Chapter 7: Introduction to Discrete Probability 7.1-7.2

7.1 Probability- LaPlace Definition

~ Possible outcomes

If S is a finite sample space of equally likely outcomes, and A is an event, where A ε S, then the probability of A is:

$$P(A) = \frac{|A|}{|S|}$$

$\{1, 2, 3, 4, 5, 6\}$ ~ S, the sample space

~ possible outcomes after rolling a die

Find P(6) for experiment of rolling a die.

1 six on a die

$$P(6) = \frac{1}{6}$$

# of possible outcomes

$$\boxed{P(6) = \frac{1}{6}}$$

A bag of marbles contains 4 green, 3 red, and 2 blue marbles. Find:

9 total

P (green) = $\frac{4}{9}$ P(red) = $\frac{3}{9} = \frac{1}{3}$ P(blue) = $\frac{2}{9}$

What is the probability that when two die are rolled the sum of the die is 3.

$\boxed{\begin{matrix} 1, 2 \\ 2, 1 \end{matrix}}$

Two outcomes - Could roll a 1, first then a 2, or vice-versa

$\boxed{P(3) = \frac{2}{36} = \frac{1}{18}}$

How many total possibilities are there?

$6 \cdot 6 = 36$

rolling die with six sides

Some Probability Rules

1. $P(S) = 1$ $\longrightarrow$ $\{H, T\}$ $P(H) = \frac{1}{2}$ $P(T) = \frac{1}{2}$ $P(S) = \frac{1}{2} + \frac{1}{2} = 1$

   - Always $P(S) = 1$
     add up to !

2. $0 <= P(A) <= 1$

3. $P(\bar{A}) = 1 - P(A)$ Complement Rule

   Example: Probability that you dont roll a 3

   $\{1, 2, 3, 4, 5, 6\}$ $P(\bar{3}) = 1 - P(3)$

   $P(\bar{3}) = \frac{5}{6}$ $P(\bar{3}) = 1 - \frac{1}{6} = \frac{5}{6}$

4. P (at least 1) = 1- P (none)

   P(at least 1 H in 3 flips) $\rightarrow \frac{7}{8}$

   HHH HHT HTH HTT

   THH THT TTH TTT

   $2 \cdot 2 \cdot 2 = 8$

   $= 1 - \frac{1}{8} = 7/8$ ✓

5. $P(A \cup B) = P(A) + P(B) \boxed{- P(A \cap B)}$ ~ disjoint

   One event, more than one out come

   $P(2 \cup 4) = P(2) + P(4) - 0$

   rolling a 2 or a 4, only roll die one time

   $P(2 \cup 4) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$ ✓

6. $P(A \cap B) = P(A) * P(B|A)$ ↖ Conditional

   Two events

   $P(2 \cap 4) = P(2) \cdot P(4|2)$

   $= \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$

Discrete Probability Practice

You will win the lottery if you match 5 digits, in order to 5 randomly drawn digits. What is the probability of winning the jackpot? 0-9 10 Digits

$10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$

10^5

$$P(\text{winning}) = \frac{1}{10^5} = \frac{1}{100{,}000} = 0.00001$$

You will get your money back if you match at least one digit. What is the probability that you don't lose your money?

$$P(\text{at least } 1) = 1 - P(\text{none})$$

$$= 1 - \left(\frac{9}{10}\right)^5$$

$$= 1 - 0.59049$$

$$= 0.40951$$

$= 41.5\%$ you don't lose your money

What is the probability of selecting a hand using a deck of cards of A, K, Q, J, 10 ?

52 cards in a deck

4 of each type/ 4 suits

13 of each suit

$$P(A \cap B) = P(A) \cdot P(B|A)$$

4 ACE's King Queen Jack 10

People always forget!

$$\boxed{5!\left(\frac{4}{52}\right)\left(\frac{4}{51}\right)\left(\frac{4}{50}\right)\left(\frac{4}{49}\right)\cdot\left(\frac{4}{48}\right)}$$

4^5

a correct Answer

$$\binom{4}{1} = \frac{4!}{(4-1)!\,1!} = \frac{4\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{3}\cdot\cancel{2}\cdot\cancel{1}\cdot 1} = 4$$

$$\binom{52}{5} = \frac{52!}{(52-5)!\cdot 5!} = \frac{52!}{47!\,5!} = \frac{52\cdot 51\cdot 50\cdot 49\cdot 48\cdot\cancel{47}}{\cancel{47!}\cdot 5!}$$

There's 4 Kings, I choose 1,...
....

$$\frac{\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}} = \frac{4^5}{\frac{52\cdot 51\cdot 50\cdot 49\cdot 48}{5!}} = \boxed{5!\left(\frac{4^5}{52\cdot 51\cdot 50\cdot 49\cdot 48}\right)}$$

52 Cards in the deck, I pick 5 of them.

$= 0.000394$

What is the probability of selecting an Ace four times in row if cards are put back in the deck after each draw?

$$\left(\frac{4}{52}\right)\left(\frac{4}{52}\right)\left(\frac{4}{52}\right)\left(\frac{4}{52}\right) \approx 0.000035$$

~ put card back in

What is the probability the hand contains a full house?

3 of one type of Card and 2 of another
example: 3 Jacks, 2 Tens

For that card there are 4 of each suit and I am going to choose 3 of them out of the deck

13 different Cards Choose 1 of them

Then theres 12 different Cards left, and we will Choose one of them

3 of Same suit

Two of Same suit

$$\frac{\underbrace{\binom{13}{1}\binom{4}{3}}_{\text{3 of Same suit}}\underbrace{\binom{12}{1}\binom{4}{2}}_{\text{Two of Same suit}}}{\binom{52}{3}}$$

For that card going to Choose 2 of them

$$\frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{3}} = \frac{13 \cdot 4 \cdot 12 \cdot 6}{2,598,960} \approx 0.00144$$

A sequence of 10 bits is randomly generated. What is the probability at least one of these bits is zero?

$P(0) + \boxed{P(1) + P(2) + P(3) + P(4) + P(5) + P(6) + P(7) + P(8) + P(9) + P(10)}$

Everything we have above here

$P(\text{at least one zero}) = 1 - P(\text{no zeros})$

$= 1 - \dfrac{1}{2^{10}}$

\# of total bit strings availble

$= 1 - \dfrac{1}{1024}$

$\boxed{= \dfrac{1023}{1024}}$

What is the probability that a positive integer not exceeding 100 is divisible by either 2 or 5.

1

Going to have addition

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Or

The ones that are divisible by both 2 and 5: Take $2 \cdot 5 = 10$

$P(2 \cup 5) = \dfrac{\lfloor \frac{100}{2} \rfloor + \lfloor \frac{100}{5} \rfloor - \lfloor \frac{100}{10} \rfloor}{100}$

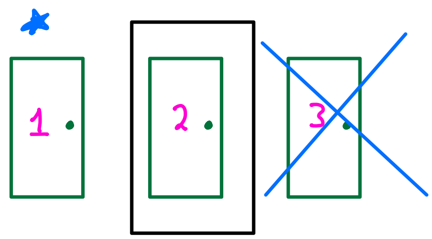
Floor function will give us the # of integer divisible by 2

$P(2 \cup 5) = \dfrac{\lfloor \frac{100}{2} \rfloor + \lfloor \frac{100}{5} \rfloor + \lfloor \frac{100}{10} \rfloor}{100} = \dfrac{50 + 20 - 10}{100} = \dfrac{60}{100} = 0.6$

$\boxed{P(2 \cup 5) = 0.6}$

Monte Hall Puzzle

Let's Make a Deal! There is a prize behind one of 3 doors. If you Choose the correct door, you get to keep the prize. Once you choose a door, Monte Hall, the host, opens one of the other doors (a non- winner) and gives you the option to change your selection. What should you do?



Let's say we choose door number 1, and number 3 was opened, is it better to stick with door number 1 or pick door number 2? Use the Probability theorem.

- You should door number 2! (switch)

Before picking a door: $P(\text{Winning}) = \boxed{\frac{1}{3}}$

2^{nd} $P(\text{Winning}) = \frac{2}{3} = 1 - \frac{1}{3} = \frac{2}{3}$

7.2 Probability Theory

Probability for Events Not Equally Likely

Let S be a sample space of an experiment with a finite number of outcomes. We assign p(s) to each outcome S, so that

a. $0 \leq p(s) \leq 1$ for each $s \in S$ and

b. $\sum_{s \in S} p(s) = 1$

The function p from the set of all outcomes of the sample space S is called a probability distribution or probability model.

A coin is biased so that tails lands face up twice as often as heads. Find the probability distribution.

$P(T) = 2\,P(H)$

$1 = P(H) + P(T)$

$1 = P(H) + 2\,P(H)$

$1 = 3(H)$

$P(H) = \frac{1}{3}$

$P(T) = \frac{2}{3}$

Probability distribution

$P(H) = \frac{1}{3}$

$P(T) = \frac{2}{3}$

A formal 'Union' Definition

If $E_1, E_2, \ldots$ is a sequence of pairwise disjoint events in a sample space S, then:

no overlap

$$P\left(\bigcup_i E_i\right) = \sum_i p(E_i)$$

$$\boxed{P(A \cup B) = P(A) + P(B)} - P(\cancel{A \cap B})$$

Suppose a die is biased so that 2 appears twice as often as the other 5 outcomes that are equally likely. What is p (even) ?

1	1/7
2	2/7
3	1/7
4	1/7
5	1/7
6	1/7
	7

$$P(\text{even}) = \frac{2}{7} + \frac{1}{7} + \frac{1}{7} = \frac{4}{7}$$

Conditional Probability

Let A and B be events, with p (B) > 0. The conditional probability of A given B is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

A bit string of length 4 is generated at random so that each of the 16 bit strings of length 4 (2^4) is equally likely. What is the probability that the bit string contains at least two consecutive 0's given the first bit is a zero ?

$$P(\underbrace{\text{at least 2 0's}}_{E} \mid \underbrace{\text{first bit zero}}_{F}) = \frac{P(\text{both})}{P(\text{first bit zero})} = \frac{5/16}{1/2} = \boxed{\frac{5}{8}}$$

16 ways, and half of them will start with 0.

What is the numerator?

Both starts with a 0 and has at least two consecutive 0's.

↓ So 5 ways

Gotta List them:

0000
0001
0010
0011
0100

options

Independence

Events A and B are independent if and only if

$$\boxed{p(A \cap B) = p(A) \cdot p(B)}$$

-Outcome of one event effects another event

Assume a family has two children, {BB, BG, GB, GG}. Are having two boys and having at least one boy independent?

A

B

Intersect says both have occured

$$P(A \cap B) \stackrel{?}{=} P(A) \cdot P(B)$$

$$\frac{1}{4} = \frac{1}{4} \cdot \frac{3}{4}$$

$$\frac{1}{4} \neq \frac{3}{16}$$

$\boxed{\therefore \text{ Not Independent}}$

Random Variables and the Binomial Distribution

Random Variables

A random variable is a function from the sample space of an experiment to the set of real numbers. The distribution of a random variable X on a sample space S is the set of pairs (r, p (X = r)) $\forall$ r $\in$ S, where P (X = r) is the probability that X takes on the value of r.

Suppose you flip a fair coin 3 times. Let X(t) be the random variable that equals the number of heads flipped for each outcome t. Find the distribution of X (t)

Probability head gets picked x = times

Outcomes:

HHH
HTH HHT THH
HTT THT TTH
TTT

$P(x=3) = \frac{1}{8}$

$P(x=2) = \frac{3}{8}$

$P(x=1) = \frac{3}{8}$

$P(x=0) = \frac{1}{8}$

Suppose the probability of passing Discrete Math is 92%. What is the probability that exactly 4 of 5 students pass ?

$1 - 0.92 = 0.08$

How to write random variable: $P(X=4)$

$$= \binom{5}{4}(0.92)^4 \cdot (0.08)^1$$

$$\underbrace{PPPP\ \ F}_{\text{Order}}$$

$$= \frac{5!}{4!\,1!}(0.92)^4(0.08)^1$$

$$\boxed{= 0.2866}$$

Binomial Distribution

A probability distribution based on Bernoulli Trials, where an experiment has 2 outcomes, success and failure. Let p = probability of success and q = 1 – p = probability of failure. Then the probability of k successes in n trials can be found by:

$$\boxed{\binom{n}{k} p^k q^{n-k}}$$

$b(k:n, p)$
$b(4:5, 0.92)$] On calculator

Suppose the probability of passing Discrete Math is 92%....

What is the probability that exactly 4 of 5 students pass ?

$$\boxed{P(x=4) = \binom{5}{4}(0.92)^4(0.08)^1}$$

Suppose the probability of having O- negative blood is 6 %. What is the probability that exactly 1 donor in the first 5 donors at a blood drive has O – negative blood ?

$q = 1 - 0.06 = 0.94$

$P(x=1) = \binom{5}{1}(0.06)^1(0.94) =$

$\boxed{P(x=1) = 0.2342}$

TI CALC:
binompdf(5, 0.06, 1) =
n, p, k

What is the probability that at least 2 donors in the first 5 have O- negative blood.

One method:

$P(x \geq 2) = \binom{5}{2}(0.06)^2(0.94)^3 + \binom{5}{3}(0.06)^3(0.94)^2 + \binom{5}{4}(0.06)^3(0.94)^2 + \binom{5}{5}(0.06)^3(0.94)^2$

Another way:

$1 - P(x \leq 1) = \boxed{1 - \left(\binom{5}{1}(0.06)^1(0.94)^4 + \binom{5}{0}(0.06)^0(0.94)^5\right)}$

$\boxed{1 - P(x \leq 1) = 0.0319}$