Homework #5 : sections 2.3-2.5

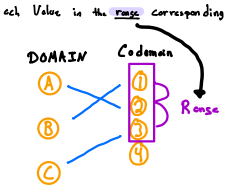
2.3

12. Determine whether each of these functions from **Z** to **Z** is one-to-one.

a) $f(n) = n - 1$ **b)** $f(n) = n^2 + 1$
c) $f(n) = n^3$ **d)** $f(n) = \lceil n/2 \rceil$

Injective (one-to-one) Functions

- Each value in the range corresponding to exactly one element in the domain.



a) $f(n) = n - 1$

if $f(a) = f(b)$ then $a = b$ ← To prove one-to-one problems

$f(a) = 3$ $f(b) = 3$

$a = 4$ $b = 4$

Yes one-to-one

b) $f(n) = n^2 + 1$

if $f(a) = f(b)$ then $a = b$

$f(a) = 5$ $f(b) = 5$

$a = 2$ $b = -2$ ×

$f(a) = f(b)$ but $a \neq b$

Not one-to-one

c) $f(n) = n^3$

if $f(a) = f(b)$ then $a = b$

One-to-one

$f(a) = 8$ $f(b) = 8$ ✓

$a = 2$ $b = 2$ ✓

d) $f(n) = \lceil n/2 \rceil$

if $f(a) = f(b)$ then $a = b$

Not one-to-one $f(a) = 2$ $f(b) = 2$

$\lceil \; \rceil$ ← Will round #'s

2. Determine whether f is a function from **Z** to **R** if

a) $f(n) = \pm n$.

b) $f(n) = \sqrt{n^2 + 1}$.

c) $f(n) = 1/(n^2 - 4)$.

a) $f(n) = \pm n$.

NO

Because each # is mapped to two integers

b) $f(n) = \sqrt{n^2 + 1}$.

Yes

Because each # is mapped to one number

c) $f(n) = 1/(n^2 - 4)$.

NO

Because the function is undefined at -2 and 2

8. Find these values.

a) $\lfloor 1.1 \rfloor$ **b)** $\lceil 1.1 \rceil$
c) $\lfloor -0.1 \rfloor$ **d)** $\lceil -0.1 \rceil$
e) $\lceil 2.99 \rceil$ **f)** $\lceil -2.99 \rceil$
g) $\lfloor \frac{1}{2} + \lceil \frac{1}{2} \rceil \rfloor$ **h)** $\lceil \lfloor \frac{1}{2} \rfloor + \lceil \frac{1}{2} \rceil + \frac{1}{2} \rceil$

Floor Function $f(x) = \lfloor x \rfloor$

Largest integer less than or equal to x.

Ceiling Function $f(x) = \lceil x \rceil$

Smallest integer greater than or equal to x.

a) $\lfloor 1.1 \rfloor$

1

b) $\lceil 1.1 \rceil$

2

c) $\lfloor -0.1 \rfloor$

-.2

d) $\lceil -0.1 \rceil$

0

e) $\lceil 2.99 \rceil$

3

f) $\lceil -2.99 \rceil$

-2

g) $\lfloor \frac{1}{2} + \lceil \frac{1}{2} \rceil \rfloor$

$\lfloor \frac{1}{2} + \lceil \frac{1}{2} \rceil \rfloor$

$\lceil \frac{1}{2} \rceil = 1$

$\lfloor \frac{1}{2} + 1 \rfloor$

$\lfloor 1.5 \rfloor$

1

h) $\lceil \lfloor \frac{1}{2} \rfloor + \lceil \frac{1}{2} \rceil + \frac{1}{2} \rceil$

$\lfloor \frac{1}{2} \rfloor$ $\lceil \frac{1}{2} \rceil$

0 1

$\lceil 0 + 1 + \frac{1}{2} \rceil$

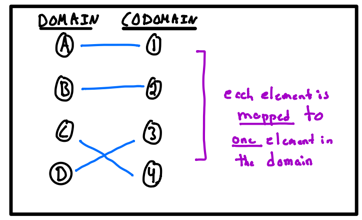
$\lceil 1.5 \rceil$

2

13. Which functions in Exercise 12 are onto?

Surjective (onto) Functions

Every element in the codomain maps to at least one element in the domain



a) $f(n) = n - 1$

onto

- because every integer is 1 less than some integer

b) $f(n) = n^2 + 1$

Not onto

- Because everything is positive (n^2), so the range cannot include any negative integer

c) $f(n) = n^3$

Not onto

$2^{\frac{1}{3}} \neq$ Integer

d) $f(n) = \lceil n/2 \rceil$

onto

22. Determine whether each of these functions is a bijection from **R** to **R**.

a) $f(x) = -3x + 4$
b) $f(x) = -3x^2 + 7$
c) $f(x) = (x + 1)/(x + 2)$
d) $f(x) = x^5 + 1$

Bijective (one-to-one Correspondence) Functions

Functions that are both one-to-one and onto, or both surjective and injective.

a) $f(x) = -3x + 4$

One-to-one ✓
Onto ✓

$f(a) = f(b) \quad a = b$
$-3a + 4 = -3b + 4$
$-3(2) + 4 = -3(2) + 4$
$-2 = -2$ ✓ $\quad a = b$ ✓

Onto:
$f(x) = -3x + 4$
$f\left(\frac{4-x}{3}\right) = x$ ✓

Yes, Bijection

b) $f(x) = -3x^2 + 7$

if $f(a) = f(b)$ then $a = b$
$f(-a) \neq f(b)$ X

Not Bijection

c) $f(x) = (x + 1)/(x + 2)$

$f(x) = \frac{(x+1)}{(x+2)} = 1$? There is no real # for x to equal 1. So not onto

Not Bijection

· Plus when $x = -2$ it is undefined, so it doesn't have a image, just another observation.

d) $f(x) = x^5 + 1$

Yes, Bijection

~ Increasing function

26. **a)** Prove that a strictly increasing function from **R** to itself is one-to-one.

b) Give an example of an increasing function from **R** to itself that is not one-to-one.

a) Prove that a strictly increasing function from **R** to itself is one-to-one.

Have to prove $f(x)$ is one-to-one

If is "strictly" increasing then it implies $x<y$, the $f(x)<f(y)$

Assuming $f(a) = f(b)$

If $a<b$, then by definition of strictly increasing $f(a) < f(b)$, then it's not possible that $a<b$ when $f(a) = f(b)$

If $b<a$, then by definition of strictly increasing $f(b)<f(a)$, then it's not possible that $b<a$ when $f(a) = f(b)$

Since $a<b$ is not true and since $b<a$ is not true, a and b then have to be the same:

$$a=b$$

Then by the definition of one-to-one we've shown that f is one-to-one

b) Give an example of an increasing function from **R** to itself that is not one-to-one.

$f(x) = \lceil x \rceil$, is an increasing function, but not one-to-one, because it's possible for some values to have the same image)

e.g. $f(1) = \lceil 1 \rceil = 1$

$f(.6) = \lceil .6 \rceil = 1$

30. Let $S = \{-1, 0, 2, 4, 7\}$. Find $f(S)$ if

a) $f(x) = 1$. **b)** $f(x) = 2x + 1$.

c) $f(x) = \lceil x/5 \rceil$. **d)** $f(x) = \lfloor (x^2 + 1)/3 \rfloor$.

DEFINITION 4 Let f be a function from A to B and let S be a subset of A. The *image* of S under the function f is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$, so

$$f(S) = \{t \mid \exists s \in S\ (t = f(s))\}.$$

We also use the shorthand $\{f(s) \mid s \in S\}$ to denote this set.

Remark: The notation $f(S)$ for the image of the set S under the function f is potentially ambiguous. Here, $f(S)$ denotes a set, and not the value of the function f for the set S.

a) $f(x) = 1$.

$S = \{-1, 0, 2, 4, 7\}$

Determine the <u>image</u> of every element in the set S.

$f(-1) = 1$
$f(0) = 1$
$f(2) = 1$
$f(4) = 1$
$f(7) = 1$

f(S) is the set of all the images of every element in S.

$f(S) = \{1\}$

b) $f(x) = 2x + 1$.

$f(-1) = 2(-1) + 1 = -1$
$f(0) = 2(0) + 1 = 1$
$f(2) = 2(2) + 1 = 5$
$f(4) = 2(4) + 1 = 9$
$f(7) = 2(7) + 1 = 15$

$f(S) = \{-1, 1, 5, 9, 15\}$

c) $f(x) = \lceil x/5 \rceil$.

$f(-1) = \lceil \frac{-1}{5} \rceil = \lceil -.2 \rceil = 0$
$f(0) = \lceil \frac{0}{5} \rceil = \lceil 0 \rceil = 0$
$f(2) = \lceil \frac{2}{5} \rceil = \lceil .4 \rceil = 1$
$f(4) = \lceil \frac{4}{5} \rceil = \lceil .8 \rceil = 1$
$f(7) = \lceil \frac{7}{5} \rceil = \lceil 1.4 \rceil = 2$

$f(S) = \{0, 1, 2\}$

d) $f(x) = \lfloor (x^2 + 1)/3 \rfloor$.

$f(-1) = \left\lfloor \frac{((-1)^2 + 1)}{3} \right\rfloor = \left\lfloor \frac{2}{3} \right\rfloor = 0$

$f(0) = \left\lfloor \frac{(0^2 + 1)}{3} \right\rfloor = \left\lfloor \frac{1}{3} \right\rfloor = 0$

$f(2) = \left\lfloor \frac{(2^2 + 1)}{3} \right\rfloor = \left\lfloor \frac{5}{3} \right\rfloor = 1$

$f(4) = \left\lfloor \frac{(4^2 + 1)}{3} \right\rfloor = \left\lfloor \frac{17}{3} \right\rfloor = 5$

$f(7) = \left\lfloor \frac{(7^2 + 1)}{3} \right\rfloor = \left\lfloor \frac{50}{3} \right\rfloor = 16$

$f(S) = \{0, 1, 5, 16\}$

$\frac{17}{3} = 5.\overline{66}$

$\frac{50}{3} = 16.\overline{66}$

33. Suppose that g is a function from A to B and f is a function from B to C.

a) Show that if both f and g are one-to-one functions, then $f \circ g$ is also one-to-one.

b) Show that if both f and g are onto functions, then $f \circ g$ is also onto.

a) Show that if both f and g are one-to-one functions, then $f \circ g$ is also one-to-one.

b) Show that if both f and g are onto functions, then $f \circ g$ is also onto.

***34.** If f and $f \circ g$ are one-to-one, does it follow that g is one-to-one? Justify your answer.

***35.** If f and $f \circ g$ are onto, does it follow that g is onto? Justify your answer.

43. Let $g(x) = \lfloor x \rfloor$. Find

a) $g^{-1}(\{0\})$. **b)** $g^{-1}(\{-1, 0, 1\})$.

c) $g^{-1}(\{x \mid 0 < x < 1\})$.

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n+1$
(1b)	$\lceil x \rceil = n$ if and only if $n-1 < x \le n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x-1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x+1$
(2)	$x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x+1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x+n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x+n \rceil = \lceil x \rceil + n$

- must find all number where floor is 0

a) $g^{-1}(\{0\})$.

$$g^{-1}(\{0\}) = \{x \mid 0 \le x \le 1\} = [0, 1)$$

Such that

b) $g^{-1}(\{-1, 0, 1\})$.

$$g^{-1}(\{-1,0,1\}) = \{x \mid -1 \le x < 2\} = [-1, 2)$$

c) $g^{-1}(\{x \mid 0 < x < 1\})$.

· Since x is an integer there is no values such that g(x) is strictly b/w 0 and 1.

Section 2.4

4. What are the terms a_0, a_1, a_2, and a_3 of the sequence $\{a_n\}$, where a_n equals

a) $(-2)^n$? **b)** 3?
c) $7+4^n$? **d)** $2^n+(-2)^n$?

a) $(-2)^n$?

$a_0 = (-2)^0$ $a_1 = (-2)^1$ $a_2 = (-2)^2$ $a_3 = (-2)^3$

$a_0 = 1$ $a_1 = -2$ $a_2 = 4$ $a_3 = -8$

b) 3?

$a_0 = a_1 = a_2 = a_3 = 3$

c) $7+4^n$?

$a_0 = 7 + 4^0$ $a_1 = 7 + 4^1$ $a_2 = 7 + 4^2$ $a_3 = 7 + 4^3$

$a_0 = 7+1$ $= 7 + 4$ $= 7 + 16$

$a_0 = 8$ $a_1 = 11$ $a_2 = 23$ $a_3 = 71$

d) $2^n+(-2)^n$?

$a_0 = 2^0 + (-2)^0$ $a_1 = 2^1 + (-2)^1$ $a_2 = 2^2 + (-2)^2$ $a_3 = 2^3 + (-2)^3$

$a_0 = 1 + 1$ $a_1 = 2 + -2$ $a_2 = 4 + 4$ $a_3 = 8 + -8$

$a_0 = 2$ $a_1 = 0$ $a_2 = 8$ $a_3 = 0$

9. Find the first five terms of the sequence defined by each of these recurrence relations and initial conditions.

a) $a_n = 6a_{n-1}$, $a_0 = 2$
b) $a_n = a_{n-1}^2$, $a_1 = 2$
c) $a_n = a_{n-1} + 3a_{n-2}$, $a_0 = 1$, $a_1 = 2$
d) $a_n = na_{n-1} + n^2a_{n-2}$, $a_0 = 1$, $a_1 = 1$
e) $a_n = a_{n-1} + a_{n-3}$, $a_0 = 1$, $a_1 = 2$, $a_2 = 0$

a) $a_n = 6a_{n-1}$, $a_0 = 2$

$a_0 = 2$
$a_1 = 6(2) = 12$

$a_2 = 6(12) = 72$

$a_3 = 6(72) = 432$

$a_4 = 6(432) = 2592$

2, 12, 72, 432, 2592

b) $a_n = a_{n-1}^2$, $a_1 = 2$

2, 4, 16, 256, 65536

$a_2 = a_{2-1}^2$
$a_2 = a_1^2$
$a_2 = 2^2 = \underline{4}$

$a_3 = a_{3-1}^2$
$a_3 = a_2^2$
$a_3 = 4^2 = \underline{16}$

$a_4 = a_{4-1}^2$
$a_4 = a_3^2$
$a_4 = (16)^2 = \underline{256}$

$a_5 = a_{5-1}^2$
$a_5 = a_4^2$
$a_5 = 256^2 = \underline{65536}$

c) $a_n = a_{n-1} + 3a_{n-2}$, $a_0 = 1$, $a_1 = 2$

$a_n = a_{n-1} + 3a_{n-2}$ $\quad a_0 = 1, a_1 = 2$

$a_2 = a_1 + 3a_0 = 2 + 3(1) = 5$

1, 2, 5, 11, 26

$a_3 = a_2 + 3a_1 = 5 + 3(2) = 11$
$a_4 = a_3 + 3a_2 = 11 + 3(5) = 26$

d) $a_n = na_{n-1} + n^2a_{n-2}$, $a_0 = 1$, $a_1 = 1$

$a_2 = 2a_1 + (2)^2 \cdot a_0 = 2(1) + (2)^2 \cdot 1 = 6$
$a_3 = 3a_2 + (3)^2 \cdot a_1 = 3(6) + (3)^2 \cdot 1 = 18 + 9 = 27$
$a_4 = 4a_3 + (4)^2 \cdot a_2 = 4(27) + 4^2 \cdot 6 = 204$

1, 1, 6, 27, 204

e) $a_n = a_{n-1} + a_{n-3}$, $a_0 = 1$, $a_1 = 2$, $a_2 = 0$

$a_3 = a_2 + a_0 = 0 + 1 = 1$

1, 2, 0, 1, 3

$a_4 = a_3 + a_1 = 1 + 2 = 3$

12. Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = -3a_{n-1} + 4a_{n-2}$ if

a) $a_n = 0$. **b)** $a_n = 1$.
c) $a_n = (-4)^n$. **d)** $a_n = 2(-4)^n + 3$.

a) $a_n = 0$.

$a_n = -3a_{n-1} + 4a_{n-2}$

$0 = -3(0) + 4(0)$ ✓
$0 = 0$

b) $a_n = 1$.

$a_n = -3a_{n-1} + 4a_{n-2}$

$1 = -3(1) + 4(1)$
$1 = 1$ ✓

c) $a_n = (-4)^n$.

$a_n = -3a_{n-1} + 4a_{n-2}$

$(-4)^n = -3((-4)^n) + 4((-4)^n)$

$(-4)^n = (-4)^n$

d) $a_n = 2(-4)^n + 3$.

19. Suppose that the number of bacteria in a colony triples every hour.

a) Set up a recurrence relation for the number of bacteria after n hours have elapsed.

b) If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?

a) Set up a recurrence relation for the number of bacteria after n hours have elapsed.

b_n = number of bacteria after n hours

$$b_n = 3b_{n-1}$$

b) If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?

$b_0 = 100$ $n = 10$

$$b_n = 3b_{n-1} = 3^2 b_{n-2} = \underline{3^n b_0}$$

$$b_{10} = 3^{10}(100) =$$

$$b_{10} = 5904900$$

29. What are the values of these sums?

a) $\sum_{k=1}^{5}(k+1)$ **b)** $\sum_{j=0}^{4}(-2)^j$

c) $\sum_{i=1}^{10} 3$ **d)** $\sum_{j=0}^{8}(2^{j+1}-2^j)$

a) $\sum_{k=1}^{5}(k+1)$ $= (1+1) + (2+1) + (3+1) + (4+1) + (5+1) =$

$2 + 3 + 4 + 5 + 6 = 20$

b) $\sum_{j=0}^{4}(-2)^j$ $= (-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + (-2)^4 =$

$1 + -2 + 4 + -8 + 16 = 11$

c) $\sum_{i=1}^{10} 3$ $3 \cdot 10 = 30$

d) $\sum_{j=0}^{8}(2^{j+1}-2^j)$ $(2^{0+1} - 2^0)$

Section 2.5

3. Determine whether each of these sets is countable or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

a) all bit strings not containing the bit 0

b) all positive rational numbers that cannot be written with denominators less than 4

c) the real numbers not containing 0 in their decimal representation

d) the real numbers containing only a finite number of 1s in their decimal representation

a) all bit strings not containing the bit 0

b) all positive rational numbers that cannot be written with denominators less than 4

c) the real numbers not containing 0 in their decimal representation

d) the real numbers containing only a finite number of 1s in their decimal representation

7. Suppose that Hilbert's Grand Hotel is fully occupied on the day the hotel expands to a second building which also contains a countably infinite number of rooms. Show that the current guests can be spread out to fill every room of the two buildings of the hotel.