# Commonality Analysis for a Library of Linear Algebraic Equation Solver

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# Contents



Table 1: **Revision History**

| **Date** | **Version** | **Notes** |
|---|---|---|
| October 4, 2017 | 1.0 | Initial Draft |

# 1 Reference Material

## 1.1 Table of Units

This section does not apply to this program family.

## 1.2 Table of Symbols

The table that follows summarizes the symbols used in this document along with their units. The choice of symbols was made to be consistent with the heat transfer literature and with existing documentation for solar water heating systems. The symbols are listed in alphabetical order.

| **symbol** | **unit** | **description** |
|---|---|---|
| $A$ | - | known $m$ $x$ $n$ matrix |
| $x$ | - | $n$-vector |
| $b$ | - | $m$-vector |
| $\mathrm{M}_1$ | - | elementary elimination matrix |
| $L$ | - | lower triangular matrix |
| $U$ | - | upper triangular matrix |
| $\mathrm{A}^{-1}$ | - | inverse of matrix |
| $I$ | - | identity matrix |
| $\|A\|$ | - | determinant of matrix |

## 1.3 Abbreviations and Acronyms

| symbol | description |
|---|---|
| A | Assumption |
| DD | Data Definition |
| GD | General Definition |
| GS | Goal Statement |
| IM | Instance Model |
| LC | Likely Change |
| PS | Physical System Description |
| R | Requirement |
| SRS | Software Requirements Specification |
| T | Theoretical Model |
| O | Output |

# 2 Introduction

Many of the relationships in nature are linear, that means their effects are proportional to their causes. For example in Mechanics if we take Newtons second law of motion, $\mathbf{F} = \mathbf{ma}$ says that force is proportional to acceleration and mass is the proportionality constant. For example in Electricity if we take Ohm's Law, $\mathbf{V} = \mathbf{iR}$ voltage across the conductor is proportional to the current flowing though it and resistance is the proportionality constant. These examples shows us the importance of linear equations and solving them. In matrix notation the general form of linear equation is

$$\mathbf{Ax} = \mathbf{b}$$

where $\mathbf{A}$ is known $m\ x\ n$ matrix, $\mathbf{b}$ is an $m$-vector and $\mathbf{x}$ is an $n$-vector. If $\mathbf{x}$ is known, then such a linear relationship enables us to predict effect $\mathbf{b}$ from cause $\mathbf{x}$ by matrix-vector multiplication $\mathbf{b} = \mathbf{Ax}$.

The most important problem in technical computing is the solution of system linear equations.

## 2.1 Purpose of Document

The purpose of the document is to describe the methods and process for solving the family of linear algebraic equations. A study on the method for solution of system of linear equations and the problems effecting the effectiveness and efficiency will be detected and correct recommendations will be made. The significance of this document is to make a solution technique for solving family of linear equations easier and to reduce stress on the human brain associated with lots of reasoning when solving linear equation.

This document also describes General System Description, Commonalities, Variabilities.

## 2.2 Scope of the Family

The scope of the family is limited to the library of linear algebraic equation solvers. If the user gives proper input the linear algebraic equation solver aims to solve the equations and give the correct output.

## 2.3 Characteristics of Intended Reader

The intended readers who read this document must have basic knowledge about linear algebraic equations and the different methods of solving linear algebraic equations which are typically covered in first and second year Linear Algebra courses .

## 2.4 Organization of Document

The template for Commonality Analysis for scientific computing software is proposed by Smith (2006). The document is organized perfectly step by step right from the introduction to the appendix by explaining the important concepts like General System Description, Commonalities, Variabilities.

# 3 General System Description

This section identifies the interfaces between the system and its environment, describes the user characteristics and lists the system constraints.

## 3.1 System Context

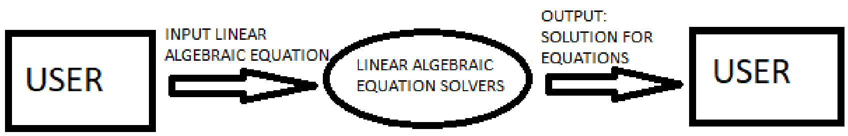


Figure 1: System Context

The figure shows the system context. The user gives the input in the form of linear equations to the solver and the solver aims to give the solution to the equations as the output.

- User Responsibilities:
  - The main responsibility of the user is to give the appropriate input.
  - User must make sure that the input given is error free.
  - User must enter the values in matrix form.
- Linear Algebraic Equation Solver Responsibilities:
  - The main responsibility of the linear algebraic equation solver is to identify the input which is improper or invalid.
  - Detect data type mismatch, such as a string of characters instead of a floating point number.
  - the ultimate responsibility of the linear algebraic equation solver is to solve the equations and to produce the output.

## 3.2 User Characteristics

The end user of Linear Algebraic Equation Solver should have an understanding of undergraduate Level of linear algebra.

## 3.3 System Constraints

There are no system constraints applicable.

# 4 Commonalities

This section first presents the problem description, which gives a high-level view of the problem to be solved. This is followed by the Terminology and Definitions, Data Definitions, Goal Statements and Theoretical Models.

## 4.1 Problem Description

Linear Algebraic Equation Solver is a software library which is developed to solve linear algebraic equations. As many relationships in nature are linear, it will be very handy for students to solve linear algebraic equations.

## 4.2 Terminology and Definitions

This subsection provides a list of terms that are used in the subsequent sections and their meaning, with the purpose of reducing ambiguity and making it easier to correctly understand the requirements:

- $\mathbf{A}$ is a known $m$ $x$ $n$ matrix
- $\mathbf{b}$ is an $m$-vector
- $\mathbf{x}$ is an $n$-vector
- $\mathbf{A}^{-1}$ is the inverse of the matrix $\mathbf{A}$
- $\text{rank}(\mathbf{A}) = n$ the rank of a matrix is the maximum number of linearly independent rows or columns it contains.
- $\det(\mathbf{A})$ is determinant of matrix $\mathbf{A}$

## 4.3 Data Definitions

| Number | DD1 |
|---|---|
| Label | **Matrix Representation of the Linear Algebraic Equation** |
| Symbol | $\mathbf{Ax} = \mathbf{b}$ |
| Equation | $$\begin{cases} ax_1 + bx_2 = b_1 \\ cx_1 + dx_2 = b_2 \end{cases}$$ |
| Description | $\mathbf{A}$ is a known $m$ $x$ $n$ matrix that is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.<br>$\mathbf{b}$ is an $m$-vector that is $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.<br>$\mathbf{x}$ is an $n$-vector that is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. |
| Source | Scientific Computing, An Introductory Survey, Second Edition by MICHEL T. HEATH, Chapter 2. |
| Ref. By | [IM1], [IM2], [T2], [T1], [A2], [A3] |

| Number | DD2 |
|---|---|
| Label | **Inverse of a Matrix** |
| Symbol | $\mathbf{A}^{-1}$ |
| Equation | $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ |
| Description | $\mathbf{A}$ is a known $m$ $x$ $n$ matrix.<br>$\mathbf{A}^{-1}$ is the inverse of matrix $\mathbf{A}$ .<br>$\mathbf{I}$ is an Identity matrix . |
| Source | Scientific Computing, An Introductory Survey, Second Edition by MICHEL T. HEATH, Chapter 2. |
| Ref. By | [IM1], [IM2], [DD3], [T2], [A4] |

| Number | DD3 |
|---|---|
| Label | **Determinant of a Matrix** |
| Symbol | $\lvert A\rvert$ |
| Equation | $$\lvert A\rvert = \sum_{i=1}^{k} a_{ij}C_{ij}$$ |
| Description | $\lvert A\rvert$ is the determinant of matrix **A**.<br>$\mathbf{C}_{ij}$ is the cofactor of $a_{ij}$ defined by $C_{ij}$ = $(-1)^{i+j}M_{ij}$ .<br>$M_{ij}$ is the minor of matrix A formed by eliminating row i and column j from A. |
| Source | http://mathworld.wolfram.com/Determinant.html |
| Ref. By | [IM1], [IM2], [DD2], [T2], [A4] |

| Number | DD4 |
|---|---|
| Label | **Rank of a Matrix** |
| Symbol | **rank(A)** |
| Equation | **rank(A)** $\leq min(m, n)$ |
| Description | rank(A) is the rank of matrix A.<br>We assume that A is an m x n matrix and we define the linear map f by f(x) = Ax .<br>The rank of an m x n matrix is a nonnegative integer and cannot be greater than either m or n . |
| Source | https://www.cliffsnotes.com/study-guides/algebra/linear-algebra/real-euclidean-vector-spaces/the-rank-of-a-matrix |
| Ref. By | [IM1], [IM2], [T2], [A4] |

## 4.4 Goal Statements

GS1: An explicit general linear algebraic equation problem is represented by $\mathbf{Ax} = \mathbf{b}$. If x is known then such a linear relationship enables us to predict effect b from cause x. If the vector b of effects is known then it would likely able to determine the corresponding vector x of cause.

## 4.5 Theoretical Models

This section focuses on the general equations and laws that Linear Algebraic Equation Solver is based on.

| Number | T1 |
|---|---|
| Label | **General Linear Algebraic Equation** |
| Equation | $\mathbf{Ax} = \mathbf{b}$ |
| Description | A linear transformation between two finite dimensional vector spaces is represented by a matrix. In matrix-vector notation, a system of linear algebraic equations has the form $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is known $m\ x\ n$ matrix, $\mathbf{b}$ is an $m$-vector and $\mathbf{x}$ is an $n$-vector. If $\mathbf{x}$ is known, then such a linear relationship enables us to predict effect $\mathbf{b}$ from cause $\mathbf{x}$ by matrix-vector multiplication $\mathbf{b} = \mathbf{Ax}$. The linear system will also enable us to "reverse engineering", if we know the vector $\mathbf{b}$ of effects, we can be able to determine the corresponding vector $\mathbf{x}$ of cause. |
| Source | http://www.sciencedirect.com/science/article/pii/0377042794903026 |
| Ref. By | [IM1], [IM2], [A2], [A3], [C1] |

| Number | T2 |
|---|---|
| Label | **Existence and Uniqueness** |
| Equation | $\mathbf{Ax} = \mathbf{b}$<br>An $n\ x\ n$ matrix $\mathbf{A}$ is said to be nonsingular if it satisfies any of the following equivalent conditions:<br>1. $\mathbf{A}$ has an inverse (there is a matrix, denoted by $\mathbf{A}^{-1}$, such that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, the identity matrix).<br>2. Determinant of $(\mathbf{A})$ is not equals to 0<br>3. $\text{rank}(\mathbf{A}) = n$ (the rank of a matrix is the maximum number of linearly independent rows or columns it contains). |
| Description | The existence of a solution to a system of linear equations $\mathbf{Ax} = \mathbf{b}$ depend on whether the matrix $\mathbf{A}$ is singular or nonsingular. If the matrix $\mathbf{A}$ is nonsingular, then its inverse $\mathbf{A}^{-1}$ exists, and the system $\mathbf{Ax} = \mathbf{b}$ always has a unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ regardless of the value for b. If the matrix $\mathbf{A}$ is singular, then the number of solutions is determined by the right hand side vector $\mathbf{b}$. |
| Source | http://www.sciencedirect.com/science/article/pii/0377042794903026 |
| Ref. By | [IM1], [IM2], [DD2], [DD3], [DD4], [A4], [C1] |

# 5 Variabilities

The instance models that govern Linear Algebraic Equation Solver are presented in Subsection 4.2.5. The information to understand the meaning of the instance models and their derivation is also presented, so that the instance models can be verified.

## 5.1 Instance Models

This section transforms the problem defined in Section 4.1 into one which is expressed in mathematical terms.

| Number | IM1 |
|---|---|
| Label | **Gaussian Elimination Method for solving Linear Algebraic Equations** |
| Input | $\mathbf{A}$, $\mathbf{b}$ |
| Output | $\mathbf{M}_1$, $\mathbf{L}$, $\mathbf{U}$, $\mathbf{x}$ |
| Description | A is a known $m$ $x$ $n$ matrix.<br>x is an $n$-vector.<br>b is an $m$-vector.<br>$\mathrm{M}_1$ is an elementary elimination matrix.<br>L is lower triangular matrix.<br>U is upper triangular matrix. |
| Sources | http://www.sciencedirect.com/science/article/pii/0377042794903026 |
| Ref. By | [A1], [A2], [A3], [A4], [A5], [A6], [A7] |

### Derivation of Gaussian Elimination Method

It is fairly simple matter to reduce a general linear system $\mathbf{Ax} = \mathbf{b}$ to upper triangular form. We first choose an elementary elimination matrix $\mathbf{M}_1$ with the first diagonal entry $\mathbf{a}_{11}$ as pivot, so that the first column of $\mathbf{A}$ becomes zero below the first row when premultiplied by $\mathbf{M}_1$. All the remaining columns of $\mathbf{A}$, as well as right-hand-side vector $\mathbf{b}$, must also be multiplied by $\mathbf{M}_1$, so the new system becomes $\mathbf{M}_1\mathbf{Ax} = \mathbf{M}_1\mathbf{b}$.

Next we use the second diagonal entry as pivot to determine a second elementary elimination matrix $\mathbf{M}_2$ that annihilates all of the entries of the second column of the new matrix, $\mathbf{M}_1\mathbf{A}$, below the second row. Again, $\mathbf{M}_2$ must be applied to the entire matrix and right-hand-side vector, so that we obtain the further modified linear system $\mathbf{M}_2\mathbf{M}_1\mathbf{Ax} = \mathbf{M}_2\mathbf{M}_1\mathbf{b}$. Note that the first column of the matrix $\mathbf{M}_1\mathbf{A}$ is not effected by $\mathbf{M}_2$ because all of its entries are zero in the relevant rows. If we define the matrix $\mathbf{M} = \mathbf{M}_{n-1}...\mathbf{M}_1$, then the transformed the linear system is

$\mathbf{MAx} = \mathbf{M}_{n-1}...\ \mathbf{M}_1\mathbf{Ax} = \mathbf{M}_{n-1}...\ \mathbf{M}_1\mathbf{b} = \mathbf{Mb}$

is upper triangular and can be solved by back-substitution to obtain the solution to the original linear system $\mathbf{Ax} = \mathbf{b}$.

The process we have just described is known as Gaussian elimination. It is also known as LU factorization or LU decomposition because it decomposes the matrix $\mathbf{A}$ into product of a unit lower triangular matrix, $\mathbf{L}$, and the upper triangular matrix, $\mathbf{U}$. To see this, recall that the product $\mathbf{L}_k\mathbf{L}_j$ is unit lower triangular if k¡j, so taht

$\mathbf{L} = \mathbf{M}^{-1} = (\mathbf{M}_{n-1}...\mathbf{M}_1)^{-1} = \mathbf{M}_1{}^{-1}...\mathbf{M}_{n-1}{}^{-1} = \mathbf{L}_1...\mathbf{L}_{n-1}$

is unit lower triangular. We have already seen that, by design, the matrix $\mathbf{U} = \mathbf{MA}$ is upper triangular. Therefore, we have expressed $\mathbf{A}$ as a product

$\mathbf{A} = \mathbf{LU}$

where $\mathbf{L}$ is unit lower triangular and $\mathbf{U}$ is upper triangular. Given such a factorization, the linear system $\mathbf{Ax} = \mathbf{b}$ can be written as $\mathbf{LUx} = \mathbf{b}$ and hence can be solved by first solving the lower triangular system $\mathbf{Ly} = \mathbf{b}$ by forward-substitution, then the upper triangular matrix $\mathbf{Ux} = \mathbf{y}$ by back-substitution.

In solving a linear system $\mathbf{Ax} = \mathbf{b}$, the necessary transformation of the right-hand-side vector $\mathbf{b}$ could be included as a part of LU factorization process, or it could be done as a separate step to solve the lower triangular system $\mathbf{Ly} = \mathbf{b}$ after $\mathbf{L}$ has been obtained. In either case, back-substitution for upper triangular matrix is then used to solve the upper triangular system $\mathbf{Ux} = \mathbf{y}$ to obtain the solution $\mathbf{x}$.

| Number | IM2 |
|---|---|
| Label | **Gauss-Jordan Elimination Method for solving Linear Algebraic Equations** |
| Input | $\mathbf{A}$, $\mathbf{b}$ |
| Output | $\mathbf{x}$ |
| Description | The algorithm of Gauss-Jordan transforms matrix A by means of elementary transformations into a diagonal matrix (or into the identity matrix), and performs similar transformations to the right-hand side vector b in order to find the solution vector x. |
| Sources | http://www.sciencedirect.com/science/article/pii/0377042794903026 |
| Ref. By | [A1], [A2], [A3], [A4], [A5], [A6], [A7] |

**Derivation of Gauss-Jordan Elimination Method**

The transformation of A is achieved in n successive elimination steps. Starting from $A^{(1)}$ = A, the $k^{th}$ elimination step, k = 1,. . . , n, transforms $A^{(k)}$ into $A^{(k+1)}$ such that the off-diagonal elements in the $k^{th}$ column, not only below but also above the main diagonal, become zero. Thus, after n steps the diagonal matrix D = $A^{(n+1)}$ is obtained. The $k^{th}$ elimination step can be formulated as follows. The pivot of the$k^{th}$ elimination step is $\delta_k$, = $A^{(k)}_{k,k}$, as in Gaussian elimination. Let $g_k$, be the column vector given by

$g_k = A^{(k)}e_k - \delta_k e_k$

therefore the vector obtained from the$k^{th}$ column of $A^{(k)}$ by replacing its diagonal element by zero. Then the$k^{th}$ elimination step, to introduce the required zeros in the $k^{th}$ column of the matrix, consists of premultiplying $A^{(k)}$ by the matrix

$T_k$ = I - $\delta_k^{-1}$ $g_k e_k^T$

In other words,$A^{(k+1)}$ is obtained from $A^{(k)}$ by means of the rank-one modification

$A^{(k+1)} = T_k A^{(k)} = A^{(k)}$ - $\delta_k^{-1} g_k e_k^T$

The corresponding transformation of right-hand side vector b proceeds as follows. Starting from $b^{(1)}$ = b, the$k^{th}$ elimination step, k = 1,. . . , n, transforms $b^{(k)}$ into $b^{(k+1)}$ according to

$b^{(k+1)} = T_k b^{(k)} = b^{(k)}$ - $\delta_k^{-1} b_k^{(k)} g_k$,

which is a vector update operation.

Thus, the given linear system is transformed into the equivalent system Dx = y, which is easily solved by calculating

x = $D^{-1}$y.

## 5.2 Assumptions

This section simplifies the original problem and helps in developing the theoretical model by filling in the missing information for the physical system. The numbers given in the square brackets refer to the theoretical model [T], general definition [GD], data definition [DD], instance model [IM], or likely change [LC], in which the respective assumption is used.

A1: It is assumed that user will not declare the method by which the linear algebraic equation is solved.
[IM1], [IM2].

A2: The entry of values for A, b will be of the explicit form such that Ax = b as described.
[T1]. [IM1], [IM2]

A3: it is assumed that the values for A, b are entered in matrix form.
[IM1], [IM2], [T2], [T1], [DD2], [DD3], [DD4], [GS1]

A4: It is assumed that the entered matrix A is non-singular.
[IM1], [IM2], [A2], [DD2], [GS1]

A5: It is assumed that the entry of values for A, b will not have any complex numbers.
[IM1], [IM2], [T2], [T1], [DD2], [DD3], [DD4], [GS1]

A6: It is assumed that the entry of values for A is from the set of $\mathbb{R}$.
[IM1], [IM2], [T2], [T1]

A7: It is assumed that the entry of values for A is from the set of $\mathbb{R}$.
[IM1], [IM2], [T2], [T1]

### 5.3 Calculation

C1: Check the inputs if they satisfy the input assumptions.
[GS1], [A2], [A3], [A5], [A6], [A7], [T2], [T1], [IM1], [IM2]

C2: Perform the linear algebraic solver model when the user calls the program.
[GS1], [A2], [A3], [A5], [A6], [A7], [T2], [T1], [IM1], [IM2]

### 5.4 Output

Not Applicable

## 6 Traceability Matrices and Graphs

The purpose of the traceability matrices is to provide easy references on what has to be additionally modified if a certain component is changed. Every time a component is changed, the items in the column of that component that are marked with an "X" should be modified as well. Table 2 shows the dependencies of theoretical models, data definitions, and instance models with each other. Table 3 shows the dependencies of theoretical models, data definitions, instance models, and likely changes on the assumptions..

| | DD1 | DD2 | DD3 | DD4 | T1 | T2 | IM1 | IM2 |
|---|---|---|---|---|---|---|---|---|
| DD1 | | | | | X | X | X | X |
| DD2 | | | X | | | X | X | X |
| DD3 | | X | | | | X | X | X |
| DD4 | | | | | | X | X | X |
| T1 | | | | | | | X | X |
| T2 | | | | | | | X | X |
| IM1 | | | | | | | | |
| IM2 | | | | | | | | |

Table 2: Traceability Matrix Showing the Connections Between Items of Different Sections

| | A1 | A2 | A3 | A4 | A5 | A6 | A7 | C1 | C2 |
|---|---|---|---|---|---|---|---|---|---|
| GS1 | | | X | X | X | | | X | X |
| DD1 | | | | X | | | | | |
| DD2 | | | X | | X | | | | |
| DD3 | | | X | | X | | | | |
| DD4 | | | X | | X | | | | |
| T1 | | X | X | | X | X | X | X | X |
| T2 | | | X | | X | X | X | X | X |
| IM1 | X | X | X | X | X | X | X | X | X |
| IM2 | X | X | X | X | X | X | X | X | X |

Table 3: Traceability Matrix Showing the Connections Between Assumptions and Other Items

# References

Spencer Smith. Systematic development of requirements documentation for general purpose scientific computing software. In *Requirements Engineering, 14th IEEE International Conference*, pages 209–218. IEEE, 2006.