2.1 – 2.5: Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

2.1 Introduction to Sets

Set Vocabulary and Notation

Set - Unordered collection of objects

$B = \{\text{Eric, Adam, Kevin}\}$

Elements - An Object in Set

$\in$ and $\notin$ Adam $\in B$ Larry $\notin B$

not in

Sets you should know....

$\mathbb{N}$ - Natural Counting #'s $\{1, 2, 3, \ldots\}$

$\mathbb{W}$ - Whole $0, \mathbb{N}$ $\{0, 1, 2, 3 \ldots\}$

$\mathbb{Z}$ - Integers $\pm \mathbb{W}$ $\{\ldots -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{Q}$ - Rational $\frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$ $\frac{a}{b}$ is in lowest terms

$\mathbb{R}$ - Real

$\mathbb{Z}^+$ positive int.

$\mathbb{C}$ - Complex $\underline{\underline{a}} + bi$

Set Notation

→ $S = \{1, 2, 3, 4, 5\}$

Roster Notation

→ Discrete Sets
Countable

$S = \{1, 2, 3, 4, 5\}$

Continous

$0 \leq x \leq 1$

interval

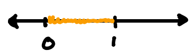
Set-Builder Notation

$S = \{x \mid x \in \mathbb{N},\ x = 5\}$ $\qquad$ $S = \{x \mid x \in \mathbb{Z},\ 1 \leq x \leq 5\}$

Such that

Interval Notation

$B = \{x \mid 0 \leq x \leq 1\}$



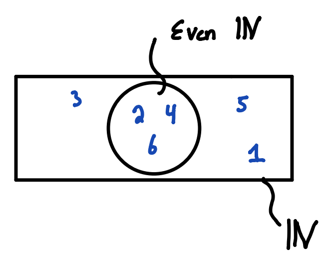
$M = \{x \mid 0 < x < 1\}$

$(0, 1)$

Special Sets

Universal Set

- The set of all elements under consideration



Empty Set

A Set with no elements

$\emptyset$, $\{\ \}$

$\cancel{\{\emptyset\}}$

Set Relationships

Set Equality

Sets are equal iff they have the same elements

$$\boxed{\forall x\,(x \in A \leftrightarrow x \in B)}$$

Notations: A = B

$$A = \{0, 1, 1, 3, 4, 4\} \qquad B = \{0, 1, 3, 4, 5\}$$

$$A \neq B$$

Subsets

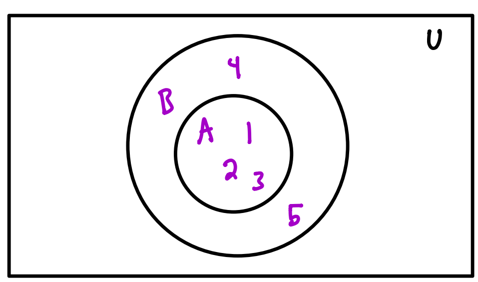
A set A is a subset of B iff every element of A is also an element of B.

$$\boxed{\forall x\,(x \in A \rightarrow x \in B)}$$

Notation: $A \subseteq B$

$$A = \{1, 2, 3\}$$

$$B = \{1, 2, 3, 4, 5\}$$



More on Subsets

Show A $\subseteq$ B

Show every element of A belongs to B if $x \in A$ then $x \in B$

Show A $\nsubseteq$ B

$\exists x \in A \to x \notin B$

Show x belongs to A but not set B

Show A $\subseteq$ B and B $\subseteq$ A

$\boxed{A = B}$

Proper Subsets

If A $\subseteq$ B but A $\neq$ B, meaning B contains an element not contained in A, then A is a proper subset of B.

$\to \quad \forall x (x \in A \to x \in B) \land \exists x (x \in B \land x \notin A)$

Notation: $A \boxed{\subset} B$

$A = \{1, 2, 3\}$

$B = \{1, 2, 3, 4\}$

$A \subseteq B$

$B \nsubseteq A \qquad A \subset B$

$A \neq B$

Cardinality

The number of distinct elements of a set.

Notation: $|A| = 4$ $A = \{1, 2, 3, 3, 3, 4\}$

$|\text{alphabet}| = 26$

$|\phi| = 0$

Power Sets

The set of all subsets of a set.

$A = \{0, 1, 2\}$

Notation P (A) $P(A) = \{\phi, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$

$|P(A)| = 8 \rightarrow$ Cardinaty of powerset of a set with n elements is 2^n

Tuples

An ordered n-tuple is an ordered collection that has a_1 as its first element, a_2 as its second and so on until a_n.

Notation: $(a_1, a_2, \ldots, a_n)$

Ordered pairs (a_1, a_2)

$(5, 2) \neq (2, 5)$

Cartesian Product

The set of ordered pairs (a, b) where a $\in$ A and b $\in$ B, resulting from A x B.

Notation: A x B = $\{(a,b) \mid a \in A,\ b \in B\}$

$A = \{0, 1\}$ $B = \{2, 3, 4\}$

$A \times B = \{(0,2), (0,3), (0,4), (1,2), (1,3), (1,4)\}$

$R = \{(0,2), (1,2)\}$

Truth Sets and Quantifiers

A truth set of P is the set of elements x in D such that P (x) is true.

Notation: $\{x \in D \mid P(x)\}$

$D = \mathbb{Z}$ $P(x)$: $|x| = 3$

$\{-3, 3\}$

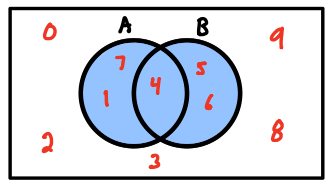
2.2 Operations on Sets

Union of Sets

"or"

Let A, B be sets. The Union of the sets A and B is:

$$A \cup B = \{ x \mid x \in A \vee x \in B \}$$



$A = \{1, 4, 7\}$

$B = \{4, 5, 6\}$

$A \cup B = \{1, 4, 5, 6, 7\}$

$$\boxed{|A \cup B| = |A| + |B| - |A \cap B| \quad \text{Inclusive - Exclusion}}$$

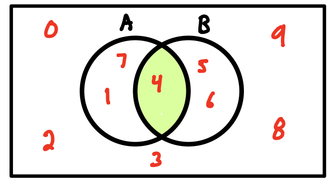
$\{1, 4, 7\} + \{\cancel{4}, 5, 6\} - \{\cancel{4}\}$

$= \{1, 4, 7, 5, 6\}$

Intersection of Sets

"and"

Let A, B be sets. The intersection of A and B is



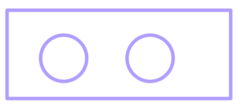
$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

If $A \cap B = \phi$ then A and B are said to be disjoint.

$$\boxed{A \cap B = \{4\}}$$

$A = \{1, 4, 7\}$

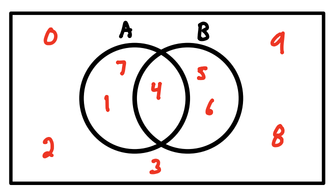
$B = \{4, 5, 6\}$



Complement

Let A be a set. The complement of set A with respective to U is U – A.

$$\bar{A} = A^c = \{x \in U \mid x \notin A\}$$



$\bar{A} = \{0, 2, 3, 5, 6, 8, 9\}$

~ Anything else that's not 1, 4, or 7

$\bar{B} = \{0, 1, 2, 3, 7, 8, 9\}$

$\bar{A} \cup \bar{B} = \{0, 2, 3, 8, 9\}$ - What's not in B

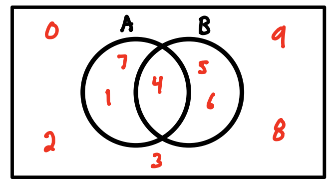
$\bar{A} \cap \bar{B} = \{0, 1, 2, 3, 5, 6, 7, 8, 9\}$ - Whats not in the middle

Difference

Let A and B be sets. The difference of A and B is the set containing elements of A that are not in B.

$$A - B = \{ x \mid x \in A \wedge x \notin B \}$$

$$= A \cap \overline{B}$$



$A - B = \{ 1, 7 \}$

Practice

Let

u = Letter of the alphabet

A = vowels = { a, e, i, o, u }

B = letters in "Discrete Math"

$B = \{ d, i, s, c, r, e, t, m, a, h \}$

$A \cup B = \{ a, e, i, o, u, d, s, c, r, t, m, h \}$

$A - B = \{ o, u \}$

$A \cap B = \{ a, e, i \}$

$B - A = \{ d, s, c, r, t, m, h \}$

$\overline{A \cap B} = \{ b, c, d, f, g, h, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z \}$

Is $A \subseteq B$?

NO !

Set Identities

Set identities

Identity Laws $A \cup \emptyset = A$ $\quad A \cap U = A$

Domination Laws

$$A \cup U = U \qquad A \cap \emptyset = \emptyset$$

Idempotent Laws

$$A \cup A = A \qquad A \cap A = A$$

Complementation law

$$\overline{(\bar{A})} = A$$

Commutative Law - Order

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

Associative Laws - Grouping

$$A \cup (B \cup C) = (A \cup B) \cup C \rightarrow \text{any}$$
$$A \cap (B \cap C) = (A \cap B) \cap C \rightarrow \text{each}$$

Distributive Laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Set Identities

De Morgan's Laws

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

Absorption Laws

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

Complement Laws

$$A \cup \bar{A} = U$$

$$A \cap \bar{A} = \phi$$

Proving Set Identities

Methods of Identity Proof

1) Prove each set in the identity is a subset of the other

2) Use propositional logic ~ 2 Column proof

3) Use a membership table showing the same combinations of sets do or don't belong to the identity.

if $A = B$
$A \subseteq B$
$B \subseteq B$

① Prove $\boxed{\overline{A \cap B}} = \boxed{\bar{A} \cup \bar{B}}$ by showing $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$ and $\boxed{\bar{A} \cup \bar{B} \subseteq A \cap B}$

Prove this

$x \in \overline{A \cap B}$	By assumption	~ like our premise, we need to show if element is in both
$x \notin A \cap B$	Definition of Complement	re-write it differently; "not"
$\neg((x \in A) \wedge (x \in B))$	Definition of intersection	
$\neg(x \in A) \vee \neg(x \in B)$	De Morgan for Prop. Logic	
$x \notin A \vee x \notin B$	Definition of Negation	
$x \in \bar{A} \vee x \in \bar{B}$	Definition of Complement	
$x \in \bar{A} \vee \bar{B}$	Definition of Union	Proved

Proving this

$x \in \bar{A} \cup \bar{B}$	By Assumption	~ x belongs, by assumption
$(x \in \bar{A}) \vee (x \in \bar{B})$	Def of Union	
$x \notin A \vee x \notin B$	Def of Complement	
$\neg(x \in A) \vee \neg(x \in B)$	Def of Negation	
$\neg((x \in A) \wedge (x \in B))$	De Morgan Law Prop. Logic	
$\neg(x \in A \cap B)$	Def of Intersection	
$x \in \overline{A \cap B}$	Def of Complement	

$\therefore \overline{A \cap B} = \bar{A} \cup \bar{B}$ ~ We proved both directions

Prove $A \cap B = \bar{A} \cup \bar{B}$ using set Builder notation and propositional logic.

$$\overline{A \cap B} = \{x \mid x \notin A \cap B\}$$ Def. of Compliment

$$= \{x \mid \neg(x \in A \cap B)\}$$ Def of $\notin$

$$= \{x \mid \neg(x \in A \wedge x \in B)\}$$ Def of intersection

$$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$$ De Morgan Prop Logic

$$= \{x \mid x \notin A \vee x \notin B\}$$ Definition of $\notin$

$$= \{x \mid x \in \bar{A} \vee x \in \bar{B}\}$$ Definition of Compliment

$$= \{x \mid x \in \bar{A} \cup \bar{B}\}$$ Definition of a Union

$$= \bar{A} \cup \bar{B}$$ Proven its True

Prove $A \cap B = \bar{A} \cup \bar{B}$ using a <u>membership table</u>

A	B	$A \cap B$	$\overline{A \cap B}$	$\bar{A}$	$\bar{B}$	$\bar{A} \cup \bar{B}$
1	1	1	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	0	1	1	1	1

Generalizetized Union and Intersection

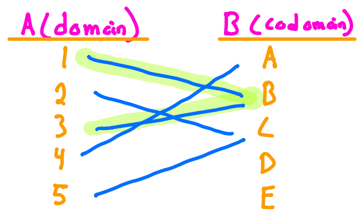
Union of several items

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \dots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \dots \cap A_n$$

2.3 Introduction to Functions

A function f from A to B, denoted $\boxed{f: A \to B}$
assigns each element of A to exactly one element of B.
Functions may also be called mappings or Transformations



$f(1) = B$ then we can say

Image: B is the image of 1 under f
$\{A, B, C\}$ image of f

$\{A, B, C\}$ image of f

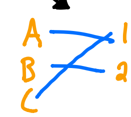
Preimage: $\{1, 3\}$ the preimage of B under f

range: $\{A, B, C\}$

All Values mapped in codomain.

Representing Functions

1. Explicit Statement $f(A) = 1$, $f(B) = 2$, $f(C) = 1$



2. Formula $f(x) = x^2 + 1$

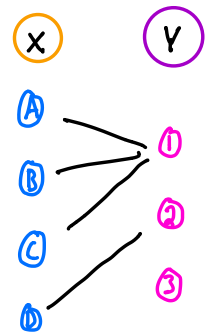
3. Computer Program

4. A relation $A \times B$

(1, 2) (1, 7)
(2, 3) (2, 6)
(4, 5)

Answer the following for $f: X \to Y$

- domain X
- Codomain Y
- range $\{1, 2\}$
- Preimage of 2 $\{D\}$
- Images of A $\{1\}$
- $f(D) = 2$



One to One and Onto Functions

Injective (one-to-one) Functions

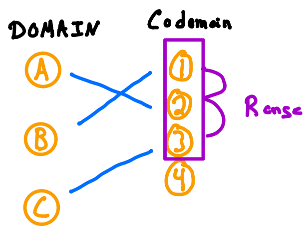
- Each Value in the range corresponding to exactly one element in the domain.

$$\forall a \forall b ((a \neq b) \rightarrow (f(a) \neq f(b)))$$

OR

$$\forall a \forall b ((f(a) = f(b)) \rightarrow (a = b))$$

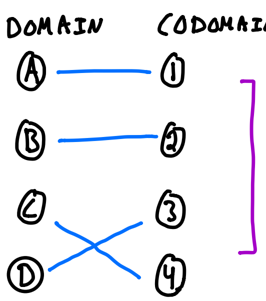
Contrapositive



Surjective (onto) Functions

Every element in the codomain maps to at least one element in the domain

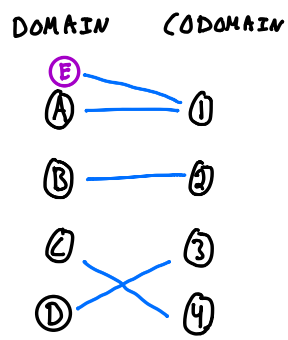
$$\forall y \exists x (f(x) = y)$$



each element is mapped to one element in the domain

Question:

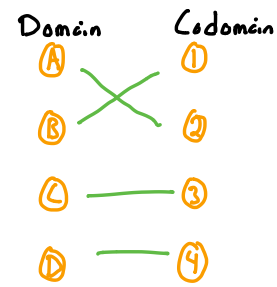
What if we add another Domain Value E that is mapped to a same codomain?



It's still Surjective but not One-to-One anymore.

Bijective (one-to-one Correspondence) Functions

Functions that are both one-to-one and onto, or both surjective and injective.

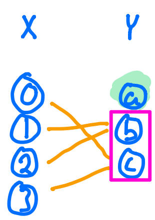


~ only one mapping for codomain

Practice:

Let f be a function from $X = \{0,1,2,3\}$ to $Y = \{a,b,c\}$ defined by $f(0)=c$, $f(1)=b$, $f(2)=b$ and $f(3)=c$. Is $f: X \to Y$ either one-to-one or onto?

mapping:



Not One-to-One

~ b is mapped to both 1 and 2, and c is mapped to both 0 and 3

Since we have a value in the codomain that isn't mapped to anything then it's not onto

Practice:

Is the function $f(x) = x^2$ from the set of integers to the set of integers either one-to-one or onto?

$f: \mathbb{Z} \to \mathbb{Z}$

One-to-one

1 way to show one-to-one: if $f(a) = f(b)$ then $a = b$

$f(a) = 4$ $f(b) = 4$ $a = 2$ $b = -2$

$f(a) = f(b)$ but $a \neq b$

Not one-to-one

Onto

Is the codomain ever going to be negative?

$f(a) \neq -1$

Won't be in the range

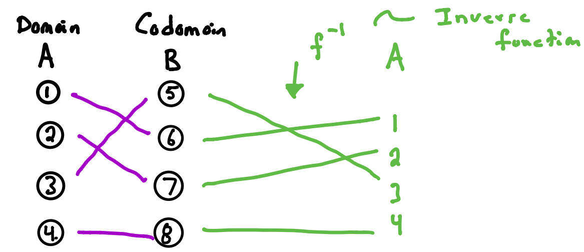
Not onto

Inverse Functions and Composition Functions

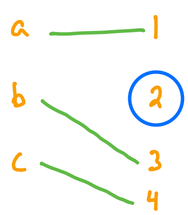
Inverse Functions

Let f be a $\boxed{\text{bijection}}$ (both one-to-one and onto) from set A to B. The inverse of f, f^{-1}, is the function from B to A defined as $f^{-1}(y) = x$ iff $f(x) = y$.

One-to-one, each element is only mapped one time.



If f is the function from $\{a, b, c\}$ to $\{1, 2, 3, 4\}$ such that $f(a) = 1$, $f(b) = 3$ and $f(c) = 4$, is f invertible? If so, what is the inverse?



not onto / not a bijection

not invertible

Let $\boxed{f: \mathbb{Z} \to \mathbb{Z}}$ such that $\boxed{f(x) = x + 3}$. Is f invertible? If so, what is the inverse?

One-to-one? $\boxed{f(a) = f(b)} \to a = b$ ~ Same Value

$7 \qquad\qquad 4$

$f(a) = 7$ $\qquad f(b) = 7$ $\qquad$ Yes

$a + 3 = 7$ $\qquad b + 3 = 7$ $\qquad$ Invertible

$a = 4$ $\qquad b = 4$

Onto? $\qquad$ Codomain $\qquad$ Yes

$4 \to 7$

Everything is going to get mapped to

$5 \to 8$

Inverse: $f(x) = x + 3$

$$x = f^{-1}(x) + 3$$

$$x - 3 = f^{-1}(x)$$

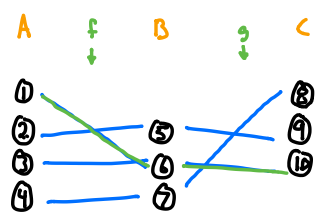
$$\boxed{f^{-1}(x) = x - 3}$$ ~ inverse has to <u>undo</u>

<u>Composition of Functions</u>

Let $f: A \to B$ and $g: B \to C$. The Composition of g with f, $g \circ f$, is the function from A to C defined by $g \circ f(x) = g(f(x))$

$g(f(1)) = 10$ - Work <u>inside out</u>

$g \circ f(1) = 10$



<u>Examples</u>:

If $f(x) = x + 3$ and $g(x) = x^2 - 2$ find:

$f \circ g(1) = f(1^2 - 2) = f(-1) = -1 + 3 = 2$

Other way to solve

$f \circ g(x) = f(x^2 - 2) = (x^2 - 2) + 3 = \boxed{x^2 + 1}$

$f \circ g(1) = 1^2 + 1 = 1 + 1 = 2$ ✓ ~ Checked

$g \circ f(1) = g(1 + 3) = g(4) = 4^2 - 2 = 14$

$g \circ f(x) = g(x + 3) = (x + 3)^2 - 2 = x^2 + 6x + 9 - 2$
$= x^2 + 6x + 7$

$g \circ f(1) = 1^2 + 6(1) + 7 = 14$ ✓ ~ Checked

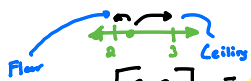
Useful Functions to Know

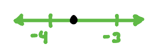
Floor Function $f(x) = \lfloor x \rfloor$

Largest integer less than or equal to x.

Ceiling Function $f(x) = \lceil x \rceil$

Smallest integer greater than or equal to x.





$\lceil 2.2 \rceil = 3$ $\qquad$ $\lceil -3.7 \rceil = -3$

$\lfloor 2.2 \rfloor = 2$ $\qquad$ $\lfloor -3.7 \rfloor = -4$

Factorial Function

$f: \mathbb{N} \to \mathbb{Z}^+$ denoted by $f(n) = n!$ is the product of the first n positive integers when n is a non-negative integer.

$f(n) = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$

$f(4) = 4 \cdot 3 \cdot 2 \cdot 1$

Stirling formula: $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

Question:

Let $f(x) = \lfloor \frac{x^2}{2} \rfloor$. Find $f(S)$ if $S = \{0, 1, 2, 3\}$

$f(0) = \lfloor \frac{0^2}{2} \rfloor = \lfloor 0 \rfloor = 0$

$f(1) = \lfloor \frac{1^2}{2} \rfloor = \lfloor \frac{1}{2} \rfloor = 0$

$f(2) = \lfloor \frac{2^2}{2} \rfloor = \lfloor 2 \rfloor = 2$

$f(3) = \lfloor \frac{3^2}{2} \rfloor = \lfloor \frac{9}{2} \rfloor = 4$

2.4 Introduction to Sequences

Section 2.4

Sequences

An ordered list of elements created by a function mapping the integers to a set S.

Notation: a_n represents the n^{th} term

Ex. If $\boxed{a_n = 2n}$, find $\{a_n\} = \{0, 2, 4, 6 \ldots\}$

$a_0 = 2(0) = 0$
$a_1 = 2(1) = 2$
$a_2 = 2(2) = 4$
$a_3 = 2(3) = 6$

~ Pattern would continue.

Arithmetic Sequence

Arithmetic Sequence → $\boxed{a_n = a + dn}$ ★ ~ How you write an Arithmetic Sequence Explicitly

A sequence formed by adding the initial term, a, and the product of the common difference, d, and the term number, n.

$$\{a_n\} = \boxed{a_1} + a \underbrace{}_{+d} + d, \; a + 2d, \; a + 3d, \ldots \; a + nd$$

Questions:

1. Let $a = 5$ and $d = 2$. Find the first 5 terms of $\{a_n\}$

Written Explicitly → $\boxed{a_n = 5 + 2n}$

Written recursively

$\{5, 7, 9, 11, 13 \ldots\}$ (+2, +2)
$a_0 \; a_1 \; a_2 \; a_3 \; a_4$

$a_4 = 5 + 2(4)$
$a_4 = 13$ ✓

2. Find a and d for the sequence $\{7, 4, 1, -2, \ldots\}$

$\{7, 4, 1, -2, \ldots\}$ (-3, -3, -3)
$a_0 \; a_1 \; a_2 \; a_3$

$a_n = 7 + dn$ → $\boxed{a_n = 7 - 3n}$

must find the common difference, which we can see is -3.

Lets check:
$a_3 = 7 - 3(3)$
$a_3 = -2$ ✓

Geometric Sequence

$\boxed{a_n = ar^n}$ ← Geometric Sequence Explicitly

A sequence formed by multiplying the initial term, a, by the common ratio r to the nth power, r^n.

$$\{a_n\} = a,\ ar,\ ar^2,\ ar^3,\ \ldots,\ ar^n$$

Questions:

1. Let $a=4$ and $r=3$. Find the first 5 terms of the geometric progression.

$a_n = 4(3)^n$

One way to do problem → $\{\underset{a_0}{4}, \underset{a_1}{12}, \underset{a_2}{36}, \underset{a_3}{108}, \underset{a_4}{324}\ldots\}$

$a_3 = 4(3)^3 = 4(27)$

$a_3 = 108$ ✓

2. Find a and r for $\{a_n\} = \{3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \ldots\}$

$a_n = 3r^n$

$\{a_n\} = \{3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \ldots\}$

~ divide by previous term

$\boxed{a_n = 3(\tfrac{1}{2})^n}$

$a = 3$
$r = \frac{1}{2}$

$$\frac{\frac{3}{2}}{3} = \frac{\cancel{3}}{2}\cdot\frac{1}{\cancel{3}} = \frac{1}{2} = r$$

Recurrence Relations

Recurrence Relation

An equation that expresses a_n in terms of one or more the previous terms of the sequence. Initial conditions are required to specify terms that precede the first term where the relation takes effect.

Example: $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 2$ where $\boxed{\begin{matrix} a_0 = 2 \\ a_1 = 5 \end{matrix}}$

$\{2, 5, 9, 19, 37\}$ When does EQ start

$a_2 = a_1 + 2a_0 = 5 + 2(2) = 5+4 = 9$

$a_3 = a_2 + 2a_1 = 9 + 2(5) = 9 + 10 = 19$

$a_4 = a_3 + 2(a_2) = 19 + 2(9) = 19 + 18 = 37$

Questions

Let a_n be the sequence that satisfies the recurrence relation $\boxed{a_n = a_{n-1} + 6 \text{ for } n \geq 1}$

a.) If $a_0 = 3$ Find a_1, a_2, a_3.

$a_1 = a_0 + 6 = 3 + 6 = 9$ +6

$a_2 = a_1 + 6 = 9 + 6 = 15$ +6

$a_3 = a_2 + 6 = 15 + 6 = 21$

$a_n = a + dn$

$\boxed{a_n = 3 + 6n}$ *

If you wanted to find a_{100}

$a_{100} = 3 + 6(100) = 3 + 600$

$a_{100} = 603$

b.) If $a_0 = -7$ Find a_1, a_2, a_3.

$a_1 = a_0 + 6 = -7 + 6 = -1$ +6

$a_2 = a_1 + 6 = -1 + 6 = 5$ +6

$a_3 = a_2 + 6 = 5 + 6 = 11$

$\boxed{a_n = -7 + 6n}$

Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ..

Expressed as a recurrence relation

2 $\begin{cases} f_0 = 0 \\ f_1 = 1 \end{cases}$

1 $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$

Solving Recurrence Relations

Finding a non-recursive formula to calculate a_n is called solving the recurrence relation. The solution is called a closed formula.

One Method for solving is called iteration involving substitution.

Let a_n be the sequence that satisfied the recurrence relation $a_n = a_{n-1} + 3$ for $n \geq 1$ with $a_0 = 2$.

$\boxed{a_n = 2 + 3n}$

$a_0 = 2$

$a_1 = 2 + 3$

iteration

$a_2 = (2+3) + 3 = 2 + 2(3)$

$a_3 = (2+3+3) + 3 = 2 + 3(3)$

$a_4 = 2 + 4(3)$

$$\boxed{a_n = 2 + 3n}$$

Summations and Sigma Notation

To express the sum of the terms of the sequence $a_n = \{a_m, a_{m+1}, \ldots a_n\}$, we write:

$$\sum_{i=m}^{n} a_i = \sum_{i=m}^{n} a_i = \sum_{m \le i \le n} a_i$$

upper limit

Limits

Lower Limit

index

Which represents $a_m + a_{m+1} + a_{m+2} + \ldots + a_n$

Use sigma notation to express the sum the first 100 terms of the sequence a_i where $a_i = \frac{1}{i}$ for i = 1,2,......

$a_1 = \frac{1}{1}$

$a_2 = \frac{1}{2}$

$a_3 = \frac{1}{3}$

$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{100}$

$$\boxed{\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \frac{1}{100}}$$

$$\boxed{\sum_{i=1}^{100} \frac{1}{i}} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} +$$

Question

What is the Value of $\sum_{i=5}^{9} i^2$?

$$\sum_{i=5}^{9} i^2 = 5^2 + 6^2 + 7^2 + 8^2 + 9^2 = 25 + 36 + 49 + 64 + 81 = \boxed{255}$$

What is the Value of $\sum_{i=7}^{10} (-1)^i$?

$$\sum_{i=7}^{10} (-1)^i = (-1)^7 + (-1)^8 + (-1)^9 + (-1)^{10} = -1 + 1 + -1 + 1 = \boxed{0}$$

Summations Properties and Formulas

Basic Properties

Constant   move to outside

$$\sum_{k=1}^{n} c a_k = c \sum_{k=1}^{n} a_k$$

$$* \quad \sum_{k=1}^{n} c = cn \qquad \overbrace{c + c + c + \dots + c}^{n}$$

~ if you have just a constant

$$\sum_{k=1}^{n} (a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$$

Two different summation

$$\sum_{k=1}^{n} 2k + 3 = \sum_{k=1}^{n} 2k$$

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{j} a_k + \sum_{k=j+1}^{n} a_k \quad \text{for} \quad 1 \le j < j+1 < n \quad \text{and } j \in \mathbb{N}$$

~ lower   upper

$$* \quad \sum_{k=1}^{n} a_k = \sum_{k=1+m}^{n+m} a_{k-m} \quad \text{for all } r \in \mathbb{N}$$

might have to "+" or "-" m

~ Complicated

Summation Formula:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Practice:

$$\sum_{i=1}^{7} i = 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$$

Summation Formula:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Summation Formula:

$$\sum_{k=0}^{n} ar^k = \frac{ar^{n+1} - a}{r-1}$$

~ Geometric Progression / Sequence

$a_n = 0$

Example:

$$\sum_{k=0}^{7} \left(2 \cdot 3^k + 5 \cdot 2^k\right) = \sum_{k=0}^{7} \overset{a}{2} \cdot \overset{r}{3}^k + \sum_{k=0}^{7} 5 \cdot 2^k$$

$$= \frac{2(3)^8 - 2}{2} + \frac{5(2)^8 - 5}{1}$$

$$= \frac{13122 - 2}{2} + 1275$$

$$= 6560 + 1275 = \boxed{7835}$$

Useful Summation Formula:

$$\sum_{k=0}^{n} ar^k = \frac{ar^{n+1} - a}{r-1}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

2.5 Cardinality of Sets

Definitions

The sets A and B have the same cardinality if and only if there is a one-to-one correspondence from A to B. When A and B have the same cardinality, we write $|A| = |B|$.

If there is a one-to-one function from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$. Moreover, when $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and we write $|A| < |B|$.

A set that is either finite or has the same cardinality as the set of positive integers is called countable. A set that is not countable is called uncountable. When an infinite set S is countable, we denote the cardinality of S by 0א (where א is aleph, the first letter of the Hebrew alphabet). We write |S| = 0א and say that S has cardinality "aleph null."

An Uncountable Set

If A and B are countable sets, then $A \cup B$ is also countable.

2.6 Matrices and Matrix Operations

Matrices

A rectangular array of numbers. A matrix with m rows and n columns is called an m x n matrix.

$$M_{3\times 2} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{32} \end{bmatrix}$$

- 3 rows
- 2 columns

element address m_{11}

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 4 & \boxed{5} \end{bmatrix}$$

dimension 2×3

$n_{23} = 5$

Matrix Addition (and Subtraction)

Matrices of the same size can be added by finding the sum of the elements of the same address.

$$\underset{2\times 2}{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \underset{2\times 2}{B} = \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix} \qquad \cancel{\underset{2\times 3}{C} = \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}}$$

- Can't add/subtract
- not same size !

$$A + B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 9 & 11 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix}$$

Matrix Multiplication

Matrices M and N can be multiplied if M is a m x k matrix and N is a k x n matrix. The resulting product matrix will be m x n matrix.

$$A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 3 & -2 \\ 5 & -1 & 0 \end{bmatrix}$$

- Can't always multiply

$$\underset{2\times 2 \quad 2\times 3}{AB} = {}^{R_1}\begin{bmatrix} -1 & 3 \\ 4 & -2 \end{bmatrix} \cdot \begin{array}{cc} -1 & 4 \\ 3 & -2 \end{array} \overset{C_1}{\begin{bmatrix} 4 & 3 & -2 \\ 5 & -1 & 0 \end{bmatrix}} = \begin{bmatrix} -4+15 & -3+3 & 2+0 \\ 16+-10 & 12+2 & -8+0 \end{bmatrix} = \begin{bmatrix} 11 & -6 & 2 \\ 6 & 14 & -8 \end{bmatrix}$$

$$\underset{2\times \boxed{\cancel{3 \quad 2}}\times 2}{BA} = \text{Not Possible!}$$

Examples:

Let $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -7 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 8 \\ 7 & 6 \end{bmatrix}$

$A - B = NP$ ~dimension are not possible

$\underset{2\times \boxed{3 \quad 2}\times 2}{AB} = NP$

$$\underset{2\times \boxed{2 \quad 2}\times 3}{BA} = \begin{bmatrix} -2 & 8 \\ 7 & 6 \end{bmatrix} \begin{array}{cc} -2 & 7 \\ 8 & 6 \end{array} \begin{bmatrix} 0 & 1 & -1 \\ 4 & -7 & 3 \end{bmatrix} = \begin{bmatrix} 0+32 & -2+56 & 2+24 \\ 0+24 & 7+-42 & -7+18 \end{bmatrix} = \begin{bmatrix} 32 & 54 & 26 \\ 24 & -35 & 11 \end{bmatrix}$$

possible

$\underset{2\times \boxed{3 \quad 2}\times 3}{A^2} = NP$

$$\underset{2\times \boxed{2 \quad 2}\times 2}{B^2} = \begin{bmatrix} -2 & 8 \\ 7 & 6 \end{bmatrix} \begin{array}{cc} -2 & 7 \\ 8 & 6 \end{array} \begin{bmatrix} -2 & 8 \\ 7 & 6 \end{bmatrix} = \begin{bmatrix} 4+56 & -16+48 \\ -14+42 & 56+36 \end{bmatrix} = \begin{bmatrix} 60 & 32 \\ 28 & 92 \end{bmatrix}$$

$$2B = 2\begin{bmatrix} -2 & 8 \\ 7 & 6 \end{bmatrix} = \begin{bmatrix} -4 & 16 \\ 14 & 12 \end{bmatrix}$$

Identity Matrix

- The Idenity Metrix is a square metrix $I = \delta_{ij}$

  Where: $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$

(first diagonal is 1's and all other elements are 0)

Example: Find AI where $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -2 & 7 \\ 0 & 1 & -1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & -2 & 7 \\ 0 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -2 & 7 \\ 0 & 1 & -1 \end{bmatrix}$$

Exact same metrix as we had before

Transpose of Metrices

The transpose of $A_{m \times n}$ denoted A^t is the $n \times m$ matrix obtained by interchanging the rows and columns of A

If $A = A^t$ the A is Symmetric

$$\underset{2\times3}{A} = \begin{bmatrix} 4 & 5 & 7 \\ 0 & 1 & -1 \end{bmatrix} \qquad A^t = \begin{bmatrix} 4 & 0 \\ 5 & 1 \\ 7 & -1 \end{bmatrix}$$

not symmetric

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A^t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

~ Symmetric

Zero – One Matrices

A matrix all of whose entries are either 0 or 1 is called a zero-one matrix. Algorithms using these structures are based on Boolean arithmetic with zero-one matrices

$$b_1 \wedge b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$b_1 \vee b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

Example:

Find the join and meet of the zero-one matrices

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

We find the join of A and B is:

$$A \vee B = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The meet of A and B is:

$$A \wedge B = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

We defined the <u>Boolean Product</u> of the two matrices