UNIT I - DIFFERENTIAL CALCULUS

Representation of functions – Limit of a function – Continuity – Derivatives – Differentiation rules – Maxima and minima of functions of one variable.

REPRESENTATION OF FUNCTIONS:-

A function is a rule that assigns to each element x in a set A to exactly one element called $f(x)$ in a set B.

★ **Domain** : Let $f: A \to B$ then set A is called the domain of the function.

★ **Co-domain**: set B is called Co-domain of the function.

★ **Range** : The set of all images of all the elements of A under the function 'f' is called the range of 'f' and it is denoted by $f(A)$.

① Find the domain of the function $f(x) = \sqrt{x+2}$.

Soln:-

The given function is $f(x) = \sqrt{x+2}$

Since the square root of a negative number is not defined, the domain of 'f' must be positive.

$\therefore x+2 \geq 0$

$\Rightarrow x \geq -2$

$x \geq -2$

-2 ∞

$\therefore$ $\boxed{\text{Domain is } [-2, \infty)}$.

② Find the domain of the function $f(x) = \sqrt{3-x} - \sqrt{2+x}$.

Soln:-

NOV / DEC 2018

Given:- $f(x) = \sqrt{3-x} - \sqrt{2+x}$

Since the square root of a negative number is not defined, the domain of 'f' must be positive.

$3-x \geq 0$ and $2+x \geq 0$

$\Rightarrow x \leq 3$ and $x \geq -2$

-2 3

$x \geq -2 \rightarrow$ $\leftarrow x \leq 3$

$\boxed{\therefore \text{Domain is } [-2, 3]}$

③ Find the domain of the function $f(x) = \dfrac{x+4}{x^2-9}$.

Soln:-

Given:- $f(x) = \dfrac{x+4}{x^2-9}$

The function is not defined at $x = 3$ and $x = -3$.

$\boxed{\text{Domain :- } \{x \mid x \neq 3, x \neq -3\}}$

(or)

$\boxed{\text{Domain :- } (-\infty, -3) \cup (-3, 3) \cup (3, \infty)}$

④ Sketch the graph and find domain & range of the function

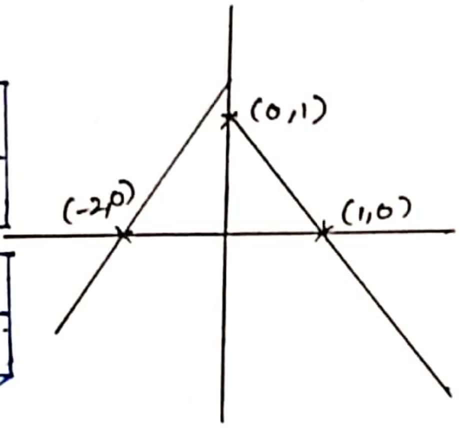
$$f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ 1-x & \text{if } x \geq 0 \end{cases}$$

Soln:-

Given:- $f(x) = \begin{cases} x+2, & x < 0 \\ 1-x, & x \geq 0 \end{cases}$

Domain : x	-1	-2	-3	...
Range : $y = x+2$	1	0	-1	...

Domain : x	0	1	2	..
Range : $y = 1-x$	1	0	-1	..



Domain :	$(-\infty, \infty)$
Range :	$(-\infty, 1)$

⑤ Sketch the graph of the function $f(x) = \begin{cases} 1+x, & x < -1 \\ x^2, & -1 \leq x \leq 1 \\ 2-x, & x \geq 1 \end{cases}$ and use it to determine the values of 'a' for which $\lim\limits_{x \to a} f(x)$ exists ?

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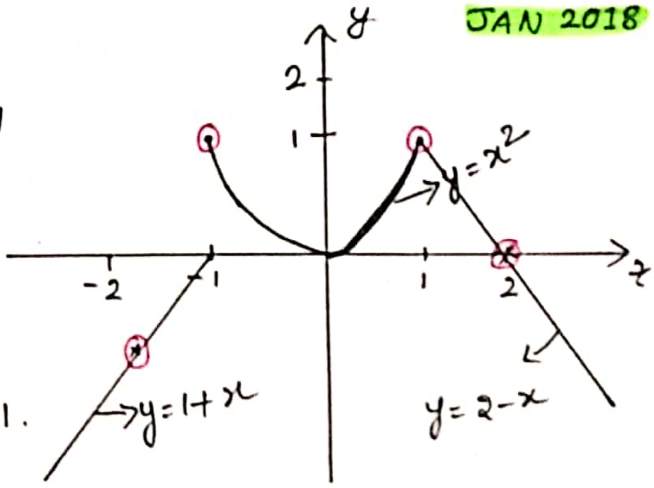
Soln:- Given:- $f(x) = \begin{cases} 1+x, & x < -1 \\ x^2, & -1 \leq x \leq 1 \\ 2-x, & x \geq 1 \end{cases}$



From the graph, it is observed that $\lim\limits_{x \to a} f(x)$ exists for all 'a' except when $a = -1$, since the right and left limits are different at $a = -1$.

LIMIT OF A FUNCTION :-

Let $f(x)$ be a function of a real variable x. Let 'a' and 'l' be a fixed numbers. If $f(x)$ approaches 'l' as x approaches 'a', then we say 'l' is the limit of $f(x)$ as x tends to 'a' & we write $\lim\limits_{x\to a} f(x) = l$

Left hand limit :-

If $f(x)$ approaches the value 'l' as x approaches 'a' from the left, then $\lim\limits_{x\to a^-} f(x) = l$.

Right hand limit :-

If $f(x)$ approaches the value 'l' as x approaches 'a' from the right, then $\lim\limits_{x\to a^+} f(x) = l$.

Result :-

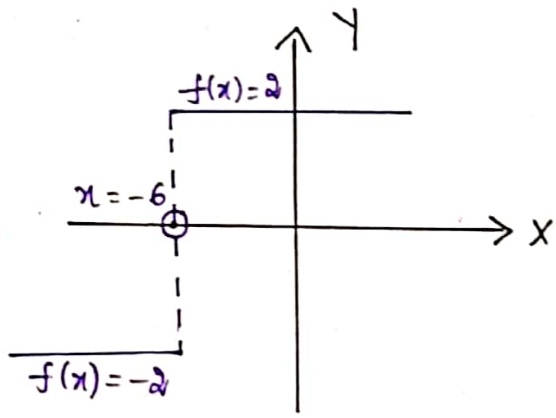
$$\boxed{\lim_{x\to a} f(x) = l \text{ if and only if } \lim_{x\to a^-} f(x) = l = \lim_{x\to a^+} f(x)}.$$

① Evaluate the Limit $\lim\limits_{x\to -6} \dfrac{2x+12}{|x+6|}$

Soln:-

We know that $\lim\limits_{x\to a} f(x) = l$ iff $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) = l$

Given:- $\lim\limits_{x\to -6} \dfrac{2x+12}{|x+6|}$



$$\text{Now, } f(x) = \begin{cases} \dfrac{2(x+6)}{x+6}, & x+6 \geq 0 \\ \dfrac{2(x+6)}{-(x+6)}, & x+6 < 0 \end{cases}$$

$$= \begin{cases} 2, & x \geq -6 \\ -2, & x < -6 \end{cases}$$

Now, $\lim\limits_{x\to -6^-} \dfrac{2x+12}{|x+6|} = -2$

& $\lim\limits_{x\to -6^+} \dfrac{2x+12}{|x+6|} = 2$

$$\therefore \lim_{x\to -6^-} \frac{2x+12}{|x+6|} \neq \lim_{x\to -6^+} \frac{2x+12}{|x+6|}$$

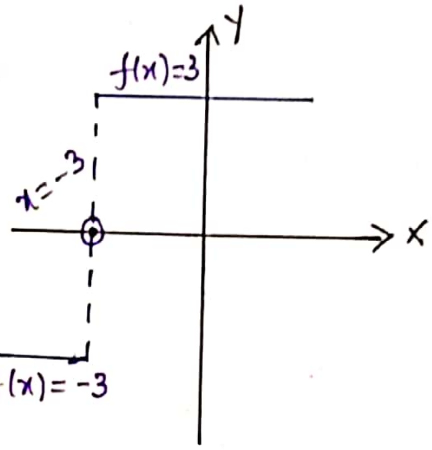
$\therefore$ Limit does not exists.

② Check whether $\lim\limits_{x \to -3} \dfrac{3x+9}{|x+3|}$ exists. APRIL / MAY 2019

Soln:-

Given:- $\lim\limits_{x \to -3} \dfrac{3x+9}{|x+3|}$, $f(x) = \begin{cases} \dfrac{3(x+3)}{x+3}, & x+3 \geq 0 \\ \dfrac{3(x+3)}{-(x+3)}, & x+3 < 0 \end{cases}$

$= \begin{cases} 3, & x \geq -3 \\ -3, & x < -3 \end{cases}$



Now, $\lim\limits_{x \to -3^+} \dfrac{3x+9}{|x+3|} = 3$

Also, $\lim\limits_{x \to -3^-} \dfrac{3x+9}{|x+3|} = -3$

$\therefore \lim\limits_{x \to -3^+} \dfrac{3x+9}{|x+3|} \neq \lim\limits_{x \to -3^-} \dfrac{3x+9}{|x+3|}$

$\therefore$ limit does not exists.

③ Guess the value of the limit (if it exists) for the function $\lim\limits_{x \to 0} \dfrac{e^{5x}-1}{x}$ by evaluating the function at the given numbers $x = \pm 0.5, \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001$ (correct to six decimal places). NOV/DEC 2018

Soln:-

Given:- $\lim\limits_{x \to 0} \dfrac{e^{5x}-1}{x}$

Here, $f(x) = \dfrac{e^{5x}-1}{x}$

x	-0.5	-0.1	-0.01	-0.001	-0.0001
$f(x)$	1.835830	3.934693	4.877058	4.987521	4.998750

x	0.5	0.1	0.01	0.001	0.0001
$f(x)$	22.364988	6.487213	5.127110	5.012521	5.001250

As x approaches to 0, the function $f(x) = \dfrac{e^{5x}-1}{x}$ approaches to 5.

$\therefore \lim\limits_{x \to 0} \dfrac{e^{5x}-1}{x} = 5.$

④ Investigate $\lim\limits_{x \to 0} \frac{1}{x^2}$.

Soln:-

Given:- $\lim\limits_{x \to 0} \frac{1}{x^2}$

Here, $f(x) = \frac{1}{x^2}$.

x	-1	-0.5	-0.1	-0.05	-0.01	-0.001	0.001	0.01	0.05	0.1	0.5	1
$f(x)$	1	4	100	400	10000	1000000	1000000	10000	400	100	4	1

As x approaches to 0, the function $f(x) = \frac{1}{x^2}$ becomes very large and does not approaches to a number.

$$\therefore \boxed{\lim_{x \to 0} \frac{1}{x^2} = \infty}$$

⑤ Evaluate $\lim\limits_{x \to 0} \frac{\sin x}{x}$.

Soln:-

Given: $\lim\limits_{x \to 0} \frac{\sin x}{x}$

Here, $f(x) = \frac{\sin x}{x}$

x	-1	-0.5	-0.1	-0.05	-0.01	-0.001	0.001	0.01	0.05	0.1	0.5	1
$f(x)$	0.8415	0.9589	0.9983	0.9996	0.9999	0.99999	0.99999	0.9999	0.9996	0.9983	0.9589	0.8415

As x approaches to 0, the function $f(x) = \frac{\sin x}{x}$ approaches to 1.

$$\therefore \boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}.$$

⑥ Prove that $\lim\limits_{x \to 0} |x| = 0$.

Soln:-

Given:- $\lim\limits_{x \to 0} |x|$

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

We know $\lim_{x\to a} f(x) = l$ iff $\lim_{x\to a^-} f(x) = l = \lim_{x\to a^+} f(x)$

Now, $\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0$

Also, $\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$.

$\therefore \lim_{x\to 0} |x| = 0$.

⑦ Determine $\lim_{x\to \frac{\pi}{2}} \frac{1+\cos 2x}{(\pi-2x)^2}$. AU JAN 2016 R-15 MA7151

Soln:-

$$\lim_{x\to \frac{\pi}{2}} \frac{1+\cos 2x}{(\pi-2x)^2} = \frac{1+\cos 2\frac{\pi}{2}}{\left(\pi-2\frac{\pi}{2}\right)^2} = \frac{1-1}{0} = \frac{0}{0}.$$

Apply L'hospital rule,

$$= \lim_{x\to \frac{\pi}{2}} \frac{+2\sin 2x}{2(\pi-2x)(+2)} = \lim_{x\to \frac{\pi}{2}} \frac{\sin 2x}{2(\pi-2x)}$$

$$= \frac{\sin 2\frac{\pi}{2}}{2\left(\pi-2\frac{\pi}{2}\right)}$$

$$= \frac{0}{0}$$

Apply L'hospital rule,

$$= \lim_{x\to \frac{\pi}{2}} \frac{2\cos 2x}{2(-2)}$$

$$= \frac{\cos 2\frac{\pi}{2}}{-2} = \frac{-1}{-2} = \frac{1}{2}.$$

⑧ Evaluate $\lim_{x\to \infty} [x\sqrt{x^2+1} - x]$.

Soln:-

$$\lim_{x\to\infty} [x\sqrt{x^2+1} - x] = \lim_{x\to\infty} x\sqrt{x^2+1} - x \times \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to\infty} \frac{x\left[(x^2+1)-x^2\right]}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}+x}$$

$$= \lim_{x\to\infty} \frac{x}{x\sqrt{1+\frac{1}{x^2}} + x}$$

$$= \lim_{x\to\infty} \frac{\cancel{x}}{\cancel{x}\left[\sqrt{1+\frac{1}{x^2}} + 1\right]} \qquad \left[\because \frac{1}{\infty} = 0\right.$$

$$= \frac{1}{\sqrt{1+\frac{1}{\infty}} + 1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}$$

(9) Evaluate $\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2+4x+1}$.

Soln:-

$$f(x) = \frac{3x^2 - x - 2}{5x^2+4x+1}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x^2-x-2}{5x^2+4x+1}$$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}\left[3 - \frac{1}{x} - \frac{2}{x^2}\right]}{\cancel{x^2}\left[5 + \frac{4}{x} + \frac{1}{x^2}\right]}$$

$$= \frac{3 - \frac{1}{\infty} - \frac{2}{\infty}}{5 + \frac{4}{\infty} + \frac{1}{\infty}} = \frac{3}{5}$$

HORIZONTAL ASYMPTOTE :-

The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.

(1) Find the horizontal asymptote of the curve $\frac{x^2-1}{x^2+1}$.

Soln:-

Given:- $f(x) = \frac{x^2-1}{x^2+1}$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2-1}{x^2+1}$$

$$= \lim_{x\to-\infty} \frac{\cancel{x^2}\left[1 - 1/x^2\right]}{\cancel{x^2}\left[1 + 1/x^2\right]}$$

$$= \frac{1-0}{1+0} = 1$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2-1}{x^2+1}$$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}\left[1-\frac{1}{x^2}\right]}{\cancel{x^2}\left[1+\frac{1}{x^2}\right]}$$

$$= \frac{1-0}{1+0} = 1$$

Hence the line $y=1$ is a horizontal asymptote of the given curve.

(2) Find the horizontal and vertical asymptotes of the curve $\frac{\sqrt{2x^2+1}}{3x-5}$.

Soln:-

Given :- $f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2\left[2+\frac{1}{x^2}\right]}}{x\left[3-\frac{5}{x}\right]}$$

$$= \lim_{x\to\infty} \frac{\cancel{x}\sqrt{2+\frac{1}{x^2}}}{\cancel{x}\left[3-\frac{5}{x}\right]}$$

$$= \frac{\sqrt{2+0}}{3}$$

$$= \frac{\sqrt{2}}{3}.$$

When $x\to\infty$,

We have $\sqrt{x^2} = -x$, $x<0$.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5}$$

$$= \lim_{x\to-\infty} \frac{-\cancel{x}\sqrt{2+\frac{1}{x^2}}}{\cancel{x}\left(3-\frac{5}{x}\right)}$$

$$= \frac{-\sqrt{2}}{3}.$$

$\therefore$ Both the line $y = -\frac{\sqrt{2}}{3}$ and $\frac{\sqrt{2}}{3}$ are horizontal asymptotes.

The vertical asymptotes occurs when the given function becomes either $-\infty$ or ∞.

For $x = \frac{5}{3}$ the function becomes ∞.

$$\therefore \lim_{x \to \left(\frac{5}{3}\right)^+} f(x) = \lim_{x \to \frac{5}{3}^+} \frac{\sqrt{2x^2+1}}{3x-5} = \infty$$

and $$\lim_{x \to \frac{5}{3}^-} f(x) = \lim_{x \to \frac{5}{3}^-} \frac{\sqrt{2x^2+1}}{3x-5} = -\infty$$

SQUEEZE THEOREM :-

If $f(x) \le g(x) \le h(x)$ when x is near 'a' (except possible at 'a') and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$ then $\lim_{x \to a} g(x) = L$.

1) Show that $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$

soln:-

LHS: $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$

Let $f(x) = \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$, $x \neq 0$

If $x = 0$, $f(x)$ is not defined.

If $x \neq 0$, $\frac{1}{x}$ is real.

$\therefore \sin\frac{1}{x}$ is defined

$$-1 \le \sin\frac{1}{x} \le 1$$

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2$$

$$\lim_{x \to 0} (-x^2) \le \lim_{x \to 0} x^2 \sin\frac{1}{x} \le \lim_{x \to 0} x^2$$

since $\lim_{x \to 0} (-x^2) = 0$ and $\lim_{x \to 0} (x^2) = 0$

$\therefore$ By squeeze theorem, $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

(2) Show that $\lim_{x \to 0} \sqrt{x^3 + x^2} \sin \frac{\pi}{x} = 0$.

Soln:-

LHS: $\lim_{x \to 0} \sqrt{x^3+x^2} \sin\frac{\pi}{x}$

Let $f(x) = \lim_{x \to 0} \sqrt{x^3+x^2} \sin\frac{\pi}{x}$

If $x = 0$, $f(x)$ is not defined.

If $x \neq 0$, $\frac{\pi}{x}$ is real.

$\therefore \sin\frac{\pi}{x}$ is defined

$$-1 \leq \sin\frac{\pi}{x} \leq 1$$

$$-\sqrt{x^3+x^2} \leq \sqrt{x^3+x^2} \sin\frac{\pi}{x} \leq \sqrt{x^3+x^2}$$

$$\lim_{x \to 0} -\sqrt{x^3+x^2} \leq \lim_{x \to 0} \sqrt{x^3+x^2} \sin\frac{\pi}{x} \leq \lim_{x \to 0} \sqrt{x^3+x^2}$$

Since $\lim_{x \to 0} -\sqrt{x^3+x^2} = 0$ & $\lim_{x \to 0} \sqrt{x^3+x^2} = 0$

$\therefore$ By squeeze theorem, $\lim_{x \to 0} \sqrt{x^3+x^2} \sin\frac{\pi}{x} = 0$

(3) Show that $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x^2}\right)$.

Soln:-

LHS $\lim_{x \to 0} x^2 \cos\frac{1}{x^2}$

$$-1 \leq \cos\frac{1}{x^2} \leq 1$$

$$-x^2 \leq x^2 \cos\frac{1}{x^2} \leq x^2$$

$$\lim_{x \to 0} (-x^2) \leq \lim_{x \to 0} x^2 \cos\frac{1}{x^2} \leq \lim_{x \to 0} x^2$$

Since $\lim_{x \to 0} (-x^2) = 0$ & $\lim_{x \to 0} (x^2) = 0$,

By squeeze theorem, $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x^2}\right) = 0$.

CONTINUITY :-

A function f is continuous at the point 'a' if $\lim_{x\to a} f(x) = f(a)$.

★ A function f is continuous from right at a point 'a' if $\lim_{x\to a^+} f(x) = f(a)$.

★ A function f is continuous from left at a point 'a' if

$\lim_{x\to a^-} f(x) = f(a)$.

RESULT :- $f(x)$ is continuous if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$.

① Show that the function $f(x) = 1-\sqrt{1-x^2}$ is continuous on the interval $[-1,1]$.

Soln:-

Given:- $f(x) = 1-\sqrt{1-x^2}$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} 1-\sqrt{1-x^2}$$

$$= 1-\sqrt{1-(-1)^2}$$

$$= 1$$

& $$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 1-\sqrt{1-x^2}$$

$$= 1-\sqrt{1-1}$$

$$= 1$$

$$\lim_{x\to a} f(x) = \lim_{x\to a} 1-\sqrt{1-x^2}$$

$$= 1-\sqrt{1-a^2}$$

$$= f(a)$$

$\therefore -1 < a < 1$, f is continuous

$$\therefore \lim_{x\to -1^+} f(x) = \lim_{x\to 1^-} f(x) = 1$$

$\therefore f(x)$ is continuous on the interval $[-1,1]$

② For what value of the constant 'c' is the function f continuous on $(-\infty,\infty)$, $f(x) = \begin{cases} cx^2+2x ; & x<2 \\ x^3-cx ; & x\geq 2 \end{cases}$

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Soln:-

Given:- $f(x) = \begin{cases} cx^2+2x ; & x<2 \\ x^3-cx ; & x\geq 2 \end{cases}$

cx^2+2x $\quad$ x^3-cx

$-\infty$ $\quad$ 2 $\quad$ ∞

The given function $f(x)$ is continuous on $(-\infty,2)$ and $(2,\infty)$.

Now, $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (cx^2 + 2x)$

$= c(2^2) + 2(2)$

$= 4c + 4.$

Also, $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (x^3 - cx)$

$= 2^3 - 2c$

$= 8 - 2c.$

We know that a function f is continuous at a point 'a' if $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x)$.

$\therefore \quad 4c + 4 = 8 - 2c$

$4c + 2c = 8 - 4$

$6c = 4$

$c = \frac{4}{6}$

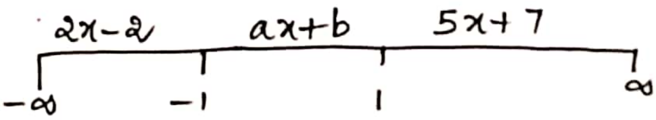
$\boxed{c = \frac{2}{3}}$

③ Let $f(x) = \begin{cases} 2x - 2, & x < -1 \\ ax + b, & -1 \le x \le 1 \\ 5x + 7, & x \geq 1 \end{cases}$ is continuous for all real x, find the values of a & b.

Soln:-

Given:- $f(x) = \begin{cases} 2x - 2, & x < -1 \\ ax + b, & -1 \le x \le 1 \\ 5x + 7, & x \geq 1 \end{cases}$

Also, given :- $f(x)$ is continuous.



$\lim\limits_{x \to (-1)^-} f(x) = \lim\limits_{x \to (-1)^-} (2x - 2)$

$= 2(-1) - 2 = -4.$

$$\lim_{x\to(-1)^+} f(x) = \lim_{x\to(-1)^+} (ax+b)$$

$$= a(-1)+b$$

$$= -a+b.$$

$\because f$ is continuous, $\lim_{x\to(-1)^-} f(x) = \lim_{x\to(-1)^+} f(x)$

$$-4 = -a+b$$

$$\Rightarrow a-b = 4 \longrightarrow (1)$$

Also, $\lim_{x\to1^-} f(x) = \lim_{x\to1^-} (ax+b)$

$$= a+b$$

and $\lim_{x\to1^+} f(x) = \lim_{x\to1^+} (5x+7)$

$$= 5+7$$

$$= 12$$

$\because f$ is continuous, $\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x)$

$$a+b = 12 \longrightarrow (2)$$

From (1) & (2)

$$a+\cancel{b} = 12$$

$$a-\cancel{b} = 4$$

$$2a = 16$$

$$\boxed{a=8}$$

$\therefore$ (1) $\Rightarrow$ $8-b=4$

$$-b = 4-8 = -4$$

$$\boxed{b=4}$$

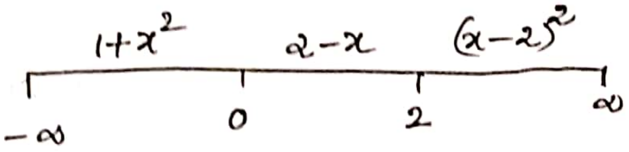
$\therefore$ $a=8$ and $b=4$.

④ Find the domain where the function f is continuous. Also find the number at which the function f is discontinuous, where $f(x) = \begin{cases} 1+x^2, & x \le 0 \\ 2-x, & 0 < x \le 2 \\ (x-2)^2, & x > 2 \end{cases}$

Soln:-

Given:- $f(x) = \begin{cases} 1+x^2, & x \le 0 \\ 2-x, & 0 < x \le 2 \\ (x-2)^2, & x > 2 \end{cases}$

At $x=0$,



$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (1+x^2)$$
$$= 1$$

and $$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (2-x)$$
$$= 2$$

$$\therefore \lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$$

$\therefore$ $f(x)$ is discontinuous at $x=0$.

At $x=2$,

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2-x)$$
$$= 0$$

and $$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x-2)^2$$
$$= 0$$

$$\therefore \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$$

$\therefore$ $f(x)$ is continuous at $x=2$

Domain : $(-\infty, 0) \cup (0, \infty)$.

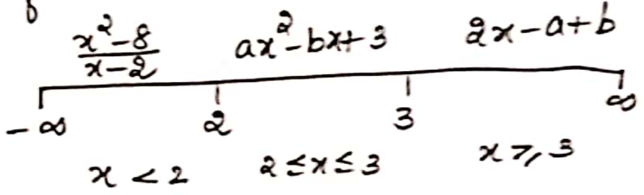
⑤ Find the values of a and b that make f continuous on $(-\infty,\infty)$

$$f(x) = \begin{cases} \dfrac{x^3-8}{x-2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \le x \le 3 \\ 2x - a + b & \text{if } x \geq 3 \end{cases}$$

NOV / DEC 2018

Soln:-

Given:- $f(x) = \begin{cases} \dfrac{x^3-8}{x-2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \le x \le 3 \\ 2x - a + b & \text{if } x \geq 3 \end{cases}$



At $x = 2$,

Now, $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} \dfrac{x^2-8}{x-2}$

$$= \frac{(2)^2-8}{2-2} = \frac{-4}{0} = \infty$$

Apply L' hospital rule,

$$= \lim_{x\to 2^-} \frac{2x}{1}$$

$$= 2(2) = 4.$$

Also, $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (ax^2 - bx + 3)$

$$= 4a - 2b + 3$$

Given:- f is continuous.

$\therefore \lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x)$

$\Rightarrow \quad 4a - 2b + 3 = 4$

$4a - 2b = 4 - 3$

$4a - 2b = 1 \longrightarrow (1)$

At $x = 3$,

Now, $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} (ax^2 - bx + 3)$

$$= 9a - 3b + 3.$$

Also, $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (2x - a + b)$

$$= 6 - a + b$$

$\because$ f is continuous, $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x)$

$$\Rightarrow 9a - 3b + 3 = 6 - a + b$$

$$\Rightarrow 9a + a - 3b - b = 6 - 3$$

$$\Rightarrow 10a - 4b = 3 \longrightarrow (2)$$

Solving (1) & (2)

$$(2) \Rightarrow 10a - \cancel{4b} = 3$$

$$(1) \times 2 \Rightarrow \underset{(-)}{8a} \underset{(+)}{- \cancel{4b}} = \underset{(-)}{2}$$

$$2a = 1$$

$$\boxed{a = 1/2}$$

$\therefore$ (1) $\Rightarrow 4\left(\frac{1}{2}\right) - 2b = 1$

$$2 - 2b = 1$$

$$-2b = -1$$

$$\boxed{b = 1/2}$$

DERIVATIVE :-

The derivative of a function f at a number 'a', denoted by $f'(a)$, is $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ if this limit exists.

① If $f(x) = \sqrt{x}$, find the derivative of $f(x)$.

Soln:-

Given: $f(x) = \sqrt{x}$

By the definition of derivative,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \lim_{h\to 0} \frac{\cancel{x}+\cancel{h}-\cancel{x}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

② If $f(x) = \sin x$, find the derivative of $f(x)$.

Soln:-

Given:- $f(x) = \sin x$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$$= \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}}{h}$$

$$= \lim_{h\to 0} \frac{\sin\frac{h}{2}}{h/2} \cos\left(x+\frac{h}{2}\right)$$

$$= \lim_{h\to 0} \frac{\sin h/2}{h/2} \times \lim_{h\to 0} \cos\left(x+\frac{h}{2}\right) \qquad \left[\because \lim_{x\to 0} \frac{\sin x}{x} = 1\right]$$

$$= 1 \times \cos x$$

$$= \cos x$$

③ Find the derivative of the function $f(x) = \frac{1}{\sqrt{x}}$ using the definition of derivative.

Soln:-

Given:- $f(x) = \frac{1}{\sqrt{x}}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h}\,\sqrt{x}}$$

$$= \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h}\,\sqrt{x}} \times \frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}$$

$$= \lim_{h\to 0} \frac{(\sqrt{x})^2 - (\sqrt{x+h})^2}{h\sqrt{x}\,\sqrt{x+h}\,[\sqrt{x}+\sqrt{x+h}]}$$

$$= \lim_{h\to 0} \frac{x-(x+h)}{h\sqrt{x}\,\sqrt{x+h}\,[\sqrt{x}+\sqrt{x+h}]}$$

$$= \lim_{h\to 0} \frac{\cancel{x}-\cancel{x}-\cancel{h}}{\cancel{h}\sqrt{x}\,\sqrt{x+h}\,[\sqrt{x}+\sqrt{x+h}]}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{x}\,\sqrt{x+h}\,[\sqrt{x}+\sqrt{x+h}]}$$

$$= \frac{-1}{\sqrt{x}\,\sqrt{x}\,[\sqrt{x}+\sqrt{x}]} = -\frac{1}{x(2\sqrt{x})} = -\frac{1}{2x\sqrt{x}}.$$

RULES OF DIFFERENTIATION :-

1. $\frac{d}{dx}(c) = 0$

2. $\frac{d}{dx}(c.u) = c\frac{du}{dx}$, u is a function of x & c is a constant.

3. Product rule, $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

4. Quotient rule, $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' + uv'}{v^2}$

5. Chain rule,

i) If y is a function of u and u itself is a function of x, then $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

ii) If y is a function of u, u is a function of v, v is a function of w and w is a function of x then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dw}\cdot\frac{dw}{dx}$$

① If $f(x) = e^x(x + x\sqrt{x})$, find the derivative of $f(x)$.

Soln:-

Given:- $f(x) = e^x(x + x\sqrt{x})$

$$= e^x[x + x^{3/2}]$$

$$f'(x) = \frac{d}{dx}\left[e^x(x + x^{3/2})\right]$$

$$= e^x\frac{d}{dx}(x + x^{3/2}) + (x + x^{3/2})\frac{d}{dx}(e^x)$$

$$= e^x\left[1 + \frac{3}{2}x^{3/2 - 1}\right] + (x + x^{3/2})e^x$$

$$= e^x\left[1 + \frac{3}{2}x^{1/2} + x + x^{3/2}\right]$$

$$= e^x\left[1 + x + \frac{3}{2}\sqrt{x} + x\sqrt{x}\right].$$

② Find $\frac{dy}{dx}$ if $y = x^2e^{2x}(x^2+1)^4$.

APRIL/MAY 2019

Soln:-

Given:- $y = x^2e^{2x}(x^2+1)^4$.

$$\frac{dy}{dx} = \frac{d}{dx}\left[x^2 e^{2x}(x^2+1)^4\right]$$

$$= x^2 e^{2x} \frac{d}{dx}(x^2+1)^4 + x^2(x^2+1)^4 \frac{d}{dx}(e^{2x}) + e^{2x}(x^2+1)^4 \frac{d}{dx}(x^2)$$

$$= x^2 e^{2x}\, 4(x^2+1)^3(2x) + x^2(x^2+1)^4(2e^{2x}) + e^{2x}(x^2+1)^4(2x)$$

$$= 8x^3 e^{2x}(x^2+1)^3 + 2x^2 e^{2x}(x^2+1)^4 + 2x e^{2x}(x^2+1)^4$$

$$= 2x e^{2x}(x^2+1)^3\left[4x^2 + x(x^2+1) + (x^2+1)\right]$$

$$= 2x e^{2x}(x^2+1)^3\left[4x^2 + x^3 + x + x^2 + 1\right]$$

$$= 2x e^{2x}(x^2+1)^3\left[x^3 + 5x^2 + x + 1\right].$$

③ If $f(x) = \dfrac{1-xe^x}{x+e^x}$, find the derivative of $f(x)$.

Soln:-

Given:- $f(x) = \dfrac{1-xe^x}{x+e^x}$

$$f'(x) = \frac{d}{dx}\left[\frac{1-xe^x}{x+e^x}\right]$$

$$= \frac{(x+e^x)\frac{d}{dx}(1-xe^x) - (1-xe^x)\frac{d}{dx}(x+e^x)}{(x+e^x)^2}$$

$$= \frac{(x+e^x)(-e^x - xe^x) - (1-xe^x)(1+e^x)}{(x+e^x)^2}$$

$$= \frac{-xe^x - x^2e^x - e^{2x} - xe^{2x} - 1 - e^x + xe^x + xe^{2x}}{(x+e^x)^2}$$

$$= \frac{-x^2e^x - e^{2x} - 1 - e^x}{(x+e^x)^2}$$

$$= \frac{-(x^2e^x + e^{2x} + e^x + 1)}{(x+e^x)^2}.$$

④ If $y = (1-x^2)^{10}$, find the derivative of y.

Soln:-

Given:- $y = (1-x^2)^{10}$

Let $u = 1-x^2 \Rightarrow y = u^{10}$

$\frac{du}{dx} = -2x \qquad \frac{dy}{du} = 10u^9.$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= 10u^9 \times (-2x)$

$= -20x(1-x^2)^9.$

⑤ If $y = \tan(\sin x)$, find the derivative of y.

Soln:-

Given:- $y = \tan(\sin x)$

Let $y = \tan u$ where $u = \sin x$.

$\frac{dy}{du} = \sec^2 u \qquad \frac{du}{dx} = \cos x.$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= \sec^2 u \cdot \cos x$

$= \cos x \sec^2(\sin x).$

⑥ If $y = \log(x + \sqrt{x^2-1})$, find the derivative of y.

Soln:-

Given:- $y = \log(x+\sqrt{x^2-1})$

Let $y = \log u \quad ; \quad u = x + \sqrt{x^2-1}$

$\frac{dy}{du} = \frac{1}{u} \quad ; \quad \frac{du}{dx} = 1 + \frac{1}{\cancel{2}\sqrt{x^2-1}}(\cancel{2}x) = 1 + \frac{x}{\sqrt{x^2-1}}$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= \frac{1}{u}\left(1 + \frac{x}{\sqrt{x^2-1}}\right)$

$= \frac{1}{x+\sqrt{x^2-1}}\left(1 + \frac{x}{\sqrt{x^2-1}}\right)$

$$= \frac{1}{x+\sqrt{x^2-1}} \times \frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}$$

$$= \frac{1}{\sqrt{x^2-1}}.$$

7. If $y = \sin(\sin \sin x)$, find the derivative of y.

Soln:-

Let $y = \sin(\sin \sin x)$

Let $y = \sin u$; $u = \sin v$; $v = \sin x$

$\frac{dy}{du} = \cos u$; $\frac{du}{dv} = \cos v$; $\frac{dv}{dx} = \cos x$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$= \cos u \times \cos v \times \cos x$$

$$= \cos(\sin \sin x) \cos(\sin x) \cos x.$$

8. If $y = \cos\sqrt{\sin(\tan \pi x)}$, find the derivative of y.

Soln:-

Given:- $y = \cos\sqrt{\sin(\tan \pi x)}$

Let $y = \cos\sqrt{u}$; $u = \sin v$; $v = \tan \pi x$

$\frac{dy}{du} = -\sin\sqrt{u}\left(\frac{1}{2\sqrt{u}}\right)$; $\frac{du}{dv} = \cos v$; $\frac{dv}{dx} = \sec^2 \pi x \ (\pi)$.

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$= -\frac{1}{2\sqrt{u}} \sin\sqrt{u} \times \cos v \times \pi \sec^2 \pi x$$

$$= -\frac{1}{2\sqrt{\sin v}} \sin\sqrt{\sin v} \times \cos(\tan \pi x)\, \pi \sec^2 \pi x$$

$$= -\frac{\pi}{2} \frac{\sin\sqrt{\sin(\tan \pi x)}}{\sqrt{\sin(\tan \pi x)}} \times \cos(\tan \pi x) \sec^2 \pi x.$$

IMPLICIT DIFFERENTIATION :-

① If $\sqrt{x}+\sqrt{y}=1$ then find $\frac{dy}{dx}$.

Soln:-

Given:- $\sqrt{x}+\sqrt{y}=1$

Diff w.r.to 'x'

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$$

$$\frac{1}{2\sqrt{y}}\frac{dy}{dx}=-\frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx}=-\frac{2\sqrt{y}}{2\sqrt{x}}=-\frac{\sqrt{y}}{\sqrt{x}}.$$

② Find y'' if $x^4+y^4=16$

JAN 2018

Soln:-

Given :- $x^4+y^4=16$

Diff w.r.to 'x'

$$4x^3+4y^3\frac{dy}{dx}=0$$

$$4y^3\frac{dy}{dx}=-4x^3$$

$$\frac{dy}{dx}=-\frac{4x^3}{4y^3}$$

$$y'=\frac{dy}{dx}=-\frac{x^3}{y^3}.$$

$$y''=\frac{d^2y}{dx^2}=-\frac{y^3(3x^2)-x^3(3y^2)\frac{dy}{dx}}{y^6}$$

$$=-\frac{3x^2y^3-3x^3y^2\left(-\frac{x^3}{y^3}\right)}{y^6}$$

$$=-\frac{3x^2y^3+3x^3\left(\frac{x^3}{y}\right)}{y^6}$$

$$=-\frac{3x^2y^4+3x^6}{y^7}$$

$$=-\frac{3x^2}{y^7}(y^4+x^4)=-\frac{3x^2}{y^7}(16)=-\frac{48x^2}{y^7}.$$

③ Find y' for $\cos(xy) = 1+\sin y$. NOV/DEC 2018

Soln:-

Given:- $\cos(xy) = 1+\sin y$

Diff w.r. to 'x'

$$\frac{d}{dx}[\cos(xy)] = \frac{d}{dx}[1+\sin y]$$

$$-\sin xy \frac{d}{dx}(xy) = 0 + \cos y \frac{dy}{dx}$$

$$-\sin xy \left[x\frac{dy}{dx} + y(1)\right] = \cos y \frac{dy}{dx}$$

$$-x\sin xy \frac{dy}{dx} - y\sin(xy) = \cos y \frac{dy}{dx}$$

$$\cos y \frac{dy}{dx} + x\sin xy \frac{dy}{dx} = -y\sin(xy)$$

$$\frac{dy}{dx}[\cos y + x\sin xy] = -y\sin(xy)$$

$$\frac{dy}{dx} = \frac{-y\sin(xy)}{\cos y + x\sin xy}.$$

④ If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, then prove that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$.

Soln:-

Given:- $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$$x\sqrt{1+y} = -y\sqrt{1+x}$$

squaring on both sides,

$$x^2(1+y) = y^2(1+x)$$

$$x^2 + x^2y = y^2 + y^2x$$

$$x^2 + x^2y - y^2 - y^2x = 0$$

$$(x^2-y^2) + xy(x-y) = 0$$

$$(x+y)(x-y) + xy(x-y) = 0$$

$$(x-y)[x+y+xy] = 0$$

If $x - y = 0$, then $1 - \frac{dy}{dx} = 0$

$$\frac{dy}{dx} = 1$$

If $x+y+xy = 0$

$$y + xy = -x$$

$$y(1+x) = -x$$

$$y = \frac{-x}{1+x}$$

$$\frac{dy}{dx} = -\frac{(1+x)(1) - x(1)}{(1+x)^2}$$

$$= -\frac{1+x-x}{(1+x)^2} = -\frac{1}{(1+x)^2}$$

DERIVATIVE OF TRIGONOMETRIC FUNCTIONS :-

FORMULAS :-

1. $\frac{d}{dx} \sin x = \cos x$
2. $\frac{d}{dx} \cos x = -\sin x$
3. $\frac{d}{dx} \tan x = \sec^2 x$
4. $\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$
5. $\frac{d}{dx} \sec x = \sec x \tan x$
6. $\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x$.

1) $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

2) $\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$

3) $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$

4) $\frac{d}{dx} \cot^{-1} x = -\frac{1}{1+x^2}$

5) $\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$

6) $\frac{d}{dx} \operatorname{cosec}^{-1} x = -\frac{1}{x\sqrt{x^2-1}}$

① Find $\frac{dy}{dx}$ if $y = \tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$.

Soln:-

Put $x = \tan\theta$

$$\therefore \frac{\sqrt{1+x^2}-1}{x} = \frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}$$

$$= \frac{\sqrt{\sec^2\theta}-1}{\tan\theta}$$

$$= \frac{\sec\theta - 1}{\tan\theta}$$

$$= \frac{1/\cos\theta - 1}{\sin\theta/\cos\theta}$$

$$= \frac{1-\cos\theta}{\cos\theta} \times \frac{\cos\theta}{\sin\theta}$$

$$= \frac{1-\cos\theta}{\sin\theta}$$

$$\therefore y = \tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right]$$

$$= \tan^{-1}\left[\frac{2\sin^{2}\theta/2}{2\sin\theta/2\,\cos\theta/2}\right]$$

$$= \tan^{-1}\left[\tan\theta/2\right]$$

$$\Rightarrow y = \theta/2$$

$$= \frac{\tan^{-1}x}{2}$$

$\because x = \tan\theta$

$\Rightarrow \theta = \tan^{-1}x$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\frac{1}{1+x^2}$$

(2) If $f(x) = \cos^{-1}\left[\frac{b+a\cos x}{a+b\cos x}\right]$, find the derivative of $f(x)$.

NOV/DEC 2018

Soln:-

Given:- $f(x) = \cos^{-1}\left[\frac{b+a\cos x}{a+b\cos x}\right]$

Let $u = \frac{b+a\cos x}{a+b\cos x}$

$$\Rightarrow y = \cos^{-1}(u)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$= -\frac{1}{\sqrt{1-u^2}}\frac{d}{dx}\left[\frac{b+a\cos x}{a+b\cos x}\right]$$

$$= -\frac{1}{\sqrt{1-u^2}}\left[\frac{(a+b\cos x)(-a\sin x)-(b+a\cos x)(-b\sin x)}{(a+b\cos x)^2}\right]$$

$$= -\frac{1}{\sqrt{1-\frac{(b+a\cos x)^2}{(a+b\cos x)^2}}}\left[\frac{-a^2\sin x - ab\sin x\cos x + b^2\sin x + ab\cos x\sin x}{(a+b\cos x)^2}\right]$$

$$= -\frac{a+b\cancel{\cos x}}{\sqrt{a^2+b^2\cos^2 x+ab\cos x - b^2 - a^2\cos^2 x - ab\cos x}} \times \left[\frac{-(a^2-b^2)\sin x}{(a+b\cos x)^{\cancel{2}}}\right]$$

$$= \frac{(a^2-b^2)\sin x}{(a+b\cos x)\sqrt{(a^2-b^2)-(a^2-b^2)\cos^2 x}}$$

$$= \frac{(a^2-b^2)\sin x}{(a+b\cos x)\sqrt{(a^2-b^2)[1-\cos^2 x]}}$$

$$= \frac{(a^2-b^2)\sin x}{(a+b\cos x)\sqrt{(a^2-b^2)\sin^2 x}}$$

$$= \frac{(a^2-b^2)\cancel{\sin x}}{(a+b\cos x)\sqrt{(a^2-b^2)}\cancel{\sin x}}$$

$$= \frac{\sqrt{a^2-b^2}}{a+b\cos x}.$$

DERIVATIVE OF LOGARITHMIC FUNCTIONS :-

① Differentiate $y = x^x$.

Soln:-

Given:- $y = x^x$.

Taking log on both sides,

$$\log y = \log x^x$$

$$\log y = x\log x$$

Diff w.r.t. 'x',

$$\frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + \log x$$

$$= 1+\log x$$

$$\frac{dy}{dx} = y(1+\log x)$$

$$= x^x(1+\log x).$$

② Differentiate $y = x^{\cos x}$.

Soln:-

Given:- $y = x^{\cos x}$

Taking log on both sides,

$$\log y = \log x^{\cos x}$$

$$\log y = \cos x \log x.$$

Diff w.r.to 'x'

$$\frac{1}{y}\frac{dy}{dx} = \cos x\left(\frac{1}{x}\right) + \log x\,(-\sin x)$$

$$= \frac{1}{x}\cos x - \sin x \log x$$

$$\frac{dy}{dx} = y\left(\frac{\cos x - x\sin x\log x}{x}\right)$$

$$= x^{\cos x}\left(\frac{\cos x - x\sin x\log x}{x}\right).$$

③ Find y' if $x^y = y^x$.

Soln:-

Given:- $x^y = y^x$

Taking log on both sides,

$$\log x^y = \log y^x$$

$$y\log x = x\log y$$

Diff w.r.to 'x'

$$y\left(\frac{1}{x}\right) + \log x\frac{dy}{dx} = x\frac{1}{y}\frac{dy}{dx} + \log y\,(1)$$

$$\frac{y}{x} + \log x\frac{dy}{dx} = \frac{x}{y}\frac{dy}{dx} + \log y$$

$$\log x\frac{dy}{dx} - \frac{x}{y}\frac{dy}{dx} = \log y - \frac{y}{x}$$

$$\frac{dy}{dx}\left[\log x - \frac{x}{y}\right] = \log y - \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{x\log y - y}{x} \times \frac{y}{y\log x - x}$$

$$= \frac{y\,[x\log y - y]}{x\,[y\log x - x]}.$$

TANGENT & NORMAL :-

- The equation of tangent at a given point (x_1,y_1) is given by
  $y-y_1 = m(x-x_1)$.
- The equation of normal at a given point (x_1,y_1) is given by
  $y-y_1 = -\frac{1}{m}(x-x_1)$.

① Find the equation of the tangent line to the curve $x^3+y^3=6xy$ at the point $(3,3)$ and what point the tangent line horizontal in the first quadrant. **JAN 2018**

Soln:-

Given:- $x^3+y^3=6xy$

D.w.r. to 'x'

$$3x^2+3y^2\frac{dy}{dx} = 6\left[x\frac{dy}{dx}+y\right]$$

$$3\left[x^2+y^2\frac{dy}{dx}\right] = 6\left[x\frac{dy}{dx}+y\right]$$

$$x^2+y^2\frac{dy}{dx} = 2x\frac{dy}{dx}+2y$$

$$y^2\frac{dy}{dx} - 2x\frac{dy}{dx} = 2y-x^2$$

$$(y^2-2x)\frac{dy}{dx} = 2y-x^2$$

$$\frac{dy}{dx} = \frac{2y-x^2}{y^2-2x}.$$

At $(3,3)$,

$$m=\left(\frac{dy}{dx}\right)_{(3,3)} = \frac{2(3)-3^2}{3^2-2(3)} = \frac{6-9}{9-6} = -1$$

$$\therefore \boxed{m=-1}$$

$\therefore$ The equation of the tangent at the point $(3,3)$ is

$$y-y_1 = m(x-x_1)$$

$$y-3 = -1(x-3)$$

$$y-3 = -x+3$$

$$\Rightarrow \quad x+y=6$$

The tangent line is horizontal if $y'=0$.

$$\Rightarrow \frac{2y-x^2}{y^2-2x}=0$$

$$\Rightarrow 2y-x^2=0$$

$$\Rightarrow 2y=x^2$$

$$\Rightarrow y=\frac{x^2}{2}.$$

Sub $y=\frac{x^2}{2}$ in $x^3+y^3=6xy$

$$x^3+\left(\frac{x^2}{2}\right)^3=6x\left(\frac{x^2}{2}\right)$$

$$x^3+\frac{x^6}{8}=\frac{6x^3}{2}$$

$$x^3+\frac{x^6}{8}=3x^3$$

$$\Rightarrow 3x^3-x^3=\frac{x^6}{8}$$

$$2x^3=\frac{x^6}{8} \Rightarrow x^6=16x^3$$

$$\Rightarrow x^3=16$$

$$\Rightarrow x=(16)^{1/3}$$

$$\Rightarrow x=2^{4/3}$$

$$\therefore y=\frac{(2^{4/3})^2}{2}=\frac{2^{8/3}}{2}=2^{5/3}. \qquad \left[\because y=\frac{x^2}{2}\right]$$

$\therefore$ The tangent is horizontal at $(2^{4/3}, 2^{5/3})$.

(2) Find the points on the curve $y=x^4-6x^2+4$ where the tangent line is horizontal?

Soln:- Given:- $y=x^4-6x^2+4$

Diff w.r.to 'x'

$$\frac{dy}{dx}=4x^3-12x.$$

The tangent line is horizontal if $y'=0$.

$\Rightarrow 4x^3 - 12x = 0$

$\Rightarrow 4x(x^2 - 3) = 0$

$\Rightarrow 4x = 0 ; \quad x^2 - 3 = 0$

$\Rightarrow x = 0 ; \quad x^2 = 3$

$\Rightarrow x = \pm\sqrt{3}$.

So the given curve has horizontal tangents when $x = 0, \sqrt{3}$ & $-\sqrt{3}$.

when $x = 0 \Rightarrow y = 0 - 6(0) + 4 \Rightarrow y = 4$.

$\Rightarrow (0, 4)$.

when $x = \sqrt{3} \Rightarrow (\sqrt{3})^4 - 6(\sqrt{3})^2 + 4 = y$.

$\Rightarrow y = -5$

$\Rightarrow (\sqrt{3}, -5)$

when $x = -\sqrt{3} \Rightarrow y = (-\sqrt{3})^4 - 6(-\sqrt{3})^2 + 4$

$= -5$.

$\Rightarrow (-\sqrt{3}, -5)$

$\therefore$ The corresponding points are $(0,4), (\sqrt{3}, -5)$ & $(-\sqrt{3}, -5)$.

MAXIMA AND MINIMA :-

Let c be a point in a domain D of the function f. Then $f(c)$ is the

- absolute maximum value of f on D if $f(c) \geq f(x)$ for all x in D.
- absolute minimum value of f on D if $f(c) \leq f(x)$ for all x in D.

(1) Find the absolute maximum and absolute minimum value of $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ on $[-2, 3]$.

Soln:-

Given:- $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ on $[-2, 3]$

$f'(x) = 12x^3 - 12x^2 - 24x$.

The critical numbers of $f(x)$ are

$f'(x) = 0 \Rightarrow 12x^3 - 12x^2 - 24x = 0$

$12x(x^2 - x - 2) = 0$

$$x(x+1)(x-2)=0$$
$$x = 0;\ -1;\ 2$$

$\therefore$ The critical points are $0, -1, 2$.

The values of $f(x)$ at these critical numbers are

$$f(0) = 3(0)^4 - 4(0)^3 - 12(0)^2 + 1 = 1$$
$$f(-1) = 3(-1)^4 - 4(-1)^3 - 12(-1)^2 + 1 = -4$$
$$f(2) = 3(2)^4 - 4(2)^3 - 12(2)^2 + 1 = -31$$

The value of $f(x)$ at the end points of $[-2, 3]$ are

$$f(-2) = 3(-2)^4 - 4(-2)^3 - 12(-2)^2 + 1 = 33.$$
$$f(3) = 3(3)^4 - 4(3)^3 - 12(3)^2 + 1 = 28$$

$\therefore$ The absolute minimum value is $f(2) = -31$

The absolute maximum value is $f(-2) = 33$

(2) Find the absolute maximum and absolute minimum value of $f(x) = x - 2\sin x$ on $[0, 2\pi]$.

Soln:-

Given:- $f(x) = x - 2\sin x$, $[0, 2\pi]$

$$f'(x) = 1 - 2\cos x$$

Critical numbers :- $f'(x) = 0$

$$\Rightarrow 1 - 2\cos x = 0$$
$$\Rightarrow \cos x = \frac{1}{2}$$
$$\Rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}.$$

$$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - 2\sin\frac{\pi}{3} = -0.68485$$
$$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} - 2\sin\frac{5\pi}{3} = 6.968039$$
$$f(0) = 0 - 2\sin 0 = 0$$
$$f(2\pi) = 2\pi - 2\sin 2\pi = 2\pi = 6.28$$

$\therefore$ The absolute minimum value is $f\left(\frac{\pi}{3}\right) = -0.6848$

The absolute maximum value is $f\left(\frac{5\pi}{3}\right) = 6.9680$

FIRST & SECOND DERIVATIVE TEST :-

INCREASING / DECREASING / CONCAVITY / INFLECTION POINTS :-

INCREASING & DECREASING FUNCTIONS :-

- ★ If $f'(x)>0$, on an interval, then 'f' is increasing on that interval.
- ★ If $f'(x)<0$, on an interval, then 'f' is decreasing on that interval.

CRITICAL NUMBER :-

A critical number of a function 'f' is a number 'c' in the domain of 'f' such that $f'(c)=0$ or $f'(c)$ doesnot exists.

FIRST DERIVATIVE TEST :-

Suppose that 'c' is a critical number of a continuous function 'f' then

- ★ If $f'(x)$ changes from positive to negative at 'c', then $f(x)$ has local maximum at 'c'.
- ★ If $f'(x)$ changes from negative to positive at 'c', then $f(x)$ has local minimum at 'c'.
- ★ If $f'(x)$ doesnot changes sign at 'c', then $f(x)$ has no local maximum or minimum at 'c'.

NOTE :- The first derivative test is a consequence of the increasing and decreasing test.

CONCAVITY :-

CONCAVE UPWARDS / CONCAVE DOWNWARDS :-

- ★ If $f''(x)>0$ in any interval, then $f(x)$ is concave upwards [convex downwards]
- ★ If $f''(x)<0$ in any interval, then $f(x)$ is convex upwards [concave downwards]

INFLECTION POINTS :-

A point 'p' on a curve $y=f(x)$ is called an inflection points if $f(x)$ is continuous and the curve changes from concave upwards to concave downwards or from concave downward to concave upward at 'p'.

SECOND DERIVATIVE TEST :-

Suppose $f''(x)$ is continuous near 'c'

- ★ If $f'(c)=0$ and $f''(c)>0$, then $f(x)$ has a local minimum at 'c'.

★ If $f'(c) = 0$ and $f''(c) < 0$, then $f(x)$ has a local maximum at 'c'.

① For the function $f(x) = 2 + 2x^2 - x^4$, find the intervals of increase or decrease, local maximum or minimum values, the intervals of concavity and the inflection points.

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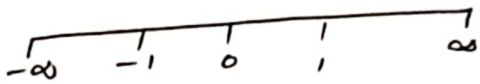
Soln:-

Given:- $f(x) = 2 + 2x^2 - x^4$

$f'(x) = 4x - 4x^3$



To find critical points :-

$$f'(x) = 0 \Rightarrow 4x - 4x^3 = 0$$

$$4x(1 - x^2) = 0$$

$$4x = 0 \quad ; \quad 1 - x^2 = 0$$

$$x = 0 \quad ; \quad x^2 = 1$$

$$\Rightarrow x = \pm 1$$

∴ The critical points are $-1, 0, 1$

INCREASING / DECREASING:-

Interval	Sign of $f'(x)$	Behaviour of $f(x)$
$(-\infty, -1)$	$f'(-2) = 4(-2) - 4(-2)^3$ $= 24$, +ive	Increasing
$(-1, 0)$	$f'(-1/2) = 4(-1/2) - 4(-1/2)^3$ $= -3/2$, -ive	Decreasing
$(0, 1)$	$f'(1/2) = 4(\frac{1}{2}) - 4(1/2)^3$ $= 3/2$, +ive	Increasing
$(1, \infty)$	$f'(2) = 4(2) - 4(2)^3$ $= 8 - 32 = -24$, -ive	Decreasing

LOCAL MAXIMUM OR MINIMUM USING FIRST DERIVATIVE TEST :-

★ ∵ $f'(x)$ changes +ive to -ive at $x = -1$

∴ $f'(x)$ has a local maximum at $x = -1$

$\therefore f(-1) = 2+2(-1)^2 - (-1)^4$
$= 3.$

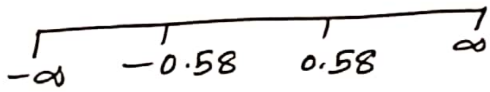
★ $\because f'(x)$ changes negative to positive at $x=0$,
$\therefore f(x)$ has a local minimum at $x=0$.
$\therefore f(0) = 2+2(0)-0$
$= 2.$

★ $\because f'(x)$ changes positive to negative at $x=1$,
$\therefore f(x)$ has local maximum at $x=1$.
$\therefore f(1) = 2+2(1)-(1)$
$= 3.$

CONCAVITY :-

$f''(x) = 4-12x^2$

$f''(x)=0 \Rightarrow 4-12x^2=0$
$4(1-3x^2)=0$
$1-3x^2=0$
$3x^2=1$
$x^2=\frac{1}{3} \Rightarrow x = \pm\frac{1}{\sqrt{3}} = \pm 0.58$



Interval	Sign of $f''(x)$	Behaviour of $f(x)$
$(-\infty,-0.58)$	$f(-1) = 4-12(-1)^2 = -8$, $-$ive	Concave downwards
$(-0.58, 0.58)$	$f(0) = 4-12(0) = 4$, $+$ive	Concave Upwards
$(0.58, \infty)$	$f(1) = 4-12(1)^2 = -8$, $-$ive	Concave downwards

LOCAL MAXIMUM OR MINIMUM USING SECOND DERIVATIVE TEST :-

★ $\because f'(-1)=0$ and $f''(-1) = 4-12(-1)^2 = -8 < 0$
$\therefore f(x)$ is local maximum at $x=-1$.

★ $\because f'(0)=0$ and $f''(0) = 4-12(0) = 4 > 0$
$\therefore f(x)$ is local minimum at $x=0$.

★ $f'(1)=0$ and $f''(1) = 4-12(1)^2 = -8 < 0$

$\therefore f(x)$ is local maximum at $x=1$.

INFLECTION POINTS :-

★ Curve changes from concave downward to concave upward at $x=-0.58$.

$\therefore f(-0.58) = 2+2(-0.58)-(-0.58)^4$

$= 2.56$

$\Rightarrow (-0.58, 2.56)$

★ Also, curve changes from concave upward to concave downward at $x=0.58$.

$\therefore f(0.58) = 2+(0.58)-(0.58)^4$

$= 2.56$.

$\Rightarrow (0.58, 2.56)$

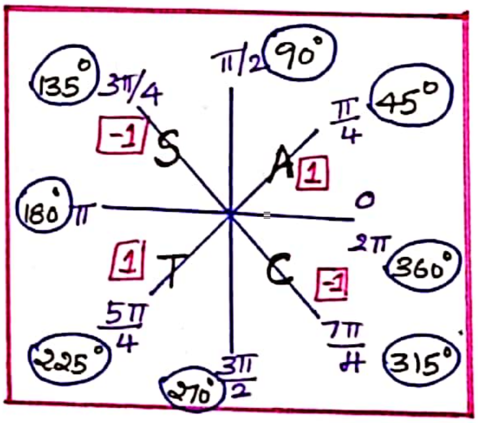
$\therefore$ The inflection points are $(\pm 0.58, 2.56)$.

(2) Given $f(x) = \sin x + \cos x$, $0 \le x \le 2\pi$, find the intervals of increase or decrease; the local maximum or minimum, intervals of concavity and the inflection points.

Soln:-

Given:- $f(x) = \sin x + \cos x$, $0 \le x \le 2\pi$

$f'(x) = \cos x - \sin x$



To find critical points :-

$f'(x)=0 \Rightarrow \cos x - \sin x = 0$

$\sin x = \cos x$

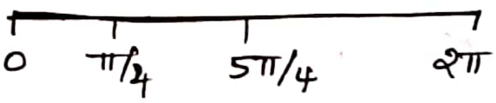
$\dfrac{\sin x}{\cos x} = 1$

$\tan x = 1$

$\Rightarrow x = \tan^{-1}(1)$

$= \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

$\therefore$ The critical points are $\dfrac{\pi}{4}, \dfrac{5\pi}{4}$.



INCREASING / DECREASING :-

Interval	Sign of $f'(x)$	Behaviour of $f(x)$
$0 \le x \le \pi/4$ $(0 \le x \le 45^\circ)$	$f'(30^\circ) = \cos 30^\circ - \sin 30^\circ$ $= 0.37$, +ive	Increasing
$\pi/4 \le x \le 5\pi/4$ $(45^\circ \le x \le 225^\circ)$	$f'(180^\circ) = \cos 180^\circ - \sin 180^\circ$ $= -1$, −ive	Decreasing
$5\pi/4 \le x \le 2\pi$ $225^\circ \le x \le 360^\circ$	$f'(270^\circ) = \cos 270^\circ - \sin 270^\circ$ $= 1$, +ive	Increasing

LOCAL MAXIMUM OR MINIMUM USING FIRST DERIVATIVE TEST :-

★ $\because f'(x)$ changes sign from positive to negative at $\pi/4$,

$\therefore f(x)$ has a local maximum at $x = \pi/4$.

$\therefore f(\pi/4) = f(45^\circ) = \sin 45^\circ + \cos 45^\circ$

$= 1.414$

★ $\because f'(x)$ changes sign from negative to positive at $\frac{5\pi}{4}$,

$\therefore f(x)$ has a local minimum at $x = \frac{5\pi}{4}$.

$\therefore f(5\pi/4) = f(225^\circ) = \sin 225^\circ + \cos 225^\circ$

$= -1.414$

CONCAVITY :-

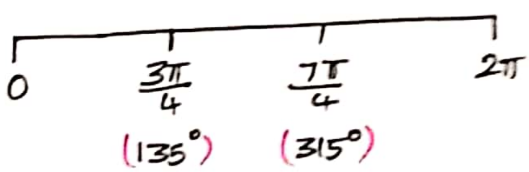
$$f''(x) = -\sin x - \cos x$$

$$f''(x) = 0 \Rightarrow -\sin x = \cos x$$

$$\Rightarrow \frac{\sin x}{\cos x} = -1 \Rightarrow \tan x = -1$$

$$\Rightarrow \tan^{-1}(-1) = x$$

$$\Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}.$$



Interval	Sign of $f''(x)$	Behaviour of $f(x)$
$0 \le x \le \frac{3\pi}{4}$ $(0 \le x \le 135^\circ)$	$f''(90^\circ) = -\sin 90^\circ - \cos 90^\circ$ $= -1$, $-$ive	Concave downwards
$\frac{3\pi}{4} \le x \le \frac{7\pi}{4}$ $(135^\circ \le x \le 315^\circ)$	$f''(180^\circ) = -\sin 180^\circ - \cos 180^\circ$ $= 1$, $+$ive	Concave upwards
$\frac{7\pi}{4} \le x \le 2\pi$ $315^\circ \le x \le 360^\circ$	$f''(320^\circ) = -\sin 320^\circ - \cos 320^\circ$ $= -0.48$, $-$ive	Concave downwards

INFLECTION POINTS :-

★ $\because f''(x)$ changes concave upwards to concave downward at $x = \frac{7\pi}{4}$.

$$\therefore f\left(\frac{7\pi}{4}\right) = f(315^\circ) = \sin 315^\circ + \cos 315^\circ$$
$$= 0$$

$$\Rightarrow \left(\frac{7\pi}{4}, 0\right)$$

★ $\because f''(x)$ changes concave downwards to concave upwards at $\frac{3\pi}{4}$.

$$\therefore f\left(\frac{3\pi}{4}\right) = f(135^\circ) = \sin 135^\circ + \cos 135^\circ$$
$$= 0$$

$$\Rightarrow \left(\frac{3\pi}{4}, 0\right)$$

$\therefore$ The inflection points are $\left(\frac{7\pi}{4}, 0\right)$ & $\left(\frac{3\pi}{4}, 0\right)$.

LOCAL MAXIMUM OR MINIMUM USING SECOND DERIVATIVE TEST :-

➤ $\because f'(\pi/4) = 0$ & $f''(\pi/4) = f''(45^\circ)$
$$= -\sin 45^\circ - \cos 45^\circ$$
$$= -1.414 < 0$$

$\Rightarrow f(x)$ has local maximum at $x = \pi/4$.

➤ $\because f'(5\pi/4) = 0$ & $f''(5\pi/4) = f''(225^\circ)$
$$= -\sin 225^\circ - \cos 225^\circ$$
$$= 1.414 > 0$$

$\Rightarrow f(x)$ has local minimum at $x = 5\pi/4$.

3) Find the local maximum or minimum values of $f(x) = \sqrt{x} - \sqrt[4]{x}$ using both first and second derivative tests.

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Soln:-

Given:- $f(x) = \sqrt{x} - \sqrt[4]{x}$

$$= x^{1/2} - x^{1/4}$$

$$f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{4}x^{-3/4}$$

$$= \frac{x^{-3/4}}{4}\left[2x^{1/4} - 1\right]$$

$$= \frac{2x^{1/4} - 1}{4x^{3/4}}$$

To find critical points :-

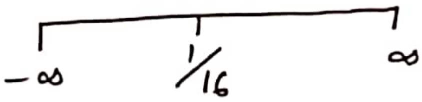
$$f'(x) = 0 \Rightarrow \frac{2x^{1/4} - 1}{4x^{3/4}} = 0$$

$$2x^{1/4} - 1 = 0$$

$$\Rightarrow 2x^{1/4} = 1$$

$$\Rightarrow x^{1/4} = 1/2$$

$$\Rightarrow x = (1/2)^4 = 1/16$$



INCREASING & DECREASING :-

Interval	Sign of $f'(x)$	Behaviour of $f(x)$
$(-\infty, 1/16)$	$f'(1/17) = f'(0.06) = \dfrac{2\sqrt[4]{0.06} - 1}{4\sqrt[4]{(0.06)^3}} < 0$	Decreasing
$(1/16, \infty)$	$f'(1) = \dfrac{2-1}{4} = 1/4$, +ive	Increasing

FIRST DERIVATIVE TEST (LOCAL MAXIMUM OR MINIMUM) :-

- $\because f''(x)$ changes sign from negative to positive at $x = 1/16$.

$\therefore f(x)$ has local minimum at $x = 1/16$

$f(1/16) = \sqrt{1/16} - \sqrt[4]{1/16} = -0.25$.

SECOND DERIVATIVE TEST :-

$$f''(x) = -\frac{1}{4}x^{-3/2} + \frac{3}{16}x^{-7/4}$$
$$= -\frac{1}{4\sqrt{x^3}} + \frac{3}{16\sqrt[4]{x^7}}$$

$\because f'(1/16) = 0$ and $f''(1/16)$

$$= -\frac{1}{4\sqrt{(1/16)^3}} + \frac{3}{16\sqrt[4]{(1/16)^7}}$$
$$= -16 + 24$$
$$= 8 > 0$$

$\therefore f(x)$ has a local minimum at $x = 1/16$.

4) Find the intervals on which $f(x) = \cos^2 x - 2\sin x$, $0 \le x \le 2\pi$ is local maximum and local minimum, Concavity, point of inflection.

Soln:- Given:- $f(x) = \cos^2 x - 2\sin x$.

$$f'(x) = 2\cos x(-\sin x) - 2\cos x$$
$$= -2\sin x\cos x - 2\cos x$$
$$= -\sin 2x - 2\cos x$$

To find the critical points,

$$f'(x) = 0 \Rightarrow -2\sin x\cos x - 2\cos x = 0$$
$$-2\cos x[1 + \sin x] = 0$$

$\cos x = 0$ (or) $1 + \sin x = 0$

$x = \cos^{-1}(0)$ $\Rightarrow \sin x = -1$

$= \frac{\pi}{2}, \frac{3\pi}{2}$ $x = \sin^{-1}(-1)$

$= \frac{3\pi}{2}$

$\therefore$ The critical points are $\frac{\pi}{2}$ & $\frac{3\pi}{2}$.

Interval	Sign of $f'(x)$	Behaviour of $f(x)$
$0 < x < \pi/2$ $(0 < x < 90°)$	$f'(45°) = -\sin 2(45°) - 2\cos 45° = -2.4$	Decreasing
$\pi/2 < x < 3\pi/2$	$f'(180°) = -\sin 2(180°) - 2\cos 180° = 2$	Increasing
$\frac{3\pi}{2} < x < 2\pi$	$f'(300°) = -\sin 2(300°) - 2\cos 300° = -0.13$	Decreasing

LOCAL MAXIMUM / MINIMUM :-

- $f'(x)$ changes from negative to positive at $x = \frac{\pi}{2}$.

  $\therefore f(x)$ has a local minimum at $x = \pi/2$

  $f(\pi/2) = \cos^2(\pi/2) - 2\sin(\pi/2) = 0 - 2(1) = -2.$

- $f'(x)$ changes from positive to negative at $x = \frac{3\pi}{2}$.

  $\therefore f(x)$ has a local maximum at $x = \frac{3\pi}{2}$.

  $f\left(\frac{3\pi}{2}\right) = \cos^2(270°) - 2\sin(270°) = 2.$

CONCAVITY :-

$$f''(x) = -2\cos 2x + 2\sin x$$
$$= -2(1 - 2\sin^2 x) + 2\sin x$$
$$= 2(2\sin x - 1)(\sin x + 1)$$

$f''(x) = 0 \Rightarrow 2(2\sin x - 1)(\sin x + 1) = 0$

$2\sin x - 1 = 0$

$2\sin x = 1$

$\sin x = 1/2 \Rightarrow x = \sin^{-1}(1/2)$

$= \frac{\pi}{6}, \frac{5\pi}{6}$

$\sin x + 1 = 0$

$\Rightarrow \sin x = -1$

$x = \sin^{-1}(-1)$

$= \frac{3\pi}{2}.$

Interval	Sign of $f''(x)$	Behaviour of $f(x)$
$0 \le x < \pi/6$	$f''(25°) = -2\cos 2(25°) + 2\sin 25° = -0.44$	Concave downward.
$\frac{\pi}{6} < x < \frac{5\pi}{6}$	$f''(90°) = -2\cos 2(90°) + 2\sin 90° = 4$	Concave upward.
$\frac{5\pi}{6} < x < \frac{3\pi}{2}$	$f''(180°) = -2\cos 2(180°) + 2\sin 180° = -2$	Concave downward
$\frac{3\pi}{2} < x < 2\pi$	$f''(300°) = -2\cos 2(300°) + 2\sin(300°) = -0.73$	Concave downward.

INFLECTION POINTS :-

- $f''(x)$ changes concave downward to concave upward at $x = \pi/6$

  $\therefore f(\pi/6) = \cos^2(30°) - 2\sin 30° = -0.25$

- $f''(x)$ changes concave upward to concave downward at $x = \frac{5\pi}{6}$.

  $\therefore f\left(\frac{5\pi}{6}\right) = \cos^2 150° - 2\sin 150° = -0.25.$

$\therefore$ The inflection points are $\left(\frac{\pi}{6}, -0.25\right)$ & $\left(\frac{5\pi}{6}, -0.25\right)$.

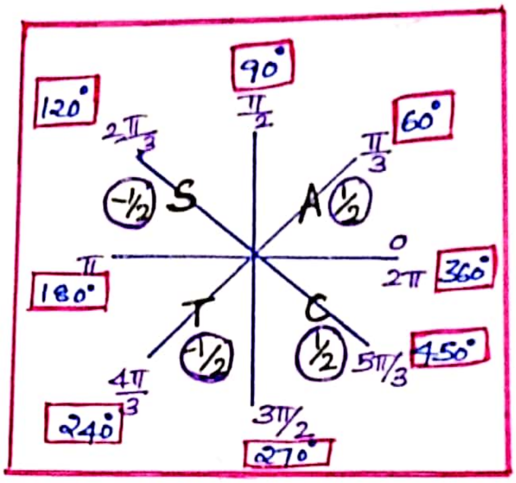
⑤ Find the local maximum & minimum values of the function $f(x) = x + 2\sin x$, $0 \le x \le 2\pi$.

Soln:-

Given:- $f(x) = x + 2\sin x$, $0 \le x \le 2\pi$

$f'(x) = 1 + 2\cos x$

To find the critical points :-



$f'(x) = 0 \Rightarrow 1 + 2\cos x = 0$

$2\cos x = -1$

$\cos x = -1/2$

$\Rightarrow x = \cos^{-1}(-1/2)$

$= \frac{2\pi}{3}$ & $\frac{4\pi}{3}$.

$\therefore$ The critical points are $\frac{2\pi}{3}$ & $\frac{4\pi}{3}$

INCREASING & DECREASING:-

Interval	Sign of $f'(x)$	Behaviour of $f(x)$
$0 < x < \frac{2\pi}{3}$ $(0 < x < 120°)$	$f'(\pi/2) = f'(90°) = 1 + 2\cos 90° = 3$, +ive	Increasing
$\frac{2\pi}{3} < x < \frac{4\pi}{3}$ $(120° < x < 240°)$	$f'(\pi) = f'(180°) = 1 + 2\cos\pi = -1$, -ive	Decreasing
$4\pi/3 < x < 2\pi$ $240° < x < 360°$	$f'(3\pi/2) = f'(270°) = 1 + 2\cos 270° = 1$, +ive	Increasing

LOCAL MAXIMUM OR MINIMUM USING FIRST DERIVATIVE TEST :-

➤ $\because f'(x)$ changes sign from positive to negative at $x = \frac{2\pi}{3}$.

$\therefore f(x)$ has local maximum at $x = \frac{2\pi}{3}$

$\Rightarrow f\left(\frac{2\pi}{3}\right) = f(120°) = \frac{2\pi}{3} + 2\sin\frac{2\pi}{3}$

$= \frac{2\pi}{3} + 2\sin 120° = 3.83$

➤ $\because f'(x)$ changes sign from negative to positive at $x = \frac{4\pi}{3}$.

$\therefore f(x)$ has local minimum at $x = \frac{4\pi}{3}$.

$\Rightarrow f\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} + 2\sin\frac{4\pi}{3}$

$= \frac{4\pi}{3} + 2\sin 240^\circ$

$= 2.46$

SECOND DERIVATIVE TEST :- $f''(x) = -2\sin x$

➤ $\because f'\left(\frac{2\pi}{3}\right) = 0$ and $f''\left(\frac{2\pi}{3}\right) = -2\sin\frac{2\pi}{3}$

$= -2\sin 120^\circ$

$= -1.732 < 0$

$\therefore f(x)$ has local maximum at $x = \frac{2\pi}{3}$.

➤ $\because f'\left(\frac{4\pi}{3}\right) = 0$ and $f''\left(\frac{4\pi}{3}\right) = -2\sin\frac{4\pi}{3}$

$= -2\sin 240^\circ$

$= 1.732 > 0$

$\therefore f(x)$ has local minimum at $x = \frac{4\pi}{3}$.

⑥ If $f(x) = 2x^3 + 3x^2 - 36x$, find the intervals on which it is increasing or decreasing, the local maximum or minimum values of $f(x)$. APRIL/MAY 2019

Soln :-

Given :- $f(x) = 2x^3 + 3x^2 - 36x$

$f'(x) = 6x^2 + 6x - 36$

To find critical points :-

$f'(x) = 0 \Rightarrow 6x^2 + 6x - 36 = 0$

$6(x^2 + x - 6) = 0$

$x^2 + x - 6 = 0$

$(x-2)(x+3) = 0$

$x = 2$ or -3

-6

$-2 \quad 3$

1

$\therefore$ The critical points are 2 and -3.

INCREASING / DECREASING :-

Interval	Sign of $f'(x)$	Behaviour of $f(x)$
$(-\infty, -3)$	$f'(-4) = 6(-4)^2 + 6(-4) - 36$ $= 36$, +ive	Increasing

$(-3, 2)$	$f'(0) = 6(0)^2 + 6(0) - 36$ $= -36 < 0$	Decreasing
$(2, \infty)$	$f'(3) = 6(3)^2 + 6(3) - 36$ $= 36 > 0$	Increasing

FIRST DERIVATIVE TEST :-
(LOCAL MAXIMUM / MINIMUM)

- $\because f'(x)$ changes positive to negative at $x = -3$.
  $\therefore f(x)$ has a local maximum at $x = -3$.
  $\Rightarrow f(-3) = 2(-3)^3 + 3(-3)^2 - 36(-3) = 81$.
- $\because f'(x)$ changes negative to positive at $x = 2$.
  $\therefore f(x)$ has local minimum at $x = 2$.
  $\Rightarrow f(2) = 2(2)^3 + 3(2)^2 - 36(2)$
  $= -44$.

SECOND DERIVATIVE TEST :-

$$f''(x) = 12x + 6$$

- $\because f'(-3) = 0$ and $f''(-3) = 12(-3) + 6 = -30 < 0$
  $\Rightarrow f(x)$ is local maximum at $x = -3$.
- $\because f'(2) = 0$ and $f''(2) = 12(2) + 6 = 30 > 0$
  $\Rightarrow f(x)$ is local minimum at $x = 2$.