Fourier Integrals & Fourier Transforms & Their Applications

Along EH, $s = ue^{\pi i}$, $\sqrt{s} = i\sqrt{u}$ and we find

$$\int_{EH} = \int_R^{\varepsilon} \frac{e^{-ut - i\sqrt{\frac{s}{k}}x}}{u(i\sqrt{u} - a)} du$$

Along KL, $s = ue^{-\pi i}$, $\sqrt{s} = -i\sqrt{u}$ and we find

$$\int_{KL} = \int_{\varepsilon}^{R} \frac{e^{-ut + i\sqrt{\frac{s}{k}}x}}{u(-i\sqrt{u} - a)} du$$

Along HJK, $s = \varepsilon e^{i\theta}$ and we find

$$\int_{HJK} = \int_{\pi}^{-\pi} \frac{e^{\varepsilon e^{i\theta} t} - \sqrt{\frac{\varepsilon e^{i\theta}}{k}}x}{\sqrt{\varepsilon e^{i\theta/2}} - a}$$

Using these results in (8), we get

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st - \sqrt{\frac{s}{k}}x}}{s(\sqrt{s} - a)} ds$$

$$= -\frac{1}{a} + \frac{1}{\pi} \int_0^{\infty} \frac{e^{-ut}}{u} \left[\frac{\sqrt{u}\cos x\sqrt{u} - a\sin x\sqrt{u}}{u + a^2}\right] du$$

Hence $u(x, t) = \frac{aU_0}{\pi} \int_0^{\infty} \frac{e^{-ut}}{u} \left[\frac{\sqrt{u}\cos x\sqrt{u} - a\sin x\sqrt{u}}{u + a^2}\right] du$

If $u = v^2$, then

$$u(x, t) = \frac{2aU_0}{\pi} \int_0^{\infty} \frac{e^{-v^2 t}}{v} \left[\frac{v\cos xv - a\sin xv}{v^2 + a^2}\right] dv$$

which is the required solution of the given boundary value Problem.

CHAPTER FOUR

FOURIER INTEGRALS AND FOURIER TRANSFORMS AND THEIR APPLICATIONS

4.1 Introduction

The purpose of this chapter is to present the fundamental concepts of Fourier series, Fourier integrals and Fourier transforms and also their important applications in the solution of the partial differential equations and boundary value problems. Also some illustrative examples and some important engineering and physical applications will be included. Fourier series is named after the French physicist JOSEPH FOURIER (1768—1830).

Definition Periodic functions.

The function $f(x)$ of real variable x is said to be **periodic** if there exists a non-zero number P, independent of x, such that $f(p + x) = f(x)$ holds for all values of x. The least value of P>0 is called **least period** or simply the **period** of $f(x)$. As for examples, $f(x) = \sin x$ and $f(x) = \cos x$ are periodic functions having period 2π. Also $f(x) = \tan x$ is a periodic function having period π.

Defintion : Trigonometric series

Any series of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

where the coefficients a_n and b_n are constants,

is called the **trigonometric series.**

4.2 Fourier series

Definition The trigonometric series

$$f(x) = a_0 + a_1\cos x + a_2\cos 2x + \ldots\ldots + a_n\cos nx + \ldots + b_1\sin x + b_2\sin 2x + \ldots\ldots + b_n\sin nx + \ldots\ldots$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n\cos nx + b_n\sin nx) \quad (1)$$

is a **Fourier series** if its coefficients a_0, a_n and b_n are given by the following formulas.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(v)\,dv$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(v)\cos nv\,dv$$

$$(n = 1, 2, 3, \ldots\ldots)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(v)\sin nv\,dv$$

$$(n = 1, 2, 3, \ldots\ldots)$$

where $f(x)$ is any single-valued function defined on the interval $(-\pi, \pi)$.

The Fourier series can also be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\text{where } a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\ (n = 0, 1, 2, 3, \ldots\ldots)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,\ (n = 1, 2, 3, \ldots\ldots)$$

4.3 Change of intervals

(A) If $f(x)$ be difined in the interval $(-c,c)$ having period $2c$, the **Fourier series** of $f(x)$ in the interval $(-c, c)$ is given by.

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c}\right) \quad (1)$$

$$\text{where } a_0 = \frac{1}{2c}\int_{-c}^{c} f(x)\,dx\ (n = o)$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx\ (n = 1, 2, 3, \ldots\ldots)$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\,dx\ (n = 1, 2, 3, \ldots\ldots)$$

(B) If $f(x)$ is defined in the interval $(\alpha, \alpha + 2c)$ having period $2c$, the **Fourier series** of $f(x)$ in the interval $(\alpha, \alpha + 2c)$ is given loy.

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c}\right)$$

$$\text{where } a_0 = \frac{1}{2c}\int_{\alpha}^{\alpha+2c} f(x)\,dx\ (n = 0)$$

$$a_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\cos\frac{n\pi x}{c}\,dx\ (n = 1, 2, 3, \ldots\ldots)$$

$$b_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\sin\frac{n\pi x}{c}\,dx\ (n = 1, 2, 3, \ldots\ldots)$$

4.4 Complex form of Fourier series.

From trigonometry, we have

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \text{ and } \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right).$$

Now the complex form of the Fourier series is obtained by expressing $\cos\frac{n\pi x}{c}$ and $\sin\frac{n\pi x}{c}$ in exponential form.

That is, the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{c}\right) + b_n \sin\left(\frac{n\pi x}{c}\right)\right] \qquad -c < x < c \quad (1)$$

can be written in the form

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[\frac{a_n}{2}\left(e^{\frac{in\pi x}{c}} + e^{-\frac{in\pi x}{c}}\right) + \frac{b_n}{2i}\left(e^{\frac{in\pi x}{c}} - e^{-\frac{in\pi x}{c}}\right)\right]$$

$$= a_0 + \sum_{n=1}^{\infty}\left[\frac{a_n}{2}\left(e^{\frac{in\pi x}{c}} + e^{-\frac{in\pi x}{c}}\right) - \frac{ib_n}{2}\left(e^{\frac{in\pi x}{c}} - e^{-\frac{in\pi x}{c}}\right)\right]$$

$$= a_0 + \sum_{n=1}^{\infty}\left[\frac{1}{2}(a_n - ib_n)\,e^{\frac{in\pi x}{c}} + \frac{1}{2}(a_n + ib_n)\,e^{-\frac{in\pi x}{c}}\right]$$

$$= C_0 + \sum_{n=1}^{\infty}\left[C_n e^{\frac{in\pi x}{c}} + C_{-n} e^{-\frac{in\pi x}{c}}\right] \quad (2)$$

where $C_0 = a_0$, $C_n = \frac{1}{2}(a_n - ib_n)$ and $C_{-n} = \frac{1}{2}(a_n + ib_n)$

Equation no (2) is known as the **Complex form of the Fourior series** which can also be written as

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{c}}, \quad -c< x<c$$

where $C_n = \frac{1}{2c}\int_{-c}^{c} f(x)\, e^{-\frac{in\pi x}{c}}\, dx$ and

$$n = 0, \pm 1, \pm 2, \ldots\ldots$$

4.5 Double Fourier series

The idea of a Fourier series expansion for a function of a single variable x can be extended to the case of functions of two variables x and y i.e $f(x,y)$. As for example, we can expand $f(x,y)$ into a **double Fourier**

sine series $f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \sin\frac{m\pi x}{c_1}\sin\frac{n\pi y}{c_2}$,

where $B_{mn} = \frac{4}{c_1 c_2}\int_0^{c_1}\int_0^{c_2} f(x, y)\, \sin\frac{m\pi x}{c_1}\sin\frac{n\pi y}{c_2}\, dxdy$

Similarly, we can expand $f(x, y)$ into a **double Fourier cosine series**

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \cos\frac{m\pi x}{c_1}\cos\frac{n\pi y}{c_2}$$

where $B_{mn} = \frac{4}{c_1 c_2}\int_0^{c_1}\int_0^{c_2} f(x, y)\cos\frac{m\pi x}{c_1}\cos\frac{n\pi y}{c_2}\, dxdy.$

Similar results can be obtained for series having both sines and cosines and also these ideas can be generalised to **triple** Fourier series, etc.

4.6 The Fourier cosine and sine series.

Difiniton Even function

A function $f(x)$ is called even if $f(-x) = f(x)$.

Graphically, an even function is symmetrical about the y-axis.

If $f(x)$ is an even function, then

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx$$

$$= \int_{\pi}^{0} f(-x)d(-x) + \int_{0}^{\pi} f(x)dx$$

$$= -\int_{\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx$$

$$= \int_{0}^{\pi} f(x)dx + \int_{0}^{\pi} f(x)dx = 2\int_{0}^{\pi} f(x)dx.$$

Thus if $f(x)$ is even, we have

$$a_0 = \frac{1}{2\pi}.2\int_0^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} f(v)dv.$$

Also if (x) is even i; e $f(-x) = f(x)$

then $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx\ (n = 1, 2, 3, \ldots)$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} f(-x)\cos n(-x)d(-x)$$

$$= -\frac{1}{\pi}\int_{\pi}^{-\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(v)\cos nv\,dv.$$

but $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{\pi}^{-\pi} f(-x)\sin n(-x)\,d(-x)$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} f(x).-\sin nx.-dx$$

$$= -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = -b_n$$

$\therefore\ 2b_n = 0$ or, $b_n = 0$.

Therefore, if $f(x)$ is even, then we have

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{1}{\pi}\int_0^{\pi} f(v)\,dv + \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\int_0^{\pi} f(v)\cos nv\,dv\right\}\cos nx$$

which represents the function $f(x)$ in a series of cosines and therefore it is known as **Fourier cosine series** in the interval $(0, \pi)$.

Definition A function $f(x)$ is called **odd** if $f(-x) = -f(x)$.

Graphically, an odd function is symmetrical about the origin. When $f(x)$ is odd, we have

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{\pi}^{-\pi} f(-x)\,d(-x)$$

$$= \frac{1}{2\pi}\int_{\pi}^{-\pi} f(x)dx = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = -a_0$$

$\therefore\ 2a_0 = 0$ or, $a_0 = 0$.

Also if $f(x)$ is odd i. e $f(-x) = -f(x)$, then

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} f(-x)\cos n(-x)\,d(-x)$$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} -f(x)\cos nx.-dx$$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} f(x)\cos nx\,dx = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = -a_n$$

$\therefore\ 2a_n = 0$, or, $a_n = 0$.

Finally, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} f(-x)\sin n(-x)d(-x)$$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} -f(x)-.\sin x.-dx$$

$$= \frac{1}{-\pi}\int_{\pi}^{-\pi} f(x).-\sin x.-dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$

$$= \frac{2}{\pi}\int_0^{\pi} f(v)\sin nv\,dv.$$

Therefore, if $f(x)$ is odd, then we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\int_0^{\pi} f(v)\sin nv\,dv\right\}\sin nx$$

which represents the function $f(x)$ in a series of sines in the interval $(0, \pi)$ and therefore, it is known as **Fourier sine series** in the interval $(0, \pi)$.

These Fourier cosine and sine series are also called **half range Fourier cosine series** and **half range Fourier sine**

series respectively since in these cases the function $f(x)$ is defined in the interval $(0, \pi)$ which is the half of the interval $(-\pi, \pi)$;

4.7 Dirichlet's conditions.

Suppose that

(i) $f(x)$ is defined and single - valued except possibly at a finite number of points in $(-c, c)$

(ii) $f(x)$ is periodic with period $2c$

(iii) $f(x)$ and $f'(x)$ are Piecewise continuous in $(-c, c)$.

Then the Fourier series

$$a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{c} + b_n \sin \frac{n\pi x}{c} \right)$$

with coefficients

defined in 4.3 converges to

(a) $f(x)$ if x is a point of continuity

(b) $\dfrac{f(x+0)+f(x-0)}{2}$ if x is a point of discontinuity.

The conditions (i), (ii) and (iii) imposed on $f(x)$ are sufficient but not necessary, i, e, if the conditions are satisfied, ;the convergence is guaranted, However if they are not satisfied the series may or may not converge.

4.8 Parseval's Formula .

One of the most important properties of Fourier series is **Parseval's Formula** or **the completeness relation** which gives a relation between the average of the square (or absolute square) of the function $f(x)$ and the co-efficients in Fourier series of $f(x)$.

(A) Particular Case

Let $f(x)$ be a real valued function of period 2π whose Fourier series is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)dx,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

and $b_n = \displaystyle\int_{-\pi}^{\pi} f(x) \sin nx\, dx$

Now the **average of the square of** $f(x)$ over

$(-\pi, \pi)$ is $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} [f(x)]^2 dx$.

Thus we have average of $[f(x)]^2$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx\right]^2 dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} a_0^2\,dx + \frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} a_n^2\cos^2 nx\,dx$$

$$+ \frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} b_n^2 \sin^2 nx\,dx + \text{other terms (which}$$

vanish when average is taken).

$$= \frac{1}{2\pi}a_0^2\Big[x\Big]_{-\pi}^{\pi} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{2}a_n^2\int_{-\pi}^{\pi}(1+\cos 2nx)dx$$

$$+ \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{2}b_n^2\int_{-\pi}^{\pi}(1-\cos 2nx)\,dx + 0$$

$$= a_0^2 + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{2}a_n^2 . 2\pi + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{2}b_n^2 . 2\pi$$

$$= a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Hence $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}[f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$.

This is one form of **Parseval's Formula.** One can easil verify that the formula is unchanged if $f(x)$ has period 2 in place of 2π and its square is averaged over any period of length 2c. Then we have

$$\frac{1}{2c}\int_{-c}^{c} |f(x)|^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

Let $f(x)$ be a complex-valued function of period 2π, whose Fourier series is

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$\text{where } c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} \quad (n = 0, \pm 1, \pm 2, \ldots)$$

Then the average square of $f(x)$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \sum_{n=-\infty}^{\infty} |c_n|^2$$

(B) General case.

A general form of Parseval's Formula states that if $f(x)$ and $g(x)$ are two real valued functions of period 2π, whose Fourier series are

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \text{ and}$$

$$a_0' + \sum_{n=1}^{\infty} (a_n' \cos nx + b_n' \sin nx) \text{ respectively}$$

$$\text{where } a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx,\quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$\text{and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

Similarly, a_0', a_n' and b_n' are defined in terms of g. Then we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,g(x)\,dx = a_0 a_0' + \frac{1}{2}\sum_{n=1}^{\infty} (a_n a_n' + b_n b_n')$$

When $f(x)$ and $g(x)$ are two complex valued functions of period 2π, whose Fourier series are

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} \text{ and } \sum_{n=-\infty}^{\infty} c_n' e^{inx} \text{ respectively}$$

$$\text{where } c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx \quad (n = 0, \pm 1, \pm 2, \ldots)$$

$$\text{and } c_n' = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x) e^{-inx}\,dx \quad (n = 0, \pm 1, \pm 2, \ldots)$$

Then we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx = \sum_{n=\infty}^{\infty} c_n \overline{c_n'}$$

4.9 Fourier Integral

The **Fourier Intergal** is very useful in the field of electrical communication and forms the basis of Cauchy's method for the solution of the partial differential equation.

General Fourier series of a periodic function $f(x)$ in the interval $(-c, c)$ is given by.

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \sum_{n=1}^{\infty} \frac{1}{c}\int_{-c}^{c} f(t)\cos\frac{n\pi t}{c}\,dt \cos\frac{n\pi x}{c}$$

$$+\sum_{n=1}^{\infty} \frac{1}{c}\int_{-c}^{c} f(t)\sin\frac{n\pi t}{c}\,dt \sin\frac{n\pi x}{c} \qquad (1)$$

$$= \frac{1}{2c}\int_{-c}^{c} f(t)\,dt$$

$$+\frac{1}{c}\int_{-c}^{c} f(t)\left[\sum_{n=1}^{\infty}\left\{\cos\frac{n\pi x}{c}\cos\frac{n\pi t}{c} + \sin\frac{n\pi x}{c}\sin\frac{n\pi t}{c}\right\}\right] dt$$

$$= \frac{1}{2c}\int_{-c}^{c} f(t)\, dt$$

$$+ \frac{1}{2c}\int_{-c}^{c} f(t)\left[2\sum_{n=1}^{\infty} \cos\left\{\frac{n\pi(x-t)}{c}\right\}\right] dt$$

$$= \frac{1}{2c}\int_{-c}^{c} f(t)\left[1 + 2\sum_{n=1}^{\infty} \cos\left\{\frac{n\pi(x-t)}{c}\right\}\right] dt.$$

$$= \frac{1}{2\pi}\int_{-c}^{c} f(t)\left[\frac{\pi}{c} + \sum_{n=1}^{\infty} 2.\frac{\pi}{c}.\ \cos\left\{\frac{n\pi(x-t)}{c}\right\}\right] dt.$$

$$= \frac{1}{2\pi}\int_{-c}^{c} f(t)\left[\frac{\pi}{c}\cos\left\{0\,\frac{\pi}{c}(x-t)\right\}\right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{\pi}{c}\cos\left\{\frac{n\pi(x-t)}{c}\right\} + \sum_{n=1}^{\infty} \frac{\pi}{c}\cos\left\{\frac{-n\pi(x-t)}{c}\right\}\right] dt$$

$$= \frac{1}{2\pi}\int_{-c}^{c} f(t)\,\frac{\pi}{c}\sum_{n=0}^{\infty}\left[\cos\left\{\frac{n\pi(x-t)}{c}\right\}\right] + \cos\left[\left\{\frac{-n\pi(x-t)}{c}\right\}\right]$$

$$= \frac{1}{2\pi}\int_{-c}^{c} f(t)\left[\lim_{n\to\infty}\sum_{r=-n}^{n} \frac{\pi}{c}\cos\left\{\frac{r\pi}{c}(x-t)\right\}\right] dt$$

$$= \frac{1}{2\pi}\int_{-c}^{c} f(t)\left[\lim_{n\to\infty}\sum_{r=-n}^{n}.\ \frac{1}{\frac{c}{\pi}}\cos\left\{\frac{r}{\frac{c}{\pi}}(x-t)\right\}\right] dt \qquad (2)$$

Now when $c\to\infty, \frac{c}{\pi}\to\infty$ and we have

$$\lim_{c\to\infty}\sum_{r=-\infty}^{\infty} \frac{1}{\frac{c}{\pi}}\cos\left\{\frac{r}{\frac{c}{\pi}}(x-t)\right\}$$

$$= \lim_{\Delta u\to 0}\sum_{r=-\infty}^{\infty} \cos r\,\Delta u(x-t)\,\Delta u \text{ where } \Delta u = \frac{1}{\frac{c}{\pi}}$$

$$= \int_{-\infty}^{\infty} \cos\{u(x-t)\}du \quad \left\{\begin{array}{r}\text{Writing } r\,\Delta u = u \text{ and}\\ \Delta u = du\end{array}\right\}$$

by the definition of the integral as the limit of a sum. Substituting this value of the sum in the equation (2),

we get $f(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} f(t)\, dt. \int_{-\infty}^{\infty} \cos\{u(x-t)\}\, du \qquad (3)$

This double integral is known as **Fourier integral** and holds if x is a point of continuity of $f(x)$.

The second integral in the equation (3) can be written as

$$\int_{-\infty}^{\infty} \cos\{u(x-t)\}\, du = \int_{-\infty}^{0} \cos\{u(x-t)\}\, du +$$

$$\int_{0}^{\infty} \cos\{u(x-t)\}\, du = 2\int_{0}^{\infty} \cos\{u(x-t)\}\, du.$$

Thus equation (3) can also be written as

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)dt. \int_{0}^{\infty} \cos\{u(x-t)\}\, du$$

$$= \frac{1}{\pi}\int_{0}^{\infty} du \int_{-\infty}^{\infty} f(t)\cos\{u(x-t)\}dt \qquad (4)$$

which gives another form of the **Fourier integral.**

When $f(x)$ is an **even function** of x, that is, $f(-x) = f(x)$,

we have $\displaystyle\int_{-\infty}^{\infty} f(t)\cos\{u(x-t)\}\, dt$

$$= \int_{-\infty}^{0} f(t)\cos u(x-t)\, dt + \int_{0}^{\infty} f(t)\cos\{u(x-t)\}\, dt.$$

[replacing t by (− t) in the first integral]

$$= -\int_{\infty}^{0} f(-t)\cos\{u(x+t)\}\, dt + \int_{0}^{\infty} f(t)\cos\{u(x-t)\}\, dt$$

$$= \int_{0}^{\infty} f(t)\cos u(x+t)dt + \int_{0}^{\infty} f(t)\cos\{u(x-t)\}\, dt$$

$$= 2\int_0^\infty f(t)\cos ut \cos ux dt.$$

Substituting this result in equation (4)

we get $f(x) = \frac{2}{\pi}\int_0^\infty du \int_0^\infty f(t) \cos ut \cos ux dt$

$$= \frac{2}{\pi}\int_0^\infty f(t)\, dt \int_0^\infty \cos ut \cos ux du \quad (5)$$

which gives **Fourier integral of an even function.** Similarly, when $f(x)$ is an **odd function** of x, that is,

$f(-x) = -f(x)$ we have $\int_{-\infty}^{\infty} f(t)\cos\{u(x-t)\} dt$

$$= \int_{-\infty}^{0} f(t)\cos\{u(x-t)\} dt + \int_0^\infty f(t)\cos\{u(x-t)\} dt$$

On replacing t by (−t) in the first integral on the right hand side, we have

$$\int_{-\infty}^{0} f(t)\cos\{u(x-t)\}\, dt = \int_{\infty}^{0} f(-t)\cos\{u(x+t)\}.\,-dt$$

$$= -\int_0^\infty f(t)\{u(x+t)\} dt$$

Thus $\int_{-\infty}^{\infty} f(t)\cos\{u(x-t)\} dt$

$$= \int_0^\infty f(t)\left[\cos\{u(x-t)\} - \cos\{u(x+t)\}\right] dt$$

$$= 2\int_0^\infty f(t)\sin ux \sin ut\, dt \quad (6)$$

Substituting this relation in equation (4)

we get $f(x) = \frac{2}{\pi}\int_0^\infty du \int_0^\infty \sin ut \sin ux dt$

$$= \frac{2}{\pi}\int_0^\infty f(t)\, dt \int_0^\infty \sin ut \sin ux\, du$$

which is the **Fourier Integral of an odd function.**

4.10 Finite Fourier sine and cosine transforms.

The **finite Fourier sine trfansform** of $F(x)$, $0<x<l$, is defined as

$$f_s(n) = \int_0^l F(x)\sin\frac{n\pi x}{l}\, dx \quad (1)$$

where n is an integer.

The function $F(x)$ is then called the **inverse finite Fourier sine transform** of $f_s(n)$ and is given by.

$$F(x) = \frac{2}{l}\sum_{n=1}^{\infty} f_s(n)\sin\frac{n\pi x}{l} \quad (2)$$

The finite Fourier cosine transform of $F(x)$, $0<x<l$, is defined as

$$f_c(n) = \int_0^l F(x)\cos\frac{n\pi x}{l}\, dx \quad (3)$$

where n is an integer.

The function $F(x)$ is then called the **inverse finit Fourier cosine transform** of $f_c(n)$ and is given by

$$F(x) = \frac{1}{l} f_c(0) + \frac{2}{l}\sum_{n=1}^{\infty} f_c(n)\cos\frac{n\pi x}{l} \quad (4)$$

4.11 (Infinite) Fourier sine and cosine transforms

The **(infinite) Fourier sine transform** of a function F of x such that $0 < x < \infty$ is denoted by $f_s(n)$ and is defi as

$$f_s(n) = \int_0^\infty F(x)\sin nx\, dx \quad (1)$$

The function F (x) is then called the **inverse Fourier sine transform** of $f_s(n)$ and is given by

$$F(x) = \frac{2}{\pi}\int_0^{\infty} f_s(n)\sin nx\,dn \quad (2)$$

The (infinite) Fourier cosine transform of a function F(x) of x for $0 < x < \infty$ is denoted by $f_c(n)$ and is defined as

$$f_c(n) = \int_0^{\infty} F(x)\cos nx\,dx \quad (3)$$

The function F(x) is then called the **inverse Fourier cosine transform** of $f_c(n)$ and is given by

$$F(x) = \frac{2}{\pi}\int_0^{\infty} f_c(n)\cos nx\,dn \quad (4)$$

Note 1. The infinite Fourier sine transform and the infinite Fourier cosine transform are generally known as **Fourier sine transform** and **Fourier cosine transform** respectively.

Note 2. Some authors also define **Fourier sine transform** and **Fourier cosine transform** in the following ways respectively :

(i) $F_s\{f(x)\} = f_s(n) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin nx\,dx.$

(ii) $F_c\{f(x)\} = f_c(n) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos nx\,dx.$

Note 3. Some authors also define **inverse Fourier sine transform** and **inverse Fourier cosine transform** in the following ways respectively :

(i) $f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f_s(n)\sin nx\,dn.$

(ii) $f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f_c(n)\cos nx\,dn.$

4.12 Complex form of the Fourier integral and Fourier transforms

From the definition of **Fourier integral**, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt\int_{-\infty}^{\infty}\cos u(x-t)\,du \quad (1)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\cos\{u(x-t)\}\,dt\right]du \quad (2)$$

One can easily show that

$$\frac{i}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\sin\{u(x-t)\}\,dt\right]du = 0 \quad (3)$$

Adding (2) and (3), we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\{\cos u(x-t) + i\sin u(x-t)\}\,dt\right]du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\,e^{iu(x-t)}\,dt\right]du \quad (4)$$

which is called the **complex form of the Fourier integral.** Also (4) can be written as

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{-iut}\,dt\right]e^{iux}\,du$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} C(u)e^{iux}\,du \text{ where } C(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-iut}\,dt$$

f(x) is called the **Fourier transform** of C(u) and C(u) is called the **inverse Fourier transform** of f(x) .

Alternative forms of Fourier transform

(A) If $F(u) = \int_{-\infty}^{\infty} f(x)e^{-iux}\,dx \quad (1)$

then $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\,e^{iux}\,du \quad (2)$

The function F(u) is called the **Fourier transform** of $f(x)$ and is sometimes written as $F(u) = F\{f(x)\}$.

The function $f(x)$ is called the **inverse Fourier transform** of F(u) and is sometims written as

$f(x) = F^{-1}\{F(u)\}$.

(B) If $F(u) = \int_{-\infty}^{\infty} f(x)\, e^{iux} dx$ (1)

then $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{-iux} du$ (2)

The function F(u) is called the **Fourier transform** of $f(x)$ and is sometimes written as $F(u) = F\{f(x)\}$.

The function f (x) is called the **inverse Fourier transform** of F(u) and is sometimes written as

$f(x) = F^{-1}\{F(u)\}$.

(C) ; If $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} C(u)\, e^{-iux} dx$ (1)

then $C(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{iut} dt$ (2)

The function $f(x)$ is called the **Fourier transform** of C (u) and C (u) is called the **inverse Fourier transform** of $f(x)$.

4.13 Convolution and Convolution theorem

Definition Convolution

The function $H(x) = F * G = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(u)\, G(x-u)\, du$

is called the **Convolution** or **Faltung** of two integrable functions F and G over the interval $(-\infty, \infty)$.

The Convolution (or Faltung) theorem for Fourier transforms :

Statement : If $F\{f(x)\}$ and $F\{g(x)\}$ are the Fourier transforms of the functions $f(x)$ and $g(x)$ respectively then the Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms.

i.e. $F\{f(x) * g(x)\} = F\{f(x)\}.\, F\{g(x)\}$

Proof :

By definition of convolution of two functions $f(x)$ and $g(x)$, we have

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, g(x-u)\, du.$$

$$\therefore\ F\{f(x) * g(x)\} = F\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, g(x-u)\, du\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, g(x-u)\, du\right] e^{ipx} dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} g(x-u)\, e^{ipx} dx\right] du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} g(x-u)\, e^{ip(x-u)}\, .\, e^{ipu} d(x-u)\right] du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[e^{ipu}\int_{-\infty}^{\infty} g(v)\, e^{ipv} dv\right] du \text{ where } x-u$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\left[e^{ipu}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)\, e^{ipx} dx\right] du$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,\{e^{ipu}\, F\{g(x)\}\}\, du$$

$$= \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, e^{ipu} du\right] F\{g(x)\}$$

$$= \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{ipx} dx\right] F\{g(x)\}$$

$= F\{f(x)\}\,.\,F\{g(x)\}$

Hence $F\{f(x) * g(x)\} = F\{f(x)\}.\,F\{g(x)\}$.

Alternative form of definition of convolution

The **convolution** of the two functions $f(x)$ and $g(x)$ is defined by $f * g = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$.

Alternative proof of the convolution theorem

By definition of Fourier transform, we have

$$\left.\begin{aligned} F(u) &= \int_{-\infty}^{\infty} f(t)\, e^{-iut}\, dt \\ \text{and } G(u) &= \int_{-\infty}^{\infty} g(s)\, e^{-ius}\, ds \end{aligned}\right\} \quad (1)$$

$$\text{Then } F(u)\, G(u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, g(s)\, e^{-iu(t+s)}\, dt\, ds \quad (2)$$

Let $t + s = x$ in the double integral (2) which we wish to transform from the variables (t, s) to the variables (t, x). From advanced calculus, we have

$$dt\,ds = \frac{\partial(t, s)}{\partial(t, x)}\, dt\, dx \quad (3)$$

where the **Jacobian** of the transformation is given by

$$\frac{\partial(t, s)}{\partial(t, x)} = \begin{vmatrix} \frac{\partial t}{\partial t} & \frac{\partial t}{\partial x} \\ \frac{\partial s}{\partial t} & \frac{\partial s}{\partial x} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1)$$

Then (2) becomes

$$F(u)\, G(u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, g(x-t)\, e^{-iux}\, dt\, dx$$

$$= \int_{-\infty}^{\infty} e^{-iux} \left[\int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt\right] dx$$

$$= F\left\{\int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt\right\}$$

$= F(f * g)$ Since

$f * g = \int_{-\infty}^{\infty} f(t)\, g(x-t)\,dt$ is the **convolution** of f and g.

Hence $F(f * g) = F(u)\, G(u) = F(f)\,.\,G(g)$.

4.14 Parseval's identity for Fourier transforms (Rayleigh's theorem or plancherel's theorem)

Statement : If $F(u)$ and $G(u)$ are the complex Fourier transforms of $f(x)$ and $g(x)$ respectively, then

$$\text{(i)} \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, \bar{G}(u)\, du = \int_{-\infty}^{\infty} f(x)\, \bar{g}(x)\, dx$$

$$\text{(ii)} \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(u)|^2\, du = \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

where bar denotes the complex conjugate.

Proof : (i) Using the inverse Fourier transform, we have

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(u)\, e^{iux}\, du \quad (1)$$

Taking complex conjugates on both sides of (1), we get

$$\overline{g(x)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{G(u)}\, e^{-iux}\, du \quad (2)$$

$$\therefore \int_{-\infty}^{\infty} f(x)\, \overline{g(x)}\, dx = \int_{-\infty}^{\infty} f(x)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{G(u)}\, e^{-iux}\, du\right] dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{G(u)}\left[\int_{-\infty}^{\infty} f(x)\, e^{-iux}\, dx\right] du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{G(u)}\, F(u)\, du$$

Since $F(u) = \int_{-\infty}^{\infty} f(x) e^{iux}\, dx$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\overline{G(u)}\, du.$$

Hence $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\overline{G(u)}\, du. = \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\, dx.$

Proof : (ii) From part (i) we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\overline{G(u)}\, du = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx \quad (3)$$

Taking $g(x) = f(x)$ in (3), we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\overline{G(u)}\, du. = \int_{-\infty}^{\infty} f(x)\,\overline{f(x)}\, dx.$$

Since $g(x) = f(x) \Rightarrow \overline{G(u)} = \overline{F(u)}$

or, $$\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(u)|^2\, du = \int_{-\infty}^{\infty} |f(x)|^2\, dx.$$

Note : One can easily prove the following parseval's identities for Fourier cosine and sine transforms :

(iii) $$\frac{2}{\pi}\int_{0}^{\infty} F_c(u)\, G_c(u)\, du = \int_{0}^{\infty} f(x)\, g(x)\, dx.$$

(iv) $$\frac{2}{\pi}\int_{0}^{\infty} F_s(u)\, G_s(u)\, du = \int_{0}^{\infty} f(x)\, g(x)\, dx$$

(v) $$\frac{2}{\pi}\int_{0}^{\infty} [F_c(u)]^2\, du = \int_{0}^{\infty} [f(x)]^2\, dx$$

(vi) $$\frac{2}{\pi}\int_{0}^{\infty} [F_s(u)]^2\, du = \int_{0}^{\infty} [f(x)]^2\, dx$$

WORKED OUT EXAMPLES

Example 1. The function x^2 is periodic with period $2l$ on the interval $[-l, l]$. Find its Fourier series.

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Solution : $f(x) = x^2$. $f(-x) = (-x)^2 = x^2 = f(x)$ so $f(x)$ is an even function and hence sine terms will vanish. i.e $b_n = 0$.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where $a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)dx \; (n = 0)$

and $a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx \; (n = 1, 2, 3 \;......)$

Since $f(x)$ is even.

$$a_0 = \frac{1}{l}\int_{0}^{l} f(v)dv,\; a_n = \frac{2}{l}\int_{0}^{l} f(v)\cos\frac{n\pi v}{l}dv$$

$$a_0 = \frac{1}{l}\int_{0}^{l} x^2 dx = \frac{1}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{1}{3}l^2.$$

$$a_n = \frac{2}{l}\int_{0}^{l} x^2\cos\frac{n\pi x}{l}dx. \text{ (Integrating by parts)}$$

$$= \frac{2}{l}\left[x^2.\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right]_0^l - \frac{2}{l}.2\int_{0}^{l} x.\frac{l}{n\pi}\sin\frac{n\pi x}{l}dx$$

$$= 0 - \frac{4}{n\pi}\int_{0}^{l} x\sin\frac{n\pi x}{l}dx$$

$$= \frac{4}{n\pi}.\frac{l}{n\pi}\left[x\cos\frac{n\pi x}{l}\right]_0^l - \frac{4}{n\pi}.\frac{l}{n\pi}\int_{0}^{l}\cos\frac{n\pi x}{l}dx$$

$$= \frac{4l}{n^2\pi^2}[l\cos n\pi - 0] - \frac{4l^2}{n^3\pi^3}\left[\sin\frac{n\pi x}{l}\right]_0^l$$

$$=\frac{4l^2}{n^2\pi^2}(-1)^n-0=\frac{4l^2}{n^2\pi^2}(-1)^n.\therefore a_n=\frac{4l^2}{n^2\pi^2}(-1)^n.$$

Therefore, $f(x)=\frac{l^2}{3}+\sum_{n=1}^{\infty}\frac{4l^2}{n^2\pi^2}(-1)^n\cos\frac{n\pi x}{l}$

$$=\frac{l^2}{3}+\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^n.\cos\frac{n\pi x}{l}$$

$$=\frac{l^2}{3}+\frac{4l^2}{\pi^2}\left[-\frac{1}{1^2}\cos\frac{\pi x}{l}+\frac{1}{2^2}\cos\frac{2\pi x}{l}-\frac{1}{3^2}\cos\frac{3\pi x}{l}+\frac{1}{4^2}\cos\frac{4\pi x}{l}..\right]$$

$$=\frac{l^2}{3}-\frac{4l^2}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l}-\frac{1}{2^2}\cos\frac{2\pi x}{l}+\frac{1}{3^2}\cos\frac{3\pi x}{l}-\frac{1}{4^2}\cos\frac{4\pi x}{l}+..\right]$$

Example 2. Obtain the Fourier series of the function

$$f(x)=\begin{cases}0, -\pi\le x\le 0\\ 1, 0\le x\le\pi\end{cases}$$

and verify the result by assuming the complex form of Fourier series. $\boxed{\textbf{D. U. M. SC (F) 1991}}$

Solution : Def^n : The **complex form of the Fourier series** can be written as.

$$f(x)=\sum_{n=-\infty}^{\infty}C_n e^{\frac{in\pi x}{c}}, -c<x<c$$

where $C_n=\frac{1}{2c}\int_{-c}^{c}f(x)\,e^{-\frac{in\pi x}{c}}\,dx$

and $n=0,\pm1,\pm2,\ldots\ldots$

Ist portion

By def^n, we have $f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$ (1)

where $a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx.$$

Now $a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{2\pi}\left[\int_{-\pi}^{0}f(x)\,dx+\int_{0}^{\pi}f(x)dx\right]$

$$=\frac{1}{2\pi}\left[\int_{-\pi}^{0}o\,dx+\int_{0}^{\pi}1dx\right]$$

$$=\frac{1}{2\pi}[0+\pi]=\frac{1}{2}\therefore\boxed{a_0=\frac{1}{2}}$$

Again $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx\,(n\neq o)$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0}f(x)\cos nx\,dx+\int_{0}^{\pi}f(x)\cos nx\,dx\right]$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0}o\cos nx\,dx+\int_{0}^{\pi}1.\cos nx\,dx\right]$$

$$=\frac{1}{\pi}[o]+\frac{1}{\pi n}\left[\sin nx\right]_0^{\pi}=0+0=0\therefore\boxed{a_n=0}$$

Finally, $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0}f(x)\sin nx\,dx+\int_{0}^{\pi}f(x)\sin nx\,dx\right]$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0}o\sin nx\,dx+\int_{0}^{\pi}1.\sin nx\,dx\right]$$

$$=\frac{1}{\pi}[0]+\frac{1}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi}$$

$$=0-\frac{1}{n\pi}(\cos n\pi-\cos o)$$

$$=-\frac{1}{n\pi}\left[(-1)^n-1\right]$$

$$= \frac{1}{n\pi}\left[1-(-1)^n\right] = \begin{cases} 0 \text{ when n is even} \\ \frac{2}{n\pi} \text{ when n is odd.} \end{cases}$$

Now putting the values of a_0, a_n and b_n in (1) we get

$$f(x) = \frac{1}{2} + 0 + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{1}{2} + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x + b_5 \sin 5x + \ldots\ldots$$

$$= \frac{1}{2} + \frac{2}{\pi}\sin x + 0 + \frac{2}{3\pi}\sin 3x + 0 + \frac{2}{5\pi}\sin 5x + \ldots\ldots$$

$$= \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\ldots\right)$$

Second portion.

Now we have to expand the function $f(x)$ in the complex Fourier series. By def^n, we have

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{c}}, \quad -c < x < c.$$

$$\text{where } C_n = \frac{1}{2c}\int_{-c}^{c} f(x)\, e^{-\frac{in\pi x}{c}}\, dx, \; n = 0, \pm 1, \pm 2, \pm 3, \ldots\ldots$$

Here in our given problem $c = \pi$.

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}, \; -\pi < x < \pi$$

$$\text{where } C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx,$$

$n = 0, \pm 1, \pm 2, \pm 3, \ldots\ldots$

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)\, dx + \int_{0}^{\pi} f(x)\, dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\, dx + \int_{0}^{\pi} 1\, dx\right]$$

$$= \frac{1}{2\pi}\left[0 + \pi\right] = \frac{1}{2} \;\therefore \boxed{C_0 = \frac{1}{2}}$$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx} dx \; (n \neq 0)$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)\, e^{-inx} dx + \int_{0}^{\pi} f(x)\, e^{-inx} dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\, e^{-inx} dx + \int_{0}^{\pi} 1.\, e^{-inx} dx\right]$$

$$= 0 + \frac{1}{2\pi}\left[\frac{e^{-inx}}{-in}\right]_0^{\pi}$$

$$= -\frac{1}{2n\pi i}\left[e^{-in\pi} - e^{0}\right]$$

$$= -\frac{1}{2n\pi i}\left[\cos n\pi - i \sin n\pi - 1\right]$$

$$= -\frac{1}{2n\pi i}\left[(-1)^n - 1\right] \text{ since } \sin n\pi = 0$$

$$= \begin{cases} \frac{1}{n\pi i} \text{ when } n = \pm 1, \pm 3, \pm 5, \ldots\ldots\ldots \\ 0 \text{ when } n = \pm 2 \pm 4 \pm 6 \ldots\ldots\ldots \end{cases}$$

$$\text{Thus } f(x) = \frac{1}{2} + \frac{1}{\pi i}\left(\frac{e^{ix}}{1} + 0 + \frac{e^{i3x}}{3} + 0 + \frac{e^{i5x}}{5} + \ldots\right)$$

$$+ \frac{1}{\pi i}\left(\frac{e^{-ix}}{-1} + 0 + \frac{e^{-i3x}}{-3} + 0 + \frac{e^{-i5x}}{-5} + \ldots\ldots\right)$$

$$= \frac{1}{2} + \frac{1}{\pi i}\left[(e^{ix} - e^{-ix}) + \frac{1}{3}(e^{i3x} - e^{-i3x}) + \frac{1}{5}(e^{i5x} - e^{-i5x}) + \ldots\right]$$

$$= \frac{1}{2} + \frac{1}{\pi i}\left[2i \sin x + \frac{1}{3} 2i \sin 3x + \frac{1}{5} 2i \sin 5x + \ldots\ldots\right]$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\ldots\right]$$

Hence $f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\ldots\right]$

which is same as in the real form.

Example 3. Show that $\int_0^\infty \frac{\cos ux}{u^2+1}\,du = \frac{\pi}{2}e^{-x}, x>0.$

Proof : Let $f(x) = \begin{cases} e^{-x}, & x > 0 \\ e^{x}, & x < 0 \end{cases}$

Then $f(-x) = \begin{cases} e^{x}, & x < 0 \\ e^{-x}, & x > 0. \end{cases}$

Therefore, $f(x) = f(-x)$. Thus $f(x)$ is an even function. Now by definition of Fourier integral of an even function for $x > 0$, we have

$$f(x) = \frac{2}{\pi}\int_0^\infty du \int_0^\infty f(t)\cos ut \cos ux\,dx \qquad (1)$$

Putting $f(x) = e^{-x}$ in (1), we get

$$e^{-x} = \frac{2}{\pi}\int_0^\infty du \int_0^\infty e^{-t}\cos ut \cos ux\,dt$$

$$= \frac{2}{\pi}\int_0^\infty \cos ux\,du \int_0^\infty e^{-t}\cos ut\,dt$$

$$= \frac{2}{\pi}\int_0^\infty \cos ux\,du\left[\frac{e^{-t}}{1+u^2}(-\cos ut + u\sin ut)\right]_0^\infty$$

$$= \frac{2}{\pi}\int_0^\infty \cos ux\left\{0 + \frac{1}{1+u^2}\right\}du = \frac{2}{\pi}\int_0^\infty \frac{\cos ux}{u^2+1}\,du$$

$$\therefore \int_0^\infty \frac{\cos ux}{u^2+1}\,du = \frac{\pi}{2}e^{-x}, x > 0$$

Note 1. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

In example 3, $a = 1$, $b = u$ and $x = t$

$$\therefore \int e^{-t}\cos ut\,dt = \frac{e^{-t}}{1+u^2}(-\cos ut + u\sin ut)$$

Note 2. When $x = 0$, the result of example 3 reduces to

$$\int_0^\infty \frac{du}{1+u^2} = \frac{\pi}{2}$$

Example 4. Prove that $\int_0^\infty \frac{x\sin mx}{x^2+1}\,dx = \frac{\pi}{2}e^{-m}, m > 0.$

Proof : Let $f(x) = \begin{cases} e^{-m}, & m > 0 \\ -e^{m}, & m < 0 \end{cases}$

Then $f(-m) = \begin{cases} e^{m}, & m < 0 \\ -e^{-m}, & m > 0 \end{cases}$

Again $-f(m) = \begin{cases} -e^{-m}, & m > 0 \\ e^{m}, & m < 0 \end{cases}$

$\therefore f(-m) = -f(m)$

Thus $f(x)$ is an odd function. Now by definition of Fourier integral of an odd function for $m > 0$, we have

$$f(m) = \frac{2}{\pi}\int_0^\infty dx \int_0^\infty f(t)\sin xt \sin mx\,dt \quad (2)$$

Putting $f(m) = e^{-m}$ in (2), we get

$$e^{-m} = \frac{2}{\pi}\int_0^\infty dx \int_0^\infty e^{-t}\sin xt \sin mx\,dt$$

$$= \frac{2}{\pi}\int_0^\infty \sin mx\,dx \int_0^\infty e^{-t}\sin xt\,dt$$

$$= \frac{2}{\pi}\int_0^\infty \sin mx\,dx\left[\frac{e^{-t}(-\sin xt - x\cos xt)}{1+x^2}\right]_0^\infty$$

$$= \frac{2}{\pi}\int_0^\infty \sin mx\,dx\left[0 + \frac{x}{1+x^2}\right]$$

$$= \frac{2}{\pi}\int_0^\infty \frac{x\sin mx}{x^2+1}\,dx$$

$$\text{Or, } \int_0^\infty \frac{x\sin mx}{x^2+1}\,dx = \frac{\pi}{2}e^{-m}, m > 0.$$

Example 5. Find the Fourier integral of the function $f(x) = e^{-kx}$ when $x > 0$ and $f(-x) = f(x)$ for $k > 0$, and hence prove that $\int_0^\infty \frac{\cos ux}{k^2+u^2} du = \frac{\pi}{2k} e^{-kx}$. **D. U. M. SC (F) 1991**

Solution : Since $f(-x) = f(x)$, so $f(x)$ is even and for even function we have the Fourier integral

$$f(x) = \frac{2}{\pi}\int_0^\infty f(t)\,dt\int_0^\infty \cos ut \cos ux\,du$$

$$= \frac{2}{\pi}\int_0^\infty \left[\int_0^\infty f(t)\cos ut\,dt\right]\cos ux\,du \quad (1)$$

Now $\int_0^\infty f(t)\cos ut\,dt = \int_0^\infty e^{-kt}\cos ut\,dt$

$$= \left[\frac{e^{-kt}}{k^2+u^2}(-k\cos ut + u\sin ut)\right]_0^\infty$$

$$= 0 + \frac{k}{k^2+u^2} = \frac{k}{k^2+u^2}.$$

Thus from (1), we get

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{k}{k^2+u^2}\cos ux\,du$$

$$= \frac{2k}{\pi}\int_0^\infty \frac{\cos ux}{k^2+u^2}du \; (x > 0, k > 0) \quad (2)$$

which is the required Fourier integral of the function $f(x) = e^{-kx}$.

Again putting $f(x) = e^{-kx}$ in (2), we get $e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{\cos ux}{k^2+u^2} dx$

$$\therefore \int_0^\infty \frac{\cos ux}{k^2+u^2} du = \frac{\pi}{2k} e^{-kx}.$$

Example 6. Find the Fourier integral of the function $f(x) = e^{-kx}$ when $x > 0$ and $f(-x) = -f(x)$ for $k > 0$ and hence prove that

$$\int_0^\infty \frac{u\sin ux}{k^2+u^2} du = \frac{\pi}{2} e^{-kx},\; k > 0.$$

Solution : Since $f(-x) = -f(x)$, so $f(x)$ is an odd function for which we have Fourier integral $f(x) = \frac{2}{\pi}\int_0^\infty f(t)\,dt\int_0^\infty \sin ut \sin ux\,du$

$$= \frac{2}{\pi}\int_0^\infty \left[\int_0^\infty f(t)\sin ut\,dt\right]\sin ux\,du \quad (1)$$

Now $\int_0^\infty f(t)\sin ut\,dt = \int_0^\infty e^{-kt}\sin ut\,dt$

$$= \left[\frac{e^{-kt}(-k\sin ut - u\cos ut)}{k^2+u^2}\right]_0^\infty$$

$$= 0 + \frac{u}{k^2+u^2} = \frac{u}{k^2+u^2}$$

Thus from (1), we have

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{u}{k^2+u^2}\sin ux\,du$$

$$= \frac{2}{\pi}\int_0^\infty \frac{u\sin ux}{k^2+u^2}du \; (x > 0, k > 0) \quad (2)$$

which is the required Fourier integral of the function $f(x) = e^{-kx}$.

putting $f(x) = e^{-kx}$ in (2), we get

$$f(x) = e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{u\sin ux}{k^2+u^2}du.$$

or, $$\int_0^\infty \frac{u\sin ux}{k^2+u^2}du = \frac{\pi}{2}e^{-kx}.$$

Example 7. Find the Fourier integral of the function

$$f(x) = \begin{cases} 0 & \text{when } x < 0 \\ \frac{1}{2} & \text{when } x = 0 \\ e^{-x} & \text{when } x > 0. \end{cases}$$

Solution : By the definition of Fourier integral in general, we have

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\, dt \int_{0}^{\infty} \cos u(x-t)\, du$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\, dt \int_{0}^{\infty} \cos u(x-t)\, du$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\, dt \int_{0}^{\infty} (\cos ux \cos ut + \sin ux \sin ut)\, du$$

$$= \frac{1}{\pi}\left[\int_{0}^{\infty}\left\{\int_{-\infty}^{\infty} f(t)\cos ut\, dt\right\}\cos ux\, du + \int_{0}^{\infty}\left\{\int_{-\infty}^{\infty} f(t)\sin ut\, dt\right\}\sin ux\, du\right] \quad (1)$$

Now $\int_{-\infty}^{\infty} f(t)\cos ut\, dt = \int_{-\infty}^{0} f(t)\cos ut\, dt + \int_{0}^{\infty} f(t)\cos ut\, dt$

$$= \int_{-\infty}^{0} 0 \cos ut\, dt + \int_{0}^{\infty} e^{-t}\cos ut\, dt$$

$$= 0 + \left[\frac{e^{-t}}{1+u^2}(-\cos ut + u \sin ut)\right]_{0}^{\infty}$$

$$= \frac{1}{1+u^2} + 0 = \frac{1}{1+u^2}.$$

Similarly, $\int_{-\infty}^{\infty} f(t)\sin ut\, dt = \int_{-\infty}^{0} f(t)\sin ut\, dt + \int_{0}^{\infty} f(t)\sin ut\, dt$

$$= \int_{-\infty}^{0} 0 \sin ut\, dt + \int_{0}^{\infty} e^{-t}\sin ut\, dt$$

$$= 0 + \int_{0}^{\infty} e^{-t}\sin ut\, dt$$

$$= \left[\frac{e^{-t}(-\sin ut - u\cos ut)}{1+u^2}\right]_{0}^{\infty}$$

$$= 0 + \frac{u}{1+u^2} = \frac{u}{1+u^2}$$

Putting these values in (1), we get

$$f(x) = \frac{1}{\pi}\left[\int_{0}^{\infty} \frac{\cos ux}{1+u^2}\, du + \int_{0}^{\infty} \frac{u \sin ux}{1+u^2}\, du\right]$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\left(\frac{\cos ux + u \sin ux}{1+u^2}\right) du \quad (2)$$

putting $x = 0$ in (2), we get

$$f(0) = \frac{1}{\pi}\int_{0}^{\infty} \frac{du}{1+u^2} = \frac{1}{\pi}\left[\tan^{-1} u\right]_{0}^{\infty}$$

$$= \frac{1}{\pi}\left[\tan^{-1}\infty - \tan^{-1} 0\right]$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} - 0\right) = \frac{1}{2}$$

so $f(x) = \frac{1}{2}$ for $x = 0$ is satisfied.

Hence $f(x) = \frac{1}{\pi}\int_{0}^{\infty} \frac{(\cos ux + u \sin ux)}{1+u^2}\, du$

which is the required Fourier integral of the given function.

Example 8. Find the Fourier sine transform of e^{-x}, $x \geq 0$.

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Solution : By definition of Fourier sine transform of $f(x)$ for $0 < x < \infty$, we have

$$f_s(n) = \int_{0}^{\infty} F(x) \sin nx\, dx$$

$$\therefore \quad f_s(n) = \int_{0}^{\infty} e^{-x} \sin nx\, dx \quad \text{Since } f(x) = e^{-x}$$

$$= \left[-\frac{e^{-x}}{n}\cos nx\right]_{0}^{\infty} - \int_{0}^{\infty} \frac{e^{-x}\cos nx}{n}\, dx$$

$$= 0 + \frac{1}{n} - \left[\frac{e^{-x}\sin nx}{n^2}\right]_{0}^{\infty} - \int_{0}^{\infty} \frac{e^{-x}\sin nx}{n^2}\, dx$$

$$= \frac{1}{n} - 0 - \frac{1}{n^2} f_s(n)$$

or, $f_s(n) + \frac{1}{n^2} f_s(n) = \frac{1}{n}$

or, $\left(1 + \frac{1}{n^2}\right) f_s(n) = \frac{1}{n}$

or, $\left(\frac{n^2+1}{n^2}\right) f_s(n) = \frac{1}{n}$

or, $f_s(n) = \frac{1}{n} \cdot \frac{n^2}{n^2+1} = \frac{n}{n^2+1}$

$\therefore \quad f_s(n) = \frac{n}{n^2+1}$.

Hence the Fourier sine transform of e^{-x} is $\frac{n}{n^2+1}$.

Example 9. Find the inverse Fourier sine transform of

$f_s(n) = \frac{n}{1+n^2}$

Solution : By definition of the inverse Fourier sine transform, we have

$$F(x) = \frac{2}{\pi} \int_0^{\infty} f_s(n) \sin nx \, dn$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{n}{1+n^2} \sin nx dn \quad (1)$$

From the Fourier integral formula of an odd function, we have

$$f(x) = \frac{2}{\pi} \int_0^{\infty} du \int_0^{\infty} f(t) \sin ut \sin ux \, dt$$

$$\therefore \quad f(x) = \frac{2}{\pi} \int_0^{\infty} dn \int_0^{\infty} f(t) \sin nt \sin nx \, dt \quad (2)$$

Taking $f(x) = e^{-x}$ in (2), we have

$$e^{-x} = \frac{2}{\pi} \int_0^{\infty} dn \int_0^{\infty} e^{-t} \sin nt \sin nx dt$$

$$= \frac{2}{\pi} \int_0^{\infty} \sin nx \left\{ \int_0^{\infty} e^{-t} \sin nt \, dt \right\} dn$$

$$= \frac{2}{\pi} \int_0^{\infty} \sin nx \left(\frac{n}{1+n^2}\right) dn$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{n \sin nx}{1+n^2} dn$$

or, $$\int_0^{\infty} \frac{n}{1+n^2} \sin nx \, dn = \frac{\pi}{2} e^{-x} \quad (3)$$

Combining (1) and (3), we get

$$F(x) = \frac{2}{\pi} \cdot \frac{\pi}{2} e^{-x} = e^{-x}.$$

Hence $F(x) = e^{-x}$ which is the required inverse Fourier sine transform of $f_s(n) = \frac{n}{1+n^2}$.

Example 10. Find the Fourier cosine transform of e^{-x}, $x \geq 0$.

Solution : By definition of Fourier cosine transform of for $0 < x < \infty$, we have

$$f_c(n) = \int_0^{\infty} F(x) \cos nx \, dx \quad (1)$$

$$\therefore \quad f_c(n) = \int_0^{\infty} e^{-x} \cos nx \, dx \quad \text{Since } F(x) = e^{-x}$$

$$= \left[\frac{e^{-x} \sin nx}{n}\right]_0^{\infty} + \int_0^{\infty} \frac{e^{-x} \sin nx}{n} dx$$

$$= 0 - \frac{1}{n^2} \left[e^{-x} \cos nx\right]_0^{\infty} - \frac{1}{n^2} \int_0^{\infty} e^{-x} \cos nx \, dx$$

$$= 0 + \frac{1}{n^2} - \frac{1}{n^2} f_c(n)$$

or, $f_c(n) + \frac{1}{n^2} f_c(n) = \frac{1}{n^2}$

or, $\frac{(n^2+1)}{n^2} f_c(n) = \frac{1}{n^2}$

or, $f_c(n) = \frac{1}{n^2+1}$

Hence the Fourier cosine transform of e^{-x} is $\frac{1}{n^2+1}$.

Example 11. Find the inverse Fourier cosine transform of $f_c(n) = \frac{1}{1+n^2}$

Solution : By definition of the inverse Fourier cosine transform, we have

$$F(x) = \frac{2}{\pi}\int_0^\infty f_c(n)\cos nx\,dn$$

$$= \frac{2}{\pi}\int_0^\infty \frac{1}{1+n^2}\cos nx\,dn$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\cos nx}{1+n^2}\,dn \quad (1)$$

From the Fourier integral formula of an even function, we have

$$f(x) = \frac{2}{\pi}\int_0^\infty du\int_0^\infty f(t)\cos ut\cos ux\,dt$$

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty dn\int_0^\infty f(t)\cos nt\cos nx\,dt \quad (2)$$

Taking $f(x) = e^{-x}$ in (2) we get

$$e^{-x} = \frac{2}{\pi}\int_0^\infty dn\int_0^\infty e^{-t}\cos nt\cos nx\,dt$$

$$= \frac{2}{\pi}\int_0^\infty \cos nx\left\{\int_0^\infty e^{-t}\cos nt\,dt\right\}dn$$

$$= \frac{2}{\pi}\int_0^\infty \cos nx\cdot\frac{1}{1+n^2}\,dn$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\cos nx}{1+n^2}\,dn$$

$$\therefore \quad \int_0^\infty \frac{\cos nx}{1+n^2}\,dn = \frac{\pi}{2}e^{-x} \quad (3)$$

Combining (1) and (3), we get

$$F(x) = \frac{2}{\pi}\cdot\frac{\pi}{2}e^{-x} = e^{-x}.$$

Hence $F(x) = e^{-x}$ which is the required inverse Fourier cosine transform of $F_c(n) = \frac{1}{1+n^2}$

Example 12. Find the (a) finite Fourier sine transform (b) finite Fourier cosine transform of the function

$F(x) = 2x,\ 0 < x < 4.$ **D. U. M. SC (F) 1987**

Solution : (a) Since $l = 4$, we have

$$f_s(n) = \int_0^l F(x)\sin\frac{n\pi x}{l}\,dx = \int_0^4 F(x)\sin\frac{n\pi x}{4}\,dx$$

$$= \int_0^4 2x\sin\frac{n\pi x}{4}\,dx$$

$$= \left[-2x\cdot\frac{4}{n\pi}\cos\frac{n\pi x}{4}\right]_0^4 + \frac{8}{n\pi}\int_0^4 \cos\frac{n\pi x}{4}\,dx$$

$$= -\frac{32}{n\pi}\cos n\pi + 0 + \frac{32}{n^2\pi^2}\left[\sin\frac{n\pi x}{4}\right]_0^4$$

$$= -\frac{32}{n\pi}\cos n\pi + \frac{32}{n^2\pi^2}(0-0)$$

$= -\frac{32}{n\pi}\cos n\pi$ which is the finite Fourier sine transform of $F(x) =$

(b) If $n > 0$, $f_c(n) = \int_0^l F(x)\cos\frac{n\pi x}{l}\,dx$

$$= \int_0^4 2x\cos\frac{n\pi x}{4}\,dx$$

$$= \left[2x\cdot\frac{4}{n\pi}\sin\frac{n\pi x}{4}\right]_0^4 - \frac{8}{n\pi}\int_0^4 \sin\frac{n\pi x}{4}\,dx$$

$$= 0 - \frac{8}{n\pi}\left(-\frac{4}{n\pi}\right)\left[\cos\frac{n\pi x}{4}\right]_0^4$$

$= \frac{32}{n^2\pi^2}(\cos n\pi - 1)$ which is the finite Fourier cosine transform of $F(x) = 2x$.

If $n = 0$, $f_c(n) = f_c(0) = \int_0^4 2x\,dx$

$$= 2\left[\frac{x^2}{2}\right]_0^4 = (4^2 - 0) = 16.$$

$\therefore \quad f_c(n) = f_c(0) = 16.$

Example 13. Find the Fourier transform of $f(x)$ defined by

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

and hence evaluate (i) $\int_{-\infty}^{\infty} \frac{\sin ua \cos ux}{u}\,du$ (ii) $\int_0^{\infty} \frac{\sin u}{u}\,du$.

Solution : By definition of Fourier transform, we have

$$F(u) = \int_{-\infty}^{\infty} f(t)\, e^{-iut}dt.$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{-iux}dx$$

$$= \int_{-\infty}^{-a} f(x)e^{-iux}\,dx + \int_{-a}^{a} f(x)\, e^{-iux}\,dx + \int_{a}^{\infty} f(x)e^{-iux}\,dx$$

$$= \int_{-\infty}^{-a} 0e^{-iux}dx + \int_{-a}^{a} 1e^{-iux}\,dx + \int_{a}^{\infty} 0e^{-iux}dx$$

$$= 0 + \int_{-a}^{a} e^{-iux}dx + 0$$

$$= \frac{1}{-iu}\left[e^{-iux}\right]_{-a}^{a}$$

$$= -\frac{1}{iu}\left(e^{-iua} - e^{iua}\right)$$

$$= \frac{1}{iu}\left(e^{iua} - e^{-iua}\right)$$

$$= \frac{2}{u}\cdot\frac{1}{2i}\left(e^{iua} - e^{-iua}\right)$$

$$= \frac{2}{u}\sin ua.$$

Hence $F(u) = \frac{2\sin ua}{u}$, $u \neq 0$.

For $u = 0$, we obtain $F(u) = \int_{-a}^{a} dx = 2a$.

Solution of second portion :

By the definition of the inverse Fourier transform, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{iux}du$$

$$\therefore \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{iux}\,du = \begin{cases} 1, & |x| < a \\ 0, & |x| > a. \end{cases} \quad (1)$$

$$\text{Now } \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{iux}\,du = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin ua}{u}\, e^{iux}\,du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin ua}{u}(\cos ux + i\sin ux)\,du$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin ua \cos ux}{u}\,du + \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{\sin ua \sin ux}{u}\,du \quad (2)$$

The integrand in the second integral of (2) on the right side is odd and so this integral is zero.

Hence combining (1) and (2), we get

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin ua \cos ux}{u}\,du = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

or, $\int_{-\infty}^{\infty} \frac{\sin ua \cos ux}{u} du = \begin{cases} \pi, & |x| < a \\ 0, & |x| > a \end{cases}$

If $x = 0$ and $a = 1$, then from the above result we have

$$\int_{-\infty}^{\infty} \frac{\sin u}{u} du = \pi$$

or, $2\int_{0}^{\infty} \frac{\sin u}{u} du = \pi$, since the integrand is even.

or, $\int_{0}^{\infty} \frac{\sin u}{u} du = \frac{\pi}{2}$.

Example 14. Find the Fourier transform of $F(x) = e^{-|x|}$ where x belongs to $(-\infty, \infty)$.

Solution : By the definition of Fourier transform, we have

$$F(u) = \int_{-\infty}^{\infty} f(t)\, e^{iut} dt.$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{iux} dx$$

$$= \int_{-\infty}^{0} f(x)\, e^{iux} dx + \int_{0}^{\infty} f(x)\, e^{iux} dx$$

$$= \int_{-\infty}^{0} e^{x} e^{iux} dx + \int_{0}^{\infty} e^{-x} e^{iux} dx$$

Since $e^{-|x|} = \begin{cases} e^{-x}, & x > 0 \\ e^{x}, & x < 0. \end{cases}$

$$= \int_{-\infty}^{0} e^{(1+iu)x} dx + \int_{0}^{\infty} e^{-(1-iu)x} dx$$

$$= \frac{1}{1+iu}\left[e^{(1+iu)x}\right]_{-\infty}^{0} + \frac{1}{-(1-iu)}\left[e^{-(1-iu)x}\right]_{0}^{\infty}$$

$$= \frac{1}{1+iu}(1-0) - \frac{1}{1-iu}(0-1)$$

$$= \frac{1}{1+iu} + \frac{1}{1-iu}$$

$$= \frac{1-iu+1+iu}{1-i^2u^2} = \frac{2}{1+u^2}$$

Hence $F(u) = \frac{2}{1+u^2}$ which is the required Fourier transform of $f(x) = e^{-|x|}$.

Example 15. Find the Fourier transform of

$$f(x) = \begin{cases} 1-x^2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

Solution : By definition of the Fourier transform

$$F(u) = \int_{-\infty}^{\infty} f(t)\, e^{iut} dt.$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{iux} dx$$

$$= \int_{-\infty}^{-1} f(x) e^{iux} dx + \int_{-1}^{1} f(x)\, e^{iux} dx + \int_{1}^{\infty} f(x)\, e^{iux} dx$$

$$= \int_{-\infty}^{-1} 0\, e^{iux} dx + \int_{-1}^{1} (1-x^2)\, e^{iux} dx + \int_{1}^{\infty} 0\, e^{iux} dx$$

$$= 0 + \int_{-1}^{1} (1-x^2)\, e^{iux} dx + 0$$

$$= \left[(1-x^2).\frac{1}{iu} e^{iux}\right]_{-1}^{1} - \int_{-1}^{1} (0-2x).\frac{1}{iu} e^{iux} dx$$

$$= 0 + \frac{2}{iu}\int_{-1}^{1} x\, e^{iux} dx$$

$$= \frac{2}{iu}\left[\frac{x}{iu}e^{iux}\right]_{-1}^{1} - \frac{2}{i^2u^2}\int_{-1}^{1} e^{iux}dx$$

$$= \frac{2}{i^2u^2}\left[1e^{iu} + 1e^{-iu}\right] + \frac{2}{u^2}\left[\frac{1}{iu}e^{iux}\right]_{-1}^{1}$$

$$= -\frac{2}{u^2}\left(e^{iu}+e^{-iu}\right) + \frac{2}{iu^3}\left(e^{iu}-e^{-iu}\right)$$

$$= -\frac{4}{u^2}\cdot\frac{1}{2}\left(e^{iu}+e^{-iu}\right) + \frac{4}{u^3}\cdot\frac{1}{2i}\left(e^{iu}-e^{-iu}\right)$$

$$= -\frac{4}{u^2}\cos u + \frac{4}{u^3}\sin u$$

$$= \frac{4}{u^3}(\sin u - u\cos u)$$

Hence $F(u) = \frac{4}{u^3}(\sin u - u\cos u)$.

Example 16. Prove that $\int_0^\infty \frac{(\sin x - x\cos x)}{x^3}\cos\frac{x}{2}\,dx = \frac{3\pi}{16}$.

Proof : From the result of example 15, we have

$$F(u) = \frac{4}{u^3}(\sin u - u\cos u)$$

By definition of the inverse Fourier transform, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{iux}du$$

$$\text{or, } 2\pi f(x) = \int_{-\infty}^{\infty} \frac{4}{u^3}(\sin u - u\cos u)\, e^{iux}du$$

$$= 4\int_{-\infty}^{\infty} \frac{(\sin u - u\cos u)}{u^3}(\cos ux + i\sin ux)\,du$$

$$\text{Or, } \frac{\pi}{2}f(x) = \int_{-\infty}^{\infty} \frac{(\sin u - u\cos u)}{u^3}(\cos ux + i\sin ux)\,du \quad (1)$$

Equating real part only from both sides of (1), we get.

$$\frac{\pi}{2}f(x) = \int_{-\infty}^{\infty} \frac{(\sin u - u\cos u)}{u^3}\cos ux\,du$$

$$\text{Or, } 2\int_0^\infty \frac{(\sin u - u\cos u)}{u^3}\cos ux\,du = \begin{cases}\frac{\pi}{2}(1-x^2), & |x| \le 1\\ 0, & |x| > 1.\end{cases}$$

Putting $x = \frac{1}{2}$, we get.

$$2\int_0^\infty \frac{(\sin u - u\cos u)}{u^3}\cos\frac{1}{2}u\,du = \frac{\pi}{2}\left(1-\frac{1}{4}\right) = \frac{3\pi}{8}$$

$$\text{or, } \int_0^\infty \frac{\sin u - u\cos u}{u^3}\cos\frac{1}{2}u\,du = \frac{3\pi}{16}$$

$$\text{or, } \int_0^\infty \frac{(\sin x - x\cos x)}{x^3}\cos\frac{x}{2}\,dx = \frac{3\pi}{16}$$

Example 17. Find the Fourier cosine transform of the function $F(x) = e^{-x^2}$.

Solution : By definition of Fourier cosine transform, we have

$$f_c(n) = \int_0^\infty F(x)\cos nx\,dx$$

$$= \int_0^\infty e^{-x^2}\cos nx\,dx = I\ (\text{Say}) \quad (1)$$

Differentiating with respect to n we have

$$\frac{dI}{dn} = \int_0^\infty e^{-x^2}(-x\sin nx)\,dx = \frac{1}{2}\int_0^\infty \sin nx\,(-2x\,e^{-x^2})\,dx$$

$$= \frac{1}{2}\left[\sin nx\, e^{-x^2}\right]_0^\infty - \frac{n}{2}\int_0^\infty \cos nx\, e^{-x^2}dx$$

$$= 0 - \frac{n}{2}\int_0^\infty e^{-x^2}\cos nx\,dx = -\frac{nI}{2}$$

$$\text{or, } \frac{dI}{I} = -\frac{1}{2}n\,dn \quad (2)$$

Integrating both sides, of (2) we get $\log I = -\frac{1}{2}\frac{n^2}{2} + \log C$

or, $\log I = -\frac{n^2}{4} + \log C = \log e^{-\frac{n^2}{4}} + \log C$

or, $\log I = \log C e^{-\frac{n^2}{4}}$

$$\therefore I = Ce^{-\frac{n^2}{4}}$$

or, $\int_0^\infty e^{-x^2} \cos nx\,dx = Ce^{-\frac{n^2}{4}}$ (3)

Putting n = 0 in (3), we get $\int_0^\infty e^{-x^2}dx = C \therefore C = \frac{\sqrt{\pi}}{2}$

Thus from (3), we have $\int_0^\infty e^{-x^2} \cos nx\,dx = \frac{\sqrt{\pi}}{2} e^{-\frac{n^2}{4}}$

which is the required Fourier cosine transform of e^{-x^2}.

Example 18. Solve the integral equation

$$\int_0^\infty F(x) \cos ux\,dx = \begin{cases} 1-u, & 0 \le u < 1 \\ 0, & u > 1 \end{cases}$$

Hence deduce that $\int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$.

Proof : By definition of the Fourier cosine transform of $F(x)$, the given integral equation gives

$$F_c\{F(x)\} = f_c(u) = \begin{cases} 1-u, & 0 \le u < 1 \\ 0, & u > 1. \end{cases}$$

Using the corresponding inverse Fourier cosine formula, we have

$$F(x) = \frac{2}{\pi} \int_0^\infty f_c(u) \cos ux\,du$$

$$= \frac{2}{\pi}\left[\int_0^1 f_c(u) \cos ux\,du + \int_1^\infty f_c(u) \cos ux\,du\right]$$

$$= \frac{2}{\pi}\left\{\int_0^1 (1-u) \cos ux\,du + \int_1^\infty 0 \cos ux\,du\right\}$$

$$= \frac{2}{\pi}\int_0^1 (1-u) \cos ux\,du + 0$$

$$= \frac{2}{\pi}\left[(1-u)\frac{\sin ux}{x}\right]_0^1 - \frac{2}{\pi}\int_0^1 (0-1)\frac{\sin ux}{x}du$$

$$= 0 + \frac{2}{\pi}\int_0^1 \frac{\sin ux}{x}du$$

$$= \frac{2}{\pi x}\int_0^1 \sin ux\,du$$

$$= \frac{2}{\pi x^2}\left[(-\cos ux)\right]_0^1$$

$$= -\frac{2}{\pi x^2}(\cos x - 1) = \frac{2(1-\cos x)}{\pi x^2}.$$

Hence $F(x) = \frac{2(1-\cos x)}{\pi x^2}$ which is the required solution.

Proof of the second portion

From the given integral equation we have

$$\int_0^\infty F(x) \cos ux\,dx = \begin{cases} 1-u, & 0 \le u < 1. \\ 0, & u > 1 \end{cases}$$

where $F(x) = \frac{2(1-\cos x)}{\pi x^2}$.

$$\therefore \frac{2}{\pi}\int_0^\infty \frac{(1-\cos x)}{x^2} \cos ux\,dx = \begin{cases} 1-u & 0 \le u < 1 \\ 0, & u > 1. \end{cases}$$

Letting $u \to 0$, we get

$$\frac{2}{\pi}\int_0^\infty \frac{1-\cos x}{x^2}dx = 1.$$

$$\text{Or.} \int_0^\infty \frac{1-\cos x}{x^2} = \frac{\pi}{2}$$

$$\text{or,} \int_0^\infty \frac{2\sin^2 x/2}{x^2} dx = \frac{\pi}{2}$$

Putting $x = 2t$ so that $dx = 2dt$, then we have

$$\int_0^\infty \frac{2\sin^2 t}{4t^2} . 2dt = \frac{\pi}{2}$$

$$\text{or,} \int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}.$$

Example 19. If $f(x) = \begin{cases} 1 , & |x| < a \\ 0 , & |x| > a \end{cases}$ and $F(u) = \dfrac{2\sin au}{u}$; $u \neq 0$, then using the Parseval's identity for Fourier transform prove that $\displaystyle\int_0^\infty \frac{\sin^2 ax}{x^2} dx = \frac{\pi a}{2}$.

Proof : From Parseval's identity for Fourier transform, we have $\displaystyle\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(u)|^2 du.$

$$\text{or,} \int_{-\infty}^{-a} |f(x)|^2 dx + \int_{-a}^{a} |f(x)|^2 dx + \int_{a}^{\infty} |f(x)|^2 dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(u)|^2 du.$$

$$\text{or,} \int_{-\infty}^{-a} 0dx + \int_{-a}^{a} 1dx + \int_{a}^{\infty} 0dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4\sin^2 au}{u^2} du$$

$$\text{or, } 0 + \Big[x\Big]_{-a}^{a} + 0 = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 au}{u^2} du.$$

$$\text{or, } 2a = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 au}{u^2} du.$$

$$\text{or, } \pi a = \int_{-\infty}^{\infty} \frac{\sin^2 au du.}{u^2}$$

$$\text{or, } \pi a = 2\int_0^\infty \frac{\sin^2 au}{u^2} du.$$

$$\text{or, } \int_0^\infty \frac{\sin^2 au}{u^2} du = \frac{\pi a}{2}$$

$$\text{or, } \int_0^\infty \frac{\sin^2 ax}{x^2} dx = \frac{\pi a}{2}.$$

4.15. Relation between Fourier and Laplace transforms.

Let us consider the function.

$$f(t) = \begin{cases} e^{-xt} g(t) & , t > 0 \\ 0 & , t < 0 \end{cases} \qquad (1)$$

The Fourier transform of $f(t)$ is given by $F\{f(t)\} = \displaystyle\int_{-\infty}^{\infty} f(t)\, e^{-iut} dt.$

$$= \int_{-\infty}^{0} f(t)\, e^{-iut} dt + \int_0^\infty f(t)\, e^{-iut} dt$$

$$= \int_{-\infty}^{0} 0\, e^{-iut} dt + \int_0^\infty e^{-xt} g(t)\, e^{-iut} dt.$$

$$= 0 + \int_0^\infty e^{-(x+iu)t} g(t)\, dt$$

$$= \int_0^\infty e^{-(x+iu)t} g(t)\, dt$$

Letting $x + iu = s$

$$= \int_0^\infty e^{-st} g(t) dt$$

$$= \mathcal{L}\{g(t)\}$$

Hence the Fourier transform of the function $f(t)$ defined by (1) is the Laplace transform of the function $g(t)$.

4.16. Fourier transform of the derivatives of a function

If $F^n(x)$ is the nth derivative of $F(x)$ and the first (n-1) derivatives of $F(x)$ vanish as $x \to \pm\infty$, then

$$F\{F^n(x)\} = (-iu)^n F\{F(x)\}.$$

4.17. Multiple Fourier transforms.

Let $F(x, y)$ be a function of two variables. Then the complex Fourier transform of $F(x, y)$ is given by

$$\bar{F}(u, v) = \int_{-\infty}^{\infty} F(x, y)\, e^{-i(ux+vy)}\, dx dy$$ and the corresponding inverse Fourier transform is given by

$$F(x, y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \bar{F}(u,v)\, e^{i(ux+vy)}\, dudv.$$

EXERCISES 4(A)

1. Expand the function $f(x) = kx$ in the interval $-\frac{p}{2} < x < \frac{p}{2}$ in a complex Fourier series.

Answer : $f(x) = \frac{kp}{\pi}\left[\sin\frac{2\pi x}{p} - \frac{1}{2}\sin\frac{4\pi x}{p} + \frac{1}{3}\sin\frac{6\pi x}{p} - \frac{1}{4}\sin\frac{8\pi x}{p} + \dots\dots\right]$

2. Find the complex form of the Fourier series of the periodic function $f(x) = \cosh x$ over the interval $-1 < x < 1$.

Answer : $f(x) = \sinh 1 - 2\sinh 1\left[\frac{\cos\pi x}{1+\pi^2} - \frac{\cos 2\pi x}{1+2^2\pi^2} + \frac{\cos 3\pi x}{1+3^2\pi^2} - \dots\dots\right]$

3. Find the Fourier integral of the function

$$f(x) = \begin{cases} 1 & \text{when } |x| < 1 \\ 0 & \text{when } |x| > 1. \end{cases}$$

Answer :

4. By applying the Fourier sine integral formula to the

$$\text{function } f(x) = \begin{cases} 1 & \text{when } 0 < x < k \\ \frac{1}{2} & \text{when } x = k \\ 0 & \text{when } x > k. \end{cases}$$

prove that $f(x) = \frac{2}{\pi}\int_0^{\infty}\frac{1-\cos ku}{u}\sin u x\, du.$

5. Using the Fourier cosine integral formula prove that

$$e^{-x}\cos x = \frac{2}{\pi}\int_0^{\infty}\frac{u^2+2}{u^4+4}\cos ux\, du.$$

6. Prove that $e^{-|x|} = \int_0^{\infty}\frac{\cos ux}{1+u^2}du$ where $-\infty < x < \infty$.

7. Prove that if $f(x) = \sin x$ when $0 \le x \le \pi$ and $f(x) = 0$ when $x < 0$ and when $x > \pi$.

Then $f(x) = \frac{1}{\pi}\int_0^{\infty}\frac{\cos ux + \cos\{u(\pi - x)\}}{1-u^2}du, \ (-\infty < x < \infty)$

and hence prove that $\int_0^{\infty}\frac{\cos\left(\frac{\pi u}{2}\right)}{1-u^2}du = \frac{\pi}{2}.$

8. Find the (a) finite Fourier sine transform and (b) finite Fourier cosine transform of $F(x) = 1$ where $0 < x < l$.

Answers : (a) $\frac{l(1-\cos n\pi)}{n\pi}$.

(b) 0 if $n = 1, 2, 3, \dots$; l if $n = 0$.

9. Find the (a) finite Fourier sine transform and (b) finite Fourier cosine transform of $F(x) = x^2$ where $0 < x < b$.

Answers : (a) $\frac{2b^3}{n^3\pi^3}(\cos n\pi - 1) - \frac{b^3}{n\pi}\cos n\pi$ if $n = 1, 2, 3;$

$\frac{b^3}{3}$ if $n = 0$.

(b) $\frac{2b^3}{n^2\pi^2}(\cos n\pi - 1)$.

10. Find the finite Fourier sine and cosine transform of $f(x) = x$ where $0 < x < \pi$.

Answers : $\frac{\pi(-1)^{n+1}}{n}$, $\frac{(-1)^n - 1}{n^2}$ if $n = 1, 2, 3, \dots$; $\frac{\pi^2}{2}$ if $n = 0$.

11. Find the finite Fourier cosine transform of f(x) where

$$f(x) = \begin{cases} 1, & 0 < x < \pi/2 \\ -1, & \pi/2 < x < \pi \end{cases}$$

Answer : $\frac{2}{n} \operatorname{Sin} \frac{n\pi}{2}$, $n > 0$ and 0 if $n = 0$.

12. Find the finite Fourier sine transform of f(x) if

$$f(x) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 < x < \pi \end{cases}$$

Answer : $\frac{2}{n^2} \sin \frac{n\pi}{2}$.

13. Find the finite Fourier cosine transform of $f(x) = \sin ax$ where $0 < x < \pi$.

Answer. $f_c(n) = 0$ or $\frac{2a}{a^2 - n^2}$ according as $a - n$ is even or odd.

14. Find the Fourier transform of $f(x) = \begin{cases} x, & |x| < a \\ 0, & |x| > a \end{cases}$

Answer : $\frac{2i \sin au - au \cos au}{u^2}$

15. Find the Fourier transform of $f(x) = e^{-\lambda|x|}$ where $\lambda > 0$ and x belongs to $(-\infty, \infty)$.

Answer : $\frac{2\lambda}{\lambda^2 + u^2}$.

16. Find the Fourier cosine transform of a function

$$f(x) = \begin{cases} 1, & 0 < x < a \\ 0, & x \geq a. \end{cases}$$

Answer : $\frac{\sin au}{u}$.

17. What is the function whose Fourier cosine transform is $\frac{\sin au}{u}$?

Answer : [illegible] $= \begin{cases} 1, & x < a \\ 0, & [illegible] \end{cases}$

18. If $f(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 0, & x \geq 1. \end{cases}$

Find the (a) Fourier sine transform and (b) Fourier cosine transform of f(x).

Answer : (a) $\frac{1 - \cos u}{u}$ (b) $\frac{\sin u}{u}$.

19. Find the Fourier transform of $f(x) = \begin{cases} \frac{1}{2a}, & |x| < a \\ 0, & |x| > a. \end{cases}$

Answer : $\frac{\sin ua}{ua}$.

20. Show that the Fourier transform of $e^{-\frac{x^2}{2}}$ is $\sqrt{2\pi}\, e^{-\frac{u^2}{2}}$.

21. Find the Fourier sine transform of $\frac{e^{-ax}}{x}$.

Answer : $\tan^{-1}\left(\frac{u}{a}\right)$.

22. Find the Fourier sine transform of f(x), if

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x > 2. \end{cases}$$

Answer : $\frac{2(1 - \cos u) \sin u}{u^2}$.

23. Find the Fourier cosine transform of $\frac{1}{1 + x^2}$.

Answer : $\frac{\pi}{2} e^{-u}$.

24. Find the Fourier sine transform of $\frac{x}{1 + x^2}$.

Answer : $\frac{\pi}{2} e^{-u}$.

25. Find the Fourier sine transform of $f(x) = \frac{1}{x}$.

Answer : $\frac{\pi}{2}$.

26. Solve for F(x) the integral equation

$$\int_0^{\infty} F(x)\sin xt\,dx = \begin{cases} 1, & 0 \le t < 1 \\ 2, & 1 \le t < 2 \\ 0, & x \ge 2. \end{cases}$$

Answer : $F(x) = \frac{2}{\pi x}(1 + \cos x - 2\cos 2x)$.

27. Using Parseval's identity for Fourier transform prove the followings :

(i) $\displaystyle\int_0^{\infty} \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$.

(ii) $\displaystyle\int_0^{\infty} \frac{\sin at}{t(a^2+t^2)}\,dt = \frac{\pi}{2}\cdot\frac{(1-e^{-a^2})}{a^2}$.

4.18 Applications of Fourier transforms in solving boundary value Problems.

Example 1 (a) . Find the finite Fourier sine transform and the finite Fourier cosine transform of $\frac{\partial U}{\partial x}$ where U is a function of x and t for $0 < x < l$, $t > 0$.

Solution : (i) By def^n of finite Fourier sine transform of F(x),

$0 < x < l$, we have

$$f_s[F(x)] = f_s(n) = \int_0^l F(x)\sin\frac{n\pi x}{l}\,dx.$$

$$\therefore \quad f_s\left(\frac{\partial U}{\partial x}\right) = \int_0^l \frac{\partial U}{\partial x}\sin\frac{n\pi x}{l}\,dx.$$

$$= \left[\sin\frac{n\pi x}{l}.U(x,t)\right]_0^l - \frac{n\pi}{l}\int_0^l \cos\frac{n\pi x}{l}U(x,t)\,dx$$

$$= 0 - \frac{n\pi}{l}\int_0^l U(x,t)\cos\frac{n\pi x}{l}\,dx$$

$$= -\frac{n\pi}{l} f_c\,U(x,t) = -\frac{n\pi}{l} f_c(U) \quad (1)$$

$$\text{Since } f_c(n) = \int_0^l F(x)\cos\frac{n\pi x}{l}\,dx$$

$$\therefore f_c[F(x)] = f_c(n).$$

(ii) By def^n of finite Fourier cosine transform of F(x), $0 < x < l$, we have

$$f_c(n) = \int_0^l F(x)\cos\frac{n\pi x}{l}\,dx,$$

$$\therefore \quad f_c\left\{\frac{\partial U}{\partial x}\right\} = \int_0^l \frac{\partial U}{\partial x}\cos\frac{n\pi x}{l}\,dx$$

$$= \left[\cos\frac{n\pi x}{l}U(x,t)\right]_0^l + \frac{n\pi}{l}\int_0^l U(x,t)\sin\frac{n\pi x}{l}\,dx$$

$$= U(l,t)\cos n\pi - U(0,t) + \frac{n\pi}{l}f_s[U(x,t)]$$

$$= U(l,t)\cos n\pi - U(0,t) + \frac{n\pi}{l}f_s(U) \quad (2)$$

Example 1(b). Find the finite Fourier sine transform and the finite Fourier cosine transform of $\frac{\partial^2 U}{\partial x^2}$ where U is a function of x and t for $0 < x < l$, $t > 0$.

Solution : (iii) Replacing U by $\frac{\partial U}{\partial x}$ in (1)

$$\text{we get } f_s\left\{\frac{\partial^2 U}{\partial x^2}\right\} = -\frac{n\pi}{l} f_c\left\{\frac{\partial U}{\partial x}\right\}$$

$$= -\frac{n\pi}{l}\left[U(l,t)\cos n\pi - U(0,t) + \frac{n\pi}{l}f_s(U)\right]$$

$$= -\frac{n\pi}{l}U(l,t)\cos n\pi + \frac{n\pi}{l}U(0,t) - \frac{n^2\pi^2}{l^2}f_s(U)$$

(iv) Replacing U by $\frac{\partial U}{\partial x}$ in (2), we get

$$f_c\left\{\frac{\partial^2 U}{\partial x^2}\right\} = \frac{\partial U(l,t)}{\partial x}\cos n\pi - \frac{\partial U(o,t)}{\partial x} + \frac{n\pi}{l} f_s\left\{\frac{\partial U}{\partial x}\right\}$$

$$= \frac{\partial U(l,t)}{\partial x}\cos n\pi - \frac{\partial U(o,t)}{\partial x} - \frac{n^2\pi^2}{l^2} f_c(U).$$

Since $f_s\left\{\frac{\partial U}{\partial x}\right\} = -\frac{n\pi}{l} f_c(U)$.

Example 2. Prove that the solution of the boundary value problem $\frac{\partial U}{\partial t} = 3\frac{\partial^2 U}{\partial x^2}$

$U(o,t) = U(2,t) = o,\ t > o$

$U(x, o) = x,\ o < x < 2$

is $U(x, t) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin\frac{n\pi x}{2} e^{-\frac{3}{4}n^2\pi^2 t}$. **D. U. H. 1990**

Proof : The given partial differential equation is

$$\frac{\partial U}{\partial t} = 3\frac{\partial^2 U}{\partial x^2} \quad (1)$$

Taking the finite Fourier sine transform (with $l = 2$) of both sides of (1), we get

$$\int_0^2 \frac{\partial U}{\partial t}\sin\frac{n\pi x}{2}\,dx = \int_0^2 3\frac{\partial^2 U}{\partial x^2}\sin\frac{n\pi x}{2}\,dx \quad (2)$$

Let $u = u(n,t) = \int_0^2 U(x,t)\sin\frac{n\pi x}{2}\,dx$

then $\frac{du}{dt} = \int_0^2 \frac{\partial U}{\partial t}\sin\frac{n\pi x}{2}\,dx$

$$= \int_0^2 3\frac{\partial^2 U}{\partial x^2}\sin\frac{n\pi x}{2}\,dx \text{ using (2)}$$

(on integrating by parts)

$$= 3\left[\sin\frac{n\pi x}{2}\cdot\frac{\partial U}{\partial x}\right]_0^2 - \frac{3n\pi}{2}\int_0^2 \cos\frac{n\pi x}{2}\cdot\frac{\partial U}{\partial x}\,dx$$

$$= 0 - \frac{3n\pi}{2}\left[\cos\frac{n\pi x}{2}\cdot U(x,t)\right]_0^2 - \frac{3n^2\pi^2}{4}\int_0^2 \sin\frac{n\pi x}{2}\cdot U(x,t)\,dx$$

$$= 0 - \frac{3n^2\pi^2}{4}\int_0^2 U(x,t)\sin\frac{n\pi x}{2}\,dx$$

Since $U(o, t) = U(2, t) = 0$

$$= -\frac{3n^2\pi^2}{4}u, \text{ Since } u = \int_0^2 U(x,t)\sin\frac{n\pi x}{2}\,dx$$

$\therefore \frac{du}{dt} = -\frac{3n^2\pi^2}{4}u$ where $u = u(n, t)$.

or, $\frac{du}{u} = -\frac{3n^2\pi^2}{4}dt$

Integrating both sides, we get

$\log u = -\frac{3n^2\pi^2}{4}t + \log A$, where A is an arbitrary constant.

or, $\log u = \log e^{-\frac{3n^2\pi^2 t}{4}} + \log A = \log A e^{-\frac{3n^2\pi^2}{4}t}$

$$\therefore \quad u = u(n, t) = Ae^{-\frac{3n^2\pi^2}{4}t} \quad (3)$$

when $t = 0$, $u(n, o) = Ae^o = A$

$$\therefore \boxed{A = u(n, o)} \quad (4)$$

Now $u(n,t) = \int_0^2 U(x,t)\sin\frac{n\pi x}{2}\,dx$

$$\therefore \quad u(n, o) = \int_0^2 U(x, o)\sin\frac{n\pi x}{2}\,dx$$

$$= \int_0^2 x \sin \frac{n\pi x}{2}\, dx, \text{ Since } U(x, o) = x$$

$$= \left[-\frac{2x}{n\pi}\cos\frac{n\pi x}{2}\right]_0^2 + \frac{2}{n\pi}\int_0^2 \cos\frac{n\pi x}{2}\, dx$$

$$= -\frac{4}{n\pi}\cos n\pi + 0 + \frac{4}{n^2\pi^2}\left[\sin\frac{n\pi x}{2}\right]_0^2$$

$$= -\frac{4}{n\pi}\cos n\pi + 0 = -\frac{4}{n\pi}\cos n\pi.$$

Thus from (4), we have $\boxed{A = -\frac{4}{n\pi}\cos n\pi}$

putting the value of A in (3), we get

$$u(n, t) = -\frac{4}{n\pi}\cos n\pi\, e^{-\frac{3n^2\pi^2}{4}t} \qquad (5)$$

Now taking the inverse finite Fourier sine transform, we get

$$U(x,t) = \frac{2}{2}\sum_{n=1}^{\infty} -\frac{4}{n\pi}\cos n\pi .\, e^{-\frac{3}{4}n^2\pi^2 t} \sin\frac{n\pi x}{2}$$

$$= \sum_{n=1}^{\infty} -\frac{4}{n\pi}(-1)^n e^{-\frac{3}{4}n^2\pi^2 t} \sin\frac{n\pi x}{2}$$

$$= \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi}\sin\frac{n\pi x}{2} .\, e^{-\frac{3}{4}n^2\pi^2 t}$$

which is the required solution.

Example 3. Use finite Fourier transforms to solve

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2},\ U(o, t) = o;$$

$$U(\pi, t) = 0,\ U(x, o) = 2x$$

where $o < x < \pi$, $t > 0$.

D. U. M.SC (F) 1986

Solution : The given partial differential equation is

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \qquad (1)$$

Taking the finite Fourier sine transform of both sides of (1), we get

$$\int_0^\pi \frac{\partial U}{\partial t}\sin nx\, dx = \int_0^\pi \frac{\partial^2 U}{\partial x^2}\sin nx\, dx \qquad (2)$$

Let $u = u(n, t) = \int_0^\pi U(x, t)\sin nx\, dx$

then $\frac{du}{dt} = \int_0^\pi \frac{\partial U}{\partial t}\sin nx\, dx$

$$= \int_0^\pi \frac{\partial^2 U}{\partial x^2}\sin nx\, dx \text{ using (2)}$$

(on integrating by parts)

$$= \left[\sin nx \frac{\partial U}{\partial x}\right]_0^\pi - n\int_0^\pi \cos nx \frac{\partial U}{\partial x}\, dx$$

$$= 0 - n\int_0^\pi \cos nx \frac{\partial U}{\partial x}\, dx.$$

$$= -n\left[\cos nx\, U(x, t)\right]_0^\pi - n^2\int_0^\pi \sin nx\, U(x, t)\, dx$$

$$= 0 - n^2\int_0^\pi U(x, t)\sin nx\, dx, \text{ Since } U(\pi, t)$$

and $U(o, t) = o$.

$$= -n^2 u, \text{ Since } u = \int_0^\pi U(x, t)\sin nx\, dx.$$

$$\therefore \quad \frac{du}{dt} = -n^2 u$$

$$\text{or,} \quad \frac{du}{u} = -n^2 dt$$

Integrating both sides, we get $\log u = -n^2 t + \log A$, A being some constant of integration.

or, $\log u = \log e^{-n^2 t} + \log A = \log A e^{-n^2 t}$

$\therefore \quad u = Ae^{-n^2 t} \quad (3)$

Now $u = u(n, t) = \int_0^\pi U(x, t) \sin nx\, dx$

$$\therefore \quad u(n, o) = \int_0^\pi U(x, o) \sin nx\, dx$$

$$= \int_0^\pi 2x \sin nx\, dx, \text{ Since } U(x, o) = 2x$$

$$= 2\left[-\frac{x\cos nx}{n}\right]_0^\pi + \frac{2}{n}\int_0^\pi \cos nx\, dx$$

$$= -\frac{2\pi}{n}\cos n\pi + 0 + \frac{2}{n^2}[\sin nx]_0^\pi$$

$$= -\frac{2\pi}{n}\cos n\pi \quad \boxed{\therefore u(n, o) = -\frac{2\pi}{n}\cos n\pi}$$

When $t = 0$, $u(n, o) = Ae^0 = A$

$$\therefore \quad \boxed{A = -\frac{2\pi}{n}\cos n\pi}$$

Putting the value of A in (3), we get

$$u(n,t) = u = -\frac{2\pi}{n}\cos n\pi e^{-n^2 t}$$

Applying the inversion formula for finite Fourier sine transform, we get

$$U(x, t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(-\frac{2\pi}{n}\cos n\pi e^{-n^2 t}\right)\sin nx.$$

For physical interpretation, $U(x, t)$ may be regarded as the temperature at any point x at an instant of time t in a solid bounded by the planes $x = o$ and $x = \pi$. The boundary

conditions $U(o, t) = o$ and $U(\pi, t) = o$ give the zero temperature at the ends while $U(x, o) = 2x$ represents that the initial temperature is a function of x.

Example 4. Use finite Fourier transforms to solve

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, \; U(o, t) = o\, ; U(4, t) = o;$$

$$U(x, o) = 2x \text{ where } o < x < 4, \; t > o.$$

Solution : The given partial differential equation is

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \quad (1)$$

Taking the finite Fourier sine transform (with $l = 4$) of both sides of (1), we get

$$\int_0^4 \frac{\partial U}{\partial t}\sin\frac{n\pi x}{4}dx = \int_0^4 \frac{\partial^2 U}{\partial x^2}\sin\frac{n\pi x}{4}dx \quad (2)$$

Let $u = u(n, t) = \int_0^4 U(x, t)\sin\frac{n\pi x}{4}dx.$

Then $\frac{du}{dt} = \int_0^4 \frac{\partial U}{\partial t}\sin\frac{n\pi x}{4}dx$

$$= \int_0^4 \frac{\partial^2 U}{\partial x^2}\sin\frac{n\pi x}{4}dx \text{ using (2)}$$

(on integrating by parts)

$$= \left[\sin\frac{n\pi x}{4}\cdot\frac{\partial U}{\partial x}\right]_0^4 - \frac{n\pi}{4}\int_0^4 \cos\frac{n\pi x}{4}\frac{\partial U}{\partial x}dx$$

$$= 0 - \frac{n\pi}{4}\int_0^4 \cos\frac{n\pi x}{4}\cdot\frac{\partial U}{\partial x}dx$$

$$= -\frac{n\pi}{4}\left[\cos\frac{n\pi x}{4}\cdot U(x,t)\right]_0^4 - \frac{n^2\pi^2}{16}\int_0^4 U(x, t)\sin\frac{n\pi x}{4}dx$$

$$= 0 - \frac{n^2\pi^2}{16}\int_0^4 U(x,t)\sin\frac{n\pi x}{4}dx \text{ Since } U(o,t) = U(4,t) = o$$

$$= -\frac{n^2\pi^2}{16} u, \text{ Since } u = \int_0^4 U(x, t) \sin\frac{n\pi x}{4} dx$$

$$\therefore \quad \frac{du}{dt} = -\frac{n^2\pi^2}{16} u \text{ where } u = u(n, t).$$

or, $\frac{du}{u} = -\frac{n^2\pi^2}{16} dt$

Integrating both sides, we get

$\log \quad u = -\frac{n^2\pi^2 t}{16} + \log A$, A being some constant of integration.

or, $\log u = \log e^{-\frac{n^2\pi^2 t}{16}} + \log A = \log A e^{-\frac{n^2\pi^2 t}{16}}$

$$\therefore \quad u = A e^{-\frac{n^2\pi^2 t}{16}} \quad (3)$$

or, $u(n, t) = A e^{-\frac{n^2\pi^2 t}{16}}$

When $t = o$, $u(n, o) = A e^0 = A$

$$\therefore \quad \boxed{A = u(n, o)} \quad (4)$$

Now $u(n, t) = \int_0^4 U(x, t) \sin\frac{n\pi x}{4} dx$

$$\therefore \quad u(n, o) = \int_0^4 U(x, o) \sin\frac{n\pi x}{4} dx$$

$$= \int_0^4 2x \sin\frac{n\pi x}{4} dx$$

$$= \left[-2x\frac{4}{n\pi}\cos\frac{n\pi x}{4}\right]_0^4 + \frac{8}{n\pi}\int_0^4 \cos\frac{n\pi x}{4} dx$$

$$= -\frac{32}{n\pi}\cos n\pi + 0 + \frac{32}{n^2\pi^2}\left[\sin\frac{n\pi x}{4}\right]_0^4$$

$$= -\frac{32}{n\pi}\cos n\pi$$

Thus from (4), we have $\boxed{A = -\frac{32}{n\pi}\cos n\pi}$

Putting the value of A in (3), we get

$$u(n, t) = -\frac{32}{n\pi}\cos n\pi \, e^{-\frac{n^2\pi^2 t}{16}} \quad (5)$$

Now applying the inversion formula for finite Fourier sine transform, we get

$$U(x, t) = \frac{2}{4}\sum_{n=1}^{\infty} -\frac{32}{n\pi}\cos n\pi \, e^{-\frac{n^2\pi^2 t}{16}} . \sin\frac{n\pi x}{4}$$

$$= \frac{16}{\pi}\sum_{n=1}^{\infty} (-1)^{n+1}\frac{e^{-\frac{n^2\pi^2 t}{16}}}{n} . \sin\frac{n\pi x}{4}$$

which is the required solution.

Physical interpretation :

Physically, $U(x, t)$ represents the temperature at any point x at any time t in solid bounded by the planes $x = o$ and $x = 4$ (or a bar on the x – axis with the ends $x = o$ and $x = 4$, whose surface is insulated laterally).The condition $U(o, t) = o$ and $U(4, t) = o$ implies that the ends are kept at zero temperature while $U(x, o) = 2x$ implies that the initial temperature is a function of x.

Example 5. Solve $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$, $0 < x < 6$, $t > 0$, subject to the conditions $U(0, t) = 0$, $U(6, t) = 0$, $U(x, 0) = \begin{cases} 1, & 0 < x < 3 \\ 0, & 3 < x < 6 \end{cases}$

and interpret physically.

Solution : The given partial differential equation is $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$ (1)

Taking the finite Fourier sine transform (with $l = 6$) of both sides of (1), we get

$$\int_0^6 \frac{\partial U}{\partial t} \sin\frac{n\pi x}{6} dx = \int_0^6 \frac{\partial^2 U}{\partial x^2} \sin\frac{n\pi x}{6} dx \quad (2)$$

Let $u = u(n, t) = \int_0^6 U(x, t) \sin\frac{n\pi x}{6} dx$

Then $\frac{du}{dt} = \int_0^6 \frac{\partial U}{\partial t} \sin\frac{n\pi x}{6} dx$

$$= \int_0^6 \frac{\partial^2 U}{\partial x^2} \sin\frac{n\pi x}{6} dx \text{ using (2)}$$

(On integrating by parts)

$$= \left[\sin\frac{n\pi x}{6} \cdot \frac{\partial U}{\partial x}\right]_0^6 - \frac{n\pi}{6}\int_0^6 \cos\frac{n\pi x}{6} \cdot \frac{\partial U}{\partial x} dx$$

$$= 0 - \frac{n\pi}{6}\left[\cos\frac{n\pi x}{6} \cdot U(x, t)\right]_0^6 - \frac{n^2\pi^2}{36}\int_0^6 \sin\frac{n\pi x}{6} U(x, t)\, dx$$

$$= 0 - \frac{n\pi}{6}\left[\cos n\pi\, U(6, t) - U(0, t)\right] - \frac{n^2\pi^2}{36}\int_0^6 U(x, t) \sin\frac{n\pi x}{6} dx$$

$$= 0 - \frac{n^2\pi^2}{36}\int_0^6 U(x, t) \sin\frac{n\pi x}{6} dx, \text{ Since } U(6,t) = U(0,t) = 0$$

$$= -\frac{n^2\pi^2}{36} u, \text{ Since } u = \int_0^6 U(x,t) \sin\frac{n\pi x}{6} dx$$

$\therefore \frac{du}{dt} = -\frac{n^2\pi^2}{36} u$, where $u = u(n, t)$

or, $\frac{du}{u} = -\frac{n^2\pi^2}{36} dt$

Integrating both sides, we get

$\log u = -\frac{n^2\pi^2}{36} t + \log A$, A being some constant of integration

or, $\log u = \log e^{-\frac{n^2\pi^2 t}{36}} + \log A = \log A e^{-\frac{n^2\pi^2 t}{36}}$.

$$\therefore \quad u = A e^{-\frac{n^2\pi^2 t}{36}} \quad (3)$$

When $t = 0$, $u(n, 0) = Ae^0 = A$

$\therefore \boxed{A = u(n, 0)}$ (4)

Now $u(n, t) = \int_0^6 U(x, t) \sin\frac{n\pi x}{6} dx$

$$\therefore \quad u(n, 0) = \int_0^6 U(x, 0) \sin\frac{n\pi x}{6} dx$$

$$= \int_0^3 U(x, 0) \sin\frac{n\pi x}{6} dx + \int_3^6 U(x, 0) \sin\frac{n\pi x}{6} dx$$

$$= \int_0^3 1. \sin\frac{n\pi x}{6} dx + \int_3^6 0. \sin\frac{n\pi x}{6} dx$$

$$= \int_0^3 \sin\frac{n\pi x}{6} dx + 0$$

$$= -\frac{6}{n\pi}\left[\cos\frac{n\pi x}{6}\right]_0^3$$

$$= -\frac{6}{n\pi}\left[\cos\frac{n\pi}{2} - 1\right]$$

$$= \frac{6}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right).$$

Thus from (4), we have

$$A=\frac{6}{n\pi}\left(1-\cos\frac{n\pi}{2}\right) \quad (5)$$

Putting the value of A in (3), we get

$$u(n,t)=\frac{6}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)e^{-\frac{n^2\pi^2t}{36}}$$

Taking the inverse Fourier sine transform we get

$$U(x,t)=\frac{2}{6}\sum_{n=1}^{\infty}\frac{6}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)e^{\frac{-n^2\pi^2t}{36}}.\sin\frac{n\pi x}{6}$$

$$\text{or,}\quad U(x,t)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1-\cos\frac{n\pi}{2}\right)e^{-\frac{n^2\pi^2t}{36}}.\sin\frac{n\pi x}{6}.$$

Physical interpretation

Physically $U(x, t)$ represents the temperature at any point x at any time t in a bar with the ends $x = 0$ and $x = 6$ kept at zero temperature which is insulated laterally. Initially the temperature in the half bar from $x = 0$ to $x = 3$ is constant equal to 1 unit while the half bar from $x = 3$ to $x = 6$ is at zero temperature.

Example 6 : Solve $\frac{\partial U}{\partial t}=\frac{\partial^2 U}{\partial x^2}, x>0, t>0$

Subject to the conditions $U(0, t) = 0$,

$$U(x,0)=\begin{cases}1 & 0<x<1\\ 0 & x\geq 1\end{cases}$$

and $U(x, t)$ is bounded.

Solution : Given partial differential equation is $\frac{\partial U}{\partial t}=\frac{\partial^2 U}{\partial x^2}$ (1)

Taking the Fourier sine transform of both sides of (1), we get

$$\int_0^{\infty}\frac{\partial U}{\partial t}\sin nx\,dx=\int_0^{\infty}\frac{\partial^2 U}{\partial x^2}\sin nx\,dx \quad (2)$$

Let $u=u(n,t)=\int_0^{\infty}U(x,t)\sin nx\,dx$

$$\text{then }\frac{du}{dt}=\int_0^{\infty}\frac{\partial U(x,t)}{\partial t}\sin nx\,dx$$

$$=\int_0^{\infty}\frac{\partial^2 U}{\partial x^2}\sin nx\,dx \text{ by (2)}$$

(on integrating by parts)

$$=\left[\sin nx\frac{\partial U}{\partial x}\right]_0^{\infty}-n\int_0^{\infty}\cos nx\frac{\partial U}{\partial x}dx$$

$$=0-n\int_0^{\infty}\cos nx\frac{\partial U}{\partial x}dx \quad \text{Since }\frac{\partial U}{\partial x}\to 0 \text{ as } x\to\infty$$

$$=-n\left[\cos nx\,U(x,t)\right]_0^{\infty}-n^2\int_0^{\infty}\sin nx\,U(x,t)\,dx.$$

$$=-n\left[0-U(0,t)\right]-n^2u \quad \text{Since } U\to 0 \text{ as } x\to\infty$$

$$=n\,U(0,t)-n^2u.$$

$$\therefore\quad \frac{du}{dt}=n\,U(0,t)-n^2u \quad (3)$$

From the given condition, we have $U(0, t) = 0$

$\therefore$ from (2), we have $\frac{du}{dt}=-n^2u$

or, $\frac{du}{u}=-n^2\,dt$

Integrating both sides, we have

$\log u=-n^2t+\log A$, A being some constant of integration.

or, $\log u=\log e^{-n^2t}+\log A=\log A\,e^{-n^2t}$

$$\therefore\quad u=Ae^{-n^2t} \quad (4)$$

Now $u(n, t) = \int_0^{\infty} U(x, t) \sin nx\,dx$

$$\therefore \quad u(n, o) = \int_0^{\infty} U(x, o) \sin nx\,dx$$

$$= \int_0^1 U(x, o) \sin nx\,dx + \int_1^{\infty} U(x, o) \sin nx\,dx$$

$$= \int_0^1 1.\sin nx\,dx + o \text{ Since } U(x, o) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \geq 1 \end{cases}$$

$$= \int_0^1 \sin nx\,dx = -\frac{1}{n}[\cos nx]_0^1$$

$$= -\frac{1}{n}(\cos n - \cos o)$$

$$= \frac{1}{n}(1 - \cos n)$$

Therefore initially, when $t = o$, $u(n, t) = u(n,o) = \dfrac{1 - \cos n}{n}$

Thus from (4), we get

$$\frac{1 - \cos n}{n} = Ae^0 = A \quad \therefore \boxed{A = \frac{1 - \cos n}{n}}$$

putting the value of A in (4), we get

$$u = u(n, t) = \frac{1 - \cos n}{n} e^{-n^2 t}$$

Note : Inverse Fourier sine transform of $f_s(n)$ is defined as

$$F(x) = \frac{2}{\pi}\int_0^{\infty} f_s(n) \sin nx\,dn$$

Now applying the inversion formula for Fourier sine transform, we have

$$U(x, t) = \frac{2}{\pi}\int_0^{\infty} u(n, t) \sin nx\,dn$$

$$= \frac{2}{\pi}\int_0^{\infty} \frac{1 - \cos n}{n} e^{-n^2 t} \sin nx\,dn$$

which gives the required solution, physically interpreted as the temperature at any point x at any time t in a solid $x > o$.

Example 7. Solve the boundary value problem $\dfrac{\partial U}{\partial t} = \dfrac{\partial^2 U}{\partial x^2}$,

$U(o, t) = 1$, $U(\pi, t) = 3$, $U(x, o) = 2$, where $o < x < \pi$, $t > o$.

Solution : The given partial differential equation is

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \quad . (1)$$

Taking the finite Fourier sine transform of both sides of (1), we get

$$\int_0^{\pi} \frac{\partial U}{\partial t} \sin nx\,dx = \int_0^{\pi} \frac{\partial^2 U}{\partial x^2} \sin nx\,dx \quad (2)$$

Let $u = u(n, t) = \int_0^{\pi} U(x, t) \sin nx\,dx$

then $\dfrac{du}{dt} = \displaystyle\int_0^{\pi} \frac{\partial U}{\partial t} \sin nx\,dx$

$$= \int_0^{\pi} \frac{\partial^2 U}{\partial x^2} \sin nx\,dx \quad \text{using (2)}$$

(on integrating by parts)

$$= \left[\sin nx \frac{\partial U}{\partial x}\right]_0^{\pi} - n\int_0^{\pi} \cos nx \frac{\partial U}{\partial x}\,dx$$

$$= 0 - n\,[\cos nx\,U(x, t)]_0^{\pi} - n^2\int_0^{\pi} \sin nx\,U(x, t)\,dx$$

$$= -n\,[\cos n\pi\,U(\pi, t) - U(o, t)] - n^2\int_0^{\pi} U(x, t) \sin nx\,dx$$

$$= -n\,[3\cos n\pi - 1] - n^2 u$$

$$\therefore \quad \frac{du}{dt} = n\,(1 - 3\cos n\pi) - n^2 u$$

$$\text{or,} \quad \frac{du}{dt} + n^2 u = n(1 - 3\cos n\pi) \quad (3)$$

which is a linear differential equation of first order.

$$I.F = e^{\int n^2 dt} = e^{n^2 t}.$$

Therefore, solution of (3) is

$$ue^{n^2t} = n(1-3\cos n\pi)\int e^{n^2t}\,dt$$

$$= \frac{n(1-3\cos n\pi)}{n^2}e^{n^2t} + A$$

$$= \frac{(1-3\cos n\pi)}{n}e^{n^2t} + A.$$

or, $u = u(n, t) = \frac{1-3\cos n\pi}{n} + Ae^{-n^2t}$ (4)

When $t = o$, $u(n, o) = \frac{1-3\cos n\pi}{n} + A$ (5)

$$u = u(n, t) = \int_0^\pi U(x, t)\sin nx\,dx$$

$$\therefore \quad u(n, o) = \int_0^\pi U(x, o)\sin nx\,dx$$

$$= \int_0^\pi 2\sin nx\,dx$$

$$= \frac{-2}{n}\left[\cos nx\right]_0^\pi$$

$$= -\frac{2}{n}(\cos n\pi - 1) = \frac{2}{n}(1-\cos n\pi)$$

Thus from (5), we get

$$\frac{2}{n}(1-\cos n\pi) = \frac{1-3\cos n\pi}{n} + A$$

$$\therefore \quad A = \frac{1}{n}(2 - 2\cos n\pi - 1 + 3\cos n\pi)$$

or, $A = \frac{1}{n}(1+\cos n\pi)$

putting the value of A in (4), we get

$$u = (n, t) = \frac{1-3\cos n\pi}{n} + \frac{1}{n}(1+\cos n\pi)e^{-n^2t}$$

Taking inverse finite Fourier sine transform we have

$$U(x, t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-3\cos n\pi}{n}\sin nx$$

$$+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(1+\cos n\pi)e^{-n^2t}\sin nx.$$

Example 8. Solve the boundary value problem

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2},\ U(o, t) = 1,\ U(\pi, t) = 3$$

$U(x, o) = 1$, where $o < x < \pi$, $t > o$. **D. U. H. 1990, 1993**

Solution : The given partial differential equation is

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \quad (1)$$

Taking the finite Fourier sine transform of both sides of (1), we get

$$\int_0^\pi \frac{\partial U}{\partial t}\sin nx\,dx = \int_0^\pi \frac{\partial^2 U}{\partial x^2}\sin nx\,dx \quad (2)$$

Let $u = u(n, t) = \int_0^\pi U(x, t)\sin nx\,dx$

then $\frac{du}{dt} = \int_0^\pi \frac{\partial U}{\partial t}\sin nx\,dx$

$$= \int_0^\pi \frac{\partial^2 U}{\partial x^2}\sin nx\,dx \quad \text{using (2)}$$

(on integrating by parts)

$$= \left[\sin nx\frac{\partial U}{\partial x}\right]_0^\pi - n\int_0^\pi \cos nx\frac{\partial U}{\partial x}\,dx$$

$$= 0 - n\left[\cos nx\, U(x, t)\right]_0^\pi - n^2\int_0^\pi \sin nx\, U(x, t)\,dx$$

$$= -n\left[\cos n\pi\, U(\pi, t) - U(o, t)\right] - n^2\int_0^\pi U(x, t)\sin nx\,dx$$

$= -n(3\cos n\pi - 1) - n^2u$

$= n(1 - 3\cos n\pi) - n^2u.$

or, $\frac{du}{dt} = n(1 - 3\cos n\pi) - n^2u$

or, $\frac{du}{dt} + n^2u = n(1 - 3\cos n\pi)$ (3)

which is a linear differential equation of first order.

I. F = $e^{\int n^2 dt} = e^{n^2t}$.

Therefore, solution of (3) is

$ue^{n^2t} = n(1 - 3\cos n\pi)\int e^{n^2t}\,dt$

$= \frac{n(1 - 3\cos n\pi)}{n^2}e^{n^2t} + A$

or, $u = u(n, t) = \frac{1}{n}(1 - 3\cos n\pi) + Ae^{-n^2t}$ (4)

When $t = o$, $u(n, o) = \int_0^\pi U(x, o)\sin nx\,dx$

$= \int_0^\pi 1.\sin nx\,dx$

$= -\frac{1}{n}[\cos nx]_0^\pi$

$= -\frac{1}{n}(\cos n\pi - 1)$

$= \frac{1}{n}(1 - \cos n\pi).$

Again, when $t = o$, from (4), we get

$u(n, o) = \frac{1}{n}(1 - 3\cos n\pi) + A$

$\therefore \frac{1}{n}(1 - \cos n\pi) = \frac{1}{n}(1 - 3\cos n\pi) + A.$

or, $A = \frac{1}{n}(1 - \cos n\pi - 1 + 3\cos n\pi) = \frac{2\cos n\pi}{n}$

putting the value of A in (4), we get

$u = u(n, t) = \frac{1}{n}(1 - 3\cos n\pi) + \frac{2\cos n\pi}{n}e^{-n^2t}$

Taking inverse finite Fourier sine transform, we get

$U(x, t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(1 - 3\cos n\pi)}{n}\sin nx$

$+ \sum_{n=1}^{\infty}\frac{2\cos n\pi}{n}e^{-n^2t}\sin nx.$

$= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(1 - 3\cos n\pi)}{n}\sin nx + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\pi}{n}e^{-n^2t}\sin nx$ (5)

The finite Fourier sine transform of F(x), $0 < x < l$, is defined as $f_s(n) = \int_0^l F(x)\sin\frac{n\pi x}{l}\,dx.$

Here $l = \pi$

$\therefore f_s(n) = \int_0^\pi F(x)\sin nx\,dx$

$f_s\{F(x)\} = \int_0^\pi F(x)\sin nx\,dx.$

$\boxed{F(x) = 1}$ $f_s\{1\} = \int_0^\pi 1\sin x\,dx$

$= -\frac{1}{n}[\cos nx]_0^\pi$

$= -\frac{1}{n}(\cos n\pi - 1) = \frac{1}{\pi}(1 - \cos n\pi).$

$\therefore$ Taking inverse finite Fourier sine transform, we g

$1 = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1 - \cos n\pi}{n}\sin nx.$

$\boxed{F(x) = x}$ $f_s\{x\} = \int_0^\pi x\sin nx\,dx$

$$= \left[-x\frac{\cos nx}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}\,dx$$

$$= -\frac{\pi\cos n\pi}{n} + 0 + \frac{1}{n^2}\left[\sin nx\right]_0^{\pi}$$

$$= -\frac{\pi\cos n\pi}{n}$$

Taking inverse finite Fourier sine transform we get

$$x = \frac{2}{\pi}\sum_{n=1}^{\infty} -\frac{\pi\cos n\pi}{n}\sin nx$$

Therefore, $\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-3\cos n\pi}{n}\sin nx$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos n\pi}{n}\sin nx + \frac{2}{\pi}\sum_{n=1}^{\infty} -\frac{2\cos n\pi}{n}\sin nx$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos n\pi}{n}\sin nx + \frac{2}{\pi}\cdot\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-\pi\cos n\pi}{n}\sin nx$$

$$= 1 + \frac{2}{\pi}\cdot x$$

Thus $\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-3\cos n\pi}{n}\sin nx = 1 + \frac{2x}{\pi}$

Hence from (5), we have

$$U(x,t) = 1 + \frac{2x}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\pi}{n}e^{-n^2t}\sin nx$$

$$= 1 + \frac{2x}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2t}\sin nx.$$

Important notes

(A) Fourier's integral theorem can be written in the form.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{i\alpha(x-u)}\,du\,d\alpha.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x}\,d\alpha\int_{-\infty}^{\infty} f(u)\,e^{-i\alpha u}\,du.$$

Then $F(\alpha) = \int_{-\infty}^{\infty} f(u)\,e^{-i\alpha u}\,du$ (1)

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{i\alpha x}\,d\alpha \quad (2)$$

The function $F(\alpha)$ is called the **Fourier transform** of $f(x)$ and is some times written as $F(\alpha) = \mathfrak{F}\{f(x)\}$.

The function $f(x)$ is called the **Inverse Fourier transform** of $F(\alpha)$ & is written as $f(x) = \mathfrak{F}^{-1}\{F(\alpha)\}$.

(B) Convolution theorem

$\mathfrak{F}(f * g) = \mathfrak{F}\{f\}\,\mathfrak{F}\{g\} = F(\alpha)\,G(\alpha)$ where

$$f * g = \int_{-\infty}^{\infty} f(u)\,g(x-u)\,du.$$

Example 9. Use the complex form of the Fourier transform to solve the boundary value problem

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2},\; U(x,0) = f(x),$$

$$|U(x,t)| < M, \text{ where } -\infty < x < \infty,\; t > 0.$$

Solution : The given partial differential equation is

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \quad (1)$$

Taking the complex form of the Fourier transform with respect to x of both sides of (1) we get

$$\int_{-\infty}^{\infty} \frac{\partial U}{\partial t} e^{i\alpha x} dx = \int_{-\infty}^{\infty} \frac{\partial^2 U}{\partial x^2} e^{i\alpha x} dx \quad (2)$$

$$\text{Let } \mathcal{F}(U) = \int_{-\infty}^{\infty} U(x,t)\, e^{i\alpha x}\, dx$$

$$\therefore \quad \frac{d}{dt}\mathcal{F}(U) = \int_{-\infty}^{\infty} \frac{\partial U}{\partial t} e^{i\alpha x} dx$$

$$= \int_{-\infty}^{\infty} \frac{\partial^2 U}{\partial x^2} e^{i\alpha x} dx \text{ using (2)}$$

$$= \left[e^{i\alpha x}\frac{\partial U}{\partial x}\right]_{-\infty}^{\infty} - i\alpha \int_{-\infty}^{\infty} e^{i\alpha x}\frac{\partial U}{\partial x} dx$$

$$= 0 - i\alpha \int_{-\infty}^{\infty} e^{i\alpha x}\frac{\partial U}{\partial x} dx, \text{ since } \frac{\partial U}{\partial x} \to 0 \text{ as } x \to \infty,$$

$$= -i\alpha\left[e^{i\alpha x}\, U(x,t)\right]_{-\infty}^{\infty} + i^2\alpha^2 \int_{-\infty}^{\infty} e^{i\alpha x}\, U(x,t)\, dx$$

$$= 0 - \alpha^2 \int_{-\infty}^{\infty} e^{i\alpha x}\, U(x,t) dx \text{ since } U \to 0 \text{ as } x \to \infty$$

$$= -\alpha^2 \mathcal{F}(U).$$

$$\therefore \quad \frac{d\mathcal{F}(U)}{dt} = -\alpha^2 \mathcal{F}(U)$$

$$\text{or,} \quad \frac{d\mathcal{F}(U)}{F(U)} = -\alpha^2 dt \quad (3)$$

Integrating both sides of (3), we get

$\log \mathcal{F}(U) = -\alpha^2 t + \log C$, C being some constant of integration.

$$\text{or,} \quad \log \mathcal{F}(U) = \log e^{-\alpha^2 t} + \log C = \log Ce^{-\alpha^2 t}$$

$$\therefore \mathcal{F}(U) = Ce^{-\alpha^2 t} \quad (4)$$

$$\text{i.e. } \mathcal{F}[U(x,t)] = Ce^{-\alpha^2 t} \quad (5)$$

When $t = 0$, $U(x, o) = f(x)$

$$\therefore \quad \mathcal{F}[U(x, o)] = \mathcal{F}[f(x)] = Ce^{0} = C$$

$$\therefore C = \mathcal{F}[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx.$$

$$= \int_{-\infty}^{\infty} f(u)\, e^{i\alpha u}\, du.$$

Thus from (4), we have $\mathcal{F}[U(x,t)] = \mathcal{F}[f(x)]\, e^{-\alpha^2 t}$ (6)

Taking the inverse Fourier transform, we get

$$U(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}(f)\, e^{-\alpha^2 t}\, e^{-i\alpha x}\, d\alpha.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx\right] e^{-\alpha^2 t}\, e^{-i\alpha x}\, d\alpha.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)\, e^{i\alpha u}\, du\right] e^{-\alpha^2 t}\, e^{-i\alpha x}\, d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-\alpha^2 t - i\alpha(x-u)}\, d\alpha\right] du.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-t\left\{\alpha + \frac{i(x-u)}{2t}\right\}^2 - \frac{(x-u)^2}{4t}}\, d\alpha\right] du.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-t\left\{\alpha + \frac{i(x-u)}{2t}\right\}^2}\, d\alpha\right] e^{\frac{-(x-u)^2}{4t}}\, du \quad (7)$$

Taking the complex form of the Fourier transform with respect to x of both sides of (1) we get

$$\int_{-\infty}^{\infty} \frac{\partial U}{\partial t} e^{i\alpha x} dx = \int_{-\infty}^{\infty} \frac{\partial^2 U}{\partial x^2} e^{i\alpha x} dx \quad (2)$$

Let $\mathcal{F}(U) = \int_{-\infty}^{\infty} U(x,t)\, e^{i\alpha x}\, dx$

$$\therefore \quad \frac{d}{dt}\mathcal{F}(U) = \int_{-\infty}^{\infty} \frac{\partial U}{\partial t} e^{i\alpha x} dx$$

$$= \int_{-\infty}^{\infty} \frac{\partial^2 U}{\partial x^2} e^{i\alpha x} dx \text{ using (2)}$$

$$= \left[e^{i\alpha x}\frac{\partial U}{\partial x}\right]_{-\infty}^{\infty} - i\alpha \int_{-\infty}^{\infty} e^{i\alpha x}\frac{\partial U}{\partial x} dx.$$

$$= 0 - i\alpha \int_{-\infty}^{\infty} e^{i\alpha x}\frac{\partial U}{\partial x} dx, \text{ since } \frac{\partial U}{\partial x} \to 0 \text{ as } x \to \infty,$$

$$= -i\alpha \left[e^{i\alpha x}\, U(x,t)\right]_{-\infty}^{\infty} + i^2\alpha^2 \int_{-\infty}^{\infty} e^{i\alpha x}\, U(x,t)\, dx.$$

$$= 0 - \alpha^2 \int_{-\infty}^{\infty} e^{i\alpha x}\, U(x,t) dx \text{ since } U \to 0 \text{ as } x \to \infty$$

$$= -\alpha^2 \mathcal{F}(U).$$

$$\therefore \quad \frac{d\mathcal{F}(U)}{dt} = -\alpha^2 \mathcal{F}(U)$$

or, $\quad \dfrac{d\mathcal{F}(U)}{F(U)} = -\alpha^2 dt \quad (3)$

Integrating both sides of (3), we get

$\log \mathcal{F}(U) = -\alpha^2 t + \log C$, C being some constant of integration.

or, $\log \mathcal{F}(U) = \log e^{-\alpha^2 t} + \log C = \log Ce^{-\alpha^2 t}$

$\therefore \mathcal{F}(U) = Ce^{-\alpha^2 t} \quad (4)$

i.e $\mathcal{F}\{U(x,t)\} = Ce^{-\alpha^2 t} \quad (5)$

When $t = 0$, $U(x, o) = f(x)$

$\therefore \quad \mathcal{F}\{U(x, o)\} = \mathcal{F}\{f(x)\} = Ce^0 = C$

$$\therefore C = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx.$$

$$= \int_{-\infty}^{\infty} f(u)\, e^{i\alpha u}\, du.$$

Thus from (4), we have $\mathcal{F}\{U(x, t)\} = \mathcal{F}\{f(x)\}\, e^{-\alpha^2 t} \quad (6)$

Taking the inverse Fourier transform, we get

$$U(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}(f)\, e^{-\alpha^2 t}\, e^{-i\alpha x}\, d\alpha.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx\right] e^{-\alpha^2 t}\, e^{-i\alpha x}\, d\alpha.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)\, e^{i\alpha u}\, du\right] e^{-\alpha^2 t}\, e^{-i\alpha x}\, d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-\alpha^2 t - i\alpha(x-u)}\, d\alpha\right] du.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-t\left\{\alpha + \frac{i(x-u)}{2t}\right\}^2 - \frac{(x-u)^2}{4t}}\, d\alpha\right] du.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-t\left\{\alpha + \frac{i(x-u)}{2t}\right\}^2}\, d\alpha\right] e^{\frac{-(x-u)^2}{4t}}\, du \quad (7)$$

Putting $\sqrt{t}\left\{\alpha + \frac{i(x-u)}{2t}\right\} = y$

so that $\sqrt{t}\, d\alpha = dy$

$\therefore d\alpha = \frac{1}{\sqrt{t}} dy.$

Limits $\left.\begin{matrix}\alpha = \infty \\ y = \infty\end{matrix}\right\}$ $\left.\begin{matrix}\alpha = -\infty \\ y = -\infty\end{matrix}\right\}$

$$\therefore \int_{-\infty}^{\infty} e^{-t\left\{\alpha + \frac{i(x-u)}{2t}\right\}^2} d\alpha.$$

$$= 2\int_0^{\infty} e^{-\left[\sqrt{t}\left\{\alpha + \frac{i(x-u)}{2t}\right\}\right]^2} d\alpha.$$

$$= 2\int_0^{\infty} e^{-y^2} \frac{1}{\sqrt{t}} dy.$$

$$= \frac{2}{\sqrt{t}}\int_0^{\infty} e^{-y^2} dy = \frac{2}{\sqrt{t}}\cdot\frac{\sqrt{\pi}}{2} = \sqrt{\frac{\pi}{t}}.$$

Since $\int_0^{\infty} e^{-y^2} dy = \frac{\sqrt{\pi}}{2}.$

Therefore, from (7), we get

$$U(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\cdot\frac{\sqrt{\pi}}{\sqrt{t}}\cdot e^{-\frac{(x-u)^2}{4t}} du.$$

$$= \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} f(u)\cdot e^{\frac{-(x-u)^2}{4t}} du.$$

Example 10. Use the complex form of the Fourier transform to solve the boundary value problem $\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}$, $U(x, o) = f(x)$, $|U(x, t)| < M$,

where $-\infty \angle x \angle \infty$, $t > 0$.

Solution : The given partial differential equation is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} \quad (1)$$

Taking the complex form of the Fourier transform with respect to x of both sides of (1), we get

$$\int_{-\infty}^{\infty}\frac{\partial U}{\partial t}e^{i\alpha x}\cdot dx = \int_{-\infty}^{\infty} k\frac{\partial^2 U}{\partial x^2}e^{i\alpha x} dx \quad (2)$$

Let $\mathfrak{F}(U) = \int_{-\infty}^{\infty} U(x, t)\, e^{i\alpha x} dx.$

$$\therefore \frac{dF(U)}{dt} = \int_{-\infty}^{\infty}\frac{\partial U}{\partial t}e^{i\alpha x} dx$$

$$= \int_{-\infty}^{\infty} k\frac{\partial^2 U}{\partial x^2}e^{i\alpha x} dx \text{ using (2)}$$

$$= k\left[e^{i\alpha x}\frac{\partial U}{\partial x}\right]_{-\infty}^{\infty} - k\, i\alpha\int_{-\infty}^{\infty} e^{i\alpha x}\frac{\partial U}{\partial x} dx.$$

$$= 0 - ki\alpha\int_{-\infty}^{\infty} e^{i\alpha x}\frac{\partial U}{\partial x} dx.$$

Since $\frac{\partial U}{\partial x} \to 0$ as $x \to \infty$

$$= -ki\alpha\left[e^{i\alpha x}\, U(x, t)\right]_{-\infty}^{\infty} + ki^2\alpha^2\int_{-\infty}^{\infty} e^{i\alpha x}\, U(x, t)\, dx.$$

$$= 0 - k\alpha^2\int_{-\infty}^{\infty} e^{i\alpha x}\, U(x, t)\, dx.$$

Since $U \to 0$ as $x \to \infty$, and $i^2 = -1$

$$= -k\alpha^2\int_{-\infty}^{\infty} U(x, t)\, e^{i\alpha x} dx$$

$$= -k\alpha^2\, \mathfrak{F}(U)$$

$$\therefore \quad \frac{d\mathcal{F}(U)}{dt} = -k\alpha^2 \mathcal{F}(U)$$

$$\text{or,} \quad \frac{d\mathcal{F}(U)}{\mathcal{F}(U)} = -k\alpha^2 dt \quad (3)$$

Integrating both sides of (3), we get

$\log \mathcal{F}(U) = -k\alpha^2 t + \log C$, C being some constant of integration.

$$\text{or,} \quad \log \mathcal{F}(U) = \log e^{-k\alpha^2 t} + \log C = \log Ce^{-k\alpha^2 t}$$

$$\therefore \mathcal{F}(U) = Ce^{-k\alpha^2 t} \quad (4)$$

That is, $\mathcal{F}\{U(x,t)\} = = Ce^{-k\alpha^2 t}$ (5)

When $t = o$, $U(x, o) = f(x)$

$$\therefore \mathcal{F}[U(x, o)] = \mathcal{F}\{f(x)\} = Ce^{0} = C$$

$$\text{Therefore, } C = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} U(x, o)\, e^{i\alpha x}\, dx$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx$$

$$= \int_{-\infty}^{\infty} f(u)\, e^{i\alpha u}\, du.$$

Thus from (4) we have

$$\mathcal{F}\{U(x,t)\} = \mathcal{F}\{f(x)\}\, e^{-k\alpha^2 t} \quad (6)$$

Taking the inverse Fourier transform, we get

$$U(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}\{f\} e^{-k\alpha^2 t}\, e^{-i\alpha x}\, d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)\, e^{i\alpha u} du\right] e^{-k\alpha^2 t}\, e^{-i\alpha x}\, d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-k\alpha^2 t - i\alpha(x-u)}\, d\alpha\right] du.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-t\left[\sqrt{k}\alpha + \frac{i(x-u)}{2\sqrt{kt}}\right]^2 - \frac{(x-u)^2}{4kt}}\, d\alpha\right] du.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{-\left[\sqrt{t}\left\{\sqrt{k}\alpha + \frac{i(x-u)}{2\sqrt{kt}}\right\}\right]^2} d\alpha\right] e^{-\frac{(x-u)^2}{4kt}}\, du \quad (7)$$

Putting $\sqrt{t}\left\{\sqrt{k}\alpha + \frac{i(x-u)}{2\sqrt{kt}}\right\} = y$

so that $\sqrt{kt}\, d\alpha = dy \;\therefore\; d\alpha = \frac{1}{\sqrt{kt}} dy.$

$$\text{Now} \int_{-\infty}^{\infty} e^{-\left[\sqrt{t}\left\{\sqrt{k}\alpha + \frac{i(x-u)}{2\sqrt{kt}}\right\}\right]^2} d\alpha$$

$$= 2\int_{0}^{\infty} e^{-\left[\sqrt{t}\left\{\sqrt{k}\alpha + \frac{i(x-u)}{2\sqrt{kt}}\right\}\right]^2} d\alpha$$

$$= 2\int_{0}^{\infty} e^{-y^2} \frac{dy}{\sqrt{kt}}$$

$$= \frac{2}{\sqrt{kt}}\int_{0}^{\infty} e^{-y^2} dy = \frac{2}{\sqrt{kt}} \cdot \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{\sqrt{kt}}$$

Therefore, from (7) we get

$$U(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u) \cdot \frac{\sqrt{\pi}}{\sqrt{kt}} \cdot e^{\frac{-(x-u)^2}{4kt}}\, du.$$

$$= \frac{1}{2\pi} \cdot \frac{\sqrt{\pi}}{\sqrt{kt}}\int_{-\infty}^{\infty} f(u)\, e^{-\frac{(x-u)^2}{4kt}}\, du.$$

$$= \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} f(u)\, e^{-\frac{(x-u)^2}{4kt}}\, du \quad (8)$$

Now if we change variables from u to z according to the transformation

$$\frac{(x-u)^2}{4kt} = z^2 \text{ or, } \frac{x-u}{2\sqrt{kt}} = z$$

$$u = (x - 2\sqrt{kt}z) \quad \textbf{Limits.}$$

$$du = -2\sqrt{kt}\,dz \qquad \left.\begin{matrix} u=\infty \\ z=-\infty \end{matrix}\right\} \left.\begin{matrix} u=-\infty \\ z=\infty \end{matrix}\right\}$$

Thus from (8) we have

$$U(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{+\infty}^{-\infty} f(x-2z\sqrt{kt})\, e^{-z^2} . -2\sqrt{kt}\, dz.$$

$$= -\frac{2\sqrt{kt}}{2\sqrt{\pi}\sqrt{kt}} \int_{\infty}^{-\infty} f(x-2z\sqrt{kt})\, e^{-z^2}\, dz.$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f\left(x - 2z\sqrt{kt}\right) e^{-z^2}\, dz.$$

Note : $-k\alpha^2 t - i\alpha(x-u)$

$$= -t\left[\sqrt{k}\alpha + \frac{i(x-u)}{2\sqrt{k}t}\right]^2 - \frac{(x-u)^2}{4kt}$$

$$= -t\left\{k\alpha^2 + \frac{i\alpha}{t}(x-u) + i^2\frac{(x-u)^2}{4kt^2}\right\} - \frac{(x-u)^2}{4kt}$$

$$= -k\alpha^2 t - i\alpha(x-u) + \frac{(x-u)^2}{4kt} - \frac{(x-u)^2}{4kt}$$

$$= -k\alpha^2 t - i\alpha(x-u)$$

$$\therefore \quad -k\alpha^2 t - i\alpha(x-u) = -t\left[\sqrt{k}\alpha + \frac{i(x-u)}{2\sqrt{k}t}\right]^2 - \frac{(x-u)^2}{4kt}.$$

Example 11. Solve $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$, $0 \angle x \angle 6$, $t > 0$.

$U_x(0,t) = 0$, $U_x(6, t) = 0$, $U(x, 0) = 2x$.

Solution : The given partial differential equation is

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \quad (1)$$

According to the given boundary conditions, here the finite Fourier cosine transform is more useful. Taking the finite Fourier cosine transform($l = 6$) of both sides of (1), we get.

$$\int_0^6 \frac{\partial U}{\partial t}\cos\frac{n\pi x}{6}\,dx = \int_0^6 \frac{\partial^2 U}{\partial x^2}\cos\frac{n\pi x}{6}\,dx \quad (2)$$

$$\text{Let } v = v(n,t) = \int_0^6 U(x,t)\cos\frac{n\pi x}{6}\,dx = f_c(U)$$

$$\therefore \quad \frac{dv}{dt} = \int_0^6 \frac{\partial U}{\partial t}\cos\frac{n\pi x}{6}\,dx$$

$$= \int_0^6 \frac{\partial^2 U}{\partial x^2}\cos\frac{n\pi x}{6}\,dx.$$

Note : $f_c\left\{\frac{\partial^2 U}{\partial x^2}\right\} = \int_0^l \frac{\partial^2 U}{\partial x^2}\operatorname{Cos}\frac{n\pi x}{l}\,dx$

$$= \frac{\partial U(l,t)}{\partial x}\cos n\pi - \frac{\partial U(0,t)}{\partial x} - \frac{n^2\pi^2}{l^2} f_c(U)$$

$$\therefore \quad \int_0^6 \frac{\partial^2 U}{\partial x^2}\cos\frac{n\pi x}{6}\,dx = \frac{\partial U(6,t)}{\partial x}\cos n\pi - \frac{\partial U(0,t)}{\partial x} - \frac{n^2\pi^2}{6^2} f_c(U)$$

$$= U_x(6,t)\cos n\pi - U_x(0,t) - \frac{n^2\pi^2}{6^2} f_c(U)$$

$$= U_x(6,t)\cos n\pi - U_x(0,t) - \frac{n^2\pi^2}{6^2} v$$

$$\text{Since } f_c(U) = \int_0^6 U(x,t)\cos\frac{n\pi x}{6}\,dx.$$

$$\therefore \boxed{f_c(U) = v}$$

Therefore, $\frac{dv}{dt} = U_x(6,t) - U_x(0,t) - \frac{n^2\pi^2}{6^2} v$

or, $\frac{dv}{dt} = 0 - 0 - \frac{n^2\pi^2}{6^2} v$ $\boxed{\text{Since } U_x(0, t) = U_x(6, t) = 0}$

$$\frac{dv}{v} = -\frac{n^2\pi^2}{6^2} dt \quad (3)$$

Integrating both sides, we get $\log v = -\frac{n^2\pi^2}{6^2} t + \log A$, A being some constant of integration.

Or, $\log v = \log e^{-\frac{n^2\pi^2 t}{6^2}} + \log A = \log Ae^{-\frac{n^2\pi^2 t}{6^2}}$.

$$v = Ae^{-\frac{n^2\pi^2}{6^2}t} \quad (4)$$

When $t = 0$, R. H. S of (4) $= Ae^0 = A$.

When $t = 0$, L. H. S of (4) $= \int_{-\infty}^{\infty} U(x, 0) \cos \frac{n\pi x}{6} dx$,

$$= \int_0^6 2x \cos \frac{n\pi x}{6} dx$$

$$= \left[2x \cdot \frac{6}{n\pi} \sin \frac{n\pi x}{6}\right]_0^6 - 2 \cdot \frac{6}{n\pi} \int_0^6 \sin \frac{n\pi x}{6} dx$$

$$= 0 - \frac{12}{n\pi} \int_0^6 \sin \frac{n\pi x}{6} dx$$

$$= \frac{12}{n\pi} \cdot \frac{6}{n\pi} \left[\cos \frac{n\pi x}{6}\right]_0^6$$

$$= \frac{72}{n^2\pi^2} (\cos n\pi - 1)$$

Therefore, from (4), we get $A = \frac{72}{n^2\pi^2} (\cos n\pi - 1)$

Putting the value of A in (4), we get

$$v = \frac{72}{n^2\pi^2} (\cos n\pi - 1) e^{-\frac{n^2\pi^2 t}{6^2}}.$$

Taking the inverse Fourier cosine transform, we get

$$F(x) = \frac{1}{l} f_c(0) + \frac{2}{l} \sum_{n=1}^{\infty} f_c(n) \cos \frac{n\pi x}{l}.$$

That is, $U(x, t) = \frac{1}{6} f_c(0) + \frac{2}{6} \sum_{n=1}^{\infty} v \cos \frac{n\pi x}{6}$

or, $U(x, t) = \frac{1}{6} f_c(0) + \frac{1}{3} \sum_{n=1}^{\infty} \frac{72}{n^2\pi^2} (\cos n\pi - 1) e^{\frac{-n^2\pi^2 t}{6^2}} \cos \frac{n\pi x}{6}$ (5)

$$f_c(n) = \int_0^l F(x) \cos \frac{n\pi x}{l} dx = \int_0^6 F(x) \cos \frac{n\pi x}{6} dx$$

$$\therefore f_c(0) = \int_0^6 F(x) dx = \int_0^6 U(x,0) dx.$$

$$= \int_0^6 2x dx = [x^2]_0^6 = 36.$$

Thus from (5), we get

$$U(x, t) = \frac{36}{6} + \frac{24}{\pi^2} \sum_{n=1}^{\infty} \frac{(\cos n\pi - 1)}{n^2} e^{-\frac{n^2\pi^2 t}{36}} \cos \frac{n\pi x}{6}.$$

or, $U(x, t) = 6 + \frac{24}{\pi^2} \sum_{n=1}^{\infty} \frac{(\cos n\pi - 1)}{n^2} e^{-\frac{n^2\pi^2 t}{36}} \cos \frac{n\pi x}{6}$.

Example 12. Use the method of Fourier transform to detemline the displacement $y(x,t)$ of an infinite string, given that the string is initially at rest and that the initial displacement is $f(x)$, $-\infty < x < \infty$. Also show that the solution can be put in the form

$$y(x,t) = \frac{1}{2} [f(x+ct) + f(x-ct)].$$

Solution : Here we have to solve the one-dimensional wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, -\infty < x < \infty, t > 0$

subject to the following initial conditions

$y(x,o)$ = Initial displacement $= f(x)$

and $y_t(x,o)$ = Initial velocity $= 0$.

The given partial differential equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad (1)$$

Taking the complex Fourier transform of both sides of (1), we have

$$\int_{-\infty}^{\infty} \frac{\partial^2 y}{\partial t^2} e^{-iux}\,dx = c^2 \int_{-\infty}^{\infty} \frac{\partial^2 y}{\partial x^2} e^{-iux}\,dx \quad (2)$$

By the Fourier transform of the derivative of a function we have if $F^n(x)$ is the nth-derivative of $F(x)$ and the first $(n-1)$ derivatives of $F(x)$ vanish as $x \to \pm\infty$, then

$$F\{F^n(x)\} = (-iu)^n F\{F(x)\}.$$

Thus from (2), we have

$$\frac{d^2}{dt^2} \int_{-\infty}^{\infty} y e^{-iux}\,dx = c^2 (-iu)^2 F\{y(x,t)\}$$

or, $$\frac{d^2}{dt^2} \int_{-\infty}^{\infty} y(x,t) e^{-iux}\,dx = c^2(-u^2) F\{y(x,t)\}$$

$$\frac{d^2}{dt^2} F\{y(x,t)\} = -c^2u^2 F\{y(x,t)\}$$

or, $$\frac{d^2\bar{y}}{dt^2} = -c^2u^2 \bar{y}$$

where $\bar{y} = \bar{y}(u,t) = F\{y(x,t)\} = \int_{-\infty}^{\infty} y(x,t) e^{-iux}\,dx.$

or, $\frac{d^2\bar{y}}{dt^2} + c^2u^2 \bar{y} = 0$ (3) which is ordinary second order differential equation whose solution

is $\bar{y} = \bar{y}(u,t) = A\cos(cut) + B\sin(cut)$ (4)

Differentiating both sides with respect to t,

we get $\bar{y}_t(u,t) = -Acu\sin(cut) + Bcu\cos(cut)$ (5)

Also from the initial given conditions, we have

$y(x,o) = f(x)$ (6)

$y_t(x,o) = 0$ (7)

Taking the Fourier transform of (6) & (7) we get

$$\bar{y}(u,o) = \int_{-\infty}^{\infty} y(x,o) e^{-iux}\,dx = \int_{-\infty}^{\infty} f(x) e^{-iux}\,dx = \bar{f}(u) \text{ say)}$$

$\therefore \bar{y}(u,o) = \bar{f}(u)$ (8)

and $\bar{y}_t(u,o) = \int_{-\infty}^{\infty} y_t(x,o) e^{-iux}\,dx$

$$= \int_{-\infty}^{\infty} 0\, e^{-iux}\,du = 0$$

$\therefore \bar{y}_t(u,o) = 0$ (9)

Putting t=o in (5), we have $\bar{y}_t(u,o) = Bcu$.

or, $Bcu = 0$ using (9)

$\therefore B = 0$ Since $cu \neq 0$

Again putting t=o in (4), we get

$\bar{y}(u,o) = A$

$\therefore \bar{f}(u) = A$ using (8)

or, $A = \bar{f}(u)$.

Putting the values of A and B in (4), we get

$\bar{y} = \bar{y}(u,t) = \bar{f}(u)\cos(cut)$ (10)

Taking the inverse Fourier transform of (10) we have

$$y(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(u) \cos(cut) e^{iux}\,du$$

or, $$y(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(u) \left(\frac{e^{icut} + e^{-icut}}{2} \right) e^{iux}\,du$$

$$=\frac{1}{2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{f}(u)\,e^{iu(x+ct)}du\right.$$

$$\left.+\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{f}(u)\,e^{iu(x-ct)}du\right]$$

$$=\frac{1}{2}[f(x+ct)+f(x-ct)]$$

(using the definition of inverse Fourier transform)

$$\therefore\quad y(x,t)=\frac{1}{2}[f(x+ct)+f(x-ct)].$$

Example 13. A thin membrane of great extent is released from rest in the position $z=f(x,y)$. show that the displacement at any subsequent time is given by

$$z(x,y,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(u,v)\cos ct\sqrt{(u^2+v^2)}e^{-i(ux+vy)}\,dudv$$

where $F(u,v)$ is the double Fourier transform of $f(x,y)$.

proof : Here the displacement of the membrane is governed by two dimensional wave equation

$$\frac{\partial^2 z}{\partial t^2}=c^2\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)\quad(1)\text{ where } c^2=\frac{T}{\rho}$$

Taking the double Fourier transform of both sides of (1) we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial^2 z}{\partial t^2}.\,e^{i(ux+vy)}\,dxdy$$

$$=\frac{c^2}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)e^{i(ux+vy)}\,dxdy$$

$$\text{or,}\quad\frac{d^2}{dt^2}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}ze^{i(ux+vy)}\,dxdy$$

$$=\frac{c^2}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)e^{i(ux+vy)}\,dxdy$$

$$\text{or,}\quad\frac{d^2\bar{z}}{dt^2}=c^2\{(-iu)^2+(-iv)^2\}\,F\{z(x,y,t)\}$$

where $\bar{z}=\bar{z}(u,v,t)=F\{z(x,y,t)\}$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}ze^{i(ux+vy)}\,dxdy$$

$$\text{or}\quad\frac{d^2\bar{z}}{dt^2}=-c^2(u^2+v^2)\bar{z}$$

$$\text{or,}\quad\frac{d^2\bar{z}}{dt^2}+c^2(u^2+v^2)\bar{z}=0\quad(2)$$

which is an ordinary differential equation whose solution is

$$=A\cos\{c\sqrt{(u^2+v^2)}\,t\}+B\sin\{c\sqrt{(u^2+v^2)}\,t\}\quad(3)$$

The given initial conditions are

$$z=f(x,y)\text{ and }\frac{\partial z}{\partial t}=0\text{ at }t=0.$$

Taking the Fourier transform of these initial conditions, we get

$$\bar{z}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{i(ux+vy)}\,dxdy=F(u,v)\quad(4)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial z}{\partial t}\,e^{i(ux+vy)}\,dxdy$$

$$=\frac{d}{dt}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}ze^{i(ux+vy)}\,dxdy$$

$$=\frac{d\bar{z}}{dt}\quad\text{since }\bar{z}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}ze^{i(ux+vy)}\,dxdy.$$

$$\therefore\quad\frac{d\bar{z}}{dt}=0\text{ at }t=0.$$

When $t=0$, combining (3) and (4), we get

$$A=F(u,v)$$

$$\text{Also}\quad\frac{d\bar{z}}{dt}=-Ac\sqrt{(u^2+v^2)}\sin c\sqrt{(u^2+v^2)}t+Bc\sqrt{(u^2+v^2)}$$

$$\cos c\sqrt{(u^2+v^2)}t$$

$$\therefore \quad 0=\left(\frac{d\bar{z}}{dt}\right)_{t=0} = Bc\sqrt{(u^2+v^2)}$$

or, $B=0$.

putting the values of A and B in (3), we get

$$\bar{z} = F(u,v)\cos(c\sqrt{(u^2+v^2)}\,t) \qquad (5)$$

Now applying the inversion formula for double Fourier transform, we have

$$z(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\cos(c\sqrt{(u^2+v^2)}\,t)\; e^{-i(ux+vy)}\,du\,dv$$

which is the required displacment at any subsequent time t.

EXERCISES 4 (P)

1. Use finite Fourier transform to solve the following boundary value problem:

$$\frac{\partial U}{\partial t} = 2\frac{\partial^2 U}{\partial x^2},\; 0<x<4,\, t>0$$

$$U(0,t)=0,\; U(4,t)=0,\; U(x,0)=3\sin \pi x - 2\sin 5\pi x$$

Answer: $U(x,t) = 3e^{-2\pi^2 t}\sin \pi x - 2e^{-50\pi^2 t}\sin 5\pi x$.

2. Solve the following boundary value problem using Fourier transform:

$$\frac{\partial U}{\partial t} = 2\frac{\partial^2 U}{\partial x^2},\; U(0,t)=0,\; U(x,0)=e^{-x},\; x>0,$$

$U(x,t)$ is bounded where $x>0$, $t>0$.

Answer: $U(x,t) = \dfrac{2}{\pi}\displaystyle\int_0^{\infty}\frac{ne^{-2n^2 t}\sin nx}{n^2+1}\,dn$

3. Solve the following boundary value problem using Fourier transform:

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2},\; U_x(0,t)=0,\; U(x,0)=\begin{cases} x, & 0\le x\le 1 \\ 0, & x>1\end{cases}$$

$U(x,t)$ is bounded where $x>0$, $t>0$.

Answer: $U(x,t) = \dfrac{2}{\pi}\displaystyle\int_0^{\infty}\left(\frac{\sin n}{n}+\frac{\cos n-1}{n^2}\right)e^{-n^2 t}\cos nx\,dn.$

4. Using Fourier sine transform solve the partial differential equation $\dfrac{\partial U}{\partial t} = k\dfrac{\partial^2 U}{\partial x^2}$ for $x>0$, $t>0$

under the boundary conditions $U(0,t)=U_0$ and $U(x,0)=0$.

Answer: $U(x,t) = U_0\left[1-\dfrac{2}{\pi}\displaystyle\int_0^{\infty}\frac{e^{-kn^2 t}}{n}\sin nx\,dn\right]$

5. Using Fourier cosine transform solve the following boundary value problem

$\dfrac{\partial U}{\partial t} = k\dfrac{\partial^2 U}{\partial x^2}$ subject to the conditions

$U(x,0)=0$ and $U_x(0,t)=-\mu$ (a constant).

Answer: $U(x,t) = \dfrac{2\mu}{\pi}\displaystyle\int_0^{\infty}\frac{\cos nx}{n^2}(1-e^{-kn^2 t})\,dn.$

6. Using finite Fourier sine transform solve the following wave equation

$\dfrac{\partial^2 U}{\partial t^2} = 4\dfrac{\partial^2 U}{\partial x^2}$ subject to the conditions

$U(0,t)=0,\; U(\pi,t)=0,\; U(x,0)=(0.1)\sin x+(0.01)\sin 4x$

and $U_t(x,0)=0$ for $0<x<\pi$, $t>0$.

Answer: $U(x,t)=(0.1)(\cos 2t\sin x + (0.01)\cos 8t\sin 4x$.

7. Using finite Fourier sine transform solve the following two dimensional Laplace equation

$\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}=0$ under the boundary conditions

$V(0,y)=0,\; V(\pi,y)=0,\; V(x,0)=0$ and

$V(x,\pi)=V_0$ (constant).

Answer: $V(x,y) = \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{V_0}{\sinh n\pi}\left(\frac{1-\cos n\pi}{n}\right)\sinh ny\,\sin nx$

or, $V(x,y) = \dfrac{4V_0}{\pi}\displaystyle\sum_{m=0}^{\infty}\frac{\sinh(2m+1)y\,\sin(2m+1)x}{(2m+1)\sinh(2m+1)\pi}.$

8. Using the finite Fourier cosine transform solve th following boundary value problem: