Homework #8 Sections 4.4, 5.1-5.2

Section 4.4

6. Find an inverse of a modulo m for each of these pairs of relatively prime integers using the method followed in Example 2.

a) $a = 2, m = 17$
b) $a = 34, m = 89$

c) $a = 144, m = 233$
d) $a = 200, m = 1001$

a) $a = 2, m = 17$

Euclidean Algorithm

12. Solve each of these congruences using the modular inverses found in parts (b), (c), and (d) of Exercise 6.

a) $34x \equiv 77 \pmod{89}$

b) $144x \equiv 4 \pmod{233}$

c) $200x \equiv 13 \pmod{1001}$

Need to revisit this section

34. Use Fermat's little theorem to find 23^{1002} **mod** 41.

Section 5.1

4. Let $P(n)$ be the statement that $1^3 + 2^3 + \cdots + n^3 = (n(n+1)/2)^2$ for the positive integer n.

a) What is the statement $P(1)$?

b) Show that $P(1)$ is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step, identifying where you use the inductive hypothesis.

f) Explain why these steps show that this formula is true whenever n is a positive integer.

Let $P(n) = 1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

a) What is the statement $P(1)$?

b) Show that $P(1)$ is true, completing the basis step of the proof.

$$\underbrace{P(1) = (1)^3 = \left(\frac{1(1+1)}{2}\right)^2}_{\text{Statement for } P(1)} = 1 = \left(\frac{2}{2}\right)^2 = 1 = (1)^2 = 1 = 1 \checkmark$$

P(1) is True

c) What is the inductive hypothesis? $P(K) = P(K+1)$

Inductive Hypothesis: $1^3 + 2^3 + \ldots + K^3 = \left(\frac{K(K+1)}{2}\right)^2$

What we want?

* Must Show: $1^3 + 2^3 + \ldots + (K+1)^3 = \left(\frac{(K+1)(K+2)}{2}\right)^2$ - Extra Step

$1^3 + 2^3 + \ldots + (K+1)^3 = \left(\frac{K(K+1)}{2}\right)^2 + (K+1)^3$ - Add $(K+1)^3$ to both sides

$$= \frac{K^2(K+1)^2}{2^2} + \frac{4(K+1)^3}{4}$$

d) What do you need to prove in the inductive step?

If IH $P(K)$ is True, then $P(K+1)$ is also true

$$= \frac{K^2(K^2+2K+1)}{4} + \frac{4(K+1)^3}{4}$$

$$= \frac{K^4 + 2K^3 + K^2}{4} + \frac{4(K+1)^3}{4}$$

$$= \frac{(K^4 + 2K^3 + K^2 + (4(K+1)^3)}{4}$$

e) Complete the inductive step, identifying where you use the inductive hypothesis.

$$= \frac{K^4 + 6K^3 + 13K^2 + 12K + 4}{4}$$

$$= \left(\frac{(K^2 + 3K + 2)}{2}\right)^2$$

$$= \left(\frac{(K+1)(K+2)}{2} \right)^2$$

$$= \left(\frac{(K+1)((K+1)+1)}{2} \right)^2$$

$$\left(\frac{(K+1)((K+1)+1)}{2} \right)^2 = \left(\frac{(K+1)(K+2)}{2} \right)^2$$ ~ Equal to the inductive Hypothesis

f) Explain why these steps show that this formula is true whenever n is a positive integer.

If IH P(K) is True, then P(K+1) is also true

$$\boxed{\therefore P(n) \text{ is true } \forall\, n \in \mathbb{Z},\ n > 0}$$

6. Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer.

$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$$

Let $P(n): 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$

① Basis: Prove $P(1)$ is true

$$P(1) = (1) \cdot (1!) = (1+1)! - 1$$

$$1 \cdot 1! = (2)! - 1 \qquad 2! = 2 \cdot 1 = 2$$

$$1 = 1 \checkmark$$

② Inductive step: $P(k) \to P(k+1)$

Inductive Hypothesis: $1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k+1)! - 1$

Show: $1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! + \underbrace{(k+1)(k+1)!}_{\text{next integer}} = ((k+1)+1)! - 1$ ← What we want to end up with!

$$1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! + (k+1)(k+1)! = (k+1)! - 1 + (k+1) \cdot (k+1)!$$

- Got to do to both sides

$$= 1 \cdot (k+1)! + (k+1) \cdot (k+1)! - 1$$

$$= (1 + k + 1)(k+1)! - 1$$

- Factor out $(k+1)!$

$$= (k+2)(k+1)! - 1$$

$$= (k+2)! - 1$$

$$= ((k+1)+1)! - 1$$

$$\boxed{\begin{array}{l} \therefore P(n) \text{ is true} \\ \text{for all } n > 0 \\ n \in \mathbb{Z} \end{array}}$$

10. a) Find a formula for

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}$$

by examining the values of this expression for small values of n.

Summation Problem

b) Prove the formula you conjectured in part (a).

a)

$n=1$ $\quad \frac{1}{1(1+1)} = \frac{1}{2}$

$n=2$ $\quad \frac{1}{1\cdot 2}+\frac{1}{2(2+1)} = \frac{1}{2}+\frac{1}{6} = \frac{2}{3}$

$n=3$ $\quad \frac{1}{2}+\frac{1}{6}+\frac{1}{3(3+1)} = \frac{1}{2}+\frac{1}{6}+\frac{1}{12} = \frac{3}{4}$

$$\boxed{\text{notice the Pattern: } f(n) = \frac{n}{n+1}}$$

b) Prove the formula you conjectured in part (a).

Let $P(n)$ $\quad \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)} = \frac{n}{n+1}$

① Basis: Prove $P(1)$ is true

$$P(1) = \frac{1}{1(1+1)} = \frac{1}{1+1}$$

$$\frac{1}{2} = \frac{1}{2} \checkmark$$

② Inductive step: $P(K) \rightarrow P(K+1)$

Inductive Hypothesis: $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{K(K+1)} = \frac{K}{K+1}$

Show: $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{K(K+1)}+\frac{1}{(K+1)((K+1)+1)} = \frac{(K+1)}{((K+1)+1)}$ ← What we want to end up with at the end.

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{K(K+1)}+\frac{1}{(K+1)((K+1)+1)} = \frac{K}{K+1}+\frac{1}{(K+1)((K+1)+1)}$$

$$= \frac{K}{K+1}+\frac{1}{(K+1)(K+2)}$$

← We want to match the denominator

$$= \frac{K(K+2)}{(K+1)(K+2)}+\frac{1}{(K+1)(K+2)}$$

$$= \frac{K(K+2)+1}{(K+1)(K+2)}$$

$$= \frac{K(K+2)+1}{(K+1)(K+2)}$$

$$= \frac{K^2+2K+1}{(K+1)(K+2)}$$

$$= \frac{(K+1)^{\cancel{2}}}{\cancel{(K+1)}(K+2)}$$

$$= \frac{K+1}{K+2}$$

$$= \frac{K+1}{(K+1)+1}$$

Matches what we wanted to show above

$\therefore$ $\boxed{P(n) \text{ is true for } n \geq 0,\ n \in \mathbb{Z}}$

16. Prove that for every positive integer n,

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) = n(n+1)(n+2)(n+3)/4.$$

Let $P(n) = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$

① Basis: Prove P(1) is true

For $P(1) \cdot 1(1+1)(1+2) = \frac{1(1+1)(1+2)(1+3)}{4}$

$6 = 6$ ✓

② Inductive step: $P(K) \rightarrow P(K+1)$

Inductive Hypothesis: $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots K(K+1)(K+2) = \frac{K(K+1)(K+2)(K+3)}{4}$

* Must Show: $(K+1)((K+1)+1)((K+1)+2) = \frac{(K+1)(K+2)(K+3)(K+4)}{4}$ ← Want to end up at this at the end!

$K(K+1)(K+2) + (K+1)(K+2)(K+3) = \frac{K(K+1)(K+2)(K+3)}{4} + (K+1)(K+2)(K+3)$ - Got to do the same to both sides!

$= \left(\frac{K}{4} + 1\right)(K+1)(K+2)(K+3)$ ← Factored out $(K+1)(K+2)(K+3)$

$= \left(\frac{K}{4} + \frac{4}{4}\right)(K+1)(K+2)(K+3)$

$= \left(\frac{K+4}{4}\right)(K+1)(K+2)(K+3)$

$= \left(\frac{(K+1)(K+2)(K+3)(K+4)}{4}\right)$ ← Boom! matches what we wanted to show above!

$\boxed{P(n) \text{ is True}}$

18. Let $P(n)$ be the statement that $n! < n^n$, where n is an integer greater than 1.

a) What is the statement $P(2)$?
b) Show that $P(2)$ is true, completing the basis step of the proof.
c) What is the inductive hypothesis?
d) What do you need to prove in the inductive step?
e) Complete the inductive step.
f) Explain why these steps show that this inequality is true whenever n is an integer greater than 1.

* Inequlity Problem *

a) What is the statement $P(2)$?

$P(n) = n! < n^n$ $\quad$ $P(2) = 2! < 2^2$ ✓ True

b) Show that $P(2)$ is true, completing the basis step of the proof.

Basis: $P(2) = 2! < 2^2$ $\quad$ $2! = 1! \cdot 2! = 1 \cdot 2 = 2$

$P(2) = 2 < 4$ ✓ True

$P(2)$ is True

c) What is the inductive hypothesis?

$K! < K^K$

d) What do you need to prove in the inductive step?

That $P(K+1)$ is also true

$(K+1)! < (K+1)^{K+1}$

e) Complete the inductive step.

$(K+1)! = (K+1) \cdot K!$

$(K+1) \cdot K! < (K+1) \cdot K^K$

$< (K+1) \cdot (K+1)^K$ $\quad$ Since $K < K+1$ $\quad$ example for $K = 1$

$K < K+1$

$1 < 1+1$

$= (K+1)^{K+1}$ $\quad$ $1 < 2$ ✓

f) Explain why these steps show that this inequality is true whenever n is an integer greater than 1.

Principle of mathematical induction $P(n)$ is true

Section 5.2

Well-ordering Principle

4. Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for $n \geq 18$.

a) Show statements $P(18)$, $P(19)$, $P(20)$, and $P(21)$ are true, completing the basis step of the proof.

b) What is the inductive hypothesis of the proof?

c) What do you need to prove in the inductive step?

d) Complete the inductive step for $k \geq 21$.

e) Explain why these steps show that this statement is true whenever $n \geq 18$.

a) Show statements $P(18)$, $P(19)$, $P(20)$, and $P(21)$ are true, completing the basis step of the proof.

18 - 2 7-cent Stamps and 1 4-cent stamps

19 1 7-cent Stamps and 3 4-cent stamps

20 5 4-cent stamps

21 3 7-cent Stamps

b) What is the inductive hypothesis of the proof?

We assume that P(18), P(19), ... P(k) are all true, thus any postage between 18 and k cents can be formed using just 4-cent and 7-cent stamps

c) What do you need to prove in the inductive step?

need to prove P(k+1) is true

d) Complete the inductive step for $k \geq 21$.

$$k \geq 21$$
$$k-3 \geq 21-3$$
$$k-3 \geq 18$$

e) Explain why these steps show that this statement is true whenever $n \geq 18$.

If I can make postage of k-3, then I can make postage of k+1 cents by adding a 4-cent stamp

12. Use strong induction to show that every positive integer n can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, and so on. [*Hint:* For the inductive step, separately consider the case where $k + 1$ is even and where it is odd. When it is even, note that $(k + 1)/2$ is an integer.]

$P(n)$: Positive integer can be written as a sum of distinct powers of 2

Need to show $P(n)$ is true for all $n \geq 1$.

<u>Basis step:</u>

$P(1)$: $1 = 2^0$ ✓

$P(2)$: $2 = 2^1$ ✓

<u>Inductive step:</u>

Assuming $P(n)$ is true for all $n \leq k$, we need to show $P(k+1)$ is true

Case 1:

?

Case 2:

$\therefore$

Section 5.3

3. Find $f(2)$, $f(3)$, $f(4)$, and $f(5)$ if f is defined recursively by $f(0) = -1$, $f(1) = 2$, and for $n = 1, 2, \ldots$

a) $f(n+1) = f(n) + 3f(n-1)$.
b) $f(n+1) = f(n)^2 f(n-1)$.
c) $f(n+1) = 3f(n)^2 - 4f(n-1)^2$.
d) $f(n+1) = f(n-1)/f(n)$.

a) $f(n+1) = f(n) + 3f(n-1)$.

$f(2) = f(1) + 3f(1-1)$
$f(2) = f(1) + 3f(0)$
$f(2) = 2 + 3(-1)$
$f(2) = -1$

$f(3) = f(2) + 3f(1)$
$f(3) = -1 + 3(2)$
$f(3) = 5$

$f(4) = f(3) + 3f(2)$
$= 5 + 3(-1)$
$f(4) = 2$

$f(5) = f(4) + 3f(3)$
$f(5) = 2 + 3(5)$
$f(5) = 17$

b) $f(n+1) = f(n)^2 f(n-1)$. $f(0) = -1$ $f(1) = 2$

$f(2) = f(1)^2 \cdot f(0)$
$f(2) = 4 \cdot (-1)$
$f(2) = -4$

$f(3) = f(2)^2 \cdot f(1)$
$= (-4)^2 \cdot 2$
$f(3) = 32$

$f(4) = f(3)^2 \cdot f(2)$
$= (32)^2 \cdot -4$
$f(4) = -4096$

$f(5) = f(4)^2 \cdot f(3)$
$= (-4096)^2 \cdot (32)$
$f(5) = 536{,}870{,}912$

c) $f(n+1) = 3f(n)^2 - 4f(n-1)^2$. $f(0) = -1$ $f(1) = 2$

$f(2) = 3f(1)^2 - 4f(0)^2$
$f(2) = 3(2)^2 - 4(-1)^2$
$f(2) = 8$

$f(3) = 3f(2)^2 - 4(1)^2$
$f(3) = 3(8)^2 - 4(2)^2$
$f(3) = 176$

$f(4) = 3f(3)^2 - 4f(2)^2$
$= 3(176)^2 - 4(8)^2$
$f(4) = 92{,}672$

$f(5) = 3f(4)^2 - 4f(3)^2$
$= 3(92{,}672)^2 - 4(176)^2$
$f(5) = 25{,}764{,}174{,}848$

d) $f(n+1) = f(n-1)/f(n)$. $f(0) = -1$ $f(1) = 2$

$f(2) = \dfrac{f(0)}{f(1)}$
$= \dfrac{-1}{2}$
$f(2) = -1/2$

$f(3) = \dfrac{f(1)}{f(2)}$
$f(3) = \dfrac{2}{-1/2}$
$f(3) = -4$

$f(4) = \dfrac{f(2)}{f(3)}$
$f(4) = \dfrac{-1/2}{-4}$
$f(4) = 1/8$

$f(5) = \dfrac{f(3)}{f(4)}$
$f(5) = \dfrac{-4}{1/8}$
$f(5) = -32$

8. Give a recursive definition of the sequence $\{a_n\}$, $n = 1, 2, 3, \ldots$ if

a) $a_n = 4n - 2$. **b)** $a_n = 1 + (-1)^n$.
c) $a_n = n(n + 1)$. **d)** $a_n = n^2$.

a) $a_n = 4n - 2$.

for n of:

a_n	
1	$a_1 = 4(1) - 2 = 2$
2	$a_2 = 4(2) - 2 = 6$
3	$a_3 = 4(3) - 2 = 10$

Base

+4

+4

Inital Condition (s)

$a_0 = 2$

Recursive Function

~ for what values of n

$a_n = a_{n-1} + 4$

$n \geq 2$

b) $a_n = 1 + (-1)^n$.

1 $a_1 = 1 + (-1)^1 = 1 + -1 = 0$
2 $a_2 = 1 + (-1)^2 = 1 + 1 = 2$
3 $a_3 = 1 + (-1)^3 = 1 + -1 = 0$
$a_4 = 1 + (-1)^4 = 1 + 1 = 2$

Patterns ~ its either 2 or 0

Inital Condition (s)

$a_1 = 0$
$a_2 = 2$

Recursive Function

$a_n = a_{n-2}$ when $n \geq 3$

c) $a_n = n(n + 1)$.

$a_1 = 1(1+1) = 2$
$a_2 = 2(2+1) = 6$
$a_3 = 3(3+1) = 12$
$a_4 = 4(4+1) = 20$
$a_5 = 5(5+1) = 30$

+4 +6 +8 +10

$a_2 = a_1 + 2(2)$
$= 2 + 4$
$a_2 = 6$ ✓

$a_3 = a_2 + 2n$
$= 6 + 2(3)$
$a_3 = 12$ ✓

Inital Condition (s)

$a_1 = 2$

Recursive Function

$a_n = a_{n-1} + 2n$

When $n \geq 2$

d) $a_n = n^2$.

$a_1 = 1^2 = 1$
$a_2 = 2^2 = 4$
$a_3 = 3^2 = 9$
$a_4 = 4^2 = 16$

?

Inital Condition (s)

Recursive Function

24. Give a recursive definition of

a) the set of odd positive integers.
b) the set of positive integer powers of 3.
c) the set of polynomials with integer coefficients.

Basis step: Specifies an inital collection of elements

Recursive step: Gives rules for forming new elements from those already in the set.

a) the set of odd positive integers.

$S = \{1, 3, 5, \ldots\}$

Basis: $1 \in S$

Recursive: $s + 2 \in S$ whenever $s \in S$

b) the set of positive integer powers of 3.

3^1

Basis: $3 \in S$

Recursive: $3s \in S$ whenever $s \in S$

c) the set of polynomials with integer coefficients.

Basis:

Recursive: