has visited Canada or Mexico.

Domain: Students in class

m(x) Rep: "x has visited Mexico"

C(x) Rep: "x has visited Canada"

$$\boxed{\forall x\,(C(x) \vee m(x))}$$

One way to write this.

- many ways to solve these kind of problems

Domain: All people

m(x) Rep: "x has visited Mexico"

C(x) Rep: "x has visited Canada"

$$\boxed{\forall x\,(S(x) \rightarrow (C(x) \vee m(x)))}$$

~ using "if, then"

For all people, $\underbrace{\text{if they're a student in the class}}_{S(x)}$, then they have visited $\underbrace{\text{Canada}}_{C(x)}$ or $\underbrace{\text{Mexico}}_{M(x)}$

Propositional Functions Examples

1. $\exists x S(x, \text{open})$
2. $\forall x (S(x, \text{malfunctioning}) \vee S(x, \text{diagnostic}))$
3. $\exists x S(x, \text{open}) \vee \exists x S(x, \text{diagnostic})$
4. $\exists x \neg S(x, \text{availble})$
5. $\forall x \neg S(x, \text{working})$

$S(x, y)$ = x is in state y

Domain: All system x and states y

1. There exists a System that is open.
2. Every system is either malfuncting or diagnostic.
3. Some systems are open, or some systems are diagnostic
4. There exist a system that is unavialble
5. None of these systems are working.

★ ~ Can/should Rewrite ~ ★

Example 1:

$\forall x$

Every user has acess to an electronic mailbox

Propositional Function

$M(x)$ = x can acess a mailbox.

need to "Quantify" it.

$\forall x\, M(x)$

$M(x, y)$ = x can acess system y

$\forall x\, M(x, \text{mailbox})$

Example 2:

then

The system mailbox can be accessed by everyone in the group if the file system is locked.

Define predicate

$L(y)$ = system y is Locked

$M(x)$ = x can acess a mailbox.

$L(\text{file system}) \rightarrow \forall x\, M(x)$

Example 3:

The fire wall is in a diagnostic state only if the proxy server is in a diagnostic state.

D(y) = system y diagnostic

$$D(\text{proxy}) \rightarrow D(\text{fire wall})$$

Example 4:

At least one router is functioning normally if the throughput is between 100 Kbps and 500 Kbps and the proxy server is not in diagnostic mode.

y = routers

Define Domain

D(y) = x is in proxy

N(y) = y is functioning normally

$$\left(T(100, 500) \land \neg D(\text{proxy})\right) \rightarrow \exists y\, N(y)$$

T(a, b) = throughput is between a & b

1.5 Nested Quantifiers and Negations

Every real number has an additive inverse.

$$\forall x \exists y (x + y = 0)$$

Is this true?

TRUE

Domain ~ all real #'s , reasoning + proof

$x = \{0, 1, 2, 3$

Problems:

Let $P(x,y)$ denote "$xy = yx$". Assume the domain is the real numbers.

Read as: For all real numbers x, and all for all real numbers y,

① is $\forall x \forall y\, P(x,y)$ true?

$\boxed{xy = yx}$

True, yes the commutative property of multiplication, that the order doesnt matter, i'm going to end up at the same product

② is $\forall y \forall x\, P(x,y)$ true? For all real numbers y, and all for all real numbers x,

True, by the same property

Let $Q(x,y)$ denote "$x + y = 5$". Assume the domain is the real numbers.

For all real #'s x there exists a real # y such that $x + y = 5$.

Is $\forall x \exists y\, Q(x,y)$ true?

for every single real #

True

e.g. $2 + \_\_ = 5$?
$2 + 3 = 5$

Dealing with real #'s
$\frac{1}{2} + 4\frac{1}{2} = 5$

Is $\exists y \forall x\, Q(x,y)$ true?

There exists a real # y such that for all real #s x, $x + y = 5$.

Let $y = 0$

$x + y = 5$
$1 + 0 \neq 5$
$1 + 4 = 5$ ✓

False

Practice

Let U be the real numbers and $P(x,y)$ denote "$x \cdot y = 0$". Find the truth values for the following:

$\forall x \forall y\, P(x,y)$ False

Counter example: $2 \cdot 3 \neq 0$

$\forall x \exists y\, P(x,y)$ True

$13 \cdot \underline{0} = 0$ $\quad -\frac{1}{2} \cdot \underline{0} = 0$

Tricky ?

$\exists x \forall y\, P(x,y)$ True

there exist some x, that for all of y, $P(x,y)$ is true

Going to be using zero

$\underline{0} \cdot 2 = 0$

$\exists x \exists y\, P(x,y)$ True

$-37 \cdot 0 = 0$ ✓, $0 \cdot -\frac{4}{3} = 0$

Let U be the real numbers and $P(x,y)$ denote "$\frac{x}{y} = 1$". Find the truth values for the following:

$\forall x \forall y\, P(x,y)$ False

counter example: $\frac{3}{4} \neq 1$

$\forall x \exists y\, P(x,y)$ False

- dealing with real numbers

counter example: $\frac{0}{1} \neq 1$

b/c zero is an real number

$\exists x \forall y\, P(x,y)$ False

counter example: $\frac{15}{y} = 1$ $\quad \frac{15}{15} = 1$ $\quad \frac{15}{32} \neq 1$

~ Won't Work

$\exists x \exists y\, P(x,y)$ True

$\frac{13}{13} = 1$

Let $P(x,y)$ denote $(x=-y)$. Find the negation of $\forall x \exists y\, P(x,y)$.

no negation to proceed a quantifier.

$$\neg(\forall x \exists y\ P(x,y))$$

$$\exists x\, \neg(\exists y\, P(x,y))$$

now negating this

$$\exists x\ \forall y\ \neg P(x,y)$$

Read as:

There exists a real # x such that for all real #'s y

$$x \neq -y$$

Translating With Nested Quantifiers

Translate "The sum of two positive integers is always positive" into a logical expression.

(">0" is written above "always positive".)

★ - lots of ways to write these ★

For all positive integers $\underline{x}$ and $\underline{y}$, $x+y>0$

① $\forall x\,\forall y\big(((x>0)\wedge(y>0))\rightarrow \underbrace{\cancel{(x+y>0)}}_{P(x,y)}\big)$, domain $\mathbb{Z}$ ← All integers

② $\forall x\,\forall y\ \overbrace{\cancel{((x+y)>0)}}^{P(x,y)}$, domain $\mathbb{Z}^+$

③ Let $P(x,y)$ denote $(x+y>0)$

~ Alot of correct ways to do these problems

Person recieving email

Let E(x, y) denote " x is sent y and email " and T (x, y) denote " x sent y a text ".

Person sending email

Translate the following into predicate logic, with domain of students in the class.

$\forall$

a. Every student in the class send an email to Joe.

$\forall x\, E(x, Joe)$

$\forall x\,((x \neq Joe) \to E(x, Joe))$

$\exists$

b. There is a student in class who has not received a text or email from any other student in class.

negation

$\exists y\, \forall x\,((x \neq y) \to \neg(E(x,y) \vee T(x,y)))$

$\exists y\, \forall x\,((x \neq y)) \to (\neg E(x,y) \wedge \neg T(x,y))$

Let T (x) denote “ x uses TikTok “ and F (x,y) denote “ x and y are friends “. Translate the following English, again with a domain of students in class

$$\forall x\,(\,T(x) \lor \exists y\,(T(y) \land F(x,y))$$

For all students in class, the student uses tik tok or there exists another student who uses TikTok and is friends with our original student.

Simplfy it:

All students in class use tik tok or are friends with another student who uses Tik Tok.

Let S(x,y) denote student x has taken class y for the domain of all students a BU and all classes at BU. Translate the following:

a. $\exists x S(x, MA315)$

There exists a student x who has taken MA315.

b. $\exists x\, \exists y\, \forall z\, ((x \neq y) \land (S(x,z)) \leftrightarrow S(y,z)))$

Student

There exists two distinct students at BU who have taken the exact same classes.

Translate the Statement:

There is a man that has taken flight on every airline in the world.

Let $P(x,y)$ denote x has taken flight y.

Let $Q(y,z)$ denote y is a flight on airline z.

$$\exists m\, \forall a\, \exists f\, (P(m,f) \land Q(f,a))$$

There exists a man on all airlines f represents a flight

Now negate that statement:

$$\neg \exists m\, \forall a\, \exists f\, (P(m,f) \land Q(f,a))$$

$$\forall m\, \neg \forall a\, \exists f\, (P(m,f) \land Q(f,a))$$

$$\forall m\, \exists a\, \neg \exists f\, (P(m,f) \land Q(f,a))$$

$$\forall m\, \exists a\, \forall f\, \neg (P(m,f) \land Q(f,a))$$

~ distribute, DeMorgan's Law

$$\forall m\, \exists a\, \forall f\, (\neg P(m,f) \lor \neg Q(f,a))$$

What does it mean?

For all men there exist or there is an airline such that for all flights the man has not taken the flight or that flight is not on that airline.

1.6 Rules of Inference for Propositional Logic

Argument - a sequence of propositions $p_1, p_2 ..$ Premises

$\boxed{(p_1 \land p_2 \land p_3 \cdots \land p_n) \to q}$ Conclusion

If this is a tatoulgy then this is a valid argument.

Valid Argument - The premises imply the conclusion

If it is raining, I will need an umbrella. $p \to q$

It is raining. p

∴ I will need an umbrella $\therefore\ q$

Therefore q is true you will need an umbrella

Different Rules of Inference

Modus Ponens

Unbrella Example

If $\left\{ \begin{array}{l} p \to q \\ p \end{array} \right.$

Then $\to$ $\therefore q$

$$\begin{array}{l} R \to U \\ R \\ \hline \therefore U \end{array}$$

Written as a tautology $\to$ $\boxed{((p \to q) \wedge p) \to q}$

Modus Tollens

Premises $\left\{ \begin{array}{l} p \to q \\ \neg q \end{array} \right.$ $\equiv \neg q \to \neg p$

$\therefore \neg p$

$$\begin{array}{l} R \to U \\ \neg U \\ \hline \therefore \neg R \end{array}$$

$\boxed{((p \to q) \wedge \neg q) \to \neg p}$

$p \to q$ $\quad$ $\begin{array}{l} \neg q \to \neg p \\ \neg q \\ \hline \therefore \neg p \end{array}$

Hypothetical Syllogism

~ Similar to the transitive property

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array} \qquad \begin{array}{l} R \to U \\ U \to G \\ \hline \therefore R \to G \end{array}$$

$\boxed{(p \to q) \wedge (q \to r) \to (p \to r)}$

Disjunctive Syllogism

"or"

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

$\boxed{((p \vee q) \wedge \neg p) \to q}$

Addition

P is true $\to$

$$\begin{array}{l} p \\ \hline \therefore p \vee q \end{array}$$

$\boxed{p \to (p \vee q)}$

Simplification

$$\begin{array}{l} p \wedge q \\ \hline \therefore q \end{array}$$

$\boxed{(p \wedge q) \to q}$

Conjunction

P is true $\to$ p

q is true $\to$ q

$\therefore p \wedge q$

$\boxed{((p) \wedge (q)) \to (p \wedge q)}$

Resolution

$\neg p \vee r$ $\leftarrow$ One of these is true

$$\begin{array}{l} \neg p \vee r \\ p \vee q \\ \hline \therefore q \vee r \end{array}$$

$\boxed{((\neg p \vee r) \wedge (p \vee q)) \to (q \vee r)}$

Question:

From the proposition $\boxed{p \wedge (p \to q)}$, show that q is a conclusion.

$$\overset{T}{p} \wedge \overset{T}{(p \to q)}$$

1. $p \wedge (p \to q)$ Premise

Seperate

2. p Simplification on 1
3. $(p \to q)$ Simplification on 1
4. q Modus Ponens using 2 and 3

Modus Ponens

$$\begin{array}{l} p \to q \\ \underline{p} \\ \therefore q \end{array}$$

Question:

Use the rules inference to show the premises "John work hard" If John works hard then he isn't having any fun" and "If John isn't having any fun, then he won't make any friends" Imply the conclusion, "John will not make any friends."

p q r

$$(p)\ (p \to \neg q)\ (\neg q \to \neg r)$$

1. p Hypothesis
2. $p \to \neg q$ Hypothesis

3. $\neg q$ Modus Ponens using ① and ②
4. $\neg q \to \neg r$ Hypothesis

5. $\neg r$ Modus Ponens using ③ and ④

Rules of Interference for Quantified Statements

Universal Instantiation (UI)

$$\frac{\forall x\, P(x)}{\therefore\ P(c)}$$

For all of a domain, so fact is true.

All dogs are cute.
Oliver is my dog.
$\therefore P(o)$

Universal Generalization (UG)

$c \in U$

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore\ \forall x\, P(x)}$$

Existential Instantiation (EI)

$$\frac{\exists x\, P(x)}{\therefore\ P(c) \text{ for some } c}$$

Existential Generalization (EG)

$$\frac{P(c) \text{ for some element } c}{\therefore\ \exists x\, P(x)}$$

Universal Modus Ponens

$\forall x\, (P(x) \to Q(x))$

$P(a)$ where a is a particular element in the domain

$\therefore\ Q(a)$

Question:

Construct a Valid argument to show that "Oliver has 4 legs" is a consequence of the premises "Every dog has 4 legs" and "Oliver is a dog".

Steps:

1) Come up with predicate statements to construct argument

$D(x)$ denote x is a dog

$F(x)$ denote x has 4 legs

Two Premises

$\forall x\,(D(x) \to F(x))$

$D(o)$

Now were going to construct are argument:

1.	$\forall x\,(D(x) \to F(x))$	Premise
2.	$D(o) \to F(o)$	UI from 1
3.	$D(o)$	Premise
4.	$F(o)$	MP 2 and 3

Question:

Use the rules of Inference to construct a Valid argument showing that the conclusion "Someone who passed Discrete Mathematics has not read the book" follows from "A student in Discrete Math hasn't read the book" and "Every one in Discrete Math passed the class."

$D(x)$ denote x is a student in discrete Math

$B(x)$ denote x read the book

$P(x)$ denote x passed the class

$$\exists x\,(D(x) \land \neg B(x))$$

$$\forall x\,(D(x) \to P(x))$$

$$\therefore \exists x\,(P(x) \land \neg B(x))$$

What is the Conclusion going to be?

Solve the Proof:

1	$\exists x\,(D(x) \land \neg B(x))$	Premise
2	$D(a) \land \neg B(a)$	EI 1
3	$D(a)$	Simplification
4	$\forall x\,(D(x) \to P(x))$	Premise
5	$D(a) \to P(a)$	UI 4
6	$P(a)$	MP 3 5
7	$\neg B(a)$	Simpl 2
8	$P(a) \land \neg B(a)$	Conju 6 7
9	$\exists x\,(P(x) \land \neg B(x))$	EG 8

Chapter 1.7: Direct Proof

Direct Proof

In a direct proof, we assume the antecedent is true, then use rules of inference, axioms, definitions and/or previously proven theorems to show the consequent is true.

$$\boxed{p \to q}$$

Prove "If n is an odd integer, then n^2 is odd"

$p \rightarrow q$ We're assuming p is true

Even Integer

$n = 2K, \quad K \in \mathbb{Z}$

Odd Integer

$n = 2K + 1, \quad K \in \mathbb{Z}$

Start of Proof:

Assume n is an odd integer. Then $n = 2K+1$, $K \in \mathbb{Z}$

By definition of an odd integer.

$$(n)^2 = (2K+1)^2$$
$$n^2 = (2K+1)(2K+1)$$
$$n^2 = 4K^2 + 4K + 1$$
$$n^2 = 2(2K^2 + 2K) + 1$$
$$n^2 = 2r + 1 \quad \text{where } r = 2K^2 + 2K,\ r \in \mathbb{Z}$$

odd integer

$\boxed{\therefore n^2 \text{ is odd}}$

Prove "The sum of two even integers is even."

$p \rightarrow q$

If I add two even integers, then the sum is even.

Start Proof:

Assume a and b are even integers. Then $a = 2K$ for some $K \in \mathbb{Z}$ and $b = 2m$ for some $m \in \mathbb{Z}$.

$$a + b = 2K + 2m$$
$$= 2(K + m)$$
$$= 2r \quad \text{where } r = K + m,\ r \in \mathbb{Z}$$

By Direct Proof!

$\boxed{\therefore \text{The sum of two even integers is even}}$

Contraposition

A type of indirect proof that makes use of the face that $p \to q$ is equivalent to its contrapositive $\neg q \to \neg p$

So we assume $\neg q$ is true, then work to prove $\neg p$ is true.

Prove "If n is an integer and 3n+2 is odd, then n is odd."

p q

$\neg q \to \neg p$ $p \to q$

Assume n is even. So $n = 2k$, $k \in \mathbb{Z}$ by def. of an even integer.

$$3n+2 = 3(2k)+2$$
$$= 6k+2$$
$$= 2(3k+1)$$
$$= 2r \quad \text{when } r = 3k+1,\ r \in \mathbb{Z}$$

∴ 3n+2 is even
Since $\neg q \to \neg p$ is true, then $p \to q$ is true by Contrapositive

QED

Prove "If n is an integer and 3n+2 is even, then n is even."

p q

$\neg q \to \neg p$ $p \to q$

Assume n is odd. Then $n = 2k+1$, $k \in \mathbb{Z}$

$$3n+2 = 3(2k+1)+2$$
$$= 6k+3+2$$
$$= 2(3k+2)+1$$
$$= 2r+1, \quad r = 3k+2,\ r \in \mathbb{Z}$$

∴ 3n+2 is odd. So since $\neg q \to \neg p$, Then $p \to$ is true by Contraposition

Proof by Contradiction

Proof by Contradiction

In this method of proof, we assume a proposition is not true, then through the premise and logic find a contradiction that shows our original premise must have been incorrect, and therefore, the proposition was true.

1. For one proposition, p, assume $\neg p$ is true, then find a contradiction that shows $\neg p$ is false, so p is true.

2. For an implication, $p \rightarrow q$, assume p and $\neg q$ are true, then find a contradiction that shows either $p \rightarrow q$ or $\neg q \rightarrow p$

Prove by contradiction "$\underbrace{\sqrt{2} \text{ is a irrational}}_{p}$"

Assume $\neg p$

$\sqrt{2}$ is rational. Then there exists 2 integers a and b such that $\sqrt{2} = \frac{a}{b}$, $b \neq 0$ and a and b have no common factors

$$\sqrt{2} = \frac{a}{b}$$

Therefore a must be even so $a = 2c$ for some $c \in \mathbb{Z}$.

$$2 = \frac{a^2}{b^2}$$

$$2b^2 = a^2$$

$$2b^2 = (2c)^2$$

$$2b^2 = 4c^2$$

$$b^2 = 2c^2$$

Therefore b must be even.

Since a and b are both even, they have a common factor.

Prove "If $\underbrace{\text{n is an integer and 3n+2 is even}}_{p}$, then $\underbrace{\text{n is even}}_{q}$.

$p \to q$ Assume p and $\neg q$

Assume $3n+2$ is even, $n \in \mathbb{Z}$, and n is odd. Since $3n+2$ is even, then $3n$ is even by subtraction of 2 even integer. Then $3n-n$ must be odd by subtraction of an even and odd integer. However $3n-n = 2n$, which is even by definition which is a condradition of our original assumptions.

$\therefore$ If $n \in \mathbb{Z}$ and $3n+2$ is even, then n is even

1.8 Proof by Cases

Proof by Cases:

In a proof by cases, we must cover all possible cases that arise and cover all of the domain.

Example: Prove $\underbrace{\text{if n is an integer}}_{p}$, $\underbrace{n \leq n^2}_{q}$

Case 1 : $n \leq -1$

Case 2 : $n = 0$

Case 3 : $n \geq 1$

Must cover all possible cases.

Prove " if n is an integer, $n \leq n^2$ "

Case 1 : $n \leq -1$ — $n^2 > 0$, it follows $n \leq n^2$

Case 2 : $n \boxed{=} 0$ — $0 \leq 0^2$, $0 \leq 0$ it follows $n \leq n^2$

Case 3 : $n \geq 1$ — $n \geq 1$, $n^2 \geq n$ it follows $n \leq n^2$

Since $n \leq n^2$ for all values in our domain, we can conclude if $n \in \mathbb{Z}$, then $n \leq n^2$

Prove " if n is an integer, $n^2 + 3n + 2$ is even. "

$p \rightarrow q$

Case 1: n is even

$n = 2k, \quad k \in \mathbb{Z}$

$= (2k)^2 + 3(2k) + 2$

$= 4k^2 + 6k + 2$

$= 2(2k^2 + 3k + 1)$

$= 2r$ where $r = 2k^2 + 3k + 1$

$\boxed{\text{even}}$

Case 2: n = 0

$= 0^2 + 3(0) + 2$

$= 0 + 0 + 2$

$= 2$

$\boxed{\text{even}}$

$\therefore n^2 + 3n + 2$ is even if $n \in \mathbb{Z}$

Case 3: n is odd

$n = 2m + 1 \quad m \in \mathbb{Z}$

$= (2m+1)^2 + 3(2m+1) + 2$

$= 4m^2 + 4m + 1 + 6m + 3 + 2$

$= 4m^2 + 10m + 6$

$= 2(2m^2 + 5m + 3)$

$= 2\ell$ where $\ell = (2m^2 + 5m + 3)$

$\boxed{\text{even}}$

Prove " if x and y are integers and both x·y and x + y are even, then both x and y are even.

p: x·y is even

q: x+y is even

r: both x+y are even

$(p \wedge q) \rightarrow$ domain: integers

$p \rightarrow q$ $\neg q \rightarrow \neg p$

$\neg r \rightarrow \neg(p \wedge q)$

$\neg r \rightarrow \underline{\underline{(\neg p \vee \neg q)}}$

Case	X	Y
Case 1	o	o
Case 2	o	e
Case 3	e	o
Case 4	e	e

o: odd

e: even

$3 \cdot 2 = 6$
$2 \cdot 3 = 6$

$3 + 2 = 5$
$2 + 3 = 5$

WLOG I will assume x is odd

Prove " if x and y are integers and both x·y and x + y are even, then both x and y are even.

Assume, without loss of generality that x is odd. So $x = 2m+1$ for some $m \in \mathbb{Z}$.

Case 1: y is odd, $y = 2n+1$

$x \cdot y$

$(2m+1)(2n+1)$

$4mn + 2m + 2n + 1$

$2(2mn + m + n) + 1$

$2r + 1, \; r = (2mn + m + n)$

$\boxed{\text{Odd}}$

$x + y$

$2m+1 + 2n+1$

$2m + 2n + 2$

$2(m + n + 1)$

even

Case 2: y is even $y = 2n$

$x \cdot y$

$(2m+1)(2m)$

$4m \cdot n + 2n$

$2(2m \cdot n + n)$

even

$\neg x + y$

$2m+1 + 2n$

$2(m+n) + 1$

$\boxed{\text{odd}}$

Prove " There exist a pair of consecutive integers such that one integer is a perfect square and the other is a perfect cube.

	1	2	3	4	5	6
Sq.	1	4	9			
Cubes	1	8	27			

$2^3 = 8$ and $3^2 = 9$ then there exists a pair of integers such that are a perfect Cube and one is a perfect square.

Constructive Proof: Find value for which P (C) is true.

Prove " There is a rational number x and irrational number y such that x is irrational.

Let $x = 4$ (rational) and $y = \sqrt{2}$ (irrational)

$\rightarrow 4^{\sqrt{2}}$ is irrational

Nonconstructive: Assume no values make P (c) true, then contradict.

Prove " If a and b are real numbers, and a $\neq$ 0, then there is a unique real number r such that ar + b = 0.

Existence: We show an element x with that property exist.

Let $r = \frac{-b}{a}$ Then r is a solution to $ar + b = 0$

$\cancel{a}\left(\frac{-b}{\cancel{a}}\right) + b = 0 \quad -b + b = 0 \quad 0 = 0$

Uniqueness: We show x = y, then y does not have that property.

Suppose s is a real number such that $as + b = 0$.

Then $ar + b = as + b$

$ar = as$

$r = s$ ✓