MPCS 50103 Discrete Mathematics—Autumn 2019

Homework 5: This problem set is due Monday November 4 at 11:59 pm.

Reading: Rosen 7e, chapter 7, sections 7.1–7.3.

Written assignment:

- Solve the following "DO" exercises and homework problems "HW".
- "DO" exercises are strongly recommended to check your understanding of the concepts. **Do not submit them**.
- **Problems labeled "HW" must be submitted**.
- You are responsible for the material covered in **both** "DO" exercises and HW problems.

"Do" Exercises (*not* to be submitted):

1. **"DO"** Rosen 7e, section 7.1, exercises 25a, 27a, 29, 33, 35, 37, 38, and 41, on pages 451–452.

2. **"DO"** Rosen 7e, section 7.2, exercises 1, 5, 7, 15, 17, 19, 23, and 27, on pages 466–467.

3. **"DO"** Rosen 7e, section 7.3, exercises 7 and 16, on pages 475–476.

4. **"DO"** Rosen 7e, Chapter 7 Supplementary exercises, exercises 25, 27, and 28, on pages 497–498.

Assigned problems: DUE Monday November 4 at 11:59 pm

- **Collaboration policy**: There is no penalty for acknowledged collaboration. To acknowledge collaboration, give the names of students with whom you worked at the beginning of your homework submission and mark any solution that relies on your collaborators' ideas. Note: you must work out and write up each homework solution by yourself without assistance.
  DO NOT COPY or rephrase someone else's solution.
  The same requirement applies to books and other written sources: you should acknowledge all sources that contributed to your solution of a homework problem. Acknowledge specific ideas you learned from the source.
  DO NOT COPY or rephrase solutions from written sources.
- **Internet policy**: Looking for solutions to homework problems on the internet, even when acknowledged, is STRONGLY DISCOURAGED. If you find a solution to a homework problem on the internet, do not not copy it. Close the website, work out and write up your solution by yourself, and cite the url of website in your writeup. Acknowledge specific ideas you learned from the website.
  Copied solutions obtained from a written source, from the internet, or from another person will receive ZERO credit and will be flagged to the attention of the instructor.
- Write out your work for every problem. If you just write your answer without showing your work, you will not receive credit.

1. **HW** In the card game of bridge, a standard 52 card deck is dealt to four players called North, East, South, and West, so that each player receives 13 cards.
   - What is the probability that North holds all the aces?
   - What is the probability that each player holds one of the aces?

   These questions refer to uniform probability spaces. Show your work and explain your answers. (3 points each)

2. **HW** Consider a sequence of 7 independent tosses of an unbiased coin.
   - Calculate the probabilities of the following events:
     - A: The number of heads is even.
     - B: The number of heads is at least 4.

  - C: The first three tosses are heads.
  - $A \cap C$
  - $B \cap C$

  Show your work. (1 point each)
- Are A and C independent? Comment briefly. (1 point)
- Are B and C independent? Comment briefly. (1 point)

3. **HW**
   - A coin is tossed twice. Bob claims that the event of two heads is at least as likely if we know that the first toss is a head than if we know that at least one of the tosses is a head. Is Bob correct? Does it make a difference if the coin is fair or unfair? Show work and justify your answers. (3 points)
   - Suppose you repeatedly flip a fair coin until you see the sequence `HTT` or the sequence `HHT`. What is the probability that you will see the sequence `HTT` before you see `HHT`? (Hint: find the probability that `HHT` comes before `HTT` conditioning on whether you first toss an `H` or a `T`.) Show your work and justify your answer. (3 points)

4. **HW** Consider the following game. Start with one red marble and one blue marble in a bag. Repeatedly do the following: choose one marble from the bag uniformly at random, and then put the marble back in the bag along with a new marble of the same color. Repeat until there are n marbles in the bag. Show that the number of blue marbles is equally likely to be any number between 1 and $n - 1$. Hint: Use mathematical induction. (5 points)

5. **HW** Supposing that the integers $1, 2, \dots, n$ are permuted uniformly at random, what is the probability that the permutation is an increasing sequence followed by a decreasing sequence? It is allowed for either of these sequences to be "empty".

   Example:
   $\langle 1, 2, 5, 4, 3 \rangle$ works since $\langle 1, 2, 5 \rangle$ is increasing and $\langle 4, 3 \rangle$ is decreasing.
   $\langle 1, 2, 3, 4, 5 \rangle$ works since $\langle 1, 2, 3, 4, 5 \rangle$ is increasing and the remaining "empty" sequence is assumed to be decreasing. (Alternatively, $\langle 1, 2, 3, 4 \rangle$ is increasing, and $\langle 5 \rangle$ is decreasing.)
   $\langle 5, 4, 3, 2, 1 \rangle$ works, by a similar argument to the previous example.

   Show your work and explain your answer. (5 points)

6. **HW** Three professors—A, B, and C—have been nominated for two awards, and the scientific community believes that each professor has an equal chance to win an award. Professor A has a friend on the awards committee who knows which two professors will win the awards. Professor A thinks that it would be unethical to ask her friend if she, Professor A, is a winner, but is okay with asking her friend for the name of one professor *other than herself* who has been selected to win. She believes that before she asks her friend, her chances of receiving an award are 2/3. She reasons that if her friend tells her that Professor B is an award winner (if both Professors B and C have been selected, her friend will name one of them with equal probability), her own chances of being selected will now go down to 1/2, because either Professors A and B or Professors B and C are to receive the awards. And so Professor A decides not to reduce her chances by asking her friend.
   Is Professor A's reasoning correct or incorrect? Justify your answer by describing a sample space for the problem and computing the conditional probabilities involved. (6 points)

Gerry Brady
Thursday October 31 20:08:03 CDT 2019