Review of Vectors

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Introduction

- Imagine having to describe something — anything at all — to another person only using a pen and paper.
- If there's any movement at all involved in what you have to describe, chances are you'll soon find yourself drawing arrows.
- And what characterizes an arrow?
- Well, it's the direction it's pointing in and its length.
- This is a vector: an object that has a direction and a magnitude.

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What is a point?

- A geometrical **point** is a location in space. It has no other characteristics. It has no length, width, or thickness. It is pure location.
- Consider two points P=(1,3) and Q= (4,1).
- The displacement from P to Q is a vector v having components (3, -2)
- To "get from" P to Q , we shift down by 2 and to the right by 3.
- Since a vector is a displacement , it has a size and a direction, but no inherent location.
- The difference between two points is a VECTOR v = Q - P
- The sum of a point and a vector is a POINT Q = P + v

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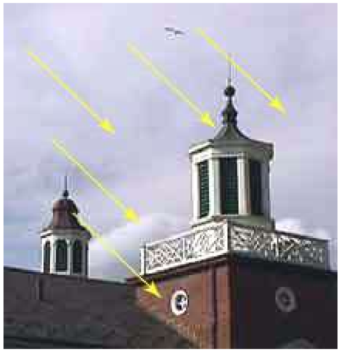
Examples

- The direction of the sunlight and its intensity can be represented as a **vector**.
- Light from the sun has two properties: brightness and direction
- Make the length of the vector proportional to the brightness of the light.
- Wind, also, is represented as a vector.
- Wind has direction and speed.
- A particle of dust travels in the direction of the wind.
- The length of the vector is proportional to the speed of the dust particle.



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Vectors

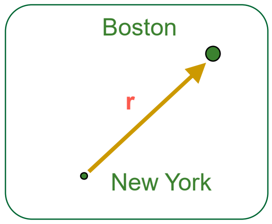
- In 1 dimension, we can specify direction with a + or - sign.
- In 2 or 3 dimensions, we need more than a sign to specify the direction of something.
- To illustrate this, consider the ***position* vector r** in 2 dimensions.
- (A vector which expresses the position of a point with respect to the origin.)

Example: Where is Boston?

- Choose **origin** at New York
- Choose coordinate system

Boston is 212 miles northeast of New York [in (r,θ)] OR

Boston is 150 miles north and 150 miles east of New York [in (x,y)]



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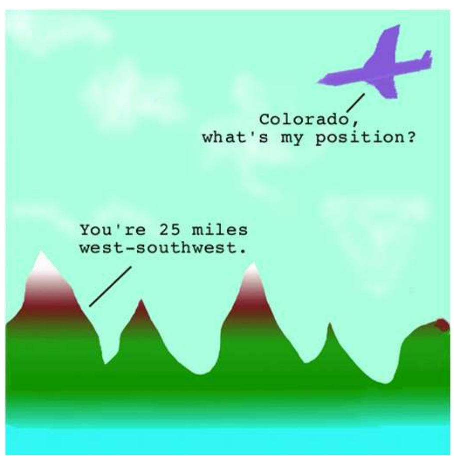

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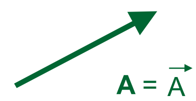
Vectors...

- There are two common ways of indicating that something is a vector quantity:
  - Boldface notation: ***A***



$\vec{A}$

  - "Arrow" notation:

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Vectors have rigorous definitions

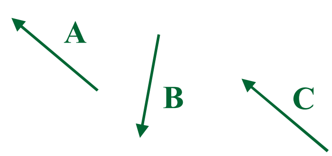
- A vector is composed of a magnitude and a direction
  - Examples: displacement, velocity, acceleration
  - Magnitude of **A** is designated |**A**|
  - Usually vectors include units (m, m/s, m/s^2)
- A vector has no particular position

(Note: the position vector reflects displacement from the origin)

- Two vectors are <u>equal</u> if their <u>directions</u>, <u>magnitudes</u> and <u>units</u> match.

$\mathbf{A} = \mathbf{C}$

$\mathbf{A} \neq \mathbf{B},\ \mathbf{B} \neq \mathbf{C}$



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Comparing Vectors and Scalars

- A scalar is an ordinary number.
  - A magnitude **without** a direction
  - May have units (kg) or be just a number
  - Usually indicated by a regular letter, no bold face and no arrow on top.

  Note: the lack of specific designation of a scalar can lead to confusion

- The product of a vector and a scalar is another vector in the same "direction" but with modified magnitude.

$$\mathbf{A} = -0.75\ \mathbf{B}$$

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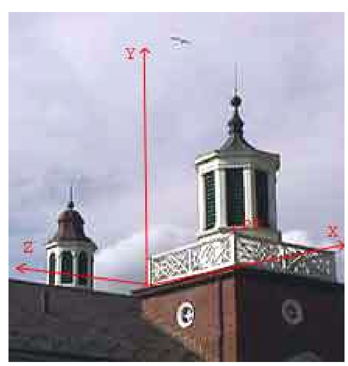
Column Matrices

- Any point in our 3D scene can be assigned a representation by measuring its distance along the three axes. These three distances are put into a column matrix as follows:

$$\begin{pmatrix} X \\ Y \\ z \end{pmatrix}$$

- For example, the coordinates of the corner of the building that lies along the X axis are (about):

$$\begin{pmatrix} 8.0 \\ 0.0 \\ 0.0 \end{pmatrix}$$



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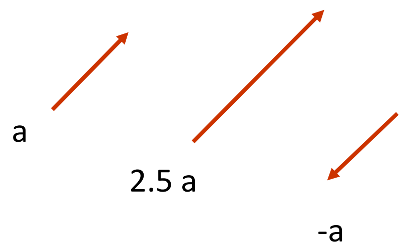
Operations with Vector

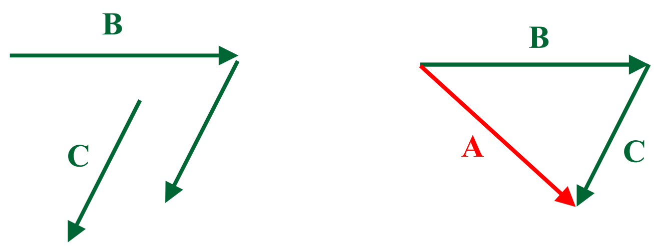
- Vectors permit two fundamental operations.
  - Addition (a+b where a and b are vectors)
  - Scaling (multiplication of a vector by a scalar value)



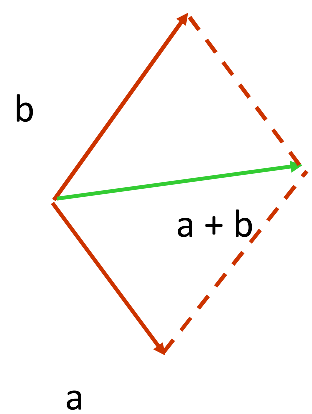
Vector addition (First Method)

- The sum of two vectors is another vector.

$$\mathbf{A} = \mathbf{B} + \mathbf{C}$$



Vector addition (Second Method)



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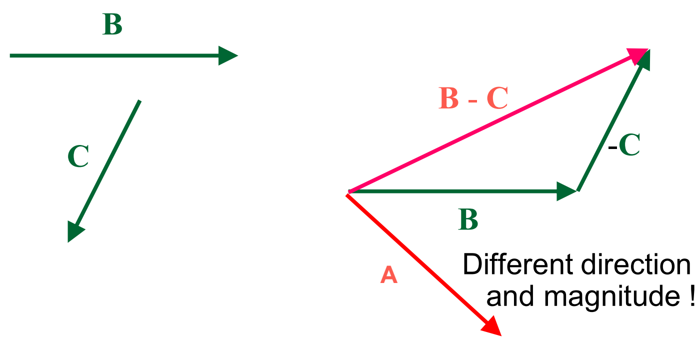
Vector subtraction

- Vector subtraction can be defined in terms of addition.

B - C = **B** + (-1)**C**



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The Magnitude of a Vector

- If a vector w is represented by the n-tuple (w_1,w_2,…,w_n) its magnitude denoted by |w| is defined as the distance from the tail to the head of the vector.
- Mathematically this can be represented as :

$$|w| = \sqrt{w_1^2 + w_2^2 + \ldots + w_n^2}$$

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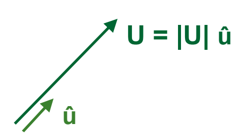
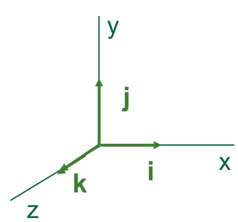
Unit Vectors

- A **Unit Vector** is a vector having length 1 and no units
- It is used to specify a direction.
- Unit vector ***u*** points in the direction of ***U***
  - Often denoted with a "hat": ***u* = *û***



- Useful examples are the cartesian unit vectors [**i, j, k**]
  - Point in the direction of the x, y and z axes.

$R = r_x\,\mathbf{i} + r_y\,\mathbf{j} + r_z\,\mathbf{k}$



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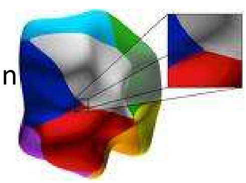
Linear Combinations of Vectors

- A linear combination of n vectors $v_1,v_2,\ldots v_n$ is a vector of the form
- $W=a_1v_1+a_2v_2+\ldots+a_nv_n$

1) Point 1 – Point 2 = A vector
2) A point + A Vector = Another point
3) Vector1 + Vector 2 = Another vector
4) Any linear combination of vectors is also a vector. (Ex.: Scaling a vector.)

– Useful when we seek to represent curves an using spline functions



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Spline Functions

- Spline functions are formed by joining polynomials together at fixed points called knots. That is, we divide the interval extending from lower limit tL to upper limit tU over which we wish to approximate a curve into L+1 sub-intervals separated by L interior boundaries

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Affine Combinations of Vectors

- A linear combination of vectors is an affine combination if the coefficients a1,a2…an add up to unity

$a_1+a_2+\ldots a_n=1$

- Affine transformations preserve collinearity and flatness so the image of a straight line is another straight line.

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Convex Combinations of Vectors

- A convex combination of vectors require not only the sum of the coefficients to be unity but also that none of them should be negative.
- Any color of unit brightness can be considered to be a convex combination of three primary colors.

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Vectors in sports

- In the 1950s a group of talented Brazilian footballers invented the swerving free kick.
- By kicking the ball in just the right place, they managed to make it curl around the wall of defending players and, quite often, go straight into the back of the net.
- When a ball is in flight it's acted upon by various forces and some of these depend on the way the ball is spinning around its own axis.
- If you manage to give it just the right spin, the forces will interact in just the right way to deflect the ball while it's flying, resulting in a curved flight path.
- The forces at work here can be described by vectors. Understanding their interaction requires vector maths.

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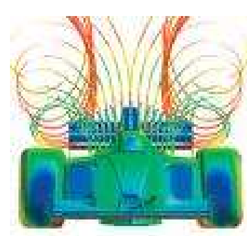
- Today science is increasingly used to improve the performance of athletes' equipment.
- The exact shape of a football can have important implications on how it moves through the air and teams of scientists are employed to work out how to make the perfect football.
- When the equipment is more complicated than a football then the use of science is even more important.
- Formula One teams, for example, always employ physicists and mathematicians to help build perfect cars.
- Tiny differences in the shape of the car can make a difference to its speed that can determine the outcome of a race.



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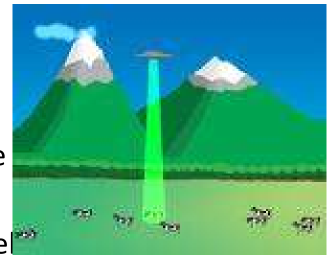
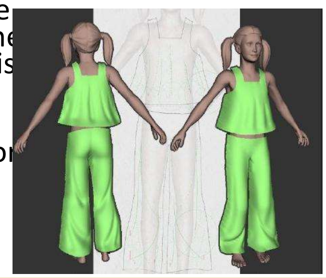
Vectors and visuals

- Vector maths is used extensively in computer graphics.
- Suppose you want to create an image on a computer screen.
- One way of doing this is to tell the computer the exact color of each pixel on the screen.
- This requires a lot of memory and has another disadvantage: if you'd like the image to move, for example to give the viewer the impression that he or she is moving around a scene, you need to constantly

renew the information of the pixel color from scratch.





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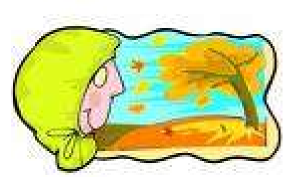
- For example, that you're creating a scene lit by sunlight and ruffled by a strong wind. The sunlight and wind both come from a specific direction and have a certain intensity — so both can be represented by vectors.
- Using these vectors you can create a program that calculates exactly how an object in the scene should be colored and move to give a realistic impression of lighting and wind.
- Even better, you can write your program so that the vectors representing sun and wind constantly change their direction and magnitude — thus you can create gusts of wind and clouds passing overhead.



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Dot Product of Two Vectors Definition

- **The dot product *d* of two n dimensional vectors v and w is denoted as v·w and its value is given as**

d = v·w = $\sum_{i=1}^{n} v_i w_i$

- **If n=3, v=(v_1,v_2,v_3) and w=(w_1,w_2,w_3)**
  v.w= $v_1 \cdot w_1 + v_2 \cdot w_2 + v_3 \cdot w_3$

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Example

- For two vectors **v**=(2,3,1) and **w**=(0,4,-1)
  Dot product is given as
  v·w=(2·0) + (3·4) – (1·1)
  = 11

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Properties of the Dot Product

- Symmetry (or commutative)

  a · **b** = **b** · **a**
- Linearity

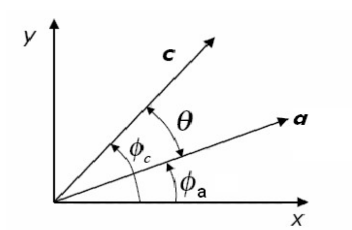
  a ·(**b** + **c**) = **a** · **b** + **a** · **c**
- Homogeneity

  (s**a**) · **b** = s(**a** · **b**)
- |**a**|^2 = **a** · **a**

  Length of a vector a is given as

  |a|=√a·a

Angle Between Two Vectors



- $\boldsymbol{a} = (|\boldsymbol{a}| cos\o_a,\ |\boldsymbol{a}| sin\o_a)$
- $\boldsymbol{c} = (|\boldsymbol{c}| cos\o_c,\ |\boldsymbol{c}| sin\o_c)$

- $\boldsymbol{a.c} = |\boldsymbol{a}||\boldsymbol{c}| cos\o_c cos\o_a + |\boldsymbol{a}||\boldsymbol{c}| sin\o_a sin\o_c$
- $\boldsymbol{a.c} = |\boldsymbol{a}||\boldsymbol{c}| cos(\o_c - \o_a)$
- *Multiplying both sides by* $|\boldsymbol{a}||\boldsymbol{c}|$
- $cos(\theta) = \hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{c}}$ *, where θ is the angle from **a** to **c**.*

Example

- *Angle b/w the vectors **b** = (3,4) and **c** = (5,2)*
- *|**b**|= 5, |**c**|= 5.385*
- $\hat{\mathbf{b}} = \left(\frac{3}{5}, \frac{4}{5}\right)$ $\quad \hat{\mathbf{b}} \bullet \hat{\mathbf{c}} = 0.85422 = \cos\theta$
  $\theta = 31.326^\circ$
- *A*

- ***Application In Computer Graphics***
  - *Angle between light source and surface for shading*

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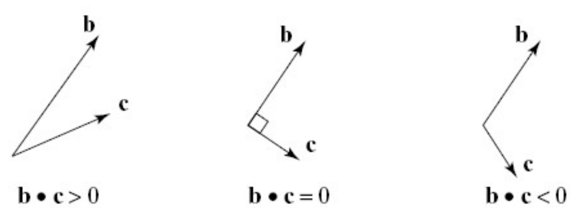
Sign of b·c and Perpendicularity



- ***b** and **c** are perpendicular if **b**·**c** = 0*

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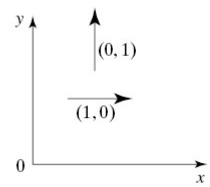
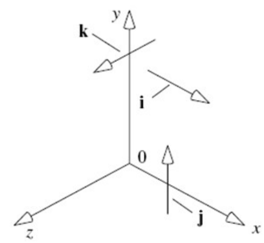
Most Familiar Orthogonal Vectors

- *2D vectors(1,0) and (0,1) are mutually perpendicular unit vector*

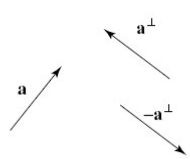


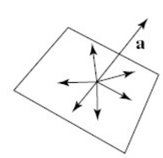
- *The standard unit vectors in 3D*
- ***i**=(1,0,0), **j**=(0,1,0), **k**=(0,0,1)*
- *Any 3D vector (a,b,c) can be written as a**i**+b**j**+c**k**.*



The 2D “Perp” vector

- *Let* $\mathbf{a} = (a_x, a_y)$ *then* $\mathbf{a}^{\perp} = (-a_y, a_x)$ *is the counterclockwise perpendicular to* $\mathbf{a}$.





- *2D*
- *3D*

Properties of 2D Prep Vector

$(\mathbf{a}+\mathbf{b})^{\perp} = \mathbf{a}^{\perp} + \mathbf{b}^{\perp}$

$(A\mathbf{a})^{\perp} = A\mathbf{a}^{\perp}$

$\mathbf{a}^{\perp\perp} = (\mathbf{a}^{\perp})^{\perp} = -\mathbf{a}$

$\mathbf{a}^{\perp} \cdot \mathbf{b} = -\mathbf{b}^{\perp} \cdot \mathbf{a}$

$\mathbf{a}^{\perp} \cdot \mathbf{b} = a_x b_y - a_y b_x$

$\mathbf{a}^{\perp} \cdot \mathbf{a} = 0$

$|\mathbf{a}^{\perp}|^2 = |\mathbf{a}|^2$

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Vector and Raster Images

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Vector Graphics

- Vector graphics are made up of
  - lines and curves defined by mathematical objects called vectors.
- These vectors can be filled with
  - solid colors
  - gradients
  - patterns

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Vector Graphics (advantages)

- Can be scaled up or down without distortion.
- Lines and curves will remain crisp and sharp.
- Attributes such as stroke thickness, fill color, fill gradient, stroke color, stroke type etc. can be changed at any time.
- Not restricted to rectangular shapes like bitmaps.
- Smaller file size.
- Ideal for graphics that frequently have to be displayed at varying sizes and colors (e.g. logos.)

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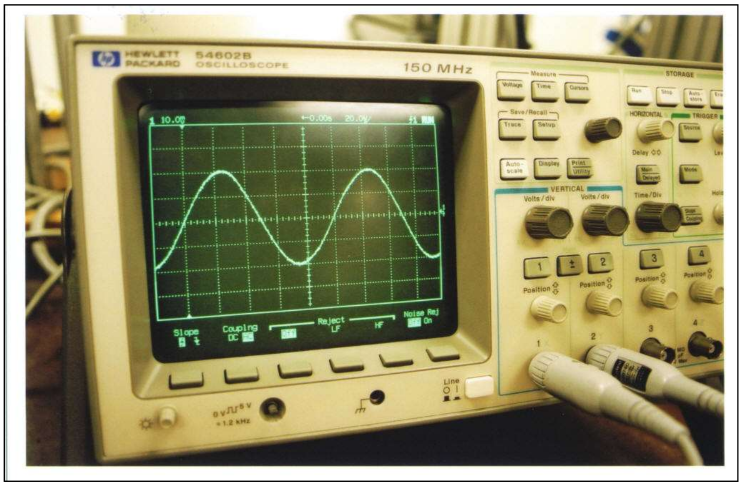
Vector Displays



HP Oscilloscope

Vector Graphics (defining characteristics)

- Cartoony (non-realistic) appearance
- Solid or smoothly gradated fill colors
- Crisp edges without anti-aliasing
  - Anti-aliasing is a method by which you can eliminate jaggies that appear in objects

Examples of vector images

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How Vector Displays Work

- They just draw line segments
  - User/computer:
    - Define start and end points
  - Display:
    - Move electron gun to start point
    - Turn electron gun on
    - Move electron gun to end point
    - Turn electron gun off

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Advantages of Vector Displays

- Require very little memory
  - Important on a 64K system
  - Conceptually very simple
  - No aliasing of lines/curves
  - No fixed timing
    - Refresh rate can be very high

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Disadvantages of Vector Displays

- Really just one: Can only draw line segments
  - Time needed to draw a screen increases with number of lines drawn
    - If most of the image isn't black, you won't be able to finish drawing it in time!

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Vector Graphics (disadvantages)

- Unsuitable for photo-realistic imagery as they cannot depict the continuous subtle tones of a photograph. That's why most of the vector images you see tend to have a cartoon-like appearance.

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Vector Graphics Software

- Adobe Illustrator
- CAD software
- Adobe Flash
- 3ds max
- All vector graphics are eventually displayed as raster graphics on displays. The transformation of a vector image into a raster on is called rasterizing.

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What is “rasterization”?

- Definition 1: The process of converting a vector image (shapes) to a raster image (dots)

- Why?
  - Dots are the only things modern displays can understand!

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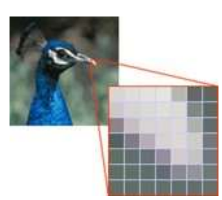
Raster Graphics (bitmaps)

- Use a grid of small squares known as pixels to represent images.
- Each pixel is assigned a specific location and color value.



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Raster Graphics (advantages)

- Superior for creating photo-realistic images and areas of inconsistent tone.
- Great for manipulating and compositing photographic images.
- Best medium for fine artists with backgrounds in sketching, painting, etc. when used in conjunction with a drawing tablet.
- Refresh rate is not dependent on the amount of pixels drawn

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Raster Graphics (defining characteristics)

- Can appear realistic or cartoony
- Takes on the appearances of traditional illustration techniques such as sketchy lines, a painterly appearance, fine detail.
- Drastic changes in tone including realistic shading and shadows.
- Blurred anti-aliased edges

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Raster Graphics (disadvantages)

- The image will be appear blurry when you scale a raster graphic up.
- Since you are saving the color data of every pixel in a raster graphic the file sizes are larger than vector graphics.
- The edges of raster graphics are not crisp. To avoid seeing the “stair-stepping” of pixels (aliasing), the surrounding pixel colors are averaged to create smoother transitions (anti-aliasing).

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Raster Graphics (created by)

- Digital Cameras Scanners Raster graphics software (e.g. Adobe Photoshop, The GIMP.)
- Any image found on the Internet (.JPG,.GIF,.PNG, is a raster graphic).

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Disadvantages of Raster Displays

- Need a fairly large amount of memory
  - Draws the whole screen "at once"
  - Need a frame buffer that can hold the information for a whole image
- Aliasing!

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Aliasing

- This is what causes "jaggies"
- A signal processing problem:
  - The incoming signal (the desired image) can only be sampled at pixel centers on the display
  - Demonstrations

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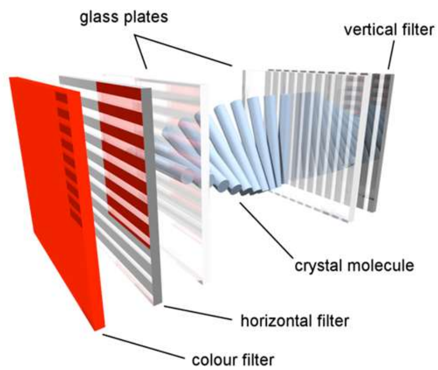
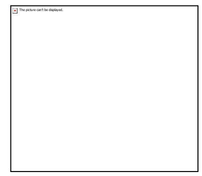
Raster displays

- Liquid Crystal Displays (LCD)



A single subpixel of an LCD display



Close-up of pixels on an LCD display

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What is “rasterization”? (Part 2)

- Virtually all displays used today are raster displays
- So, technically, anything that produces an image on a screen “rasterizes”
- <u>Definition 2:</u> The rendering method used by current graphics cards

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How do we draw an image?

1 - Start with geometry
  - We have some 3D model and/or environment in the system, and we want to draw it on the screen
- Problem: The display is, virtually always, only 2D
  - Need to transform the 3D model into 2D

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3D to 2D (Projection)

- Problem: The display is, virtually always, only 2D
  - Need to transform the 3D model into 2D
- We do this with a virtual camera
  - Represented mathematically by a 3x4 projection (or P) matrix

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2 Shading

- Problem: How do we determine the color of a piece of geometry?
- In the real world, color depends on the object's surface color and the color of the light
  - It is the same way in computer graphics
- "Shading" is the process by which color is assigned to geometry

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3 Clipping

- Problem: The camera doesn't see the whole scene
  - In particular, the camera might only see parts of objects
- Solution: Find objects that cross the edge of the viewing volume, and "clip" them
  - Clip: Cut a polygon into multiple parts, such that each is entirely inside or outside the display area

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4 Rasterization

- Problem: How to convert these polygons on a plane to pixels on a screen?
- Have to figure out which pixels a polygon covers
  - A part of a polygon that covers one pixel is called a “fragment

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5 Fragment Processing

- Problem: How do we know which fragment to use to color a given pixel?
- Need to know which fragment is in front (not counting transparency)

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Rendering process

- Start: Geometric model
- Compute color of geometry (Shading)
  - Based on lighting and surface color
- Project geometry (Projection)
- Clip geometry (Clipping)
- Generate fragments from geometry (Rasterization)
- Compute pixel colors from fragments (Fragment processing)
- End: Display pixels

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Rendering pipeline

- Shading, projection, and clipping all operate on the original geometry
  - Combined into one unit, or “shader”
    - “Vertex shader” or “geometry shader”
- So this is a basic graphics pipeline:

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