Rule of Sum and Rule of Product

- How many ways can you roll an even number on a die? 6 sided

3 2, 4, 6

- How many ways can you draw a face card in a standard 52-card deck?

J Q K
♡ ◇ ♣ ♠ 12

Addition Rule

For two seperate tasks A and B,
if A can be done in m ways, and
B can be done in n ways,
Then A and B can be done in $m+n$ ways.

How many ways can you roll an even number on a die or draw a face card?

A — Die: 2, 4, 6 — 3

B — Card: ♡J ♡Q ♡K, ◇J ◇Q ◇K, ♠J ♠Q ♠K, ♣J ♣Q ♣K — 12

$3 + 12$

15

Rule of Product

In subsequent tasks, A and B,

if there are m ways to do task A, and n ways to do task B,

then there are mn ways to do A then B.

I want to make a sandwich with one thing from each category

A Top bread: White, Whole wheat, sourdough 3

B meat: Ham, bologna, pork, steak, salami, rabbit, human 7

C "Vegetable": Lettuce, tomatoe, cucumber, turnip 4

D Bottom Bread: White, Whole wheat, sourdough 3

How many ways can I do that?

$$3 \times 7 \times 4 \times 3 = 21 \times 12$$

$$= 252 \text{ ways}$$

Questions

How many ways can we make a license plate with.....

a.) 3 even numbers and 3 letters?

0 2 4 6 8

$$5 \times 5 \times 5 \times 26 \times 26 \times 26 = 5^3 26^3$$

b.) 2 numbers, 2 letters, 1 odd number, 1 even number, and 2 vowels?

$$10 \times 10 \times 26 \times 26 \times 5 \times 5 \times 5 \times 5 = 10^2 26^2 5^4$$

Ten #'s to pick from 1-9

c.) 6 letters that are not the same.

$$\boxed{26 \times 25 \times 24 \times 23 \times 22 \times 21}$$

~~a~~ b c d

4×3

Rule of Sum and Rule of Product Example

8 women and 5 men are elected for president.

(i) How many ways to choose a winner?

$$8 + 5 = 13$$

(ii) How many men vs women pairs can we make?

$$5 \times 8 = 40$$

Rule of Product

(iii) How many ways can we choose a product and a female vice president?

$$\underset{M\quad F}{(5 \times 8)} + \underset{F\quad F}{(8 \times 7)} = 96$$

Rule of Sum

Questions

Went to buy a sandwich.

- 3 types of bread
- 5 types of meat
- 6 types of cheese

(1) How many ways to make a 1 meat, 1 cheese sandwich?

$$3 \times 5 \times 6 = 90$$

(2.) How many ways to make a 3 meat, 2 cheese sandwich? (NO REPEATS)

$$3 \times (5 \times 4 \times 3) \times (6 \times 5) = 5{,}400$$

Factorials and Permutations — Order matters

VIDEO 29

Factorials

if $n \in \mathbb{Z}^+$ (Positive #), then "n factorial" is denoted

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots 3 \cdot 2 \cdot 1$$

Examples

$3!$ $\quad 3! = 3 \times 2 \times 1 = 6$

$6!$ $\quad 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

$0!$ $\quad 0! = 0$

How many ways can we write a list of 7 numbers (1,2,3,4,5,6,7) without repeating any?

$$7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = \boxed{7!}$$

What if the first three numbers in the list must be even? $\quad 2 \quad 4 \quad 6$

Must be Even

$$\underbrace{3 \times 2 \times 1}_{1^{st}\text{ 3 spots}} \times \underbrace{4 \times 3 \times 2 \times 1}_{\text{odd \#'s}} = \boxed{4!\,3!}$$

(2, 4, 6)

List of numbers

$(\cancel{1}, \cancel{2}, \cancel{3}, \cancel{4}, \cancel{5}, \cancel{6}, \cancel{7})$

- When we have conditions we have to break these problems into steps

The number of permutations of length K from a list of n elements without repetition is:

$$\frac{n!}{(n-K)!} = \frac{7!}{(7-4)!} = \frac{7!}{3!} \qquad K \downarrow 4$$

SAME THING

How many ways can we write a list of 4 numbers from (1,2,3,4,5,6,7) without repeating any?

$$7 \times 6 \times 5 \times 4$$

Matches

$$\frac{7!}{3!} = \frac{7 \times 6 \times 5 \times 4 \times \cancel{3 \times 2 \times 1}}{\cancel{3 \times 2 \times 1}}$$

Questions

How many ways can we order the letters in the word "BASKET"?

1 2 3 4 5 6

$$6! \qquad \frac{6!}{1!\,1!\,1!\,1!\,1!\,1!}$$

How many ways can we order the letters in the word "ERR"?

1 2 2

$$\boxed{\begin{array}{c} E\,R_1\,R_2 \\ E\,R_2\,R_1 \end{array}}$$

3

$$\boxed{\begin{array}{c} R_1\,R_2\,E \\ R_2\,R_1\,E \end{array}} \quad \boxed{\begin{array}{c} R_1\,E\,R_2 \\ R_2\,E\,R_1 \end{array}}$$

- Same thing

$$\frac{3!}{1!\,2!} = \frac{3!}{2!} = \frac{3 \times \cancel{2 \times 1}}{\cancel{2 \times 1}}$$

$$= 3$$

What Formula could we use?

Divide by the # of times every single letter occurs.

E occurs 1, and R occurs 2

How many ways can we order the letters in the word "MISSISSIPPI"?

$$\frac{11!}{1!\,4!\,4!\,2!}$$

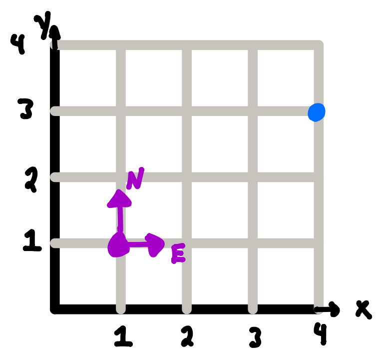
M	I	S	P
	I	S	P
	I	S	
	I	S	
1	4	4	2

How many paths are there from $(1,1)$ to $(4,3)$ if we can only make moves:

$$(x,y) \mapsto (x+1,y)$$

or

$$(x,y) \mapsto (x,y+1)$$

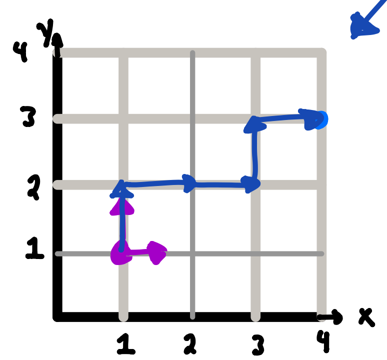


EEENN

NEENE

$$\frac{5!}{3!\,2!}$$

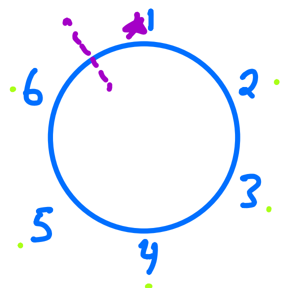
Let's use this sequence



Permutation Practice

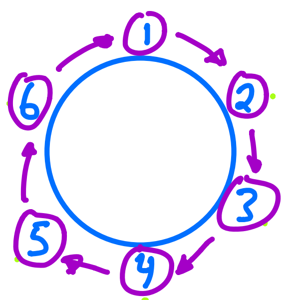
Video 29

How many ways can we arrange 6 people at a circular table?



$6!$

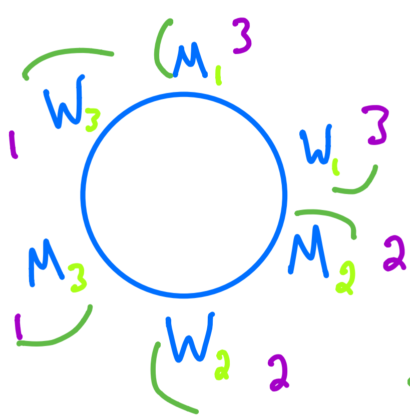
$5!$



- Can start at 6 possible locations

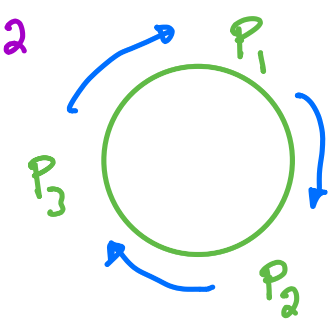
$$\frac{6!}{6} = 5!$$

With 3 men and 3 women, how many can we do it so that the sexes alternate?



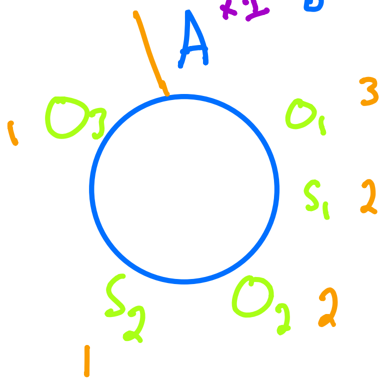
$$3 \times 3 \times 2 \times 2 \times 1 \times 1 = \frac{36}{3}$$

$= 12$ ways



pairs of M & W

Another way to look at the Problem:



$\times 2$ ← pick a anchor position

$3 \times 2 \times 2 = 12$ ways

Problems

Evaluate $P(6,3)$:

$$\boxed{P(n,k)=\frac{n!}{(n-k)!}}$$

$$P(6,3)=\frac{6!}{(6-3)!}=\frac{6!}{3!}=\frac{6\times5\times4\times\cancel{3}\times\cancel{2}\times\cancel{1}}{\cancel{3}\times\cancel{2}\times\cancel{1}}=120$$

Expanded

$P(12,3)$:

$$P(12,3)=\frac{12!}{9!}=\frac{12\times11\times10\times\cancel{9!}}{\cancel{9!}}=1{,}320$$

How many Integer n can we form with 3, 4, 4, 5, 5, 6, 7 if we want n to exceed 5,000,000?

for Example 5, 434, 567

<u>Check Each Case:</u>

Case:

1) first is 5   5, xxx, xxx $\frac{6!}{2!}$ Check for duplicates (4)

2) first 6   6, xxx, xxx

$$\frac{6!}{2!\,2!}$$

Duplicates (4 and 5)

3) first 7 $\frac{6!}{2!\,2!}$

now we need to <u>add them</u> together.

$$= \frac{6!}{2!} + \frac{6!}{2!\,2!} + \frac{6!}{2!\,2!}$$

or

$$= \frac{6!}{2!} + 2\left(\frac{6!}{2!\,2!}\right)$$

$$= \frac{6!}{2!} + \frac{6!}{2!}$$

$$= 2\left(\frac{6!}{2!}\right)$$

$$= 6!$$

Challenging Permutation Problems

Combinations

Video 30

← order doesnt matter

From a set {a. b. c}, how many different subsets can we make with 2 elements?

Permutation	Combination
abc acb 2	abc = acb 1

ab — ba
ac — ca
bc — cb

this map to another

3 different combinations we can make

If n and k are integers, then $\binom{n}{k}$ is the amount of subsets that can be made using k elements in an n element set.

$$\binom{n}{k} = n \text{ choose } k$$

$$= C(n,k)$$

Capital C

$$= \frac{n!}{k!(n-k)!} = \frac{P(n,k)}{k!}$$

Permutation Function

Divided by the number of elements were making.

$$\boxed{\binom{n}{k} = \frac{n!}{k!(n-k)!}}$$

In the previous Example:

$$\binom{3}{2} = \frac{n!}{k!(n-k)!} = \frac{3!}{2!(1!)} = 3$$

We want 2 element combination out of a three total set

Read "3 choose 2"

Practice

Evaluate 1) $\binom{9}{4}$

$$\boxed{\binom{n}{k} = \frac{n!}{k!(n-k)!}}$$

$$\binom{9}{4} = \frac{n!}{k!(n-k)!} \qquad \frac{n!}{a!+b!} \quad a+b=n$$

$$= \frac{9!}{4!(9-4)!}$$

Note: Always try to cancel largest factorial

$$= \frac{9!}{4!(5!)} = \frac{9\times8\times7\times6\times\cancel{5!}}{4!(\cancel{5!})}$$

Two bottom #'s should equal the top

$$= \frac{\cancel{9}^{3}\times\cancel{8}\times7\times6}{\cancel{4}\times\cancel{3}\times\cancel{2}\times1} \sim \text{simplfy}$$

$$= 3\times7\times6$$

$$\boxed{= 126}$$

2) $\binom{6}{3}$

$$\binom{6}{3} = \frac{n!}{k!(n-k)!}$$

$$= \frac{6!}{3!(6-3)!} = \frac{6!}{3!\,3!} = \frac{6\times5\times4\times\cancel{3}\times\cancel{2}\times\cancel{1}}{3!\,\cancel{3}\times\cancel{2}\times\cancel{1}} = \frac{6\times5\times4}{3!} = \frac{6\times5\times4}{3\times2} = \boxed{20}$$

Application Questions

1) How many 5 card hands can be drawn from a deck of 52 cards?

Is it a permutation or combination question?
- Combination

formula: $\boxed{\binom{n}{k} = \frac{n!}{k!(n-k)!}}$

$$\binom{52}{5} = \frac{n!}{k!(n-k)!} = \frac{52!}{5!(52-5)!} = \frac{52!}{47!\,5!} = \frac{52\times51\times50\times49\times48\times\cancel{47!}}{\cancel{47!}\,5!}$$

$$= \frac{52\times51\times50\times49\times48}{5!}$$

$$= \frac{\cancel{52}\times\cancel{51}\times\cancel{50}\times49\times\cancel{48}}{\cancel{5}\times\cancel{4}\times\cancel{3}\times\cancel{2}}$$

Simplfy w/o Calculator

$$\boxed{= 26\times17\times10\times12\times49}$$

$\frac{52}{2} = 26$

$\frac{51}{3} = 17$

$\frac{50}{5} = 10$

$\frac{48}{4} = 12$

2.) How many ways to draw 3 diamonds and 2 clubs?

diamonds = 13
Clubs = 13

$$\binom{13}{3} \times \binom{13}{2}$$

NOTE

$$\binom{13}{3}\binom{13}{2} \neq \binom{13}{5}$$

3.) What are the odds of winning a lottery where all 6 numbers chosen out of 49 must be picked to win?

$$\frac{\text{\# Situation Win}}{\text{\# Total Situation}} = \boxed{\frac{1}{\binom{49}{6}}}$$

49 numbers
pick 6

4.) Given the word TALLAHASSEE, how many arrangements can be made with no consecutive vowels?

T $\boxed{\begin{matrix}A\\A\\A\end{matrix}}$ L L H S S $\boxed{\begin{matrix}E\\E\end{matrix}}$

~ lets leave vowels out for now

$$\times T \times L \times L \times H \times S \times S \times = \left(\frac{6!}{2!\,2!}\right)\binom{7}{5}\left(\frac{5!}{3!\,2!}\right)$$

Divided by repeats (L & S)

Remember order does not matter

7 position we can choose to put the vowels and 5 vowels

One more step to the puzzle

$$AAAEE = \frac{5!}{3!\,2!}$$

still have to order these

Permutation and Combinations Examples

Video 31

1) How many ways can 6 men and 4 women stand in Line?

$\boxed{10!}$

- How many without 2 women next to each other?

$$\begin{array}{ccccccccccccc} & W_1 & & W_2 & & W_3 & & & & W_4 & & & \\ \times & M & \times & M & \times & M & \times & M & \times & M & \times & M & \times \end{array}$$

$\boxed{6!\binom{7}{4}4!}$

1) Arrange the men (M)
2) pick where the women would go
3) Order the women

- How many without 2 men next to each other?

$\boxed{0}$

-impossible, theres 2 more men than there are women

2) How many binary strings contain exactly five 0's and fourteen 1's when each 0 must be followed immediately by two 1's?

0	1	1
0	1	1
0	1	1
0	1	1
0	1	1
	1	1
	1	1

→ What were actually ordering here

$$5\left\{\begin{matrix}011\\011\\011\\011\\011\end{matrix}\right. \qquad \left.\begin{matrix}1\\1\\1\\1\end{matrix}\right\}4$$

$$\boxed{\frac{9!}{4!\,5!}}$$

9! ~ # of objects

are "1" ~ 4!

5! — are "011"

$$\frac{9!}{4!\,5!} = \binom{9}{4,5} = \binom{9}{5}$$

Permutation and Combinations Examples 2

Video 32

- How many permutations are there for letters a, c, e, g, i, j, m, p, t?

3 Vowels
6 Consonants

$$9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = \boxed{9!}$$

- How many start with the letter j and then a vowel?

$$1 \times 3 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = \boxed{3 \times 7!}$$

j a, e, i

- How many Cons Vowel Cons words can be made?

CVC

$$6 \times 3 \times 5 = \boxed{90}$$

How many paths from (2,7) to (6,12) where either $x \mapsto x+1$ or $y \mapsto y+1$ in each step?

$\langle x, y \rangle$

$$\boxed{\frac{9!}{4!\,5!}}$$

total choices

Same as like we did in example above

EEEE NNNNN

- $(-1,0,6) \to (3,2,8)$ where $x \mapsto x+1$, $y \mapsto y+1$, or $z \mapsto z+1$ in each step

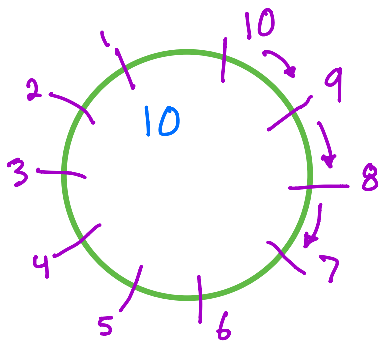
$\langle x, y, z \rangle$

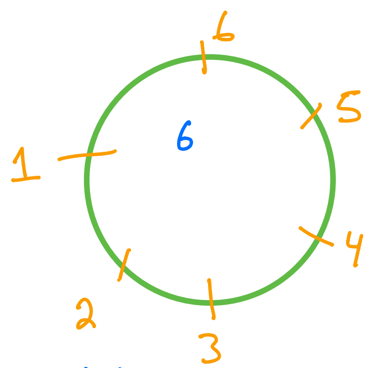
xxxxyyzz

find # of Permutations

$$\boxed{\frac{8!}{4!\,2!\,2!}}$$

How many ways can we sit 16 people around two circular tables?
One holds 10, the other holds 6.





1st Step: Choose people to go to the table.

$$\binom{16}{10} \times \frac{10!}{10} \times \binom{6}{6} \times \frac{6!}{6}$$

2nd Step: Order them

Ten possible rotations that are the same.

3rd Step: Look at last table

4th Step: Order them
- 6 possible rotations

simplify!

$$\boxed{\binom{16}{10}(9!)(5!)}$$

Binomial Theorem and Pascal's Triangle

Video 33

Binomial Theorem

if $n \in \mathbb{Z}^+$, then

Positive Integers

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y^1 + \dots + \binom{n}{n}x^0y^n$$

$$= \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

Summation

$$\binom{n}{0}x^{n-0}y^0 + \binom{n}{1}x^{n-1}y^1$$

Something we know!

$$(x+y)^2 = (x+y)(x+y) = x^2 + xy + xy + y^2 = x^2 + 2xy + y^2$$

Same thing

Use the Binomial Theorem

$$(x+y)^2 = \underbrace{\binom{2}{0}}x^2y^0 + \underbrace{\binom{2}{1}}x^1y^1 + \underbrace{\binom{2}{2}}x^0y^2$$

$$= \boxed{1x^2 + 2xy + 1y^2}$$

$$\boxed{\binom{2}{0}} = \boxed{\frac{2!}{0!\,2!}} = 1$$

$$\binom{2}{1} = \frac{2!}{1!\,1!} = 2$$

$$\boxed{\binom{2}{2}} = \boxed{\frac{2!}{0!\,2!}} = 1$$

$$\binom{n}{k} = \binom{n}{n-k} \qquad \binom{49}{6} = \binom{49}{43}$$

Practice Problem

What is $(2a-b)^4$?

$x = 2a$
$y = -b$

$$\boxed{(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i}$$

$\binom{4}{1} =$ ___

$$\begin{aligned}
&= \binom{4}{0} \underbrace{(2a)^4} \underbrace{(-b)^0} && = 1\,(16a^4) \\
&+ \binom{4}{1} (2a)^3 (-b)^1 && -4\,(8a^3)(b) \\
&+ \binom{4}{2} (2a)^2 (-b)^2 && +6\,(4a^2)(b)^2 \\
&+ \binom{4}{3} (2a)^1 (-b)^3 && -4\,(2a)(b)^3 \\
&+ \binom{4}{4} (2a)^0 (-b)^4 && +1\,(b)
\end{aligned}$$

$$\boxed{= 16a^4 - 32a^3b + 24a^2b^2 - 8ab^3 + b^4}$$

$$\boxed{\binom{n}{k} = \frac{n!}{k!(n-k)!}}$$

$$\binom{4}{1} = \frac{4!}{1!(4-1)!} = \frac{4!}{1!\,3!} = \frac{4 \cdot \cancel{3!}}{1!\,\cancel{3!}} = \frac{4}{1!} = 4 \checkmark$$

Pascals Triangle

$$\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \\
\vdots & & & \vdots & & & \vdots
\end{array}$$

$$\boxed{\text{Claim: } \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}}$$

Coefficients

$$\begin{array}{c|ccccccccc}
0 & & & & & 1 & & & & \\
1 & & & & 1 & & 1 & & & \\
2 & & & 1 & & 2 & & 1 & & \\
3 & & 1 & & 3 & & 3 & & 1 & \\
4 & 1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

1 — 1
1 1 — 2
1 2 1 — 3
1 3 3 1 — 4

$(x+y)^n$ takes coefficients of row $n+1$

$$(x+y)^2 = x^2 + 2xy + y^2$$

What are the coefficients of $(x+y)^6$?

- Draw out the triangle

1 — 1
1 1 — 2
1 2 1 — 3
1 3 3 1 — 4
1 4 6 4 1 — 5
1 5 10 10 5 1 — 6
1 6 15 20 15 6 1 — 7

$$(x+y)^6 = \boxed{1x^6 + 6xy^5 + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6}$$

Combinations and Repetition

Video 34

Given n objects, we want to select r objects with replacement

$$= \binom{n+r-1}{r}$$

Example:
$n = 4$
$r = 3$

We're going to have 4 objects:

1 ~~2~~ 3 ~~4~~ 2 ~~4~~

2 4 4

Put in a bin

When we select things with a combination we cross it out. We can't take this again.

- But we can take it again, so we add (2) back
- We selected 3 objects from the original 4 with repetition.
- What we really did here: Out of 5 total objects we chose 3, so this is equal to 6 choose 3.
- The intuition here is that you're just replacing and you're filling in an empty spot for whatever the first one you chose is and then you're filling an empty spot for w/e the second is

$$= \binom{6}{3}$$

If we plugged into the formula:

$$= \binom{n+r-1}{r}$$

$$= \binom{4+3-1}{3}$$

$$= \binom{6}{3}$$

EXAMPLE:

In a donut Shop, there are 20 types of donuts. How many ways can we select 12 different donuts to take home?

Add an extra slot for 2 so we can take it again

$1 \quad \cancel{2} \quad 3 \quad \ldots \quad 19 \quad 20 \quad 2$

2

We have,

$n = 20$

$r = 12$ ← We want 12 different donuts

$$= \binom{n+r-1}{r}$$

$$= \binom{20+12-1}{12}$$

$$= \binom{31}{12}$$

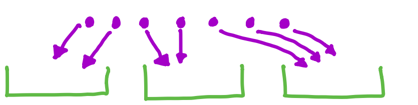
One way to look at this types of problems

EXAMPLE 2

$X_1 + X_2 + X_3 = 7$ Where $X_1, X_2, X_3, \geq 0$

How many positive integer solutions are there?

We have 3 different bins
We have 7 objects that we want to put in the bins.



$x_1 \quad x_2 \quad x_3$

$2 + 2 + 3 = 7$

Another way to look at the problem:

7 Objects
Insert bins using lines

$$\underbrace{x \quad x}_{x_1} \; \Big| \; \underbrace{x \quad x \quad x}_{x_2} \; \Big| \; \underbrace{x \quad x}_{x_3}$$

What if we want to have one of the bins to have 0?

$$\underset{x_1}{\underline{\quad}}\ \Big|\ \underbrace{x\ \ x\ \ x\ \ x\ \ x\ \ x\ \ x}_{x_2}\ \Big|\ \underset{x_3}{\underline{\quad}}$$

What are we actually doing here?

$$x\ \ x\ \ x\ \ x\ \ x\ \ x\ \ x\ \ x\ \ x$$

- To get three bins we need <u>two dividers</u> so we're actually working with <u>9 different spots</u>
- We just put our dividers into two of our spots (||)

↓ Example

$$\underbrace{x\ \ x\ \ x\ \ x\ \ x\ \ x\ \ x}_{7}\ \ \cancel{x}\ \ \cancel{x}\qquad 0\quad 0$$

- You take the amount of dividers you need $(n-1)$

So for $x_1 + x_2 + x_3 = 7$ where $x_1, x_2, x_3, \geq 0$

$$x_1 + x_2 + \ldots x_n = r \qquad x_i \geq 0$$

$$\binom{n+r-1}{r}$$

So we have:

$$\binom{3+7-1}{7} = \binom{9}{7}$$

- Same thing as saying we have 7 objects, but I need two dividers so I added two more options (x's) (went from 7 objects (x's) to 9 x's)

$$x\ \ x\ \ \cancel{x}\ \ x\ \ x\ \ \cancel{x}\ \ x\ \ x\ \ x$$

- And I am going to pick <u>two of the objects</u> to be divider

$$\binom{9}{7} = \binom{9}{2}$$ ← So we know this is equal to 9 choose 2 by the symmetry property

- Same thing as saying we choose 2 of our objects to be dividers.

EXAMPLE 3

How many ways can we put 10 identical balls into 6 distinct bins?

Same as saying

$$\underbrace{X_1 + X_2 + \dots + X_6}_{\text{Bins}} = 10 \leftarrow \text{Total Balls}$$

Formula:

$$= \binom{n+r-1}{r}$$

$$\binom{6+10-1}{10} = \binom{15}{10} \text{ or } = \binom{15}{5}$$

b/c we need 5 dividers

EXAMPLE 4

$x_1 + x_2 + x_3 = 9 \quad x_1 \geq 0,\ x_2 \geq 0,\ x_3 > 1$

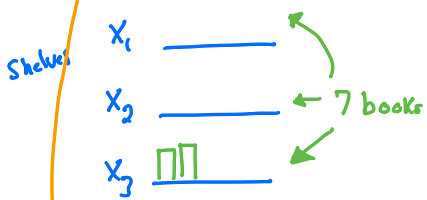
One of our Conditions is greater than another number. x_3 has to be greater then 1

$x_3 \geq 2$

Same thing ~ What this really means

I'm saying it has to be greater than 1 so it can be greater or equal to 2.

Books



We have 7 books Left to distribute

Shelf 3 needs to have at least 2 books

$x_1 + x_2 + x_3 = 7 \quad x_i \geq 0$

Which we know is just $\binom{9}{7}$ from the previous Example

$x_1 + x_2 + x_3 = 9 \quad x_3 \geq 2$

$x_3' = x_3 - 2 \quad x_3' \geq 0$ ← Were going to make a Subsitution

$x_3' + 2 = x_3$

$x_1 + x_2 + (x_3' + 2) = 9$

Substitute in $x_3' + 2$ b/c we want to shift $x_3 \geq 2 \to 0$

$x_1 + x_2 + x_3' = 7 \qquad x_i \geq 0$

- Find Solutions b/c all of our x_i are greater than zero

$\binom{9}{7}$ ~ Which we know is 9 choose 7

or

$\binom{9}{2}$ ~ or 9 choose 2 with dividers

- Really just Combinations with repetitions

Combinations and Repetition Examples

Video 35

EXAMPLE 1

Determine the number of solutions to $x_1 + x_2 + x_3 + x_4 = 36$ where $x_i \geq 4 \ \forall i$

We can't solve this ≥ 4 so, we want to reduce it to $x_i' \geq 0$

+4 +4

$$(x_1'+4) + (x_2'+4) + (x_3'+4) + (x_4'+4) = 36$$

$$x_1' + x_2' + x_3' + x_4' = 20$$

$$= \binom{4+20-1}{20}$$

Formula: $= \binom{n+r-1}{r}$

$$= \binom{23}{20}$$

EXAMPLE 2

For which n will the following have the same # of positive integer solutions?

$x_1 + x_2 + \dots x_{19} = n$ ~ need to have same # of solutions

$y_1 + y_2 + \dots y_{64} = n$ ~ When $n = 82$

So it means $x_i \geq 1 \rightarrow x_i' \geq 0$

$y_i \geq 1 \rightarrow y_i' \geq 0$

$x_1' + \dots x_{19}' = n - 19$ ~ subtract 19

$y_1' + \dots + y_{64}' = n - 64$

$$\binom{19 + (n-19) - 1}{n-19} = \binom{64 + (n-64) - 1}{n-64}$$ ~ Solutions have to be equal

$$\binom{n-1}{n-19} = \binom{n-1}{n-64}$$

$$\binom{n}{n-k} = \binom{n}{k}$$ Find that K value!

$$\frac{(n-1)!}{(n-19)!\ 18!}$$

$$\frac{n!}{k!(n-k)!}$$ ~ Two bottom have to add up to the top

$n - 19 + \underline{18} = n - 1$

since $\binom{n}{n-k} = \binom{n}{k}$

then $\binom{n-1}{n-19} = \binom{n-1}{18} = \binom{n-1}{n-64}$

$18 = n - 64$

$\boxed{n = 82}$

EXAMPLE 3

How many times is print executed?

```
for i := 1 to 20 do
 for j := 1 to i do
  for k := 1 to j do
   for m := 1 to k do
     print(i*j*k*m)
```

4 loops

```
for i := 1 to 20 do
 for j := 1 to i do
  for k := 1 to j do
   for m := 1 to k do
     print(i*j*k*m)
```

$i = 4$
$j = 3 \to 4$
$k = 3$
$m = 1 \to 2 \to 3$

Bounds $1 \le m \le k \le j \le i \le 20$

$\{1, 2, 3, 4, \dots, 20\}$

- Were picking $\underline{4 \text{ numbers}}$

We picked 4 numbers $\to$ 1, 3, 3, 20

Order them m k j i

$$\binom{20+4-1}{4}$$

$$\boxed{= \binom{23}{4}}$$

Counting Practice

Video 36

Let $A = \{1,2,3,4,5,6\}$ where we select 3 objects

(a) How many ordered sequences without repetition?

- when you hear ordered you think permutations

$P(6,3) = 6 \times 5 \times 4$

Formula?

(b) With repetition?

$6 \times 6 \times 6 = 6^3$

(c) Unordered without repetition?

$\binom{6}{3}$

(d) With repetition?

$\binom{n+r-1}{r} = \binom{6+3-1}{3} = \binom{8}{3}$

$A = \{1,2,3,4,5,6\}$ Pick 3

How many strictly increasing sequences can be chosen?

$(\langle 1,4,5\rangle, \langle 2,3,6\rangle \ldots$ NOT $\langle 2,4,4\rangle)$

No > than previous number

$\binom{6}{3}$

Go Back!

Consider the word "aeemrry"

(a) How many arrangements can we make?

$\frac{7!}{2!\,2!}$

(b) How many contain the word "eye"?

[eye] a m rr

- Treat eye as a single element

$\frac{5!}{2!}$

(c) How many contain the word "ram" and "eye"?

1 ram 2 eye 3 r

$3!$