Prescriptive Analytics

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PORTFOLIO OPTIMIZATION

Welte Mutual Funds, Plc.

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Question 1 | Set-Up of the optimization model

a) Identification of the constraints

The constraints of the optimization model are listed below, with indication which constraints are binding in the base model.

Constraint	Formula	Binding?
Total Oil Invest	A+P ≤ 50	Yes
Total Steel Invest	M+H ≤ 50	No
Pacific Oil max. 60%	P ≤ 0.6 * (A+P)	Yes
T-Bills at least 25% of Steel	G ≥ 0.25 * (M+H)	Yes
Total Invest of 100 million	Total Invest ≤ 100	Yes
Non-Negativity	A, P, M, H, G ≥ 0	Yes

b) Solver window base model

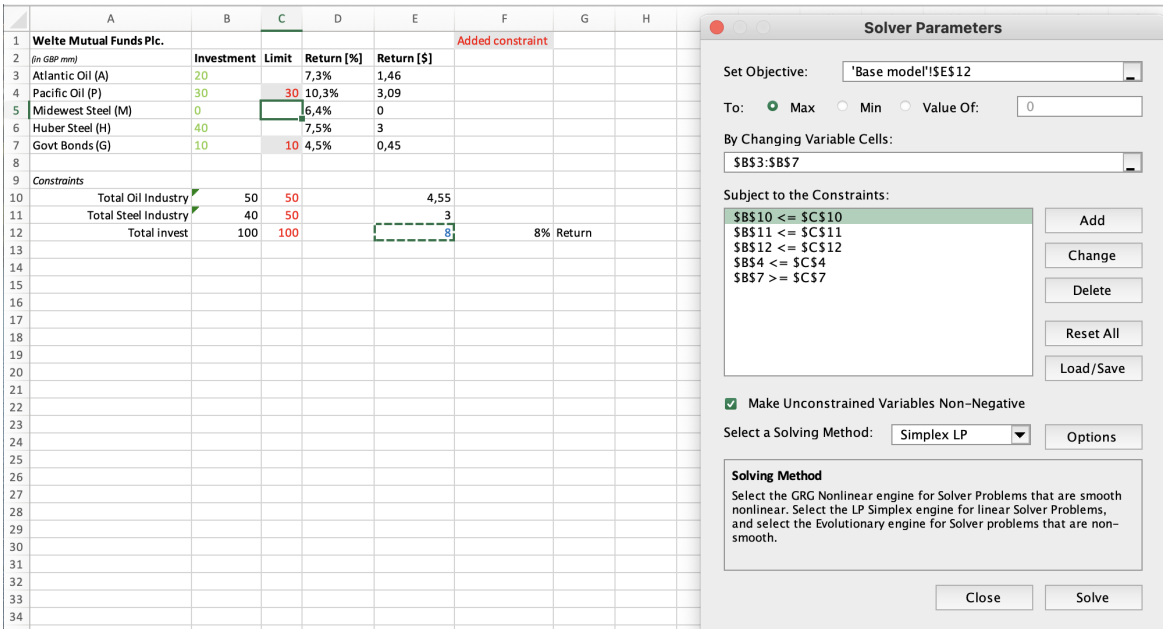


Figure 1: Optimization of base model

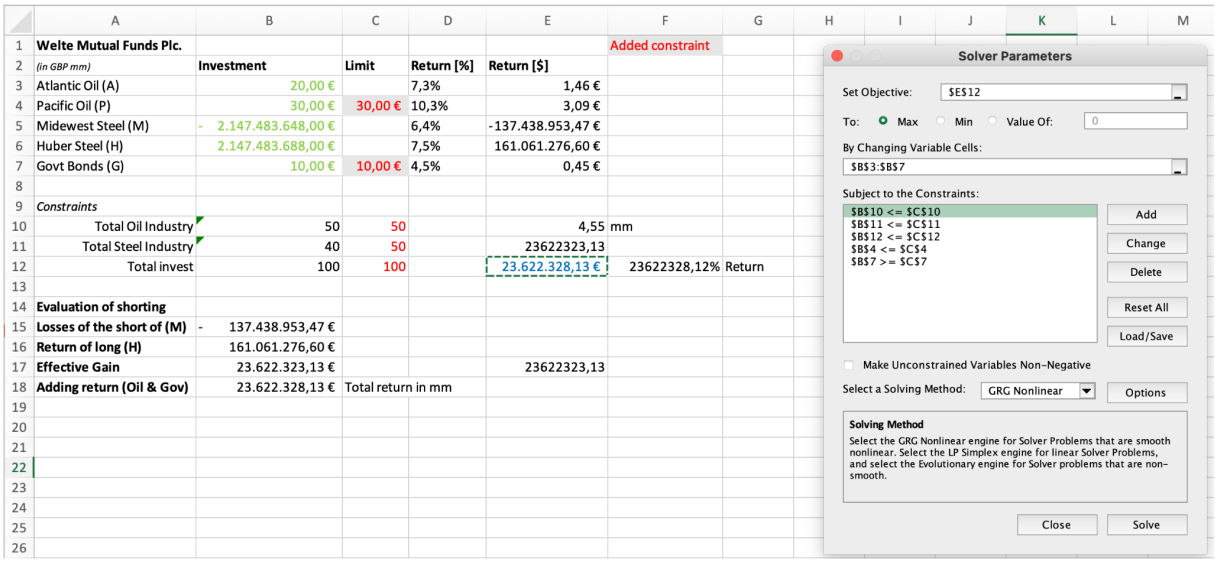
	A	B	C	D	E	F	G
1	**Welte Mutual Funds Plc.**					Added constraint	
2	*(in GBP mm)*	**Investment**	**Limit**	**Return [%]**	**Return [$]**		
3	Atlantic Oil (A)	20		0,073	=+D3*B3		
4	Pacific Oil (P)	30	=0,6*B10	0,103	=+D4*B4		
5	Midewest Steel (M)	0		0,064	=+D5*B5		
6	Huber Steel (H)	40		0,075	=+D6*B6		
7	Govt Bonds (G)	10	=0,25*B11	0,045	=+D7*B7		
8							
9	*Constraints*						
10	Total Oil Industry	=+SUM(B3:B4)	50		=+SUM(E3:E4)		
11	Total Steel Industry	=+SUM(B5:B6)	50		=+SUM(E5:E6)		
12	Total invest	=+B7+B11+B10	100		=+E10+E11+E7	=8/100	Return

Figure 2: Optimization of base model - Formulas

c) Explanation of the non-negativity constraint

The non-negativity constraint prevents the decision variables from taking negative values. For Welte Funds investment decisions, negative values would imply shorting of a security. Short-selling means borrowing a stock Welte doesn't own, selling it and hoping to buy it back (hopefully) at a lower price. As Midwest Steel has a positive return, Welte Funds actually incurs a loss of £137,438,953.47mm with the short. However, as the proceeds of the sale of Midwest Steel (£2,147,483,648mm) would be used to buy Huber Steel, which has a 1,1% higher return then Midwest, the effective gain of the trade would be positive and sum up to £23,622,323.13mm *(Figure 3)*. If we add the return of the investments in the oil industry (£4.55mm) and government bonds (£0,45mm), the total return would come up to £23,622,328.13mm.

However, this solution should not be considered by Welte Funds, and non-negativity constraints should hold for their portfolio, as borrowing of £2.14e+15 to short Midwest Steel is simply not possible and would probably exceed the market capitalization of that company by far. While the GRG nonlinear algorithm calculates this solution as a local optimum to the maximization problem, it is clearly not applicable for the investment fund.



	A	B	C	D	E	F	G
1	**Welte Mutual Funds Plc.**					Added constraint	
2	*(in GBP mm)*	**Investment**	**Limit**	**Return [%]**	**Return [$]**		
3	Atlantic Oil (A)	20,00 €		7,3%	1,46 €		
4	Pacific Oil (P)	30,00 €	30,00 €	10,3%	3,09 €		
5	Midewest Steel (M)	- 2.147.483.648,00 €		6,4%	-137.438.953,47 €		
6	Huber Steel (H)	2.147.483.688,00 €		7,5%	161.061.276,60 €		
7	Govt Bonds (G)	10,00 €	10,00 €	4,5%	0,45 €		
8							
9	*Constraints*						
10	Total Oil Industry	50	50		4,55	mm	
11	Total Steel Industry	40	50		23622323,13		
12	Total invest	100	100		23.622.328,13 €	23622328,12%	Return
13							
14	**Evaluation of shorting**						
15	**Losses of the short of (M)**	- 137.438.953,47 €					
16	**Return of long (H)**	161.061.276,60 €					
17	**Effective Gain**	23.622.323,13 €			23622323,13		
18	**Adding return (Oil & Gov)**	23.622.328,13 €	Total return in mm				

Figure 3: Alternative investment proposal without non-negative constraints

	A	B	C	D	E	F
1	**Welte Mutual Funds Plc.**					Added constraint
2	*(in GBP mm)*	**Investment**	**Limit**	**Return [%]**	**Return [$]**	
3	Atlantic Oil (A)	20		0,073	=+D3*B3	
4	Pacific Oil (P)	30	=0,6*B10	0,103	=+D4*B4	
5	Midewest Steel (M)	-2147483648		0,064	=+D5*B5	
6	Huber Steel (H)	2147483688		0,075	=+D6*B6	
7	Govt Bonds (G)	10,0000000238419	=0,25*B11	0,045	=+D7*B7	
8						
9	*Constraints*					
10	Total Oil Industry	=+SUM(B3:B4)	50		=+SUM(E3:E4)	mm
11	Total Steel Industry	=+SUM(B5:B6)	50		=+SUM(E5:E6)	
12	Total invest	=+B7+B11+B10	100		=+E10+E11+E7	=E12/B12
13						
14	**Evaluation of shorting**					
15	**Losses of the short of (M)**	=B5*D5				
16	**Return of long (H)**	=B6*D6				
17	**Effective Gain**	=SUM(B15:B16)			23622323,128	
18	**Adding return (Oil & Gov)**	=E7+E10+B17	Total return in mm			

Figure 4: Alternative investment proposal without non-negative constraints - formulas

d) Optimal portfolio allocation within the base model

With respect to our optimal base model (*Figure 1*), we infer the optimal capital allocation, as illustrated in the table below:

Investment in	% of £100 million investment
Atlantic Oil	20 %
Pacific Oil	30 %
Midwest Steel	0 %
Huber Steel	40 %
Gov. Bonds	10 %

With this portfolio, Welte Funds achieves a return of £8 million, representing an 8% return for the £100 million investment.

Question 2 | Analysis of the optimization model

a) Re-allocation of £10 million

Analyzing the sensitivity report of the base model (Figure 5), we infer that the allowable increase of the total investment is £12,5 million with a shadow price of 0,069. This means that for an increase of the Investment by £1 million there will be an extra return of 6,9%. This 6,9% increase in return is greater than the demanded return of 6,5% of Welte Funds. Therefore they should invest the £10 million. Out of the additional £10 million investment, £8 million would be allocated to Huber Steel and £2 million would be allocated to Government Bonds, to satisfy the existing constraints.

As an additional investment of £15 million would exceed the allowable increase of £12,5 million, we cannot answer the question whether to invest with our sensitivity report as our model would change. To provide an answer for the question, we had to build a new model incorporating the additional investable £15 million in the beginning.

Microsoft Excel 16.74 Sensitivity Report
Worksheet: [WF.xlsx]Welte Fund
Report Created: 03.07.23 16:35:16

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B3	Atlantic Oil (A) Investment	20	0	0,073	0,03	0,055
B4	Pacific Oil (P) Investment	30	0	0,103	1E+30	0,03
B7	Midewest Steel (M) Investment	0	-0,011	0,064	0,011	1E+30
B8	Huber Steel (H) Investment	40	0	0,075	0,0275	0,011
B11	Govt Bonds (G) Investment	10	0	0,045	0,03	0,345

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
B11	Govt Bonds (G) Investment	10	-0,024	0	50	12,5
B13	Total Investment	100	0,069	100	12,5	50
B4	Pacific Oil (P) Investment	30	0,03	0	20	30
B5	Total Oil Industry Investment	50	0,022	50	50	12,5
B9	Total Steel Industry Investment	40	0	50	1E+30	10

Figure 5: Sensitivity Report Base Model - Increase of Total Investment

b) Allocation of additional £1 or £15 million to oil investments

The allowable increase for the total oil investment is £50 million with a shadow price of 0,022 *(Figure 6)*. Following, if Welte Mutual Funds would increase their investment in the oil industry by £1 million, they would earn an additional return on this investment of 2,2%, summing up to £22,000 and a total return of £8,022,000.

The shadow price is made up of changes in the allocation of the 1 million investment. If the constraint of Total Oil Investment increases by 1 million, Welte would only invest an additional 600k into Pacific Oil to satisfy the limit constraint and 400k into Atlantic Oil. To satisfy the Total Investment constraint of 100 million, the investment in Huber Steel decreases from 40 million to 39,2 million and the investment in government bonds decreases from 10 million to 9,8 million. Thereby all the other remaining constraints are met. The allocation is illustrated in *Figure 7.*

As an increase of the Total Oil Investment constraint by £15mm also lies in the range of the allowable increase, the same shadow price as in the consideration above holds. By investing an extra £15mm, the company would generate an additional £330,000 and a total return of £8,33 mm. The extra return would be made up of an increase of Oil Investment in Atlantic and Pacific Oil with a ratio of 2:3 and a decrease from the base-case Investments in Huber Steel and Government Bonds of £12mm and £3mm respectively *(Figure 8)*.

Microsoft Excel 16.74 Sensitivity Report
Worksheet: [WF.xlsx]Welte Fund
Report Created: 03.07.23 16:35:16

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B3	Atlantic Oil (A) Investment	20	0	0,073	0,03	0,055
B4	Pacific Oil (P) Investment	30	0	0,103	1E+30	0,03
B7	Midewest Steel (M) Investment	0	-0,011	0,064	0,011	1E+30
B8	Huber Steel (H) Investment	40	0	0,075	0,0275	0,011
B11	Govt Bonds (G) Investment	10	0	0,045	0,03	0,345

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
B11	Govt Bonds (G) Investment	10	-0,024	0	50	12,5
B13	Total Investment	100	0,069	100	12,5	50
B4	Pacific Oil (P) Investment	30	0,03	0	20	30
B5	Total Oil Industry Investment	50	0,022	50	50	12,5
B9	Total Steel Industry Investment	40	0	50	1E+30	10

Figure 6: Sensitivity Report Base Model - Investment in Oil Industry

	A	B	C	D	E	F	G
1	**Welte Mutual Funds Plc.**					Added constraint	
2	*(in GBP mm)*	**Investment**	**Limit**	**Return [%]**	**Return [$]**		
3	Atlantic Oil (A)	20,40 €		7,3%	1,49 €		
4	Pacific Oil (P)	30,60 €	30,60 €	10,3%	3,15 €		
5	Midewest Steel (M)	- €		6,4%	- €		
6	Huber Steel (H)	39,20 €		7,5%	2,94 €		
7	Govt Bonds (G)	9,80 €	9,80 €	4,5%	0,44 €		
8							
9	*Constraints*						
10	Total Oil Industry	51	51		4,641		
11	Total Steel Industry	39,2	50		2,94		
12	Total invest	100	100		8,022 €	8,022%	Return

Figure 7: Model with additional £1mm Investment in Oil Industry

	A	B	C	D	E	F	G
1	**Welte Mutual Funds Plc.**					Added constraint	
2	*(in GBP mm)*	**Investment**	**Limit**	**Return [%]**	**Return [$]**		
3	Atlantic Oil (A)	26,00 €		7,3%	1,90 €		
4	Pacific Oil (P)	39,00 €	39,00 €	10,3%	4,02 €		
5	Midewest Steel (M)	- €		6,4%	- €		
6	Huber Steel (H)	28,00 €		7,5%	2,10 €		
7	Govt Bonds (G)	7,00 €	7,00 €	4,5%	0,32 €		
8							
9	*Constraints*						
10	Total Oil Industry	65	65		5,915		
11	Total Steel Industry	28	50		2,1		
12	Total invest	100	100		8,330 €	8,330%	Return

Figure 8: Model with additional £15mm Investment in Oil Industry

c) Shadow price Total Steel Industry

A shadow price of zero implies that changing the right hand side constraint does not affect the maximization of the total return of the portfolio. This is because the constraint is not exhausted and thereby is not binding for our optimization problem. Referring to our Sensitivity Report *(Figure 9)*, we observe that the final value of the Total Steel Industry constraint is 40, whereas the constraint is 50. Therefore, by increasing the investment in the Steel Industry, Welte Funds would not increase their return.

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B3	Atlantic Oil (A) Investment	20	0	0,073	0,03	0,055
B4	Pacific Oil (P) Investment	30	0	0,103	1E+30	0,03
B7	Midewest Steel (M) Investment	0	-0,011	0,064	0,011	1E+30
B8	Huber Steel (H) Investment	40	0	0,075	0,0275	0,011
B11	Govt Bonds (G) Investment	10	0	0,045	0,03	0,345

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
B11	Govt Bonds (G) Investment	10	-0,024	0	50	12,5
B13	Total Investment	100	0,069	100	12,5	50
B4	Pacific Oil (P) Investment	30	0,03	0	20	30
B5	Total Oil Industry Investment	50	0,022	50	50	12,5
B9	Total Steel Industry Investment	40	0	50	1E+30	10

Figure 9: Sensitivity Report Base Model - Shadow Price Steel Industry

d) Allowable Increase/Decrease Atlantic Oil

To evaluate the allowable in-/decrease of our decision variable Atlantic Oil, we have to acknowledge that the objective coefficient (7,3%) represents the return for that investment *(Figure 10)*. If the return for Atlantic Oil would increase by 3% to 10,3% or decrease by 5,5% to 1,8%, the capital allocation in our optimization problem would not change. For a decrease in return of 5,5% to 1,8%, the allocation would remain equivalent to our base model. However, even though the capital allocation does not change, when decreasing the return of Atlantic Oil to 1,8%, we observe that the total return of the portfolio would decrease from 8% to 6,9%.

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B3	Atlantic Oil (A) Investment	20	0	0,073	0,03	0,055
B4	Pacific Oil (P) Investment	30	0	0,103	1E+30	0,03
B7	Midewest Steel (M) Investment	0	-0,011	0,064	0,011	1E+30
B8	Huber Steel (H) Investment	40	0	0,075	0,0275	0,011
B11	Govt Bonds (G) Investment	10	0	0,045	0,03	0,345

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
B11	Govt Bonds (G) Investment	10	-0,024	0	50	12,5
B13	Total Investment	100	0,069	100	12,5	50
B4	Pacific Oil (P) Investment	30	0,03	0	20	30
B5	Total Oil Industry Investment	50	0,022	50	50	12,5
B9	Total Steel Industry Investment	40	0	50	1E+30	10

Figure 10: Sensitivity Report Base Model - Allowable In-/Decrease Atlantic Oil

e) Reduced cost for Midwest Steel

In our base model, we identify the final value for Midwest Steel to be zero, meaning that Welte Funds would not invest in this company under the given constraints, as it has a lower return than Huber Steel. The reduced cost represents the amount by which the coefficient (return) of Huber Steel had to increase, for it to be considered in the optimization. By increasing the return of Midwest Steel by 1,1% it would have a return of 7,5% and thus equal the return of Huber Steel *(Figure 11)*. With the same return for Huber Steel, Welte should be indifferent between an investment in Huber or Midwest Steel, and therefore the Midwest Steel *could* get a positive value in our optimization model. The allowable increase is equal to the reduced cost, as an increase above 0,011 would change the portfolio allocation as Midwest would have a higher return then Huber Steel, and therefore would be preferred by Welte Funds.

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B3	Atlantic Oil (A) Investment	20	0	0,073	0,03	0,055
B4	Pacific Oil (P) Investment	30	0	0,103	1E+30	0,03
B7	Midewest Steel (M) Investment	0	-0,011	0,064	0,011	1E+30
B8	Huber Steel (H) Investment	40	0	0,075	0,0275	0,011
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Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
B11	Govt Bonds (G) Investment	10	-0,024	0	50	12,5
B13	Total Investment	100	0,069	100	12,5	50
B4	Pacific Oil (P) Investment	30	0,03	0	20	30
B5	Total Oil Industry Investment	50	0,022	50	50	12,5
B9	Total Steel Industry Investment	40	0	50	1E+30	10

Figure 11: Sensitivity Report Base Model - Reduced cost of Midwest Steel

Question 3 | Less constrained model

a) New model in solver window & model results

After removing the constraints associated with Government Bonds and Pacific Oil, the model and the Solver box would change to what is shown on *Figure 12* below. The remaining constraints target the 50:50 distribution of the investment between the oil and steel industry. Furthermore the constraint of a maximum total investment of £100 million is binding.

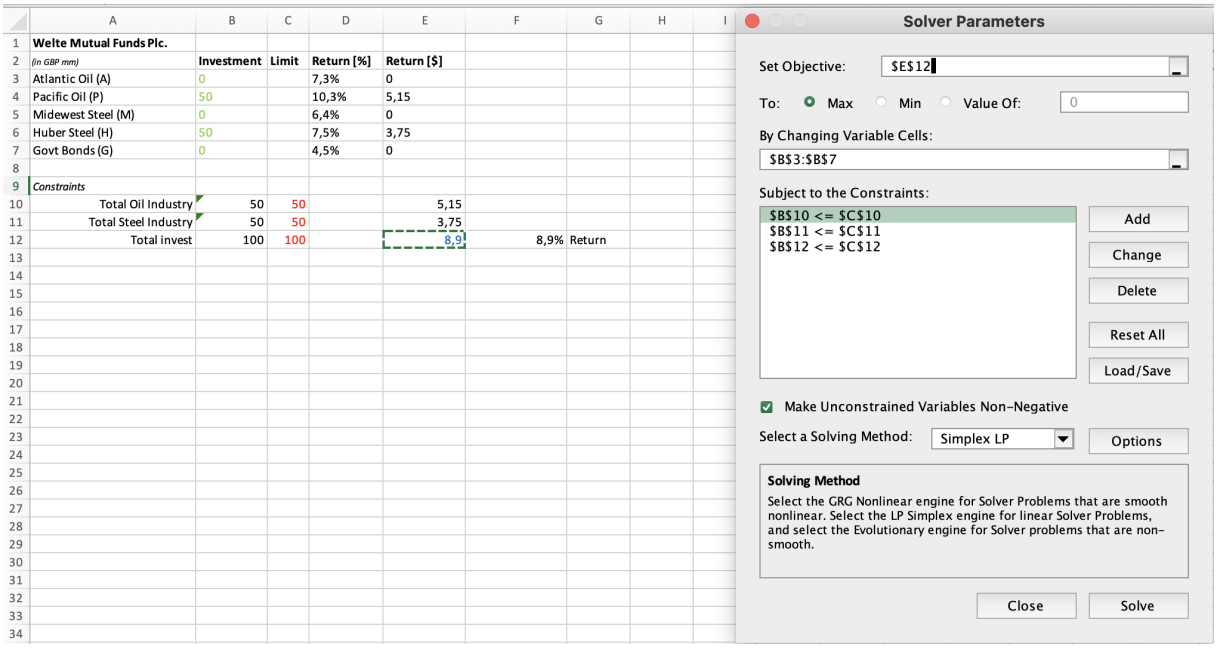

Figure 12: Optimization without G minimum and P maximum percentages.

	A	B	C	D	E	F	G
1	**Welte Mutual Funds Plc.**						
2	*(in GBP mm)*	**Investment**	**Limit**	**Return [%]**	**Return [$]**		
3	Atlantic Oil (A)	0		0,073	=+D3*B3		
4	Pacific Oil (P)	50		0,103	=+D4*B4		
5	Midewest Steel (M)	0		0,064	=+D5*B5		
6	Huber Steel (H)	50		0,075	=+D6*B6		
7	Govt Bonds (G)	0		0,045	=+D7*B7		
8							
9	*Constraints*						
10	Total Oil Industry	=+SUM(B3:B4)	50		=+SUM(E3:E4)		
11	Total Steel Industry	=+SUM(B5:B6)	50		=+SUM(E5:E6)		
12	Total invest	=+B7+B11+B10	100		=+E10+E11+E7	=E12/100	Return

Figure 13: Optimization showing formulas.

b) Which investments are being made?

Obeying to the new set of constraints, the portfolio would be maximized by investing £50 million of the capital into Pacific Oil (P), and £50 million into Huber Steel (H). This makes sense, as those two investments yield the greatest return, and still satisfy the given constraints. Thereby the return is just the average of the two investment opportunities: P and H.

c) Return of the new portfolio

In this new scenario, the absolute return would be £8,9 million or 8,9%. As the return is higher than £8.8 million, Welte Funds should proceed with the presented portfolio in *Figure 12* and disregard the alternative investment opportunity in the capital market.