Homework #9 Sections 6.1 - 7.1

Section 6.1

1. There are 18 mathematics majors and 325 computer science majors at a college.

a) In how many ways can two representatives be picked so that one is a mathematics major and the other is a computer science major?

b) In how many ways can one representative be picked who is either a mathematics major or a computer science major?

a) In how many ways can two representatives be picked so that one is a mathematics major and the other is a computer science major?

Product Rule Applies here

$18 \cdot 325 = 5850$

b) In how many ways can one representative be picked who is either a mathematics major or a computer science major?

Sum Rule Applies

$18 + 325 = 343$

6. There are four major auto routes from Boston to Detroit and six from Detroit to Los Angeles. How many major auto routes are there from Boston to Los Angeles via Detroit?

Product Rule

$4 \cdot 6 = 24$

7. How many different three-letter initials can people have?

Product Rule

$26 \times 26 \times 26$

or

26^3

17,576

Section 6.2

3. A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in the dark.

a) How many socks must he take out to be sure that he has at least two socks of the same color?

b) How many socks must he take out to be sure that he has at least two black socks?

Black Socks = 12 Broun Socks = 12

a) How many socks must he take out to be sure that he has at least two socks of the same color?

Colors are the Pigeonholes $\left[\frac{x}{2} \right]$

$X = 3$

b) How many socks must he take out to be sure that he has at least two black socks?

first 12 could be broun so 14 socks

6. Let d be a positive integer. Show that among any group of $d + 1$ (not necessarily consecutive) integers there are two with exactly the same remainder when they are divided by d.

Thereom

Section 6.3

11. How many bit strings of length 10 contain

a) exactly four 1s?
b) at most four 1s?
c) at least four 1s?
d) an equal number of 0s and 1s?

$$C(n,r) = \binom{n}{r} = \frac{n!}{(n-r)! \cdot r!}$$

a) exactly four 1s?

Combination Problem

$$C(10,4) = \frac{10!}{(10-4)! \cdot 4!} = \frac{10!}{6! \cdot 4!} = 210$$

b) at most four 1s?

Combination Problem with addtn

$$C(10,4) = \frac{10!}{(10-4)! \cdot 4!} = \frac{10!}{6! \cdot 4!} = 240$$

$$C(10,3) = \frac{10!}{(10-3)! \cdot 3!} = \frac{10!}{7! \cdot 3!} = 120$$

$$C(10,2) = \frac{10!}{(10-2)! \cdot 2!} = \frac{10!}{8! \cdot 2!} = 45$$

$$C(10,1) = \frac{10!}{(10-1)! \cdot 1!} = \frac{10!}{9! \cdot 1!} = 10$$

$$C(10,0) = \frac{10!}{(10-0)! \cdot 0!} = \frac{10!}{0! \cdot 0!} = 1$$

$$210 + 120 + 45 + 10 + 1$$

$$= 386$$

c) at least four 1s? Combination Problem

$$C(10,4) = \frac{10!}{(10-\ )! \cdot \ !} = \frac{10!}{6! \cdot 4!} = 240$$

$$C(10,5) = \frac{10!}{(10-5)! \cdot 5!} = \frac{10!}{5! \cdot 5!} = 252$$

$$C(10,6) = \frac{10!}{(10-6)! \cdot 6!} = \frac{10!}{6! \cdot 4!} = 210$$

$$C(10,7) = \frac{10!}{(10-7)! \cdot 7!} = \frac{10!}{7! \cdot 3!} = 120$$

$$C(10,8) = \frac{10!}{(10-8)! \cdot 8!} = \frac{10!}{8! \cdot 2!} = 45$$

$$C(10,9) = \frac{10!}{(10-9)! \cdot 9!} = \frac{10!}{9! \cdot 1!} = 10$$

$$C(10,10) = \frac{10!}{(10-10)! \cdot 10!} = \frac{10!}{10! \cdot 0!} = 1$$

$$210 + 252 + 210 + 120 + 45 + 10 + 1 = 848$$

d) an equal number of 0s and 1s? Combination Problem

So there must be 5 0's and 5 1's.

$$C(n,r) = \binom{n}{r} = \frac{n!}{(n-r)! \cdot r!}$$

$$C(10,5) = \frac{10!}{(10-5)! \cdot 5!} = \frac{10!}{10!\, 5!} = 252$$

19. A coin is flipped 10 times where each flip comes up either heads or tails. How many possible outcomes
a) are there in total?
b) contain exactly two heads?
c) contain at most three tails?
d) contain the same number of heads and tails?

Combination: (order is not important)

$$C(n,r) = \binom{n}{r} = \frac{n!}{(n-r)! \cdot r!}$$

a) are there in total?

$$2^{10} = 1024$$

Heads or Tails

b) contain exactly two heads?

$$C(10,2) = \frac{10!}{(10-2)!2!} = \frac{10!}{8! \cdot 2!} = 45$$

c) contain at most three tails?

$$C(10,0) + C(10,1) + C(10,2) + C(10,3)$$

$$= 176$$

d) contain the same number of heads and tails?

5 heads and 5 tails

$$C(10,5) = \frac{10!}{(10-5)! \cdot 5!} = \frac{10!}{5! \cdot 5!} = 252$$

Permutation: (order is important)

$$P(n,r) = \frac{n!}{(n-r)!}$$

21. How many permutations of the letters *ABCDEFG* contain

a) the string *BCD*?
b) the string *CFGA*?
c) the strings *BA* and *GF*?
d) the strings *ABC* and *DE*?
e) the strings *ABC* and *CDE*?
f) the strings *CBA* and *BED*?

a) the string *BCD*?

A, BCD, E, F, G
1, 2, 3, 4, 5

$$P(n,r) = \frac{n!}{(n-r)!}$$

$$P(5,5) = \frac{5!}{(5-5)!} = \frac{5!}{\underset{1}{0!}} = 5! = 120$$

b) the string *CFGA*?

CFGA, B, D, E

$n=4$
$r=4$

$$P(4,4) = \frac{4!}{(4-4)!} = \frac{4!}{0!} = 4! = 24$$

c) the strings *BA* and *GF*?

BA, GF, C, D, E

$n=5$
$r=5$

$$P(5,5) = \frac{5!}{(5-5)!} = \frac{5!}{\underset{1}{0!}} = 5! = 120$$

d) the strings *ABC* and *DE*?

ABC, DE, F, G

$n=4$
$r=4$

$$P(4,4) = \frac{4!}{(4-4)!} = \frac{4!}{0!} = 4! = 24$$

e) the strings *ABC* and *CDE*?

ABC CDE, F, G ~ C can only occur once so they must be combined

$n=3$
$r=3$

$$P(3,3) = \frac{3!}{(3-3)!} = \frac{3!}{0!} = 3! = 6$$

f) the strings *CBA* and *BED*?

- B appears more than once and cannot follows directly after B

0

33. Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?

So we need 3 men and 3 women

men

$$C(10,3) = \frac{10!}{(10-3)! \cdot 3!} = \frac{10!}{7! \cdot 3!} = 120$$

Women

$$C(15,3) = \frac{15!}{(15-3)! \cdot 3!} = \frac{15!}{12! \cdot 3!} = 455$$

$$120 \cdot 455 = 54{,}600$$

Combination: (order is not important)

$$C(n,r) = \binom{n}{r} = \frac{n!}{(n-r)! \cdot r!}$$

Section 6.4

8. What is the coefficient of x^8y^9 in the expansion of $(3x + 2y)^{17}$?

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$\sum_{j=0}^{17} \binom{17}{j} (3x)^{17-j} \cdot (2y)^j$$

What is j?

$j = 9$

$$\sum_{j=0}^{17} \binom{17}{9} (3x)^{17-9} \cdot (2y)^9$$

$$= \frac{17!}{9!(17-9)!} \cdot (3x)^8 \cdot (2y)^9$$

$$= \frac{17!}{9!\,8!} \cdot 3^8 \cdot x^8 \cdot 2^9 \cdot y^9$$

$$= 24310 \cdot 3^8 \cdot x^8 \cdot 2^9 \cdot y^9$$

$$\boxed{= \underbrace{81{,}662{,}929{,}920}_{\text{Coefficent}}\, x^8 y^9}$$

9. What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x-3y)^{200}$?

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$= \sum_{j=0}^{200} \binom{200}{99} 2x^{200-99} (-3 \cdot y)^{99}$$

$$= -\frac{200!}{99!(200-99)!} (2x)^{101} \cdot (3y)^{99}$$

$$= -\frac{200!}{99! \cdot 100!} \cdot (2x)^{101} \cdot (3y)^{99}$$

$$= -\frac{200!}{99! \cdot 100!} \cdot 2^{101} \cdot x^{101} \cdot 3^{99} \cdot y^{99}$$

$$\boxed{= \underbrace{-39 \times 10^{135}}_{\text{Coefficent}} x^{101} y^{99}}$$

12. The row of Pascal's triangle containing the binomial coefficients $\binom{10}{k}$, $0 \le k \le 10$, is:

1 10 45 120 210 252 210 120 45 10 1

Use Pascal's identity to produce the row immediately following this row in Pascal's triangle.

	1	10	45	120	210	252	210	120	45	10	1
1	11	55	165	330	462	462	330	165	55	11	1

- just add values

- Super Easy

21. Prove that if n and k are integers with $1 \leq k \leq n$, then $k\binom{n}{k} = n\binom{n-1}{k-1}$,

a) using a combinatorial proof. [*Hint:* Show that the two sides of the identity count the number of ways to select a subset with k elements from a set with n elements and then an element of this subset.]

b) using an algebraic proof based on the formula for $\binom{n}{r}$ given in Theorem 2 in Section 6.3. ~ Pascal Identity

Section 6.5

5. How many ways are there to assign three jobs to five employees if each employee can be given more than one job?

$n = 5$ Employees
$r = 3$ Jobs

Combination

Not repetion ?

?

Type	Repetitions?	
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!(n-1)!}$

8. How many different ways are there to choose a dozen donuts from the 21 varieties at a donut shop?

$n = 21$ ~ Set of donuts

$r = 12$

Repetition is allowed

$$\frac{(n+r-1)!}{r!(n-1)!} = \frac{(21+12-1)!}{12!(21-1)!} = \frac{32!}{12! \cdot 20!} = \boxed{225,792,840 \text{ Combinations}}$$

9. A bagel shop has onion bagels, poppy seed bagels, egg bagels, salty bagels, pumpernickel bagels, sesame seed bagels, raisin bagels, and plain bagels. How many ways are there to choose

8 Bagels

a) six bagels?
b) a dozen bagels?
c) two dozen bagels?
d) a dozen bagels with at least one of each kind?
e) a dozen bagels with at least three egg bagels and no more than two salty bagels?

a) six bagels?

$n = 8,\ r = 6$

Repetition is allowed

$$\frac{(n+r-1)!}{r!(n-1)!} = \frac{(8+6-1)!}{6!(8-1)!} = \frac{13!}{6! \cdot 7!} = 1716$$

b) a dozen bagels?

$n = 8,\ r = 12$

Repetition is allowed

$$\frac{(n+r-1)!}{r!(n-1)!} = \frac{(8+12-1)!}{12!(8-1)!} = \frac{19!}{12!\,7!} = 50{,}388$$

c) two dozen bagels?

$n = 8,\ r = 24$

Repetition is allowed

$$\frac{(n+r-1)!}{r!(n-1)!} = \frac{(8+24-1)!}{24!(8-1)!} = \frac{31!}{24! \cdot 7!} = 2{,}629{,}575$$

d) a dozen bagels with at least one of each kind?

$n = 8,\ r = 4$

Repetition is allowed

$$\frac{(n+r-1)!}{r!(n-1)!} = \frac{(8+4-1)!}{4!(8-1)!} = \frac{11!}{4! \cdot 7!} = 330$$

e) a dozen bagels with at least three egg bagels and no more than two salty bagels?

?

15. How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21,$$

where x_i, $i = 1, 2, 3, 4, 5$, is a nonnegative integer such that

a) $x_1 \geq 1$?
b) $x_i \geq 2$ for $i = 1, 2, 3, 4, 5$?
c) $0 \leq x_1 \leq 10$?
d) $0 \leq x_1 \leq 3$, $1 \leq x_2 < 4$, and $x_3 \geq 15$?

?

revist

18. How many strings of 20-decimal digits are there that contain two 0s, four 1s, three 2s, one 3, two 4s, three 5s, two 7s, and three 9s?

- if a digit occurs twice we divide by 2!, if it occurs 3 times, we divide by 3!, .. 4 then by 4!

$$\boxed{\frac{20!}{(2!)^3\cdot(3!)^3\cdot 4!} = 58{,}663{,}725{,}120{,}000}$$

27. There are 10 questions on a discrete mathematics final exam. How many ways are there to assign scores to the problems if the sum of the scores is 100 and each question is worth at least 5 points?

from Section "6.5 or 6.4"

"Box Question"

$n = 10$

$r = 50$

Repetition is allowed $\frac{(n+r-1)!}{r!(n-1)!} =$

$$\frac{(10+50-1)!}{50!(10-1)!} = \frac{59!}{50!\cdot 9!} =$$

32. How many different strings can be made from the letters in *AARDVARK*, using all the letters, if all three *A*s must be consecutive?

AAA R D V R K

1 2 3 4 5 6

$$\frac{6!}{1!\cdot \underset{\text{R's}}{2!}\cdot 1!\cdot 1!\cdot 1!\cdot 1!} = \frac{6!}{2!} = 360$$

Section 7.1

6. What is the probability that a card selected at random from a standard deck of 52 cards is an ace or a heart?

4 aces
13 heart
52 Cards in Deck

ace of hearts Doubly occure

$\left(\frac{17-1}{52}\right) = \frac{16}{52} = 0.31$

14. What is the probability that a five-card poker hand contains cards of five different kinds?

$$C(13,5) = \frac{n!}{(n-r)!\,r!} = \frac{13!}{(13-5)!\,5!} = \frac{13!}{5!\,8!} = 1287$$

Then there is 4 different suites : 4^5

$$1287 \cdot 4^5 = 1,317,888$$

18. What is the probability that a five-card poker hand contains a straight flush, that is, five cards of the same suit of consecutive kinds?

~ 10 ways to form a straight flush

$$C(4,1) = \frac{4!}{1!\,(4-1)!} = \frac{4!}{1!\cdot 3!} = 4$$

$$10 \cdot 4 = 40$$

22. What is the probability that a positive integer not exceeding 100 selected at random is divisible by 3?

$$P(E) = \frac{33}{100}$$

$$\boxed{P(E) = 0.3\overline{3}}$$