Scale Invariant and Noise Robust Interest Points with Shearlets

Miguel A. Duval-Poo, Nicoletta Noceti, Francesca Odone, and Ernesto De Vito

Abstract—Shearlets are a relatively new directional multi-scale framework for signal analysis, which have been shown effective to enhance signal discontinuities such as edges and corners at multiple scales even in the presence of a large quantity of noise. In this work we consider blob-like features in the shearlets framework. We derive a measure which is very effective for blob detection and, based on this measure, we propose a blob detector and a keypoint description, whose combination outperforms the state-of-the-art algorithms with noisy and compressed images. We also demonstrate that the measure satisfies the perfect scale invariance property in the continuous case. We evaluate the robustness of our algorithm to different types of noise, including blur, compression artifacts, and Gaussian noise. Furthermore, we carry on a comparative analysis on benchmark data, referring in particular to tolerance to noise and image compression.

Index Terms—Shearlets, multi-scale image analysis, scale selection, image features, blob detection, keypoint description.

I. INTRODUCTION

FEATURE detection consists in the extraction of perceptually interesting low-level features over an image, in preparation of higher level processing tasks. In the last decade a considerable amount of work has been devoted to the design of effective and efficient local feature detectors able to associate scale and orientation information with a given interesting point. Scale-space theory has been one of the main sources of inspiration for this line of research, providing an effective framework for detecting features at multiple scales and for devising scale invariant image descriptors.

In this work we refer in particular to blob features, image regions which are approximately uniform. In early works the Laplacian of the Gaussian (LoG) operator has been proposed as a way of enhancing blob-like structures [1]. Later, difference of Gaussians (DoG) has been introduced as an efficient approximation of the Laplacian [2], while the Hessian determinant [1] was suggested as an alternative operator with a higher sensitivity and better invariance properties. Later on, computationally efficient variants have also been devised [3], [4]. Since feature detection often precedes feature matching, local features need to be associated with an appropriate descriptor. For a reliable feature matching, it is important to identify a descriptor able to deal with geometric transformations, illumination changes, and the presence of noise. Therefore

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over the years there has been a lot of work in devising feature descriptors able to address different types of variations [2], [3], [5]–[8]. It should be noticed how, in this context, much effort has been devoted to reducing the computational cost — it is worth mentioning the well known SIFT [2] and SURF [3] feature descriptors, obtained from scale-normalized derivatives in a scale-space representation based on image pyramids to speed up computation. Recently, binary descriptors such as BRIEF [9], BRISK [10], FREAK [11] lead to more compact and fast to compute representations, to the price of a higher sensitivity to image transformations. In this framework, the challange of noisy data have seldom been considered (see, the recent paper [12]), while it appears to be an important issue when dealing compressed images and video frames taken by moving or low resolution cameras.

The effects of noise are a central issue in wavelet analysis and, unsurprisingly, image feature detection at multiple scales has also been addressed in the context of wavelet theory [13]–[19]. Indeed, this framework allows for a natural derivation of the feature scale [13], [19] and for the design of perfect scale-invariant measurements [20], but it also guarantees an optimal sparse representation robust to the presence of noise [21]. Wavelets may be seen as a generalization of scale-space , since the scale-space representation is equivalent to wavelet representation for the specific choice of a mother wavelet equals to the derivative of the Gaussian [13]. In other words, the expressive power of wavelets is greater, indeed, by choosing a different mother wavelet one may enhance different specific features.

While for 1D signals, wavelets and space-scale theory are the canonical multi-scale representations, for 2D signals there is a large class of representations with a further sensitivity to directional information, useful to deal with rotation invariance — here it is worth mentioning directional wavelets [22], contourlets [23], complex wavelets [24], ridgelets [25], curvelets [26], and shearlets [27].

In this paper we focus on shearlets representation and we show how the use of shearlet coefficients may enhance blob structures in an image providing [28]:

a) A clear definition of these interest points;
b) A good localization in image space;
c) A local image structure with a rich directional information content;
d) A stable procedure with a high degree of repeatability against noise and deformations;
e) The capability of detecting other interesting points, like edges and corners, with a different choice of the generating function [29];

f) An automatic scale selection with a scale invariant descriptor.

Indeed, shearlets enjoy different interesting properties which are meaningful to feature detection and description:

1) As in the space-scale approach, the filters give rise to a *coarse-to-fine multi-scale representation*, but for shearlets two consecutive scales are related by an anisotropic dilation with ratio $1/\sqrt{2}$ and this anisotropy is the key property to have (optimal) sparse representation for natural images [30]. Among the multi-scale representations, only shearlets and curvelets ensure this kind of optimality, so that shearlets are appealing in applications dealing with signal compression.

2) The shearlet coefficients *directly encode directional information*, unlike scale-space representations and traditional wavelets where one could derive directional information only as a post-processing, for example by computing the ratio between the partial derivatives.

3) Shearlets provide an optimal sparse representation for two-dimensional signals having singularity along curves [30]. On the contrary, the coefficients of the noise are uniformly distributed over all the components. Hence, an accurate processing of the shearlet coefficients ensures both a *sparse representation* stable under compression and an effective denoising without adding artifacts [31].

4) The *rotational invariance* of the representation is given by a shearing which preserves the rectangular lattice of the digital image, so that a faithful digital implementation is easy to obtain.

5) In contrast to the scale-space approaches, with shearlets we have a *large choice of admissible templates* allowing to tune the shearlet transform to specific applications, e.g, the Gaussian derivative to locate edges or corners as in [29], [32], or the Mexican hat to analyze blob structures or ridge points.

6) Shearlets also appear to have a potential in providing meaningful descriptions, although this capability has not been largely explored so far (see, for instance, [33], [34]).

In this paper, first, we provide an analysis of perfect scale invariance properties in the continuous case, similar to the study carried out by Lindberg for the scale-space [35]. Then, we derive a discretized formulation of the problem, obtaining a discrete measure which will be the main building block of our algorithms. This measure, obtained by summing up coefficients over all the shearing, is naturally isotropic, but we can easily recover the directional information by looking at the single coefficient. By the property of shearlets highlighted in item 3) above the measure is stable with respect to noise and compression. We confirm this theoretical issue by reporting an analysis of the stability of the measure to different types of signal perturbations, including Gaussian noise, image blur, JPEG compression.

Next, we propose an algorithm for detecting and describing blob-like features. The main peculiarity of our approach is in the fact it fully exploits the expressive power of the shearlet transform. Indeed, each detected feature is associated with a scale, an orientation, and a position, directly related with the dilation, the shearing and the translation provided by the underlying transformation. In the description phase we also use shearlets coefficients, orienting the contributions with respect to the estimated feature orientation. We underline how all the steps of our procedure are based on the same image transformation. In this sense the procedure is elegant and clean and has a potential in computational efficiency.

We present an experimental assessment on benchmark data where we show that the proposed method outperforms the state-of-the-art algorithms with noisy and compressed images. We also present a further experiment on a larger set of images, where we underline the appropriateness of the method to address image matching at different compression and noise levels. In this specific aspect resides one of the main contribution of our work from the application standpoint: the sparsity properties of the shearlet transform are very appropriate to deal with noise and compression artefacts. With other kind of transformations, like viewpoint and scale changes, our procedure has a performance comparable with classical feature extraction techniques. As for the computational cost, it is also comparable, in the order of magnitude, to classical methods such as SIFT, but admittedly not good as the one of appropriately designed binary detectors.

The paper is organized as follows. In Section II we review the shearlet transform. Section III provides an analysis of scale selection in multi-scale image representations also in the presence of signal perturbations and the theoretical justifications of scale invariance for feature detection by shearlets. In Section IV we propose the shearlet-based blob detection algorithm, while the descriptor is introduced in Section V. Section VI reports an experimental analysis of the proposed blob detector and descriptor following the Oxford evaluation procedure. In Section VII we evaluate the proposed methods for image matching at different compression and noise levels. Section VIII is left to a final discussion.

II. A REVIEW OF THE SHEARLET TRANSFORM

A shearlet is generated by the dilation, shearing and translation of a function $\psi \in L^2(\mathbb{R}^2)$, called the *mother shearlet*, in the following way

$$\psi_{a,s,t}(x) = a^{-3/4}\psi(A_a^{-1}S_s^{-1}(x-t)) \qquad (1)$$

where $t \in \mathbb{R}^2$ is a translation, A_a is a *dilation* matrix and S_s a *shearing* matrix defined respectively by

$$A_a = \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix} \qquad S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \qquad (2)$$

with $a \in \mathbb{R}^+$ and $s \in \mathbb{R}$. The anisotropic dilation A_a controls the scale of the shearlets, by applying a different dilation factor along the two axes. The shearing matrix S_s, not expansive, determines the orientation of the shearlets. The normalization factor $a^{-3/4}$ ensures that $\|\psi_{a,s,t}\| = \|\psi\|$, where $\|\psi\|$ is the norm in $L^2(\mathbb{R}^2)$. The *shearlet transform* $\mathcal{SH}(f)$ of a signal $f \in L^2(\mathbb{R}^2)$ is defined by

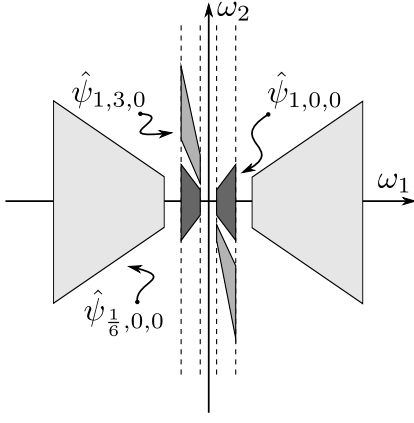$$\mathcal{SH}(f)(a,s,t) = \langle f, \psi_{a,s,t} \rangle \qquad (3)$$

Fig. 1. Support of the shearlets $\hat{\psi}_{a,s,t}$ (in the frequency domain) for different values of a and s.

where $\langle f, \psi_{a,s,t} \rangle$ is the scalar product in $L^2(\mathbb{R}^2)$.

In the classical setting the *mother shearlet* ψ is assumed to factorize in the Fourier domain as

$$\hat{\psi}(\omega_1, \omega_2) = \hat{\psi}_1(\omega_1)\hat{\psi}_2(\frac{\omega_2}{\omega_1}) \tag{4}$$

where $\hat{\psi}$ is the Fourier transform of ψ, ψ_1 is a one dimensional wavelet and $\hat{\psi}_2$ is any non-zero square-integrable function.

With the choice of Eq. (4) the shearlet definition in the frequency domain (see Fig. 1) becomes

$$\hat{\psi}_{a,s,t}(\omega_1, \omega_2) = a^{3/4}\hat{\psi}_1(a\omega_1)\hat{\psi}_2\left(\frac{\omega_2 + s\omega_1}{\sqrt{a}\,\omega_1}\right)e^{-2\pi i(\omega_1,\omega_2)\cdot t}. \tag{5}$$

As a consequence of the Plancherel formula, Eq. (3) can be rewritten as

$$\mathcal{SH}(f)(a, s, t_1, t_2) = a^{3/4}\int_{\widehat{R}^2} \hat{f}(\omega_1, \omega_2)\hat{\psi}_1(a\omega_1)$$
$$\times \hat{\psi}_2\left(\frac{\omega_2 + s\omega_1}{\sqrt{a}\,\omega_1}\right)e^{2\pi i\xi(t_1\omega_1 + t_2\omega_2)}d\omega_1 d\omega_2. \tag{6}$$

A different approach is proposed in [36] where the mother shearlet is of the form $\psi_1(x_1)\psi_2(x_2)$, where ψ_1 is a one dimensional wavelet and ψ_2 is a scaling function. This property allows to have compact support shearlets in the space domain. However, as noted in [37], " *The property [of* (4)*] does indeed not only improve the frame bounds of the associated system, but also improves the directional selectivity significantly*". To overcome this problem, in [37] the mother shearlet is multiplied by a suitable 2D fan filter in order to approximate the property (4). For sake of simplicity, in this short review we consider only classical shearlets — the ones we have adopted in our work.

A. Cone-adapted Shearlets

A major limitation of the shearlets defined in the previous section is the directional bias of shearlet elements associated with large shearing parameters. To deal with this problem [30] introduces the notion of *cone-adapted shearlets* whose construction is based on a partition of the Fourier into two

cones and a square centered around the origin. The two conic regions are defined as

$$\mathcal{C}_{\mathrm{h}} = \{(\omega_1, \omega_2) \in \mathbb{R}^2 : |\omega_2/\omega_1| \leq 1, |\omega_1| > 1\} \tag{7}$$
$$\mathcal{C}_{\mathrm{v}} = \{(\omega_1, \omega_2) \in \mathbb{R}^2 : |\omega_1/\omega_2| \leq 1, |\omega_2| > 1\}. \tag{8}$$

A shearlet ψ suitable for the horizontal cone is

$$\hat{\psi}^{\mathrm{h}}(\omega_1, \omega_2) = \hat{\psi}_1(\omega_1)\hat{\psi}_2\left(\frac{\omega_2}{\omega_1}\right)\chi_{\mathcal{C}_{\mathrm{h}}}(\omega_1, \omega_2). \tag{9}$$

where $\chi_{\mathcal{C}_{\mathrm{h}}}(\omega)$ is equal to 1 for $\omega \in \mathcal{C}_{\mathrm{h}}$ and 0 outside. Likewise the shearlet for the vertical cone is defined by interchanging the roles of ω_1 and ω_2.

The square region is the low-frequency part

$$\{(\omega_1, \omega_2) \in \mathbb{R}^2 : |\omega_1|, |\omega_2| \leq 1\}. \tag{10}$$

Since the interest points of an image are associated with high frequencies, for space reason we do not consider the low-frequency contribution, see [30] for further details.

B. Digital Shearlets

Digital shearlet systems are defined by sampling continuous shearlet systems on a discrete subset of the space of parameters $\mathbb{R}_+ \times \mathbb{R}^3$ and by sampling the signal on a grid. In the literature there are many different discretization schemes, see [37], [38] and reference therein. In this work we adopt the Fast Finite Shearlet Transform (FFST) [39] which performs the entire shearlet construction in the Fourier domain. It is possible to choose as ψ_1 wavelets whose analytic form is given in the Fourier domain, whereas in [32] and [37] are restricted to wavelets associated with multiresolution analysis. In this scheme, the signal is discretized on a square grid of size N, which is independent on the dilation and shearing parameter, whereas the scaling, shear and translation parameters are discretized as

$$a_j = 2^{-j}, \quad j = 0, \ldots, j_0 - 1, \tag{11}$$
$$s_{j,i} = i2^{-j/2}, \quad -\lfloor 2^{j/2} \rfloor \leq i \leq \lfloor 2^{j/2} \rfloor, \tag{12}$$
$$t_m = \left(\frac{m_1}{N}, \frac{m_2}{N}\right), \quad m \in \mathcal{I} \tag{13}$$

where j_0 is the number of considered scales and $\mathcal{I} = \{(m_1, m_2) : m_1, m_2 = 0, \ldots, N-1\}$. With respect to the original implementation we use a dyadic scale 2^{-j} instead of 4^{-j} to reduce the difference among two consecutive scales, which is consistent with the discretization lattice in [37].

With these notations the shearlet system becomes

$$\psi^{\mathrm{x}}_{j,i,m}(x) = \psi^{\mathrm{x}}_{a_j, s_{j,i}, t_m}(x) \tag{14}$$

where x = h or x = v and the discrete shearlet transform of a digital image $\mathcal{I}$ is

$$\mathcal{SH}(\mathcal{I})(j, \mathrm{x}, i, m) = \langle \mathcal{I}, \psi^{\mathrm{x}}_{j,i,m} \rangle \tag{15}$$

where $j = 0, \ldots, j_0 - 1$, x = h, v, $|i| \leq \lfloor 2^{j/2} \rfloor$, $m \in \mathcal{I}$. Based on the Plancherel formula $\langle f, g \rangle = \frac{1}{N^2}\langle \hat{f}, \hat{g} \rangle$, the discrete shearlet transform can be computed by applying the 2D Fast
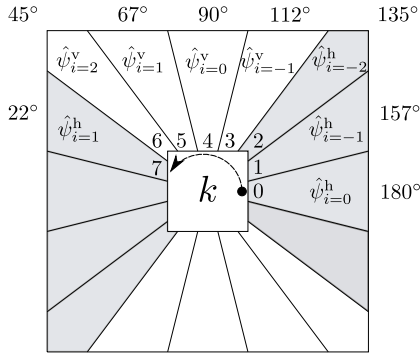
Fig. 2. A visualization, in the Fourier domain for scale $j = 2$, of the shearing indices i on the horizontal and vertical cone (gray and white areas respectively), the complete ordering of index k, and the associated angles θ_k.

Fourier Transform (fft) and its inverse (ifft). For example, for the horizontal cone $\mathcal{SH}(\mathcal{I})(j, \mathrm{h}, i, m)$ is given by

$$2^{-\frac{3j}{4}} \mathtt{ifft}(\hat{\psi}_1(2^{-j}\omega_1)\hat{\psi}_2(2^{j/2}\frac{\omega_2}{\omega_1} + i)\mathtt{fft}(\mathcal{I}))(m). \quad (16)$$

We conclude by observing, the computational complexity of the FFST is $\mathcal{O}(N^2 \log N)$, i.e. the computational complexity of the 2D Fast Fourier Transform. A more detailed analysis can be found in [29].

C. Shearing and Orientation

As we can notice, all the shearing associated with a scale j are distributed along the two vertical and horizontal cones x. For the sake of clarity, in order to provide a simple access to a specific shearing at a scale j, we introduce an index k (similar to [39]) which replaces the shearing parameter i and the cone parameters $\mathrm{x} = \{\mathrm{h}, \mathrm{v}\}$ and simplifies the notation. For each scale j, the index k iterates counter-clockwise the shearlets in the Fourier domain. It starts in the horizontal cone for each $i = -1, \ldots, -\lfloor 2^{j/2} \rfloor$ starting from $i = 0$. Then, it continues iterating the vertical cone for $i = -\lfloor 2^{j/2} \rfloor + 1, \ldots, \lfloor 2^{j/2} \rfloor$. Once the vertical cone is completed, it starts on the remaining of the horizontal cone, $i = \lfloor 2^{j/2} \rfloor - 1, \ldots, 1$. Fig. 2 illustrates the relationship between the original index i and k. Hence, from now on, we will consider the following formulation of the discrete shearlet transform

$$\mathcal{SH}(\mathcal{I})(j, k, m) = \mathcal{SH}(\mathcal{I})(j, \mathrm{x}, i, m), \quad k = 0, \ldots, 4\lfloor 2^{j/2} \rfloor - 1,$$

where $4\lfloor 2^{j/2} \rfloor$ is the total number of shearlets for a scale j.

Now, we may associate an orientation θ_k with each index k:

$$\theta_k = \pi \left(1 - \frac{k}{4\lfloor 2^{j/2} \rfloor} \right). \quad (17)$$

III. SCALE SELECTION WITH SHEARLETS

Multi-scale frameworks, like scale-space, wavelets and shearlets, represent image structures at multiple scales and are thus appropriate for detecting structures or features with different spatial extent. They may also be effective in estimating an appropriate scale to a given detected feature, useful for further tasks such as matching or recognition. According to Lindeberg [40], the formal definition of *scale selection* refers

to the estimation of characteristic scales in image data and the automatic selection of locally appropriate scales in a scale-space representation.

A particularly useful methodology for computing estimates of characteristic scales is by detecting local extrema over scales of differential expressions in terms of γ-normalized derivatives [1]. Following this approach, it can be shown that different types of scale invariant feature detectors can be used for different types of visual features, like blobs, corners, etc. Furthermore, the scale levels obtained from the scale selection can be used for computing image descriptors.

Detected feature points are considered *scale invariant* if the points are preserved under scaling transformations and the selected scale levels are transformed in accordance with the amount of scaling [1]. In addition, *perfect scale invariance* is considered when the extrema over scales are equal.

In this section, we start from the continuous setting and first discuss scale invariance properties reviewing Lindeberg approach in the framework of space-scale theory [1]. Then, we show how shearlet coefficients can also detect the correct scale while providing directional information. In the second part of the section we discuss how we can obtain a measure of scale invariance in the discrete setting.

A. Scale Invariance in the Continuous Setting

When a signal is subject to a scale-space smoothing, the spatial derivatives on the smoothed data are expected to decrease. This is a well-known property of the scale-space representation, according to which the amplitude of its spatial derivatives decreases with scale. To obtain a multi-scale signal representation whose amplitude is independent on the scale Lindeberg proposed a γ-normalized derivative operator [1]. In this section, we will show that shearlets share a similar behaviour, but they directly encode the directional information. We observe how a similar analysis could be carried out considering directional wavelets [22], which are not included in our study but will be taken into account in future works.

1) Scale-space: In this section we briefly recall the main properties of scale-space theory for two dimensional signals developed by Lindeberg [1].

In the scale-space theory, the filters are given by the family of 2D-Gaussian kernels

$$g_a(x) = \frac{1}{2\pi a^2} e^{-\frac{x^2}{2a^2}} \qquad x \in \mathbb{R}^2 \quad (18)$$

parametrised by the scale $a \in \mathbb{R}^+$. Each signal f is mapped to its scale-space transform by convolution with g_a

$$L[f](a, z) = g_a * f(az) = \int_{\mathbb{R}^2} g_a(y)f(az - y)dy \quad (19)$$

where $z = x/a$ is the "scale invariant" space variable. In the original paper [1] the scale is $t = a^2$ and a more general rescaling is considered (as for example $z = x/t^{\gamma/2}$). This representation has two main properties, which are at the root of the theory and are usually referred to as perfect scale invariance. First, the representation $L[f]$ is invariant under dilations, *i.e.* if $f_\alpha(x) = f(x/\alpha)$ with $\alpha \in \mathbb{R}^+$, then

$$L[f_\alpha](a, z) = L[f](\alpha^{-1}a, z). \quad (20)$$

Furthermore, for 2D sinusoidal signals

$$f(x_1, x_2) = \cos(\alpha x_1)\cos(\beta x_2), \qquad (21)$$

the corresponding transform is able to detect the scale by applying a suitable differential operator

$$\mathcal{D} = P\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right) \qquad (22)$$

where P is a given polynomial in two variables. Indeed, define the quantity

$$L_{\mathcal{D},\max}(a) = \max_{z\in\mathbb{R}^2} |\mathcal{D}L[f](z,a)|, \qquad (23)$$

since

$$L[f](a,z) = e^{-\frac{a^2(\alpha^2+\beta^2)}{2}} f(az), \qquad (24)$$

then

$$L_{\mathcal{D},\max}(a) = P(a\alpha, a\beta)e^{-\frac{a^2}{2}(\alpha^2+\beta^2)}. \qquad (25)$$

For example, if $\mathcal{D} = \frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2}$ is the z-Laplacian, then

$$L_{\mathcal{D},\max}(a) = a^2(\alpha^2+\beta^2)e^{-\frac{a^2}{2}(\alpha^2+\beta^2)}, \qquad (26)$$

which takes its maximum at $a^* = 1/\sqrt{\alpha^2 + \beta^2}$ with value independent on the scale. Hence, the extrema of the scale-space representation across the scale a allows to detect the scale $1/\sqrt{\alpha^2 + \beta^2}$ of the signal. However, to extract the ratio β/α associated with the directional information there is the need to compute other quantities as the determinant of the Hessian [28]. In the next section we show that shearlets have essentially the same behaviour, but the transform directly detects both the parameters α and β.

2) Shearlets: Since the dilation matrix defining the shearlets is not isotropic, we can not expect that the shearlet transform itself is invariant under (isotropic) scale changes. However, we will show how a related quantity has the perfect scale invariance property, as demonstrated by the following result, whose proof can be found in the appendix.

Theorem. *The cumulative shearlet transform*

$$B[f](a,z) = a^{-5/4} \int_{\mathbb{R}} \mathcal{SH}(f)(a,s,az)ds, \qquad (27)$$

with $a \in \mathbb{R}_+$ and $z \in \mathbb{R}^2$, is scale invariant, i.e. *for all $f \in L^2(\mathbb{R}^2)$*

$$B[f_\alpha](a,z) = B[f](\alpha^{-1}a, z). \qquad (28)$$

Furthermore, if f is the sinusoidal signal given by (21)*, then*

$$B[f](a,z) = \overline{\psi_2(0)}\,\overline{\hat{\psi}_1\left(\frac{a\alpha}{2\pi}\right)} f(az), \qquad (29)$$
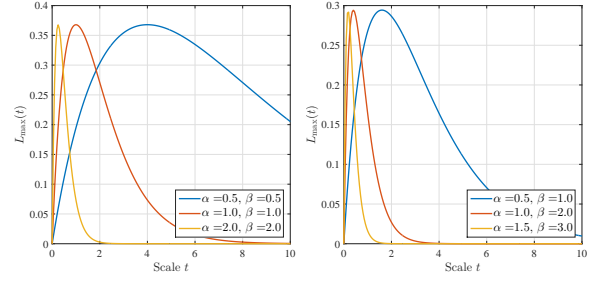
provided that ψ_1 is even.

As we did for scale-space, if f is the 2D sinusoidal signal as in (21) a simple calculation shows that the maximum of the modulus over z is
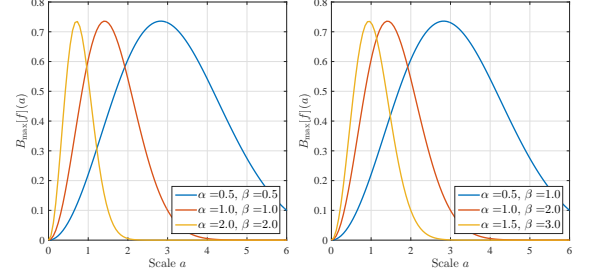
$$B_{\max}[f](a) = |\psi_2(0)|\,|\hat{\psi}_1\left(\frac{a\alpha}{2\pi}\right)|. \qquad (30)$$

By choosing $\hat{\psi}_1$ as the *1D-Mexican hat* wavelet

$$\hat{\psi}_1(\omega) = \omega^2 e^{-2\pi^2\omega^2}, \qquad (31)$$

(a) Scale-space

(b) Shearlets

Fig. 3. The plot of (a) $L_{\max}(t)$ Eq. (25) and (b) $B_{\max}[f](a)$ Eq. (32) as function of scale for 2D sinusoidal signals at different frequencies α and β.

we can rewrite Eq. (30) as

$$B_{\max}[f](a) = \frac{|\psi_2(0)|}{4\pi^2}(a\alpha)^2 e^{-\frac{(a\alpha)^2}{2}}, \qquad (32)$$

which shares the same behaviour of $L_{\max}[f]$, but the maximum of $B_{\max}[f]$ is now at $a^* = 1/\alpha$. Hence, for shearlets the selected scale a^* only depends on the frequency α, as shown in Fig. 3. However, if we consider the shearlet coefficient, a computation as above shows that

$$\max_{t\in\mathbb{R}^2} |\mathcal{SH}(f)(a,s,t)| = a^{3/4}|\hat{\psi}_1\left(\frac{a\alpha}{2\pi}\right)||\hat{\psi}_2\left(\frac{s+\beta\alpha^{-1}}{\sqrt{a}}\right)|, \qquad (33)$$

provided that both ψ_1 and ψ_2 are even. Since $\hat{\psi}_2$ is a bump function, fixed the scale a, the shearlet coefficients have a maximun around an interval centered at $s = -\beta/\alpha$. If $\hat{\psi}_2$ is a Gaussian bump and Ψ_1 is as in (31)

$$a^{-3/4}\max_{t\in\mathbb{R}^2}|\mathcal{SH}(f)(a,s,t)| = C(a\alpha)^2 e^{-\frac{(a\alpha)^2}{2}} e^{-\frac{(s+\beta\alpha^{-1})^2}{2a}}. \qquad (34)$$

Fig. 3 presents the actual plots of Eq. (25) for the scale-space and Eq. (32) for shearlets at different combinations of the frequencies parameters α and β. In the plots on the left, the frequencies α and β are equally increased by a factor of 2 (isotropic structures), while on the right, α is increased by a step equal to 0.5 and $\beta = 2\alpha$ (anisotropic structures). As we can observe, the plots associated with both transformations are capable to produce perfect scale invariance for the two types of frequency combinations in the sinusoidal function f.

Finally, we stress that the choice of ψ_1 and ψ_2 influences the type of local features that are enhanced by the shearlet transform. Thus, in order to detect blob features, as suggested by Equation (31) we selected ψ_1 as the Mexican hat wavelet and ψ_2 as a smooth function with compact support whose analytic form is given in [39].

(a) 2D Gaussian functions with different σ_x, σ_y (b) Laplacian   (c) Shearlets
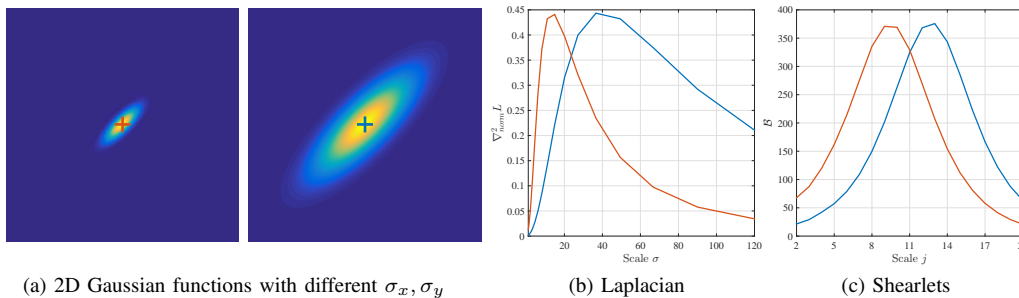
Fig. 4. Scale decomposition of the synthetic images in (a) at their center point (color-coded) by using the (b) Laplacian of Gaussian, and (c) shearlet $\mathcal{B}$ measure.

B. Scale Invariance in the Discrete Setting

In the previous section, we defined a scale invariant shearlet transform in the continuous setting. Now, let us formally define the discrete counterpart of Eq. (27), which we call the $\mathcal{B}$ measure.

Definition. *The $\mathcal{B}$ measure is the scale-normalized sum of the discrete shearlet transform coefficients across the shearing parameter,*

$$\mathcal{B}(m,j) = \frac{2^{\frac{5j}{4}}}{C_j} \sum_k \mathcal{SH}(\mathcal{I})(j,k,m), \qquad (35)$$

where j, k, m are the discretized scaling, shearing and translation parameters.

Comparing with Eq. (27), the normalization factor C_j takes into account that for each scale j there is a different number of orientations.

We now briefly discuss the concept of perfect scale invariance on a discrete synthetic signal. Fig. 4 (a) shows two 2D Gaussian functions with different σ_x, σ_y and with an orientation of $\pi/4$. The σ_y has a step of $1/3$ while $\sigma_x = \sigma_y/3$, thus producing an elliptical structure at different scales. In the rest of Fig. 4, we computed the scales at the center point of each image with the following configurations. For scale-space, we used the scale-normalized Laplacian of Gaussian (LoG). For shearlets we plotted the $\mathcal{B}$ measure where ψ_1 was chosen to be the 2-nd derivative of the Gaussian. By analyzing Fig. 4, we can observe that, by performing direct calculations, perfect scale-invariance still holds for the two discussed frameworks on syntetic images.

Fig. 5 (top) illustrates instead the behavior of LoG and $\mathcal{B}$ across scales for different key points of a real image. We consider in particular five locations corresponding to blob structures of different size and one texturized region. It is easy to observe in both cases that although there is no perfect scale invariance, the peaks are clearly visible and their position reflect the different spatial extents of the corresponding image structures. Notice that the two scale parameters j and σ have an inverse relationship with respect to the image structures, thus plots are to be read accordingly.

C. Stability with respect to noise

It is well known that wavelets provide an optimal sparse representation of 1D signals having local discontinuities [41]

in which the noise can be efficiently reduced by thresholding the coefficients [21]. However, in the 2D case they fail to provide a comparable efficiency. For images with singularities, an optimally sparse representation is provided instead by shearlets [42], [43] and the thresholding procedure (or one of its variants) gives an outperforming denoising/deconvolution algorithm as shown in many papers, see [27], [44]–[46].

Given the scale j and the location m, the measure $\mathcal{B}(m,j)$ defined by (35) is the sum of the shearlet coefficients over the orientations k – with only a few significant shearing contributions. Blobs correspond to high values of $\mathcal{B}$, so that it is quite natural to expect that the local maxima of $\mathcal{B}$ are very stable under noise and blurring.

Indeed, by the linearity of the shearlet coefficients, clearly $\mathcal{B} = \mathcal{B}_f + \mathcal{B}_\gamma$, where the first term is the contribution of the signal and the second term of the random noise vector γ. Based on Lemma 1 of [44], we claim that the expected mean $\mathbb{E}[|\mathcal{B}_\gamma(m,j)|^2]$ only depends on the scale j.

This intuition is confirmed by some illustrative experiments. Fig. 5 (bottom) shows the behavior of LoG and $\mathcal{B}$ for different color-coded points on the same (top) image corrupted by Gaussian noise and JPEG compression. Fig. 6 reports the effect on $\mathcal{B}$ of adding increasing Gaussian noise to the image. The perturbation starts affecting the smallest flower which soon becomes indistinguishable from a texture or noisy pattern. Instead the three larger flowers continue to be visible at coarser scales and the scale invariance property still holds for them, although the value of the peaks tend to decrease. This can also be observed in Fig. 7, where we show the behavior of $\mathcal{B}$ with an increasing amount of noise. The left plot refers to Gaussian noise, the other plots describe the situations where Gaussian blur is added (here notice how larger structures decay more slowly), images have a reduced contrast factor obtained by computing a weighted average with a constant average gray image, and the signals are compressed with JPEG2000 algorithm (notice here the negligible effect on the measure). Finally, Fig. 8 illustrates the effect of imposing large amounts of different perturbations. The results are coherent with what we previously observed: (i) different compression algorithms do not effect the behavior of $\mathcal{B}$, (ii) image blur produces effects that depend on the size of the observed structures, (iii) a low contrast reduces the magnitude of peaks, but retains scale invariance.
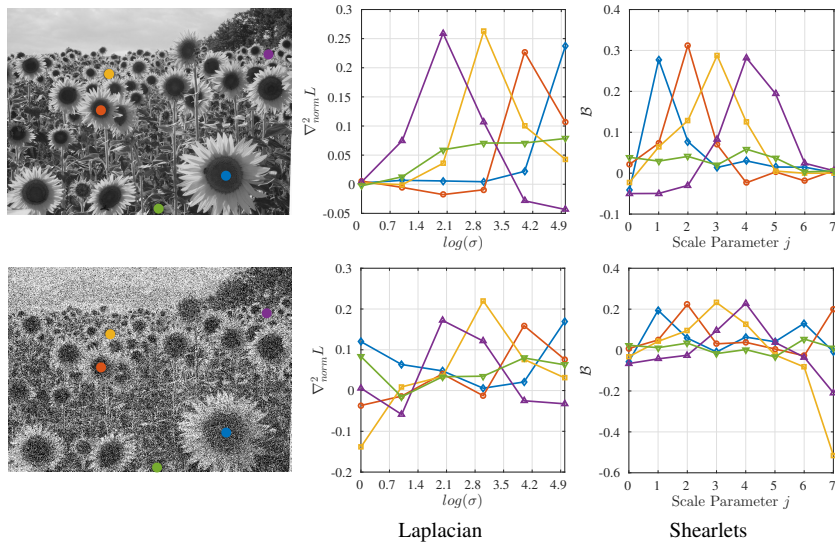
Fig. 5. The behavior of the Laplacian of Gaussian, and shearlet $\mathcal{B}$ measure across scales for different points, coherently color-coded, on a real image (top) and the same image corrupted by noise and JPEG compression (bottom).
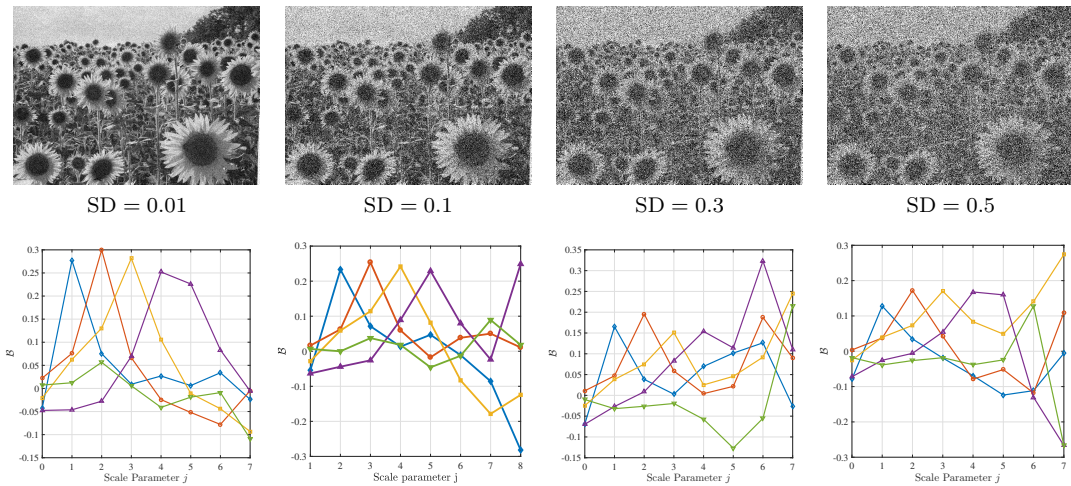

Fig. 6. The behavior of $\mathcal{B}$ for points highlighted in Fig. 5 on Gaussian noise with increasing values of standard deviation (SD).
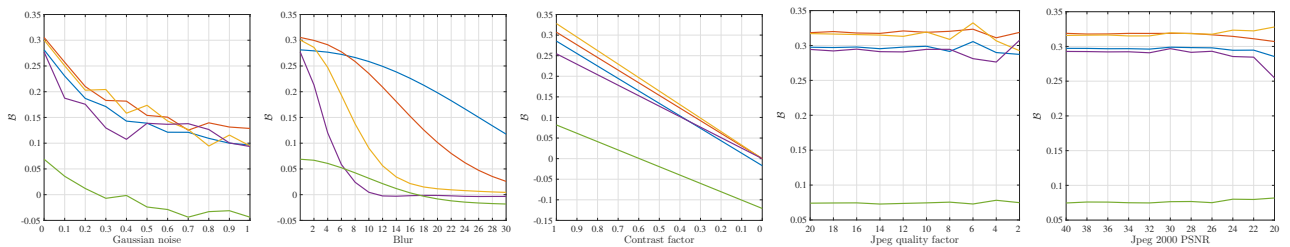

Fig. 7. The effect of different perturbations on the magnitude of $\mathcal{B}$ peaks, for increasing perturbations. From the left: additive Gaussian noise (with an increasing standard deviation), Gaussian blur, contrast reduction (with an increasing weight on a constant image), JPEG compression for a decreasing quality factor (QF), JPEG 2000 compression for a decreasing PSNR.

IV. BLOB DETECTION WITH SHEARLETS

In this section we deal with the problem of automatically detecting blobs and describe our *Shearlet Blob Detector (SBD)* algorithm.

Similarly to the method proposed by Lowe to extract DoG features [2], our approach consists of different steps of measures, computation and refinement. In the reminder of

the section we detail the three steps of the blob detection algorithm. Fig. 9 shows the intermediate steps results on a sample image.

A. Accurate feature point localization

A location m at a certain scale j is recognized as a candidate keypoint if the function $\mathcal{B}(m, j)$, computed over a spatial
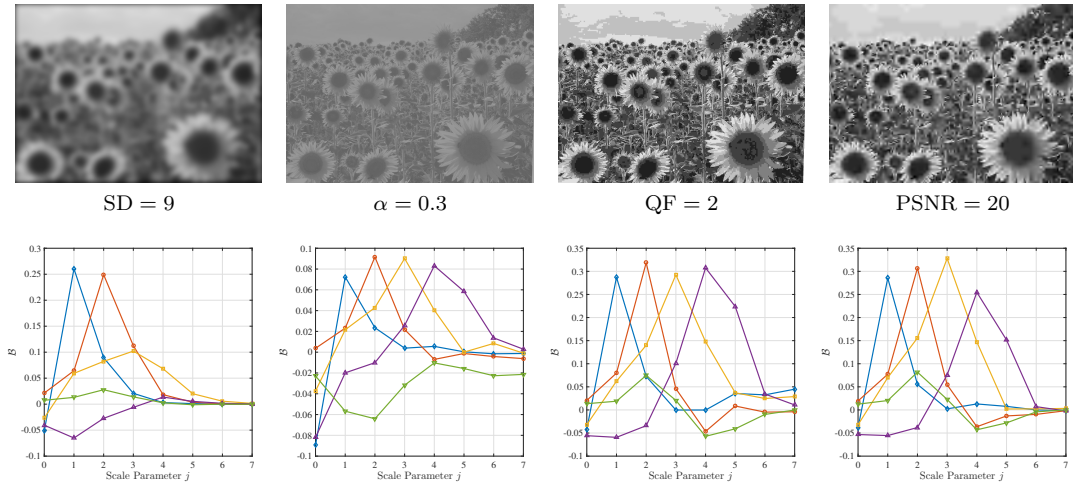
Fig. 8. The effect of different signal perturbations on the measure $\mathcal{B}$ for points highlighted in Fig. 5. From the left: Gaussian blur, reduced contrast, JPEG compression, JPEG2000 compression.

$3 \times 3 \times 3$ (2D space $\times$ scales) neighborhood centered on m assumes a local extremum (maximum or minimum) in it and its value is above a threshold.

$$(\bar{m}, \bar{j}) = \arg\max_{m,j}\min \text{ local } \mathcal{B}(m, j). \qquad (36)$$

Then, the local extrema of the $\mathcal{B}$ function are interpolated in space and scale with the Brown and Lowe method [47] to reduce the effect of considering a limited number of scales. The outcome of this step is a set of feature points with their associated scales.

Notice that since the $\mathcal{B}$ measure is isotropic, only rotationally invariant features will be detected.

B. Edge responses elimination

The function $\mathcal{B}$ has strong responses along edges, especially at fine scales. Therefore, in order to increase stability of the detected points, we need to eliminate the feature points that have high edge responses.

Given a point m at scale j, to evaluate how spread out are the corresponding orientation responses with respect to the predominant orientation response we introduce the following measure

$$\frac{1}{4\lfloor 2^{j/2} \rfloor} \sum_k \left(\mathcal{SH}(\mathcal{I})(j, k, m) - \mathcal{SH}(\mathcal{I})(j, k_{\max}, m) \right)^2, \quad (37)$$

where $4\lfloor 2^{j/2} \rfloor$ is the total number of shearings for scale j, and $k_{\max}$ is the shearing with largest shearlet response,

$$k_{\max} = \arg\max_k |\mathcal{SH}(\mathcal{I})(j, k, m)|. \qquad (38)$$

High values of (37) correspond to situations where $k_{\max}$ is the only orientation with strong support, that is, m is an edge or close to an edge point. Conversely, the value of (37) starts decreasing when the point m has more than one, or none, predominant orientations. The second case corresponds to blob points. Therefore, points m with a high edge response may be rejected by an appropriate thresholding.

C. Accurate orientation assignment

In this step an orientation is assigned to each feature point. This is an important step in view of the computation of rotation invariant local feature descriptors. By means of the shearlet transform, the predominant orientation at a point m and scale j is easily obtained by finding the index $k_{\max}$ given by Equation (38). However, the orientation estimation at coarse scales may have low accuracy since for small j a few shearings are employed. The effects can be attenuated by finding the extremum of an interpolated parabola for the following three points:

$$[\theta_{k_{\max}-1}, \mathcal{SH}(\mathcal{I})(j, k_{\max} - 1, m)] \qquad (39)$$

$$[\theta_{k_{\max}}, \mathcal{SH}(\mathcal{I})(j, k_{\max}, m)] \qquad (40)$$

$$[\theta_{k_{\max}+1}, \mathcal{SH}(\mathcal{I})(j, k_{\max} + 1, m)], \qquad (41)$$

where θ_k is the angle associated with the shearing k, as in Eq. (17).

Fig. 10 reports examples of output of our blob detector on a variety of scenarios. For all images, blobs have been detected using a shearlet transform with 8 scales, and a rather permissive threshold for edge points elimination. The circles indicating the presence of blobs have a radius proportional to the estimated optimal scale. As observed, such estimates are very close to the effective spatial extent of image structures.

V. Feature Description with Shearlets

In this section we propose a local feature descriptor based on the shearlet transform, the Shearlet Local Description algorithm (SLD).

The idea behind our descriptor is to encode the shearlet coefficients computed from the SBD, and thus complete the full detection-description pipeline with a single main computation, the shearlet transform in this case.

Given a feature $F = (m_1, m_2, j, \theta)$, where $m = (m_1, m_2)$ is its location, j its estimated scale and θ its predominant orientation, our descriptor encodes the shearlet coefficients

Fig. 9. The different steps of the shearlet blob detector algorithm. Left: feature point localization; Centre: edge response elimination; Right: orientation assignment

Fig. 10. Qualitative results of our blob detector on different scenarios.

information from a square region centered on (m_1, m_2), scaled with respect to j and rotated according to θ (Fig. 12).

A. Shearings with common orientations

Since in our representation different scales are associated with different amount of shearings, we first need to fix a common number of shearings across scales, in order to obtain descriptions of equal size from keypoints at different scales. Moving towards finer scales, there is an inclusion relation between the corresponding range of shearings, thus the orientations associated with the shearings at coarser scales are also available at fine scales, see Fig. 11.

Let O be a set of orientations that can be associated with the shearings of *all* the employed scales. The cardinality of the set is $|O| = c$, where c is the only parameter[1] of this method and will influence the size of the descriptor. By default, $O = \{180°, 135°, 90°, 45\}$ since these are the orientations associated with the four shearlets of the coarser scales $j = 0, 1$.

We refer to S_j as the sequence of all $4\lfloor 2^{j/2}\rfloor$ shearings for scale j, and to S_j^O as the sequence of shearings in scale j with respect to the common orientations O. Fig. 11 shows an example. For scale $j = 2$, S_2 is composed by all the shearlets (white and grey), while S_2^O is only composed by the gray shearlets. Notice that for $j = 1$, S_1^O is equal to S_1.

B. Spatial sampling

We sample a regular grid of 24 points per side around (m_1, m_2) with a sampling step of $p = 2^{j_0 - j}$, i.e. the inverse

[1]Notice that c must be a power of 2 to be coherent with the number of shearlets on a scale.
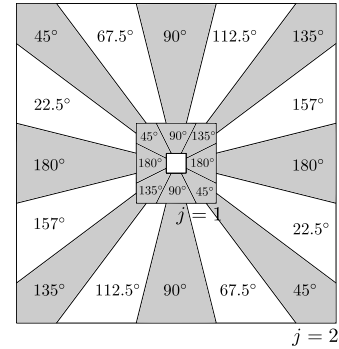

Fig. 11. Visualization of the orientations at scales $j = 1, 2$ and $O = \{180°, 135°, 90°, 45\}$. In gray: sequence of common orientations S_j^O. In white: orientations S_j. (see text).

shearlet continuous scale of the feature, covering a length of $24p$.

We divide the regular grid in 16 overlapped subregions of size $9p \times 9p$ (hence including 81 shearlet coefficients). We refer to each subregion using the centroid (thick red points in Fig. 12 (b)), which may be described by its relative position with respect to the local keypoint reference system. More formally, the subregions can be referred to as $\{\mathcal{G}_{e,f}\}$ where $e, f \in \{\pm 1, \pm 2\}$. Notice that the overlap allows us to cope with small spatial keypoint shifts.

C. Region rotation

In order to gain rotation invariance, we compute the actual descriptor on a region which is rotated according to the keypoint main orientation θ. To this purpose, we perform a
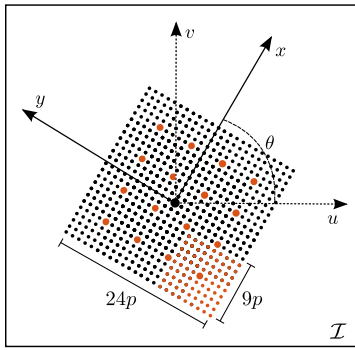
Fig. 12. Illustration of the change of coordinates for rotation invariance and the scaled and orientated shearlet sampled points used for the construction of the SLD descriptor.

change of reference system (see Fig. 12),

$$\begin{pmatrix} x \\ y \end{pmatrix} = p \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}. \tag{42}$$

To maintain the rotation invariance, the shearing parameter k also has to be aligned according to θ. To this purpose, we perform a circular shift following the shearing indexing i_k described on Section II-C,

$$t(k) = (i_k \pm n_\theta) \bmod |S_j| \tag{43}$$

where $n_\theta = \lfloor \theta |S_j|/\pi \rfloor$ is the number of shifts required to align with respect to θ.

D. Descriptor construction

The SLD descriptor concatenates statistics on shearlets co-efficients in each subregion. We start by describing a subregion $\mathcal{G}_{e,f}$ of scale j and shearing k by a 2-D vector $\mu(e,f,j,k)$

$$\mu(e,f,j,k) = \begin{pmatrix} \sum_{(u,v)\in\mathcal{G}_{e,f}} M_j(u,v,k) g(u,v,2.5s), \\ \sum_{(u,v)\in\mathcal{G}_{e,f}} |M_j(u,v,k)| \, g(u,v,2.5s) \end{pmatrix}. \tag{44}$$

where

$$M_j(u,v,k) = \mathcal{SH}(\mathcal{I})(j,t(k),(x,y)) \tag{45}$$

and g is a 2D Gaussian filter with $\sigma = 2.5p$. Then, within the subregion, we concatenate $\mu(e,f,j,k)$ for each shearing $k \in S_j^O$ following the order induced by the circular shift: let us denote it as $\mu(e,f,j) \in \mathbb{R}^{2\times c}$.

Remark: The rotation invariance can be improved by per-forming a weighted sum over the shearing neighbourhoods of the sampled point, instead of sampling directly into the shearlet coefficient. That is,

$$M_j(u,v,k) = \sum_{r=-|S_j/2|}^{|S_j/2|} \mathcal{SH}(\mathcal{I})(j,t(k+r),(x,y))g(r,\sigma) \tag{46}$$

where g in this case is a 1D Gaussian with $\sigma = |S_j|/5$. This way small misalignments in the orientation can be overcome. However, the drawback is the additional computation.

Next, the contribution of each subregion is weighted using a Gaussian with $\sigma = 1.5$. and then concatenated to build the descriptor $\mu \in \mathbb{R}^{2\times c\times 16}$ as

$$\mu = [\mu(e,f,j)g(e,f,1.5)]_{e,f\in\{\pm1,\pm2\}} \tag{47}$$

where we do not report the dependence on j as it refers to the fixed estimated scale. Both Gaussian weighting increase robustness towards geometric deformations and localization errors [3].

E. Descriptor normalization

Finally, in order to gain invariance to linear contrast changes, we normalized the descriptor to a unit vector, using the ℓ_2 normalization,

$$SLD(F) = \mu/\|\mu\|_2. \tag{48}$$

VI. Method assessment on benchmarks

In this section we provide an experimental assessment of our shearlet-based method for blob detection and description with respect to a standard benchmark. Our goal is to show the method is comparable with state of the art approaches on a variety of transformations. For the sake of replicability, our evaluation follows the Mikolajczyk's protocol and image sequences[2] implemented on VLBechmarks [48]. Each sequence includes 6 images of natural textured scenes with increasing geometric and photometric transformations.

A. Detection Evaluation

We evaluate the detection performances using the *repeata-bility score* (RS) [49], i.e. the ratio of the number of corre-spondences and the number of detected features. We compare the proposed Shearlet Blob Detector (SBD) with SIFT [2], Harris-Laplace [50], Hessian-Laplace [50], SURF [3] and BRISK [10] feature detectors. As for SIFT, Harris-Laplace and Hessian-Laplace we relied on the implementations provided with the VLBechmarks, while for SURF and BRISK, we adopted the implementation available in MATLAB. For a fair comparison, we adjust the thresholds of the detectors so that the number of detected keypoints is similar in the first image of the sequences. As for the number of scales, in the case of the SBD we considered a Shearlet decomposition with 7 scales, while in the other methods we set the default number of octaves.

Fig. 13 summarizes the obtained results and shows how our detector is appropriate under general circumstances, while there is no method which is clearly and uniformly superior to the others.

B. Descriptor Evaluation

The Shearlet Local Descriptor (SLD) is evaluated using *recall* (number of correct matches / number of correspon-dences) vs *1-precision* (number of false matches / number of matches) curves obtained by matching pairs of images (1st and 4th) from each sequence (as in [5]). As a comparative evaluation, we also report the results obtained using SIFT, SURF and LIOP [8] descriptors, along with the recent BRISK and FREAK [11] (implementation available in MATLAB). For all the descriptors we employed their default parameters included our SLD, for which we set $c = 4$. Our default choice

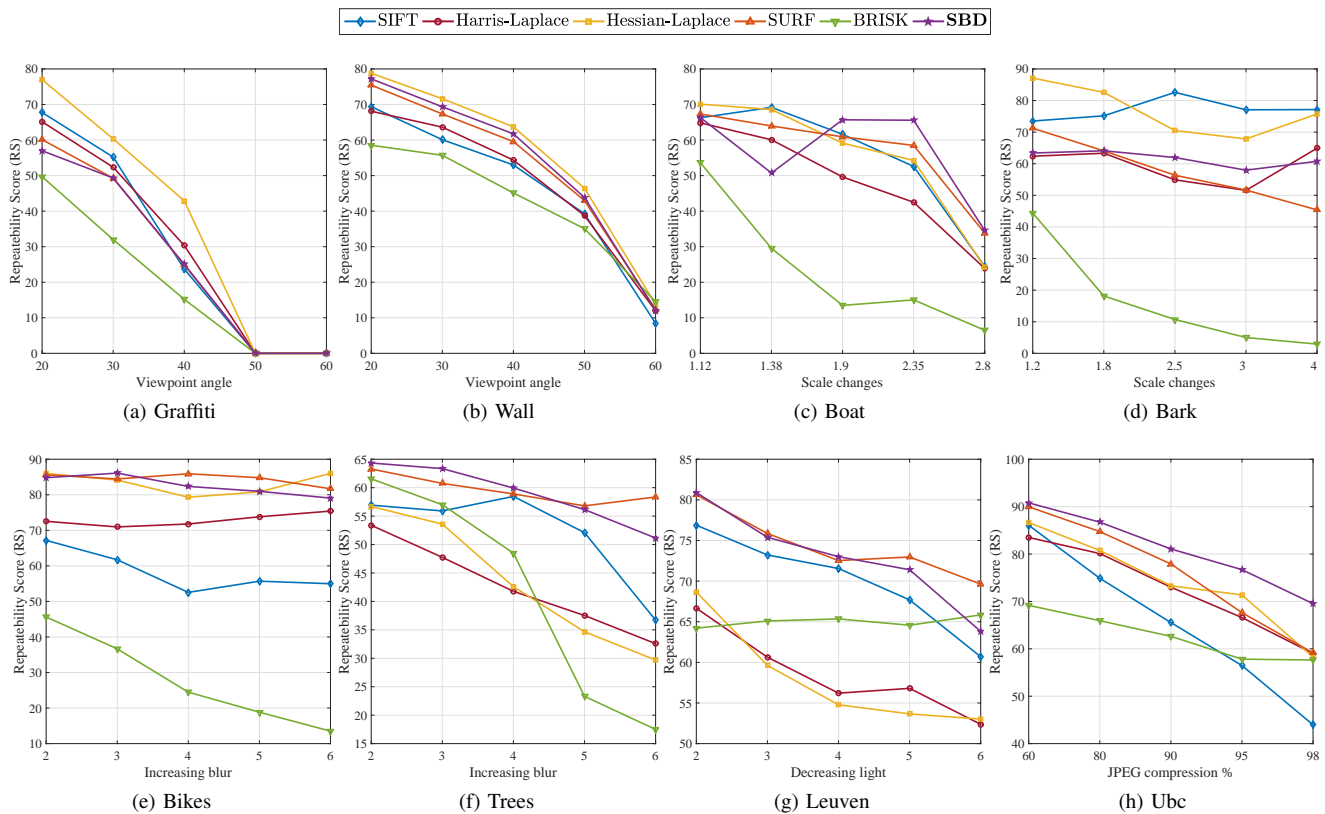[2]http://www.robots.ox.ac.uk/~vgg/research/affine/

Fig. 13. Comparison of multi-scale feature detection using repeatability scores (40% overlap error) with respect to viewpoint change (a) and (b), scale change (c) and (d), image blur (e) and (f), light change (g), and image compression (h).

represents a compromise between computational efficiency and quality of the results. As for matching strategy, we used a threshold-based approach – where two detected blobs are matched if the distance between their descriptors is below a threshold – which is known to be indicative of the distribution of the descriptors in the space [5]. We employed the Euclidean distance for SLD, SIFT, SURF and LIOP, while we compare BRISK and FREAK (binary descriptors) using the Hamming distance. In order to maintain an unbiased comparison, we evaluate all the descriptors in combination with the DoG feature detector (the one usually coupled with SIFT) using the default parameters for all the image sequences. We now discuss the obtained results, reported in Fig. 14.

Here again, we do not observe a clear superiority of a method over the others. It should be noticed how our SLD behaves consistently well in the presence of blur, compression effects, illumination changes. This is explicable in terms of the properties of shearlets which provide us with an optimal sparse representation for natural images.

C. Computational Performance Evaluation

The computational cost of the proposed algorithm heavily depends on the computation of the shearlet transform [51]. More efficient implementations than FFST are today available [27], [52]. In this work we consider FFST as it allows us to change modularly the mother wavelet ψ_1 to design different feature detection algorithms.

Table I shows a comparison of the computational performances of different detectors and descriptors on the first image

TABLE I
COMPUTATIONAL PERFORMANCES (DETECTION AND DESCRIPTION) ON THE FIRST GRAFFITI SEQUENCE IMAGE. TIME IN MS.

Detector	Keypoints	Time	Descriptor	Time	Total
DoG	1583	113	SIFT	202	315
Harris-Lapl.	1458	630	LIOP	102	732
Hessian-Lapl.	1432	226	LIOP	100	326
Fast Hess	1537	66	SURF	48	114
BRISK	1246	39	BRISK	13	52
SBD	1586	1903	SLD	147	2050
SBD*	1538	526	SLD	126	652

of the Graffiti sequence (of size 640x800). The experiments ware carried on a 8 core Intel i7-2600 CPU using MATLAB 2016a on Ubuntu 16.04.

As a proof of concept on the potential for significant improvements, we reimplemented the FFST by approximating the shearlet transform with a boundary on the maximum number of shears at fine scales. A similar approach is also followed by other shearlet transform implementations [27], [52]. To this purpose we tested the new implementation which is reported in the table as SBD*. As Table I shows, this new implementation is 3 times faster that the initial one.

Since the process of computing the shearlet transform has an intrinsically parallel nature, further implementations can benefit from the use of a GPU, as shown in [51] for image denoising.

(a) Graffiti

(b) Boat
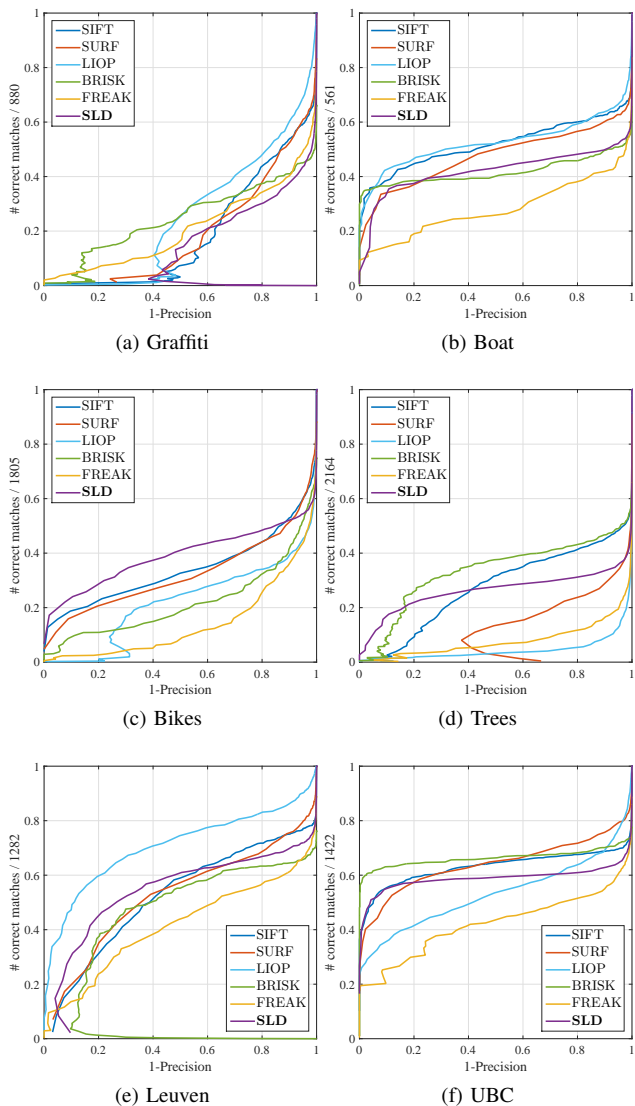
(c) Bikes

(d) Trees

(e) Leuven

(f) UBC

Fig. 14. Comparison of feature descriptors using Precision-Recall curves with respect to viewpoint change (a), scale change (b), image blur (c) and (d), light change (e), and image compression (f).

(a) JPEG compression

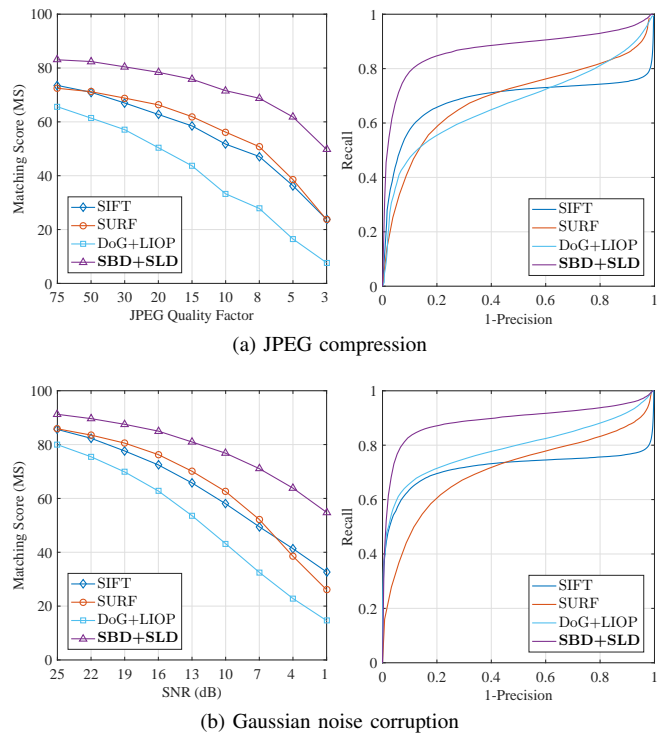(b) Gaussian noise corruption

Fig. 15. Comparison of blob detectors with their respective descriptor on the *INRIA Copydays dataset*. Left: matching score against amount of compression (a) and noise corruption (b). Right: recall vs 1-precision curve between untransformed and 15 QF compressed (a) and 13 dB of SNR noise corrupted images.

VII. EXPERIMENTS ON NOISY IMAGES

Copydays dataset: In this section we report the core experimental analysis of our method, where we highlight the superiority of Shearlet representations in the presence of noise. We consider the *INRIA Copydays* dataset[3], which contains 157 natural images that are progressively compressed, from 3 (very low quality) to 75 (typical web quality) quality factor (QF). For the evaluation in noisy environments, the images were also progressively corrupted with Gaussian noise.

We compare our full pipeline (SBD+SLD) with SIFT and SURF methods (DoG+SIFT and fastHessian+SURF, respectively), along with the DoG+LIOP detector-descriptor combination. For the evaluation, we consider the *Matching Score* (MS) [49] which is the ratio between the number of correct matches and the number of detected features. For a visual impression of the overall performances, average values are

reported. Fig. 15 (left) shows the comparison, where the matching superiority of our approach can be appreciated in both JPEG compression (a) and noise corruption (b).

As a further evidence, we also provide the *recall* vs *1-precision* curve (see Fig. 15, right) obtained when matching the (untransformed) images with the compressed instances (15 quality factor) in (a) and noisy instances (13 dB of Signal to Noise Ratio) in (b). Note that our approach consistently outperforms the competitors.

The results we obtained are in good agreement with the theoretical intuition that shearlets are an appropriate choice in particular when dealing with noisy and compressed signals and with the empirical evidence provided in Section III.

Object recognition: In a last set of experiments we consider a collection of data (30 image pairs of 11 object instances) acquired in-house by a wearable device meant to recognize objects in real time. In the experiments we considered image pairs taken from a video sequence, where at least one image is affected by a significant amount of motion blur and out of focus (see Fig. 16). Notice that these effects are quite common in real video processing applications.

Fig. 17 reports precision-recall curves obtained on average on all the pairs, following the same protocol of previous experiments. The advantage of the Shearlet representation is apparent. The quality of the detected features can be also appreciated in Fig. 18

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[3]The datasets is available at https://lear.inrialpes.fr/~jegou/data.php

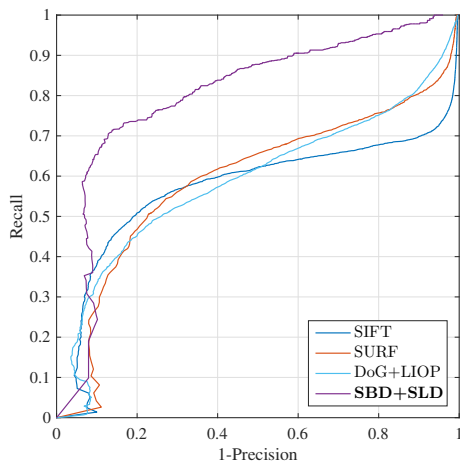Fig. 16. Sample image pairs showing different levels of noise

Fig. 17. Comparison of blob detectors with their respective descriptor on object recognition data (recall vs 1-precision curve).
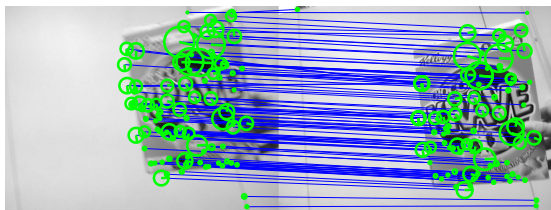

Fig. 18. An example of feature matching using (SBD+SLD).

VIII. Conclusions

In this paper we considered the shearlet representation as a multi-scale framework for the detection and the description of scale-invariant interest points tolerant to the effect of noise. We first provided a comparative analysis of scale invariance in the scale-space and shearlets domains in the continuous case — following the reasoning proposed by Lindeberg [28]. Then, we considered a discrete setting where we discussed scale invariance and tolerance to noise. Also, we addressed the problem of detecting and describing blob-like features and proposed a Shearlet Blob Detector (SBD) algorithm and a Shearlet Local Descriptor (SLD) algorithm, which we experimentally assessed by a thorough evaluation on a benchmark dataset. Our algorithm compared favorably with the state of the art, showing a very good tolerance to blur, illumination variations and compression in particular. We also considered two experiments on noisy images: a first one a larger dataset of images affected by different degrees of noise or compression degradation, a second one on images acquired by a moving camera mounted on a wearable device where the images included out of focus and motion blur effects. We observed how our shearlet-based pipeline provided superior results to SIFT, SURF, and LIOP.

In future works different shearlet transform alternatives will be worth investigating. In particular, compactly supported shearlets in the space domain [36] have been recently shown to have nice properties for edge detection [53] since they could allow us to capture effectively the spatial locality of image features. Since our proposed methods follows exactly the theoretical conceptual path of shearlets, another future work will be to provide a more efficient implementation of our methods based on approximation strategies and optimizations as in [2], [3].

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Appendix

Proof of Theorem III-A2

Proof. Let $\mathcal{SH}(f)(a,s,t_1,t_2)$ be the continuous shearlet transform of f given by (6). With the change of variables $\omega_1 = \xi$ and $\omega_2 = v\xi$ whose Jacobian is

$$\left| \frac{\partial(\omega_1, \omega_2)}{\partial(\xi, v)} \right| = \left| \det \begin{bmatrix} 1 & 0 \\ v & \xi \end{bmatrix} \right| = |\xi| \qquad (49)$$

Eq. (6) can be rewritten as

$$\mathcal{SH}(f)(a,s,t_1,t_2) = a^{3/4}$$
$$\times \int_{\mathbb{R}} \left\{ \overline{\hat{\psi_2}\left(\frac{v-s}{\sqrt{a}}\right)} \left[\int_{\hat{\mathbb{R}}} \hat{f}(\xi, v\xi) \overline{\hat{\psi_1}(a\xi)} e^{2\pi i \xi(t_1 + vt_2)} |\xi| d\xi \right] \right\} dv$$

where the inner integral

$$I(a,s,t_1,t_2,v) = \int_{\hat{\mathbb{R}}} \hat{f}(\xi, v\xi) \overline{\hat{\psi_1}(a\xi)} e^{2\pi i \xi(t_1 + vt_2)} |\xi| d\xi \quad (50)$$

is independent on s. Now, by recalling the definition of $B[f]$ given by (27) and by interchanging the integrals over v and s

we obtain,

$$B[f](a, z_1, z_2) =$$

$$a^{-1/2} \int_{\hat{\mathbb{R}}} \left\{ I(a, s, az_1, az_2, v) \int_{\mathbb{R}} \overline{\hat{\psi}_2 \left(\frac{v-s}{\sqrt{a}} \right)} ds \right\} dv. \quad (51)$$

The change of variable $w = \frac{v-s}{\sqrt{a}}$ gives

$$\int_{\mathbb{R}} \overline{\hat{\psi}_2 \left(\frac{v-s}{\sqrt{a}} \right)} ds = \sqrt{a} \int_{\mathbb{R}} \overline{\hat{\psi}_2(w)} dw = \sqrt{a} \overline{\hat{\psi}_2(0)} \neq 0, \tag{52}$$

since $\hat{\psi}$ is a bump function. Next, by plugging Eq. (52) into Eq. (51) we obtain

$$B[f](a, z_1, z_2) =$$

$$\overline{\hat{\psi}_2(0)} \int_{\mathbb{R}} \left[\int_{\hat{\mathbb{R}}} \hat{f}(\xi, v\xi) \overline{\hat{\psi}_1(a\xi)} e^{2\pi i a \xi(z_1 + vz_2)} |\xi| d\xi \right] dv. \quad (53)$$

Given $\alpha > 0$, let now f_α be the isotropic dilation of f, then Since $\hat{f}_\alpha(\omega_1, \omega_2) = \alpha^2 \hat{f}(\alpha\omega_1, \alpha\omega_2)$, we obtain

$$B[f_\alpha](a, z_1, z_2) =$$

$$\overline{\hat{\psi}_2(0)} \int_{\mathbb{R}} \left[\int_{\hat{\mathbb{R}}} \alpha^2 \hat{f}(\alpha\xi, v\alpha\xi) \hat{\psi}_1(a\xi) e^{2\pi i a \xi(z_1 + vz_2)} |\xi| d\xi \right] dv. \tag{54}$$

Next, the change of variable $\xi' = \alpha\xi$ in the inner integral gives

$$B[f_\alpha](a, z_1, z_2) = \overline{\hat{\psi}_2(0)} \tag{55}$$

$$\int_{\mathbb{R}} \left[\int_{\hat{\mathbb{R}}} \alpha \hat{f}(\xi', v\xi') \hat{\psi}_1(a\alpha^{-1}\xi') e^{2\pi i \xi' \alpha^{-1} a(z_1 + vz_2)} |\alpha^{-1}\xi'| d\xi' \right] dv$$

$$= B[f](\alpha^{-1}a, z_1, z_2).$$

If f is given by (21), its Fourier transform is the sum of four Dirac delta at $(\pm\alpha, \pm\beta)/2\pi$. Taking into account that ψ_1 is even, from (53) we get

$$B[f](a, z) = \overline{\hat{\psi}_2(0)} \overline{\hat{\psi}_1(\frac{a\alpha}{2\pi})} f(az) \tag{56}$$

$\square$

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