Logic
The Trev Tutor

Introduction to Propositional Logic

Propositional Logic

A statement is a declarative sentence that can be True (1) or False (0)

Boolean logic in computer science

Examples of statements

- Milk is white T
- $|\emptyset| = 0$ ∴ T
- Human are just fish with legs ∴ F

The point is not ~~whethere~~ whether or not these statements are True or False but rather that they are statements, which means in propositional logic we can express them.

- We can't express things like
  - Question, imperatives

Syntax

Propositions are denoted with capital letters P, Q, R, ...

P = 1 Cheated
Q = 1 Wrote an exam

Lowercase letters p, q, r... are used: for general propositions that have no meaning.

→ Used for general proofs

Connectives (some notations) · to change meanings

Notations

- p is a well-formed formula (wff)
- $\neg p$ is a wff — not p
- $p \wedge q$ is a wff — p and q
- $p \vee q$ is a wff — p or q
- $p \to q$ is a wff — if p then q

Example

Translate the following into English

P = I cheat R = I write an Exam

Q = I will get caught S = I will fail

- $(R \wedge P) \to (Q \wedge S)$

if I write an exam and I cheat

then I will get caught and I will fail

Example:

Translate into Propositional Logic:

if James does not die then Mary will not get any money and James' family will be happy.

$$(\neg P) \to (\neg Q \wedge R)$$

define our Keys

* not is a connective *

P = James dies

Q = Mary will get money

R = James' family will be happy

Statement and Translation Examples

Translate the following sentences, given the following statements:

p: I finish writing my computer program before lunch
q: I shall play tennis in the afternoon
r: The sun is shining
s: The humidity is low

usually most difficult

Same as saying: p is necessary for q
$q \to p$

p is sufficient for q
$p \to q$

Problems:

1.) If the sun is shining, I shall play tennis this afternoon.

if statement $r \to q$ - 2 statements

2.) Finishing the writing of my computer program before lunch is necessary for my playing tennis this afternoon.

p, connective, q

p is necessary for q
$q \to p$

Same as saying

$$\boxed{q \to p}$$

- 2 statements

3.) Low Humidity and sunshine are sufficient for me to play tennis this afternoon.

s, r, connectives, q

$$(s \wedge r) \to q$$

Sufficient for playing tennis this afternoon? (q)

P is sufficient for q
$p \to q$

Question 2

Determine which is a statement.

of the form

1) In 1999, Barack Obama released a K-pop album P

yes, it's a statement b/c it has a truth value, it is either True or False

2.) $17y + 20x$ is an integer $P(x, y)$

? of the form

no, not a statement b/c y and x are variables
it's not a statement if it has a variable in it.
~ tricky

3.) Tell me the time

No, this is a command

4.) I can't live without you. P

yes, a statement
~ can be True or False

Truth Tables

Logic

Recall: Each statements is True or False
1 0

~ Truth Tables will tell us all possible outcomes of statements and connectives !

- 32 is even T/1
or
- $A \subseteq B$ iff $x \in B$ implies $x \in A$ F/0

★ All ~~connection~~ connectives take a ~~limit~~ Truth value and output a truth value. ★

Taking a look at each Connective:

Negation ($\neg$, ~)

Truth Tables shows all possible combinations of truth conditions

Truth Table

P	$\neg$P
1	0
0	1

Shows all possible combination of truth conditions (P can either be True or False)

What does Negation do?
if P is True then the negation of P is going to be false (Vice-Verse)

Neg P = 1 − P
example: P = 1 $\neg P = 1 - P$
$= 0$

nice way to find the value of the negation.

Mathematical way of looking at the problem

Conjunction ($\wedge$, *, -)

and

Conjunction: compound statement formed by joining two statements with the connector "and"

mostly use the carrot ($\wedge$) symbol

p and q

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

$p \wedge q = \min(p, q)$

The min of $p \wedge q$
← lowest value in row is 1

lowest value in row is 0

Mathematical Way of looking at this.

# of rows = $2^{\#\text{ of statements}}$

All possible Combinations

What does p and q do to the truth value?
~ only true when both p and q are true
~ in every other scenario it's false

Disjunction ~ "or"

Disjunction $(\vee, +)$

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

← neither P or q are true so $p \vee q$ is false

mathmatically ↓

$p \vee q = \max(p, q)$

P and q are true if at least one of p, q are true.

Conditional $(\rightarrow, \supset)$

if p then q

p	q	$p \rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

it is only false if P is true and Q is false, the rest of the times it will be true

Practice statement

if its sunny I will wear sunscreen

mathmatically

$p \rightarrow q = 1$

iff

$p \leq q$

2^{nd} senerio : $1 \nleq 0 = 0$

KEY

When am I lying to you?

I am lying to you if it is sunny out (which would be P is true) and I'm not wearing sunscreen.

This is like my promise to you I'm saying look it it's sunny out I promise I will wear sunscreen and if it is sunny and I'm not wearing sunscreen I lied to you therefore the conditional is false. But if it's not sunny outside so I am not lying to you am I

It doesnt matter if I'm wearing sunscreen or not it's not sunny outside therefore I'm not violating any truth condition therefore it is true in the bottom two cases.

2 more connectives

Biconditional ($\leftrightarrow$, $\equiv$) iff $\rightarrow$ if and only if

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

$p = q$ then $p \leftrightarrow q = 1$

if they (p and q) are the same value then it's True.

Exclusive Or ($\oplus$, $\underline{\vee}$)

p	q	$p \oplus q$
1	1	0
1	0	1
0	1	1
0	0	0

- opposite of the biconditional

$p \neq q$
then $p \oplus q = 1$

Proofs with Truth Tables

Formula: p and q are logically equivalent iff the truth conditions of p are the same as the truth conditions of q.

$p \Longleftrightarrow q$ iff

p	q
x	x
y	y

the same, so p and q are logically equivalent.

Example:

logically equivalent

Is $(p \wedge q) \Longleftrightarrow \neg(p \vee q)$?

NO! why?

~ The truth conditions for P and q $(p \wedge q)$ and the truth conditions for $\neg(p \vee q)$ are not identical, there these are not logically equivalent.

"or"

p	q	$p \wedge q$	$p \vee q$	$\neg(p \vee q)$
1	1	1	1	0
1	0	0	1	0
0	1	0	1	0
0	0	0	0	1

Steps to solve logically Equivalent Problems:

1st: Build truth table for p and q

- Then solve each column

For $\neg(p \vee q)$ we take the negation of the previous column $(p \vee q)$

2nd: Check statement

$\neg$: not
~reverse value

example 2:

Is $\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$? Yes, they're logically equivalent

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
1	1	1	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	0	1	1	1	1

∴ After doing the truth table, you can look at the truth conditions for the two formulas we want to compare

- By being logically equivalent if I ever have $\overbrace{\text{not P and q}}^{\neg(p \wedge q)}$ in a proof somewhere I can substitute that in with $\underbrace{\text{not P or not q}}_{(\neg p \vee \neg q)}$ b/c they're exactly the same thing.

Exercise:

Show that $(p \vee \neg p)$ is always true.
~ a tautology.

~ every single value in our truth table is going to be 1

P	$\neg P$	$P \vee \neg P$
1	0	1
0	1	1

→ Every single output is a 1

p or not p could be more elaborate

$\underbrace{(a \wedge b)}_{p} \vee \underbrace{\neg(a \wedge b)}_{\neg p}$] combination of composition

Exercise:

Show that $(p \vee \neg p)$ is always False. (otherwise known as a contradiction)
~ a contradiction

P	$\neg P$	$P \wedge \neg P$
1	0	0
0	1	0

→ Every single output is a 0

Proofs using Truth Tables

Truth Table Examples

Example 1:

Draw the Truth Table for $\neg(p \vee \neg q) \rightarrow \neg p$

p	q	$\neg p$	$\neg q$	$p \vee \neg q$	$\neg(p \vee \neg q)$	$\neg(p \vee \neg q) \rightarrow \neg p$
1	1	0	0	1	0	1
1	0	0	1	1	0	1
0	1	1	0	0	1	1
0	0	1	1	1	0	1

for this column

Tautology ~ all outcomes in truth table is True (1).

$(\neg p \wedge q) \rightarrow \neg p$

Example 2:

$[p \wedge (p \rightarrow q)] \rightarrow q$

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$(p \wedge (p \rightarrow q)) \rightarrow q$
1	1	1	1	1
1	0	0	0	1
0	1	1	0	1
0	0	1	0	1

Tautology

Exclusive Or Example.

Example 1:

Write the Truth table for Exclusive or, $(p \oplus q)$, then state whether $(p \oplus p) \oplus p$ is a tautology, contradiction or neither.

Xor

p	q	$p \oplus q$
1	1	0
1	0	1
0	1	1
0	0	0

p	$p \oplus p$	$(p \oplus p) \oplus p$
1	0	1
0	0	0

not different

- We can say $(p \oplus p) \oplus p = p$, logically equivalent, neither a tautology nor a contradiction.

Example 2:

Give the Truth table for $(p \oplus q) \vee (p \oplus \neg q)$

p	q	$\neg q$	$p \oplus q$ (x)	$p \oplus \neg q$ (y)	$x \vee y$
1	1	0	0	1	1
1	0	1	1	0	1
0	1	0	1	0	1
0	0	1	0	1	1

- Tautology

$(p \oplus q) \vee (p \oplus \neg q) = 1$

Sheffer Stroke Examples

~ Another logical operator

$p \uparrow q$ is to be read as "p nand q", and is equivalent to $\neg(p \wedge q)$.

$\neg p \vee \neg q$

not p or not q

Provide the truth table for $(p \uparrow q)$ and $(p \uparrow p)$.

p	q	$p \uparrow q$
1	1	0
1	0	1
0	1	1
0	0	1

at least one of them in the columns are false, so $p \uparrow q$ is True

equivalent

p	$p \uparrow p \Leftrightarrow \neg p$
1	0
0	1

$\neg \varphi \Leftrightarrow \varphi \uparrow \varphi$ ← Sheffer stroke

Questions

1.) Using the sheffer stroke, provide the definition for $(p \wedge q)$

$$p \uparrow q \Leftrightarrow \neg(p \wedge q)$$

$$(p \wedge q) \Leftrightarrow \neg\neg(p \wedge q)$$

$$\Leftrightarrow \neg(p \uparrow q)$$

$$\Leftrightarrow (p \uparrow q) \uparrow (p \uparrow q)$$

$\neg \varphi \Leftrightarrow \varphi \uparrow \varphi$

$\varphi = p \uparrow q$

2.) find definition for $p \vee q$

$(p \vee q)$ $\neg\neg p \vee \neg\neg q$

$\neg(\neg p \wedge \neg q)$ Similar to $\underbrace{\neg(p \wedge q)}_{\text{the sheffer stroke}}$

Something as

need to define these in terms of the sheffler stroke

This "not", "and" is the sheffler stroke itself so our definition of $(p \vee q)$ is just

$$(p \uparrow p) \uparrow (q \uparrow q)$$

Logic Laws

- We can use logical equivalences to reduce complex formulas into simpler ones

★ logical Equivalence is denoted ★ ↳ $\Longleftrightarrow$ or $\equiv$

- Two new symbols

Truth table (always look like)

T	F
1	0
1	0
1	0
1	0
1	0

T : Tautology (always 1)

F : Contradiction (always 0)

first laws

Idenity

"and" ~ The Tautology is always true

$$p \wedge T \Longleftrightarrow p$$

~ logically equivalent to p

note: solved proofs on next page.

$$p \vee F \Longleftrightarrow p$$

"or"

Let's reduce this Problem:

$(p \vee F) \wedge (q \vee T)$ Rule we did

$p \wedge (q \vee T)$ Idenity Law

$p \wedge T$ Domination Law

p Idenity Law

Domination

$$p \vee T \Longleftrightarrow T$$

~ True is always True

~ in domination P, doesnt matter

$$p \wedge F \Longleftrightarrow F$$

~ False is always False

- We can see how the Truth and the False are dominating over the formula

Double Negation

$$\neg\neg p \Longleftrightarrow p$$

~ crosses out ★

Example:

$\neg(\neg p \wedge \neg q)$

$\neg\neg p \vee \neg\neg q$ DeMorgans Law

$p \vee q$ Double Negation

★ ### DeMorgan's Law ~ Used everywhere !

flip

distributed

$$\neg(p \wedge q) \Longleftrightarrow \neg p \vee \neg q$$

$$\neg(p \vee q) \Longleftrightarrow \neg p \wedge \neg q$$

$3 \times (1 + 2)$ → Another way to look
$(3 \times 1) + (3 \times 2)$

Distributive law

$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

Absorption Law

$p \wedge (p \vee q) \Leftrightarrow p$

$p \vee (p \wedge q) \Leftrightarrow p$

example:

$\neg\neg p \vee ((p \vee F) \wedge \neg\neg q)$

$p \vee ((p \vee F) \wedge q)$ Double Negation ×2

$p \vee (p \wedge q)$ Idenity

p Absorption

Commutativity $(\vee, \wedge)$

$p \wedge q \Leftrightarrow q \wedge p$
$p \vee q \Leftrightarrow q \vee p$

~ We can flip the order

Inverse Laws

$p \wedge \neg p \Leftrightarrow F$
$p \vee \neg p \Leftrightarrow T$

Associativity $(\vee, \wedge)$

$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

~ can do "ors" as well

~ make sure the signs stay the same.

Conditional Law

$p \rightarrow q \Leftrightarrow \neg p \vee q$

★ Truth Table ★

p	q	p→q	¬p	¬p ∨ q
1	1	1	0	1
1	0	0	0	0
0	1	1	1	1
0	0	1	1	1

∴ logically Equivalent

Exercise:

Show $(\neg(p \wedge q) \wedge q)$ is logically equivolent to $(\neg p \wedge q)$

$\neg(p \wedge q) \wedge q$ — whenever I see a $\neg$ (not) outside of brackets I want to DeMorgenr them together

$(\neg p \vee \neg q) \wedge q$ DeMorgen's Law

~ see these $\vee$ (or's) and $\wedge$ (ands) are different so use

$(q \wedge \neg p) \vee (q \wedge \neg q)$ Distributivity

$(q \wedge \neg p) \vee F$ Inverse

What happens if I have anything or False? ~ that's left with anything ive had before

$q \wedge \neg p$ Identity Law : $(p \vee F \Leftrightarrow p)$

$\neg p \wedge q$ Commutativity Law

~ can flip anything around

Proofs ☆

example $p \wedge T \Leftrightarrow p$ Truth Table

p	T	$p \wedge T$
T	T	T
F	T	F

We see that p and $p \wedge T$ are logically Equivalent

Idenity Laws

Example $p \vee F \Leftrightarrow p$ Truth Table

p	F	$p \vee F$
T	F	T
F	F	F

logically Equivalent

absorption Law

example: $p \wedge (p \vee q) \Leftrightarrow p$

Truth Table:

p	q	$p \vee q$	$p \wedge (p \vee q)$
1	1	1	1
1	0	1	1
0	1	1	0
0	0	0	0

p and $p \wedge (p \vee q)$ are logically equivalent.

Domination law proofs

$p \vee T \iff T$

Truth Table

p	T	p ∨ T
1	1	1
0	1	1

$p \wedge F \iff F$

p	F	p ∧ F
1	0	0
0	0	0

Inverse Law Proofs

$p \wedge \neg p \iff F$

p	¬p	p ∧ ¬p	F
1	0	0	0
0	1	0	0

$p \vee \neg p \iff T$

p	¬p	p ∨ ¬p	T
1	0	1	1
0	1	1	1

DeMorgan Law Proofs

$\neg(p \wedge q) \iff \neg p \vee \neg q$

p	q	(p ∧ q)	¬(p ∧ q)	¬p	¬q	¬p ∨ ¬q
1	1	1	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	0	1	1	1	1

$\neg(p \vee q) \iff \neg p \wedge \neg q$

p	q	(p ∨ q)	¬(p ∨ q)	¬p	¬q	¬p ∧ ¬q
1	1	1	0	0	0	0
1	0	1	0	0	1	0
0	1	1	0	1	0	0
0	0	0	1	1	1	1

Logic Laws Example 1.

1.) Use Laws to reduce: $\neg[(p \wedge q) \rightarrow r]$

$p \rightarrow q \Leftrightarrow \neg p \vee q$

1.) $\neg(\neg(p \wedge q) \vee r)$ ← Conditional Law ~ perform this law 1st.

2.) $\neg\neg(p \wedge q) \wedge \neg r$ DeMorgan's Law

3.) $p \wedge q \wedge \neg r$ Double Negation

2.) Prove that $(\neg p \vee q) \wedge (p \wedge (p \wedge q)) \Leftrightarrow (p \wedge q)$

$\rightarrow (p \wedge (p \wedge q)) \rightarrow (p \wedge p \wedge q)$

$r \vee s \quad \wedge$

1.) $(\neg p \vee q) \wedge (p \wedge q)$ ~p identity law

2.) $((p \wedge q) \wedge q) \vee ((p \wedge q) \wedge \neg p)$ Distribution law $p \wedge (r \vee s) \Leftrightarrow (p \wedge r) \vee (p \wedge s)$

reduction

3.) $(p \wedge q) \vee ((p \wedge q) \wedge \neg p)$ idenity law ~ all conjunctions

4.) $(p \wedge q) \vee F$ Negation

5) $p \wedge q$

~ if we have something that could be True or something that's always False it's always going to take the upper value.

Want this to reduce to True

3.) Show that $(p \vee q) \rightarrow (q \rightarrow q)$ is a tautology $\Leftrightarrow T$

1) $\neg(p \vee q) \vee (q \rightarrow q)$ definition of $\rightarrow$

2.) $\neg(p \vee q) \vee \underbrace{(\neg q \vee q)}$ do the $\rightarrow$ again

3.) $\neg(p \vee q) \vee T$ - b/c this is the law of the excluded middle

4.) T - if we have something ~ always be True or True, its always going to take the upper value with the or, and True is always True

If we have this statement: $p \rightarrow q$ ①

1 this is always True, then that means the conditional will always be a 1.

$\varphi \rightarrow (q \rightarrow q)$

1 1

- We know its a 1 b/c if we have $q \rightarrow q$ this is always True

- always a Tautology!

Logic Laws Examples 2

Show that $p \to q$ and $\neg q \to \neg p$ are logically equivalent without using truth tables or a "contrapositive" law.

1. $p \to q$
2. $\neg p \vee q$ — definition of $\to$ (conditional law)
3. $\neg\neg q \vee \neg p$ — Community law
4. $\neg q \to \neg p$ — def of $\to$

Show that $(p \to r) \wedge (q \to r)$ and $(p \vee q) \to r$ are logically equivalent

1. $(p \to r) \wedge (q \to r)$
2. $(\neg p \vee r) \wedge (\neg q \vee r)$ — $\to$ def
3. $r \vee (\neg q \wedge \neg p)$ — Distributed law
4. $(\neg q \wedge \neg p) \vee r$ — communitive law
5. $\neg(\neg q \wedge \neg p) \to r$ — $\to$ def
6. $(\neg\neg q \vee \neg\neg p) \to r$ — DeMorgan's
7. $(q \vee p) \to r$ — Double Negation

Conditionals

~ Challenging ~

Conditionals

★ The Conditional is equaled logically to the contrapositive ~~and~~ nothing else ★

$p \to q$

$\boxed{\neg p \vee q}$

logically equivalent

Converse

$q \to p$

~ flipping the order

$\boxed{\neg q \vee p}$

logically Equivalent

~ logically Equivalent

~ logically Equivalent

Inverse

$\neg p \to \neg q$

Just adding the negations to p and q

$\neg\neg p \vee \neg q$ Double Negation

$\boxed{p \vee \neg q}$ logically Equivalent

Contrapositive

$\neg q \to \neg p$

$\neg\neg q \vee \neg p$ using double negation

$\boxed{q \vee \neg p}$ logically equivalent

Biconditional

- conjunction of two conditionals

not a standard operator

$$\underbrace{p \leftrightarrow q}_{\text{if and only if}} \iff \overbrace{(p \to q)}^{\text{p only if q}} \wedge \overbrace{(q \to p)}^{\text{p if q}}$$

Truth Table Proof →

p	q	$p \leftrightarrow q$	$(p \to q)$	$q \to p$	$(p \to q) \wedge (q \to p)$
1	1	1	1	1	1
1	0	0	0	1	0
0	1	0	1	0	0
0	0	1	1	1	1

logically equivalent

Translation Examples

"I will pass if I get a good exam mark." if p then q

$G \rightarrow P$

"I will pass only if I get a good exam mark."

$P \rightarrow G$

"I will pass if and only if I get a good exam score."

$P \longleftrightarrow G$

~ closely related in terms of the biconditional.

This means if you get a good exam mark and you don't pass then it's false, because it's a lie.

If I pass the course $P \rightarrow G$ if P is a 1 then I better be getting a good exam mark, so I will pass only if I get a good exam mark.

$P \rightarrow G$, ← FALSE

- if I pass and don't get a good exam mark then this is going to be false, I will only pass if I get a good exam mark. This says I passed, even though I didn't get a good exam mark, which is a lie.

~ Think of it when someone is lying.

Examples

Mark is writing an exam on propositional logic. During the Exam, Dr. Cheatmaster notices that Mark is acting ~~up~~ rather suspicious. Suspecting Mark of cheating, Dr. Cheatmaster walks up behind Mark and noticinges a cheat sheet. Dr. Cheatmoster says

"If you don't give me your cheat sheet, then you will fail the course."

Becau Mark doesnt want to fail, he gives Dr. Cheatmaster his cheatsheet. After reviewing the Cheatsheet, Dr. Cheatmaster fails Mark.

NO

Did Dr. Cheatmister lie to Mark? Explain your answer using the truth conditions of the conditional and its logical equivalencies.

Answer:

$$\neg C \rightarrow F \Leftrightarrow \neg F \rightarrow C$$

But Mark gave Dr. Cheatmaster the cheat sheet

C

I doesnt matter if he gave him his cheatsheet, he was probably going to fail

Conditional Examples

Determine whether the conditionals are true or False.

(i) If $\underbrace{2 \times 4 = 6}_{\text{FALSE}}$, then $\underbrace{3 + 6 = 7}_{\text{FALSE}}$

$0 \rightarrow 0$ True!

(ii) If $\underbrace{1 + 1 = 2}_{\text{TRUE}}$, then $\underbrace{3 \times 6 = 18}_{\text{TRUE}}$

$1 \rightarrow 1$ True!

(iii) $\underbrace{2 \times 7 = 14}_{\text{TRUE}}$ $\boxed{\text{only if}}$ $\underbrace{6 \times 2 = 18}_{\text{FALSE}}$

$1 \rightarrow 0$ False!

When we have "only if"

if P then Q } Equivalent
P only if Q

Question #2:

"If it snows I will Cry"

$P \rightarrow Q$

Converse

$Q \rightarrow P$

~reverse

If I will cry then it snows

Inverse

$\neg P \rightarrow \neg Q$

if it doesnt snow, then I will not cry

Contra positive

$\neg Q \rightarrow \neg P$

If I won't cry, it doesnt snow

^ This is logically equivalent to the 1st conditional.

Rules of Inference

- Deducing logical outcomes

if it rains, I will get wet.
It's raining
∴ I will get wet.

if R then W

$R \to W$ Premise
R ← R is happening
$\therefore$ W
Conclusion

→ Logical Argument ←

★ We take one or two premises and we deduce a conclusion from them

$((R \to W) \land R) \to W$

Some thing

with a Truth Table this would be a Tautology

- A set of premises $P_1, P_2, \ldots P_n$ prove some conclusion q in an argument.

$$(P_1 \land P_2 \land \ldots \land P_n) \to q$$

- An argument is Valid if the premises logically entail the conclusion

~ if all our rules of Inference takes us from the premises to the conclusion then it's valid. If there are improper steps or steps that aren't logically equivalent in there, then it's invalid.

Laws ☆ Rules of Inferences ☆

① Modus Ponens MPP

↳ sometimes refered to as affirming the antecedent

$$\begin{array}{l} p \to q \\ \underline{p} \\ \therefore q \end{array}$$

like sticking the thing into the arrow and getting an output

② Modus Tollens MTT

$$\begin{array}{l} p \to q \\ \underline{\neg q} \\ \therefore \neg p \end{array}$$

$\neg p \to \neg p$ -opposite way

logically Equivalent

③ Hypothetical Syllogism HS

like transitivity

$$\begin{array}{l} p \to q \\ \underline{q \to r} \\ \therefore p \to r \end{array}$$

Take short cut

$$p \longrightarrow q \longrightarrow r$$

④ Disjunctive Syllogism DS

$$\begin{array}{l} p \vee q \\ \underline{\neg p} \\ \therefore q \end{array}$$

⑤ Addition $\vee I$

$$\begin{array}{l} \underline{p} \\ \therefore p \vee q \end{array}$$

~if p is True then p or q is True.

⑥ Simplification $\wedge E$

↳ and elimination b/c you're removing the and from it.

$$\begin{array}{l} \underline{p \wedge q} \\ \therefore p \\ \therefore q \end{array}$$

⑦ Conjunction $\wedge I$

$$\begin{array}{l} p \\ \underline{q} \\ \therefore p \wedge q \end{array}$$

if p and q are both True, the $p \wedge q$ are going to be true

① Exercise

Task:
- Prove that (1) - (3) entail $\neg J$.
- Justify each step

1. R Premise
2. $R \to D$ Premise
3. $D \to \neg J$ Premise

4. D -Using 1, 2, MPP
 - stuck R in and got D

5. $\neg J$ -Using 3, 4, MPP

Another way to do things (Hypothetical Syllosism)

4. $R \to \neg J$ -using 2, 3, HS
5. $\neg J$ - 1, 4 MPP

② Exercise

Task:
Show that (1) - (3) entail R

1. $(\neg R \vee \neg F) \to (S \wedge L)$ Premise
2. $S \to T$ Premise
3. $\neg T$ Premise

4. $\neg S$ 2, 3, MTT
5. $\neg(S \wedge L) \to \neg(\neg R \vee \neg F)$ 1, Contropositive
6. $(\neg S \vee \neg L) \to (\neg\neg R \wedge \neg\neg F)$ 5, De Morgens
7. $(\neg S \vee \neg L) \to (R \wedge F)$ 6, Double Negation
8. $\neg S \vee \neg L$ 4, Addition

9. $R \wedge F$ 7, 8 MPP

10. R 9, simplication

Predicate Logic and Negating Quantifiers

Predicate Logic and Quantifiers

↳ Can have **terms**?

- We want to talk about variables

- $E(x)$ = x is even

two-place predicate

open • $\boxed{G(x,y) = x \text{ is greater than } y}$ NOT A STATEMENT!

Can stick in constants in

closed
$G(2,1) = 2$ is greater than 1 True! ~ has a truth value
$G(3,6) = 3$ is greater than 6 False!

What's the difference b/w closed and open formula, well closed formulas have truth values and open do not have truth values. ∴ $G(x,y)$ is not a statement. Which means we can't even have it in propositional logic at all, we need predicate logic in order to say something like $G(x,y)$ is greater than y. and assigned truth values to constants plugged into these variables and that's one thing predicate logic lets us do. But the more important thing it does is gives us quantifiers

We want to introduce quantifiers

Universal quantifier

$\forall x\, P(x)$: "For all x, x is P"

(for all x)

Existential quantifier

$\exists x\, P(x)$: "For some x, x is P"

There exists x

We use this notation everywhere in mathematics

① For every real number n, there's a real number m such that $m^2 = n$.

$\underbrace{\forall n \in \mathbb{R}}_{\text{For all real number } n}$ $\underbrace{\exists m \in \mathbb{R}}_{\text{Exist an } m \text{ in the reals}}$: $m^2 = n$ or $P(m,n) = m^2 = n$

② Given two rationals, x and y, $\sqrt{ab}$ will ~~ab~~. also be rational.

$\forall x, y \in \mathbb{Q}, \sqrt{xy} \in \mathbb{Q}$

or

$\forall x \in \mathbb{Q}\ \forall y \in \mathbb{Q}$

- Translating some mathematical statements into predicate logic

Negating Quantifiers

- Define $\forall x, \exists x$ for a universe with elements $\{1, 2, \ldots, n\}$

$\forall x\, P(x) \Leftrightarrow P(1) \wedge P(2) \wedge \ldots \wedge P(n)$

This is True if every x in the universe is P

$\exists x\, P(x) \Leftrightarrow P(1) \vee \cancel{P}\ P(2) \vee \ldots \cancel{\vee P}\ \cancel{\vee B} \ldots P(n)$

Definition of Existent X

Show that $\neg \forall x [P(x)] \Leftrightarrow \exists x [\neg P(x)]$

$\neg \forall x P(x) = \neg(P(1) \wedge P(2) \wedge \ldots \wedge P(n))$

negate this Using DeMorgans Law

$= \neg P(1) \vee \neg P(2) \vee \ldots \vee \neg P(n)$

$\exists x P(x) = P(1) \vee P(2) \vee \ldots \vee P(n)$

$= \exists x [\neg P(x)]$

~ There exists an X except instead of just having p(x) we have not p(x) $(\neg p(x))$ because remember the definition of existent X

- look very similar except theres an negation in front of each term

All Equivalencies

$\forall x P(x) \Leftrightarrow \neg \exists x [\neg P(x)]$

$\exists x P(x) \Leftrightarrow \neg \forall x [\neg P(x)]$

$\neg \forall x P(x) \Leftrightarrow \exists x [\neg P(x)]$

$\neg \exists x P(x) \Leftrightarrow \forall x [\neg P(x)]$

Checking logically Equivalent

no negation as a plus (+)

negation as a minus (−)

$\neg \forall x P(x)$ || || reduced

$- \forall + P$

$+ \exists - P$

$\exists x \neg P(x)$

$\exists x P(x)$

$+ \exists + P$

$- \forall - P$

$\neg \forall x [\neg P(x)]$

Negate the Following:

$\forall x \exists y [P(x,y) \wedge Q(y)]$

$\neg [\forall x [\exists y [P(x,y) \wedge Q(y)]]]$ ~ Brackets ~ 1st negate $\forall x$

$\exists x \neg \underline{[\exists_y [P(x,y) \wedge Q(y)]]}$

~ now we need to negate

$\exists x \forall y \neg \underline{[P(x,y) \wedge Q(y)]}$

$\exists x \forall y [\neg P(x,y) \vee \neg Q(y)]$ ~ now we do Demorgan's Law

Negating Quantifiers and Translation Examples

Translate the following:

- Every student is majoring in math or computer science

$$\forall x \, [S(x) \rightarrow (M(x) \vee C(x))]$$

- For all ~~ABC~~ real x, y, $x > y$ if $x^2 > y^2$

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R} \, [(x^2 > y^2) \rightarrow (x > y)]$$

- For all $\varepsilon > 0$, there is an $\delta > 0$ such that $|f(x) - L| < \varepsilon$ when $|x - a| < \delta$.

$$\forall \varepsilon > 0, \exists \delta > 0 \, [(|x - a| < \delta) \rightarrow (|f(x) - L| < \varepsilon)]$$

Negate:

$$\exists x\,(Px \wedge Qx)$$

$$\neg(\exists x\,(Px \wedge Qx))$$

$$(\neg\exists x)(Px \wedge Qx)$$

$$\forall x\ \neg(Px \wedge Qx)$$

$$\forall x\,[\neg Px \vee \neg Qx]$$

Negate: $\forall x\,(Px \rightarrow Qx)$

$$\neg\forall x\,(Px \rightarrow Qx)$$

$$\exists x\,[\neg(Px \rightarrow Qx)]$$

$$\exists x\,[\neg(\neg Px \vee Qx)]$$

$$\exists x\,[Px \wedge \neg Qx]$$

Unique Quantifier Examples

Determine the Truth Values.

(i) $n \in \mathbb{Z}$ $\quad$ $\exists n\,(n^2 = 2)$

We know $n = \pm\sqrt{2}$

FALSE

(ii) $x \in \mathbb{R}$ $\quad$ $\forall x\,(x^2 + 2 \geq 1)$

$x^2 \geq 0$ $\quad$ $0 + 2 \geq 1$

TRUE

(iii) $x \in \mathbb{Z}$ $\quad$ $\forall x\,((-x)^2 = x^2)$

$0^2 = 0^2$

$(-3)^2 = 3^2$

$3^2 = (-3)^2$

TRUE

means unique

If $\exists! x\, P(x)$ means "there exists a unique x such that P(x)", determine the truth values of the following:

(i) $\exists! x\,(x+1 = 2x)$ for $x \in \mathbb{Z}$

(ii) $\exists! x\,(x > 1)$ for $x \in \{0, 1, 2, 3\}$

Which of the following are always True?

$\exists! x\, P(x) \rightarrow \exists x\, P(x)$

$\exists! x\, \neg P(x) \rightarrow \neg \forall x\, P(x)$

$\exists x\, P(x) \rightarrow \exists! x\, P(x)$

Conditionals ☆ Extra Notes ☆

extra logic

Conditional $P \rightarrow Q$

if P, "the hypothesis" then Q ~ "the conclusion"

Converse $Q \rightarrow P$

Inverse $\neg P \rightarrow \neg Q$

Contrapositive $\neg Q \rightarrow \neg P$

Bio conditional $P \Leftrightarrow Q$

Examples

① If you study then you will get a good grade

Whatever comes after the "If", will be the hypothesis

Whatever comes after the "then", that's your conclusion.

What would the Converse be?

- in reverse

If you get a good grade then you studied.

~ But that's not necessary true, you could've gotten a good grade b/c the teacher made the test really easy or you're just naturally good @ math.

What about the Inverse?

- Add word not in there (negated)

if you do not study, then you will not get a good grade.

What about the Contrapositive?

~ combination of the converse and inverse and negating them

if you do not get a good grade, then you did not study

2 Angles

- form line, they are supplementary, add up to 180° ∀

extra logic

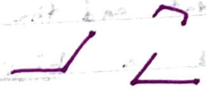
(2) If 2 Angles form a linear pair then they are supplementary

What would be the converse of the statement?

"if 2 Angles are supplementary then they form a linear pair"

Is this True?

Thats not necessary true → We could have two angles here



Example of the converse error. Just b/c you say something in reverse doesnt mean its True you know forward or a conditional statement.

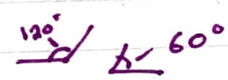
- Notice they're not adjacent to each other, they're not sharing a vertex or and a array.
- Not a linear pair

What about the Inverse of the statement?

"If 2 Angles do not form a linear pair then they are not supplementary."

True or False?

FALSE, b/c just b/c they don't form a linear pair doent mean they're not supplemenery., they could still add up to 180°.



What about the Contrapositive?

- switch the hypothesis and conclusion and we're going to negate them both.

" If 2 angles are not supplementary then they do not form a linear pair."

True or False?

TRUE, b/c if they're not supplementary, that mean they don't add up to 180° theres no way that they could be forming a straight line, a linear pair

~ Plus we know if the Conditional is True, then the contrapositive will automatically be true, they have the same truth Value.

(3.) If 2 lines form right angles then they are Perpendicular.

~ That makes sense, it's True

What about as a converse statement?

"if two lines are perpendicular then they form right angles

~ The conditional statement and the converse statement are both true, we can write this as a bi conditional statement.

Bi meaning two and conditional statement, meaning ~~as~~ its a conditional statement forward and reverse. $P \Longleftrightarrow Q$

How do we write this as a Biconditional?

~ replace with "If and Only if"

2 lines form right angles if and only if they are perpindicular