UNIT II / FUNCTIONS OF SEVERAL VARIABLES

Partial differentiation - Homogeneous functions and Euler's theorem - Total derivative - Change of variables - Jacobians - Partial differentiation of implicit functions - Taylor's series for functions of two variables - Maxima and minima for functions of two variables - Lagrange's method of undetermined multipliers.

PARTIAL DIFFERENTIATION :-

If $z = f(x,y)$ be a function of two variables x & y and if we keep y as constant and vary x alone, then z is a function of x only.

The derivative of z w.r.t x, treating y as constant is called the partial derivatives of z w.r. to x and it is denoted by $\frac{\partial z}{\partial x}$, $\frac{\partial f}{\partial x}$ or f_x.

Note :- ① $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$, $f_{xx} = \frac{\partial^2 f}{\partial x^2}$, $f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$,

$f_{yx} = \frac{\partial^2 f}{\partial y \partial x}$, $f_{yy} = \frac{\partial^2 f}{\partial y^2}$.

② $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

PROBLEMS :-

① If $u = (x-y)(y-z)(z-x)$, then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Soln

Given :- $u = (x-y)(y-z)(z-x)$

$$\frac{\partial u}{\partial x} = (y-z)\left[(x-y)(-1) + (z-x)(1)\right]$$
$$= -(y-z)(x-y) + (z-x)(y-z).$$

$$\frac{\partial u}{\partial y} = (z-x)\left[(x-y)(1) + (y-z)(-1)\right]$$
$$= (z-x)(x-y) - (z-x)(y-z).$$

$$\frac{\partial u}{\partial z} = (x-y)\left[(y-z)(1) + (z-x)(-1)\right]$$
$$= (x-y)(y-z) - (x-y)(z-x)$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

② If $u = x^y$, then show that (i) $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 y}{\partial y \partial x}$

(ii) $u_{xxy} = u_{xyx}$

Soln:- Given:- $u = x^y = e^{\log x^y} = e^{y \log x}$

$$\frac{\partial u}{\partial y} = e^{y\log x} \log x$$

$$\frac{\partial^2 u}{\partial x \partial y} = e^{y\log x}\left(\frac{1}{x}\right) + \log x \; e^{y\log x}\, y\left(\frac{1}{x}\right)$$

$$= \frac{x^y}{x} + \log x \frac{x^y}{x} \cdot y$$

$$= x^{y-1} + x^{y-1} y\log x$$

$$= x^{y-1}[1 + y\log x] \longrightarrow ①$$

Now $\frac{\partial u}{\partial x} = e^{y\log x}\, y\left(\frac{1}{x}\right) = e^{y\log x} \cdot \frac{y}{x}$

$$\frac{\partial^2 u}{\partial y \partial x} = e^{y\log x}\left(\frac{1}{x}\right) + \frac{y}{x} e^{y\log x} \log x$$

$$= \frac{x^y}{x} + \frac{x^y}{x} y\log x$$

$$= x^{y-1} + x^{y-1} y\log x$$

$$= x^{y-1}[1 + y\log x] \longrightarrow ②$$

From ① & ② $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 y}{\partial y \partial x}$.

Diff. partially w. r. to x on both sides,

$$u_{xxy} = u_{xyx}$$

③ If $f(x,y) = \log\sqrt{x^2+y^2}$, show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

Soln:-

Given:- $f = \log\sqrt{x^2+y^2}$

$$= \frac{1}{2}\log(x^2+y^2)$$

$$\frac{\partial f}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2}2x = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{(x^2+y^2)(1)-x(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}\frac{1}{x^2+y^2}2y = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{(x^2+y^2)(1)-y(2y)}{(x^2+y^2)^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

$$\therefore \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2} = 0.$$

EULER'S THEOREM FOR HOMOGENEOUS FUNCTIONS:-

A function $f(x,y)$ is said to be a homogeneous function of degree n in x and y if $f(tx,ty) = t^n f(x,y)$ for any positive t.

Euler's theorem:-

If u is a homogeneous function of degree n in x and y then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$.

PROBLEMS:-

1. If $u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

[AU M/J 2012, N/D 2014 R-13]

Soln:-

Let $u(x,y,z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$

$$u(tx,ty,tz) = \frac{tx}{ty} + \frac{ty}{tz} + \frac{tz}{tx}$$

$$= t^0 u(x,y,z)$$

$\therefore$ u is a homogeneous function of degree 0.

$\therefore$ By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu = 0.u = 0.$$

② If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.

[AU JAN 14, A/M 17 R-08]

Soln:-

Given:- $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$$\tan u = \frac{x^3+y^3}{x-y}$$

Let $f(x,y) = \tan u = \frac{x^3+y^3}{x-y}$

$$f(tx,ty) = \frac{t^3x^3 + t^3y^3}{tx - ty}$$

$$= \frac{t^3}{t}\left(\frac{x^3+y^3}{x-y}\right)$$

$$= t^2 f(x,y).$$

∴ $\tan u$ is a homogeneous function of degree 2.

∴ By Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$x\frac{\partial}{\partial x}\tan u + y\frac{\partial}{\partial y}\tan u = 2\tan u$$

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u}$$

$$= 2\frac{\sin u}{\cos u} \times \cos^2 u$$

$$= \sin 2u.$$

③ If $u = \cos^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u$.

[AU N/D 2003, A/M 2017 R-13]

Soln:-

Let $u = \cos^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$

$$\cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

Let $f(x,y) = \cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$

$$f(tx,ty) = \frac{tx+ty}{\sqrt{tx}+\sqrt{ty}} = \frac{t}{t^{1/2}}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$$

$$= t^{1-1/2} f(x,y)$$

$$= t^{1/2} f(x,y)$$

$\therefore f$ is a homogeneous function of degree $1/2$.

By Euler's theorem, $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$.

$$x\frac{\partial}{\partial x}\cos u + y\frac{\partial}{\partial y}\cos u = \frac{1}{2}\cos u.$$

$$x(-\sin u)\frac{\partial u}{\partial x} + y(-\sin u)\frac{\partial u}{\partial y} = \frac{1}{2}\cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\frac{\cos u}{\sin u} = -\frac{1}{2}\cot u.$$

(4) If u is a homogeneous function of degree n in x & y, then show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$. [AU M/J 2010]

Soln:

By Euler's theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \longrightarrow (1)$

Diff (1) p.w.r. to 'x'

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\partial y} = n\frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = n\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}$$

$$= (n-1)\frac{\partial u}{\partial x} \longrightarrow (2)$$

Diff (1) p.w.r. to 'y',

$$x\frac{\partial^2 u}{\partial y\partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = n\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial y\partial x} + y\frac{\partial^2 u}{\partial y^2} = n\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}$$

$$= (n-1)\frac{\partial u}{\partial y} \longrightarrow (3)$$

$(2)\times x + (3)\times y \Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x\partial y} + xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2}$

$$= (n-1)x\frac{\partial u}{\partial x} + (n-1)y\frac{\partial u}{\partial y}$$

$$= (n-1)\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

$$= (n-1)(nu) \qquad [\because \text{by } ①]$$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

⑤ If $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$, then find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$. [AU Nov 13, UD]

Soln:-

Given:- $u(x,y) = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{y}{x}$

$$u(tx, ty) = t^2x^2 \tan^{-1}\frac{ty}{tx} - t^2y^2 \tan^{-1}\frac{ty}{tx}$$

$$= t^2\left[x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{y}{x}\right]$$

$$= t^2 u(x,y).$$

$\therefore u$ is a homogeneous function of degree 2.

$\therefore$ By Euler's theorem,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

$$= 2(2-1)u$$

$$= 2u.$$

⑥ If $u = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, then prove that [AU A/M 2014 R-08]

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin u \cos 2u}{4\cos^3 u}$.

Soln:-

Given:- $u = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$

$$f = \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$f(tx, ty) = \frac{tx+ty}{\sqrt{tx}+\sqrt{ty}} = \frac{t}{t^{1/2}} f(x,y)$$

$= t^{1/2} f(x,y)$

$\therefore f = \sin u$ is a homogeneous function of degree $1/2$.

By Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \frac{1}{2}\sin u$$

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\sin u}{\cos u}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u. \longrightarrow (1)$$

ii) Diff (1) p. w. r. to x,

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{2}\sec^2 u\frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{2}\sec^2 u\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}$$

$$= \left[\frac{1}{2}\sec^2 u - 1\right]\frac{\partial u}{\partial x} \longrightarrow (2)$$

Diff (1) p. w. r. to 'y',

$$x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \frac{1}{2}\sec^2 u\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}\sec^2 u\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}$$

$$= \left[\frac{1}{2}\sec^2 u - 1\right]\frac{\partial u}{\partial y} \longrightarrow (3)$$

Multiplying (2) by x, (3) by y & adding,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \left[\frac{1}{2}\sec^2 u - 1\right]\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

$$= \left[\frac{1}{2\cos^2 u} - 1\right]\frac{1}{2}\tan u.$$

$$= -\left[\frac{2\cos^2 u - 1}{2\cos^2 u}\right]\frac{1}{2}\frac{\sin u}{\cos u}$$

$$= -\frac{\sin u \cos 2u}{4\cos^3 u} \quad \left[\because 2\cos^2 u - 1 = \cos 2u\right]$$

TOTAL DERIVATIVES – CHANGE OF VARIABLES – PARTIAL DIFFERENTIATION OF IMPLICIT FUNCTIONS:-

① Find $\frac{dy}{dx}$ when $x^3+y^3=3axy$

Soln:-

Let $f(x,y)=x^3+y^3-3axy$

$$\frac{dy}{dx}=\frac{-\partial f/\partial x}{\partial f/\partial y}=-\frac{3x^2-3ay}{3y^2-3ax}=-\frac{x^2-ay}{y^2-ax}.$$

② If $u=x\log(xy)$ where $x^3+y^3+3xy=1$, then find $\frac{du}{dx}$.

Soln:-

Given:- $u=x\log(xy)=x[\log x+\log y]$

$$\frac{\partial u}{\partial x}=x\left[\frac{1}{x}+0\right]+[\log x+\log y](1)$$

$$=1+\log x+\log y$$

$$\frac{\partial u}{\partial y}=x(0+1/y)+(\log x+\log y)(0)=\frac{x}{y}.$$

$$\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$=1+\log x+\log y+\frac{x}{y}\frac{dy}{dx}\longrightarrow ①$$

Given:- $x^3+y^3+3xy=1$

Diff w.r.to 'x'

$$3x^2+3y^2\frac{dy}{dx}+3\left[y(1)+x\frac{dy}{dx}\right]=0$$

$$x^2+y^2\frac{dy}{dx}+y+x\frac{dy}{dx}=0$$

$$[y^2+x]\frac{dy}{dx}=-(x^2+y)$$

$$\frac{dy}{dx}=-\frac{x^2+y}{y^2+x}$$

$$\therefore ① \Rightarrow \frac{du}{dx}=1+\log x+\log y+\frac{x}{y}\left[-\left(\frac{x^2+y}{y^2+x}\right)\right]$$

$$=1+\log x+\log y-\frac{x}{y}\left(\frac{x^2+y}{y^2+x}\right)$$

③ If $g(x,y) = \psi(u,v)$ where $u = x^2 - y^2$ and $v = 2xy$, then prove that $\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 4(x^2+y^2)\left[\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2}\right]$.

Soln:-

Given:- $g(x,y) = \psi(u,v)$

$u = x^2 - y^2$	$v = 2xy$
$\frac{\partial u}{\partial x} = 2x$ $\frac{\partial u}{\partial y} = -2y$	$\frac{\partial v}{\partial x} = 2y$ $\frac{\partial v}{\partial y} = 2x$
$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial x}$ $= \frac{\partial \psi}{\partial u}(2x) + \frac{\partial \psi}{\partial v}(+2y)$ $= 2x\frac{\partial \psi}{\partial u} + 2y\frac{\partial \psi}{\partial v}$	$\frac{\partial g}{\partial y} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial y}$ $= \frac{\partial \psi}{\partial u}(-2y) + \frac{\partial \psi}{\partial v}(2x)$ $= -2y\frac{\partial \psi}{\partial u} + 2x\frac{\partial \psi}{\partial v}$
$\frac{\partial}{\partial x} = 2x\frac{\partial}{\partial u} + 2y\frac{\partial}{\partial v}$	$\frac{\partial}{\partial y} = -2y\frac{\partial}{\partial u} + 2x\frac{\partial}{\partial v}$

$$\frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right) = \left(2x\frac{\partial}{\partial u} + 2y\frac{\partial}{\partial v}\right)\left(2x\frac{\partial \psi}{\partial u} + 2y\frac{\partial \psi}{\partial v}\right)$$

$$= 4x^2\frac{\partial^2 \psi}{\partial u^2} + 4xy\frac{\partial^2 \psi}{\partial u \partial v} + 4xy\frac{\partial^2 \psi}{\partial v \partial u} + 4y^2\frac{\partial^2 \psi}{\partial v^2} \rightarrow ①$$

$$\frac{\partial^2 g}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y}\right) = \left(-2y\frac{\partial}{\partial u} + 2x\frac{\partial}{\partial v}\right)\left(-2y\frac{\partial \psi}{\partial u} + 2x\frac{\partial \psi}{\partial v}\right)$$

$$= 4y^2\frac{\partial^2 \psi}{\partial u^2} - 4xy\frac{\partial^2 \psi}{\partial u \partial v} - 4xy\frac{\partial^2 \psi}{\partial v \partial u} + 4x^2\frac{\partial^2 \psi}{\partial v^2} \rightarrow ②$$

①^2 + ②^2

$$\Rightarrow \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 4x^2\frac{\partial^2 \psi}{\partial u^2} + \cancel{4xy\frac{\partial^2 \psi}{\partial u \partial v}} + \cancel{4xy\frac{\partial^2 \psi}{\partial v \partial u}} + 4y^2\frac{\partial^2 \psi}{\partial v^2}$$

$$+ 4y^2\frac{\partial^2 \psi}{\partial u^2} - \cancel{4xy\frac{\partial^2 \psi}{\partial u \partial v}} - \cancel{4xy\frac{\partial^2 \psi}{\partial v \partial u}} + 4x^2\frac{\partial^2 \psi}{\partial v^2}$$

$$= 4(x^2+y^2)\frac{\partial^2 \psi}{\partial u^2} + 4(x^2+y^2)\frac{\partial^2 \psi}{\partial v^2}$$

$$= 4(x^2+y^2)\left[\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2}\right].$$

④ Given the transformations $u = e^x \cos y$ and $v = e^x \sin y$ and that ϕ is a function of u and v and also of x & y, prove that $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = (u^2+v^2)\left(\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2}\right)$.

Soln:-

Given:- $u = e^x \cos y$	$v = e^x \sin y$
$\frac{\partial u}{\partial x} = e^x \cos y = u$ $\frac{\partial u}{\partial y} = -e^x \sin y = -v$	$\frac{\partial v}{\partial x} = e^x \sin y = v$ $\frac{\partial v}{\partial y} = e^x \cos y = u$
$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x}$ $= \frac{\partial\phi}{\partial u} u + \frac{\partial\phi}{\partial v} v$ $= u\frac{\partial\phi}{\partial u} + v\frac{\partial\phi}{\partial v}$	$\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial y}$ $= \frac{\partial\phi}{\partial u}(-v) + \frac{\partial\phi}{\partial v} u$ $= -v\frac{\partial\phi}{\partial u} + u\frac{\partial\phi}{\partial v}$
$\frac{\partial}{\partial x} = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$	$\frac{\partial}{\partial y} = -v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) = \left(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right)\left[u\frac{\partial\phi}{\partial u} + v\frac{\partial\phi}{\partial v}\right]$$

$$= u^2\frac{\partial^2\phi}{\partial u^2} + uv\frac{\partial^2\phi}{\partial u\partial v} + vu\frac{\partial^2\phi}{\partial v\partial u} + v^2\frac{\partial^2\phi}{\partial v^2} \longrightarrow ①$$

$$\frac{\partial^2\phi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) = \left[-v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}\right]\left[-v\frac{\partial\phi}{\partial u} + u\frac{\partial\phi}{\partial v}\right]$$

$$= v^2\frac{\partial^2\phi}{\partial u^2} - vu\frac{\partial^2\phi}{\partial u\partial v} - uv\frac{\partial^2\phi}{\partial v\partial u} + u^2\frac{\partial^2\phi}{\partial v^2} \longrightarrow ②$$

① + ②

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = u^2\frac{\partial^2\phi}{\partial u^2} + v^2\frac{\partial^2\phi}{\partial v^2} + v^2\frac{\partial^2\phi}{\partial u^2} + u^2\frac{\partial^2\phi}{\partial v^2}$$

$$= (u^2+v^2)\frac{\partial^2\phi}{\partial u^2} + (u^2+v^2)\frac{\partial^2\phi}{\partial v^2}.$$

$$= (u^2+v^2)\left[\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2}\right].$$

(5) If $Z = f(y-z, z-x, x-y)$, Show that $\frac{\partial Z}{\partial x} + \frac{\partial Z}{\partial y} + \frac{\partial Z}{\partial z} = 0$

Soln:- [AU Jan 2013, Jan 2014
AU D15 / J16 R-08, AU N/D 2016 R-08]

Let $u = y-z$, $v = z-x$, $w = x-y$

$$Z = f(u, v, w)$$

$$\frac{\partial Z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}$$

$$= \frac{\partial f}{\partial u}(0) + \frac{\partial f}{\partial v}(-1) + \frac{\partial f}{\partial w}(1) = -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}.$$

$$\frac{\partial Z}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y}$$

$$= \frac{\partial f}{\partial u}(1) + \frac{\partial f}{\partial v}(0) + \frac{\partial f}{\partial w}(-1) = \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w}.$$

$$\frac{\partial Z}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z}$$

$$= \frac{\partial f}{\partial u}(-1) + \frac{\partial f}{\partial v}(1) + \frac{\partial f}{\partial w}(0) = -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}.$$

$$\therefore \frac{\partial Z}{\partial x} + \frac{\partial Z}{\partial y} + \frac{\partial Z}{\partial z} = -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w} + \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w} - \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} = 0$$

(6) If z is a function of x and y where $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$, show that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$.

Soln:- (AU A/M 2014 U.D)

Given:- $z = f(x, y)$

$x = e^u + e^{-v}$	$y = e^{-u} - e^v$
$\frac{\partial x}{\partial u} = e^u$ $\frac{\partial x}{\partial v} = -e^{-v}$	$\frac{\partial y}{\partial u} = -e^{-u}$ $\frac{\partial y}{\partial v} = -e^v$
$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$ $= \frac{\partial z}{\partial x}e^u + \frac{\partial z}{\partial y}(-e^{-u})$ $= e^u\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y}$	$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$ $= \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^v)$ $= -e^{-v}\frac{\partial z}{\partial x} - e^v\frac{\partial z}{\partial y}$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = e^{u}\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} + e^{-v}\frac{\partial z}{\partial x} + e^{v}\frac{\partial z}{\partial y}$$

$$= (e^{u}+e^{-v})\frac{\partial z}{\partial x} - (e^{-u}-e^{v})\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

⑦ Transform the equation $z_{xx} + 2z_{xy} + z_{yy} = 0$ by changing the independent variables using $u = x-y$ and $v = x+y$.

Soln:-

(AU June 2012)

$u = x-y$	$v = x+y$
$\frac{\partial u}{\partial x} = 1$ $\frac{\partial u}{\partial y} = -1$	$\frac{\partial v}{\partial x} = 1$ $\frac{\partial v}{\partial y} = 1$
$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$ $= \frac{\partial z}{\partial u}(1) + \frac{\partial z}{\partial v}(1)$ $= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$	$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$ $= \frac{\partial z}{\partial u}(-1) + \frac{\partial z}{\partial v}(1)$ $= -\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$
$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$	$\frac{\partial}{\partial y} = -\frac{\partial}{\partial u} + \frac{\partial}{\partial v}$
$\frac{\partial^2 z}{\partial x^2} = \left(\frac{\partial}{\partial x}\right)\left(\frac{\partial z}{\partial x}\right)$ $= \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$ $= \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}$	$\frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial}{\partial y}\right)\left(\frac{\partial z}{\partial y}\right)$ $= \left(-\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$ $= \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u \partial v} - \frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}$

$$\frac{\partial^2 z}{\partial x \partial y} = \left(\frac{\partial}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= -\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} - \frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}.$$

$z_{xx} + 2z_{xy} + z_{yy} = 0$

$$\Rightarrow \cancel{\frac{\partial^2 z}{\partial u^2}} + \cancel{\frac{\partial^2 z}{\partial u \partial v}} + \cancel{\frac{\partial^2 z}{\partial v \partial u}} + \frac{\partial^2 z}{\partial v^2} - 2\cancel{\frac{\partial^2 z}{\partial u^2}} + 2\frac{\partial^2 z}{\partial u \partial v} - 2\frac{\partial^2 z}{\partial v \partial u} + 2\frac{\partial^2 z}{\partial v^2}$$

$$+ \cancel{\frac{\partial^2 z}{\partial u^2}} - \cancel{\frac{\partial^2 z}{\partial u \partial v}} - \cancel{\frac{\partial^2 z}{\partial v \partial u}} + \frac{\partial^2 z}{\partial v^2} = 0$$

$$4\frac{\partial^2 z}{\partial v^2} + 2\frac{\partial^2 z}{\partial u \partial v} - 2\frac{\partial^2 z}{\partial v \partial u} = 0 \Rightarrow 4Z_{vv} + 2Z_{uv} - Z_{vu} = 0$$

$$\Rightarrow 2Z_{vv} + Z_{uv} - Z_{vu} = 0$$

JACOBIANS:-

If u_1, u_2, u_3 are functions of three variables x_1, x_2, x_3 then

$$\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix}$$

① If $x = r\cos\theta$, $y = r\sin\theta$, find (i) $\frac{\partial(x,y)}{\partial(r,\theta)}$ (ii) $\frac{\partial(r,\theta)}{\partial(x,y)}$

[AU N/D 2014 R-08, 13, AU M/J 14, R-08, AU D15/J16 R-13, AU M/J 2016 R-13] AU N/D 2016 R-13]

Soln:-

Given:- $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

(i) $$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r\left[\cos^2\theta + \sin^2\theta\right] = r.$$

(ii) $$\frac{\partial(x,y)}{\partial(r,\theta)} \frac{\partial(r,\theta)}{\partial(x,y)} = 1$$

$$\Rightarrow r \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$$

$$\Rightarrow \frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}.$$

② If $u = \frac{y^2}{2x}$, $v = \frac{x^2+y^2}{2x}$; find $\frac{\partial(u,v)}{\partial(x,y)}$.

Soln:-

$$\frac{\partial u}{\partial x} = \frac{y^2}{2}\left(-\frac{1}{x^2}\right) = -\frac{y^2}{2x^2}.$$

$$\frac{\partial u}{\partial y} = \frac{2y}{2x} = \frac{y}{x}$$

$$v = \frac{x^2+y^2}{2x} = \frac{x^2}{2x} + \frac{y^2}{2x} = \frac{x}{2} + \frac{y^2}{2x}.$$

$$\frac{\partial v}{\partial x} = \frac{1}{2} + \frac{y^2}{2}\left(-\frac{1}{x^2}\right) = \frac{1}{2} - \frac{y^2}{2x^2} = \frac{x^2-y^2}{2x^2}.$$

$$\frac{\partial v}{\partial y} = \frac{2y}{2x} = \frac{y}{x}.$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$$

$$= \begin{vmatrix} -y^2/2x^2 & y/x \\ \frac{x^2-y^2}{2x^2} & y/x \end{vmatrix}$$

$$= -\frac{y^3}{2x^3} - \frac{y(x^2-y^2)}{2x^3} = \frac{-y^3-yx^2+y^3}{2x^3}$$

$$= \frac{-yx^2}{2x^3} = -\frac{y}{2x}.$$

③ If $u=2xy$, $v=x^2-y^2$ and $x=r\cos\theta$, $y=r\sin\theta$, Evaluate $\frac{\partial(u,v)}{\partial(r,\theta)}$ without actual substitution.

Soln:-

Given:- $u=2xy$	$v=x^2-y^2$	$x=r\cos\theta$	$y=r\sin\theta$
$\frac{\partial u}{\partial x}=2y$	$\frac{\partial v}{\partial x}=2x$	$\frac{\partial x}{\partial r}=\cos\theta$	$\frac{\partial y}{\partial r}=\sin\theta$
$\frac{\partial u}{\partial y}=2x$	$\frac{\partial v}{\partial y}=-2y$	$\frac{\partial x}{\partial \theta}=-r\sin\theta$	$\frac{\partial y}{\partial \theta}=r\cos\theta$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= (-4y^2 - 4x^2)(r\cos^2\theta + r\sin^2\theta)$$

$$= -4(x^2+y^2)\, r(\cos^2\theta + \sin^2\theta)$$

$$= -4(x^2+y^2)\, r$$

$$= -4r\,(r^2\cos^2\theta + r^2\sin^2\theta)$$

$$= -4r.\,r^2(\cos^2\theta + \sin^2\theta)$$

$$= -4r^3.$$

(4) Find the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3

$y_1 = \dfrac{x_2 x_3}{x_1}$, $y_2 = \dfrac{x_3 x_1}{x_2}$, $y_3 = \dfrac{x_1 x_2}{x_3}$. [AU N/D 2016 - R-13 M/J 14, A/M-15, AM/17]

Soln:-

Given:- $y_1 = \dfrac{x_2 x_3}{x_1}$	$y_2 = \dfrac{x_3 x_1}{x_2}$	$y_3 = \dfrac{x_1 x_2}{x_3}$
$\dfrac{\partial y_1}{\partial x_1} = -\dfrac{x_2 x_3}{x_1^2}$	$\dfrac{\partial y_2}{\partial x_1} = \dfrac{x_3}{x_2}$	$\dfrac{\partial y_3}{\partial x_1} = \dfrac{x_2}{x_3}$
$\dfrac{\partial y_1}{\partial x_2} = \dfrac{x_3}{x_1}$	$\dfrac{\partial y_2}{\partial x_2} = \dfrac{-x_3 x_1}{x_2^2}$	$\dfrac{\partial y_3}{\partial x_2} = \dfrac{x_1}{x_3}$
$\dfrac{\partial y_2}{\partial x_3} = \dfrac{x_2}{x_1}$	$\dfrac{\partial y_2}{\partial x_3} = \dfrac{x_1}{x_2}$	$\dfrac{\partial y_3}{\partial x_3} = -\dfrac{x_1 x_2}{x_3^2}$

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} & \dfrac{\partial y_1}{\partial x_3} \\[2ex] \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} & \dfrac{\partial y_2}{\partial x_3} \\[2ex] \dfrac{\partial y_3}{\partial x_1} & \dfrac{\partial y_3}{\partial x_2} & \dfrac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} -\dfrac{x_2 x_3}{x_1^2} & \dfrac{x_3}{x_1} & \dfrac{x_2}{x_1} \\[2ex] \dfrac{x_3}{x_2} & -\dfrac{x_3 x_1}{x_2^2} & \dfrac{x_1}{x_2} \\[2ex] \dfrac{x_2}{x_3} & \dfrac{x_1}{x_3} & -\dfrac{x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{-x_2x_3}{x_1^2}\left[\frac{x_1^2x_3x_2}{x_2^2x_3^2} - \frac{x_1^2}{x_2x_3}\right] - \frac{x_3}{x_1}\left[-\frac{x_1x_2x_3}{x_2x_3^2} - \frac{x_1x_2}{x_2x_3}\right]$$

$$+ \frac{x_2}{x_1}\left[\frac{x_1x_3}{x_2x_3} + \frac{x_3x_1x_2}{x_3x_2^2}\right]$$

$$= -\frac{x_2x_3x_1^2x_3x_2}{x_1^2x_2^2x_3^2} + \frac{x_2x_3x_1^2}{x_1^2x_2x_3} + \frac{x_1x_2x_3^2}{x_1x_2x_3^2} + \frac{x_1x_2x_3}{x_1x_2x_3}$$

$$+ \frac{x_2x_1x_3}{x_1x_2x_3} + \frac{x_1x_3x_2^2}{x_1x_3x_2^2}$$

$$= \cancel{-1} + \cancel{1} + 1 + 1 + 1 + 1 = 4.$$

⑤ If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$, show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$.

[AU JAN 14]

[AU D15/D16 R-08]

Soln:-

Given:- $u = \frac{yz}{x}$	$v = \frac{zx}{y}$	$w = \frac{xy}{z}$
$\frac{\partial u}{\partial x} = \frac{-yz}{x^2}$	$\frac{\partial v}{\partial x} = \frac{z}{y}$	$\frac{\partial w}{\partial x} = \frac{y}{z}$
$\frac{\partial u}{\partial y} = \frac{z}{x}$	$\frac{\partial v}{\partial y} = \frac{-zx}{y^2}$	$\frac{\partial w}{\partial y} = \frac{x}{z}$
$\frac{\partial u}{\partial z} = \frac{y}{x}$	$\frac{\partial v}{\partial z} = \frac{x}{y}$	$\frac{\partial w}{\partial z} = -\frac{xy}{z^2}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$= \frac{-yz}{x^2}\left[\frac{zx^2y}{z^2y^2} - \frac{x^2}{yz}\right] - \frac{z}{x}\left[\frac{-xyz}{yz^2} - \frac{xy}{zy}\right] + \frac{y}{x}\left[\frac{xz}{yz} + \frac{zxy}{y^2z}\right]$$

$$= \frac{-x^2y^2z^2}{x^2y^2z^2} + \frac{x^2yz}{x^2yz} + \frac{xyz^2}{xyz^2} + \frac{xyz}{xyz} + \frac{xyz}{xyz} + \frac{xy^2z}{xy^2z}$$

$$= \cancel{-1} + \cancel{1} + 1 + 1 + 1 + 1 = 4$$

⑥ If $x+y+z=u$, $y+z=uv$, $z=uvw$, prove that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$.

Soln:-

Given:-

$u = x+y+z$	$y+z=uv$	$z=uvw$
$u = x+uv$	$y = uv - z$	
$\Rightarrow x = u-uv$	$y = uv-uvw$	

$x = u-uv$	$y = uv-uvw$	$z = uvw$
$\frac{\partial x}{\partial u} = 1-v$	$\frac{\partial y}{\partial u} = v-vw$	$\frac{\partial z}{\partial u} = vw$
$\frac{\partial x}{\partial v} = -u$	$\frac{\partial y}{\partial v} = u-uw$	$\frac{\partial z}{\partial v} = uw$
$\frac{\partial x}{\partial w} = 0$	$\frac{\partial y}{\partial w} = -uv$	$\frac{\partial z}{\partial w} = uv$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= (1-v)\left[(u-uw)uv + (uv)(uw)\right] + u\left[(v-vw)uv + (uv)(vw)\right]$$

$$= (1-v)\left[u^2v - \cancel{u^2vw} + \cancel{u^2vw}\right] + u\left[uv^2 - \cancel{uv^2w} + \cancel{uv^2w}\right]$$

$$= u^2v - \cancel{u^2v^2} \cancel{+u^2v^2}$$

$$= u^2v.$$

⑦ Find the Jacobian $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$ of the transformation $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

[AU M/J 2011, AU D15/J16 R-13 AU M/J 2016 R-13]

Soln:-

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$x = r\sin\theta\cos\phi$	$y = r\sin\theta\sin\phi$	$z = r\cos\theta$
$\frac{\partial x}{\partial r} = \sin\theta\cos\phi$ $\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$ $\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi$	$\frac{\partial y}{\partial r} = \sin\theta\sin\phi$ $\frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi$ $\frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$	$\frac{\partial z}{\partial r} = \cos\theta$ $\frac{\partial z}{\partial \theta} = -r\sin\theta$ $\frac{\partial z}{\partial \phi} = 0$

$$\therefore \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

Expand using third row,

$$= \cos\theta\left[r^2\cos\theta\sin\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi\right]$$
$$+ r\sin\theta\left[r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi\right]$$
$$= \cos\theta\left[r^2\cos\theta\sin\theta\left(\cos^2\phi + \sin^2\phi\right)\right]$$
$$+ r\sin\theta\left[r\sin^2\theta\left(\cos^2\phi + \sin^2\phi\right)\right]$$
$$= r^2\cos^2\theta\sin\theta + r^2\sin^3\theta$$
$$= r^2\sin\theta\left(\cos^2\theta + \sin^2\theta\right) = r^2\sin\theta.$$

TAYLOR'S SERIES FOR FUNCTION OF TWO VARIABLES :-

FORMULA

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[h\, f_x(a,b) + k\, f_y(a,b)\right]$$
$$+ \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[h^3 f_{xxx}(a,b) + 3h^2k\, f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)\right]$$
$$+ \dots \quad \text{where } h = x-a,\ k = y-b.$$

① Expand $e^x \cos y$ about $(0, \pi/2)$ up to third degree terms using Taylor's series. [redacted]

Soln:-

Function	value at $(0, \pi/2)$
$f(x,y) = e^x \cos y$	$f(0,\pi/2) = e^0 \cos \pi/2 = 0$
$f_x = e^x \cos y$ $f_y = -e^x \sin y$	$f_x(0,\pi/2) = e^0 \cos \pi/2 = 0$ $-e^0 \sin \pi/2 = -1$
$f_{xx} = e^x \cos y$ $f_{xy} = -e^x \sin y$ $f_{yy} = -e^x \cos y$	$e^0 \cos \pi/2 = 0$ $-e^0 \sin \pi/2 = -1$ $-e^0 \cos \pi/2 = 0$
$f_{xxx} = e^x \cos y$ $f_{xxy} = -e^x \sin y$ $f_{xyy} = -e^x \cos y$ $f_{yyy} = e^x \sin y$	$f_{xxx}(0,\pi/2) = e^0 \cos \pi/2 = 0$ $f_{xxy} = -e^0 \sin \pi/2 = -1$ $-e^0 \cos \pi/2 = 0$ $e^0 \sin \pi/2 = 1$

Substitute all values in Taylor series

$$f(x,y) = 0 + \left[x(0) + (y-\pi/2)(-1)\right] + \frac{1}{2!}\left[x^2(0) + 2x(y-\pi/2)(-1) + (y-\pi/2)^2(0)\right]$$
$$+ \frac{1}{3!}\left[x^3(0) + 3x^2(y-\pi/2)(-1) + 3x(y-\pi/2)^2(0) + (y-\pi/2)^3(+1)\right]$$
$$= -y + \pi/2 - \frac{1}{2}\left(2x(y-\pi/2)\right) + \frac{1}{6}\left[-3x^2(y-\pi/2) + (y-\pi/2)^3\right]$$

② Expand $\sin xy$ in powers of $x-1$ and $y-\pi/2$ up to second degree terms by using Taylor's series. [N/D 17 R-13
AU N/D 2014, 2015 R-13]

Soln:-

Function	Value at $(1, \pi/2)$
$f(x,y) = \sin(xy)$	$f(1, \pi/2) = \sin \pi/2 = 1$
$f_x = y\cos(xy)$ $f_y = x\cos(xy)$	$\frac{\pi}{2}\cos \pi/2 = 0$ $\cos \pi/2 = 0$
$f_{xx} = -y^2 \sin(xy)$ $f_{xy} = y(-x\sin(xy)) + \cos(xy)$ $f_{yy} = -x^2 \sin(xy)$	$-\frac{\pi^2}{4}\sin \pi/2 = -\pi^2/4$ $-\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2} = -\pi/2$ $-\sin \pi/2 = -1$

Taylor series : $f(x,y) = f(a,b) + \frac{1}{1!}[h f_x(a,b) + k f_y(a,b)]$

$$+ \frac{1}{2!}[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)].$$

Here $a = 1$; $b = \pi/2$

$h = x - a = x - 1$; $k = y - b = y - \pi/2$.

$$f(x,y) = 1 + (x-1)(0) + (y-\pi/2)(0) + \frac{1}{2!}\left[(x-1)^2\left(-\frac{\pi^2}{4}\right) + 2(x-1)(y-\pi/2)(-\pi/2) + (y-\pi/2)^2(-1)\right]$$

$$= 1 + \frac{1}{2}\left[-\frac{\pi^2}{4}(x-1)^2 - \pi(x-1)(y-\pi/2) - (y-\pi/2)^2\right].$$

③ Expand $e^x \log(1+y)$ in powers of x and y up to third degree terms using Taylor's series. [[redacted]]

Soln:-

$$f(x,y) = f(a,b) + \frac{1}{1!}[h f_x(a,b) + k f_y(a,b)]$$

$$+ \frac{1}{2!}[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)]$$

$$+ \frac{1}{3!}[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)]$$

Function	Value at $(0,0)$
$f(x,y) = e^x \log(1+y)$	$f(0,0) = e^0 \log 1 = 0$
$f_x = e^x \log(1+y)$ $f_y = e^x\left(\frac{1}{1+y}\right)$	$f_x = e^0 \log 1 = 0$ $f_y = e^0(1) = 1$
$f_{xx} = e^x \log(1+y)$ $f_{xy} = e^x\left(\frac{1}{1+y}\right)$ $f_{yy} = -e^x \frac{1}{(1+y)^2}$	$f_{xx} = e^0 \log 1 = 0$ $f_{xy} = e^0(1) = 1$ $f_{yy} = -e^0 = -1$
$f_{xxx} = e^x \log(1+y)$ $f_{xxy} = e^x\left(\frac{1}{1+y}\right)$ $f_{xyy} = -e^x \frac{1}{(1+y)^2}$ $f_{yyy} = 2e^x \frac{1}{(1+y)^3}$	$f_{xxx} = e^0 \log 1 = 0$ $f_{xxy} = e^0 = 1$ $f_{xyy} = -e^0 = -1$ $f_{yyy} = e^0 = 2$

Here $a=0$ & $b=0$

$h = x-a = x$

$k = y-b = y$

$$\therefore f(x,y) = 0 + x(0) + y(1) + \frac{1}{2!}\left[x^2(0) + 2xy(1) + y^2(-1)\right]$$
$$+ \frac{1}{3!}\left[x^3(0) + 3x^2y(1) + 3xy^2(-1) + y^3(2)\right]$$
$$= y + \frac{1}{2}(2xy - y^2) + \frac{1}{6}\left[3x^2y - 3xy^2 + 2y^3\right].$$

4. Expand $e^x \sin y$ in powers of x & y up to third degree terms using Taylor's series. [JAN 16 R-13]

Soln:-

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[h f_x(a,b) + k f_y(a,b)\right]$$
$$+ \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)\right]$$

Function	Value at $(0,0)$
$f(x,y) = e^x \sin y$	$f(0,0) = e^0 \sin 0 = 0$
$f_x = e^x \sin y$ $f_y = e^x \cos y$	$f_x = 0$ $f_y = 1$
$f_{xx} = e^x \sin y$ $f_{xy} = e^x \cos y$ $f_{yy} = -e^x \sin y$	$f_{xx} = 0$ $f_{xy} = 1$ $f_{yy} = 0$
$f_{xxx} = e^x \sin y$ $f_{xxy} = e^x \cos y$ $f_{xyy} = -e^x \sin y$ $f_{yyy} = -e^x \cos y$	$f_{xxx} = 0$ $f_{xxy} = 1$ $f_{xyy} = 0$ $f_{yyy} = -1$

Here $a=0$; $b=0$

$h = x-a = x$

$k = y-b = y$

$$f(x,y) = 0 + \frac{1}{1!}[x(0) + y(1)] + \frac{1}{2!}[x^2(0) + 2xy(1) + y^2(0)]$$

$$+ \frac{1}{3!}[x^3(0) + 3x^2y(1) + 3xy^2(0) + y^3(-1)]$$

$$= y + \frac{1}{\cancel{2}}(\cancel{2}xy) + \frac{1}{6}(3x^2y) + \frac{1}{6}(-y^3)$$

$$= y + xy + \frac{1}{2}x^2y - \frac{1}{6}y^3.$$

5. Obtain the Taylor's series expansion of $x^3+y^3+xy^2$ interms of powers of $x-1$ & $y-2$ up to third degree terms.

[JAN 18 R-13 , A/M 15 R-13, A/M 17 R-13]

Soln:-

$$f(x,y) = f(a,b) + \frac{1}{1!}[hf_x(a,b) + kf_y(a,b)]$$

$$+ \frac{1}{2!}[h^2 f_{xx}(a,b) + 2hkf_{xy}(a,b) + k^2 f_{yy}(a,b)]$$

$$+ \frac{1}{3!}[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)]$$

Function	Value at (0,0)
$f(x,y) = x^3+y^3+xy^2$	$f(1,2) = 1+8+4 = 13$
$f_x = 3x^2+y^2$ $f_y = 3y^2+2xy$	$3+4 = 7$ $12+4 = 16$
$f_{xx} = 6x$ $f_{xy} = 2y$ $f_{yy} = 6y+2x$	6 4 $12+2 = 14$
$f_{xxx} = 6$ $f_{xxy} = 0$ $f_{xyy} = 2$ $f_{yyy} = 6$	6 0 2 6

$$f(x,y) = 13 + (x-1)(7) + (y-2)(16) + \frac{1}{2!}\left[(x-1)^2(6) + 2(x-1)(y-2)(4) + (y-2)^2(14)\right] + \frac{1}{3!}\left[(x-1)^3(6) + 3(x-1)^2(y-2)(0) + 3(x-1)(y-2)^2(2) + (y-2)^3(6)\right]$$

$$= 13 + 7(x-1) + 16(y-2) + \frac{1}{2}\left[6(x-1)^2 + 8(x-1)(y-2) + 14(y-2)^2\right] + \frac{1}{6}\left[6(x-1)^3 + 6(x-1)(y-2)^2 + 6(y-2)^3\right]$$

MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES :-

Necessary conditions for a maximum or minimum

$f_x(a,b) = 0$ and $f_y(a,b) = 0$

Notations :- $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$, $f_{xx} = \frac{\partial^2 f}{\partial x^2}$, $f_{yy} = \frac{\partial^2 f}{\partial y^2}$, $f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$

Sufficient conditions :- If $f_x(a,b) = 0$, $f_y(a,b) = 0$ and $f_{xx}(a,b) = A$ $f_{xy}(a,b) = B$, $f_{yy}(a,b) = C$ then

i) $f(a,b)$ is maximum value if $AC-B^2>0$ and $A<0$ (or $B<0$).

ii) $f(a,b)$ is minimum value if $AC-B^2>0$ and $A>0$ (or $B>0$).

iii) $f(a,b)$ is not extremum (saddle) if $AC-B^2<0$

iv) If $AC-B^2=0$, then the test is inconclusive.

Stationary value :-

A function $f(x,y)$ is said to be stationary at (a,b) or $f(a,b)$ is said to be a stationary value of $f(x,y)$ if $f_x(a,b)=0$ and $f_y(a,b)=0$.

PROBLEMS :-

① Find the extreme values of the function $f(x,y)=x^3+y^3-3x-12y+20$ [AU N/D 14, R-13]

Soln:-

Given :- $f(x,y) = x^3+y^3-3x-12y+20$

$f_x = 3x^2-3$

$f_y = 3y^2-12$

$A = f_{xx} = 6x$

$B = f_{xy} = 0$

$C = f_{yy} = 6y$

$AC-B^2 = (6x)(6y)-(0) = 36xy$

To find stationary points:

$f_x=0$	$f_y=0$
$3x^2-3=0$	$3y^2-12=0$
$3(x^2-1)=0$	$3(y^2-4)=0$
$x^2-1=0$	$y^2-4=0$
$x^2=1$	$\boxed{y=\pm 2}$
$\boxed{x=\pm 1}$	

$\therefore$ The stationary points are

$(1,2),\ (1,-2),\ (-1,2),\ (-1,-2)$

	$(1,2)$	$(1,-2)$	$(-1,2)$	$(-1,-2)$
$A = 6x$	$6>0$	$6>0$	$-6<0$	$-6<0$
$B = 0$	0	0	0	0
$AC-B^2 = 36xy$	$72>0$	$-72<0$	$-72<0$	$72>0$
Conclusion	minimum	Saddle	Saddle	maximum

$\therefore$ maximum value of $f(x,y)$ is

$$f(-1,-2) = (-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20$$
$$= -1 - 8 + 3 + 24 + 20$$
$$= 38$$

Minimum value of $f(x,y)$ is

$$f(1,2) = (1)^3 + (2)^3 - 3(1) - 12(2) + 20$$
$$= 1 + 8 - 3 - 24 + 20$$
$$= 2$$

(2) Find the extreme values of $f(x,y) = x^3y^2(1-x-y)$

Soln:- [AU JAN 14, R-13]

$$f(x,y) = x^3y^2(1-x-y)$$
$$= x^3y^2 - x^4y^2 - x^3y^3$$
$$f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$$
$$f_y = 2x^3y - 2x^4y - 3x^3y^2$$
$$A = f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3$$
$$B = f_{xy} = 6x^2y - 8x^3y - 9x^2y^2$$
$$C = f_{yy} = 2x^3 - 2x^4 - 6x^3y$$

To find Stationary points:-

$f_x = 0$

$3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$

$x^2y^2[3 - 4x - 3y] = 0$

$\Rightarrow x=0,\ y=0,\ 4x+3y=3$

$4x+3y=3 \rightarrow (1)$

$f_y = 0$

$2x^3y - 2x^4y - 3x^3y^2 = 0$

$x^3y[2 - 2x - 3y] = 0$

$x=0,\ y=0,\ 2x+3y=2$

$2x+3y=2 \rightarrow (2)$

$$4x + \cancel{3y} = 3$$
$$\underline{\begin{matrix} - & \cancel{\mp 3y} & = & -2 \\ 2x & & & 2 \end{matrix}}$$

$$2x = 1$$

$$\boxed{x = 1/2}$$

Substitute $x = 1/2$ in ②

$$2 \times \frac{1}{2} + 3y = 2$$
$$1 + 3y = 2$$
$$3y = 2 - 1 = 1$$
$$\boxed{y = 1/3}$$

$\therefore$ The stationary points are $(0,0)$ & $(1/2, 1/3)$.

<u>At $(1/2, 1/3)$</u>

$$A = 6(1/2)(1/3)^2 - 12(1/2)^2(1/3)^2 - 6(1/2)(1/3)^3$$
$$= 6 \times \frac{1}{2} \times \frac{1}{9} - 12 \times \frac{1}{4} \times \frac{1}{9} - 6 \times \frac{1}{2} \times \frac{1}{27}$$
$$= \cancel{\frac{1}{3}} - \cancel{\frac{1}{3}} - \frac{1}{9} = -1/9.$$

$$B = 6(1/2)^2(1/3) - 8(1/2)^3(1/3) - 9(1/2)^2(1/3)^2$$
$$= 6 \times \frac{1}{4} \times \frac{1}{3} - 8 \times \frac{1}{8} \times \frac{1}{3} - 9 \times \frac{1}{4} \times \frac{1}{9}$$
$$= \frac{1}{2} - \frac{1}{3} - \frac{1}{4} = \frac{6-4-3}{12} = -1/12$$

$$C = 2(1/2)^3 - 2(1/2)^4 - 6(1/2)^3(1/3)$$
$$= 2 \times \frac{1}{8} - 2 \times \frac{1}{16} - 6 \times \frac{1}{8} \times \frac{1}{3}$$
$$= \cancel{\frac{1}{4}} - \frac{1}{8} - \cancel{1/4} = -1/8$$

$$AC - B^2 = (-\tfrac{1}{9})(-1/8) - (-1/12)^2 = \frac{1}{72} - \frac{1}{144} = \frac{2-1}{144} = \frac{1}{144} > 0$$

$$A = -1/9 < 0.$$

$\therefore f(1/2, 1/3)$ is maximum.

Maximum value of $f(x,y)$ is $f(1/2, 1/3) = (1/2)^3(1/3)^2(1 - 1/2 - 1/3)$
$$= 1/8 \times 1/9 \left[\frac{6-3-2}{6}\right] = \frac{1}{72}(\frac{1}{6})$$
$$= \frac{1}{432}.$$

③ Find the maximum or minimum values of $f(x,y) = 3x^2 - y^2 + x^3$.

[AU J-18, R-17]

Soln:-

Given:- $f(x,y) = 3x^2 - y^2 + x^3$

$f_x = 6x + 3x^2$

$f_y = -2y$

$A = f_{xx} = 6 + 6x$

$B = f_{xy} = 0$

$C = f_{yy} = -2$

To find Stationary points:

$f_x = 0$	$f_y = 0$
$6x + 3x^2 = 0$	$-2y = 0$
$3(2x + x^2) = 0$	$y = 0$
$3x(2+x) = 0$	
$x = 0; \quad x = -2$	

$\therefore$ The stationary points are $(0,0)$ & $(-2,0)$

	$(0,0)$	$(-2,0)$
$A = 6+6x$	$6 > 0$	$-6 < 0$
$B = 0$	0	0
$C = -2$	-2	-2
$AC - B^2$	$-12 < 0$	$12 > 0$
Conclusion	Saddle	Maximum

$\therefore$ Maximum value of $f(x,y)$ is

$f(-2,0) = 3(-2)^2 - 0 + (-2)^3$

$= 3(+4) - 8 = 12 - 8 = 4$

④ Examine the maxima and minima of $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

[AU [illegible] J16, R-13 / AU M/J 16, [illegible]]

Soln:-

$f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$f_x = 3x^2 + 3y^2 - 30x + 72$

$f_y = 6xy - 30y$

$A = f_{xx} = 6x - 30$

$B = f_{xy} = 6y$

$C = f_{yy} = 6y - 30$

To find the stationary points.

$f_x = 0$

$3x^2 + 3y^2 - 30x + 72 = 0$

$f_y = 0$

$6xy - 30y = 0$

$6y(x-5) = 0$

$y = 0, \ x = 5$

when $y = 0$

$3x^2 - 30x + 72 = 0$

$3(x^2 - 10x + 24) = 0$

$x^2 - 10x + 24 = 0$

$(x-6)(x-4) = 0$

$x = 6, 4.$

$\therefore$ The stationary points are $(6,0), (4,0)$

	$(6,0)$	$(4,0)$
$A = 6x - 30$	$36 - 30 = 6 > 0$	$24 - 30 = -6 < 0$
$B = 6y$	0	0
$C = 6y - 30$	-30	-30
$AC - B^2$	$-180 < 0$	$180 > 0$
Conclusion	Saddle	maximum

$\therefore$ Maximum value of $f(x,y)$ is

$f(4,0) = (4)^3 + 0 - 15(4)^2 - 0 + 72(4)$

$= 64 - 240 + 288$

$= 328$

LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS:-

To find the maximum and minimum values of $f(x,y,z)$ where x,y,z are subject to a constraint equation $g(x,y,z)=0$

We define a function

$F(x,y,z,\lambda) = f(x,y,z) + \lambda g(x,y,z)$ where λ is called Lagrange multiplier which is independent of x,y,z. → ①

The necessary conditions for a maximum or minimum are

$$\frac{\partial F}{\partial x} = 0 \to ② \quad , \quad \frac{\partial F}{\partial y} = 0 \to ③ \quad \text{and} \quad \frac{\partial F}{\partial z} = 0 \to ④ .$$

PROBLEMS:-

① A rectangular box opens at the top is to have a volume of 32cc. Find the dimensions of the box, that requires the least material for its construction. [A/M 17 R-13, N/D 15 R-13, M/J 16 R-13]

Soln:-

Let x,y,z be the length, breadth and height of the box.

Surface area $= xy + 2yz + 2xz = f(x,y,z)$

volume $= xyz = 32 \quad = g(x,y,z)$

$F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$

$\Rightarrow F(x,y,z) = xy + 2yz + 2xz + \lambda(xyz - 32) \longrightarrow ①$

$\frac{\partial F}{\partial x} = 0$

$y + 2z + \lambda yz = 0$

$y + 2z = -\lambda yz$

$\frac{y+2z}{yz} = -\lambda$

$\frac{1}{z} + \frac{2}{y} = -\lambda \to ②$

$\frac{\partial F}{\partial y} = 0 \Rightarrow x + 2z + \lambda xz = 0$

$x + 2z = -\lambda xz$

$\frac{x+2z}{xz} = -\lambda$

$\frac{1}{z} + \frac{2}{x} = -\lambda \to ③$

$\frac{\partial F}{\partial z} = 0$

$2y + 2x + \lambda xy = 0$

$2y + 2x = -\lambda xy$

$\frac{2y+2x}{xy} = -\lambda$

$\frac{2}{x} + \frac{2}{y} = -\lambda \longrightarrow ④$

From ② & ③

$\cancel{\frac{1}{z}} + \frac{2}{y} = \cancel{\frac{1}{z}} + \frac{2}{x}$

$\frac{2}{y} = \frac{2}{x}$

$x = y \longrightarrow ⑤$

From ③ & ④

$\frac{1}{z} + \cancel{\frac{2}{x}} = \cancel{\frac{2}{x}} + \frac{2}{y}$

$\frac{1}{z} = \frac{2}{y}$

$y = 2z \longrightarrow ⑥$

From ⑤ & ⑥

$$x = y = 2z$$

Volume : $xyz = 32$

$$(2z)(2z)z = 32$$

$$4z^3 = 32$$

$$z^3 = \frac{32}{4} = 8$$

$$\boxed{z = 2}$$

$\therefore$ $x = 4$; $y = 4$; $z = 2$.

$\therefore$ Dimension of the box are 4, 4, 2.

② Find the dimensions of the rectangular box without top of maximum capacity whose surface area is 108 sq cm.

Soln:-

[Jan 18 R-17]

Let x, y, z be the length, breadth & height of the box.

Surface area : $xy + 2yz + 2zx = 108 = g$

Volume $= xyz = f$

$\therefore$ $F(x,y,z,\lambda) = xyz + \lambda(xy + 2yz + 2zx - 108) \longrightarrow$ ①

$F_x = 0$

$$yz + \lambda(y + 2z) = 0$$

$$yz = -\lambda(y + 2z)$$

$$\frac{y+2z}{yz} = -\frac{1}{\lambda}$$

$$\frac{1}{z} + \frac{2}{y} = -\frac{1}{\lambda} \longrightarrow ②$$

$F_y = 0$

$$xz + \lambda(x + 2z) = 0$$

$$xz = -\lambda(x + 2z)$$

$$\frac{x+2z}{xz} = -\frac{1}{\lambda}$$

$$\frac{1}{z} + \frac{2}{x} = -1/\lambda \longrightarrow ③$$

$F_z = 0$

$$xy + \lambda(2x + 2y) = 0$$

$$xy = -\lambda(2x + 2y)$$

$$\frac{2x+2y}{xy} = -\frac{1}{\lambda}$$

$$\frac{2}{y} + \frac{2}{x} = -1/\lambda \longrightarrow ④$$

From ② & ③

$$\cancel{\frac{1}{z}} + \frac{2}{y} = \cancel{\frac{1}{z}} + \frac{2}{x}$$

$$\frac{2}{y} = \frac{2}{x} \Rightarrow x = y \longrightarrow ⑤$$

From ③ & ④

$$\frac{1}{z} + \cancel{\frac{2}{x}} = \frac{2}{y} + \cancel{\frac{2}{x}}$$

$$\frac{1}{z} = \frac{2}{y} \Rightarrow y = 2z \longrightarrow ⑥$$

From ⑤ & ⑥

$x = y = 2z$

Surface area : $xy + 2yz + 2zx = 108$

$(2z)(2z) + 2(2z)z + 2z(2z) = 108$

$4z^2 + 4z^2 + 4z^2 = 108$

$12z^2 = 108$

$z^2 = \frac{108}{12} = 9$

$\boxed{z = 3}$

$\therefore x = 6 \; ; \; y = 6 \; ; \; z = 3$

$\therefore$ The dimensions are 6, 6, 3.

③ The temperature $u(x,y,z)$ at any point in space is $u = 400xyz^2$. Find the highest temperature on surface of the sphere $x^2+y^2+z^2=$

[N/D-17, R-13, A/M-18, R-13]

Soln:-

Given:- $u = f = 400xyz^2$

$g = x^2+y^2+z^2-1$

$F(x,y,z,\lambda) = 400xyz^2 + \lambda(x^2+y^2+z^2-1) \longrightarrow ①$

$F_x = 0$

$400yz^2 + \lambda(2x) = 0$

$400yz^2 = -2\lambda x$

$\frac{400yz^2}{2x} = -\lambda$

$\frac{200yz^2}{x} = -\lambda \longrightarrow ②$

$F_y = 0$

$400xz^2 + \lambda(2y) = 0$

$400xz^2 = -\lambda 2y$

$\frac{400xz^2}{2y} = -\lambda$

$\frac{200xz^2}{y} = -\lambda \longrightarrow ③$

$F_z = 0$

$800xyz + \lambda(2z) = 0$

$800xyz = -\lambda 2z$

$\frac{800xyz}{2z} = -\lambda$

$400xy = -\lambda \longrightarrow ④$

From ② & ③

$\frac{200yz^2}{x} = \frac{200xz^2}{y}$

$\frac{y}{x} = \frac{x}{y}$

$y^2 = x^2 \longrightarrow ⑤$

From ③ & ④

$\frac{200xz^2}{y} = 400xy$

$\frac{z^2}{y} = 2y$

$2y^2 = z^2 \longrightarrow ⑥$

From (5) & (6)

$x^2 = y^2 = \frac{1}{2}z^2$

We have $\quad x^2+y^2+z^2 = 1$

$$\frac{1}{2}z^2 + \frac{1}{2}z^2 + z^2 = 1$$

$$\frac{z^2+z^2+2z^2}{2} = 1$$

$$\frac{4z^2}{2} = 1 \Rightarrow 2z^2 = 1 \Rightarrow z^2 = \frac{1}{2} \Rightarrow z = \pm\frac{1}{\sqrt{2}}$$

$\therefore\ x^2 = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4} \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm\frac{1}{2}$

& $\ y^2 = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4} \Rightarrow y^2 = \frac{1}{4} \Rightarrow y = \pm\frac{1}{2}$

$\therefore$ Temperature $\quad u = 400xyz^2$

$$= 400\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)^2$$

$$= 400 \times \frac{1}{4} \times \frac{1}{2} = 50$$

$\therefore$ Maximum temperature is 50.

(4) Find the maximum volume of the largest rectangular parallelopiped that can be inscribed in an ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

[AU A/M 2015 R-13, AU N/D 2015]

Soln:- Let the vertex of the parallelopiped be (x, y, z)

All other vertices will be $(\pm x, \pm y, \pm z)$

Sides of the solid be $2x, 2y, 2z$

Volume $V = (2x)(2y)(2z) = 8xyz = f$

We have to maximize V subject to the condition

$$g(x,y,z) = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} - 1 = 0$$

$$F(x,y,z) = 8xyz + \lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} - 1\right) \longrightarrow (1)$$

$F_x = 0$	$F_y = 0$	$F_z = 0$
$8yz + \frac{2x\lambda}{a^2} = 0$	$8xz + \frac{2y\lambda}{b^2} = 0$	$8xy + \frac{2z\lambda}{c^2} = 0$
$8yz = -\frac{2x\lambda}{a^2}$	$8xz = -\frac{2y\lambda}{b^2}$	$8xy = -\frac{2z\lambda}{c^2}$
$\frac{8yz}{-2\lambda} = \frac{x}{a^2}$	$\frac{8xz}{-2\lambda} = \frac{y}{b^2}$	$\frac{8xy}{-2\lambda} = \frac{z}{c^2}$
$-\frac{4yz}{\lambda} = \frac{x}{a^2}$	$-\frac{4xz}{\lambda} = \frac{y}{b^2}$	$-\frac{4xy}{\lambda} = \frac{z}{c^2}$
$-\frac{4xyz}{\lambda} = \frac{x^2}{a^2}$	$-\frac{4xzy}{\lambda} = \frac{y^2}{b^2}$	$-\frac{4xyz}{\lambda} = \frac{z^2}{c^2}$
$\rightarrow (2)$	$\rightarrow (3)$	$\rightarrow (4)$

From (2), (3) & (4)

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Given:- $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$\frac{x^2}{a^2} + \frac{x^2}{a^2} + \frac{x^2}{c^2} = 1$$

$$\frac{3x^2}{a^2} = 1$$

$$x^2 = \frac{a^2}{3} \Rightarrow x = \frac{a}{\sqrt{3}}$$

Similarly, $y = \frac{b}{\sqrt{3}}$ & $z = \frac{c}{\sqrt{3}}$

$\therefore$ Extremum point is $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$, maximum point.

$\therefore$ Maximum value is $V = 8\left(\frac{a}{\sqrt{3}}\right)\left(\frac{b}{\sqrt{3}}\right)\left(\frac{c}{\sqrt{3}}\right)$

$$= \frac{8abc}{3\sqrt{3}}.$$

(5) Find the maximum Value of $x^m y^n z^p$ when $x+y+z=a$

Soln:-

Let $f = x^m y^n z^p$

$g = x+y+z-a$

$$F(x,y,z,\lambda) = x^m y^n z^p + \lambda(x+y+z-a) \longrightarrow ①$$

$F_x = 0$

$$m x^{m-1} y^n z^p + \lambda = 0$$

$$m x^{m-1} y^n z^p = -\lambda$$

$$\frac{m x^m y^n z^p}{x} = -\lambda \longrightarrow ②$$

$F_y = 0$

$$n x^m y^{n-1} z^p + \lambda = 0$$

$$n x^m y^{n-1} z^p = -\lambda$$

$$\frac{n x^m y^n z^p}{y} = -\lambda \longrightarrow ③$$

$F_z = 0$

$$p x^m y^n z^{p-1} + \lambda = 0$$

$$p x^m y^n z^{p-1} = -\lambda$$

$$\frac{p x^m y^n z^p}{z} = -\lambda \longrightarrow ④$$

From ②, ③ & ④

$$\frac{m x^m y^n z^p}{x} = \frac{n x^m y^n z^p}{y} = \frac{p x^m y^n z^p}{z}$$

$$\frac{m}{x} = \frac{n}{y} = \frac{p}{z}$$

$$\frac{m}{x} = \frac{n}{y} \Rightarrow my = nx \Rightarrow x = \frac{m}{n}y$$

$$\frac{n}{y} = \frac{p}{z} \Rightarrow nz = py \Rightarrow z = \frac{p}{n}y$$

<u>Given:-</u> $x+y+z = a$

$$\frac{m}{n}y + y + \frac{y}{n}p = a$$

$$my + ny + yp = na$$

$$y(m+n+p) = na$$

$$y = \frac{na}{m+n+p}$$

Also $x = \dfrac{ma}{m+n+p}$ & $z = \dfrac{pa}{m+n+p}$

$\therefore$ The stationary point is $\left(\dfrac{am}{m+n+p}, \dfrac{na}{m+n+p}, \dfrac{pa}{m+n+p}\right)$

$\therefore$ Maximum value of f is

$$= \left(\frac{am}{m+n+p}\right)^m \left(\frac{an}{m+n+p}\right)^n \left(\frac{ap}{m+n+p}\right)^p$$

$$= \frac{a^{m+n+p}\, m^m n^n p^p}{(m+n+p)^{m+n+p}}.$$

(6) Find the minimum values of x^2yz^3 subject to the condition $2x+y+3z=a$ [AU M/J 2007, AU A/M 2017 - R-13]

Soln:-

Let $f = x^2yz^3$

$g = 2x+y+3z-a$

$F(x,y,z,\lambda) = x^2yz^3 + \lambda(2x+y+3z-a) \longrightarrow (1)$

$F_x = 0$

$2xyz^3 + 2\lambda = 0$

$2xyz^3 = -2\lambda$

$xyz^3 = -\lambda \longrightarrow (2)$

$F_y = 0$

$x^2z^3 + \lambda = 0$

$x^2z^3 = -\lambda \longrightarrow (3)$

$F_z = 0$

$3x^2yz^2 + 3\lambda = 0$

$3x^2yz^2 = -3\lambda$

$x^2yz^2 = -\lambda \longrightarrow (4)$

From (2) & (3)

$xyz^3 = x^2z^3$

$y = x \longrightarrow (5)$

From (3) & (4)

$x^2z^3 = x^2yz^2$

$z = y \longrightarrow (6)$

From (5) & (6)

$x = y = z$

Given:- $2x+y+3z = a$

$2z + z + 3z = a$

$6z = a$

$z = \frac{a}{6}$

$\therefore \quad x = y = \frac{a}{6}$

$\therefore$ The stationary point is $\left(\frac{a}{6}, \frac{a}{6}, \frac{a}{6}\right)$.

$\therefore$ Minimum value of f is $\left(\frac{a}{6}\right)^2\left(\frac{a}{6}\right)\left(\frac{a}{6}\right)^3 = \frac{a^6}{6^6} = \left(\frac{a}{6}\right)^6$.

(7) Find the shortest and the longest distances from the point $(1,2,-1)$ to the sphere $x^2+y^2+z^2=24$ using Lagrange's method of constrained maxima and minima. [AU N/D 2016 R-13]

Soln:-

Let (x,y,z) be any point of the sphere.

Distance of the point (x,y,z) from $(1,2,-1)$ is given by

$d = \sqrt{(x-1)^2+(y-2)^2+(z+1)^2}$

We have to find the maximum and minimum values of d.

$d^2 = (x-1)^2+(y-2)^2+(z+1)^2$ subject to $x^2+y^2+z^2-24=0$

Let $f = (x-1)^2+(y-2)^2+(z+1)^2$

$g = x^2+y^2+z^2-24=0$

$F(x,y,z,\lambda) = (x-1)^2+(y-2)^2+(z+1)^2 + \lambda(x^2+y^2+z^2-24) \longrightarrow (1)$

$F_x = 0$

$2(x-1)+2\lambda x = 0$

$x-1+\lambda x = 0$

$(1+\lambda)x = 1$

$x = \frac{1}{1+\lambda} \longrightarrow (2)$

$F_y = 0$

$2(y-2)+2\lambda y = 0$

$y-2+\lambda y = 0$

$(1+\lambda)y = 2$

$\frac{y}{2} = \frac{1}{1+\lambda} \longrightarrow (3)$

$F_z = 0$

$2(z+1)+2\lambda z = 0$

$z+1+\lambda z = 0$

$(1+\lambda)z = -1$

$\frac{z}{-1} = \frac{1}{1+\lambda} \longrightarrow (4)$

From (2) & (4)

$x = -z \longrightarrow (5)$

From (3) & (4)

$\frac{y}{2} = -z \Rightarrow y = -2z \longrightarrow (6)$

Given :- $x^2+y^2+z^2 = 24$

$(-z)^2+(-2z)^2+z^2 = 24$

$z^2+4z^2+z^2 = 24$

$6z^2 = 24$

$z^2 = 4$

$z = \pm 2$

If $z=2$, then $x=-2$ and $y=-4$

$\therefore$ The point is $(-2,-4,2)$; $d = \sqrt{1^2+2^2+(-1)^2} = \sqrt{6}$

If $z=-2$, then $x=2$ and $y=4$

$\therefore$ The point is $(2,4,-2)$; $d = \sqrt{(-3)^2+(-6)^2+3^2} = 3\sqrt{6}$

$\therefore$ Shortest and longest distances are $\sqrt{6}$ and $3\sqrt{6}$.

① A thin closed rectangular box is to have one edge equal to twice the other and constant volume $72\,m^3$. Find the least surface area of the box.

[Nov/Dec -2019]

(Problem under Legrange's topic).

Soln:

Let $x, y, 2y$ be the length, breadth and height of the box respectively.

$$\text{Surface Area} = 2(x)(y) + 2(y)(2y) + 2(x)(2y)$$

$$= 2xy + 4y^2 + 4xy = 6xy + 4y^2 \quad \text{— (A)}$$

Volume : $xyz = 72$

$$x(y)(2y) = 72$$

$$2xy^2 = 72 \Rightarrow xy^2 = 36 \quad \text{— (B)}$$

∴ The auxillary Function F be

$$F(x, y, z, \lambda) = (6xy + 4y^2) + \lambda(xy^2 - 36)$$

$F_x = 6y + \lambda y^2$; $F_y = 8y + 6x + 2\lambda xy$

To find the extremum,

$F_x = 0$

$6y + \lambda y^2 = 0$

$6y = -\lambda y^2$

$\frac{6}{y} = -\lambda$ — ①

$F_y = 0$

$6x + 8y + 2\lambda xy = 0$

$6x + 8y = -2\lambda xy$

$-\lambda = \frac{3x + 4y}{xy}$

$$\frac{3}{y} + \frac{4}{x} = -\lambda \quad \text{—(2)}$$

From (1) & (2)

$$\frac{6}{y} = \frac{3}{y} + \frac{4}{x} \Rightarrow \frac{3}{y} = \frac{4}{x}$$

$$3x = 4y \Rightarrow y = \frac{3}{4}x \quad \text{—(3)}$$

(B) $\Rightarrow xy^2 = 36$

$$\Rightarrow x\left(\frac{3}{4}x\right)^2 = \frac{9}{16}x^3 = 36$$

$$\Rightarrow x^3 = 64$$

$$\Rightarrow x = 4.$$

$\therefore$ (3) $\Rightarrow$ $y = \left(\frac{3}{4}\right)(4) = 3$

$\therefore$ f is minimum at $(4, 3)$

$\therefore$ The minimum surface $f = (6)(4)(3) + 4(3)^2$

$$= 108.$$

(8) Verify Eulers Theorem for the function $u = x^2 + y^2 + 2xy$

(Nov/Dec - 2019)

2 marks

Soln

Given $u = x^2 + y^2 + 2xy$

$$u(tx, ty) = (tx)^2 + (ty)^2 + 2(tx)(ty)$$

$$= t^2x^2 + t^2y^2 + 2t^2xy.$$

$$= t^2 u(x, y)$$

$\therefore$ u is a homogeneous fnctn of degree 2.

To verify Euler's Theorem we show that

$$x.\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = nu = 2(x^2+y^2+2xy)$$
$$= 2x^2+2y^2+4xy$$

LHS

$$x(2x+2y)+y(2y+2x) = 2x^2+2y^2+4xy$$

RHS

$$nu = 2x^2+2y^2+4xy.$$

$\therefore$ Euler's Theorem is verified for this function.

(3) Find the Taylor Series expansion of the function $f(x) = \sqrt{1+x^2+y^2}$ in powers of $(x-1)$ and y upto second terms.

[Nov/Dec-2018]

Soln Given $f(x,y) = \sqrt{1+x^2+y^2}$

We have to write Taylor's series in terms of $x-1$ and y.

To expand $f(x,y)$ about the point $(1,0)$

(i.e) $(a,b) = (1,0)$

Taylors series about (a,b)

$$f(x,y) = f(a,b) + \left[(x-a)f_x(a,b)+(y-b)f_y(a,b)\right]$$
$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$

$f(x,y) = \sqrt{1+x^2+y^2}$ $\qquad$ $(a,b) = (1,0)$

$$f_x = \frac{1}{2\sqrt{1+x^2+y^2}} \times 2x = \frac{x}{\sqrt{1+x^2+y^2}}$$

$$f_y = \frac{1}{2\sqrt{1+x^2+y^2}} \times 2y = \frac{y}{\sqrt{1+x^2+y^2}}.$$

$$f_{xx} = \frac{\sqrt{1+x^2+y^2}\,(1) - x \cdot \frac{1}{2\sqrt{1+x^2+y^2}} \times 2x}{1+x^2+y^2}$$

$$= \frac{1+x^2+y^2-x^2}{(1+x^2+y^2)^{3/2}} = \frac{1+y^2}{(1+x^2+y^2)^{3/2}}.$$

$$f_{xy} = \frac{\sqrt{1+x^2+y^2}\,(0) - x \cdot \frac{1}{2\sqrt{1+x^2+y^2}} \times 2y}{1+x^2+y^2}$$

$$= \frac{-xy}{(1+x^2+y^2)^{3/2}}$$

|||ly

$$f_{yy} = \frac{1+x^2}{(1+x^2+y^2)^{3/2}}$$

At $(1,0)$

$$f(1,0) = \sqrt{2} \;;\; f_x(1,0) = \frac{1}{\sqrt{2}},\quad f_y(1,0) = 0.$$

$$f_{xx}(1,0) = \frac{1}{2\sqrt{2}} \;;\; f_{yy}(1,0) = \frac{1}{\sqrt{2}}.$$

$$\therefore f(x,y) = \sqrt{1+x^2+y^2} = \sqrt{2} + (x-1)\frac{1}{\sqrt{2}} + \frac{1}{2!}\left[(x-1)^2 \cdot \frac{1}{2\sqrt{2}} + y^2 \cdot \frac{1}{\sqrt{2}}\right]$$

(4) Find the minimum distance from the point $(1,2,0)$ to the cone $z^2 = x^2+y^2$

[Nov/Dec - 2018]

Soln:

Let $P(x,y,z)$ be a point on the cone $z^2 = x^2+y^2$ and let A be $(1,2,0)$

$$\text{Then } AP = \sqrt{(x-1)^2+(y-2)^2+(z-0)^2}$$
$$= \sqrt{(x-1)^2+(y-2)^2+z^2}$$

Let $f(x,y,z) = (x-1)^2+(y-2)^2+z^2$

A.P is minimum if $f(x,y,z)$ is minimum.

Hence we have to minimize AP subject to $x^2+y^2+z^2=0$

Let $\phi(x,y,z) = x^2+y^2-z^2$.

Let form the auxillary function.

$$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$$
$$= (x-1)^2+(y-2)^2+z^2+\lambda(x^2+y^2-z^2)$$

$F_x = 2(x-1) + 2\lambda z$

$F_y = 2(y-2) + 2\lambda y$

$F_z = 2z - 2\lambda z$

To find the stationary points we solve

$F_x = 0$, $F_y = 0$, $F_z = 0$, $\phi = 0$

$F_x = 0 \Rightarrow 2(x-1) + 2\lambda x = 0 \Rightarrow \lambda x = -(x-1)$

$$\lambda \cancel{x} = -\frac{(x-1)}{x}.$$

$F_y = 0 \Rightarrow \cancel{2z - 2\lambda z}$ $2(y-2) + 2\lambda y = 0 \Rightarrow \lambda = -\frac{(y-2)}{y}.$

$F_z = 0 \Rightarrow 2z - 2\lambda z = 0 \Rightarrow \lambda = 1$

$\therefore \frac{(x-1)}{x} = 1 \Rightarrow x = -x+1 \Rightarrow x = \frac{1}{2}.$

And $-\frac{(y-2)}{y} = 1 \Rightarrow \boxed{y=1}$

Subst in $\phi = 0$

$\Rightarrow z^2 = \frac{1}{4} + 1 = \frac{5}{4}$

$\Rightarrow z = \pm \frac{\sqrt{5}}{2}$

$\therefore$ the stationary points are

$P\left(\frac{1}{2}, 1, -\frac{\sqrt{5}}{2}\right), \; P\left(\frac{1}{2}, 1, -\frac{\sqrt{5}}{2}\right)$

$AP = \sqrt{\left(\frac{1}{2}-1\right)^2 + (1-2)^2 + \frac{5}{4}}$

$= \sqrt{\frac{1}{4}+1+\frac{5}{4}} = \sqrt{\frac{6}{4}+1} = \frac{\sqrt{10}}{2}.$

and $AP' = \frac{\sqrt{10}}{2}.$

$\therefore$ the minimum distance is $\frac{\sqrt{10}}{2}.$

(5) Find the maximum or minimum value of the function $f(x,y) = x^2+y^2+6x+12.$ [Nov/Dec - 2019]

Soln The given function is $f = x^2+y^2+6x+12.$

$f_x = 2x+6, \; f_y = 2y, \; f_{xx} = 2, \; f_{xy} = 0, \; f_{yy} = 2$

To find the stationary points

$f_x = 0 \Rightarrow 2x+6 = 0 \Rightarrow x = -3$

$f_y = 0 \Rightarrow 2y = 0 \Rightarrow y = 0.$

$\therefore (-3, 0)$ is a stationary point

Now $A = f_{xx} = 2$, $B = f_{xy} = 0$, $C = f_{yy} = 2$

$AC - B^2 = 4 > 0$ and $A > 0.$

$\therefore$ the function attain minimum at $(-3, 0)$ and

hence the minimum value is

$f(-3, 0) = (-3)^2 + 0 + 6(-3) + 12 = 3.$

(b) Expand $x^2y^2 + 2x^2y + 3xy^2$ in powers of $(x+2)$ and $(y-1)$ using Taylor's series upto third degree terms..

Soln Given function $f(x, y) = x^2y^2 + 2x^2y + 3xy^2.$

Taylor series at (a, b)

$$f(x,y) = f(a,b) + \frac{1}{1!}[hf_x(a,b) + kf_y(a,b)] + \frac{1}{2!}[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)]$$
$$+ \frac{1}{3!}[h^3 f_{xxx}(a,b) + 3h^2 f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)] + \cdots$$

$h = x + 2$, $k = y - 1$

Function	at $(-2, 1)$
$f(x,y) = x^2y^2 + 2x^2y + 3xy^2$	$f(-2, 1) = 6.$
$f_x(x,y) = 2x^2 + 4xy + 3y^2$	$f_x = -9$
$f_y(x,y) = 2x^2y + 2x^2 + 6xy$	$f_y = 4.$
$f_{xx} = 2y^2 + 4y$	$f_{xx} = 6$
$f_{xy} = 4xy + 4x + 6y$	$f_{xy} = -10$

$f_{yy}(x,y) = 2x^2+6x$ $\qquad$ $f_{yy}(-2,1) = -4.$

$f_{xxx} = 6$ $\qquad$ $f_{xxx} = 0$

$f_{xxy} = 4y+4$ $\qquad$ $f_{xxy} = \cancel{-2}\,8$

$f_{xyy} = 4x+6$ $\qquad$ $f_{xyy} = -2$

$f_{yyy} = 0$ $\qquad$ $f_{yyy} = 0$

$$f(x,y) = 6 - 9(x+2) + 4(y-1) + \frac{1}{2!}\left[6(x+2)^2 - 20(x+2)(y-1) - 4(y-1)^2\right]$$
$$+ \frac{1}{3!}\left[24(x+2)^2(y-1) - 6(x+2)(y-1)^2\right] + \cdots$$
$$= 6 - 9(x+2) + 4(y-1) + \cancel{\tfrac{1}{2}}\cancel{[}\, 3(x+2)^2 - 10(x+2)(y-1) - (y-1)^2$$
$$+ 4(x+2)^2(y-1) - 3(x+2)(y-1)^2 + \cdots$$