MPCS 50103 Discrete Mathematics—Autumn 2019

Homework 6. This problem set is due Monday November 11 at 11:59 pm.

Reading: Rosen 7e, chapter 7, sections 7.2, 7.4.

Homework assignment:

- Solve the following "DO" exercises and homework problems "HW".
- "DO" exercises are strongly recommended to check your understanding of the concepts. **Do not submit them**.
- **Problems labeled "HW" must be submitted**.
- You are responsible for the material covered in **both** "DO" exercises and HW problems.

"Do" Exercises (*not* to be submitted):

1. **"DO"** Rosen 7e, section 7.2, exercises 33 and 35, on page 468.

2. **"DO"** Rosen 7e, section 7.4, exercises 5, 7, 13, 16, 19, 21, 23, 27, 33, 35, 37, and 39, on pages 492–494.

Homework Problems (to be submitted Monday November 14 at 11:59 pm):

- **Collaboration policy**: There is no penalty for acknowledged collaboration. To acknowledge collaboration, give the names of students with whom you worked at the beginning of your homework submission and mark any solution that relies on your collaborators' ideas. Note: you must work out and write up each homework solution by yourself without assistance.
  DO NOT COPY or rephrase someone else's solution.
  The same requirement applies to books and other written sources: you should acknowledge all sources that contributed to your solution of a homework problem. Acknowledge specific ideas you learned from the source.
  DO NOT COPY or rephrase solutions from written sources.
- **Internet policy**: Looking for solutions to homework problems on the internet, even when acknowledged, is STRONGLY DISCOURAGED. If you find a solution to a homework problem on the internet, do not not copy it. Close the website, work out and write up your solution by yourself, and cite the url of website in your writeup. Acknowledge specific ideas you learned from the website.
  Copied solutions obtained from a written source, from the internet, or from another person will receive ZERO credit and will be flagged to the attention of the instructor.
- Write out your work for every problem. If you just write your answer without showing your work, you will not receive credit.

1. **HW** Consider 10 independent tosses of a biased coin with heads probability p at each toss, where $0 \le p \le 1$.
   - Let A be the event that there are 6 heads in the first 8 tosses. Let B be the event that the 7th toss results in heads. Find $P(B \mid A)$. Express your answer in terms of p. Explain. (2 points)
   - Find the probability that there are 2 heads in the first 4 tosses and 2 heads in the last 3 tosses. Express your answer in terms of p. Show your work. (2 points)
   - Given that there were 4 heads in the first 7 tosses, find the probability that the 2nd head occurred at the 4th toss. Show your work. (2 points)
   - Find the probability that that there are 5 heads in the first 8 tosses and 3 heads in the last 5 tosses. Express your answer in terms of p. Show your work and explain. (2 points)

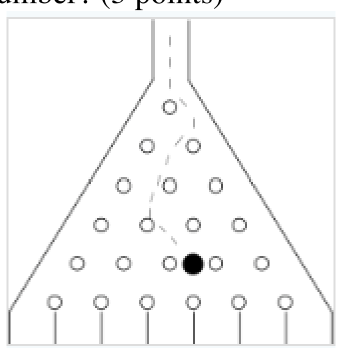
2. **HW** Consider the probability space with 3 possible outcomes, a, b, c, each of which occurs with probability 1/3. Suppose that X and Y are random variables such that $X(a) = -1$, $X(b) = 0$, $X(c) = 1$, and $Y(a) = 0$, $Y(b) = 1$, $Y(c) = 0$.
   - Show that X and Y are not independent. (1 point)

- Show that $E(XY) = E(X)E(Y)$. Note this shows that the theorem proved in class is not an "if and only if". (2 points)

3. **HW** In a sequence of coin tosses, a *run* is a maximal sequence of consecutive tosses with the same result. For instance, if H stands for heads and T for tails in a sequence of coin tosses, then `HHHTTHTH` has 5 runs: `HHH-TT-H-T-H`. A coin with bias 0.7 is flipped 100 times. What is the expected number of runs that occur? Show your work and explain your answer. (Hint: Use indicator random variables I_j denoting that a run starts on flip j.) (5 points)

4. **HW** In the hatcheck problem n people check their hats and have their hats returned at random. We showed in class that the expected number of people who get their own hat back is 1, regardless of the value of n. In this problem you will calculate the variance of the number of people who get their own hat back. Again let R_i be a random variable such that $R_i = 1$ if the ith person receives their own hat back and $R_i = 0$ otherwise, for $1 \le i \le n$, and let $R = R_1 + \ldots + R_n$, so R is the total number of people who get their own hat back.
   - What is $E(R_i^2)$? Show your work. (2 points)
   - Find a simple formula for $E(R_iR_j)$ for $i \ne j$. Show your work and explain your answer. (2 points)
   - What is $E(R^2)$? Show your work and explain your answer. (2 points)
   - What is $\mathrm{Var}(R)$? Show your work. (2 points)
   - Use Chebyshev's inequality to show that the probability that more than 10 people get their own hat back does not exceed 1/100 no matter how many people check their hats. (2 points)

5. **HW** Consider a board as the one in the picture below. It has 6 rows, each with a peg more than the previous rows. Suppose you drop a ball into this board from the top. On colliding at a given peg, the ball has equal chance of going either left or right. Number the bins at the bottom from left to right starting from 0 all the way to 6.
   - What is the probability that the ball falls into bin i? (2 points)
   - What bin is the most likely for the ball to fall into? (1 point)
   - What is the expected value for the bin number? (2 points)
   - What is the variance for the bin number? (3 points)



6. **HW** Consider a point (X_0, Y_0) on the plane starting at $(0, 0)$. At each time step, the point moves a distance of one unit up, down, left, or right, each with probability 1/4, independent of any other moves. In other words, for all positive integers n, (X_n, Y_n) is drawn independently and uniformly at random from the set $\{(X_{n-1} + 1, Y_{n-1}), (X_{n-1} - 1, Y_{n-1}), (X_{n-1}, Y_{n-1} + 1), (X_{n-1}, Y_{n-1} - 1)\}$.
   - Find the expected value of X_n and Y_n. (2 points)
   - Find the variance of X_n and Y_n. (2 points)
   - Find $E[X_n^2 + Y_n^2]$. (2 points)

7. **HW** Every box of Crackerjack includes a small and exciting toy. There are n different types of toys, and each box is equally likely to contain any given type. You buy one box each day.
   - What is the expected number of days which elapse between your acquiring the kth new type of toy and the $(k + 1)$st new type? Show your work and explain your answer. (2 points)

- Find the expected number of days which elapse before you have a full set of n toys. Use linearity of expectation. Show your work and explain your answer. Use may use the approximation $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + 1/n \approx \ln(n) + \gamma$, where "ln" denotes natural log and $\gamma = 0.577221\ldots$ is Euler's constant, in your answer. (2 points)
- For $n = 5$, what is the expected number of days which elapse before you have a full set of toys? (1 point)

Gerry Brady
Thursday November 7 15:27:05 CST 2019