Section 2.1 in Book

Intro to Sets [Discrete Mathmatics] Set Theory

The trev tutor

Set Theory

A set is a collection of objects called elements

 $= \{1, 2, 3\} = A$

visual

list notation

Sets can be finite or infinite

Sets defined use uppercase #'s

$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$\mathbb{Z}^+ = \{1, 2, 3, 4 \ldots\}$

"positive integers"

Applied Pattern forever!

additional Points

- Repeated elements are listed once

  $\{a, b, a, c, b, a\} = \{a, b, c\}$

- There is no order in a set.

  $\{3, 2, 1\} = \{1, 2, 3\}$
  $= \{2, 1, 3\}$

Common Sets

- Natural Numbers

  $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$ or $\{1, 2, 3, \ldots\} \to \mathbb{Z}^+$

- Integers

  $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Rational #s

$\mathbb{Q} = \{\ldots, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \ldots\}$ ~ # you can write as a fraction.

DEFINITION 1 A *set* is an unordered collection of objects, called *elements* or *members* of the set. A set is said to *contain* its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

These sets, each denoted using a boldface letter, play an important role in discrete mathematics:

$\mathbf{N} = \{0, 1, 2, 3, \ldots\}$, the set of **natural numbers**
$\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of **integers**
$\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$, the set of **positive integers**
$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, \text{ and } q \neq 0\}$, the set of **rational numbers**
$\mathbf{R}$, the set of **real numbers**
$\mathbf{R}^+$, the set of **positive real numbers**
$\mathbf{C}$, the set of **complex numbers**.

DEFINITION 2 Two sets are *equal* if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

DEFINITION 5 A set is said to be *infinite* if it is not finite.

Theorem 1 shows that every nonempty set S is guaranteed to have at least two subsets, the empty set and the set S itself, that is, $\emptyset \subseteq S$ and $S \subseteq S$.

THEOREM 1 For every set S, (i) $\emptyset \subseteq S$ and (ii) $S \subseteq S$.

Size

Elements and Cardinality

Let $C = \{ \text{yellow}, \text{blue}, \text{red} \}$

yellow is in the Set

"yellow is an element of C" — yellow $\in C$

"Green is not an element of C" — Green $\notin C$

"The cardinality (size) of C is 3" — $|C| = 3$

3 different Elements in C!

absolute value

The Empty Set (or null set)

- Special set with no elements.

symbol $\emptyset = \{ \}$

$|\emptyset| = 0$

$\{ \emptyset \}$ element

$|\{ \}| = 0$ empty set

$|\{\emptyset\}| = 1$

set has an empty set as an element

Set-Builder Notation

- How to build a set

· $\mathbb{Q} = \{ .., 1/1, 1/2, 1/3, 2/3, .. \}$

Better way to express the set?

$$= \left\{ \frac{m}{n} \,\middle|\, m, n \in \mathbb{Z}, n \neq 0 \right\}$$

and n can't be zero!

have to be integers

Another Example: even integers

$$2\mathbb{Z} = \{ \ldots, -4, -2, 0, 2, 4, \ldots \}$$

$$= \{ 2n \mid n \in \mathbb{Z} \}$$

n is an integer

Such that

Read as

More on the Empty Set:

THE EMPTY SET There is a special set that has no elements. This set is called the **empty set**, or **null set**, and is denoted by $\emptyset$. The empty set can also be denoted by $\{\ \}$ (that is, we represent the empty set with a pair of braces that encloses all the elements in this set). Often, a set of elements with certain properties turns out to be the null set. For instance, the set of all positive integers that are greater than their squares is the null set.

$\{\emptyset\}$ has one more element than $\emptyset$.

A set with one element is called a **singleton set**. A common error is to confuse the empty set $\emptyset$ with the set $\{\emptyset\}$, which is a singleton set. The single element of the set $\{\emptyset\}$ is the empty set itself! A useful analogy for remembering this difference is to think of folders in a computer file system. The empty set can be thought of as an empty folder and the set consisting of just the empty set can be thought of as a folder with exactly one folder inside, namely, the empty folder.

A Set is a collection of objects

What about a set with NO objects?

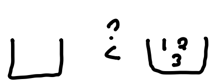
~ still a collection
~ empty set

Notation: $\{\}$ or $\emptyset$

What is $\{\emptyset\}$?

- set that contains the empty set
~ another set inside it

Is $\emptyset \subset \{1,2,3\}$?



Recall: $A \subset B$ means:
if $x \in A$, then $x \in B$

This is Vacuous Truth.

☆ Real-World Example ☆

Desk = {drink, laptop, microphone}

$= \{ x \mid x \text{ is on my desk} \}$

Such that

Exercises

1) List the Elements of D: $D = \{ x \in \mathbb{Z}^+ \mid x < 6 \}$

such that

1, 2, 3, 4, 5

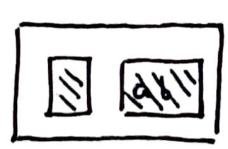
2.) What is the cardinality of D?

$|D| = 5$

"elements" in set

3.) What is the cardinality of $|\{ \phi, \{a, b\} \}| = 2$

like looking into a box



Note: look @ comma's !

How does in differ from this set here? $|\{ \phi, \{a\}, \{b\} \}| = 3$

3 elements in our box

$|\{ \phi, a, \{b\} \}| = 3$

Cartesian Products and Order Pair

Cartesian Products

- An ordered pair (a,b) is a set

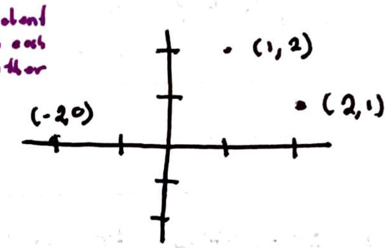
$$\{\{a\}, \cancel{\{b\}} \{a,b\}\}$$

- You've seen ordered pair before or graph coordinates

$(1,2) = \{\{1\}, \{1,2\}\}$ equivalent to each other

$(2,1) = \{\{2\}, \{2,1\}\}$

$(-2,0)$



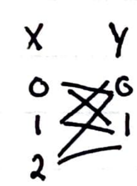
The Cartesian Product, $A \times B$, is the Set Cross Product

$$\{(a,b) \mid a \in A \text{ and } b \in B\}$$

Given $X = \{0,1,2\}$ and $Y = \{0,1\}$

Cross Product

$$X \times Y = \{(0,0), (0,1), (1,0), (1,1), (2,0), (2,1)\}$$



$$Y \times X = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\}$$

1st 2nd

DEFINITION 7 The *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, …, and a_n as its nth element.

DEFINITION 8 Let A and B be sets. The *Cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \land b \in B\}.$$

What is the Cardinality of $A \times B$?

if $|A| = m$ and $|B| = n$ then

$|A \times B| = mn$

$|X| = 3$ $\quad |X \times Y| = 3 \cdot 2$

$|Y| = 2$ $\quad = 6$

Cartesian Products can generalize to n-tuples

3-tuple

$A \times B \times C = \{(a, b, c) \mid a \in A, b \in B, c \in C\}$

·n-tuple

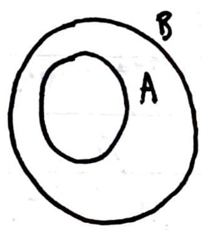
$A_1 \times A_2 \times .. \times A_n = \{(a_1, a_2, .. a_n) \mid a_1 \in A_1, a_2 \in A_2, ..., a_n \in A_n\}$

DEFINITION 9 The *Cartesian product* of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$. In other words,

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}.$$

Subsets and Powersets

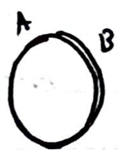
If A is a subset of B, then every element in A must also be in B.



subset of

$A \subseteq B$ — can equal equivalent

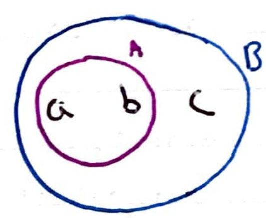
$A < B$ A must strictly be smaller than B



· if A and B are the same size (the little bar _ at the bottom)

Questions:

True or False? Examples



$\{a, b, a, a\} \subseteq \{a, b, c\}$

-repeated elements do not matter $A\ \{a, b\} \subseteq B\ \{a, b, c\}$ True

$\{c, d\} \subseteq \{c, d\}$ True

list out all the elements

A $\{a\}$ $\not\subseteq$ B $\{\{a\}\}$ False   A a   B $\{a\}$

We dont see a in B,

We just see the set containing a

line thru it means not a subset

$\emptyset \subseteq \{x, y, z\}$ True

an empty set is a subset of every set!

DEFINITION 3 The set A is a *subset* of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.

Power Sets

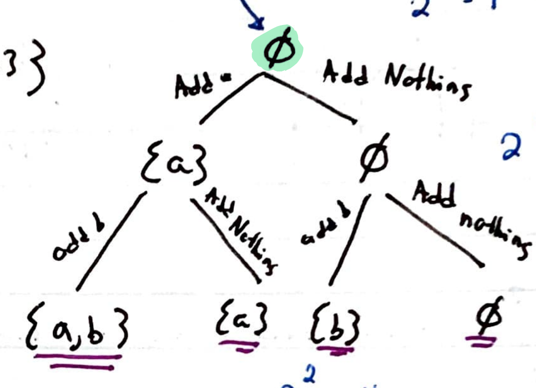
A **power set** of A, $p(A)$, is the **set containing all possible subsets of A.**

$A = \{a, b\}$

$p(A) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$

Power Set, contain all possible subsets of A.

START with empty set



= possible subset of set containing a and b

$2^0 = 1$ — Imagine if our original Set A had no element in it. We could have 1 possible subset (empty set)

$2^1 = 2$

★ Tree doesn't work when you get more and more elements ★

$2^2 = 4$ ~ possible subsets ★

★ power set grows by powers of 2.

Size of Power Set

★ if $|A| = n$, then $|p(A)| = 2^{|A|} = 2^n$

- for each element, we can add it to a subset, or we cannot add it to a subset.
  → 2 Choices per element ~ add or do not add

$|A| = 6 \qquad |p(A)| = 2^{|A|} = 2^6 = 64$ ~ possible subsets

The Empty set is an element of every single Power set

Tricky Questions

★ Set contains nothing in it. ★

Size of the Empty set

$|\emptyset| = 0$

power set of the empty set → $p(\emptyset) = \{\emptyset\} = \{\{\}\}$

So the power set of the empty set is just the set containing the empty set

$|p(\emptyset)| = 2^0 = 1$

power set of the set containing nothing ↗ $p(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$ "set containing" the empty set

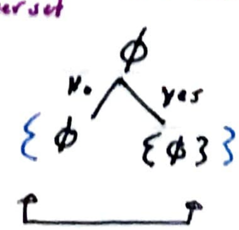
so there are a set inside containing no element in the power set

$|\{\emptyset\}| = 1 \qquad |p(\{\emptyset\})| = 2^1 = 2$

size of the set containing nothing

should have 2 elements in it.

Can draw tree for this example



∴ The Powerset is just going to be these two possible Subsets in a set

$p(\{\emptyset\}) = \{\emptyset, \underline{\underline{\{\emptyset\}}}_{A}\}$

We see A is an element

$\emptyset$

No / Yes

$\{\emptyset \quad \underline{\underline{\{\emptyset\}}}_{A}\}$

Is $A \subseteq p(A)$ for any A? No

is A a subset of the power set of A for any A?

Is $A \in p(A)$ for any A? Yes

Element Notation

Exercises

1.) · Let $|C| = K$ and $|D| = j$, what is $|p(C \times D)|$?

$|C \times D| = |C| \times |D|$
$= Kj$

$|p((C \times D))| = 2^{|C \times D|} = 2^{Kj}$

2.) List the elements of $p(p(\emptyset))$

$p(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$

3.) if $|A| = m$, what is $|p(p(p(A)))|$? $= 2^{|p(p(A))|}$

$= 2^{2^{|p(A)|}}$

$= 2^{2^{2^{|A|}}}$

$= 2^{2^{2^{m}}}$

bar indicated size

ie What is the size of the powerset of the powerset of the powerset of A?

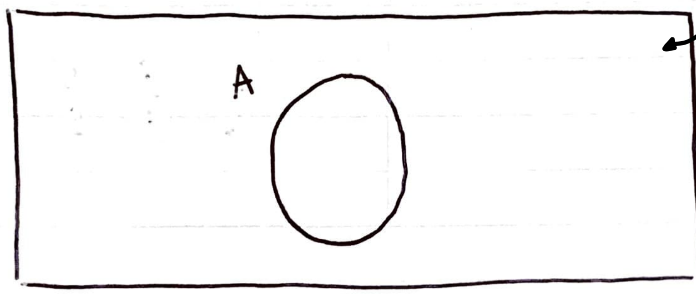
DEFINITION 6 Given a set S, the *power set* of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

Set Operations

Set Operations and Venn Diagrams → used to show relationships b/w sets.

Every Set A exists within some universe U.

Box represents U, some universe



$U = \mathbb{Z}$
$= \mathbb{R}$
$= \mathbb{Z}^+$

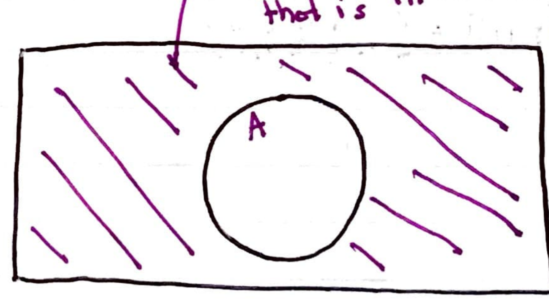
The Complement of A is written as $\bar{A}$.

$\bar{A} = \{a \in U \mid a \notin A\}$

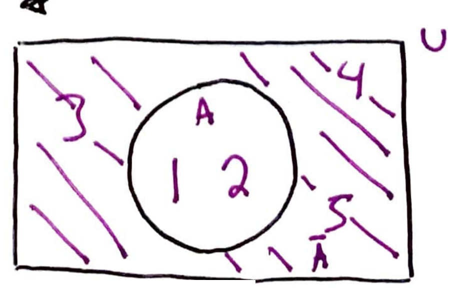
Can be written as A^c or A' as well.

· in order to define a complement you need a universe to talk about.

The Complement of A, is everything outside of the set A that is in the universe



Example: $A = \{1,2\}$
$U = \{1,2,3,4,5\}$
$\bar{A} = \{3,4,5\}$



DEFINITION 5 Let U be the universal set. The *complement* of the set A, denoted by $\overline{A}$, is the complement of A with respect to U. Therefore, the complement of the set A is $U - A$.

Intersection

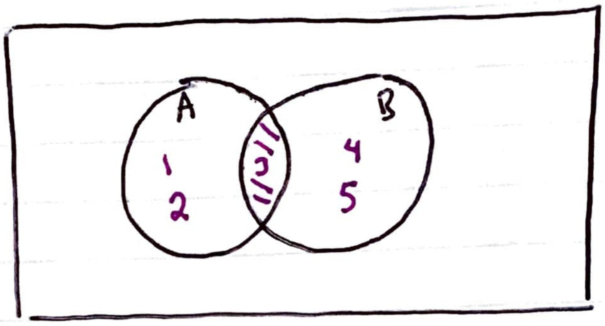
Given sets A and B, the intersection of A and B is:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

$A = \{1,2,3\}$ ∩ upside down U

$B = \{3,4,5\}$

$A \cap B = \{3\}$



Union

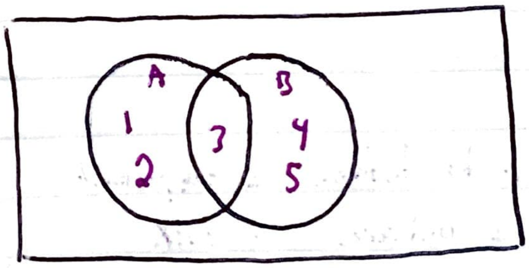
Given sets A and B, the union of A and B is

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Union is wether the element is in A or B, can be in both or just one

$A \cup B = \{1,2,3,4,5\}$

take all elements and put them in a set!



DEFINITION 1 Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

DEFINITION 2 Let A and B be sets. The *intersection* of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

DEFINITION 3 Two sets are called *disjoint* if their intersection is the empty set.

Difference

Set Operations Video

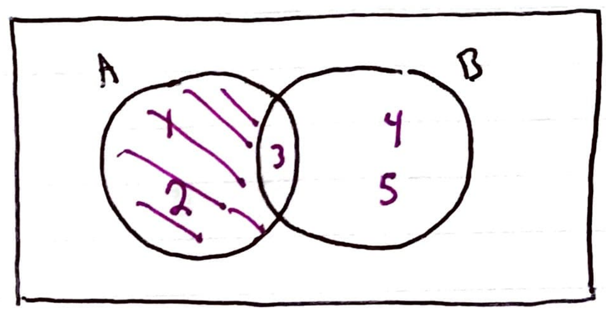
Given the sets A and B, the difference, A-B, is defined as

$$A - B = \{ x \mid x \in A \text{ and } x \notin B \}$$

or $A \setminus B$

or $A \cap \overline{B}$

$A - B = \{1, 2\}$



You take everything in A and subtract whatever is in B.

Exercises

$A = \{1,3,5,7,9\}$ $B = \{4,8,12,16\}$ $C = \{1,4,9,16\}$

$A \cup B = \{1,3,5,7,9,4,8,12,16\}$

$C \cap B = \{4, 16\}$

$C - B = \{1, \cancel{4}, 9, \cancel{16}\} = \{1, 9\}$ — C difference B

$\emptyset \cap B = \emptyset$

b/c there's no elements in the empty set!

DEFINITION 4 Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called the *complement of B with respect to A*.

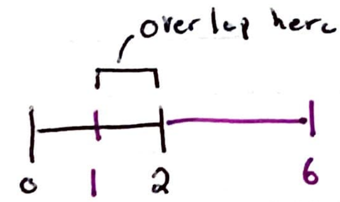
Useful definitions

Union $A \cup B$: All elements that are either in A OR in B
Intersection $A \cap B$: All elements that are both in A AND in B.
Difference $A - B$: All elements in A that are NOT in B (complement of B with respect to A).
Complement $\overline{A}$: All elements in the universal set U NOT in A.

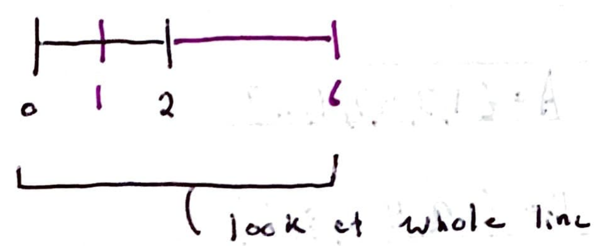
Negation $\neg p$: not p
Disjunction $p \vee q$: p or q
Conjunction $p \wedge q$: p and q

Set Operations Examples

If $A = [0, 2]$ and $B = [1, 6]$, determine the following where $\mathcal{U} = \mathbb{R}$



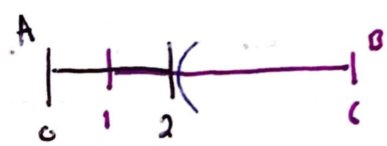
- $A \cap B = [1, 2]$



- $A \cup B = \cancel{[0,6]} = [0, 6)$

open interval

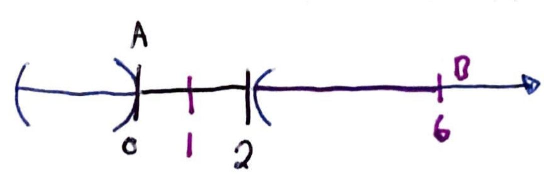
- $B - A = (2, 6)$



- $\bar{A} = (-\infty, 0) \cup (2, \infty)$

have to use a union to describe this set.

everything in the universe − A

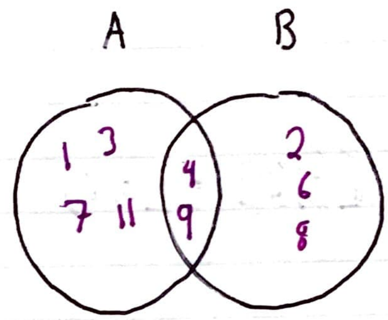


Find Sets A and B where:

$A - B = \{1, 3, 7, 11\}$

$B - A = \{2, 6, 8\}$

$B \cap A = \{4, 9\}$



$A = \{1, 3, 4, 7, 9, 11\}$

$B = \{2, 4, 6, 8, 9\}$

Prove: If $\underline{A \subseteq B}$ and $\underline{C \subseteq D}$, then:

(i) $\underline{A \cap C} \subseteq B \cap D$

Let $x \in A \cap C$

x is $x \in A$ and $x \in C$

$x \in B$ and $x \in D$

$x \in B \cap D$

(ii) $\underline{A \cup C} \subseteq \underline{B \cup D}$

Let $x \in A \cup C$

$x \in A$ or $x \in C$

$x \in B$ or $x \in D$

$x \in B \cup D$

Exercise : Let $A = \{a,b\}$ and $B = \{c,d\}$

$A \times B = \{(a,c), (a,d), (b,c), (b,d)\}$

$B^2 = B \times B = \{(c,c), (c,d), (d,c), (d,d)\}$

$\emptyset \times A = \emptyset$ ← you get the empty set back

empty set

$|\emptyset \times A| = |\emptyset| \cdot |A| = 0 \cdot 2 = 0$

more Questions:

if $|B| = m$ and $|A| = n$ then find... What is the size of these

$|A \times B| = n \cdot m$

$|A^2| = n \cdot n = n^2$

$|B^{32} \times A^{19}| = m^{32} n^{19}$

Set Identites

TABLE 1 **Set Identities.**

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

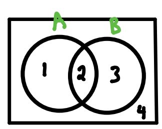
In a Union you're look for any thing

End up back where you started

Can take in either order

Grouping
- Value stay in the same order

De Morgans Law Example:



$A = \{1,2\}$
$B = \{2,3\}$
$U = \{1,2,3,4\}$

$A \cup B = \{1,2,3\}$
- We want the Compliment, which is any thing not included: which is 4

$\bar{A} = \{3,4\}$
$\bar{B} = \{1,4\}$

Union

$\overline{A \cup B} = \bar{A} \cap \bar{B}$
$4 = 4$

Proving Set Identities

Section 2.2

Methods of Identity Proof

if $A = B$
$A \subseteq B$
$B \subseteq B$

1) Prove each set in the identity is a subset of the other
2) Use propositional logic ~ 2 Column proof
3) Use a membership table showing the same combinations of sets do or don't belong to the identity.

① Prove $\boxed{\overline{A \cap B}} = \boxed{\bar{A} \cup \bar{B}}$ by showing $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$ and $\boxed{\bar{A} \cup \bar{B} \subseteq A \cap B}$

Prove this

$x \in \overline{A \cap B}$	By assumption	~ like our premise, we need to show if element is in both
$x \notin A \cap B$	Definition of Complement	"not" / re-write it differently
$\neg((x \in A) \wedge (x \in B))$	Definition of intersection	
$\neg(x \in A) \vee \neg(x \in B)$	De Morgan for Prop. Logic	
$x \notin A \vee x \notin B$	Definition of Negation	
$x \in \bar{A} \vee x \in \bar{B}$	Definition of Complement	
Proved $x \in \bar{A} \vee \bar{B}$	Definition of Union	

Proving this

$x \in \bar{A} \cup \bar{B}$	By Assumption	~ x belongs, by assumption
$(x \in \bar{A}) \vee (x \in \bar{B})$	Def of Union	
$x \notin A \vee x \notin B$	Def of Complement	
$\neg(x \in A) \vee \neg(x \in B)$	Def of Negation	
$\neg((x \in A) \wedge (x \in B))$	De Morgans Law Prop Logic	
$\neg(x \in A \cap B)$	Def of Intersection	
$x \in \overline{A \cap B}$	Def of Complement	

$\therefore \overline{A \cap B} = \bar{A} \cup \bar{B}$ ~ We proved both directions

Prove $A \cap B = \bar{A} \cup \bar{B}$ using set Builder notation and propositional logic.

$\overline{A \cap B} = \{ x \mid x \notin A \cap B \}$	Def. of Compliment
$= \{ x \mid \neg (x \in A \cap B) \}$	Def of $\notin$
$= \{ x \mid \neg (x \in A \wedge x \in B) \}$	Def of intersection
$= \{ x \mid \neg (x \in A) \vee \neg (x \in B) \}$	De Morgan Prop Logic
$= \{ x \mid x \notin A \vee x \notin B)$	Definition of $\notin$
$= \{ x \mid x \in \bar{A} \vee x \in \bar{B} \}$	Definition of Compliment
$= \{ x \mid x \in \bar{A} \cup \bar{B} \}$	Definition of a Union

$= \bar{A} \cup \bar{B}$ Proven its True

Prove $A \cap B = \bar{A} \cup \bar{B}$ using a membership table.

A	B	$A \cap B$	$\overline{A \cap B}$	$\bar{A}$	$\bar{B}$	$\bar{A} \cup \bar{B}$
1	1	1	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	0	1	1	1	1

Generalizetized Union and Intersection

Union of several items

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

Section 2.2

Indexed Sets and Well Ordering Principle

SET THEORY

Can we shorten $A_1 \cap (A_2 \cap (A_3 \cap (A_4 \cap A_5)))$? ~ Yes!
What about Unions?

Intersection notation → $\bigcap_{i=0}^{n} A_i = A_0 \cap A_1 \cap A_2 \cap \dots \cap A_n$

intersection from i=0 to n

Similar to $\sum_{i=0}^{n} a_i = a_1 + a_2 + \dots a_n$

Union Notation → $\bigcup_{i=0}^{n} A_i = A_0 \cup A_1 \cup \dots \cup A_n$

Example:

Brackets

$$\bigcap_{i=0}^{n+1} A_i = \left(\bigcap_{i=0}^{n} A_i\right) \cap A_{n+1}$$

Well ording Principle

↳ Any non empty subset of $\mathbb{N}$ has a least element.
Used in the Divison Algorithm which will be shown when we get to number Theory.

$\mathbb{N} = \{1, 2, 3, \dots\}$ ←

$K < j < i$ K is the smallest element.

$A = \{i, j, K\} \subseteq \mathbb{N}$

→ However: set of full integer

$\mathbb{Z} = \dots -\infty$, we can have negative ∞

* We can't say there's a small element

However if you have $\mathbb{Z}^+$ ✓ + is positive integers ✓

$\{ \dots, 1, 2, 3, 3, \dots \}$

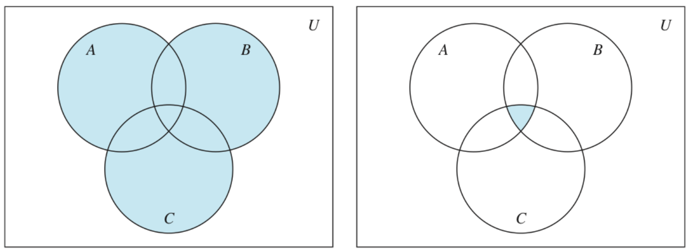
$$\bigcap_{i=0}^{n+m} A_i = \underbrace{\left(\bigcap_{k=0}^{n} A_k\right)} \cap \underbrace{\left(\bigcap_{j=n+1}^{n+m} A_j\right)}$$

$$(A_0 \cap \ldots \cap A_n) \cap (A_{n+1}, \cap \ldots \cap A_{n+m})$$

Section 2.2

Generalized Unions and Intersections

Because unions and intersections of sets satisfy associative laws, the sets $A \cup B \cup C$ and $A \cap B \cap C$ are well defined; that is, the meaning of this notation is unambiguous when A, B, and C are sets. That is, we do not have to use parentheses to indicate which operation comes first because $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$. Note that $A \cup B \cup C$ contains those elements that are in at least one of the sets A, B, and C, and that $A \cap B \cap C$ contains those elements that are in all of A, B, and C. These combinations of the three sets, A, B, and C, are shown in Figure 5.



(a) $A \cup B \cup C$ is shaded.

(b) $A \cap B \cap C$ is shaded.

FIGURE 5 **The Union and Intersection of A, B, and C.**

EXAMPLE 15 Let $A = \{0, 2, 4, 6, 8\}$, $B = \{0, 1, 2, 3, 4\}$, and $C = \{0, 3, 6, 9\}$. What are $A \cup B \cup C$ and $A \cap B \cap C$?

~ Contain elements in at least one of A, B, and C.

$A \cup B \cup C = \{0, 1, 2, 3, 4, 6, 8, 9\}$

$A \cap B \cap C = \{0\}$

Element contained in <u>all 3 sets.</u>

DEFINITION 6 The *union* of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

We use the notation

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

to denote the union of the sets $A_1, A_2, \ldots, A_n$.

Section 2.2

DEFINITION 7 The *intersection* of a collection of sets is the set that contains those elements that are members of all the sets in the collection.

We use the notation

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

EXAMPLE 16 For $i = 1, 2, \ldots$, let $A_i = \{i, i+1, i+2, \ldots\}$. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{1, 2, 3, \ldots\},$$

and

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{n, n+1, n+2, \ldots\} = A_n.$$

We can extend the notation we have introduced for unions and intersections to other families of sets. In particular, we use the notation

$$A_1 \cup A_2 \cup \cdots \cup A_n \cup \cdots = \bigcup_{i=1}^{\infty} A_i$$

to denote the union of the sets $A_1, A_2, \ldots, A_n, \ldots$. Similarly, the intersection of these sets is denoted by

$$A_1 \cap A_2 \cap \cdots \cap A_n \cap \cdots = \bigcap_{i=1}^{\infty} A_i.$$

More generally, when I is a set, the notations $\bigcap_{i \in I} A_i$ and $\bigcup_{i \in I} A_i$ are used to denote the intersection and union of the sets A_i for $i \in I$, respectively. Note that we have $\bigcap_{i \in I} A_i = \{x \mid \forall i \in I\ (x \in A_i)\}$ and $\bigcup_{i \in I} A_i = \{x \mid \exists i \in I\ (x \in A_i)\}$.

EXAMPLE 17 Suppose that $A_i = \{1, 2, 3, \ldots, i\}$ for $i = 1, 2, 3, \ldots$. Then,

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \{1, 2, 3, \ldots, i\} = \{1, 2, 3, \ldots\} = \mathbf{Z}^+$$

and

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} \{1, 2, 3, \ldots, i\} = \{1\}.$$

To see that the union of these sets is the set of positive integers, note that every positive integer n is in at least one of the sets, because it belongs to $A_n = \{1, 2, \ldots, n\}$, and every element of the sets in the union is a positive integer. To see that the intersection of these sets is the set $\{1\}$, note that the only element that belongs to all the sets $A_1, A_2, \ldots$ is 1. To see this note that $A_1 = \{1\}$ and $1 \in A_i$ for $i = 1, 2, \ldots$. ◀

Functions Notes

domain

$f : X \xrightarrow{\text{maps}} Y$ ~ codomain

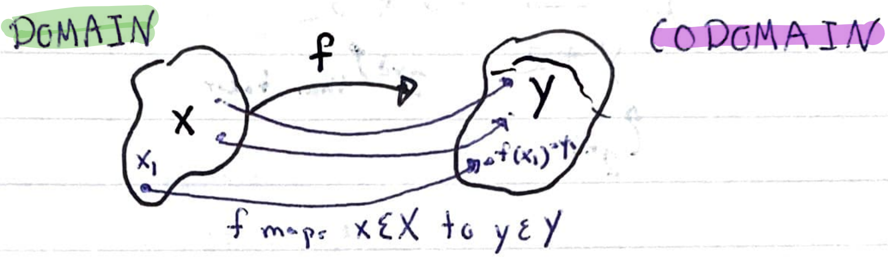
$x \in X$

x maps to y · preimage $y \in Y$ such that $f(x) = y$
- range
- image

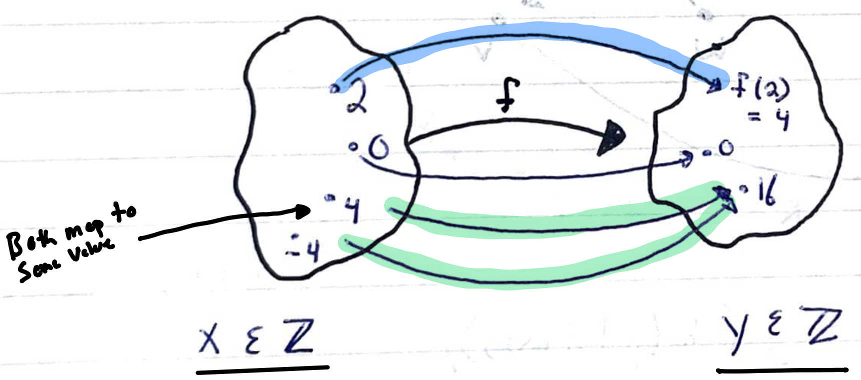
$x \in X \qquad y \in Y$

$f(x) = y$ ~ range image

Big Picture



$f(x) = x^2 \qquad \mathbb{Z}$



$f : \mathbb{Z} \to \mathbb{Z}^+$

maps integer into a positive integer

Codomain: $\mathbb{Z}^+$ ~ Since x^2 will always return a + #

Range: $\{x^2 \mid \sqrt{x} \text{ is an integer}\}$

Characteristic Function

$$f(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}$$

$A = \{a, b, c, \dots\}$

$f(1) = 1$

$f(2) = 0$

$f(3) = 1$

$f(4) = 0$

$f : \mathbb{Z} \to \{0, 1\}$

Checks if # is odd

Kind of like truth tables ?

P(x)	X
1	1
0	2
1	3
0	4
1	5

Composite functions

X, Y, Z are se

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

$f: X \to Y$

$g: Y \to Z$

$h: X \to Z$

$h: ? = g \circ f = g(f(x))$

$X \to Y$ 1^{st}

$\to X \to Z$

$Y \to Z$ 3^{rd} then take

2^{nd}

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[rrrr, bend right, "K"'] & Y \arrow[r, bend left, "g"] & Z \arrow[r, bend left, "h"] & W \arrow[r, bend left, "j"] & V \end{tikzcd}$$

- Take functions in reverse

$K = j \circ (h \circ (g \circ f)) = j(h(g(f(x))))$

$X \to V$

$= j(h(g(y)))$

$= j(h(z))$

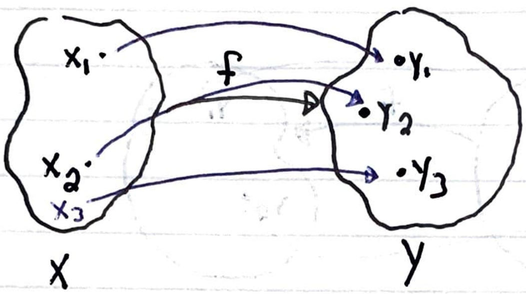
$= j(w)$

$= v$

Surjective (onto)

Let $f: X \to Y$

f is surjective iff $\forall y \in Y, \exists x \in X$ such that $f(x) = y$



Codomain = range

- For all possible values of Y there is some X that gives you Y

Show $f(x) = 5x+2$ is surjective for $\forall x \in \mathbb{R}$. What about $\forall x \in \mathbb{Z}$?
↑ real numbers

$y = f(x)$

[Get EQ in terms of X]

$y = 5x+2$

$y - 2 = 5x$

$\frac{y-2}{5} = x$

if we pick $y=0$ $x = -2/5$

$y=1$ $x = \frac{-1}{5}$

$f: \mathbb{Z} \to \mathbb{Z}$

$y = 7 \to x = 1$

$y = 5 \to x = 3/5 \notin \mathbb{Z}$ not in the integers

Not in the range

Lets say we have $f: \mathbb{R} \to \mathbb{Z}$

x any real #'s   y only integers

Yes so surjective!

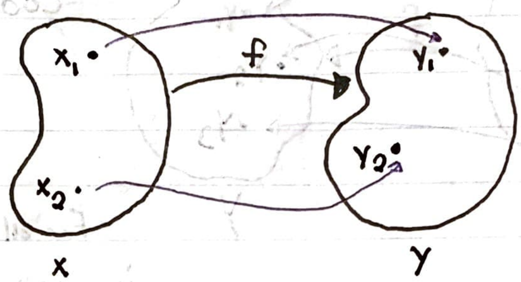
Injective, Surjective, Bijective Functions

Injective (one-to-one)

if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$

function from x to y

$f: x \to y$



x's going to different y values then x_2 value

Prove Equation is Injective:

Show $f(x) = 3x - 2$ is injective.

Implies

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

$\boxed{f(x_1) = f(x_2) \Rightarrow x_1 = x_2}$

easier to prove the contrapositive

Proof

$f(x_1) = f(x_2)$ ~ Assume

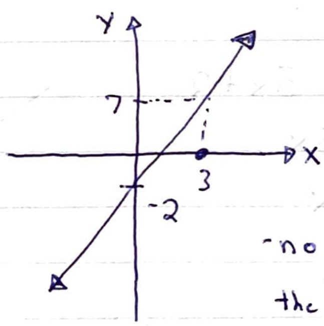
$3x_1 - 2 = 3x_2 - 2$

$3x_1 = 3x_2$

$x_1 = x_2$

proven it's injective or one-to-one

Injective



Graphed

- no other x that gives the same y

Is $f(x) = x^2$ injective?

$f(x_1) = f(x_2)$

$x_1^2 = x_2^2$

take the square root

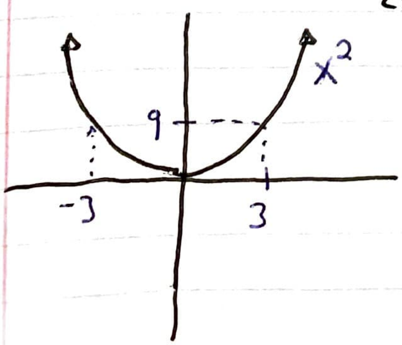
$\pm x_1 = \pm x_2$ does it equal? No!

Why? b/c of the $\pm$

$+x_1 = -x_2$

Example $3 \cancel{=} -3$

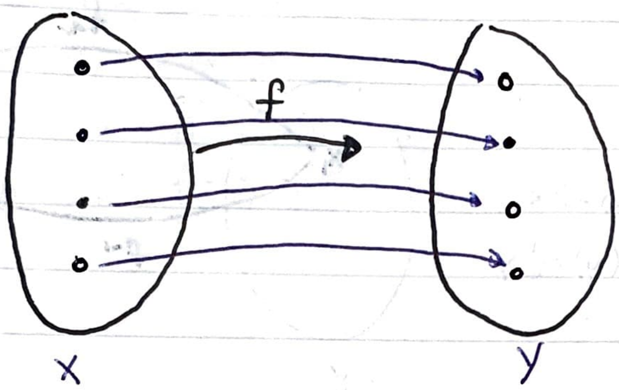
not injective!



- no other x-value should give you the y-value
- but -3 and 3 both give the y-value of 9

Bijective: Injective AND Surjective

for $f: X \to Y$, Each $x \in X$ maps to exactly one unique $y \in Y$



What does this say about $|X|$ and $|Y|$?
size size

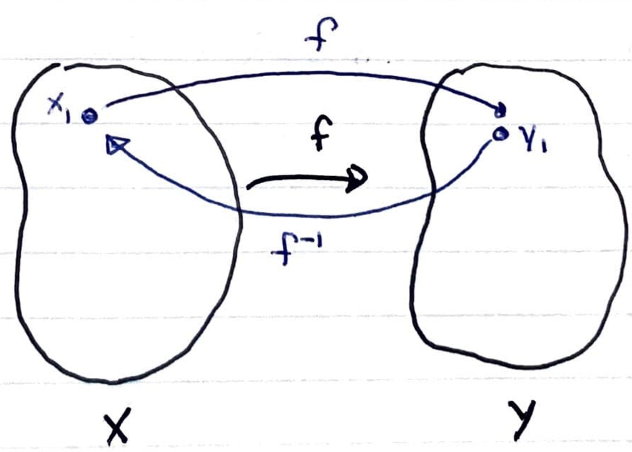
$|X| = |Y|$

Size of domain is equal to the codomain

Inverse

Inverse — Given $f: X \to Y$, we define the inverse of

$f^{-1}: Y \to X$



$f(x) = y$

$f^{-1}(y) = x$

Find the inverse of $f(x) = 5x + 2$

↓

implies it's bijective → prove injective + Surjective

When you prove a function is surjective, you always set the inverse

Quertin

$f(x) = y$
$f^{-1}(y) = x$

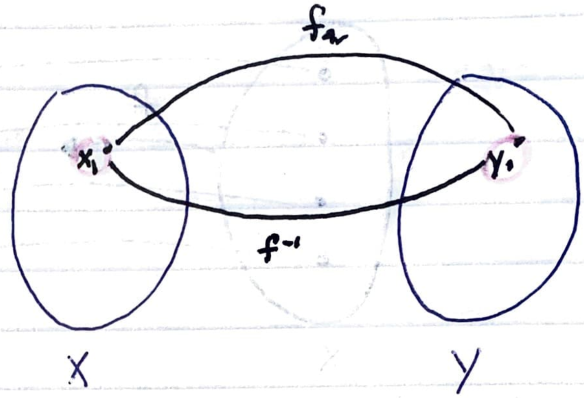
$$\underbrace{f(f^{-1}(y))}_{\text{Internal}} = f(x) = y$$

"Cycle"

Kinda like:

$$\overset{f}{3} \cdot \overset{f^{-1}}{\tfrac{1}{3}x} = x$$

$f(x) = 3x$
$f^{-1}(y) = \frac{1}{3}y$



1st take y_1 and map it to x_1
2nd take f^{*} back and set to y

Function Examples

is the function injective? What is the range?

a.) $f: \mathbb{R} \to \mathbb{R}$ where $f(x) = e^x$

$f(x) = f(y) \Rightarrow x = y$

Prove

$e^x = e^y$

$\ln e^x = \ln e^y$ YES!

$x = y$

range of function

$(0, +\infty)$

could be factored

b.) $g: \mathbb{Z} \to \mathbb{Z}$ where $g(x) = x^3 - x$ $\;x((x-1)(x+1))$

0 1 -1

$\boxed{g(0)} = 0^3 - 0 = 0$

$\boxed{g(1)} = 1^3 - 1 = 0$

same element in codomain

We have two elements in the domain mapping to each other, therefore not injective

Range?

$\{ n^3 - n \mid n \in \mathbb{Z} \}$

Give an Example of a function $f: A \to B$ and $A_1 A_2 \subseteq A$ where $f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2)$

intersection

$A = \{1, 2\}$ $\quad$ $B = \{3\}$

$A_1 = \{1\}$

$A_2 = \{2\}$

$f(1) = 3$

$f(2) = 3$

$f(A_1) = \{3\}$

$f(A_2) = \{3\}$

$f(A_1 \cap A_2) = f(\phi) = \phi$

empty set

When you put the empty set into a function you get the empty set back.

Not Equal to

$f(A_1) \cap f(A_2) = \{3\} \cap \{3\} = \{3\}$

- To prove work with simple sets

Surjective Function Examples

Show that $f: \mathbb{R} \to \mathbb{R}$ where $f(x) = x^3$ is surjective

$$y = x^3$$

all y's are going to be produce from values of x

$$\boxed{\sqrt[3]{y} = x}$$

$$\sqrt[3]{y} \in \mathbb{R}$$

is the cube root of y in the real #'s?

Yes, Surjective

Yes it is, we can put in any y in and get an x back

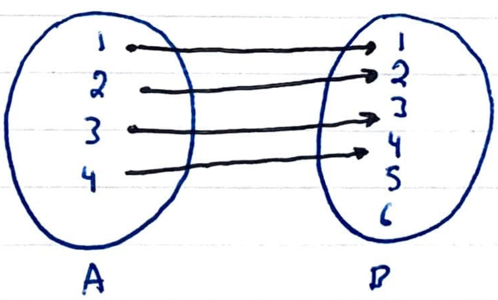
Let $|A| = 4$ and $|B| = 6$

i) How many functions $f: A \to B$ are there?

$$6 \times 6 \times 6 \times 6 = 6^4$$

\# of option it can map to.

$$= |B|^{|A|}$$



ii) How many are injective?

$$4!\binom{6}{4} = \frac{6!\,\cancel{4!}}{2!\,\cancel{4!}} = \frac{6!}{2!}$$

Everything in our domain must map to something in our codomain.

iii) How many are surjective?

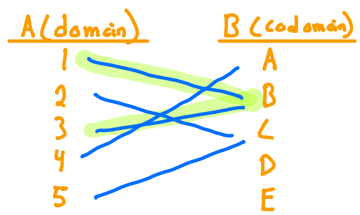
↳ means the codomain is equal to the range, we can't do that we can only map to 4 things.

0

Introduction to functions

Section 2.3.1

A function f from A to B, denoted $\boxed{f: A \to B}$
assigns each element of A to exactly one element of B.
Functions may also be called mappings or Transformations



$f(1) = B$ then we can say

Image: B is the image of 1 under f $\{A, B, C\}$ image of f

$\{A, B, C\}$ image of f

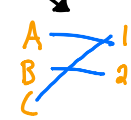
Preimage: $\{1, 3\}$ the preimage of B under f

range: $\{A, B, C\}$

All Values mapped

Representing Functions

1. Explicit Statement $f(A) = 1$, $f(B) = 2$, $f(C) = 1$



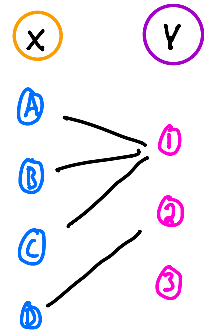
2. Formula $f(x) = x^2 + 1$

3. Computer Program

4. A relation $\underline{A \times B}$

$(1,2)$ $(1,7)$
$(2,3)$ $(2,6)$
$(4,5)$

Answer the following for $f: X \to Y$



- domain X
- Codomain Y
- range $\{1, 2\}$
- Preimage of 2 $\{D\}$
- Images of A $\{1\}$
- $f(D) = 2$

One-One and Onto Functions

Section 2.3.1

Injective (one-to-one) Functions

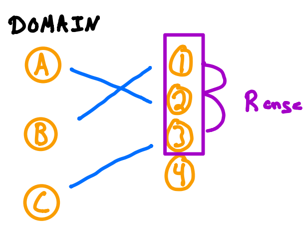
- Each Value in the range corresponding to exactly one element in the domain.

$$\forall a \forall b ((a \neq b) \rightarrow (f(a) \neq f(b)))$$

OR

Contrapositive

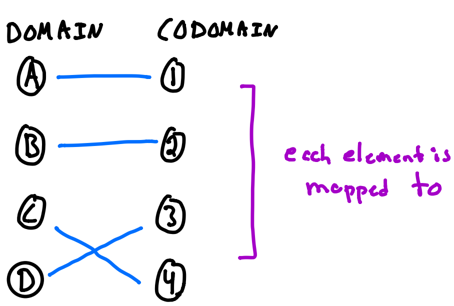
$$\forall a \forall b ((f(a) = f(b)) \rightarrow (a = b))$$



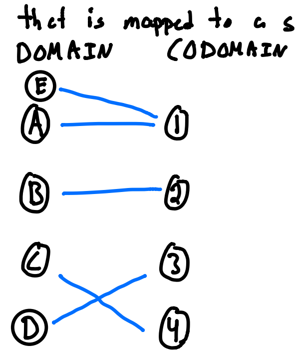
Surjective (onto) Functions

Every element in the codomain maps to at least one element in the domain

$$\forall y \exists x (f(x) = y)$$



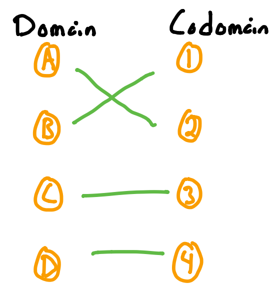
What if we add another Domain Value that is mapped to a same codomain?



It's still Surjective but not One-to-One anymore.

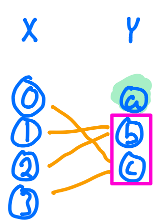
Bijective (one-to-one Correspondence) Functions

Functions that are both one-to-one and onto, or both surjective and injective.



Practice:

Let f be a function from $X = \{0, 1, 2, 3\}$ to $Y = \{a, b, c\}$ defined by $f(0) = c$, $f(1) = b$, $f(2) = b$ and $f(3) = c$. Is $f: X \to Y$ either one-to-one or onto?



Not One-to-One

Since we have a value in the codomain that isn't mapped to anything then its not onto

Practice:

Is the function $f(x) = x^2$ from the set of integers to the set of integers either one-to-one or onto?

$f : \mathbb{Z} \to \mathbb{Z}$

<u>One-to-one</u>

if $f(a) = f(b)$ then $a = b$

Not one-to-one

$f(a) = 4$ $f(b) = 4$ $a = 2$ $b = -2$

$f(a) = f(b)$ but $a \neq b$

<u>Onto</u>

Is the codomain ever going to be negative

$f(a) \neq -1$ ~ Won't be in the range

Not onto

Inverse Functions and Composition Functions

Section 2.3.1

Inverse Functions

Let f be a bijection (both one-to-one and onto) from set A to B. The inverse of f, f^{-1}, is the function from B to A defined as $f^{-1}(y) = x$ iff $f(x) = y$.

Domain A	Codomain B
1	5
2	6
3	7
4	8

If f is the function from $\{a, b, c\}$ to $\{1, 2, 3, 4\}$ such that $f(a) = 1$, $f(b) = 3$ and $f(c) = 4$, is f invertible? If so, what is the inverse?

Let $f: \mathbb{Z} \to \mathbb{Z}$ such that $f(x) = x + 3$. Is f invertible? If so, what is the inverse?

Composition of Functions

Let $f: A \to B$ and $g: B \to C$. The Composition of g with f, $g \circ f$, is the function from A to C defined by $g \circ f(x) = g(f(x))$

A	B	C
1		8
2	5	9
3	6	10
4	7	

If $f(x) = x+3$ and $g(x) = x^2-2$ find:

$f \circ g$

$g \circ f$

Useful Functions to Know

Section 2.3.1

Floor Function $f(x) = \lfloor x \rfloor$

Largest integer less than or equal to x.

Ceiling Function $f(x) = \lceil x \rceil$

Smallest integer greater than or equal to x.

Factorial Function

$f: \mathbb{N} \to \mathbb{Z}^+$ denoted by $f(n) = n!$ is the product of the first n positive integers when n is a non-negative integer.

$$f(n) = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

Stirling formula: $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

Let $f(x) = \lfloor \frac{x^2}{2} \rfloor$. Find $f(S)$ if $S = \{0,1,2,3\}$.

Introduction to Sequences

Section 2.4

Matrices and Matrix Operations

Section 2.6