Unit - II - DIFFERENTIAL CALCULUS

Representation of functions - Limit of a function - continuity - Derivatives - Differentiation Rules (sum, product, quotient, chain rules) - Implicit differentiation - Logarithmic differentiation - Applications: Maxima and minima of functions of one variable.

REPERENTAION OF FUNCTIONS:

A function is a rule that assigns to each element 'x' in a set 'A' to exactly one element called '$f(x)$' in a set 'B'

* Domain: Let $f: A \to B$ then set 'A' is called the domain of the function.
* Co-Domain: Set 'B' is called co-domain of the function.
* Range: The set of all Images of all the elements of 'A' under the function 'f' is called the range of 'f' and it is denoted by $f(A)$.

PROBLEMS

1. Find the domain of the function $f(x) = \sqrt{x+2}$

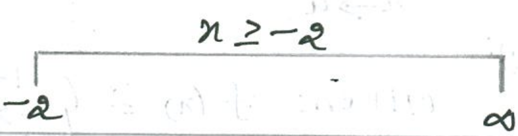
Soln:

Since the square root of a negative number is not defined, the domain of 'f' must be positive.

$\therefore x + 2 \geq 0$

$\Rightarrow x \geq -2$

$\therefore$ $\boxed{\text{Domain is } [-2, \infty)}$



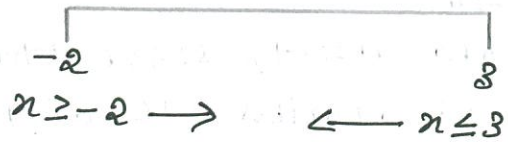
2. Find the domain of the function $f(x) = \sqrt{3-x} - \sqrt{2+x}$

Soln: NOV/DEC 2018

Let, $3 - x \geq 0$ and $2 + x \geq 0$

$\Rightarrow x \leq 3$ and $x \geq -2$

$\therefore$ $\boxed{\text{Domain is } [-2, 3]}$



[Since the square root of a negative number is not defined, the domain of 'f' must be positive]

3. Find the domain of the function $f(x) = \frac{x+4}{x^2-9}$

Soln:

Given: $f(x) = \frac{x+4}{x^2-9}$

The function is not defined at $x=3$ and $x=-3$

$\boxed{\text{Domain}: \{x \mid x \neq 3, x \neq -3\}}$

(or)

$\boxed{\text{Domain}: (-\infty,-3) \cup (-3,3) \cup (3,\infty)}$

4. Find the domain, range also sketch the graph for the following functions, $f(x) = \begin{cases} x+2, & x<0 \\ 1-x, & x \geq 0 \end{cases}$

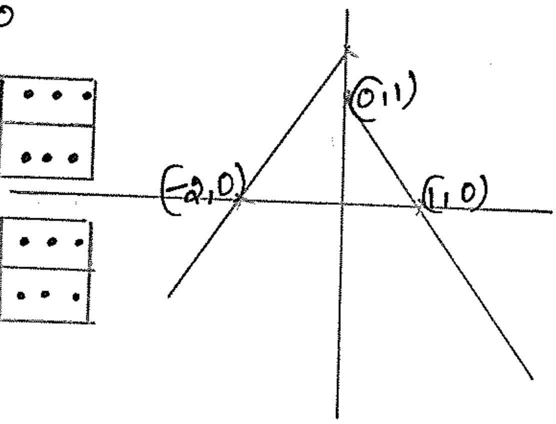
Soln:

Given: $f(x) = \begin{cases} x+2, & x<0 \\ 1-x, & x \geq 0 \end{cases}$

Domain : x	-1	-2	-3	...
Range : $y=x+2$	1	0	-1	...

Domain : x	0	1	2	...
Range : $y=1-x$	1	0	-1	...

Domain : $(-\infty, \infty)$
Range : $(-\infty, 1)$



5. Sketch the graph of the function $f(x) = \begin{cases} 1+x, & x<-1 \\ x^2, & -1 \leq x \leq 1 \\ 2-x, & x \geq 1 \end{cases}$ and use it to determine the values of 'a' for which $\lim_{x \to a} f(x)$ exists?

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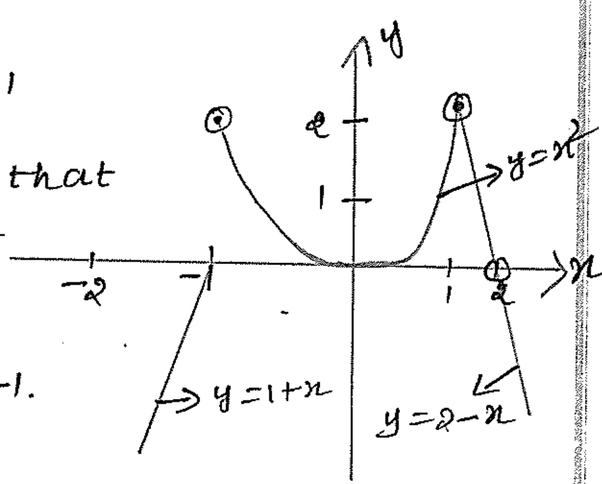
Soln:

Given: $f(x) = \begin{cases} 1+x, & x<-1 \\ x^2, & -1 \leq x \leq 1 \\ 2-x, & x \geq 1 \end{cases}$

From the graph, it is observed that $\lim_{x \to a} f(x)$ exists for all 'a' except when $a=-1$, since the right and left limits are different at $a=-1$.



LIMIT OF A FUNCTION:

Let $f(x)$ be a function of a real variable 'x'. Let 'a' and 'l' be a fixed numbers. If '$f(x)$' approaches 'l' as 'x' approaches 'a', then we say 'l' is the limit of $f(x)$ as 'x' tends to 'a' & we write $\lim_{x \to a} f(x) = l$.

LEFT HAND LIMIT:

If $f(x)$ approaches the value 'l' as 'x' approaches 'a' from the left, then $\lim_{x \to a^-} f(x) = l$.

RIGHT HAND LIMIT:

If $f(x)$ approaches the value 'l' as 'x' approaches 'a' from the right, then $\lim_{x \to a^+} f(x) = l$.

Result:

$$\lim_{x \to a} f(x) = l \text{ if and only if } \lim_{x \to a^-} f(x) = l = \lim_{x \to a^+} f(x).$$

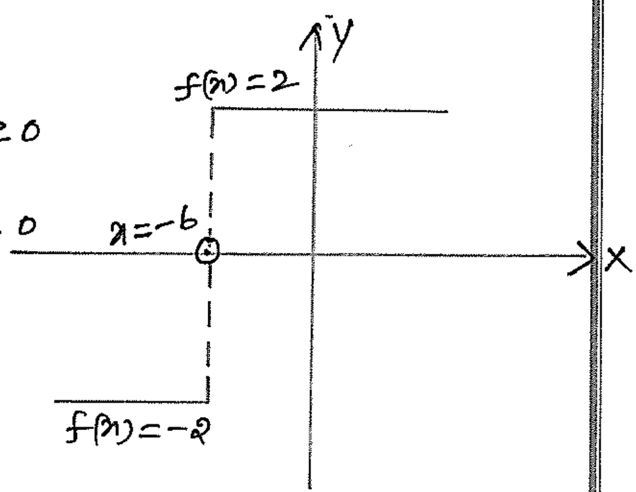
Problems

1. Evaluate $\lim_{x \to -6} \frac{2x+12}{|x+6|}$

Soln:

We know that $\lim_{x \to a} f(x) = l$ iff $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = l$

Given: $\lim_{x \to -6} \frac{2x+12}{|x+6|}$



$$\text{Now, } f(x) = \begin{cases} \frac{2(x+6)}{x+6}, & x+6 \geq 0 \\ \frac{2(x+6)}{-(x+6)}, & x+6 < 0 \end{cases}$$

$$= \begin{cases} 2, & x \geq -6 \\ -2 & x < -6 \end{cases}$$

$$\therefore \lim_{x \to 6^-} \frac{2x+12}{|x+6|} = -2 \quad \text{——} (1)$$

$$\& \lim_{x \to 6^+} \frac{2x+12}{|x+6|} = 2 \quad \text{——} (2)$$

(1) $\neq$ (2) $\therefore$ Limit does not exists.

2. Check whether $\lim_{x \to -3} \frac{3x+9}{|x+3|}$ exists.

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Soln:

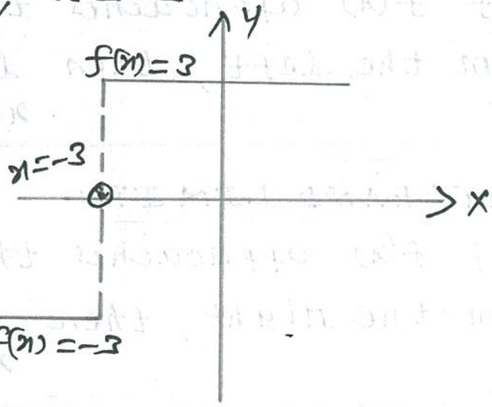
Given: $\lim_{x \to -3} \frac{3x+9}{|x+3|}$, $f(x) = \begin{cases} \frac{3(x+3)}{x+3}, & x+3 \geq 0 \\ \frac{3(x+3)}{-(x+3)}, & x+3 < 0 \end{cases}$

$= \begin{cases} 3, & x \geq -3 \\ -3, & x < -3 \end{cases}$

Now, $\lim_{x \to 3^+} \frac{3x+9}{|x+3|} = 3$ ——— ①

Also, $\lim_{x \to 3^-} \frac{3x+9}{|x+3|} = -3$ ——— ②

① ≠ ② ∴ limit does not exists.



3. Guess the value of the limit (if it exists) for the function $\lim_{x \to 0} \frac{e^{5x}-1}{x}$ by evaluating the function at the given numbers $x = \pm 0.5, \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001$ [correct to six decimal places]

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Soln:

Let $f(x) = \frac{e^{5x}-1}{x}$

x	-0.5	-0.1	-0.01	-0.001	-0.0001
$f(x)$	1.835830	3.934693	4.877058	4.987521	4.998750

x	0.5	0.1	0.01	0.001	0.0001
$f(x)$	22.364988	6.487213	5.127110	5.012521	5.001250

As 'x' approaches to '0', the function $f(x) = \frac{e^{5x}-1}{x}$ approaches to '5'.

$$\therefore \lim_{x \to 0} \frac{e^{5x}-1}{x} = 5.$$

4. Investigate $\lim\limits_{x \to 0} \frac{1}{x^2}$

Soln:

Let $f(x) = \frac{1}{x^2}$

x	-1	-0.5	-0.1	-0.05	-0.01	-0.001	0.001	0.01	0.05	0.1	0.5	1
$f(x)$	1	4	100	400	10000	1000000	1000000	10000	400	100	4	1

As 'x' approaches to '0', the function $f(x) = \frac{1}{x^2}$ becomes very large and does not approaches to a number.

$$\therefore \lim_{x \to 0} \frac{1}{x^2} = \infty.$$

5. Evaluate $\lim\limits_{x \to 0} \frac{\sin x}{x}$

Soln:

Given: $\lim\limits_{x \to 0} \frac{\sin x}{x}$, Here $f(x) = \frac{\sin x}{x}$

x	-1	-0.5	-0.1	-0.05	-0.01	-0.001	0.001	0.01	0.05	0.1	0.5	1
$f(x)$	0.8415	0.9589	0.9983	0.9996	0.9999	0.99999	0.99999	0.9999	0.9996	0.9983	0.9589	0.8415

As 'x' approaches to '0', the function $f(x) = \frac{\sin x}{x}$ approaches to 1.

$$\therefore \lim_{x \to 0} \frac{\sin x}{x} = 1.$$

6. Prove that $\lim\limits_{x \to 0} |x| = 0$

Proof:

Let $f(x) = |x|$

$$= \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

We know that $\lim\limits_{x \to a} f(x) = L$ iff $\lim\limits_{x \to a^-} f(x) = L = \lim\limits_{x \to a^+} f(x)$

Now, $\lim\limits_{x \to 0^-} |x| = \lim\limits_{x \to 0^-} (-x) = 0$

Also, $\lim\limits_{x \to 0^+} |x| = \lim\limits_{x \to 0^+} x = 0$ $\therefore \lim\limits_{x \to 0} |x| = 0$

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7. Determine $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{1+\cos 2x}{(\pi - 2x)^2}$.

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Soln:

Let, $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{1+\cos 2x}{(\pi - 2x)^2} = \dfrac{1+\cos \cancel{2} \frac{\pi}{\cancel{2}}}{\left(\pi - \cancel{2}\frac{\pi}{\cancel{2}}\right)^2} = \dfrac{1-1}{0} = \dfrac{0}{0}$ ①

Apply L'Hospital rule,

$$① \Rightarrow \lim_{x \to \frac{\pi}{2}} \frac{-2\sin 2x}{2(\pi - 2x)(-2)} = \lim_{x \to \frac{\pi}{2}} \frac{\sin 2x}{2(\pi - 2x)} \quad ②$$

$$= \frac{\sin 2\frac{\pi}{2}}{2\left(\pi - 2\frac{\pi}{2}\right)} = \frac{0}{0}$$

Apply L'Hospital rule,

$$② \Rightarrow \lim_{x \to \frac{\pi}{2}} \frac{2\cos 2x}{2(-2)} = \frac{\cos 2\frac{\pi}{2}}{-2} = \frac{-1}{-2}$$

$$= \frac{1}{2}$$

8. Evaluate $\lim\limits_{x \to \infty} \left[x\sqrt{x^2+1} - x\right]$

Soln:

Let, $\lim\limits_{x \to \infty} \left[x\sqrt{x^2+1} - x\right] = \lim\limits_{x \to \infty} x\sqrt{x^2+1} - x \left[\dfrac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right]$

$$= \lim_{x \to \infty} \frac{x\left[x^2+1-x^2\right]}{\sqrt{x^2+1}+x}$$

$$= \lim_{x \to \infty} \frac{x}{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}+x}$$

$$= \lim_{x \to \infty} \frac{x}{x\sqrt{1+\frac{1}{x^2}}+x}$$

$$= \lim_{x \to \infty} \frac{x}{x\left[\sqrt{1+\frac{1}{x^2}}+1\right]}$$

$$= \frac{1}{\sqrt{1+\frac{1}{\infty}}+1} = \frac{1}{2}$$

9. Evaluate $\lim\limits_{x\to\infty} \dfrac{3x^2-x-2}{5x^2+4x+1}$

Soln:

Let $f(x) = \dfrac{3x^2-x-2}{5x^2+4x+1}$

$$\therefore \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x^2-x-2}{5x^2+4x+1}$$

$$= \lim_{x\to\infty} \frac{x^2\left(3-\frac{1}{x}-\frac{2}{x^2}\right)}{x^2\left(5+\frac{4}{x}+\frac{1}{x^2}\right)}$$

$$= \frac{3-\frac{1}{\infty}-\frac{2}{\infty}}{5+\frac{4}{\infty}+\frac{1}{\infty}} = \frac{3}{5}$$

HORIZONTAL ASYMPTOTE:

The line $y=L$ is called a horizontal asymptote of the curve $y=f(x)$ if either $\lim\limits_{x\to\infty} f(x)=L$ (or) $\lim\limits_{x\to-\infty} f(x)=L$.

PROBLEMS

1. Find the horizontal asymptote of the curve $\dfrac{x^2-1}{x^2+1}$

Soln:

Given: $f(x) = \dfrac{x^2-1}{x^2+1}$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2-1}{x^2+1}$$

$$= \lim_{x\to-\infty} \frac{x^2\left(1-\frac{1}{x^2}\right)}{x^2\left(1+\frac{1}{x^2}\right)}$$

$$= \frac{1-0}{1+0} = 1.$$

and $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{x^2-1}{x^2+1} = \lim\limits_{x\to\infty} \dfrac{x^2\left(1-\frac{1}{x^2}\right)}{x^2\left(1+\frac{1}{x^2}\right)}$

$$= \frac{1-0}{1+0} = 1.$$

Hence the line $y=1$ is a Horizontal asymptote of the given curve.

2. Find the horizontal and vertical asymptotes of the curve $\frac{\sqrt{2x^2+1}}{3x-5}$

Soln:

Let $f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2\left(2+\frac{1}{x^2}\right)}}{x\left(3-\frac{5}{x}\right)}$$

$$= \lim_{x\to\infty} \frac{x\sqrt{2+\frac{1}{x^2}}}{x\left[3-\frac{5}{x}\right]}$$

$$= \frac{\sqrt{2+0}}{3}$$

$$= \frac{\sqrt{2}}{3}.$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5}$$

when $x \to \infty$ we have $\sqrt{x^2} = -x,\ x<0$

$$= \lim_{x\to-\infty} \frac{-x\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)}$$

$$= \frac{-\sqrt{2}}{3}$$

$\therefore$ Both the line $y = \frac{-\sqrt{2}}{3}$ and $\frac{\sqrt{2}}{3}$ are horizontal asymptotes. The vertical asymptotes occurs when the given function becomes either $-\infty$ or ∞.

For $x = \frac{5}{3}$ the function becomes ∞.

$$\therefore \lim_{x\to\left(\frac{5}{3}\right)^+} f(x) = \lim_{x\to\frac{5}{3}^+} \frac{\sqrt{2x^2+1}}{3x-5} = \infty$$

and
$$\lim_{x\to\frac{5}{3}^-} f(x) = \lim_{x\to\frac{5}{3}^-} \frac{\sqrt{2x^2+1}}{3x-5} = -\infty.$$

9

SQUEEZE THEOREM:

If $f(x) \le g(x) \le h(x)$ when 'x' is near to 'a' [except possible at 'a'] and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ then $\lim_{x\to a} g(x) = L$.

PROBLEMS

1. Show that $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$

Proof:

Let, $f(x) = \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right),\ x \neq 0.$

If $x = 0$, $f(x)$ is not defined.

If $x \neq 0$, $\frac{1}{x}$ is real.

$\therefore \sin\frac{1}{x}$ is defined

$$-1 \le \sin\frac{1}{x} \le 1$$

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2$$

$$\lim_{x\to 0}(-x^2) \le \lim_{x\to 0} x^2 \sin\frac{1}{x} \le \lim_{x\to 0} x^2$$

since $\lim_{x\to 0}(-x^2) = 0$ and $\lim_{x\to 0}(x^2) = 0$

$\therefore$ By squeeze theorem,

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

2. Show that $\lim_{x\to 0} \sqrt{x^3 + x^2}\, \sin\frac{\pi}{x} = 0$

Proof:

Let, $f(x) = \lim_{x\to 0} \sqrt{x^3 + x^2}\, \sin\left(\frac{\pi}{x}\right)$

If $x = 0$, $f(x)$ is not defined

If $x \neq 0$, $\frac{\pi}{x}$ is real.

$\therefore \sin\frac{\pi}{x}$ is defined.

$$-1 \le \sin\frac{\pi}{x} \le 1$$

$$-\sqrt{x^3+x^2} \le \sqrt{x^3+x^2}\sin\frac{\pi}{x} \le \sqrt{x^3+x^2}$$

$$\lim_{x\to 0} -\sqrt{x^3+x^2} \le \lim_{x\to 0}\sqrt{x^3+x^2}\sin\frac{\pi}{x} \le \lim_{x\to 0}\sqrt{x^3+x^2}$$

since $\lim_{x\to 0} -\sqrt{x^3+x^2} = 0$ & $\lim_{x\to 0}\sqrt{x^3+x^2} = 0$.

$\therefore$ By squeeze theorem, $\lim_{x\to 0}\sqrt{x^3+x^2}\sin\frac{\pi}{x} = 0$.

8. Show that $\lim_{x\to 0} x^2\cos\left(\frac{1}{x^2}\right)$.

Proof:

Let, $-1 \le \cos\frac{1}{x^2} \le 1$

$$-x^2 \le x^2\cos\frac{1}{x^2} \le x^2$$

$$\lim_{x\to 0} -(x^2) \le \lim_{x\to 0} x^2\cos\frac{1}{x^2} \le \lim_{x\to 0} x^2$$

since $\lim_{x\to 0}(-x^2) = 0$ & $\lim_{x\to 0}(x^2) = 0$

By squeeze theorem, $\lim_{x\to 0} x^2\cos\left(\frac{1}{x^2}\right) = 0$.

<u>CONTINUITY</u>

A function 'f' is continuous at the point 'a' if $\lim_{x\to a} f(x) = f(a)$.

* A function 'f' is continuous from right at a point 'a' if $\lim_{x\to a^+} f(x) = f(a)$.

* A function 'f' is continuous from left at a point 'a' if $\lim_{x\to a^-} f(x) = f(a)$.

<u>RESULT</u>: $f(x)$ is continuous if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$.

<u>PROBLEMS</u>

1. Show that the function $f(x) = 1-\sqrt{1-x^2}$ is continuous on the interval $[-1,1]$.

Soln:

Let, $f(x) = 1-\sqrt{1-x^2}$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} 1-\sqrt{1-x^2}$$
$$= 1-\sqrt{1-(-1)^2}$$
$$= 1$$

& $$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 1-\sqrt{1-x^2}$$
$$= 1-\sqrt{1-1}$$
$$= 1$$

$$\lim_{x\to a} f(x) = \lim_{x\to a} 1-\sqrt{1-x^2}$$
$$= 1-\sqrt{1-a^2}$$
$$= f(a)$$
$\therefore -1 < a < 1$

f is continuous.

$$\therefore \lim_{x\to -1^+} f(x) = \lim_{x\to 1^-} f(x) = 1.$$

$\therefore f(x)$ is continuous on the interval $[-1, 1]$.

2. For what value of the constant 'c' is the function 'f' continuous on $(-\infty, \infty)$, $f(x) = \begin{cases} cx^2+2x; & x < 2 \\ x^3-cx; & x \geq 2 \end{cases}$

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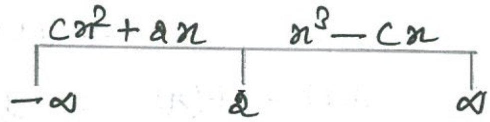
Soln:

The given function $f(x)$ is continuous on $(-\infty, 2)$ and $(2, \infty)$. Now,



$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (cx^2+2x)$$
$$= c(4) + 2(2)$$
$$= 4c+4$$

Also, $$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x^3-cx)$$
$$= 8-2c$$

We know that a function 'f' is continuous at a point 'a' if

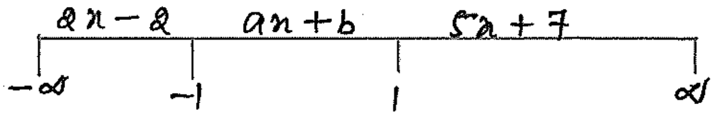
$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

$$\therefore 4c+4 = 8-2c$$
$$4c+2c = 4$$
$$c = 4/6$$
$$\boxed{c = 2/3}$$

8. Let $f(x) = \begin{cases} 2x-2, & x<-1 \\ ax+b, & -1 \le x \le 1 \\ 5x+7, & x \ge 1 \end{cases}$ is continuous for all real 'x', Find the values of 'a' & 'b'.

Soln:

Given: $f(x)$ is continuous.



$\therefore \lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^-} 2x-2$

$= -2-2 = -4$

& $\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+} ax+b$

$= a(-1)+b = -a+b$

$\because$ f is continuous, $\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^+} f(x)$

$-4 = -a+b$

$\Rightarrow a-b = 4 \longrightarrow (1)$

Also, $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} ax+b$

$= a(1)+b = a+b$

& $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} 5x+7$

$= 5+7 = 12$

$\because$ f is continuous, $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x)$

$a+b = 12 \longrightarrow (2)$

From (1) & (2)

$$\begin{array}{r} a+b = 12 \\ a-b = 4 \\ \hline 2a = 16 \end{array} \quad (+)$$

$\boxed{a = 8}$

$\therefore (1) \Rightarrow 8-b = 4$

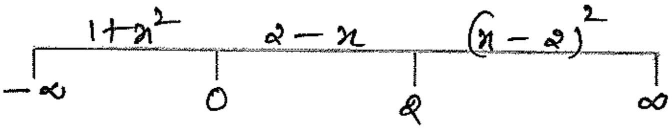
$\boxed{b = 4}$

$\therefore a = 8$

& $b = 4$.

4. Find the domain where the function 'f' is continuous. Also find the number at which the function 'f' is discontinuous where $f(x) = \begin{cases} 1+x^2, & x \le 0 \\ 2-x, & 0 < x \le 2 \\ (x-2)^2, & x > 2 \end{cases}$

Soln:



At $x=0$,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (1+x^2) = 1 \quad \text{——} \ (1)$$

& $$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (2-x) = 2 \quad \text{——} \ (2)$$

$(1) \ne (2)$ $\therefore$ $f(x)$ is discontinuous at $x=0$.

At $x=2$,

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 2-x = 0 \quad \text{——} \ (3)$$

& $$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x-2)^2 = 0 \quad \text{——} \ (4)$$

$(3) = (4)$ $\therefore$ $f(x)$ is continuous at $x=2$.

$\therefore$ Domain: $(-\infty, 0) \cup (0, \infty)$.

5. Find the values of 'a' and 'b' that make 'f' continuous on $(-\infty, \infty)$, $f(x) = \begin{cases} \dfrac{x^2-8}{x-2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \le x \le 3 \\ 2x - a + b & \text{if } x \ge 3 \end{cases}$

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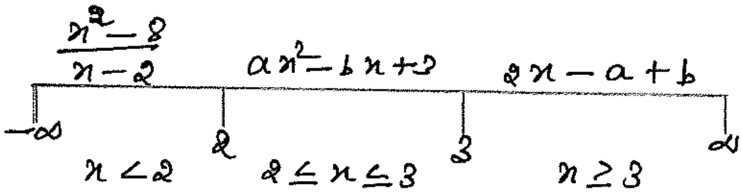
Soln:



At $x=2$,

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^2-8}{x-2} = \frac{2^2-8}{2-2} = \frac{-4}{0} = \infty$$

Apply L' Hospital's Rule,

$$= \lim_{x \to 2^-} \frac{2x}{1} = 2(2) = 4$$

Also, $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} ax^2 - bx + 3$

$$= 4a - 2b + 3$$

Given: f is continuous.

$$\therefore \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$

$$\Rightarrow \quad 4a - 2b + 3 = 4$$

$$4a - 2b = 1 \quad \text{------} \quad (1)$$

At $x = 3$,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} ax^2 - bx + 3$$

$$= 9a - 3b + 3 \quad \text{------} \quad (2)$$

Also,

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 2x - a + b$$

$$= 6 - a + b \quad \text{------} \quad (3)$$

Given: f is continuous,

$$\therefore \lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x)$$

$$9a - 3b + 3 = 6 - a + b \quad \text{By } (2) \text{ \& } (3)$$

$$10a - 4b = 3 \quad \text{------} \quad (4)$$

From (1) & (4) $\quad 10a - 4b = 3$

$$(-) \quad 8a - 4b = 2 \quad [(1) \times 2]$$

$$2a = 1$$

$$\boxed{a = 1/2}$$

$$\therefore (1) \Rightarrow 4 \frac{1}{2} - 2b = 1$$

$$2 - 1 = 2b$$

$$\boxed{b = 1/2}$$

DERIVATIVE :

The derivative of a function 'f' at a number 'a', denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ if this limit exists.

PROBLEMS

1. If $f(x) = \sqrt{x}$, find the derivative of $f(x)$.

Soln:

By the definition of derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

2. If $f(x) = \sin x$, find the derivative of $f(x)$.

Soln:

Let, $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$\because \sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}}{h}$$

$$= \lim_{h \to 0} \frac{\sin\frac{h}{2}}{h/2} \cos\left(x + \frac{h}{2}\right)$$

$$= \lim_{h \to 0} \frac{\sin h/2}{h/2} \times \lim_{h \to 0} \cos\left(x + \frac{h}{2}\right)$$

$$= 1 \times \cos x$$

$\because \lim_{x \to 0} \frac{\sin x}{x} = 1.$

$$= \cos x$$

8. Find the derivative of the function $f(x) = \frac{1}{\sqrt{x}}$.

Soln:

Let, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x}\sqrt{x+h}\,.\,h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x}\sqrt{x+h}\,.\,h} \times \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{\sqrt{x}\sqrt{x+h}\,h\left[\sqrt{x} + \sqrt{x+h}\right]}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}\sqrt{x}\sqrt{x+h}\left[\sqrt{x} + \sqrt{x+h}\right]}$$

$$= \lim_{h \to 0} \frac{-1}{\sqrt{x}\sqrt{x+h}\left[\sqrt{x} + \sqrt{x+h}\right]}$$

$$= \frac{-1}{\sqrt{x}\sqrt{x}\left(\sqrt{x} + \sqrt{x}\right)} = \frac{-1}{2x\sqrt{x}}$$

<u>RULES OF DIFFERENTIATION:</u>

1. $\frac{d}{dx}(c) = 0$ 2. $\frac{d}{dx}(c.u) = c\frac{du}{dx}$, [u is a function of x & c is a constant]
3. Product rule, $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
4. Quotient rule, $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$
5. Chain rule,
   (i) If 'y' is a function of 'u' and 'u' itself is a function of x, then $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$
   (ii) If 'y' is a function of 'u', 'u' is a function of 'v', 'v' is a function of 'w' and 'w' is a function of 'x', then $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dw}\cdot\frac{dw}{dx}$

PROBLEMS

1. If $f(x) = e^x(x + x\sqrt{x})$, find the derivative of $f(x)$.

Soln:

Given: $f(x) = e^x(x + x\sqrt{x})$

$= e^x(x + x^{3/2})$

$f'(x) = \frac{d}{dx}\left[e^x\left(x + e^{3/2}\right)\right]$

$= e^x \frac{d}{dx}(x + x^{3/2}) + (x + x^{3/2})\frac{d}{dx} e^x$

$= e^x\left(1 + \frac{3}{2}x^{\frac{3}{2}-1}\right) + (x + x^{3/2})e^x$

$= e^x\left[1 + \frac{3}{2}x^{1/2} + x + x^{3/2}\right]$

$= e^x\left[1 + x + \frac{3}{2}\sqrt{x} + x\sqrt{x}\right]$

2. Find $\frac{dy}{dx}$ if $y = x^2 e^{2x}(x^2+1)^4$. APRIL/MAY 2019

Soln:

Given: $y = x^2 e^{2x}(x^2+1)^4$

$\frac{dy}{dx} = \frac{d}{dx}\left[x^2 e^{2x}(x^2+1)^4\right]$

$= x^2 e^{2x}\frac{d}{dx}(x^2+1)^4 + x^2(x^2+1)^4\frac{d}{dx}e^{2x} + e^{2x}(x^2+1)^4\frac{d}{dx}(x^2)$

$= x^2 e^{2x}\, 4(x^2+1)^3(2x) + x^2(x^2+1)^4(2e^{2x}) + e^{2x}(x^2+1)^4(2x)$

$= 8x^3 e^{2x}(x^2+1)^3 + 2x^2 e^{2x}(x^2+1)^4 + 2xe^{2x}(x^2+1)^4$

$= 2xe^{2x}(x^2+1)^3\left[4x^2 + x(x^2+1) + x^2+1\right]$

$= 2xe^{2x}(x^2+1)^3\left[4x^2 + x^3 + x + x^2 + 1\right]$

$= 2xe^{2x}(x^2+1)^3(x^3 + 5x^2 + x + 1)$

3. If $f(x) = \frac{1 - xe^x}{x + e^x}$, find the derivative of $f(x)$.

Soln:

Let, $f'(x) = \frac{d}{dx}\left[\frac{1 - xe^x}{x + e^x}\right]$

$= \frac{(x + e^x)\frac{d}{dx}(1 - xe^x) - (1 - xe^x)\frac{d}{dx}(x + e^x)}{(x + e^x)^2}$

$$= \frac{(x+e^x)(-e^x - xe^x) - (1-xe^x)(1+e^x)}{(x+e^x)^2}$$

$$= \frac{-xe^x - x^2e^x - e^{2x} - xe^{2x} - 1 - e^x + xe^x + xe^{2x}}{(x+e^x)^2}$$

$$= \frac{-x^2e^x - e^{2x} - 1 - e^x}{(x+e^x)^2}$$

$$= \frac{-(x^2e^x + e^{2x} + e^x + 1)}{(x+e^x)^2}$$

4. If $y = (1-x^2)^{10}$, find the derivative of y.

Soln:

Given: $y = (1-x^2)^{10}$

Let $u = 1 - x^2 \quad \therefore \quad y = u^{10}$

$$\frac{du}{dx} = -2x \qquad \frac{dy}{du} = 10u^9$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= 10u^9(-2x) = -20x(1-x^2)^9.$$

5. If $y = \tan(\sin x)$, find the derivative of y.

Soln:

Let $u = \sin x \quad \therefore \quad y = \tan u$

$$\frac{dy}{du} = \sec^2 u = \sec^2(\sin x)$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \sec^2(\sin x)\cos x$$

6. If $y = \log\left(x + \sqrt{x^2-1}\right)$, find the derivative of 'y'.

Soln:

Let $y = \log u$ where $u = x + \sqrt{x^2-1}$

$$\frac{dy}{du} = \frac{1}{u} \qquad \frac{du}{dx} = 1 + \frac{1}{\cancel{2}\sqrt{x^2-1}}(\cancel{2}x)$$

$$= 1 + \frac{x}{\sqrt{x^2-1}}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{u}\left(1 + \frac{x}{\sqrt{x^2-1}}\right)$$

$$= \frac{1}{x+\sqrt{x^2-1}}\left(1+\frac{x}{\sqrt{x^2-1}}\right)$$

$$= \frac{1}{x+\sqrt{x^2-1}} \quad \frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}$$

$$= \frac{1}{\sqrt{x^2-1}}$$

7. If $y = \sin(\sin(\sin x))$, find the derivative of 'y'.

Soln:

Let $y = \sin u$, $u = \sin v$, $v = \sin x$

$$\frac{dy}{du} = \cos u \quad \frac{du}{dv} = \cos v, \quad \frac{dv}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$= \cos u \cos v \cos x$$

$$= \cos(\sin v) \cos(\sin x) \cos x$$

$$= \cos(\sin(\sin x)) \cos(\sin x) \cos x.$$

8. If $y = \cos\sqrt{\sin(\tan \pi x)}$, find the derivative of 'y'.

Soln:

Let $y = \cos\sqrt{u}$, $u = \sin v$, $v = \tan \pi x$

$$\frac{dy}{du} = -\sin\sqrt{u}\left(\frac{1}{2\sqrt{u}}\right), \quad \frac{du}{dv} = \cos v, \quad \frac{dv}{dx} = \sec^2 \pi x (\pi)$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$= \frac{-1}{2\sqrt{u}} \sin\sqrt{u} \times \cos v \times \pi \sec^2 \pi x$$

$$= \frac{-1}{2\sqrt{\sin v}} \sin\sqrt{\sin v} \times \cos(\tan \pi x)\, \pi \sec^2 \pi x$$

$$= -\frac{\pi}{2} \quad \frac{\sin\sqrt{\sin(\tan \pi x)}}{\sqrt{\sin(\tan \pi x)}} \times \cos(\tan \pi x) \sec^2 \pi x.$$

IMPLICIT DIFFERENTIATION:

PROBLEMS

1. If $\sqrt{x}+\sqrt{y}=1$ then find $\frac{dy}{dx}$

Soln:

Given: $\sqrt{x}+\sqrt{y}=1$

Differentiate with respect to 'x'

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-2\sqrt{y}}{2\sqrt{x}}=-\sqrt{\frac{y}{x}}$$

2. Find y'' if $x^4+y^4=16$

Soln: JAN 2018

Given $x^4+y^4=16$

Differentiate w.r.to 'x'.

$$4x^3+4y^3\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-x^3}{y^3}=y'$$

$$\therefore y''=\frac{d^2y}{dx^2}=-\frac{y^3(3x^2)-(x^3)(3y^2)\frac{dy}{dx}}{y^6}$$

$$=-\frac{3x^2y^3-3x^3y^2\left(\frac{-x^3}{y^3}\right)}{y^6}$$

$$=-\frac{3x^2y^3+\frac{3x^6}{y}}{y^6}$$

$$=-\frac{3x^2y^4+3x^6}{y^7}$$

$$=-\frac{3x^2}{y^7}\left[y^4+x^4\right]=\frac{-3x^2}{y^7}(16)$$

$$=\frac{-48x^2}{y^7}.$$

3. Find y' if $\cos(xy) = 1+\sin y$

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Soln:

Given $\cos xy = 1+\sin y$

Diff., w.r. to 'x'.

$$-\sin xy\left[x\frac{dy}{dx} + y(1)\right] = \cos y \frac{dy}{dx}$$

$$-\sin xy \, x\frac{dy}{dx} - y\sin xy - \cos y\frac{dy}{dx} = 0$$

$$-\frac{dy}{dx}\left[x\sin xy + \cos y\right] = y\sin xy$$

$$\therefore \frac{dy}{dx} = \frac{-y\sin xy}{x\sin xy + \cos y}$$

4. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, then prove that $\frac{dy}{dx} = \frac{-1}{(1+x)^2}$.

Soln:

Let $x\sqrt{1+y} = -y\sqrt{1+x}$

Squaring on both sides,

$$x^2(1+y) = y^2(1+x)$$

$$x^2 + x^2y = y^2 + y^2x$$

$$(x^2 - y^2) + x^2y - y^2x = 0$$

$$(x+y)(x-y) + xy(x-y) = 0$$

$$x-y\left[x+y+xy\right] = 0$$

If $x-y=0$, then $1-\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = 1.$$

$|||^{ly}$ If $x+y+xy = 0$

$$y + xy = -x$$

$$y(1+x) = -x$$

$$y = \frac{-x}{1+x}$$

$$\frac{dy}{dx} = -\frac{(1+x)(1) - x(1)}{(1+x)^2}$$

$$= -\frac{1+x-x}{(1+x)^2} = \frac{-1}{(1+x)^2}.$$

DERIVATIVE OF TRIGONOMETRIC FUNCTIONS:

FORMULA'S

1. $\frac{d}{dx}(\sin x) = \cos x$
2. $\frac{d}{dx}(\cos x) = -\sin x$
3. $\frac{d}{dx}(\tan x) = \sec^2 x$
4. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
5. $\frac{d}{dx}(\sec x) = \sec x \tan x$
6. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

1. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
2. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
3. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
4. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
5. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$
6. $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

1. If $f(x) = \cos^{-1}\left[\frac{b+a\cos x}{a+b\cos x}\right]$, find the derivative of $f(x)$.

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soln:

Let $u = \frac{b+a\cos x}{a+b\cos x}$

$\therefore$ Let $y = f(x) = \cos^{-1}(u)$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$= \frac{-1}{\sqrt{1-u^2}}\left[\frac{(a+b\cos x)(-a\sin x)-(b+a\cos x)(-b\sin x)}{(a+b\cos x)^2}\right]$$

$$= \frac{-1}{\sqrt{1-\frac{(b+a\cos x)^2}{(a+b\cos x)^2}}}\left[\frac{-a^2\sin x - \cancel{ab\sin x\cos x} + b^2\sin x + \cancel{ab\sin x\cos x}}{(a+b\cos x)^2}\right]$$

$$= \frac{-(a+b\cos x)}{\cancel{a^2}+b^2+\cancel{2ab\cos x}-b^2-\cancel{a^2}\cos^2 x - \cancel{2ab\cos x}} \quad \frac{-(a^2-b^2)\sin x}{(a+b\cos x)^2}$$

$$= \frac{(a^2-b^2)\sin x}{\sqrt{a^2(1-\cos^2 x)-b^2(1-\cos^2 x)}\,(a+b\cos x)}$$

$$= \frac{(a^2-b^2)\sin x}{\sqrt{a^2\sin^2 x-b^2\sin^2 x}\,(a+b\cos x)}$$

$$= \frac{(a^2-b^2)\cancel{\sin x}}{\cancel{\sin x}\sqrt{a^2-b^2}\,(a+b\cos x)}$$

$$= \frac{\sqrt{a^2-b^2}\;\cancel{\sin x}}{a+b\cos x}$$

2. Find $\frac{dy}{dx}$ if $y=\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$

Soln:

Let $x=\tan\theta$ ——— (1)

$$\therefore\ y=\tan^{-1}\left[\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right]$$

$$=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$=\tan^{-1}\left(\frac{1/\cos\theta-1}{\sin\theta/\cos\theta}\right)$$

$$=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{\cancel{2}\sin^{\cancel{2}}(\theta/2)}{\cancel{2}\cancel{\sin}(\theta/2)\cos(\theta/2)}\right)$$

$$=\tan^{-1}(\tan(\theta/2))$$

$$y=\theta/2$$

$$y=\frac{\tan^{-1}x}{2} \quad \text{By (1)}$$

$$\therefore\ \frac{dy}{dx}=\frac{1}{2}\left(\frac{1}{1+x^2}\right)$$

DERIVATIVE OF LOGARITHMIC FUNCTIONS:-

PROBLEMS

1. Differentiate $y = x^x$

Soln:

Given $y = x^x$

Taking 'log' on both sides,

$$\log y = \log x^x$$

$$\log y = x \log x$$

Differentiate with respect to 'x'.

$$\frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + \log x\,(1)$$

$$= 1 + \log x$$

$$\frac{dy}{dx} = y\left(1 + \log x\right)$$

$$= x^x\left(1 + \log x\right)$$

2. Find y' if $x^y = y^x$

Soln:

Given $y^x = x^y$

Taking 'log' on both sides,

$$\log x^y = \log y^x$$

$$y \log x = x \log y$$

Differentiate with respect to 'x'.

$$y\frac{1}{x} + \log x\frac{dy}{dx} = x\frac{1}{y}\frac{dy}{dx} + \log y\,(1)$$

$$\frac{dy}{dx}\left(\log x - \frac{x}{y}\right) = \log y - \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{x\log y - y}{x} \times \frac{y}{y\log x - x}$$

$$\frac{dy}{dx} = \frac{y\left(x\log y - y\right)}{x\left(y\log x - x\right)}$$

3. Differentiate $y = x^{\cos x}$

Soln:

Given: $y = x^{\cos x}$

Taking 'log' on both sides

$$\log y = \log\left(x^{\cos x}\right)$$

$\log y = \cos x \log x$

Differentiate w.r.to 'x'

$$\frac{1}{y}\frac{dy}{dx} = \cos x\left(\frac{1}{x}\right) + \log x(-\sin x)$$

$$\frac{dy}{dx} = y\left(\frac{\cos x - x\sin x \cdot \log x}{x}\right)$$

$$= x^{\cos x}\left(\frac{\cos x - x\sin x \log x}{x}\right)$$

TANGENT & NORMAL:

* The equation of tangent at a given point (x_1, y_1) is given by $y - y_1 = m(x - x_1)$.
* The equation of normal at a given point (x_1, y_1) is given by $y - y_1 = \frac{-1}{m}(x - x_1)$.

PROBLEMS

1. Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal?

Soln:

Given: $y = x^4 - 6x^2 + 4$

Differentiate w.r.to 'x',

$$\frac{dy}{dx} = 4x^3 - 12x$$

The tangent line is horizontal if $y' = 0$

$$\therefore y' = 4x^3 - 12x = 0$$

$$\Rightarrow 4x(x^2 - 3) = 0$$

$$4x = 0, \quad x^2 - 3 = 0$$

$$\boxed{x = 0} \quad \boxed{x = \pm\sqrt{3}}$$

$\therefore$ So the given curve has horizontal tangents when $x = 0, \sqrt{3}$ & $-\sqrt{3}$.

when $x = 0 \Rightarrow y = 0 - 6(0) + 4 \Rightarrow \boxed{y = 4}$

$\Rightarrow (0, 4)$

when $x = \sqrt{3} \Rightarrow (\sqrt{3})^4 - 6(\sqrt{3})^2 + 4 = y$

$\Rightarrow \boxed{y = -5} \Rightarrow (\sqrt{3}, -5)$

When $x = -\sqrt{3} \Rightarrow y = (-\sqrt{3})^4 - 6(-\sqrt{3})^2 + 4$

$\boxed{y = -5} \Rightarrow (-\sqrt{3}, -5)$.

$\therefore$ The corresponding points are $(0,4)$, $(\sqrt{3}, -5)$ & $(-\sqrt{3}, -5)$.

2. Find the equation of the tangent line to the curve $x^3 + y^3 = 6xy$ at the point $(3,3)$ and what point the tangent line horizontal in the first quadrant.

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Soln:

Given: $x^3 + y^3 = 6xy$, Diff. w.r.to 'x',

$$3x^2 + 3y^2 \frac{dy}{dx} = 6\left[x\frac{dy}{dx} + y(1)\right]$$

$$x^2 + y^2 \frac{dy}{dx} = 2x\frac{dy}{dx} + 2y$$

$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x} \quad \text{—— (1)}, \quad \text{at } (3,3) \ \frac{dy}{dx} = \frac{6-9}{9-6} = -1.$$

$\therefore \boxed{m = -1}$

The equation of the tangent at the point $(3,3)$ is

$$y - 3 = -1(x-3) \Rightarrow y - 3 = -x + 3$$

$$\boxed{x + y = 6}$$

The tangent line is horizontal if $y' = 0$

$$(1) \Rightarrow \frac{2y - x^2}{y^2 - 2x} = 0 \Rightarrow 2y - x^2 = 0$$

$$\boxed{y = x^2/2} \quad \text{—— (2)}$$

Let $x^3 + y^3 = 6xy$

$$x^3 + \left(\frac{x^2}{2}\right)^3 = 6x\frac{x^2}{2} \quad \text{By (2)}$$

$$x^3 + \frac{x^6}{8} = \frac{6x^3}{2} \Rightarrow x^3 + \frac{x^6}{8} = 3x^3$$

$$\Rightarrow 2x^3 = \frac{x^6}{8}$$

$$x^6 = 16x^3$$

$$\boxed{x^3 = 16}$$

$$\therefore x = 16^{1/3}$$

$$\Rightarrow \boxed{x = 2^{4/3}}$$

$$\therefore (2) \Rightarrow y = \frac{(2^{4/3})^2}{2} = \frac{2^{8/3}}{2} = 2^{5/3}.$$

$\therefore$ The tangent is horizontal at $(2^{4/3}, 2^{5/3})$.

Applications: Maxima and minima of functions of one variable.

Let 'C' be a point in a domain 'D' of the function 'f'.
Then $f(c)$ is the * absolute maximum value of 'f' on 'D' if $f(c) \geq f(x)$, for all 'x' in 'D'.
* absolute minimum value of 'f' on 'D' if $f(c) \leq f(x)$ for all 'x' in 'D'.

Absolute Maximum and Minimum of $f(x)$:

Step(i): Find the critical numbers of $f(x)$.
i.e $\boxed{f'(x) = 0}$

Step(ii): Substitute the critical points at $f(x)$.

Step(iii): Then the maximum value of $f(x)$ is absolute maximum and the minimum value of $f(x)$ is absolute minimum.

1. Find the absolute maximum and absolute minimum value of $f(x) = x - 2\sin x$ on $[0, 2\pi]$.

Soln:

Given: $f(x) = x - 2\sin x$

$\therefore f'(x) = 1 - 2\cos x$

Critical numbers: $f'(x) = 0$

$\Rightarrow 1 - 2\cos x = 0$

$\cos x = 1/2$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

$\therefore f(\pi/3) = \frac{\pi}{3} - 2\sin\left(\frac{\pi}{3}\right) = -0.68485$ [Radian mode]

$f(5\pi/3) = \frac{5\pi}{3} - 2\sin\left(\frac{5\pi}{3}\right) = 6.968039$

At end points, $f(0) = 0 - 2\sin 0 = 0$

$f(2\pi) = 2\pi - 2\sin 2\pi = 2\pi = 6.28$.

$\therefore$ The absolute minimum value is $f\left(\frac{\pi}{3}\right) = -0.68485$

The absolute maximum value is $f\left(\frac{5\pi}{3}\right) = 6.9680$.

2. Find the absolute maximum and absolute minimum value of $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ on $[-2, 3]$.

Soln:

Given: $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$

$f'(x) = 12x^3 - 12x^2 - 24x$

The critical numbers of $f(x)$ are occured at

$f'(x) = 0 \Rightarrow 12x^3 - 12x^2 - 24x = 0$

$12x(x^2 - x - 2) = 0$ $\quad -2 \quad -1$ $\quad -2\times1 \quad -2+1$

$x = 0,\ x = 2,\ x = -1$

The values of $f(x)$ at these critical numbers are

$f(0) = 0 + 1 = 1$

$f(2) = 3(16) - 4(8) - 12(4) + 1 = -4$

$f(-1) = 3(1) - 4(-1) - 12(1) + 1 = -31$

At end points,

$f(-2) = 33$

$f(3) = 28$

$\therefore$ The absolute minimum value is $f(2) = -31$

The absolute maximum value is $f(-2) = 33$.

FIRST AND SECOND DERIVATIVE TEST:

INCREASING / DECREASING / CONCAVITY / INFLECTION POINTS :-

INCREASING & DECREASING FUNCTIONS:

* If $f'(x) > 0$ in an Interval (a, b), then 'f' is increasing $\uparrow$.
* If $f'(x) < 0$ in an Interval (a, b), then 'f' is decreasing $\downarrow$.

CRITICAL NUMBER:

A critical number of a function 'f' is a number 'c' in the domain of 'f' such that $f'(c) = 0$ (or) $f'(c)$ does not exists.

FIRST DERIVATIVE TEST:

Suppose that 'c' is a critical number of a continuous function 'f' then

* If $f'(x)$ changes from positive to negative at 'c', then $f(x)$ has local maximum at 'c'.
* If $f'(x)$ changes from negative to positive at 'c', then $f(x)$ has local minimum at 'c'.

* If $f'(x)$ does not changes sign at 'c', then $f(x)$ has no local maximum or minimum at 'c'.

Note: The first derivative test is a consequence of the increasing and decreasing test.

CONCAVITY: CONCAVE UPWARDS / CONCAVE DOWNWARDS

* If $f''(x) > 0$ in any interval, then $f(x)$ is concave upwards [convex downwards].
* If $f''(x) < 0$ in any interval, then $f(x)$ is convex upwards [concave downwards].

INFLECTION POINTS:

A point 'P' on a curve $y = f(x)$ is called an inflection point if $f(x)$ is continuous and the curve changes from concave upwards to concave downwards (or) from concave downward to concave upward at 'P'.

SECOND DERIVATIVE TEST:

Suppose $f''(x)$ is continuous near 'c',

* If $f'(c) = 0$ and $f''(c) > 0$, then $f(x)$ has a local minimum at 'c'.
* If $f'(c) = 0$ and $f''(c) < 0$, then $f(x)$ has a local maximum at 'c'.

PROBLEMS

1. The profit function of acrosonic company is given by $P(x) = -0.02x^2 + 300x - 200{,}000$, dollars, where the function 'P' is increasing and where it is decreasing.

Soln:

The derivative p' of the function p is

$$p'(x) = -0.04x + 300$$

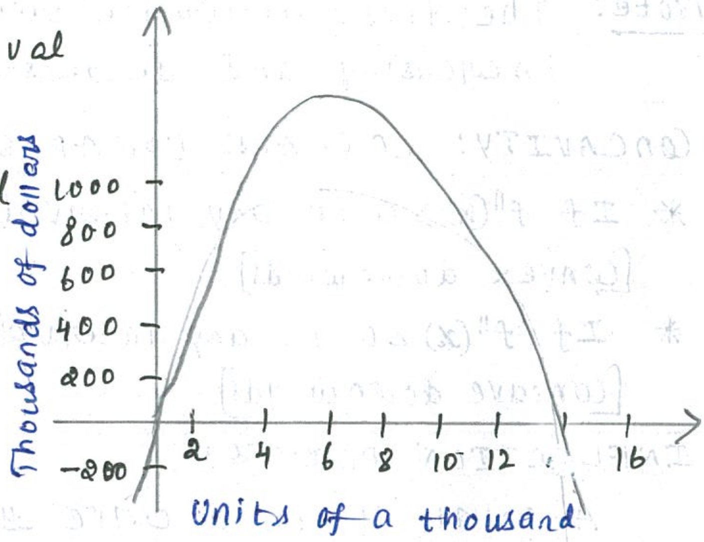
$$p'(x) = -0.04(x - 7500)$$

To find critical points: $p'(x) = 0$

$$-0.04(x - 7500) = 0$$

$$\boxed{x = 7500}$$

$0 \qquad 7500 \qquad \infty$



* $P'(x) > 0$ for 'x' in the interval $(0, 7500)$.
* $P'(x) < 0$ for 'x' in the interval $(7500, \infty)$.

This means that the profit function 'P' is increasing on $(0, 7500)$ and decreasing on $(7500, \infty)$.

2. For the function $f(x) = 2 + 2x^2 - x^4$, find the intervals of Increasing (or) decreasing, Local maximum (or) minimum values and the intervals of concavity also the inflection points.

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Soln:

Given: $f(x) = 2 + 2x^2 - x^4$

$f'(x) = 4x - 4x^3$

To find the critical points:-

$f'(x) = 0 \Rightarrow 4x - 4x^3 = 0$

$4x(1 - x^2) = 0$

$x = 0, \quad 1 = x^2$

$\boxed{x=0}, \boxed{x = \pm 1}$

$-\infty \quad -1 \quad 0 \quad 1 \quad \infty$

$\therefore$ The critical points are $-1, 0, 1$.

Interval	Sign of $f'(x)$	Behaviour of $f(x)$
$(-\infty, -1)$	$f'(-2) = 4(-2) - 4(-2)^3 = 24$ (+ive)	Increasing.
$(-1, 0)$	$f'(-1/2) = 4(-1/2) - 4(-1/2)^3 = -3/2$ (-ive)	Decreasing
$(0, 1)$	$f'(1/2) = 4(1/2) - 4(1/2)^3 = 3/2$, (+ive)	Increasing
$(1, \infty)$	$f'(2) = 4(2) - 4(2)^3 = -24$ (-ive)	Decreasing

Local maximum (or) minimum using First derivative Test:

From the above table,

* $f'(x)$ changes from $+$ive to $-$ive at $x=-1$.

  $\therefore$ $f(x)$ has a local maximum at $x=-1$.

  $\therefore$ Local maximum value at $x=-1$, $f(-1) = 2+2(-1)^2-(-1)^4$
  $= 3.$

* $f'(x)$ changes from $-$ive to $+$ive at $x=0$.

  $\therefore$ $f(x)$ has a local minimum at $x=0$.

  $\therefore$ Local minimum value at $x=0$, $f(0) = 2+2(0)-0$
  $=2$

* $f'(x)$ changes from $+$ive to $-$ive at $x=1$.

  $\therefore$ $f(x)$ has a local maximum at $x=1$.

  $\therefore$ Local maximum value at $x=1$, $f(1) = 2+2(1)-1$
  $=3.$

Concavity:

$$f''(x) = 4-12x^2$$

$$f''(x) = 0 \Rightarrow 4-12x^2 = 0$$

$$4 = 12x^2$$

$$x^2 = 1/3 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$$

$$= \pm 0.58$$

Interval	Sign of $f''(x)$	Behaviour of $f(x)$
$(-\infty, -0.58)$	$f(-1) = 4-12(-1)^2 = -8$, ($-$ive)	concave downwards
$(-0.58, 0.58)$	$f(0) = 4-12(0) = 4$, ($+$ive)	concave upwards
$(0.58, \infty)$	$f(1) = 4-12(1)^2 = -8$, ($-$ive)	concave downwards

Local maximum (or) minimum using second derivative Test:

* $f'(-1) = 0$ and $f''(-1) = 4-12(-1)^2 = -8 < 0$ (From the above table)

  $\therefore$ $f(x)$ is Local maximum at $x=-1$.

* $f'(0) = 0$ and $f''(0) = 4-12(0) = 4 > 0$

  $\therefore$ $f(x)$ is Local minimum at $x=0$.

* $f'(1) = 0$, $f''(1) = 4 - 12(1)^2 = -8 < 0$

$\therefore f(x)$ is local maximum at $x=1$.

Inflection points:

* Curve changes from concave downward to concave upward at $x = -0.58$.

$$\therefore f(-0.58) = 2 + 2(-0.58) - (-0.58)^4$$
$$= 2.56$$

$\therefore$ Inflection pts are $(-0.58, 2.56)$.

* Also, curve changes from concave upward to concave downward at $x = 0.58$.

$$\therefore f(0.58) = 2 + (0.58) - (0.58)^4$$
$$= 2.56.$$

$\therefore$ Inflection pts are $(0.58, 2.56)$.

2. Find the local maximum (or) minimum values of $f(x) = \sqrt{x} - \sqrt[4]{x}$ using both first and second derivative tests.

Soln:

Given: $f(x) = \sqrt{x} - \sqrt[4]{x}$

$$= x^{1/2} - x^{1/4}$$

$$f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{4}x^{-3/4}$$

$$= \frac{x^{-3/4}}{4}\left(2x^{1/4} - 1\right)$$

$$= \frac{2x^{1/4} - 1}{4x^{3/4}}$$

To find critical points:

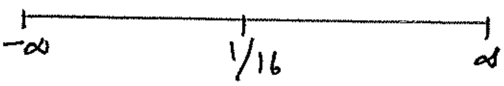
$$f'(x) = 0 \Rightarrow \frac{2x^{1/4} - 1}{4x^{3/4}} = 0$$

$$\Rightarrow 2x^{1/4} - 1 = 0$$

$$\Rightarrow 2x^{1/4} = 1$$

$$x^{1/4} = 1/2$$

$$\Rightarrow x = (1/2)^4 = 1/16$$



Increasing & Decreasing:

Interval	sign of $f'(x)$	Behaviour of $f(x)$
$(-\infty, 1/16)$	$f'(1/17) = f'(0.06) = \dfrac{2\sqrt[4]{0.06}-1}{4\sqrt[4]{(0.06)^3}} < 0$	Decreasing.
$(1/16, \infty)$	$f'(1) = \dfrac{2-1}{4} = 1/4$, +ive > 0	Increasing.

First Derivative Test (Local maximum (or) minimum):

* $f''(x)$ changes sign from negative to positive at $x = 1/16$.

$\therefore$ $f(x)$ has local minimum at $x = 1/16$.

$$f(1/16) = \sqrt{\frac{1}{16}} - \sqrt[4]{1/16} = -0.25$$

Second Derivative Test:

$$f''(x) = \frac{-1}{4} x^{-3/2} + \frac{3}{16} x^{-7/4}$$

$$= \frac{-1}{4\sqrt{x^3}} + \frac{3}{16\sqrt[4]{x^7}}$$

$$\therefore f''(1/16) = \frac{-1}{4\sqrt{(1/16)^3}} + \frac{3}{16\sqrt[4]{(1/16)^7}}$$

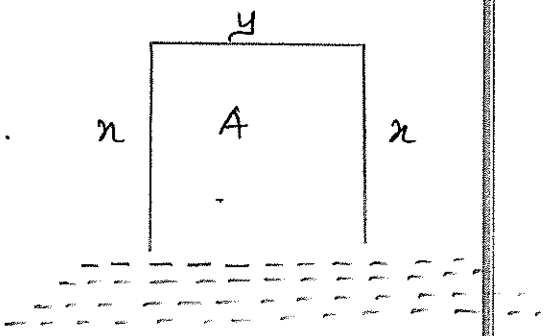
$$= -16 + 24$$

$$= 8 > 0.$$

$\therefore$ $f(x)$ has a local minimum at $x = 1/16$.

3. A farmer has 2400 ft. of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fencing along the river. What are the dimensions of the field that has the largest area?

Soln:



Given: Surface area $= 2400$ ft.

$$\therefore S = 2x + y$$

$$S = 2x + y = 2400$$

$$\boxed{y = 2400 - 2x}$$

Area $= xy$

$A = xy = x(2400 - 2x)$

$\boxed{A = 2400x - 2x^2}$

To find critical numbers:

$A'(x) = 0$

$\Rightarrow 2400 - 4x = 0$

$\boxed{x = 600}$

To find the dimensions of field,

$y = 2400 - 2(600)$

$= 1200$ ft.

$\therefore$ The dimensions of the field is 600×1200 ft.

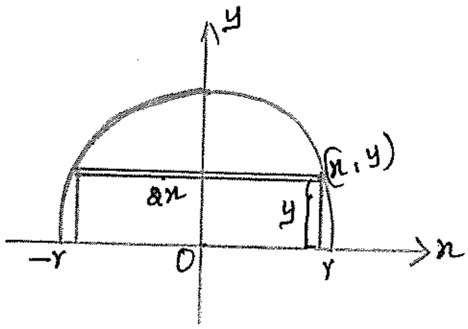
4. Find the area of the largest rectangle that can be inscribed in a semicircle of radius 'r'.

Soln:

The constraint is:

Area of circle: $x^2 + y^2 = r^2$ ——— (1)

& Area of rectangle: $A = 2xy$. ——— (2)



(1) $\Rightarrow y^2 = r^2 - x^2$

$y = \pm\sqrt{r^2 - x^2}$

$\therefore A = 2x\sqrt{r^2 - x^2}$

To find critical numbers:

$$A'(x) = 2x \frac{1}{2\sqrt{r^2 - x^2}}(-2x) + \sqrt{r^2 - x^2}\,(2) = 0$$

$$\Rightarrow \frac{-4x^2}{2\sqrt{r^2 - x^2}} + 2\sqrt{r^2 - x^2} = 0$$

$$\frac{-4x^2 + 4(r^2 - x^2)}{2\sqrt{r^2 - x^2}} = 0$$

Let $-4x^2 + 4(r^2 - x^2) = 0$ ——— (1)

$-x^2 + r^2 - x^2 = 0 \Rightarrow 2x^2 = r^2$

$\Rightarrow \boxed{x = \frac{\pm r}{\sqrt{2}}}$

Test critical numbers:

$$(1) \Rightarrow A'(x) = -4x^2 + 4r^2 - 4x^2$$

$$A''(x) = -16x$$

$$\therefore A''(r/\sqrt{2}) = -16\left(\frac{r}{\sqrt{2}}\right) < 0, \text{ since } r > 0$$

$\therefore$ A is local maximum at $x = \frac{r}{\sqrt{2}}$.

To find the area of the largest rectangle,

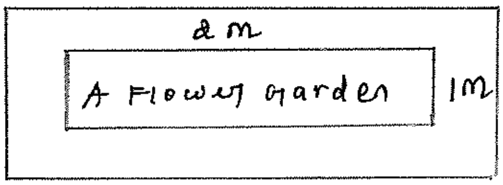
$$A\left(\frac{r}{\sqrt{2}}\right) = 2\left(\frac{r}{\sqrt{2}}\right)\sqrt{r^2 - \left(\frac{r}{\sqrt{2}}\right)^2}$$

$$= \frac{2r}{\sqrt{2}}\sqrt{r^2 - \frac{r^2}{2}}$$

$$= \frac{2r}{\sqrt{2}} \cdot \frac{r}{\sqrt{2}} = r^2$$

5. A rectangular flower garden with an area of $30m^2$ is surrounded by a grass border 1m wide on two sides and 2m wide on the other two sides. What dimensions of the garden minimize the combined area of the garden and borders?

Soln:



$$\text{Let } A = xy = 30$$

$$C = (x+4)(y+2)$$

$$\therefore xy = 30$$

$$y = 30/x$$

$$\therefore C = (x+4)\left(\frac{30}{x} + 2\right)$$

$$= 30 + 2x + \frac{120}{x} + 8$$

$$\therefore C(x) = 2x + \frac{120}{x} + 38$$

$$\text{Let } C'(x) = 2 - \frac{120}{x^2} = 0 \quad \text{——— (1)}$$

$$\Rightarrow 2x^2 - 120 = 0$$

$$x^2 = 60 \Rightarrow \boxed{x = \pm\sqrt{60}}$$

Second Derivative Test:

(1) $\Rightarrow C''(x) = \frac{240}{x^3}$

$\Rightarrow C''\sqrt{60} > 0.$

Hence 'c' is a maximum at $x = \sqrt{60}$

$\therefore y = \frac{30}{x} = \frac{30}{\sqrt{60}}$ m.

6. If $f(x) = 2x^3 + 3x^2 - 36x$, find the intervals on which it is increasing (or) decreasing, the local maximum (or) minimum values of $f(x)$.

Soln:

Given $f(x) = 2x^3 + 3x^2 - 36x$ April/May 2019

$f'(x) = 6x^2 + 6x - 36$

Let $f'(x) = 0 \Rightarrow x^2 + x - 6 = 0$ -6 1 -2×3 $-2+3$

$\boxed{x = 2, -3}$

Increasing / Decreasing:

Interval	Sign of $f'(x)$	Behaviour of
$(-\infty, -3)$	$f'(-4) = 6(-4)^2 + 6(-4) - 36$ $= 36$ (+ive)	Increasing
$(-3, 2)$	$f'(0) = 6(0^2) + 6(0) - 36$ $= -36$ (-ive)	Decreasing
$(2, \infty)$	$f'(3) = 6(3^2) + 6(3) - 36$ $= 36 > 0$	Increasing

First Derivative Test: [Local maximum / minimum]

* $f'(x)$ changes positive to negative at $x = -3$

$\therefore f(x)$ has a local maximum at $x = -3$.

$\therefore \boxed{f(-3) = 81}$

* $f'(x)$ changes negative to positive at $x = 2$

$\therefore f(x)$ has a local minimum at $x = 2$.

& $\boxed{f(2) = -44}$.

Second Derivative Test: $f''(x) = 12x + 6$

* $f'(-3) = 0$ and $f''(-3) = -30 < 0$

$\therefore f(x)$ is local maximum at $x = -3$.

* $f'(2) = 0$ & $f''(2) = 30 > 0$ $\therefore f(x)$ is local minimum at $x = 2$.