Homework 6

Section 2.6

1. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 0 & 4 & 6 \\ 1 & 1 & 3 & 7 \end{bmatrix}$.

a) What size is $\mathbf{A}$?
b) What is the third column of $\mathbf{A}$?
c) What is the second row of $\mathbf{A}$?
d) What is the element of $\mathbf{A}$ in the (3, 2)th position?
e) What is $\mathbf{A}^t$?

a) What size is $\mathbf{A}$?

3 rows and 4 columns 3×4

b) What is the third column of $\mathbf{A}$?

$$\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

c) What is the second row of $\mathbf{A}$?

$$\begin{bmatrix} 2 & 0 & 4 & 6 \end{bmatrix}$$

d) What is the element of $\mathbf{A}$ in the (3, 2)th position?

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e) What is $\mathbf{A}^t$?

$$A^t = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 4 & 3 \\ 3 & 6 & 7 \end{bmatrix}$$

5. Find a matrix **A** such that

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \mathbf{A} = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}.$$

[*Hint:* Finding **A** requires that you solve systems of linear equations.]

System of linear EQ:

$$2a_{11} + 3a_{21} = 3$$

$$2a_{12} + 3a_{22} = 0$$

$$1a_{11} + 4a_{21} = 1 \rightarrow 1a_{11} = 1 - 4a_{21}$$

$$1a_{12} + 4a_{22} = 2 \rightarrow a_{12} = 2 - 4a_{22}$$

$$2(1 - 4a_{21}) + 3a_{21} = 3$$
$$2 - 8a_{21} + 3a_{21} = 3$$
$$-5a_{21} = 1$$
$$a_{21} = \frac{1}{-5}$$

$$1a_{11} = 1 - 4a_{21}$$
$$1a_{11} = 1 - 4(-1/5)$$
$$a_{11} = \frac{9}{5}$$

$$2a_{12} + 3a_{22} = 0$$
$$2(2 - 4a_{22}) + 3a_{22} = 0$$
$$4 - 8a_{22} + 3a_{22} = 0$$
$$4 - 5a_{22} = 0$$
$$4 = 5a_{22}$$
$$a_{22} = \frac{4}{5}$$

$$2a_{12} + 3a_{22} = 0$$
$$2a_{12} + 3(4/5) = 0$$
$$2a_{12} + 12/5 = 0$$
$$a_{12} = \frac{(-12/5)}{2}$$
$$a_{12} = -6/5$$

$$\boxed{\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 9/5 & -6/5 \\ -1/5 & 4/5 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}}$$

15. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Find a formula for $\mathbf{A}^n$, whenever n is a positive integer.

Computing A^n for $n = 1$ to $n = 5$:

$$A^1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A^3 \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$$

$$\boxed{A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}}$$

19. Let **A** be the 2 × 2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Show that if $ad - bc \neq 0$, then

$$\mathbf{A}^{-1} = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}.$$

A^{-1} is the inverse of A if $A^{-1}A = AA^{-1} = I_n$

$$A^{-1} \cdot A = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{b}{ad-bc} \\ -\dfrac{c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The (i,j)th element of the product AB is the sum of the products of each element in the ith row of A with the corresponding element in the jth Column of B

$$= \begin{bmatrix} \dfrac{ad}{ad-bc} - \dfrac{bc}{ad-bc} & \dfrac{bd}{ad-bc} - \dfrac{bd}{ad-bc} \\ -\dfrac{ac}{ad-bc} + \dfrac{ac}{ad-bc} & \dfrac{bc}{ad-bc} + \dfrac{ad}{ad-bc} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{ad-bc}{ad-bc} & 0 \\ 0 & \dfrac{ad-bc}{ad-bc} \end{bmatrix}$$

$$\boxed{I_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

27. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find

a) $\mathbf{A} \vee \mathbf{B}$. **b)** $\mathbf{A} \wedge \mathbf{B}$. **c)** $\mathbf{A} \odot \mathbf{B}$.

a) $\mathbf{A} \vee \mathbf{B}$. or

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \vee \mathbf{B} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\vee 0, & 0 \vee 1, & 1 \vee 1 \\ 1 \vee 1, & 1 \vee 0, & 0 \vee 1 \\ 0 \vee 1, & 0 \vee 0, & 1 \vee 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}}$$

"and"

b) $\mathbf{A} \wedge \mathbf{B}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \wedge \mathbf{B} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\wedge 0, & 0 \wedge 1, & 1 \wedge 1 \\ 1 \wedge 1, & 1 \wedge 0, & 0 \wedge 1 \\ 0 \wedge 1, & 0 \wedge 0, & 1 \wedge 1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

c) $\mathbf{A} \odot \mathbf{B}$.

28. Find the Boolean product of **A** and **B**, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (1\wedge 1) \vee (0\wedge 0) \vee (0\wedge 1) \vee (1\wedge 1) & (1\wedge 0) \vee (0\wedge 1) \vee (0\wedge 1) \vee (1\wedge 0) \\ (0\wedge 1) \vee (1\wedge 0) \vee (0\wedge 1) \vee (1\wedge 1) & (0\wedge 0) \vee (1\wedge 1) \vee (0\wedge 1) \vee (1\wedge 0) \\ (1\wedge 1) \vee (1\wedge 0) \vee (1\wedge 1) \vee (1\wedge 1) & (1\wedge 0) \vee (1\wedge 1) \vee (1\wedge 1) \vee (1\wedge 0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 \vee 0 \vee 1 & 0 \vee 0 \vee 0 \vee 0 \\ 0 \vee 0 \vee 0 \vee 1 & 0 \vee 1 \vee 0 \vee 0 \\ 1 \vee 0 \vee 1 \vee 1 & 0 \vee 1 \vee 1 \vee 0 \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}}$$

Section 3.2

8. Find the least integer n such that $f(x)$ is $O(x^n)$ for each of these functions.

a) $f(x) = 2x^2 + x^3 \log x$
b) $f(x) = 3x^5 + (\log x)^4$
c) $f(x) = (x^4 + x^2 + 1)/(x^4 + 1)$
d) $f(x) = (x^3 + 5 \log x)/(x^4 + 1)$

$|f(x)| \leq C|g(x)|$ whenever $x > k$

a) $f(x) = 2x^2 + x^3 \log x$

$f(x) = \underbrace{2x^2}_{} + x^3 \log x$

Can ignore this since x^3 dominates x^2

$x^3 \log x$ We can round up $\log x$ to x) so $x^3 \log x \to x^3 \cdot x \to x^4$

$|f(x)| \leq |2x^2 + x^3 \log x|$

$\leq |2x^2 + x^4|$

$C = 2$ $O(x^4)$ $n = 4$

b) $f(x) = 3x^5 + (\log x)^4$

$f(x) = 3x^5 + (\log x)^4$

$|f(x)| \leq |3x^5 + (\log x)^4|$ $\log x \to x$

$|f(x)| \leq |3x^4 + (x)^4|$

$n = 5$

c) $f(x) = (x^4 + x^2 + 1)/(x^4 + 1)$

$f(x)$

14. Determine whether x^3 is $O(g(x))$ for each of these functions $g(x)$.

a) $g(x) = x^2$ **b)** $g(x) = x^3$
c) $g(x) = x^2 + x^3$ **d)** $g(x) = x^2 + x^4$
e) $g(x) = 3^x$ **f)** $g(x) = x^3/2$

a) $g(x) = x^2$

$|x^3| \cancel{\leq} C|x^2|$ NO!

b) $g(x) = x^3$

$|x^3| \leq C|x^3|$ YES

c) $g(x) = x^2 + x^3$

$|x^3| \leq C|x^2 + x^3|$ Yes

d) $g(x) = x^2 + x^4$

$|x^3| \leq C|x^2 + x^4|$ Yes

e) $g(x) = 3^x$

$|x^3| \leq |3^x|$ with $C > 1$ YES

f) $g(x) = x^3/2$

$|x^3| \leq C\left|\frac{x^3}{2}\right|$ Yes with $C > 2$

22. Arrange the function $(1.5)^n$, n^{100}, $(\log n)^3$, $\sqrt{n}\log n$, 10^n, $(n!)^2$, and $n^{99} + n^{98}$ in a list so that each function is big-O of the next function.

30. Show that each of these pairs of functions are of the same order.

a) $3x + 7$, x
b) $2x^2 + x - 7$, x^2
c) $\lfloor x + 1/2 \rfloor$, x
d) $\log(x^2 + 1)$, $\log_2 x$
e) $\log_{10} x$, $\log_2 x$

a) $3x + 7$, x

b) $2x^2 + x - 7$, x^2

c) $\lfloor x + 1/2 \rfloor$, x

d) $\log(x^2 + 1)$, $\log_2 x$

e) $\log_{10} x$, $\log_2 x$

32. Show that if $f(x)$ and $g(x)$ are functions from the set of real numbers to the set of real numbers, then $f(x)$ is $O(g(x))$ if and only if $g(x)$ is $\Omega(f(x))$.