Mathematical Induction Chapter 5.1-5.4

5.1 Proof Using Mathematical Induction – Summation Formula

Mathematical Induction

To prove P(x) is true for $x \in \mathbb{Z}^+$, where P(x) is a propositional function, we complete two steps:

1) Basis Step - Verify $\boxed{P(1)}$ is true

2) Inductive step - Verify if P(K) is true, then P(K+1) is true $\forall K \in \mathbb{Z}^+$

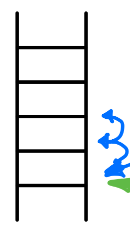
* Inductive Hypothesis : P(K) is true

* Must Show: $P(K) \rightarrow P(K+1)$

Conclusion : $\boxed{P(x) \text{ is true } \forall\ K \in \mathbb{Z}^+}$

Other instructors might not write this

like a ladder with ∞ # of runs



Proving a Summation Formula

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Show $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

Let $P(n)$: $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

① Basis: Prove $P(1)$ is true

$P(1) = 1 = \frac{1(1+1)}{2} \quad 1 = \frac{1(2)}{2} \quad 1 = 1$ ✓

~ Prove Basis Step
~ easy its just math

② Inductive step: $P(K) \rightarrow P(K+1)$

Inductive Hypothesis: $1 + 2 + 3 + \ldots + K = \frac{K(K+1)}{2}$

★ Must Show: $1 + 2 + 3 \ldots + K + (K+1) = \frac{(K+1)((K+1+1))}{2} = \boxed{\frac{(K+1)(K+2)}{2}}$

Exactly What We Wanted

$1 + 2 + 3 + \ldots + K + (K+1) = \frac{K(K+1)}{2} + (K+1)$

Write as fraction

1^{st} Step: Added $(K+1)$ to both sides

$$= \frac{K(K+1)}{2} + \frac{2(K+1)}{2}$$

$$\boxed{= \frac{(K+1)(K+2)}{2}}$$

2^{nd} Step: simplify solution

∴ $P(n)$ is true $\forall\, n \in \mathbb{Z}$

Problem 2:

Use mathematical induction to show that for all non-negative integer n, $1+2+2^2+\ldots+2^n = 2^{n+1}-1$

$n>0$

$$1+2+2^2+\ldots+2^n = 2^{n+1}-1$$

Let $P(n)$: $1+2+2^2+\ldots+2^n = 2^{n+1}-1$

① Basis: Prove $P(0)$ is true

$$2^0 = 2^{0+1}-1$$
$$1 = 1 \checkmark$$

② Inductive step: $P(K) \rightarrow P(K+1)$

Inductive Hypothesis: $1+2+2^2+\ldots+2^K = 2^{K+1}-1$

★ Must Show: $1+2+2^2+\ldots+2^K+2^{K+1} = 2^{(K+1)+1}-1 = \boxed{2^{K+2}-1}$ ← We want to get to this

"next sequence in pattern"

$$1+2+2^2+\ldots+2^K+2^{K+1} = 2^{K+1}-1+2^{K+1}$$ ~ Add to both sides

$$= 2(2^{K+1})-1$$

$$\boxed{= 2^{K+2}-1}$$

can add exponent

$3^2 \cdot 3^4 = 3^6$

$\therefore$ $P(n)$ is True
for all $n \geq 0$
$n \in \mathbb{Z}$

Problem 3

Conjective and prove a summation formula for the sum of the first n positive odd integers.

$n = 1$	$1 = 1$	1^2
$n = 2$	$1+3 = 4$	2^2
$n = 3$	$1+3+5 = 9$	3^2
$n = 4$	$1+3+5+7 = 16$	4^2
$\vdots$		$\vdots$

looking for a Pattern

$$\boxed{n = n \qquad 1+3+5+\ldots+(2n-1) = n^2} \sim P(n)$$

Prove $1+3+5+\ldots+(2n-1) = n^2$

Let $P(n)$: the proposition that $1+3+5+\ldots+(2n-1) = n^2$

① Basis: Prove $P(1)$ is true

$$1 = (1)^2$$
$$1 = 1 \checkmark$$

② Inductive step: $P(k) \rightarrow P(k+1)$

Inductive Hypothesis: $1+3+5+\ldots+(2k+1) = k^2$

Show: $1+3+5+\ldots+(2k-1) + (2k+1) = (k+1)^2$

if we added the next odd integer, which is $+2$.

$$1+3+5+\ldots+(2k-1) + (2k+1) = k^2 + (2k+1)$$

$$1+3+5+\ldots+(2k-1) + (2k+1) = (k+1)(k+1) \qquad \text{- Factor}$$

$$1+3+5+\ldots+(2k-1) + (2k+1) = (k+1)^2$$

$\therefore P(n)$ is true

Proof Using Mathematical Induction – Inequalities

Proving an Inequality:

Prove $n < 2^n \quad \forall n \in \mathbb{Z}^+$ using mathematical induction.

Let $P(n)$: $n < 2^n \quad \forall n \in \mathbb{Z}^+$

① Basis: $P(1)$

$P(1)$: $1 < 2^1$

$1 < 2^1$ ✓

② Inductive $P(k) \rightarrow \underline{P(k+1)}$

IH: $k < 2^k$

Show: $k+1 < 2^{k+1}$

$k+1 < 2^k + 1$

~ Since its an inequality that don't have same rules as an equals

~ We just have to make sure the <u>inequality</u> <u>still holds</u>

We know that: $1 < 2^k \quad \forall k \in \mathbb{Z}^+$

~ Proved this in Basis step ①

$$k+1 < 2^k + 1 < 2^k + 2^k = 2(2^k)$$

- How can we do this?

B/c $1 < 2^k$

- it's ok to do this b/c we're not breaking any <u>mathmatical</u> <u>law</u>.

- Replacing, inequality still works!

$$\boxed{k+1 < 2^{k+1}}$$

Example 2:

Prove $2^n < n!$ $\forall n \in \mathbb{Z}^+$ and $n \geq 4$.

Let $P(n): 2^n < n!$

① Basis $P(4)$

$2^4 < 4! \rightarrow 4 \cdot 3 \cdot 2 \cdot 1$

$16 < 24$ ✓

② Inductive $P(k) \rightarrow P(k+1)$

IH: $2^k < k!$

SHOW: $2^{k+1} < \underline{(k+1)!}$

Trying to get here!

$2^k \cdot 2^1 < k! \cdot 2$

$k+1 > 2$

- Just here to make sure the <u>inequality</u> still holds

$2^{k+1} < 2k!$

$2^{k+1} < (k+1)k!$

$(k+1)(k)(k-1)\ldots(2)(1) = (k+1)!$

$$\boxed{2^{k+1} < (k+1)!}$$

Proof Using Mathematical Induction – Divisibility

Use mathematical induction to prove $7^{n+2} + 8^{2n+1}$ is divisible by 57 for all non-negative integers n.

Let P(n): $7^{n+2} + 8^{2n+1}$ is divisible by 57

① Basis: P(0)

P(0): $7^{0+2} + 8^{2(0)+1} = 7^2 + 8 = 49 + 8 = \underline{\underline{57}}$ ✓ ~ is divisible by 57

② Inductive:

IH: $7^{k+2} + 8^{2k+1}$ is divisible by 57

Show: $7^{k+3} + 8^{2k+3}$ is divisible by 57

$7 \cdot 7^{k+2} + 8^{2k+1} \cdot 8^2 = 7^{k+3} + 8^{2k+3}$

~ Shows we can turn the IH into what we're suppose to Show.

$7 \cdot 8^{2k+1} + 57 \cdot 8^{2k+1}$ ← If you multiply out

$= 7 \cdot 7^{k+2} + \boxed{(7+57)\,8^{2k+1}}$

- lets re write the right-hand side.

$= 7(7^{k+2} + 8^{2k+1}) + 57 \cdot 8^{2k+1}$

$= 7(7^{k+2} + 8^{2k+1}) + 57 \cdot 8^{2k+1}$

- By Inductive Hypothesis this is divisible by 57

- Then this portion is also divisible by 57

Since $7^{k+2} + 8^{2k+1}$ is divisible by 57 and $57 \cdot 8^{2k+1}$ is divisible by 57,

then $7^{k+3} + 8^{2k+3}$ is divisible by 57.

$\boxed{\therefore\ 7^{n+2} + 8^{2n+1} \text{ is divisible by 57 for all } n \geq 0\ \ n \in \mathbb{Z}}$

5.2 The Well-Ordering Principle and Strong Induction

The Well-Ordering Principle

Every non-empty set of non-negative integers has a least element

* Works with non-negative Integers *

All have a least element

Z^+ → $n = 1$

$n \in Z^+$ $n \geq 4$ $n = 4$

non-negative int $n = 0$

In mathematical induction, we start with the least element.

If we used $n \in Q$ no least element for rational #s

Proof By Induction

Prove that every amount of postage of $0.12 or more can be formed using $0.04 and $0.05 stamps.

Basis

P(n) is the statement that postage of n-cents can be formed using 4-cent and 5-cent stamps if $n \geq 12$.

Basis: show postage of 12-cents can be made

12 cents, 3 4 cent stamps ✓

Inductive: show if P(K) is true, then P(K+1) is true for $K \geq 12$

IH: We can have postage of K-cent using 4 and 5 cent stamps

Show: I can make postage of K+1 cents using 4 and 5 cent stamps

Case 1: I've used one or more 4-cent stamps

If I've used a 4-cent stamp for K-cent postage, then I can replace my 4-cent stamp with a 5-cent stamp. ∴ K+1 postage was formed.

Case 2: I havent used an 4-cent stamp

If no 4-cent stamp was used, then I've used at least 3 5-cent stamps because $n \geq 12$, so I can replace 3 5 cent stamps with 4 4-cent stamps making postage of K+1.

Proof by Strong Induction

Prove that every amount of postage of $0.12 or more can be formed using $0.04 and $0.05 stamps.

P(n) is the statement that postage of n-cents can be formed using 4-cent and 5-cent stamps if n=> 12.

Basis: Show postage of 12,13,14 and 15-cents can be made

12 - 3 4-cent stamps
13 - 2 4-cent stamps and 1 5-cent stamp
14 - 1 4-cent stamps and 2 5-cent stamps
15 - 3 5-cent stamps

Inductive: Show that if P(j) is true, for 12 =< j =< k, where k => 15, then P(k+1) is true.

-3 -3

$K-3 \geq 12$

IH: $P(K-3)$ is true

Show: $P(K+1)$ is true

If I can make postage of K-3, then I can make postage of K+1 cents by adding a 4-cent stamp

Revisiting Recursive Definitions

Fibonacci Numbers

Recall the set of Fibonacci numbers:

0, 1, 1, 2, 3, 5, 8, 13, 21, ...

How each number found?

add 2 previous terms

Recursive Definition:

1. Initial conditions

$f_0 = 0$

$f_1 = 1$

2. Function

$f_n = f_{n-1} + f_{n-2}, \quad n \geq 2$ ~ How to find next term in the sequence

Recursive Defined Functions

- Has two parts

Basis Step: Specifies the value of the function for the first term(s)

Initial Condition

Recursive Step: Gives a rule for finding subsequent values using a previous value(s) beginning at those defined in the basis step.

Function

Example:

If f is defined recursively by $\boxed{f(0) = 2}$ (Basis) and $\boxed{f(n+1) = 3f(n) - 1}$ (Function), find $f(1)$, $f(2)$, $f(3)$, and $f(4)$.

$$f(n) = 3(f(n-1)) - 1$$

$$f(1) = 3f(0) - 1 = 3(2) - 1 = 5$$

$$f(2) = 3f(1) - 1 = 3(5) - 1 = 14$$

$$f(3) = 3f(2) - 1 = 3(14) - 1 = 41$$

$$f(4) = 3f(3) - 1 = 3(41) - 1 = 122$$

Examples

Give a recursive definition for a^n for $a \in \mathbb{R}$ and non-negative and $n \in \mathbb{Z}^+$

Initial Condition

$\boxed{a^0 = 1}$

$a^1 = a \cdot a$

$a^2 = a \cdot a$

$a^3 = a^2 \cdot a$

$a^4 = a^3 \cdot a$

Notice Pattern

- Recursive part

2 Parts to Solve these Problems:

Inital Condition (s)

$a^0 = 1$

Recursive Function

~ for what values of n

$a^n = a^{n-1} \cdot a$

for $n \geq 1$

Note that a recursive definition is well-defined, meaning that the value of the function is defined for every positive integer. As such, we are able to use a form of Mathematical Induction to prove recursive definitions. We will look at these in our next video.

Give a recursive definition of the sequence $\{a_n\}$, $n = 1, 2, 3 \ldots$ if $a_n = 2n + 1$

$a_0 = 2(0) + 1 = 0 + 1 = 1$

$a_1 = 2(1) + 1 = 2 + 1 = 3$

$a_2 = 2(2) + 1 = 4 + 1 = 5$

$a_3 = 2(3) + 1 = 6 + 1 = 7$

$a_4 = 2(4) + 1 = 8 + 1 = 9$

+2 +2 +2 +2

Pattern

Inital Condition (s)

$a_0 = 1$

Recursive Function

~ for what values of n

$a_n = a_{n-1} + 2$

$n \geq 1$

$a_1 = a_0 + 2 = 1 + 2 = 3$ Checking our function

Recursively Defined Sets and Structures

Basis step: Specifies an inital collection of elements

Recursive step: Gives rules for forming new elements from those already in the set.

Example:

The natural numbers $\mathbb{N} = \{1, 2, 3, 4 \dots\}$

+1 +1

Basis: $1 \in \mathbb{N}$

Recursive: If $n \in \mathbb{N}$ then $n+1 \in \mathbb{N}$

1, 2, 3, 4

Start at Basis

Recursively Defined Set

Let S be a subset of the integers defined recursively by:

Basis step: $7 \in S$

Recursive Step: If $x \in S$ and $y \in S$, then $x+y \in S$

List the elements of S produced by the first 5 application of the recursive definition.

$\{7, 14, 21, 28, 35\}$

Basis Step

① 7

② $7+7=14$

③ $7+14=21$

④ $7+21=28$

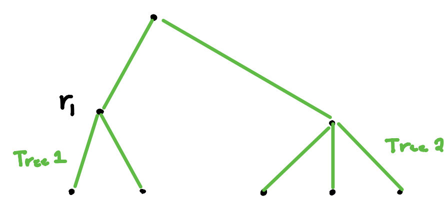
⑤ $7+28=35$

Recursively Defined Structure

A set of rooted trees, where a rooted tree consists of a set of vertices containing a distinguished vertex, called the root, and edges containing these vertices can be defined recursively by:

Basis Step: A single vertex r is a rooted tree

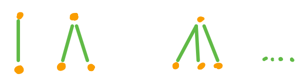
Recursive Step: Suppose that T1, T2, … Tn are disjointed rooted trees with roots r1, r2, …rn respectively. The graph formed by starting with a root r, not contained in any tree and adding an edge from r to each of r1,r2,….rn is also a rooted tree.



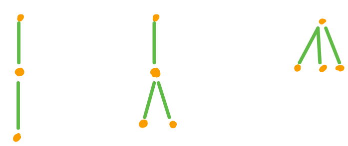
Example of Rooted Trees Defn:

Basis Step:

Recursive Step 1:



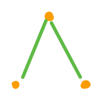
Step 2:



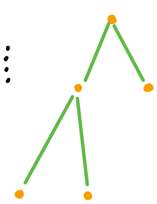
Full Binary Tree Example:

Basis Step:

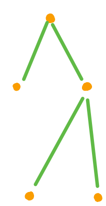
Recursive Step 1:



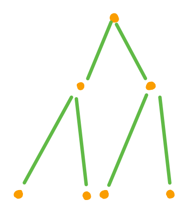
Step 2:



- Add to Left Vertis



- Add to Right



- Add to both

Structural Induction

Mathematical Induction

Let S be a subset of the integers defined recursively by:

Basis Step: $7 \in S$

Recursive Step: If $x \in S$ and $y \in S$, then $x + y \in S$

Show that S is the set of all positive integers divisibly by 7.

- Show two sets are equal to one - another

Let A be the set of all integers divisible by 7. To prove $\boxed{A = S}$, we must show $A \subseteq S$ and $S \subseteq A$.

$A = 7n$

Proof by Math Induction

Let P(n) be the statement $7n \in S$ The basis holds since $7 \cdot 1 = 7$ E S from the basis step of the recursive definition. Assume P(k) is true, or that 7K E S. If 7K E S then 7K + 7 E S since 7, 7K E S.

$\therefore \underbrace{7k + 7}_{P(k+1)} = 7(k+1) \in S$ and $A \subseteq S$.

Going in other direction

Since 7·1 belongs to A

$x + y \in S$ ABOVE

divides

For our Basis Step: $7 \in A$ since $\boxed{7 \cdot 1 \in A}$ for our recursive step, $x + y \in S$, and since $7 | x$ and $7 | y$, then $x = 7a$ and $y = 7b$ for some integers a and b.

$x + y = 7a + 7b = 7(a + b)$, therefore $x + y \in A$.

Shown in both directions

Structural Induction

Basis Step: Show the result holds for all elements specified in the basis step of the recursive definition

Recursive Step: Show that if the statement is true for all elements used to construct new elements in the recursive step of the definition, the result holds for these new elements.

- Same idea, but specific to sets

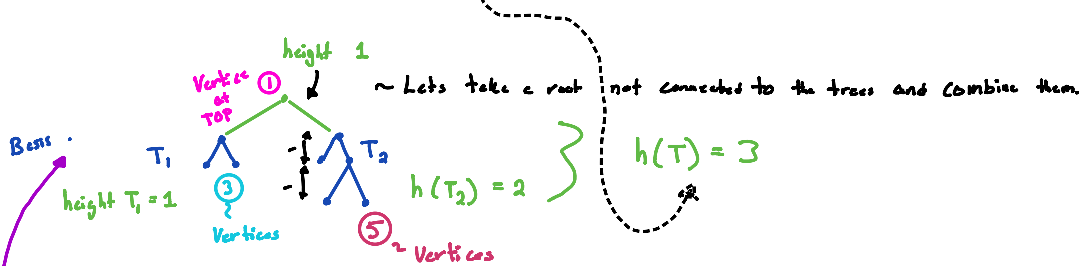
A Structural Induction proof is just a twist on proof using mathematical induction

Full Binary Trees

The height h(t) of a full binary tree is defined recursively as:

Basis Step: The height of a full binary tree T consisting of only a root r is h(T) = 0

Recursive Step: If T1 and T2 are full binary trees, then the full binary tree T = T1*T2 has a height h(t) = 1 + max(h(T1)), h(T2)).



The number of vertices of a full binary tree is defined recursively as:

Basis Step: The number of vertices of a full binary tree T consisting of only a root is n(T) = 1.

Recursive Step: If T1 and T2 are full binary trees, then the Full Binary tree T =T1*T2 has n(T)= 1+ n (T1) + n (T2) vertices.

# of total
Vertices in each Tree structure

Structural Proof

If T is a full binary tree, then $n(T) \le 2^{h(T)+1} - 1$

Prove by Structural Induction.

Basis Step:

The result holds for a full binary tree consisting of only a root, $n(T) = 1$ and $h(T) = 0$, so

$n(T) = 1$

$h(T) = 0$

$$n(T) = 1 \le 2^{0+1} - 1 = 1 \qquad 1 \le 1 \; \checkmark$$

So basis step holds

Recursive Step:

Assume $n(T_1) \le 2^{h(T_1)+1} - 1$ and $n(T_2) \le 2^{h(T_2)+1} - 1$

Whenever T_1 T_2 are full binary trees.

Proof:

Replace with what we defined them up here

Using IH, if True replace with $\le$

$$n(T) = 1 + n(T_1) + n(T_2) \qquad \text{- Recursive defn. of } n(T)$$

$$\le 1 + \left(2^{h(T_1)+1} - 1\right) + \left(2^{h(T_2)+1} - 1\right) \qquad \text{- Inductive hypothesis}$$

$$= 2 \cdot \max\left(2^{h(T_1)+1}, 2^{h(T_2)+1}\right) - 1 \qquad \text{- B/c the sum of 2 terms is at most 2 times the larger}$$

"Replaced"

$$= 2 \cdot 2^{\max(h(T_1), h(T_2)) + 1} - 1 \qquad \text{- } \max(2^x, 2^y) = 2^{\max(x,y)}$$

$$= 2 \cdot 2^{h(T)} - 1 \qquad \text{- By rec. definition of } h(T)$$

$$\boxed{= 2^{h(T)+1} - 1}$$

5.4 Recursive Algorithms

An Algorithm is called recursive if it solves a problem by reducing it to an instance of the same problem with a smaller input.

Recall $\gcd(14, 20)$ $\quad\quad$ $20 = 14(1) + 6$

$= \gcd(14, 6)$ $\quad\quad$ $14 = 6(2) + 2$

$= \gcd(6, 2)$ $\quad\quad$ $6 = 2(3) + 0$

$= \gcd(2, 0) = 2$ $\quad\quad$ $2 = 0(0) + 2$

Give a recursive algorithm for computing the greatest common divisor of two non-negative integers a and b with a < b .

Sudo Code:

procedure gcd(a, b: nonnegative integers, a < b)

if a = 0 then return b

else return gcd(b mod a, a)

→ {Output is gcd(a, b)}

$\gcd(\overset{a}{14}, \overset{b}{20})$; $20 = 14(1) + 6$

$= \gcd(\overset{a}{6}, \overset{b}{14})$; $14 = 6(2) + 2$

$= \gcd(\overset{a}{2}, \overset{b}{6})$; $6 = 2(3) + 0$

$= \gcd(\overset{a}{0}, \overset{b}{2}) = 2$

Give a recursive algorithm for computing n!, where n is a non-negative integer.

procedure factorial (n : nonnegative integer)

if n = 0 **then return** 1

else return n factorial (n-1)

{output is n!}

$n = 4 \qquad 4!$

$n = 3 \quad = 4 \cdot 3!$

$n = 2 \quad = 4 \cdot 3 \cdot 2!$

$n = 1 \quad = 4 \cdot 3 \cdot 2 \cdot 1!$

$= 4 \cdot 3 \cdot 2 \cdot 1 \cdot 0!$

$= 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1$

Give a recursive algorithm for computing a^n where a is a non-zero real number and n is a non-negative integer.

procedure power (a is a nonzero real number, n non-negative integer)

If n = 0 then return 1

else return a $\cdot$ power (a, n-1)

{ output is a^n }

$n = 4$ $\quad 5^4 = 5 \cdot 5^3$

$= 5 \cdot 5 \cdot 5^2$

$n = 3$ $\quad = 5 \cdot 5 \cdot 5 \cdot 5^1$

$= 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5^0$

$= 5 \cdot 5 \cdot 5 \cdot 5 \cdot 1$

Prove the algorithm correct which computes , where a is a non-zero real number and n is a non-negative integer.

Basis Step: $a^0 = 1$ for every non-zero real number a and $\boxed{\text{power (a, 0) =1}}$

Inductive Step: Assume $\boxed{\text{power (a, k) =}a^k}$ for all $a \neq 0$.

Then power (a, k + 1) $= a \cdot \text{power}(a, k)$

$= a \cdot a^k$

$\boxed{= a^{k+1}}$