Common Kinematic Constraints in 3D

This document presents some of the most common kinematic joints in 3D-multibody systems and the associated constraint equations. Some of the constraint equations can be formulated in several forms but here is only one version presented pr joint. It should be noted that the jacobian matrix found in this document are only used in the velocity and acceleration constraints. This Jacobian matrix is not the same as the one used in a Newton Raphson iterative solver.

Spherical Joint (Ball Joint)

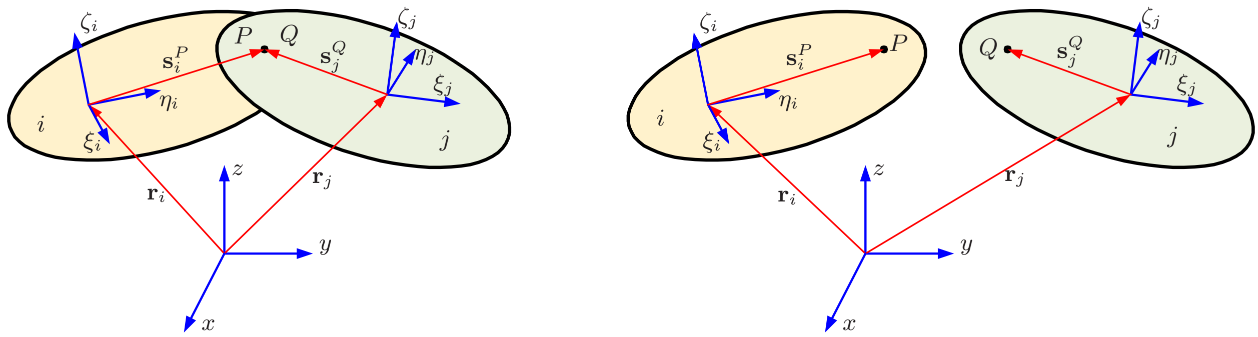

Figure 1: Spherical joint (ball joint) - removes 3 degrees of freedom.

The constraint for the spherical joint is that the points P and Q must coincide to satisfy the equation. Position constraints:

$$\boldsymbol{\Phi} = \mathbf{r}_i + \mathbf{A}_i \mathbf{s}_i^{P'} - \mathbf{r}_j - \mathbf{A}_j \mathbf{s}_j^{Q'} = \mathbf{0} \tag{1}$$

Velocity constraints:

$$\begin{aligned} \dot{\boldsymbol{\Phi}} &= \dot{\mathbf{r}}_i + \mathbf{A}_i \tilde{\boldsymbol{\omega}}'_i \mathbf{s}_i^{P'} - \dot{\mathbf{r}}_j - \mathbf{A}_j \tilde{\boldsymbol{\omega}}'_j \mathbf{s}_j^{Q'} \\ &= \dot{\mathbf{r}}_i - \mathbf{A}_i \tilde{\mathbf{s}}_i^{P'} \boldsymbol{\omega}'_i - \dot{\mathbf{r}}_j + \mathbf{A}_j \tilde{\mathbf{s}}_j^{Q'} \boldsymbol{\omega}'_j = \mathbf{0} \quad \Downarrow \end{aligned}$$

$$\underbrace{\begin{bmatrix} \mathbf{I} & -\mathbf{A}_i \tilde{\mathbf{s}}_i^{P'} & -\mathbf{I} & \mathbf{A}_j \tilde{\mathbf{s}}_j^{Q'} \end{bmatrix}}_{\text{Entries in jacobian matrix}} \begin{bmatrix} \dot{\mathbf{r}}_i \\ \boldsymbol{\omega}'_i \\ \dot{\mathbf{r}}_j \\ \boldsymbol{\omega}'_j \end{bmatrix} = \mathbf{0} \tag{2}$$

Acceleration constraints:

$$\ddot{\boldsymbol{\Phi}} = \ddot{\mathbf{r}}_i - \mathbf{A}_i \tilde{\boldsymbol{\omega}}'_i \tilde{\mathbf{s}}_i^{P'} \boldsymbol{\omega}'_i - \mathbf{A}_i \tilde{\mathbf{s}}_i^{P'} \dot{\boldsymbol{\omega}}'_i - \ddot{\mathbf{r}}_j + \mathbf{A}_j \tilde{\boldsymbol{\omega}}'_j \tilde{\mathbf{s}}_j^{Q'} \boldsymbol{\omega}'_j + \mathbf{A}_j \tilde{\mathbf{s}}_j^{Q'} \dot{\boldsymbol{\omega}}'_j = \mathbf{0} \quad \Downarrow$$

$$\underbrace{\begin{bmatrix} \mathbf{I} & -\mathbf{A}_i \tilde{\mathbf{s}}_i^{P'} & -\mathbf{I} & \mathbf{A}_j \tilde{\mathbf{s}}_j^{Q'} \end{bmatrix}}_{\text{Entries in jacobian matrix}} \begin{bmatrix} \ddot{\mathbf{r}}_i \\ \dot{\boldsymbol{\omega}}'_i \\ \ddot{\mathbf{r}}_j \\ \dot{\boldsymbol{\omega}}'_j \end{bmatrix} = \mathbf{A}_i \tilde{\boldsymbol{\omega}}'_i \tilde{\mathbf{s}}_i^{P'} \boldsymbol{\omega}'_i - \mathbf{A}_j \tilde{\boldsymbol{\omega}}'_j \tilde{\mathbf{s}}_j^{Q'} \boldsymbol{\omega}'_j \tag{3}$$

The reaction forces in the spherical joint can be found as:

$$\mathbf{g}^c = \mathbf{\Phi}^T \boldsymbol{\lambda} \quad = \quad \begin{bmatrix} \mathbf{I} \\ \left(-\mathbf{A}_i \tilde{\mathbf{s}}_i^{P'}\right)^T \\ -\mathbf{I} \\ \left(\mathbf{A}_j \tilde{\mathbf{s}}_j^{Q'}\right) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \tag{4}$$

Spherical-Spherical Joint

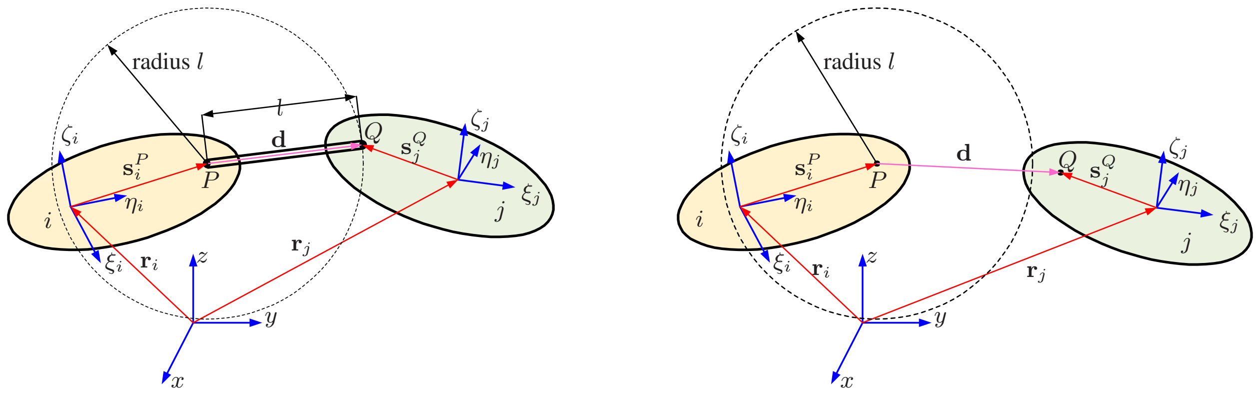

Figure 2: Spherical-spherical joint (distance constraint) - removes 1 degree of freedom.

The constraint for the spherical-spherical joint is that the distance between the points P and Q must remain constant. A helping variable $\mathbf{d}$ being a vector between the points P and Q is introduced:

$$\begin{aligned}
\mathbf{d} &= \mathbf{r}_j + \mathbf{A}_j \mathbf{s}_j^{Q'} - \mathbf{r}_i - \mathbf{A}_i \mathbf{s}_i^{P'} && (5)\\
\dot{\mathbf{d}} &= \dot{\mathbf{r}}_j + \mathbf{A}_j \tilde{\boldsymbol{\omega}}'_j \mathbf{s}_j^{Q'} - \dot{\mathbf{r}}_i - \mathbf{A}_i \tilde{\boldsymbol{\omega}}'_i \mathbf{s}_i^{P'} && (6)\\
\ddot{\mathbf{d}} &= \ddot{\mathbf{r}}_j + \mathbf{A}_j \tilde{\boldsymbol{\omega}}'_j \tilde{\boldsymbol{\omega}}'_j \mathbf{s}_j^{Q'} + \mathbf{A}_j \dot{\tilde{\boldsymbol{\omega}}}'_j \mathbf{s}_j^{Q'} - \ddot{\mathbf{r}}_i - \mathbf{A}_i \tilde{\boldsymbol{\omega}}'_i \tilde{\boldsymbol{\omega}}'_i \mathbf{s}_i^{P'} - \mathbf{A}_i \dot{\tilde{\boldsymbol{\omega}}}'_i \mathbf{s}_i^{P'} && (7)
\end{aligned}$$

Position constraints:

$$\boldsymbol{\Phi} = \mathbf{d}^T \mathbf{d} - l^2 = \mathbf{0} \qquad (8)$$

Velocity constraints:

$$\begin{aligned}
\dot{\boldsymbol{\Phi}} &= \dot{\mathbf{d}}^T \mathbf{d} + \mathbf{d}^T \dot{\mathbf{d}}\\
&= 2\mathbf{d}^T \dot{\mathbf{d}}\\
&= 2\mathbf{d}^T \left(\dot{\mathbf{r}}_j + \mathbf{A}_j \tilde{\boldsymbol{\omega}}'_j \mathbf{s}_j^{Q'} - \dot{\mathbf{r}}_i - \mathbf{A}_i \tilde{\boldsymbol{\omega}}'_i \mathbf{s}_i^{P'} \right)\\
&= 2\mathbf{d}^T \left(\dot{\mathbf{r}}_j - \mathbf{A}_j \tilde{\mathbf{s}}_j^{Q'} \boldsymbol{\omega}'_j - \dot{\mathbf{r}}_i + \mathbf{A}_i \tilde{\mathbf{s}}_i^{P'} \boldsymbol{\omega}'_i \right)
\end{aligned}$$

$$\underbrace{\begin{bmatrix} -2\mathbf{d}^T & 2\mathbf{d}^T \mathbf{A}_i \tilde{\mathbf{s}}_i^{P'} & 2\mathbf{d}^T & -2\mathbf{d}^T \mathbf{A}_j \tilde{\mathbf{s}}_j^{Q'} \end{bmatrix}}_{\text{Entries in jacobian matrix}} \begin{bmatrix} \dot{\mathbf{r}}_i \\ \boldsymbol{\omega}'_i \\ \dot{\mathbf{r}}_j \\ \boldsymbol{\omega}'_j \end{bmatrix} = \mathbf{0} \qquad (9)$$

Acceleration constraints:

$$
\begin{aligned}
\ddot{\boldsymbol{\Phi}} &= 2\dot{\mathbf{d}}^T\dot{\mathbf{d}} + 2\mathbf{d}^T\ddot{\mathbf{d}} \\
&= 2\dot{\mathbf{d}}^T\dot{\mathbf{d}} + 2\mathbf{d}^T\left(\ddot{\mathbf{r}}_j + \mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{s}_j^{Q'} + \mathbf{A}_j\dot{\tilde{\boldsymbol{\omega}}}'_j\mathbf{s}_j^{Q'} - \ddot{\mathbf{r}}_i - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{s}_i^{P'} - \mathbf{A}_i\dot{\tilde{\boldsymbol{\omega}}}'_i\mathbf{s}_i^{P'}\right) \\
&= 2\dot{\mathbf{d}}^T\dot{\mathbf{d}} + 2\mathbf{d}^T\left(\ddot{\mathbf{r}}_j + \mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{s}_j^{Q'} - \mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\dot{\boldsymbol{\omega}}'_j - \ddot{\mathbf{r}}_i - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{s}_i^{P'} + \mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\dot{\boldsymbol{\omega}}'_i\right)
\end{aligned}
$$

$$
\underbrace{\begin{bmatrix} -2\mathbf{d}^T & 2\mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{s}}_i^{P'} & 2\mathbf{d}^T & -2\mathbf{d}^T\mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'} \end{bmatrix}}_{\text{Entries in jacobian matrix}}
\begin{bmatrix} \ddot{\mathbf{r}}_i \\ \dot{\boldsymbol{\omega}}'_i \\ \ddot{\mathbf{r}}_j \\ \dot{\boldsymbol{\omega}}'_j \end{bmatrix}
= -2\dot{\mathbf{d}}^T\dot{\mathbf{d}} - 2\mathbf{d}^T\left(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{s}_j^{Q'} - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{s}_i^{P'}\right) \tag{10}
$$

The reaction forces in the spherical-spherical joint can be found as:

$$
\mathbf{g}^c = \boldsymbol{\Phi}^T\boldsymbol{\lambda} = \begin{bmatrix} -2\mathbf{d} \\ \left(2\mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\right)^T \\ 2\mathbf{d} \\ \left(-2\mathbf{d}^T\mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\right)^T \end{bmatrix} \begin{bmatrix} \lambda_1 \end{bmatrix} \tag{11}
$$

Revolute Joint

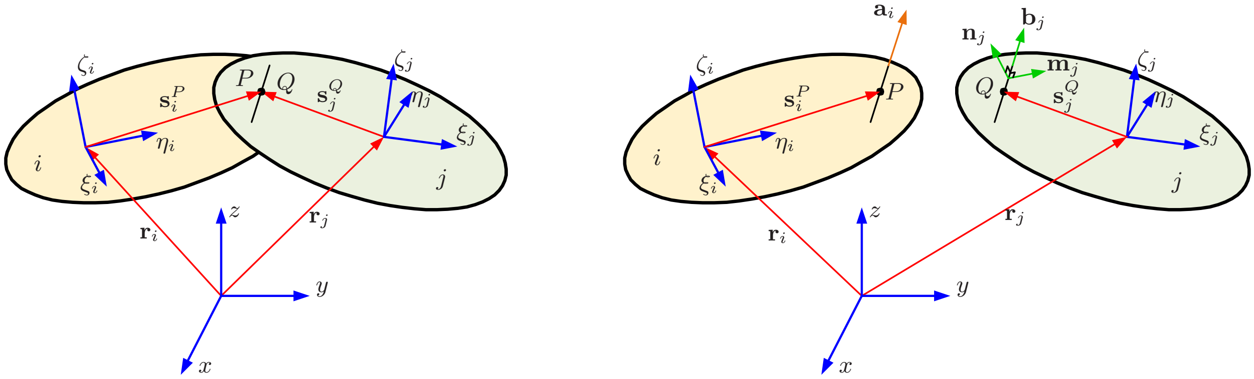

Figure 3: Revolute joint - removes 5 degrees of freedom.

The joint allows rotation around a common axis. In addition to a common point two vectors being perpendicular to the axis of rotation on body j should be perpendicular to the axis of rotation on body i.

Position constraints:

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{r}_i + \mathbf{A}_i \mathbf{s}_i^{P'} - \mathbf{r}_j - \mathbf{A}_j \mathbf{s}_j^{Q'} \\ \mathbf{a}_i^T \mathbf{m}_j \\ \mathbf{a}_i^T \mathbf{n}_j \end{bmatrix} \tag{12}$$

$$= \begin{bmatrix} \mathbf{r}_i + \mathbf{A}_i \mathbf{s}_i^{P'} - \mathbf{r}_j - \mathbf{A}_j \mathbf{s}_j^{Q'} \\ (\mathbf{A}_i \mathbf{a}_i')^T \mathbf{A}_j \mathbf{m}_j' \\ (\mathbf{A}_i \mathbf{a}_i')^T \mathbf{A}_j \mathbf{n}_j' \end{bmatrix} = \mathbf{0} \tag{13}$$

Velocity constraints:

$$\dot{\mathbf{\Phi}} = \begin{bmatrix} \dot{\mathbf{r}}_i + \mathbf{A}_i \tilde{\boldsymbol{\omega}}'_i \mathbf{s}_i^{P'} - \dot{\mathbf{r}}_j - \mathbf{A}_j \tilde{\boldsymbol{\omega}}'_j \mathbf{s}_j^{Q'} \\ (\mathbf{A}_i \tilde{\boldsymbol{\omega}}'_i \mathbf{a}_i')^T \mathbf{A}_j \mathbf{m}_j' + (\mathbf{A}_i \mathbf{a}_i')^T \mathbf{A}_j \tilde{\boldsymbol{\omega}}'_j \mathbf{m}_j' \\ (\mathbf{A}_i \tilde{\boldsymbol{\omega}}'_i \mathbf{a}_i')^T \mathbf{A}_j \mathbf{n}_j' + (\mathbf{A}_i \mathbf{a}_i')^T \mathbf{A}_j \tilde{\boldsymbol{\omega}}'_j \mathbf{n}_j' \end{bmatrix}$$

$$= \begin{bmatrix} \dot{\mathbf{r}}_i - \mathbf{A}_i \tilde{\mathbf{s}}_i^{P'} \boldsymbol{\omega}'_i - \dot{\mathbf{r}}_j + \mathbf{A}_j \tilde{\mathbf{s}}_j^{Q'} \boldsymbol{\omega}'_j \\ -\left(\mathbf{A}_j \mathbf{m}_j'\right)^T \mathbf{A}_i \tilde{\mathbf{a}}_i' \boldsymbol{\omega}'_i - (\mathbf{A}_i \mathbf{a}_i')^T \mathbf{A}_j \tilde{\mathbf{m}}_j' \boldsymbol{\omega}'_j \\ -\left(\mathbf{A}_j \mathbf{n}_j'\right)^T \mathbf{A}_i \tilde{\mathbf{a}}_i' \boldsymbol{\omega}'_i - (\mathbf{A}_i \mathbf{a}_i')^T \mathbf{A}_j \tilde{\mathbf{n}}_j' \boldsymbol{\omega}'_j \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \mathbf{I} & -\mathbf{A}_i \tilde{\mathbf{s}}_i^{P'} & -\mathbf{I} & \mathbf{A}_j \tilde{\mathbf{s}}_j^{Q'} \\ \mathbf{0} & -\left(\mathbf{A}_j \mathbf{m}_j'\right)^T \mathbf{A}_i \tilde{\mathbf{a}}_i' & \mathbf{0} & -(\mathbf{A}_i \mathbf{a}_i')^T \mathbf{A}_j \tilde{\mathbf{m}}_j' \\ \mathbf{0} & -\left(\mathbf{A}_j \mathbf{n}_j'\right)^T \mathbf{A}_i \tilde{\mathbf{a}}_i' & \mathbf{0} & -(\mathbf{A}_i \mathbf{a}_i')^T \mathbf{A}_j \tilde{\mathbf{n}}_j' \end{bmatrix}}_{\text{Entries in jacobian matrix}} \begin{bmatrix} \dot{\mathbf{r}}_i \\ \boldsymbol{\omega}'_i \\ \dot{\mathbf{r}}_j \\ \boldsymbol{\omega}'_j \end{bmatrix} = \mathbf{0} \tag{14}$$

Acceleration constraints:

$$
\begin{aligned}
\ddot{\boldsymbol{\Phi}} &= \begin{bmatrix}
\ddot{\mathbf{r}}_i - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\mathbf{s}}_i^{P'}\boldsymbol{\omega}'_i - \mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\dot{\boldsymbol{\omega}}'_i - \ddot{\mathbf{r}}_j + \mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\mathbf{s}}_j^{Q'}\boldsymbol{\omega}'_j + \mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\dot{\boldsymbol{\omega}}'_j \\
-\left(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{m}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i\boldsymbol{\omega}'_i - \left(\mathbf{A}_j\mathbf{m}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\mathbf{a}}'_i\boldsymbol{\omega}'_i - \left(\mathbf{A}_j\mathbf{m}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i\dot{\boldsymbol{\omega}}'_i - \left(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{m}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\mathbf{m}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{m}}'_j\dot{\boldsymbol{\omega}}'_j \\
-\left(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i\boldsymbol{\omega}'_i - \left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\mathbf{a}}'_i\boldsymbol{\omega}'_i - \left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i\dot{\boldsymbol{\omega}}'_i - \left(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{n}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\mathbf{n}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{n}}'_j\dot{\boldsymbol{\omega}}'_j
\end{bmatrix} \\
&= \begin{bmatrix}
\ddot{\mathbf{r}}_i - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\mathbf{s}}_i^{P'}\boldsymbol{\omega}'_i - \mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\dot{\boldsymbol{\omega}}'_i - \ddot{\mathbf{r}}_j + \mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\mathbf{s}}_j^{Q'}\boldsymbol{\omega}'_j + \mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\dot{\boldsymbol{\omega}}'_j \\
-\left(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{m}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_j\mathbf{m}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\mathbf{a}}'_i\boldsymbol{\omega}'_i - \left(\mathbf{A}_j\mathbf{m}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i\dot{\boldsymbol{\omega}}'_i - \left(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{m}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\mathbf{m}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{m}}'_j\dot{\boldsymbol{\omega}}'_j \\
-\left(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{n}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\mathbf{a}}'_i\boldsymbol{\omega}'_i - \left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i\dot{\boldsymbol{\omega}}'_i - \left(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{n}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\mathbf{n}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{n}}'_j\dot{\boldsymbol{\omega}}'_j
\end{bmatrix} \\
&= \begin{bmatrix}
\ddot{\mathbf{r}}_i - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\mathbf{s}}_i^{P'}\boldsymbol{\omega}'_i - \mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\dot{\boldsymbol{\omega}}'_i - \ddot{\mathbf{r}}_j + \mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\mathbf{s}}_j^{Q'}\boldsymbol{\omega}'_j + \mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\dot{\boldsymbol{\omega}}'_j \\
-2\left(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{m}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_j\mathbf{m}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\mathbf{a}}'_i\boldsymbol{\omega}'_i - \left(\mathbf{A}_j\mathbf{m}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i\dot{\boldsymbol{\omega}}'_i - \left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\mathbf{m}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{m}}'_j\dot{\boldsymbol{\omega}}'_j \\
-2\left(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{n}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\mathbf{a}}'_i\boldsymbol{\omega}'_i - \left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i\dot{\boldsymbol{\omega}}'_i - \left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\mathbf{n}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{n}}'_j\dot{\boldsymbol{\omega}}'_j
\end{bmatrix} = \mathbf{0} \quad \Downarrow
\end{aligned}
$$

$$
\underbrace{\begin{bmatrix}
\mathbf{I} & -\mathbf{A}_i\tilde{\mathbf{s}}_i^{P'} & -\mathbf{I} & \mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'} \\
\mathbf{0} & -\left(\mathbf{A}_j\mathbf{m}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i & \mathbf{0} & -\left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{m}}'_j \\
\mathbf{0} & -\left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i & \mathbf{0} & -\left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{n}}'_j
\end{bmatrix}}_{\text{Entries in jacobian matrix}}
\begin{bmatrix} \ddot{\mathbf{r}}_i \\ \dot{\boldsymbol{\omega}}'_i \\ \ddot{\mathbf{r}}_j \\ \dot{\boldsymbol{\omega}}'_j \end{bmatrix}
=
\begin{bmatrix}
\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\mathbf{s}}_i^{P'}\boldsymbol{\omega}'_i - \mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\mathbf{s}}_j^{Q'}\boldsymbol{\omega}'_j \\
2\left(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{m}}'_j\boldsymbol{\omega}'_j + \left(\mathbf{A}_j\mathbf{m}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\mathbf{a}}'_i\boldsymbol{\omega}'_i + \left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\mathbf{m}}'_j\boldsymbol{\omega}'_j \\
2\left(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{n}}'_j\boldsymbol{\omega}'_j + \left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\mathbf{a}}'_i\boldsymbol{\omega}'_i + \left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\mathbf{n}}'_j\boldsymbol{\omega}'_j
\end{bmatrix} \tag{15}
$$

The reaction forces and moments in the revolute joint can be found as:

$$
\mathbf{g}^c = \boldsymbol{\Phi}^T\boldsymbol{\lambda} \quad = \begin{bmatrix}
\mathbf{I} & \mathbf{0} & \mathbf{0} \\
\left(-\mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\right)^T & \left(-\left(\mathbf{A}_j\mathbf{m}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i\right)^T & \left(-\left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i\right)^T \\
-\mathbf{I} & \mathbf{0} & \mathbf{0} \\
\left(\mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\right)^T & \left(-\left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{m}}'_j\right)^T & \left(-\left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{n}}'_j\right)^T
\end{bmatrix}
\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix} \tag{16}
$$

Universal Joint (Hook Joint)

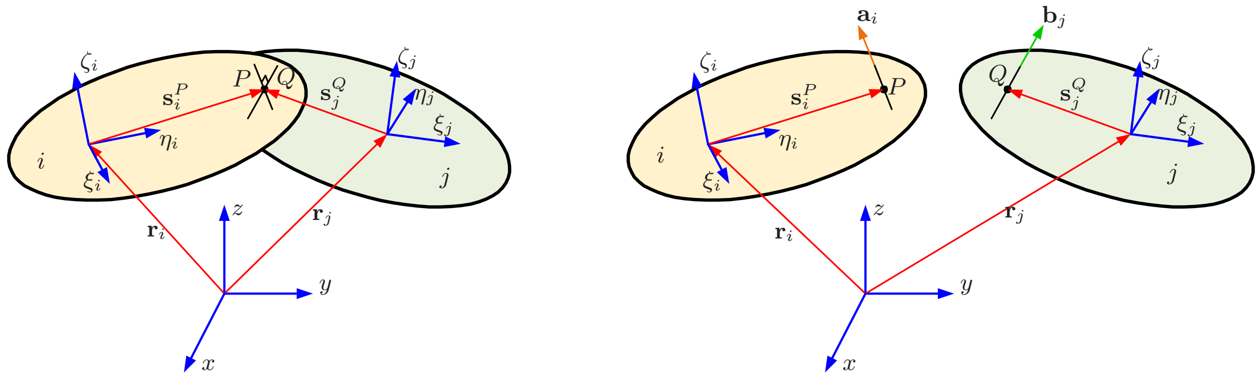

Figure 4: Universal joint (hook joint) - removes 4 degrees of freedom.

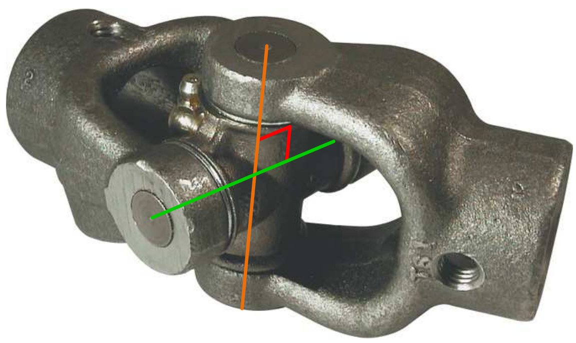

Figure 5: Universal joint (hook joint). www.princessauto.com/en/detail/1-in-r-x-1-in-r-universal-joint/A-p8087165e.

The constraint equations for the universal joint comprises a common point and a condition that two vectors - one along the rotation axis on body i and one along the rotation axis on body j - must remain perpendicular.

Position constraints:

$$\boldsymbol{\Phi} = \begin{bmatrix} \mathbf{r}_i + \mathbf{A}_i \mathbf{s}_i^{P'} - \mathbf{r}_j - \mathbf{A}_j \mathbf{s}_j^{Q'} \\ \mathbf{a}_i^T \mathbf{b}_j \end{bmatrix} \tag{17}$$

$$= \begin{bmatrix} \mathbf{r}_i + \mathbf{A}_i \mathbf{s}_i^{P'} - \mathbf{r}_j - \mathbf{A}_j \mathbf{s}_j^{Q'} \\ (\mathbf{A}_i \mathbf{a}_i')^T \mathbf{A}_j \mathbf{b}_j' \end{bmatrix} = \mathbf{0} \tag{18}$$

Velocity constraints:

$$
\begin{aligned}
\dot{\mathbf{\Phi}} &= \begin{bmatrix} \dot{\mathbf{r}}_i + \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{s}_i^{P'} - \dot{\mathbf{r}}_j - \mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{s}_j^{Q'} \\ \left(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i\right)^T\mathbf{A}_j\mathbf{b}'_j + \left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{b}'_j \end{bmatrix} \\
&= \begin{bmatrix} \dot{\mathbf{r}}_i - \mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\boldsymbol{\omega}'_i - \dot{\mathbf{r}}_j + \mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\boldsymbol{\omega}'_j \\ -\left(\mathbf{A}_j\mathbf{b}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i\boldsymbol{\omega}'_i - \left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{b}}'_j\boldsymbol{\omega}'_j \end{bmatrix}
\end{aligned}
$$

$$
\underbrace{\begin{bmatrix} \mathbf{I} & -\mathbf{A}_i\tilde{\mathbf{s}}_i^{P'} & -\mathbf{I} & \mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'} \\ \mathbf{0} & -\left(\mathbf{A}_j\mathbf{b}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i & \mathbf{0} & -\left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{b}}'_j \end{bmatrix}}_{\text{Entries in jacobian matrix}} \begin{bmatrix} \dot{\mathbf{r}}_i \\ \boldsymbol{\omega}'_i \\ \dot{\mathbf{r}}_j \\ \boldsymbol{\omega}'_j \end{bmatrix} = \mathbf{0} \tag{19}
$$

Acceleration constraints:

$$
\begin{aligned}
\ddot{\mathbf{\Phi}} &= \begin{bmatrix} \ddot{\mathbf{r}}_i - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\mathbf{s}}_i^{P'}\boldsymbol{\omega}'_i - \mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\dot{\boldsymbol{\omega}}'_i - \ddot{\mathbf{r}}_j + \mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\mathbf{s}}_j^{Q'}\boldsymbol{\omega}'_j + \mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\dot{\boldsymbol{\omega}}'_j \\ -\left(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{b}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i\boldsymbol{\omega}'_i - \left(\mathbf{A}_j\mathbf{b}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\mathbf{a}}'_i\boldsymbol{\omega}'_i - \left(\mathbf{A}_j\mathbf{b}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i\dot{\boldsymbol{\omega}}'_i - \left(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{b}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\mathbf{b}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{b}}'_j\dot{\boldsymbol{\omega}}'_j \end{bmatrix} \\
&= \begin{bmatrix} \ddot{\mathbf{r}}_i - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\mathbf{s}}_i^{P'}\boldsymbol{\omega}'_i - \mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\dot{\boldsymbol{\omega}}'_i - \ddot{\mathbf{r}}_j + \mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\mathbf{s}}_j^{Q'}\boldsymbol{\omega}'_j + \mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\dot{\boldsymbol{\omega}}'_j \\ -\left(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{b}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_j\mathbf{b}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\mathbf{a}}'_i\boldsymbol{\omega}'_i - \left(\mathbf{A}_j\mathbf{b}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i\dot{\boldsymbol{\omega}}'_i - \left(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{b}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\mathbf{b}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{b}}'_j\dot{\boldsymbol{\omega}}'_j \end{bmatrix} \\
&= \begin{bmatrix} \ddot{\mathbf{r}}_i - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\mathbf{s}}_i^{P'}\boldsymbol{\omega}'_i - \mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\dot{\boldsymbol{\omega}}'_i - \ddot{\mathbf{r}}_j + \mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\mathbf{s}}_j^{Q'}\boldsymbol{\omega}'_j + \mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\dot{\boldsymbol{\omega}}'_j \\ -2\left(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{b}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_j\mathbf{b}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\mathbf{a}}'_i\boldsymbol{\omega}'_i - \left(\mathbf{A}_j\mathbf{b}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i\dot{\boldsymbol{\omega}}'_i - \left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\mathbf{b}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{b}}'_j\dot{\boldsymbol{\omega}}'_j \end{bmatrix} = \mathbf{0} \quad \Downarrow
\end{aligned}
$$

$$
\underbrace{\begin{bmatrix} \mathbf{I} & -\mathbf{A}_i\tilde{\mathbf{s}}_i^{P'} & -\mathbf{I} & \mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'} \\ \mathbf{0} & -\left(\mathbf{A}_j\mathbf{b}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i & \mathbf{0} & -\left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{b}}'_j \end{bmatrix}}_{\text{Entries in jacobian matrix}} \begin{bmatrix} \ddot{\mathbf{r}}_i \\ \dot{\boldsymbol{\omega}}'_i \\ \ddot{\mathbf{r}}_j \\ \dot{\boldsymbol{\omega}}'_j \end{bmatrix} = \begin{bmatrix} \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\mathbf{s}}_i^{P'}\boldsymbol{\omega}'_i - \mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\mathbf{s}}_j^{Q'}\boldsymbol{\omega}'_j \\ 2\left(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{b}}'_j\boldsymbol{\omega}'_j + \left(\mathbf{A}_j\mathbf{b}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\mathbf{a}}'_i\boldsymbol{\omega}'_i + \left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\mathbf{b}}'_j\boldsymbol{\omega}'_j \end{bmatrix} \tag{20}
$$

The reaction forces and moments in the universal joint can be found as:

$$
\mathbf{g}^c = \mathbf{\Phi}^T\boldsymbol{\lambda} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \left(-\mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\right)^T & \left(-\left(\mathbf{A}_j\mathbf{b}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i\right)^T \\ -\mathbf{I} & \mathbf{0} \\ \left(\mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\right)^T & \left(-\left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{b}}'_j\right)^T \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} \tag{21}
$$

Cylindrical Joint (Translational)

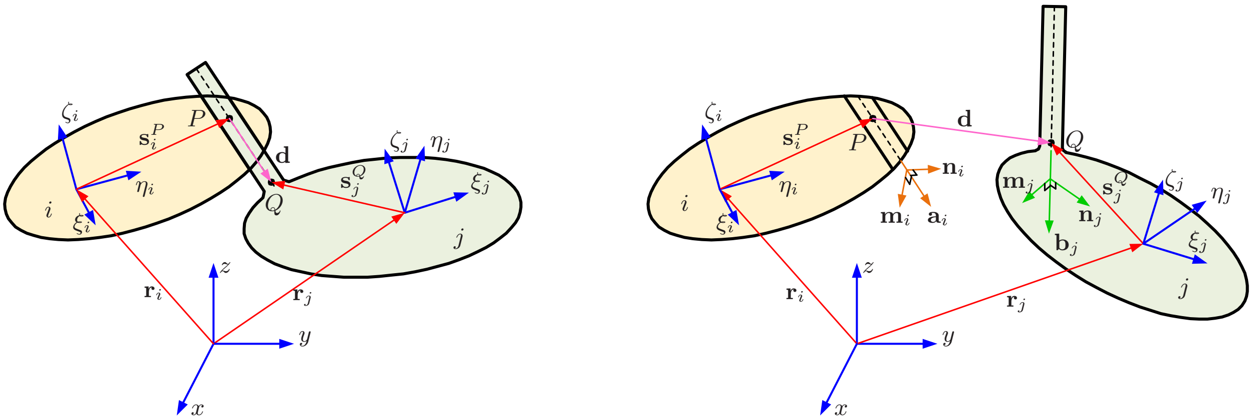

Figure 6: Cylindrical joint (translational joint) - removes 4 degrees of freedom.

The joint allows translation along and rotation around the dashed sliding line. The points P and Q are located on the sliding line on the respective bodies. A helping vector $\mathbf{d}$ between the points P and Q is introduced:

$$
\begin{aligned}
\mathbf{d} &= \mathbf{r}_j + \mathbf{A}_j\mathbf{s}_j^{Q'} - \mathbf{r}_i - \mathbf{A}_i\mathbf{s}_i^{P'} && (22)\\
\dot{\mathbf{d}} &= \dot{\mathbf{r}}_j + \mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{s}_j^{Q'} - \dot{\mathbf{r}}_i - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{s}_i^{P'} && (23)\\
\ddot{\mathbf{d}} &= \ddot{\mathbf{r}}_j + \mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{s}_j^{Q'} + \mathbf{A}_j\dot{\tilde{\boldsymbol{\omega}}}'_j\mathbf{s}_j^{Q'} - \ddot{\mathbf{r}}_i - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{s}_i^{P'} - \mathbf{A}_i\dot{\tilde{\boldsymbol{\omega}}}'_i\mathbf{s}_i^{P'} && (24)
\end{aligned}
$$

Position constraints

$$
\begin{aligned}
\boldsymbol{\Phi} &= \begin{bmatrix} \mathbf{m}_i^T\mathbf{d} \\ \mathbf{n}_i^T\mathbf{d} \\ \mathbf{m}_j^T\mathbf{a}_i \\ \mathbf{n}_j^T\mathbf{a}_i \end{bmatrix} \\
&= \begin{bmatrix} \left(\mathbf{A}_i\mathbf{m}'_i\right)^T\mathbf{d} \\ \left(\mathbf{A}_i\mathbf{n}'_i\right)^T\mathbf{d} \\ \left(\mathbf{A}_j\mathbf{m}'_j\right)^T\mathbf{A}_i\mathbf{a}'_i \\ \left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\mathbf{a}'_i \end{bmatrix} = \mathbf{0} && (25)
\end{aligned}
$$

Velocity constraints

$$\dot{\mathbf{\Phi}} = \begin{bmatrix} \left(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{m}'_i\right)^T\mathbf{d} + \left(\mathbf{A}_i\mathbf{m}'_i\right)^T\dot{\mathbf{d}} \\ \left(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i\right)^T\mathbf{d} + \left(\mathbf{A}_i\mathbf{n}'_i\right)^T\dot{\mathbf{d}} \\ \left(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{m}'_j\right)^T\mathbf{A}_i\mathbf{a}'_i + \left(\mathbf{A}_j\mathbf{m}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i \\ \left(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j\right)^T\mathbf{A}_i\mathbf{a}'_i + \left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i \end{bmatrix} = \mathbf{0}$$

$$= \begin{bmatrix} -\mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{m}}'_i\boldsymbol{\omega}'_i + \left(\mathbf{A}_i\mathbf{m}'_i\right)^T\left(\dot{\mathbf{r}}_j + \mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{s}_j^{Q'} - \dot{\mathbf{r}}_i - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{s}_i^{P'}\right) \\ -\mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{n}}'_i\boldsymbol{\omega}'_i + \left(\mathbf{A}_i\mathbf{n}'_i\right)^T\left(\dot{\mathbf{r}}_j + \mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{s}_j^{Q'} - \dot{\mathbf{r}}_i - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{s}_i^{P'}\right) \\ -\left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{m}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_j\mathbf{m}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i\boldsymbol{\omega}'_i \\ -\left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{n}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i\boldsymbol{\omega}'_i \end{bmatrix}$$

$$= \begin{bmatrix} -\mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{m}}'_i\boldsymbol{\omega}'_i + \left(\mathbf{A}_i\mathbf{m}'_i\right)^T\left(\dot{\mathbf{r}}_j - \mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\boldsymbol{\omega}'_j - \dot{\mathbf{r}}_i + \mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\boldsymbol{\omega}'_i\right) \\ -\mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{n}}'_i\boldsymbol{\omega}'_i + \left(\mathbf{A}_i\mathbf{n}'_i\right)^T\left(\dot{\mathbf{r}}_j - \mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\boldsymbol{\omega}'_j - \dot{\mathbf{r}}_i + \mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\boldsymbol{\omega}'_i\right) \\ -\left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{m}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_j\mathbf{m}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i\boldsymbol{\omega}'_i \\ -\left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{n}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i\boldsymbol{\omega}'_i \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -\left(\mathbf{A}_i\mathbf{m}'_i\right)^T & \left(-\mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{m}}'_i + \left(\mathbf{A}_i\mathbf{m}'_i\right)^T\left(\mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\right)\right) & \left(\mathbf{A}_i\mathbf{m}'_i\right)^T & \left(\mathbf{A}_i\mathbf{m}'_i\right)^T\left(-\mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\right) \\ -\left(\mathbf{A}_i\mathbf{n}'_i\right)^T & \left(-\mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{n}}'_i + \left(\mathbf{A}_i\mathbf{n}'_i\right)^T\left(\mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\right)\right) & \left(\mathbf{A}_i\mathbf{n}'_i\right)^T & \left(\mathbf{A}_i\mathbf{n}'_i\right)^T\left(-\mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\right) \\ \mathbf{0} & -\left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{m}}'_j & \mathbf{0} & -\left(\mathbf{A}_j\mathbf{m}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i \\ \mathbf{0} & -\left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{n}}'_j & \mathbf{0} & -\left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i \end{bmatrix}}_{\text{Entries in jacobian matrix}} \begin{bmatrix} \dot{\mathbf{r}}_i \\ \boldsymbol{\omega}'_i \\ \dot{\mathbf{r}}_j \\ \boldsymbol{\omega}'_j \end{bmatrix} = \mathbf{0} \quad (26)$$

Acceleration constraints

$$
\ddot{\boldsymbol{\Phi}} = \begin{bmatrix}
(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{m}'_i)^T\mathbf{d} + \left(\mathbf{A}_i\dot{\tilde{\boldsymbol{\omega}}}'_i\mathbf{m}'_i\right)^T\mathbf{d} + (\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{m}'_i)^T\dot{\mathbf{d}} + (\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{m}'_i)^T\dot{\mathbf{d}} + (\mathbf{A}_i\mathbf{m}'_i)^T\ddot{\mathbf{d}} \\
(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i)^T\mathbf{d} + \left(\mathbf{A}_i\dot{\tilde{\boldsymbol{\omega}}}'_i\mathbf{n}'_i\right)^T\mathbf{d} + (\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i)^T\dot{\mathbf{d}} + (\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i)^T\dot{\mathbf{d}} + (\mathbf{A}_i\mathbf{n}'_i)^T\ddot{\mathbf{d}} \\
(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{m}'_j)^T\mathbf{A}_i\mathbf{a}'_i + \left(\mathbf{A}_j\dot{\tilde{\boldsymbol{\omega}}}'_j\mathbf{m}'_j\right)^T\mathbf{A}_i\mathbf{a}'_i + (\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{m}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i + (\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{m}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i + (\mathbf{A}_j\mathbf{m}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i + (\mathbf{A}_j\mathbf{m}'_j)^T\mathbf{A}_i\dot{\tilde{\boldsymbol{\omega}}}'_i\mathbf{a}'_i \\
(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j)^T\mathbf{A}_i\mathbf{a}'_i + \left(\mathbf{A}_j\dot{\tilde{\boldsymbol{\omega}}}'_j\mathbf{n}'_j\right)^T\mathbf{A}_i\mathbf{a}'_i + (\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i + (\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i + (\mathbf{A}_j\mathbf{n}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i + (\mathbf{A}_j\mathbf{n}'_j)^T\mathbf{A}_i\dot{\tilde{\boldsymbol{\omega}}}'_i\mathbf{a}'_i
\end{bmatrix}
$$

$$
= \begin{bmatrix}
(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{m}'_i)^T\mathbf{d} - \mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{m}}'_i\dot{\boldsymbol{\omega}}'_i + 2(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{m}'_i)^T\dot{\mathbf{d}} + (\mathbf{A}_i\mathbf{m}'_i)^T\left(\ddot{\mathbf{r}}_j + \mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{s}_j^{Q'} + \mathbf{A}_j\dot{\tilde{\boldsymbol{\omega}}}'_j\mathbf{s}_j^{Q'} - \ddot{\mathbf{r}}_i - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{s}_i^{P'} - \mathbf{A}_i\dot{\tilde{\boldsymbol{\omega}}}'_i\mathbf{s}_i^{P'}\right) \\
(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i)^T\mathbf{d} - \mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{n}}'_i\dot{\boldsymbol{\omega}}'_i + 2(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i)^T\dot{\mathbf{d}} + (\mathbf{A}_i\mathbf{n}'_i)^T\left(\ddot{\mathbf{r}}_j + \mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{s}_j^{Q'} + \mathbf{A}_j\dot{\tilde{\boldsymbol{\omega}}}'_j\mathbf{s}_j^{Q'} - \ddot{\mathbf{r}}_i - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{s}_i^{P'} - \mathbf{A}_i\dot{\tilde{\boldsymbol{\omega}}}'_i\mathbf{s}_i^{P'}\right) \\
(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{m}'_j)^T\mathbf{A}_i\mathbf{a}'_i - (\mathbf{A}_i\mathbf{a}'_i)^T\mathbf{A}_j\mathbf{m}'_j\dot{\boldsymbol{\omega}}'_j + 2(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{m}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i + (\mathbf{A}_j\mathbf{m}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i - (\mathbf{A}_j\mathbf{m}'_j)^T\mathbf{A}_i\mathbf{a}'_i\dot{\boldsymbol{\omega}}'_i \\
(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j)^T\mathbf{A}_i\mathbf{a}'_i - (\mathbf{A}_i\mathbf{a}'_i)^T\mathbf{A}_j\mathbf{n}'_j\dot{\boldsymbol{\omega}}'_j + 2(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i + (\mathbf{A}_j\mathbf{n}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i - (\mathbf{A}_j\mathbf{n}'_j)^T\mathbf{A}_i\mathbf{a}'_i\dot{\boldsymbol{\omega}}'_i
\end{bmatrix}
$$

$$
= \begin{bmatrix}
(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{m}'_i)^T\mathbf{d} - \mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{m}}'_i\dot{\boldsymbol{\omega}}'_i + 2(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{m}'_i)^T\dot{\mathbf{d}} + (\mathbf{A}_i\mathbf{m}'_i)^T\left(\ddot{\mathbf{r}}_j + \mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{s}_j^{Q'} - \mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\dot{\boldsymbol{\omega}}'_j - \ddot{\mathbf{r}}_i - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{s}_i^{P'} - \mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\dot{\boldsymbol{\omega}}'_i\right) \\
(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i)^T\mathbf{d} - \mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{n}}'_i\dot{\boldsymbol{\omega}}'_i + 2(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i)^T\dot{\mathbf{d}} + (\mathbf{A}_i\mathbf{n}'_i)^T\left(\ddot{\mathbf{r}}_j + \mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{s}_j^{Q'} - \mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\dot{\boldsymbol{\omega}}'_j - \ddot{\mathbf{r}}_i - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{s}_i^{P'} - \mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\dot{\boldsymbol{\omega}}'_i\right) \\
(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{m}'_j)^T\mathbf{A}_i\mathbf{a}'_i - (\mathbf{A}_i\mathbf{a}'_i)^T\mathbf{A}_j\mathbf{m}'_j\dot{\boldsymbol{\omega}}'_j + 2(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{m}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i + (\mathbf{A}_j\mathbf{m}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i - (\mathbf{A}_j\mathbf{m}'_j)^T\mathbf{A}_i\mathbf{a}'_i\dot{\boldsymbol{\omega}}'_i \\
(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j)^T\mathbf{A}_i\mathbf{a}'_i - (\mathbf{A}_i\mathbf{a}'_i)^T\mathbf{A}_j\mathbf{n}'_j\dot{\boldsymbol{\omega}}'_j + 2(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i + (\mathbf{A}_j\mathbf{n}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i - (\mathbf{A}_j\mathbf{n}'_j)^T\mathbf{A}_i\mathbf{a}'_i\dot{\boldsymbol{\omega}}'_i
\end{bmatrix} = \mathbf{0} \Downarrow
$$

$$
\underbrace{\begin{bmatrix}
-(\mathbf{A}_i\mathbf{m}'_i)^T & \left(-\mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{m}}'_i + (\mathbf{A}_i\mathbf{m}'_i)^T\left(\mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\right)\right) & (\mathbf{A}_i\mathbf{m}'_i)^T & (\mathbf{A}_i\mathbf{m}'_i)^T\left(-\mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\right) \\
-(\mathbf{A}_i\mathbf{n}'_i)^T & \left(-\mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{n}}'_i + (\mathbf{A}_i\mathbf{n}'_i)^T\left(\mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\right)\right) & (\mathbf{A}_i\mathbf{n}'_i)^T & (\mathbf{A}_i\mathbf{n}'_i)^T\left(-\mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\right) \\
\mathbf{0} & -(\mathbf{A}_i\mathbf{a}'_i)^T\mathbf{A}_j\mathbf{m}'_j & \mathbf{0} & -\left(\mathbf{A}_j\mathbf{m}'_j\right)^T\mathbf{A}_i\mathbf{a}'_i \\
\mathbf{0} & -(\mathbf{A}_i\mathbf{a}'_i)^T\mathbf{A}_j\mathbf{n}'_j & \mathbf{0} & -\left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\mathbf{a}'_i
\end{bmatrix}}_{\text{Entries in jacobian matrix}}
\begin{bmatrix} \ddot{\mathbf{r}}_i \\ \dot{\boldsymbol{\omega}}'_i \\ \ddot{\mathbf{r}}_j \\ \dot{\boldsymbol{\omega}}'_j \end{bmatrix}
$$

$$
= \begin{bmatrix}
-(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{m}'_i)^T\mathbf{d} - 2(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{m}'_i)^T\dot{\mathbf{d}} - (\mathbf{A}_i\mathbf{m}'_i)^T\left(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{s}_j^{Q'} - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{s}_i^{P'}\right) \\
-(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i)^T\mathbf{d} - 2(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i)^T\dot{\mathbf{d}} - (\mathbf{A}_i\mathbf{n}'_i)^T\left(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{s}_j^{Q'} - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{s}_i^{P'}\right) \\
-\left(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{m}'_j\right)^T\mathbf{A}_i\mathbf{a}'_i - 2\left(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{m}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i - \left(\mathbf{A}_j\mathbf{m}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i \\
-\left(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j\right)^T\mathbf{A}_i\mathbf{a}'_i - 2\left(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i - \left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i
\end{bmatrix} \tag{27}
$$

The reaction forces and moments in the cylindrical joint can be found as:

$$
\begin{aligned}
\mathbf{g}^c &= \boldsymbol{\Phi}^T\boldsymbol{\lambda} \\
&= \begin{bmatrix}
-(\mathbf{A}_i\mathbf{m}'_i) & -(\mathbf{A}_i\mathbf{n}'_i) & \mathbf{0} & \mathbf{0} \\
\left(-\mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{m}}'_i + (\mathbf{A}_i\mathbf{m}'_i)^T\left(\mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\right)\right)^T & \left(-\mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{n}}'_i + (\mathbf{A}_i\mathbf{n}'_i)^T\left(\mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\right)\right)^T & \left(-(\mathbf{A}_i\mathbf{a}'_i)^T\mathbf{A}_j\mathbf{m}'_j\right)^T & \left(-(\mathbf{A}_i\mathbf{a}'_i)^T\mathbf{A}_j\mathbf{n}'_j\right)^T \\
(\mathbf{A}_i\mathbf{m}'_i) & (\mathbf{A}_i\mathbf{n}'_i) & \mathbf{0} & \mathbf{0} \\
\left((\mathbf{A}_i\mathbf{m}'_i)^T\left(-\mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\right)\right)^T & \left((\mathbf{A}_i\mathbf{n}'_i)^T\left(-\mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\right)\right)^T & \left(-(\mathbf{A}_j\mathbf{m}'_j)^T\mathbf{A}_i\mathbf{a}'_i\right)^T & \left(-(\mathbf{A}_j\mathbf{n}'_j)^T\mathbf{A}_i\mathbf{a}'_i\right)^T
\end{bmatrix}
\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix}
\end{aligned} \tag{28}
$$

Prismatic Joint

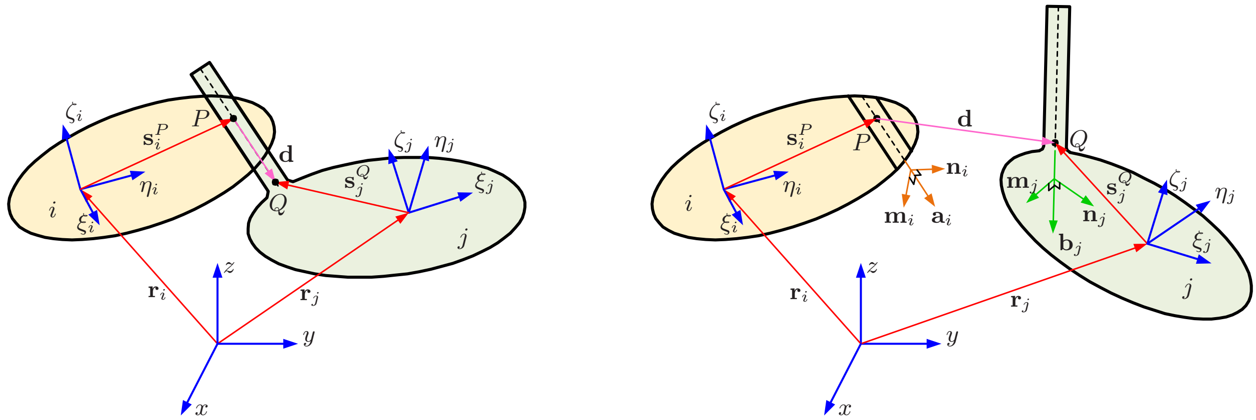

Figure 7: Prismatic joint - removes 5 degrees of freedom.

The joint allows translation along the dashed sliding line but restricts all other translations and relative rotations. This can be obtained in a real system if the common profile is not round but for example square or spline shaped. A helping vector $\mathbf{d}$ between the points P and Q is introduced:

$$\mathbf{d} = \mathbf{r}_j + \mathbf{A}_j \mathbf{s}_j^{Q'} - \mathbf{r}_i - \mathbf{A}_i \mathbf{s}_i^{P'} \tag{29}$$

$$\dot{\mathbf{d}} = \dot{\mathbf{r}}_j + \mathbf{A}_j \tilde{\boldsymbol{\omega}}'_j \mathbf{s}_j^{Q'} - \dot{\mathbf{r}}_i - \mathbf{A}_i \tilde{\boldsymbol{\omega}}'_i \mathbf{s}_i^{P'} \tag{30}$$

$$\ddot{\mathbf{d}} = \ddot{\mathbf{r}}_j + \mathbf{A}_j \tilde{\boldsymbol{\omega}}'_j \tilde{\boldsymbol{\omega}}'_j \mathbf{s}_j^{Q'} + \mathbf{A}_j \dot{\tilde{\boldsymbol{\omega}}}'_j \mathbf{s}_j^{Q'} - \ddot{\mathbf{r}}_i - \mathbf{A}_i \tilde{\boldsymbol{\omega}}'_i \tilde{\boldsymbol{\omega}}'_i \mathbf{s}_i^{P'} - \mathbf{A}_i \dot{\tilde{\boldsymbol{\omega}}}'_i \mathbf{s}_i^{P'} \tag{31}$$

Position constraints:

$$\begin{aligned}\boldsymbol{\Phi} &= \begin{bmatrix} \mathbf{m}_i^T \mathbf{d} \\ \mathbf{n}_i^T \mathbf{d} \\ \mathbf{m}_j^T \mathbf{a}_i \\ \mathbf{n}_j^T \mathbf{a}_i \\ \mathbf{n}_j^T \mathbf{n}_i \end{bmatrix} \\ &= \begin{bmatrix} (\mathbf{A}_i \mathbf{m}'_i)^T \mathbf{d} \\ (\mathbf{A}_i \mathbf{n}'_i)^T \mathbf{d} \\ \left(\mathbf{A}_j \mathbf{m}'_j\right)^T \mathbf{A}_i \mathbf{a}'_i \\ \left(\mathbf{A}_j \mathbf{n}'_j\right)^T \mathbf{A}_i \mathbf{a}'_i \\ \left(\mathbf{A}_j \mathbf{n}'_j\right)^T \mathbf{A}_i \mathbf{n}'_i \end{bmatrix} = \mathbf{0} \end{aligned} \tag{32}$$

Velocity constraints:

$$
\begin{aligned}
\dot{\boldsymbol{\Phi}} &= \begin{bmatrix}
\left(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{m}'_i\right)^T\mathbf{d} + \left(\mathbf{A}_i\mathbf{m}'_i\right)^T\dot{\mathbf{d}} \\
\left(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i\right)^T\mathbf{d} + \left(\mathbf{A}_i\mathbf{n}'_i\right)^T\dot{\mathbf{d}} \\
\left(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{m}'_j\right)^T\mathbf{A}_i\mathbf{a}'_i + \left(\mathbf{A}_j\mathbf{m}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i \\
\left(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j\right)^T\mathbf{A}_i\mathbf{a}'_i + \left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i \\
\left(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j\right)^T\mathbf{A}_i\mathbf{n}'_i + \left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i
\end{bmatrix} \\
&= \begin{bmatrix}
-\mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{m}}'_i\boldsymbol{\omega}'_i + \left(\mathbf{A}_i\mathbf{m}'_i\right)^T\left(\dot{\mathbf{r}}_j + \mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{s}_j^{Q'} - \dot{\mathbf{r}}_i - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{s}_i^{P'}\right) \\
-\mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{n}}'_i\boldsymbol{\omega}'_i + \left(\mathbf{A}_i\mathbf{n}'_i\right)^T\left(\dot{\mathbf{r}}_j + \mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{s}_j^{Q'} - \dot{\mathbf{r}}_i - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{s}_i^{P'}\right) \\
-\left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{m}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_j\mathbf{m}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i\boldsymbol{\omega}'_i \\
-\left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{n}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i\boldsymbol{\omega}'_i \\
-\left(\mathbf{A}_i\mathbf{n}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{n}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{n}}'_i\boldsymbol{\omega}'_i
\end{bmatrix} \\
&= \begin{bmatrix}
-\mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{m}}'_i\boldsymbol{\omega}'_i + \left(\mathbf{A}_i\mathbf{m}'_i\right)^T\left(\dot{\mathbf{r}}_j - \mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\boldsymbol{\omega}'_j - \dot{\mathbf{r}}_i + \mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\boldsymbol{\omega}'_i\right) \\
-\mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{n}}'_i\boldsymbol{\omega}'_i + \left(\mathbf{A}_i\mathbf{n}'_i\right)^T\left(\dot{\mathbf{r}}_j - \mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\boldsymbol{\omega}'_j - \dot{\mathbf{r}}_i + \mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\boldsymbol{\omega}'_i\right) \\
-\left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{m}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_j\mathbf{m}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i\boldsymbol{\omega}'_i \\
-\left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{n}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i\boldsymbol{\omega}'_i \\
-\left(\mathbf{A}_i\mathbf{n}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{n}}'_j\boldsymbol{\omega}'_j - \left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{n}}'_i\boldsymbol{\omega}'_i
\end{bmatrix}
\end{aligned}
$$

$$
\underbrace{\begin{bmatrix}
-\left(\mathbf{A}_i\mathbf{m}'_i\right)^T & \left(-\mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{m}}'_i + \left(\mathbf{A}_i\mathbf{m}'_i\right)^T\left(\mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\right)\right) & \left(\mathbf{A}_i\mathbf{m}'_i\right)^T & \left(\mathbf{A}_i\mathbf{m}'_i\right)^T\left(-\mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\right) \\
-\left(\mathbf{A}_i\mathbf{n}'_i\right)^T & \left(-\mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{n}}'_i + \left(\mathbf{A}_i\mathbf{n}'_i\right)^T\left(\mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\right)\right) & \left(\mathbf{A}_i\mathbf{n}'_i\right)^T & \left(\mathbf{A}_i\mathbf{n}'_i\right)^T\left(-\mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\right) \\
\mathbf{0} & -\left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{m}}'_j & \mathbf{0} & -\left(\mathbf{A}_j\mathbf{m}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i \\
\mathbf{0} & -\left(\mathbf{A}_i\mathbf{a}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{n}}'_j & \mathbf{0} & -\left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{a}}'_i \\
\mathbf{0} & -\left(\mathbf{A}_i\mathbf{n}'_i\right)^T\mathbf{A}_j\tilde{\mathbf{n}}'_j & \mathbf{0} & -\left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\mathbf{n}}'_i
\end{bmatrix}}_{\text{Entries in jacobian matrix}}
\begin{bmatrix}
\dot{\mathbf{r}}_i \\ \boldsymbol{\omega}'_i \\ \dot{\mathbf{r}}_j \\ \boldsymbol{\omega}'_j
\end{bmatrix} = \mathbf{0} \quad (33)
$$

Acceleration constraints:

$$
\ddot{\boldsymbol{\Phi}} = \begin{bmatrix}
(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{m}'_i)^T\mathbf{d} + \left(\mathbf{A}_i\dot{\tilde{\boldsymbol{\omega}}}'_i\mathbf{m}'_i\right)^T\mathbf{d} + (\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{m}'_i)^T\dot{\mathbf{d}} + (\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{m}'_i)^T\dot{\mathbf{d}} + (\mathbf{A}_i\mathbf{m}'_i)^T\ddot{\mathbf{d}} \\
(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i)^T\mathbf{d} + \left(\mathbf{A}_i\dot{\tilde{\boldsymbol{\omega}}}'_i\mathbf{n}'_i\right)^T\mathbf{d} + (\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i)^T\dot{\mathbf{d}} + (\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i)^T\dot{\mathbf{d}} + (\mathbf{A}_i\mathbf{n}'_i)^T\ddot{\mathbf{d}} \\
(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{m}'_j)^T\mathbf{A}_i\mathbf{a}'_i + \left(\mathbf{A}_j\dot{\tilde{\boldsymbol{\omega}}}'_j\mathbf{m}'_j\right)^T\mathbf{A}_i\mathbf{a}'_i + (\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{m}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i + (\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{m}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i + (\mathbf{A}_j\mathbf{m}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i + (\mathbf{A}_j\mathbf{m}'_j)^T\mathbf{A}_i\dot{\tilde{\boldsymbol{\omega}}}'_i\mathbf{a}'_i \\
(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j)^T\mathbf{A}_i\mathbf{a}'_i + \left(\mathbf{A}_j\dot{\tilde{\boldsymbol{\omega}}}'_j\mathbf{n}'_j\right)^T\mathbf{A}_i\mathbf{a}'_i + (\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i + (\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i + (\mathbf{A}_j\mathbf{n}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i + (\mathbf{A}_j\mathbf{n}'_j)^T\mathbf{A}_i\dot{\tilde{\boldsymbol{\omega}}}'_i\mathbf{a}'_i \\
(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j)^T\mathbf{A}_i\mathbf{n}'_i + \left(\mathbf{A}_j\dot{\tilde{\boldsymbol{\omega}}}'_j\mathbf{n}'_j\right)^T\mathbf{A}_i\mathbf{n}'_i + (\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i + (\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i + (\mathbf{A}_j\mathbf{n}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i + (\mathbf{A}_j\mathbf{n}'_j)^T\mathbf{A}_i\dot{\tilde{\boldsymbol{\omega}}}'_i\mathbf{n}'_i
\end{bmatrix}
$$

$$
= \begin{bmatrix}
(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{m}'_i)^T\mathbf{d} - \mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{m}}'_i\dot{\boldsymbol{\omega}}'_i + 2(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{m}'_i)^T\dot{\mathbf{d}} + (\mathbf{A}_i\mathbf{m}'_i)^T\left(\ddot{\mathbf{r}}_j + \mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{s}_j^{Q'} + \mathbf{A}_j\dot{\tilde{\boldsymbol{\omega}}}'_j\mathbf{s}_j^{Q'} - \ddot{\mathbf{r}}_i - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{s}_i^{P'} - \mathbf{A}_i\dot{\tilde{\boldsymbol{\omega}}}'_i\mathbf{s}_i^{P'}\right) \\
(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i)^T\mathbf{d} - \mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{n}}'_i\dot{\boldsymbol{\omega}}'_i + 2(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i)^T\dot{\mathbf{d}} + (\mathbf{A}_i\mathbf{n}'_i)^T\left(\ddot{\mathbf{r}}_j + \mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{s}_j^{Q'} + \mathbf{A}_j\dot{\tilde{\boldsymbol{\omega}}}'_j\mathbf{s}_j^{Q'} - \ddot{\mathbf{r}}_i - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{s}_i^{P'} - \mathbf{A}_i\dot{\tilde{\boldsymbol{\omega}}}'_i\mathbf{s}_i^{P'}\right) \\
(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{m}'_j)^T\mathbf{A}_i\mathbf{a}'_i - (\mathbf{A}_i\mathbf{a}'_i)^T\mathbf{A}_j\mathbf{m}'_j\dot{\boldsymbol{\omega}}'_j + 2(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{m}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i + (\mathbf{A}_j\mathbf{m}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i - (\mathbf{A}_j\mathbf{m}'_j)^T\mathbf{A}_i\mathbf{a}'_i\dot{\boldsymbol{\omega}}'_i \\
(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j)^T\mathbf{A}_i\mathbf{a}'_i - (\mathbf{A}_i\mathbf{a}'_i)^T\mathbf{A}_j\mathbf{n}'_j\dot{\boldsymbol{\omega}}'_j + 2(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i + (\mathbf{A}_j\mathbf{n}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i - (\mathbf{A}_j\mathbf{n}'_j)^T\mathbf{A}_i\mathbf{a}'_i\dot{\boldsymbol{\omega}}'_i \\
(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j)^T\mathbf{A}_i\mathbf{n}'_i - (\mathbf{A}_i\mathbf{n}'_i)^T\mathbf{A}_j\mathbf{n}'_j\dot{\boldsymbol{\omega}}'_j + 2(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i + (\mathbf{A}_j\mathbf{n}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i - (\mathbf{A}_j\mathbf{n}'_j)^T\mathbf{A}_i\mathbf{n}'_i\dot{\boldsymbol{\omega}}'_i
\end{bmatrix}
$$

$$
= \begin{bmatrix}
(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{m}'_i)^T\mathbf{d} - \mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{m}}'_i\dot{\boldsymbol{\omega}}'_i + 2(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{m}'_i)^T\dot{\mathbf{d}} + (\mathbf{A}_i\mathbf{m}'_i)^T\left(\ddot{\mathbf{r}}_j + \mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{s}_j^{Q'} - \mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\dot{\boldsymbol{\omega}}'_j - \ddot{\mathbf{r}}_i - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{s}_i^{P'} - \mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\dot{\boldsymbol{\omega}}'_i\right) \\
(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i)^T\mathbf{d} - \mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{n}}'_i\dot{\boldsymbol{\omega}}'_i + 2(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i)^T\dot{\mathbf{d}} + (\mathbf{A}_i\mathbf{n}'_i)^T\left(\ddot{\mathbf{r}}_j + \mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{s}_j^{Q'} - \mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\dot{\boldsymbol{\omega}}'_j - \ddot{\mathbf{r}}_i - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{s}_i^{P'} - \mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\dot{\boldsymbol{\omega}}'_i\right) \\
(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{m}'_j)^T\mathbf{A}_i\mathbf{a}'_i - (\mathbf{A}_i\mathbf{a}'_i)^T\mathbf{A}_j\mathbf{m}'_j\dot{\boldsymbol{\omega}}'_j + 2(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{m}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i + (\mathbf{A}_j\mathbf{m}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i - (\mathbf{A}_j\mathbf{m}'_j)^T\mathbf{A}_i\mathbf{a}'_i\dot{\boldsymbol{\omega}}'_i \\
(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j)^T\mathbf{A}_i\mathbf{a}'_i - (\mathbf{A}_i\mathbf{a}'_i)^T\mathbf{A}_j\mathbf{n}'_j\dot{\boldsymbol{\omega}}'_j + 2(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i + (\mathbf{A}_j\mathbf{n}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i - (\mathbf{A}_j\mathbf{n}'_j)^T\mathbf{A}_i\mathbf{a}'_i\dot{\boldsymbol{\omega}}'_i \\
(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j)^T\mathbf{A}_i\mathbf{n}'_i - (\mathbf{A}_i\mathbf{n}'_i)^T\mathbf{A}_j\mathbf{n}'_j\dot{\boldsymbol{\omega}}'_j + 2(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i + (\mathbf{A}_j\mathbf{n}'_j)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i - (\mathbf{A}_j\mathbf{n}'_j)^T\mathbf{A}_i\mathbf{n}'_i\dot{\boldsymbol{\omega}}'_i
\end{bmatrix} = \mathbf{0} \Downarrow
$$

$$
\underbrace{\begin{bmatrix}
-(\mathbf{A}_i\mathbf{m}'_i)^T & \left(-\mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{m}}'_i + (\mathbf{A}_i\mathbf{m}'_i)^T\left(\mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\right)\right) & (\mathbf{A}_i\mathbf{m}'_i)^T & (\mathbf{A}_i\mathbf{m}'_i)^T\left(-\mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\right) \\
-(\mathbf{A}_i\mathbf{n}'_i)^T & \left(-\mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{n}}'_i + (\mathbf{A}_i\mathbf{n}'_i)^T\left(\mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\right)\right) & (\mathbf{A}_i\mathbf{n}'_i)^T & (\mathbf{A}_i\mathbf{n}'_i)^T\left(-\mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\right) \\
\mathbf{0} & -(\mathbf{A}_i\mathbf{a}'_i)^T\mathbf{A}_j\mathbf{m}'_j & \mathbf{0} & -\left(\mathbf{A}_j\mathbf{m}'_j\right)^T\mathbf{A}_i\mathbf{a}'_i \\
\mathbf{0} & -(\mathbf{A}_i\mathbf{a}'_i)^T\mathbf{A}_j\mathbf{n}'_j & \mathbf{0} & -\left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\mathbf{a}'_i \\
\mathbf{0} & -(\mathbf{A}_i\mathbf{n}'_i)^T\mathbf{A}_j\mathbf{n}'_j & \mathbf{0} & -\left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\mathbf{n}'_i
\end{bmatrix}}_{\text{Entries in jacobian matrix}}
\begin{bmatrix} \ddot{\mathbf{r}}_i \\ \dot{\boldsymbol{\omega}}'_i \\ \ddot{\mathbf{r}}_j \\ \dot{\boldsymbol{\omega}}'_j \end{bmatrix}
$$

$$
= \begin{bmatrix}
-(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{m}'_i)^T\mathbf{d} - 2(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{m}'_i)^T\dot{\mathbf{d}} - (\mathbf{A}_i\mathbf{m}'_i)^T\left(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{s}_j^{Q'} - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{s}_i^{P'}\right) \\
-(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i)^T\mathbf{d} - 2(\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i)^T\dot{\mathbf{d}} - (\mathbf{A}_i\mathbf{n}'_i)^T\left(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{s}_j^{Q'} - \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{s}_i^{P'}\right) \\
-\left(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{m}'_j\right)^T\mathbf{A}_i\mathbf{a}'_i - 2\left(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{m}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i - \left(\mathbf{A}_j\mathbf{m}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i \\
-\left(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j\right)^T\mathbf{A}_i\mathbf{a}'_i - 2\left(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i - \left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{a}'_i \\
-\left(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j\right)^T\mathbf{A}_i\mathbf{n}'_i - 2\left(\mathbf{A}_j\tilde{\boldsymbol{\omega}}'_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i - \left(\mathbf{A}_j\mathbf{n}'_j\right)^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}'_i\tilde{\boldsymbol{\omega}}'_i\mathbf{n}'_i
\end{bmatrix} \tag{34}
$$

The reaction forces and moments in the cylindrical joint can be found as:

$$
\begin{aligned}
\mathbf{g}^c &= \boldsymbol{\Phi}^T\boldsymbol{\lambda} \\
&= \begin{bmatrix}
-(\mathbf{A}_i\mathbf{m}'_i) & -(\mathbf{A}_i\mathbf{n}'_i) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\left(-\mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{m}}'_i + (\mathbf{A}_i\mathbf{m}'_i)^T\left(\mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\right)\right)^T & \left(-\mathbf{d}^T\mathbf{A}_i\tilde{\mathbf{n}}'_i + (\mathbf{A}_i\mathbf{n}'_i)^T\left(\mathbf{A}_i\tilde{\mathbf{s}}_i^{P'}\right)\right)^T & \left(-(\mathbf{A}_i\mathbf{a}'_i)^T\mathbf{A}_j\mathbf{m}'_j\right)^T & \left(-(\mathbf{A}_i\mathbf{a}'_i)^T\mathbf{A}_j\mathbf{n}'_j\right)^T & \left(-(\mathbf{A}_i\mathbf{n}'_i)^T\mathbf{A}_j\mathbf{n}'_j\right)^T \\
(\mathbf{A}_i\mathbf{m}'_i) & (\mathbf{A}_i\mathbf{n}'_i) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\left((\mathbf{A}_i\mathbf{m}'_i)^T\left(-\mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\right)\right)^T & \left((\mathbf{A}_i\mathbf{n}'_i)^T\left(-\mathbf{A}_j\tilde{\mathbf{s}}_j^{Q'}\right)\right)^T & \left(-(\mathbf{A}_j\mathbf{m}'_j)^T\mathbf{A}_i\mathbf{a}'_i\right)^T & \left(-(\mathbf{A}_j\mathbf{n}'_j)^T\mathbf{A}_i\mathbf{a}'_i\right)^T & \left(-(\mathbf{A}_j\mathbf{n}'_j)^T\mathbf{A}_i\mathbf{n}'_i\right)^T
\end{bmatrix}
\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix}
\end{aligned} \tag{35}
$$