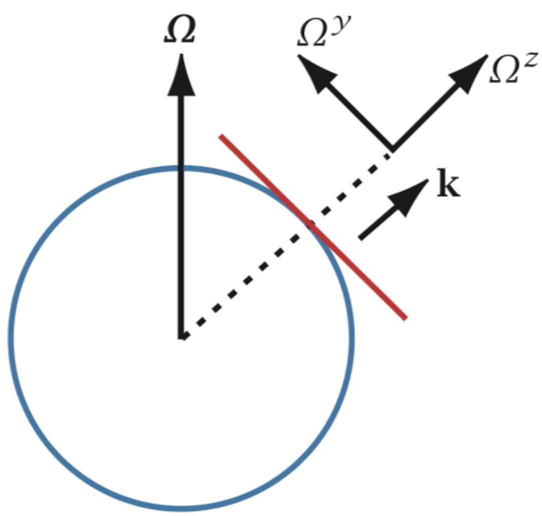
1 Tangent plane approximation

We now derive the tangent plane approximation to the momentum equation:

$$\boxed{\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p - 2\left(\vec{\Omega} \times \vec{u}\right) - \vec{g}\,.}$$

First, we construct the local frame, then we compute the Coriolis force and simplify it using scale arguments, and finally we simplify the momentum equation.

Local Frame: For a point on Earth's surface with latitude ϕ, consider the tangent plane spanned by unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$, where $\hat{\imath}$ points east, $\hat{\jmath}$ points north, and $\hat{k}$ is normal to the surface.



In the local frame, $\vec{\Omega} = \Omega^y \hat{\jmath} + \Omega^z \hat{k} = \|\vec{\Omega}\| \cos\phi \; \hat{\jmath} + \|\vec{\Omega}\| \sin\phi \; \hat{k}\,.$

Thus,

$$\vec{\Omega} \times \vec{u} = (\Omega^y w - \Omega^z v,\, \Omega^z u,\, -\Omega^y u) \approx (-\Omega^z v,\, \Omega^z u,\, 0)\,,$$

where the approximation follows from two scale analyses for large-scale flows:

1. For the $\hat{\imath}$ direction: $|\Omega^y w| \ll |\Omega^z v|$.
2. For $\hat{k}$ direction: $|\Omega^y u| \ll |g|$.

Thus, the Coriolis force becomes

$$\boxed{-2\vec{\Omega} \times \vec{u} = (fv, -fu, 0)}$$

where $f = 2\|\vec{\Omega}\| \sin\phi$ is called the Coriolis parameter.

Note that under these assumptions, the Coriolis force deflects air parcels horizontally; it has no vertical component!

Finally, plugging in the expression for the Coriolis force, the tangent plane approximation to the momentum equations is

$$\boxed{\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv\,, \qquad \frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu\,, \qquad \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g\,.}$$

2 Shallow-Water Model

Consider the momentum equations together with the continuity equation. We have five unknown quantities, u, v, w, ρ, p, and only four equations. The system is underdetermined. Typically, we close the system by introducing a thermodynamic equation, and an equation of state, so we end up with 6 unknowns u, v, w, ρ, p, T and six equations (which we will refer to as the primitive equations).

The primitive equations are very complicated. In this section, we examine a simplification called the shallow-water model, which describes a fluid with the following properties:

1. The fluid is shallow, meaning that its depth is much smaller than its horizontal scale, and consequently, the hydrostatic approximation is satisfied. (The ocean and the atmosphere are both shallow fluids.)
2. The fluid has constant density.
3. The fluid is bounded below by a rigid surface, where the fluid satisfies a no-penetration boundary condition. (This is appropriate for modeling flow over solid terrain.)
4. The fluid is bounded from above by a free surface, above which is another fluid of negligible inertia, meaning that overlying fluid exerts negligible stress (such as air over ocean, or stratosphere over troposphere.)
5. The flow is initially independent of z.

We now express these assumptions in mathematical form and derive the shallow-water equations.

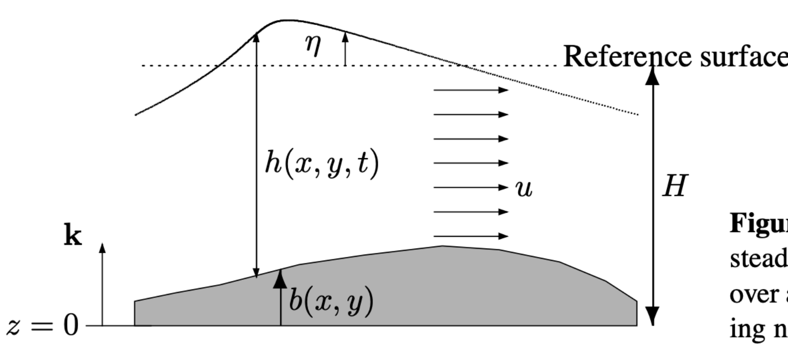


Figure 7-5 Schematic diagram of unsteady flow of a homogeneous fluid over an irregular bottom and the attending notation.

Our notation is that $\eta(x,y,t)$ is the height of the free surface, $b(x,y)$ is the height of the lower, rigid surface, and $h(x,y,t)$ is the thickness of a water column, so that $\eta(x,y,t) = b(x,y) + h(x,y,t)$.

From hydrostatic balance, $p(x, y, z, t)$ satisfies

$$\frac{\partial p}{\partial z} = -\rho g\,.$$

Integrating from z to the free surface $\eta(x, y, t)$ yields

$$p(x, y, z, t) = \rho g(\eta(x, y, t) - z) + p_\eta\,,$$

where $p_\eta = p(x, y, \eta(x, y, t), t)$ is the pressure at the free surface. We set $p_\eta = 0$ as a reference pressure without loss of generality.

It follows that

$$\frac{\partial p}{\partial x} = \rho g \frac{\partial \eta}{\partial x} \qquad \text{and} \qquad \frac{\partial p}{\partial y} = \rho g \frac{\partial \eta}{\partial y}$$

so the horizontal derivatives of pressure are independent of z. The horizontal momentum equations therefore become

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv = -g\frac{\partial \eta}{\partial x}\,, \qquad \frac{\mathrm{D}v}{\mathrm{D}t} + fu = -g\frac{\partial \eta}{\partial y}\,.$$

The right-hand sides are independent of z. Also, the left-hand sides are independent of z if $\vec{u}$ is independent of z. This tells us that if $\vec{u}$ is initially indepedent of z, it remains independent of z.

So, the velocities u and v are functions of x, y, and t only, and the horizontal momentum equations become

$$\boxed{\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -g\frac{\partial \eta}{\partial x} + fv \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -g\frac{\partial \eta}{\partial y} - fu \end{aligned}}$$

Since the fluid is assumed to have constant density, the continuity equation simplifes to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\,.$$

Integrating from $b(x, y)$ to $\eta(x, y, t)$ yields

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) h(x, y, t) + w_\eta - w_b = 0\,,$$

where

$$\begin{aligned} w_\eta &= w(x, y, \eta(x, y, t), t) \text{ is the vertical velocity at the free surface,} \\ w_b &= w(x, y, b(x, y), t) \text{ is the vertical velocity at the bottom surface.} \end{aligned}$$

We now find expresions for w_η and w_b from the boundary conditions.

At the bottom of the fluid, the no-penetration condition is $\vec{u} \cdot \vec{n} = 0$, where $\vec{n} = \vec{n}(x, y)$ is a normal vector field to the bottom surface $b(x, y)$. We can find a normal vector field to $b(x, y)$ by defining the function $f(x, y, z) = z - b(x, y)$. The bottom surface is the level set for $f(x, y, z) = 0$. So $\nabla F = (-b_x, -b_y, 1) = \vec{n}$ is normal to the surface, and the no-penetration tells us that

$$\vec{u} \cdot \vec{n} = -u\frac{\partial b}{\partial x} - v\frac{\partial b}{\partial y} + w_b = 0\,.$$

Hence,

$$w_b = u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y}\,.$$

The free surface is the level set of the function $f(x, y, z, t) = z - \eta(x, y, t)$ for $f = 0$. The boundary condition is $\frac{\mathrm{D}F}{\mathrm{D}t} = 0$.

$$\frac{\mathrm{D}F}{\mathrm{D}t} = -\frac{\partial \eta}{\partial t} - u\frac{\partial \eta}{\partial x} - v\frac{\partial \eta}{\partial y} + w_\eta = 0\,.$$

Hence,

$$w_\eta = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}\,.$$

Substituting the expressions for w_η and w_b into equation (number), combining like terms, and noting that $\partial_t \eta = \partial_t h$, the continuity equation takes on the form:

$$\boxed{\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = 0\,.}$$

Collecting up the boxed equations in this derivation, we get the shallow-water equations!

$$\boxed{\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -g\frac{\partial \eta}{\partial x} + fv \\ &\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -g\frac{\partial \eta}{\partial y} - fu \\ &\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = 0 \end{aligned}} \tag{2.1}$$

This system of PDEs contains only three unknowns: u, v, and h, so it is closed. This means that we have decoupled the horizontal momentum equations and continuity equations from the full set of primitive equations. Compared to the full primitive equations, the shallow-water equations are much simpler!

Now, we have to figure out how to solve this system numerically. One strategy is to write the shallow-water equations as a balance law, which gives us access to a class of numerical methods called finite volume methods.

3 Balance Laws

What is a balance law?

Say we have a scalar quantity $q(\vec{x}, t)$ defined for $\vec{x} \in \Omega \subset \mathbb{R}^N$.

Moreover, suppose that the total amount of q contained in Ω only changes due to the flux of q over the boundary $\partial\Omega$ and the presence of sources/sinks of q within Ω. This is the balance statement.

Mathematically, it says that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} q = - \int_{\partial\Omega} \vec{f} \cdot \vec{n} + \int_{\Omega} s \,,$$

where $\vec{f} = \vec{f}(\vec{x}, t)$ is the flux of q, $s = s(\vec{x}, t)$ is the sources of q, and $\vec{n}$ is the outward unit normal vector field on $\partial\Omega$. This is called a balance law in integral form.

By differentiating under the integral sign on the LHS, applying the divergence theorem on the RHS, and using the fact the spatial domain is arbitrary, we obtain the balance law in differential form:

$$\partial_t q + \operatorname{div} \vec{f} = s \,.$$

4 Finite Volume Methods on a 2D Spatial Domain

In this section, we will derive the finite volume method, which is useful for solving balance laws numerically.

To begin, we divide our spatial domain D into finite volumes, also called grid cells. Here, we use a uniform Cartesian grid, as shown in figure 4.1. The (i, j) grid cell is

$$\mathcal{C}_{ij} = [x_{i-1/2},\, x_{i+1/2}] \times [y_{j-1/2},\, y_{j+1/2}] \,.$$

At the time step t^n, the average of $q(x, y, t)$ over the (i, j) grid cell is

$$Q_{ij}^n \approx \frac{1}{\Delta x \Delta y} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x, y, t^n) \mathrm{d}x \, \mathrm{d}y \,.$$

In finite volume methods, we are interested in stepping the cell average Q_{ij}^n forward in time, instead of stepping the quantity q_{ij}^n forward in time.

The balance law for $q(x, y, t)$ in integral form is

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathcal{C}_{ij}} q(x, y, t) \, \mathrm{d}x \, \mathrm{d}y = - \int_{\partial\mathcal{C}_{ij}} \vec{f} \cdot \vec{n} \, \mathrm{d}s + \iint_{\mathcal{C}_{ij}} s(x, y, t) \, \mathrm{d}x \, \mathrm{d}y$$

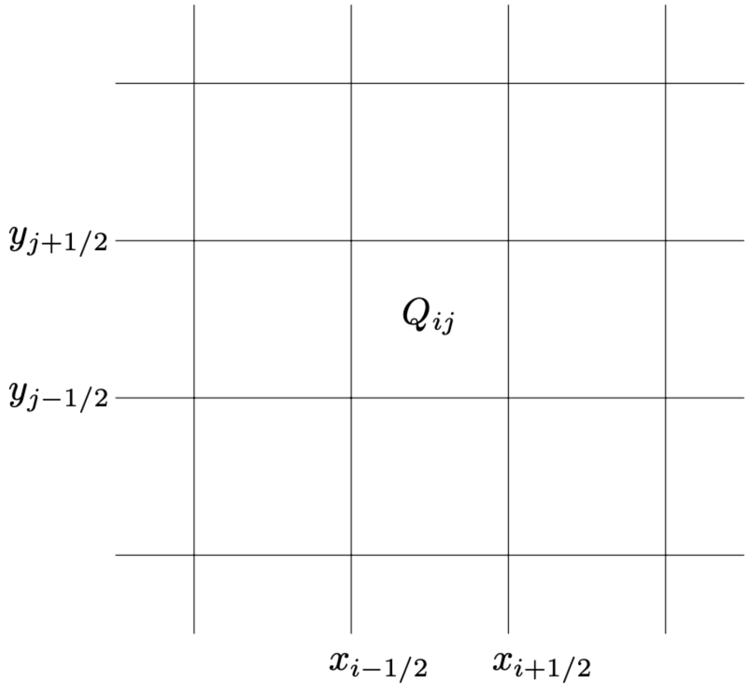


Figure 4.1: A finite volume discretization of a 2D domain.

Applied to the grid cell C_{ij}, with $\vec{f} = (f, g)$, we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathcal{C}_{ij}} q(x,y,t)\,\mathrm{d}x\,\mathrm{d}y = {} & \int_{y_{j-1/2}}^{y_{j+1/2}} f(x_{i-1/2},y,t)\,\mathrm{d}y - \int_{y_{j-1/2}}^{y_{j+1/2}} f(x_{i+1/2},y,t)\,\mathrm{d}y \\ & + \int_{x_{i-1/2}}^{x_{i+1/2}} g(x,y_{j-1/2},t)\,\mathrm{d}x - \int_{x_{i-1/2}}^{x_{i+1/2}} g(x,y_{j+1/2},t)\,\mathrm{d}x \\ & + \iint_{\mathcal{C}_{ij}} s(x,y,t)\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

Integrating both sides from t^n to t^{n+1} and dividing by $\Delta x \Delta y$ gives the finite volume update formula:

$$\boxed{Q_{ij}^{n+1} = Q_{ij}^{n} - \frac{\Delta t}{\Delta x}\left[F_{i+1/2,j}^{n} - F_{i-1/2,j}^{n}\right] - \frac{\Delta t}{\Delta y}\left[G_{i,j+1/2}^{n} - G_{i,j-1/2}^{n}\right] + \Delta t\left[S_{ij}^{n}\right]} \tag{4.1}$$

where

$$\begin{aligned}F_{i+1/2,j}^{n} &\approx \frac{1}{\Delta t \Delta y}\int_{t^n}^{t^{n+1}}\int_{y_{j-1/2}}^{y_{j+1/2}} f(x_{i+1/2},y,t)\,\mathrm{d}y\,\mathrm{d}t\,, \\ G_{i,j+1/2}^{n} &\approx \frac{1}{\Delta t \Delta x}\int_{t^n}^{t^{n+1}}\int_{x_{i-1/2}}^{x_{i+1/2}} g(x,y_{j+1/2},t)\,\mathrm{d}x\,\mathrm{d}t\,, \\ S_{ij}^{n} &\approx \frac{1}{\Delta t \Delta x \Delta y}\int_{t^n}^{t^{n+1}}\iint_{\mathcal{C}_{ij}} s(x,y,t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,.\end{aligned} \tag{4.2}$$

and the terms $F^n_{i-1/2,j}$ and $G^n_{i,j-1/2}$ follow.

5 Nice Properties of the Finite Volume Method

In the previous section, we derived the finite volume method by applying the balance law for $q(x, y, t)$ to grid cells. The finite volume method therefore inherits certain nice properties that mimic the balance law.

First of all, note that

$$\Delta x \Delta y \sum_{ij} Q^n_{ij} = \iint_D q(x, y, t^n)\,\mathrm{d}x\,\mathrm{d}y$$

that is, the sum over all the cell averages times dx dy, equals the integral of q The main question now is how to evaluate the F, G.

Recall that the balance law for q tells us that the total amount of q in the domain D

$$\iint_D q(x, y, t)\,\mathrm{d}x\,\mathrm{d}y$$

changes over time only due to fluxes of q through ∂D and sources of q within D.

We would like for the discrete sum

$$\Delta x \Delta y \sum_{ij} Q^n_{ij} = \iint_D q(x, y, t^n)\,\mathrm{d}x\,\mathrm{d}y$$

to keep track of the total amount of q in the same way, i.e., the discrete sum should only change over time due to boundary fluxes of q and sources of q.

The numerical scheme derived for Q^{n+1}_{ij} in the previous section is exactly such a scheme for which this property is satisfied. Suppose we are summing over a collection of grid cells $i = \{a, \ldots, b\}$ and $j = \{c, \ldots, d\}$. Then substituting in the finite volume scheme (4.1) and observing that the sums of flux differences are telescoping sums, we find that

$$\sum_{ij} Q^{n+1}_{ij} = \sum_{ij} Q^n_{ij} - \frac{\Delta t}{\Delta x} \sum_j \left[F^n_{b+1/2,j} - F^n_{a-1/2,j}\right] - \frac{\Delta t}{\Delta y} \sum_i \left[G^n_{i,d+1/2} - G^n_{i,c-1/2}\right] + \sum_{ij} S^n_{ij}$$

Thus, we have shown that the discrete sum for the total amount of q in the domain varies correctly due to fluxes through the boundary of the domain and sources/sinks within the domain.

6 The Main Challenge

In order to use the finite volume scheme (4.1), we need to figure out how to compute the flux terms and the source term. This is challenging because, if you look back at the definitions (4.2) of these terms, you will see that they are intended to approximate integrals from t^n to

t^{n+1}, and we only know what the fluxes/sources look like at time t^n, so we can't compute these integrals!

One idea is to note that $F^n_{i+1/2,\,j}$ is intended to approximate the average flux through the cell boundary $\{x_{i+1/2}\} \times [y_{j-1/2}, y_{j+1/2}]$, averaged from t^n to t^{n+1}.

$$F^n_{i+1/2,j} \approx \frac{1}{\Delta t \Delta y} \int_{t^n}^{t^{n+1}} \int_{y_{j-1/2}}^{y_{j+1/2}} f(x_{i+1/2}, y, t^n)\,\mathrm{d}y\,\mathrm{d}t$$

Thus, we can try approximating $F^n_{i+1/2,\,j}$ as the value of the flux f at the midpoint $(x_{i+1/2},\ y_j)$, at the midpoint in time $t^{n+1/2}$, that is,

$$F^n_{i+1/2,\,j} \approx f^{n+1/2}_{i+1/2,\,j}\,.$$

But, we don't know what $f^{n+1/2}_{i+1/2,\,j}$ is either! So, how does this help us? Well, we can try to predict this value, as we'll try in section 8.

Hence, we estimate the fluxes as:

$$F^n_{i+1/2,\,j} \approx f^{n+1/2}_{i+1/2,\,j} \qquad G^n_{i,\,j+1/2} \approx g^{n+1/2}_{i,\,j+1/2} \tag{6.1}$$

Under this approximation:

$$\boxed{Q^{n+1}_{ij} = Q^n_{ij} - \frac{\Delta t}{\Delta x}\left[f^{n+1/2}_{i+1/2,\,j} - f^{n+1/2}_{i-1/2,\,j}\right] - \frac{\Delta t}{\Delta y}\left[g^{n+1/2}_{i,\,j+1/2} - g^{n+1/2}_{i,\,j-1/2}\right] + \Delta t\left[S^n_{ij}\right]} \tag{6.2}$$

(There are also other ways to estimate the fluxes, but this is the strategy we are using.)

It's also a good idea to estimate the source term at the midpoint in time, but I haven't implemented this yet.

7 Shallow-Water Equations as a Balance Law

It turns out that the shallow-water equations can be written as a balance law as follows:

$$\boxed{\begin{aligned} \frac{\partial}{\partial t}(hu) + \frac{\partial}{\partial x}(hu^2 + gh^2/2) + \frac{\partial}{\partial y}(huv) &= -hg\frac{\partial b}{\partial x} + hfv \\ \frac{\partial}{\partial t}(hv) + \frac{\partial}{\partial x}(huv) + \frac{\partial}{\partial y}(hv^2 + gh^2/2) &= -hg\frac{\partial b}{\partial y} - hfu \\ \frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} &= 0\,. \end{aligned}} \tag{7.1}$$

We can check easily check this is equivalent to the original shallow-water system (2.1).

In the balanced equation for u, we have

$$\frac{\partial}{\partial t}(hu) = h\frac{\partial u}{\partial t} + u\frac{\partial h}{\partial t}$$

$$\frac{\partial}{\partial x}(hu^2 + gh^2/2) = hu\frac{\partial u}{\partial x} + u\frac{\partial(hu)}{\partial x} + hg\frac{\partial h}{\partial x}$$

$$\frac{\partial}{\partial y}(huv) = hv\frac{\partial u}{\partial y} + u\frac{\partial(hv)}{\partial y}$$

Adding up all of the terms on the RHS, note that some terms add up to zero according to the continuity equation, so we get

$$h\frac{\partial u}{\partial t} + hu\frac{\partial u}{\partial x} + hv\frac{\partial u}{\partial y} + hg\frac{\partial h}{\partial x} = -hg\frac{\partial b}{\partial x} + hfv$$

Dividing by h and noting that $\partial_x \eta = \partial_x h + \partial_x b$ yields the equation for u as in the original system. The balanced equation for v can be checked in the same way.

8 Fluxes in the Shallow-Water Model

Note: I'm not totally convinced of my math in this section, and I haven't found any resource that lays things out in this way.

For the shallow-water equations written as a balance law (7.1), we have

$$\vec{q} = \begin{bmatrix} hu \\ hv \\ h \end{bmatrix}, \qquad \vec{f} = \begin{bmatrix} hu^2 + gh^2/2 \\ huv \\ hu \end{bmatrix}, \qquad \vec{g} = \begin{bmatrix} huv \\ hv^2 + gh^2/2 \\ hv \end{bmatrix}, \qquad \vec{s} = \begin{bmatrix} -hg\partial_x b + hfv \\ -hg\partial_y b - hfu \\ 0 \end{bmatrix}.$$

But, wait a second, these are vector-valued functions, and we only we described the finite volume method for scalar-valued functions! Well, don't worry, it's not a problem because we can just apply the finite volume method to each component function.

Thus, we have the cell averages

$$\vec{Q}^n_{ij} = \frac{1}{\Delta x \Delta y} \begin{bmatrix} \iint_{\mathcal{C}_{ij}} hu(x,y,t)\,\mathrm{d}x\,\mathrm{d}y \\ \iint_{\mathcal{C}_{ij}} hv(x,y,t)\,\mathrm{d}x\,\mathrm{d}y \\ \iint_{\mathcal{C}_{ij}} h(x,y,t)\,\mathrm{d}x\,\mathrm{d}y \end{bmatrix} = \begin{bmatrix} (HU)^n_{ij} \\ (HV)^n_{ij} \\ (H)^n_{ij} \end{bmatrix}.$$

Now, our strategy, as described in section 6, is to compute

$$\vec{f}^{\,n+1/2}_{\,i+1/2,j} = \begin{bmatrix} (hu^2 + gh^2/2)^{n+1/2}_{i+1/2,j} \\ (huv)^{n+1/2}_{i+1/2,j} \\ (hu)^{n+1/2}_{i+1/2,j} \end{bmatrix}, \qquad \vec{g}^{\,n+1/2}_{\,i,j+1/2} = \begin{bmatrix} (huv)^{n+1/2}_{i,j+1/2} \\ (hv^2 + gh^2/2)^{n+1/2}_{i,j+1/2} \\ (hv)^{n+1/2}_{i,j+1/2} \end{bmatrix}.$$

How the heck are we going to do that? Well, to compute $\vec{f}^{\,n+1/2}_{i+1/2,\,j}$, it's sufficient to find

$$h^{n+1/2}_{i+1/2,\,j}\,, \qquad (hu)^{n+1/2}_{i+1/2,\,j}\,, \qquad (hv)^{n+1/2}_{i+1/2,\,j}\,. \tag{8.1}$$

and to compute $\vec{g}^{\,n+1/2}_{i,\,j+1/2}$, it's sufficient to find

$$h^{n+1/2}_{i,\,j+1/2}\,, \qquad (hu)^{n+1/2}_{i,\,j+1/2}\,, \qquad (hv)^{n+1/2}_{i,\,j+1/2}\,. \tag{8.2}$$

But, wait a minute, aren't we using finite volume methods? That means that after the first time step, we don't know who h, hu, and hv are, we only know who H, HU, and HV are! So, how the heck are we going to find (8.1) and (8.2)?

One idea is to use the cell averaged quantities H, HU, and HV (at t^n), to estimate the values of the conserved quantities h, hu, and hv at the cell edges (at t^n). Now that you have the conserved quantities (at t^n), you can compute the conserved quantities at $t^{n+1/2}$ using a numerical scheme of your choice, and then the fluxes follow!

Unfortunately, I haven't figured out how to implement that idea yet, so we're just going to have to do things in a different way.

Instead, we're going to express the fluxes in terms of cell averaged quantities:

$$\vec{f}^{\,n,Q}_{ij} = \begin{bmatrix} (HU^2 + gH^2/2)^n_{ij} \\ (HUV)^n_{ij} \\ (HU)^n_{ij} \end{bmatrix}, \qquad \vec{g}^{\,n,Q}_{ij} = \begin{bmatrix} (HUV)^n_{ij} \\ (HV^2 + gH^2/2)^n_{ij} \\ (HV)^n_{ij} \end{bmatrix},$$

and our hope is that $\vec{f}^{\,Q} \approx \vec{f}$ and $\vec{g}^{\,Q} \approx \vec{g}$.

Just like before, we seek

$$\vec{f}^{\,n+1/2,Q}_{i+1/2,\,j} \quad \text{and} \quad \vec{g}^{\,n+1/2}_{i,\,j+1/2}\,,$$

for which it is sufficient to find

$$\vec{Q}^{\,n+1/2}_{i+1/2,j} \quad \text{and} \quad \vec{Q}^{\,n+1/2}_{i,j+1/2}\,,$$

respectively.

We compute these quantities using a simple Lax-Friedrichs scheme:

$$\vec{Q}^{\,n+1/2}_{i+1/2,j} \approx \frac{\vec{Q}^{\,n}_{i+1,\,j} + \vec{Q}^{\,n}_{ij}}{2} - \frac{\Delta t}{2\Delta x}\left[\vec{f}^{\,n,Q}_{i+1,\,j} - \vec{f}^{\,n,Q}_{ij}\right] \tag{8.3}$$

and

$$\vec{Q}^{\,n+1/2}_{i,j+1/2} \approx \frac{\vec{Q}^{\,n}_{i,\,j+1} + \vec{Q}^{\,n}_{ij}}{2} - \frac{\Delta t}{2\Delta y}\left[\vec{g}^{\,n,Q}_{i,\,j+1} - \vec{g}^{\,n,Q}_{ij}\right]. \tag{8.4}$$

I write out (8.3) here, just to look at.

$$(HU)_{i+1/2,\,j}^{n+1/2} \approx \frac{(HU)_{i+1,j}^n + (HU)_{ij}^n}{2} - \frac{\Delta t}{2\Delta x}\Big[(HU^2 + gH^2/2)_{i+1,j}^n - (HU^2 + gH^2/2)_{ij}^n\Big]$$

$$(HV)_{i+1/2,\,j}^{n+1/2} \approx \frac{(HV)_{i+1,j}^n + (HV)_{ij}^n}{2} - \frac{\Delta t}{2\Delta x}\Big[(HUV)_{i+1,j}^n - (HUV)_{ij}^n\Big]$$

$$H_{i+1/2,\,j}^{n+1/2} \approx \frac{H_{i+1,j}^n + H_{ij}^n}{2} - \frac{\Delta t}{2\Delta x}\Big[(HU)_{i+1,j}^n - (HU)_{ij}^n\Big]$$

and I write out (8.4) here (this helped me in my coding).

$$(HU)_{i,\,j+1/2}^{n+1/2} \approx \frac{(HU)_{i,\,j+1}^n + (HU)_{ij}^n}{2} - \frac{\Delta t}{2\Delta y}\Big[(HUV)_{i,\,j+1}^n - (HUV)_{ij}^n\Big]$$

$$(HV)_{i,\,j+1/2}^{n+1/2} \approx \frac{(HV)_{i,\,j+1}^n + (HV)_{ij}^n}{2} - \frac{\Delta t}{2\Delta y}\Big[(HV^2 + gH^2/2)_{i,\,j+1}^n - (HV^2 + gH^2/2)_{ij}^n\Big]$$

$$H_{i,\,j+1/2}^{n+1/2} \approx \frac{H_{i,\,j+1}^n + H_{ij}^n}{2} - \frac{\Delta t}{2\Delta y}\Big[(HV)_{i,\,j+1}^n - (HV)_{ij}^n\Big]$$

9 The Entire Method

To summarize, we have initial data for u, v, and h, from which we can find hu and hv. Then we compute the cell averages HU, HV, and H at $t = 0$. Next, we apply the Lax-Friedrichs scheme (8.3) and (8.4) which estimates

$$H_{i+1/2,\,j}^{n+1/2}\,, \quad H_{i,\,j+1/2}^{n+1/2}\,, \quad (HU)_{i+1/2,\,j}^{n+1/2}\,, \quad (HU)_{i,\,j+1/2}^{n+1/2}\,, \quad (HV)_{i+1/2,\,j}^{n+1/2}\,, \quad (HV)_{i,\,j+1/2}^{n+1/2}\,.$$

Then we use the midpoint quantities to compute the fluxes:

$$\vec{f}_{\,i+1/2,\,j}^{\,n+1/2,\,Q} = \begin{bmatrix} (HU^2 + gH^2/2)_{i+1/2,\,j}^{n+1/2} \\ (HUV)_{i+1/2,\,j}^{n+1/2} \\ (HU)_{i+1/2,\,j}^{n+1/2} \end{bmatrix} = \begin{bmatrix} \big((HU)^2/H + gH^2/2\big)_{i+1/2,\,j}^{n+1/2} \\ \big((HU)(HV)/H\big)_{i+1/2,\,j}^{n+1/2} \\ (HU)_{i+1/2,\,j}^{n+1/2} \end{bmatrix},$$

$$\vec{g}_{\,i,\,j+1/2}^{\,n+1/2,Q} = \begin{bmatrix} (HUV)_{i,\,j+1/2}^{n+1/2} \\ (HV^2 + gH^2/2)_{i,\,j+1/2}^{n+1/2} \\ (HV)_{ij}^{n+1/2} \end{bmatrix} = \begin{bmatrix} \big((HU)(HV)/H\big)_{i,\,j+1/2}^{n+1/2} \\ \big((HV)^2/H + gH^2/2\big)_{i,\,j+1/2}^{n+1/2} \\ (HV)_{i,\,j+1/2}^{n+1/2} \end{bmatrix}.$$

Finally, we do the finite volume step.

$$\boxed{\vec{Q}_{ij}^{n+1} = \vec{Q}_{ij}^n - \frac{\Delta t}{\Delta x}\Big[\vec{f}_{\,i+1/2,\,j}^{\,n+1/2,\,Q} - \vec{f}_{\,i-1/2,\,j}^{\,n+1/2,\,Q}\Big] - \frac{\Delta t}{\Delta y}\Big[\vec{g}_{\,i,\,j+1/2}^{\,n+1/2} - \vec{g}_{\,i,\,j-1/2}^{\,n+1/2}\Big] + \Delta t\,\Big[\vec{S}_{ij}^n\Big]} \tag{9.1}$$

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