Commonality Analysis for a Library of Linear Algebraic Equation Solver

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Contents

Table 1: **Revision History**

Date	**Version**	**Notes**
October 4, 2017	1.0	Initial Draft

1 Reference Material

1.1 Table of Units

This section does not apply to this program family.

1.2 Table of Symbols

The table that follows summarizes the symbols used in this document along with their units. The choice of symbols was made to be consistent with the heat transfer literature and with existing documentation for solar water heating systems. The symbols are listed in alphabetical order.

symbol	**unit**	**description**
A	-	known $m \times n$ matrix
x	-	n-vector
b	-	m-vector
M_1	-	elementary elimination matrix
L	-	lower triangular matrix
U	-	upper triangular matrix
A^{-1}	-	inverse of matrix
I	-	identity matrix
$\|A\|$	-	determinant of matrix

1.3 Abbreviations and Acronyms

symbol	description
A	Assumption
DD	Data Definition
GD	General Definition
GS	Goal Statement
IM	Instance Model
LC	Likely Change
PS	Physical System Description
R	Requirement
SRS	Software Requirements Specification
T	Theoretical Model
O	Output

2 Introduction

Many of the relationships in nature are linear, that means their effects are proportional to their causes. For example in Mechanics if we take Newtons second law of motion, $F = ma$ says that force is proportional to acceleration and mass is the proportionality constant. For example in Electricity if we take Ohm's Law, $V = iR$ voltage across the conductor is proportional to the current flowing though it and resistance is the proportionality constant. These examples shows us the importance of linear equations and solving them. In matrix notation the general form of linear equation is

$$Ax = b$$

where A is known $m \times n$ matrix, b is an m-vector and x is an n-vector. If x is known, then such a linear relationship enables us to predict effect b from cause x by matrix-vector multiplication $b = Ax$.

The most important problem in technical computing is the solution of system linear equations.

2.1 Purpose of Document

The purpose of the document is to describe the methods and process for solving the family of linear algebraic equations. A study on the method for solution of system of linear equations and the problems effecting the effectiveness and efficiency will be detected and correct recommendations will be made. The significance of this document is to make a solution technique for solving family of linear equations easier and to reduce stress on the human brain associated with lots of reasoning when solving linear equation.

This document also describes General System Description, Commonalities, Variabilities.

2.2 Scope of the Family

The scope of the family is limited to the library of linear algebraic equation solvers. If the user gives proper input the linear algebraic equation solver aims to solve the equations and give the correct output.

2.3 Characteristics of Intended Reader

The intended readers who read this document, must have basic knowledge about linear algebraic equations and the different methods of solving linear algebraic equations, which are typically covered in first and second year Linear Algebra courses .

2.4 Organization of Document

The template for Commonality Analysis for scientific computing software is proposed by Smith (2006). The document is organized perfectly step by step right from the introduction to the appendix by explaining the important concepts like General System Description, Commonalities, Variabilities.

3 General System Description

This section identifies the interfaces between the system and its environment, describes the user characteristics and lists the system constraints.

3.1 System Context

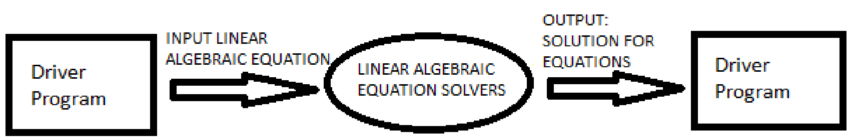

Figure 1: System Context

The figure shows the system context. The user gives the input in the form of linear equations to the solver and the solver aims to give the solution to the equations as the output.

- User Responsibilities:
  - The main responsibility of the user is to give the appropriate input.
  - User must make sure that the input given is error free.
  - User must declare the method by which the linear system must be solved.
- Linear Algebraic Equation Solver Responsibilities:
  - The main responsibility of the linear algebraic equation solver is to identify the input which is improper or invalid.
  - Detect data type mismatch, such as a string of characters instead of a floating point number.
  - the ultimate responsibility of the linear algebraic equation solver is to solve the equations and to produce the output.

3.2 User Characteristics

The end user of Linear Algebraic Equation Solver should have an understanding of undergraduate Level of Linear algebra 1 .

3.3 System Constraints

There are no system constraints applicable.

4 Commonalities

This section first presents the problem description, which gives a high-level view of the problem to be solved. This is followed by the Terminology and Definitions, Data Definitions, Goal Statements and Theoretical Models.

4.1 Problem Description

Linear Algebraic Equation Solver is a software library which is developed to solve linear algebraic equations using numerical methods for linear algebraic equation . This solver is used to solve the linear algebraic equations with different number of variables. As many relationships in nature are linear, it will be very handy for students to solve linear algebraic equations.

4.2 Terminology and Definitions

This subsection provides a list of terms that are used in the subsequent sections and their meaning, with the purpose of reducing ambiguity and making it easier to correctly understand the requirements:

- A^{-1} is the inverse of the matrix A
- rank(A) = The rank of a matrix is the maximum number of linearly independent rows or columns it contains.
- det(A) is determinant of matrix A
- square matrix = A matrix is said to be a square matrix if and only if it has the same number of rows and columns.
- singular matrix = A square matrix is said to be a singular matrix if it's determinant is "0".
- upper triangular matrix = A square matrix is said to be upper triangular matrix if and only if all the entries below the main diagonal are zero.
- lower triangular matrix = A square matrix is said to be lower triangular matrix if and only if all the entries above the main diagonal are zero.

4.3 Data Definitions

Number	DD1
Label	**Matrix Representation of the Linear Algebraic Equation**
Symbol	$Ax = b$
Equation	$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1m}x_m = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2m}x_m = b_2 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + ... + a_{nm}x_m = b_n \end{cases}$$
Description	A is a known m x n matrix that is $\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nm} \end{bmatrix}$. b is an m-vector that is $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$. x is an n-vector that is $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$.
Source	Heath (2002)
Ref. By	[IM1], [IM2], [T1], [A2]

Number	DD2
Label	**Identity Matrix**
Symbol	I
Equation	$AI = IA = A$
Description	A is a known $n\ x\ n$ matrix. I is an Identity matrix . $I_1 = \begin{bmatrix}1\end{bmatrix}, I_2 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, I_3 = \begin{bmatrix}1,0,0\\ 0,1,0\\ 0,0,1\end{bmatrix}, I_n = \begin{bmatrix}1 & 0 & 0 & \dots & 0\\ 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 1\end{bmatrix}.$
Source	https://www.colorado.edu/engineering/Aerospace/CAS/courses.d/IFEM.d/IFEM.AppD.d/IFEM.AppD.pdf
Ref. By	[IM1], [IM2], [DD3], [A4]

Number	DD3
Label	**Inverse of a Matrix**
Symbol	A^{-1}
Equation	$AA^{-1} = A^{-1}A = I$
Description	A is a known $m\ x\ n$ matrix. A^{-1} is the inverse of matrix A . I is an Identity matrix .
Source	https://www.colorado.edu/engineering/Aerospace/CAS/courses.d/IFEM.d/IFEM.AppD.d/IFEM.AppD.pdf
Ref. By	[IM1], [IM2], [DD4], [T1], [A4]

Number	DD4
Label	**Determinant of a Matrix**
Symbol	$\|A\|$
Equation	$$\|A\| = \sum_{i=1}^{k} a_{ij}C_{ij}$$
Description	$\|A\|$ is the determinant of matrix **A**. $\mathbf{C}_{ij}$ is the cofactor of a_{ij} defined by C_{ij} = $(-1)^{i+j}M_{ij}$. M_{ij} is the minor of matrix A formed by eliminating row i and column j from A.
Source	http://mathworld.wolfram.com/Determinant.html
Ref. By	[IM1], [IM2], [DD3], [T1], [A4]

Number	DD5
Label	**Rank of a Matrix**
Symbol	**rank(A)**
Equation	**rank(A)** $\leq min(m, n)$
Description	rank(A) is the rank of matrix A. The rank of a matrix can be defined as the maximum number of linearly independent column vectors in the matrix or the maximum number of linearly independent row vectors in the matrix . The rank of an m $\times$ n matrix is a nonnegative integer and cannot be greater than either m or n .
Source	https://www.cliffsnotes.com/study-guides/algebra/linear-algebra/real-euclidean-vector-spaces/the-rank-of-a-matrix
Ref. By	[IM1], [IM2], [T1],

Number	DD6
Label	**Upper Triangular Matrix**
Symbol	U
Equation	$\begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ & u_{2,2} & u_{2,3} & \cdots & u_{2,n} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & u_{n-1,n} \\ 0 & & & & u_{n,n} \end{bmatrix}.$
Description	U is a upper triangular matrix matrix. A square matrix is said to be upper triangular matrix if all the entries below the main diagonal are Zero . The sum and product of two upper triangular matrices is always a upper triangular matrix.
Source	http://mathworld.wolfram.com/UpperTriangularMatrix.html
Ref. By	[IM1], [IM2], [T2], [T3],

Number	DD7
Label	**Lower Triangular Matrix**
Symbol	L
Equation	$\begin{bmatrix} l_{1,1} & & & & 0 \\ l_{2,1} & l_{2,2} & & & \\ l_{3,1} & l_{3,2} & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ l_{n,1} & l_{n,2} & \dots & l_{n,n-1} & l_{n,n} \end{bmatrix}.$
Description	L is a lower triangular matrix. A square matrix is said to be a lower triangular matrix if all the entries above the main diagonal are zero.
Source	http://mathworld.wolfram.com/LowerTriangularMatrix.html
Ref. By	[IM1], [IM2], [T2], [T3]

4.4 Goal Statements

GS1: Given the general linear algebraic equation problem which is represented by $\mathbf{Ax} = \mathbf{b}$. The initial values are A and B. The input which are the coefficients, are entered in a matrix form. If the vector b of effects is known then it would likely able to determine the corresponding vector x of cause.

4.5 Theoretical Models

This section focuses on the general equations and laws that Linear Algebraic Equation Solver is based on.

Number	T1
Label	**General Linear Algebraic Equation Existence and Uniqueness**
Equation	$Ax = b$ An $n \times n$ matrix A is said to be nonsingular if it satisfies any of the following equivalent conditions: 1. A has an inverse (there is a matrix, denoted by A^{-1}, such that AA^{-1} $= A^{-1}A = I$, the identity matrix). 2. Determinant of (A) is not equals to 0 3. rank$(A) = n$ (the rank of a matrix is the maximum number of linearly independent rows or columns it contains).
Description	• A linear transformation between two finite dimensional vector spaces is represented by a matrix. In matrix-vector notation, a system of linear algebraic equations has the form $Ax = b$ where A is known $n \times n$ square matrix, b is an m-vector and x is an n-vector. • The existence of a solution to a system of linear equations $Ax = b$ depend on whether the matrix A is singular or nonsingular. If the matrix A is nonsingular, then its inverse A^{-1} exists, and the system $Ax = b$ always has a unique solution $x = A^{-1}b$ regardless of the value for b. If the matrix A is singular, then the number of solutions is determined by the right hand side vector b.
Source	Heath (2002)
Ref. By	[IM1], [IM2], [DD1], [DD2], [DD3], [DD4], [DD5], [A2], [A4]

Number	T2
Label	**Row Echelon Form**
Equation	$\begin{bmatrix} 1 & * & * & \dots & * \\ 0 & 1 & * & \dots & * \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & * \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$ Row echelon form of $n \times n$ matrix.
Description	The $n \times n$ matrix is said to be in row echelon form if it follows the following condition: • The first non-zero element in each row is called the leading element and it should be 1. • Each leading entry which is in a column must be to the right of the leading entry in the previous row. • If there are any rows with all zero elements, they must be below the rows with non-zero elements.
Source	http://stattrek.com/matrix-algebra/echelon-form.aspx
Ref. By	[IM1], [IM2], [DD6], [DD7]

Number	T3
Label	**Reduced Row Echelon Form**
Equation	$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$ Reduced row echelon form of $n \times n$ matrix.
Description	The $n \times n$ matrix is said to be in reduce row echelon form if it follows the following condition: • It should follow all the conditions of row echelon form. • The leading entry in each row must be the only non-zero entry in its column.
Source	http://stattrek.com/matrix-algebra/echelon-form.aspx
Ref. By	[IM1], [IM2], [DD6], [DD7]

5 Variabilities

The instance models that govern Linear Algebraic Equation Solver are presented in Subsection 4.2.5. The information to understand the meaning of the instance models and their derivation is also presented, so that the instance models can be verified.

5.1 Instance Models

This section transforms the problem defined in Section 4.1 into one which is expressed in mathematical terms.

Number	IM1
Label	**Gaussian Elimination Method for solving Linear Algebraic Equations**
Input	A, b
Output	x
Description	• A is a known $n \times n$ matrix. • $x$is an n-vector. • b is an m-vector. • In Gaussian elimination method the equations are solved by reducing the given matrix into row echelon form.
Sources	Heath (2002)
Ref. By	[A1], [A2], [A4], [T1], [T2]

Derivation of Gaussian Elimination Method

The derivation for Gaussian elimination can be referred at Heath (2002)

Number	IM2
Label	**Gauss-Jordan Elimination Method for solving Linear Algebraic Equations**
Input	A, b
Output	x
Description	• In Gauss-Jordan elimination method the equations are solved by reducing the given matrix into reduced row echelon form. • The algorithm of Gauss-Jordan transforms matrix A by means of elementary transformations into a diagonal matrix (or into the identity matrix), and performs similar transformations to the right-hand side vector b in order to find the solution vector x.
Sources	Dekker et al. (1994)
Ref. By	[A1], [A2], [A4], [T1],[T3]

Derivation of Gauss-Jordan Elimination Method

The derivation for Gauss-Jordan elimination can be referred at Dekker et al. (1994)

5.2 Assumptions

This section simplifies the original problem and helps in developing the theoretical model by filling in the missing information for the physical system. The numbers given in the square brackets refer to the theoretical model [T], general definition [GD], data definition [DD], instance model [IM], or likely change [LC], in which the respective assumption is used.

A1: It is assumed that user will declare the method by which the linear algebraic equation is solved.
[IM1], [IM2], [GS1].

A2: It is assumed that the entered input matrix A is a square matrix.
[T1], [IM1], [IM2]

A3: It is assumed that the user gives the input as a sequence of numbers.
[IM1], [IM2], [GS1]]

A4: It is assumed that the entered matrix A is non-singular.
[IM1], [IM2], [A1], [DD3], [GS1]

A5: It is assumed that the user gives the input values for the matrices A, b separately. [IM1], [IM2], [GS1]

5.3 Calculation

C1: Perform the linear algebraic solver model when the user calls the program. [GS1], [A1], [A2], [A3], [A4], [A5], [IM1], [IM2]

5.4 Output

Not Applicable

6 Traceability Matrices and Graphs

The purpose of the traceability matrices is to provide easy references on what has to be additionally modified if a certain component is changed. Every time a component is changed, the items in the column of that component that are marked with an "X" should be modified as well. Table 2 shows the dependencies of theoretical models, data definitions, and instance models with each other. Table 3 shows the dependencies of theoretical models, data definitions, instance models, and likely changes on the assumptions..

	DD1	DD2	DD3	DD4	DD5	DD6	DD7	T1	T2	T3	IM1	IM2
DD1								X			X	X
DD2			X								X	X
DD3				X				X			X	X
DD4			X					X			X	X
DD5								X			X	X
DD6									X	X	X	X
DD7									X	X	X	X
T1	X	X	X	X	X						X	X
T2						X	X				X	X
T3						X	X				X	X
IM1									X			
IM2										X	X	

Table 2: Traceability Matrix Showing the Connections Between Items of Different Sections

	A1	A2	A3	A4	A5	C1
GS1	X	X	X	X	X	X
DD1		X				
DD2				X		
DD3				X		
DD4				X		
DD5		X				
DD6						
DD7						
T1		X		X		
T2						
T3						
IM1	X	X		X		
IM2	X	X		X		

Table 3: Traceability Matrix Showing the Connections Between Assumptions and Other Items

References

Theodorus Jozef Dekker, W Hoffmann, and K Potma. Parallel algorithms for solving large linear systems. *Journal of Computational and Applied Mathematics*, 50(1-3):221–232, 1994.

Michael T Heath. *Scientific computing*. McGraw-Hill New York, 2002.

Spencer Smith. Systematic development of requirements documentation for general purpose scientific computing software. In *Requirements Engineering, 14th IEEE International Conference*, pages 209–218. IEEE, 2006.