3D Transformations

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3D Affine Transformations

- Again we use coordinate frames, and suppose that we have an origin $\mathcal{O}$ and three mutually perpendicular axes in the directions **i**, **j**, and **k** (see Figure 5.8). Point *P* in this frame is given by $P = \mathcal{O} + P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k}$, and vector V by $V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}$.

$$P = \begin{pmatrix} P_x \\ P_y \\ P_z \\ 1 \end{pmatrix}, V = \begin{pmatrix} V_x \\ V_y \\ V_z \\ 0 \end{pmatrix}$$

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3-D Affine Transformations

- The matrix representing a transformation is now 4 x 4, with Q = M P as before.

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- The fourth row of the matrix is a string of zeroes followed a lone one.

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Translation and Scaling

- Translation and scaling transformation matrices are given below. The values S_x, S_y, and S_z cause scaling about the origin of the corresponding coordinates.

$$T = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}, S = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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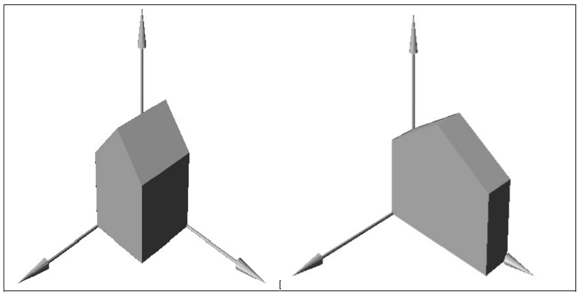
Scaling

- Figure shows the effect of scaling in the *z*-direction by 0.5 and in the *x*-direction by a factor of two.



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Shear

- The shear matrix is given below.
  - a: y along z; b: z along x; c: x along y; d: z along y; e: x along z; f: y along z
- Usually only one of {a,…,f} is non-zero.

$$H = \begin{pmatrix} 1 & a & b & 0 \\ c & 1 & d & 0 \\ e & f & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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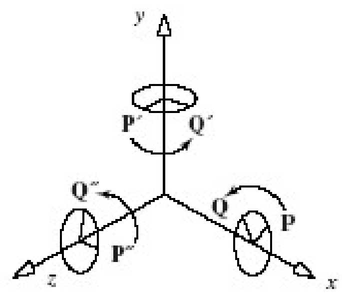
Rotations

- Rotations are more complicated. We start by defining a **roll** (rotation **counter-clockwise** around an axis looking **toward** the origin):



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Rotations (2)

- *z*-roll: the *x*-axis rotates to the *y*-axis.
- *x*-roll: the *y*-axis rotates to the *z*-axis.
- *y*-roll: the *z*-axis rotates to the *x*-axis.

$$R_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\vartheta & -\sin\vartheta & 0 \\ 0 & \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, R_y = \begin{pmatrix} \cos\vartheta & 0 & \sin\vartheta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\vartheta & 0 & \cos\vartheta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$R_z = \begin{pmatrix} \cos\vartheta & -\sin\vartheta & 0 & 0 \\ \sin\vartheta & \cos\vartheta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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Rotation About y-axis

- Rotate the point P = (3, 1, 4) through 30° about the y-axis.
- **Solution**: c = .866 and s = .5, P is transformed into

$$Q = \begin{pmatrix} c & 0 & s & 0 \\ 0 & 1 & 0 & 0 \\ -s & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4.6 \\ 1 \\ 1.964 \\ 1 \end{pmatrix}$$

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Rotations (3)

- Note that 12 of the terms in each matrix are the zeros and ones of the identity matrix.
- They occur in the row and column that correspond to the axis about which the rotation is being made (e.g., the first row and column for an x-roll).
- They guarantee that the corresponding coordinate of the point being transformed will not be altered.
- The *cos* and *sin* terms always appear in a rectangular pattern in the other rows and columns.

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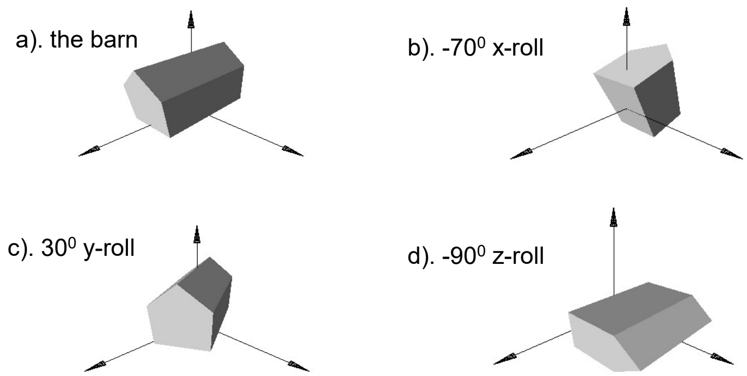
Example

- A barn in its original orientation, and after a -70° x-roll, a 30° y-roll, and a -90° z-roll.



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Composing 3D Affine Transformations

- 3D affine transformations can be composed, and the result is another 3D affine transformation.
- The matrix of the overall transformation is the product of the individual matrices M_1 and M_2 that perform the two transformations, with M_2 *pre-multiplying* M_1: $M = M_2M_1$
- Any number of affine transformations can be composed in this way, and a single matrix results that represents the overall transformation.

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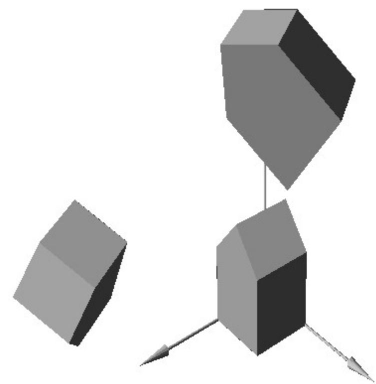
Example

- A barn is first transformed using some M_1, and the transformed barn is again transformed using M_2. The result is the same as the barn transformed once using M_2M_1.



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Building Rotations

- All 2D rotations are R_z. Two rotations combine to make a rotation given by the sum of the rotation angles, and the matrices commute.
- In 3D the situation is much more complicated, because rotations can be about different axes.
- The order in which two rotations about different axes are performed *does* matter: 3D rotation matrices do **not** commute.

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Building Rotations (2)

- We build a rotation in 3D by composing three elementary rotations: an *x*-roll followed by a *y*-roll, and then a *z*-roll. The overall rotation is given by M = $R_z(\beta_3)R_y(\beta_2)R_x(\beta_1)$.
- In this context the angles β_1, β_2, and β_3 are often called **Euler angles**.

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Euler's Theorem

- ***Euler's Theorem: Any rotation (or sequence of rotations) about a point is equivalent to a single rotation about some axis through that point.***
- What is the matrix associated with an *x*-roll of 45^o followed by a *y*-roll of 30^o followed by a *z*-roll of 60^o? Direct multiplication of the three component matrices yields:

$$\begin{pmatrix} .5 & -.866 & 0 & 0 \\ .866 & .5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} .866 & 0 & .5 & .0 \\ 0 & 1 & 0 & 0 \\ -.5 & 0 & .866 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & .707 & -.707 & 0 \\ 0 & .707 & .707 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} .433 & -.436 & .789 & 0 \\ .75 & .66 & -.047 & 0 \\ -.5 & .612 & .612 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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Building Rotations (3)

- *Any* 3D rotation around an axis (passing through the origin) can be obtained from the product of five matrices for the appropriate choice of Euler angles; we shall see a method to construct the matrices.
- This implies that three values are required (and only three) to completely specify a rotation!

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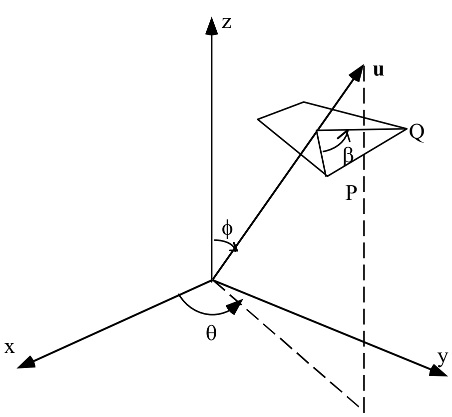
Rotating about an Arbitrary Axis

- We wish to rotate around axis **u** to make P coincide with Q.
- **u** can have any direction; it appears difficult to find a matrix that represents such a rotation.
- But it can be found in two ways, a classic way and a constructive way.



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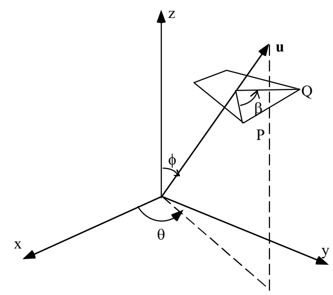
Rotating about an Arbitrary Axis (2)

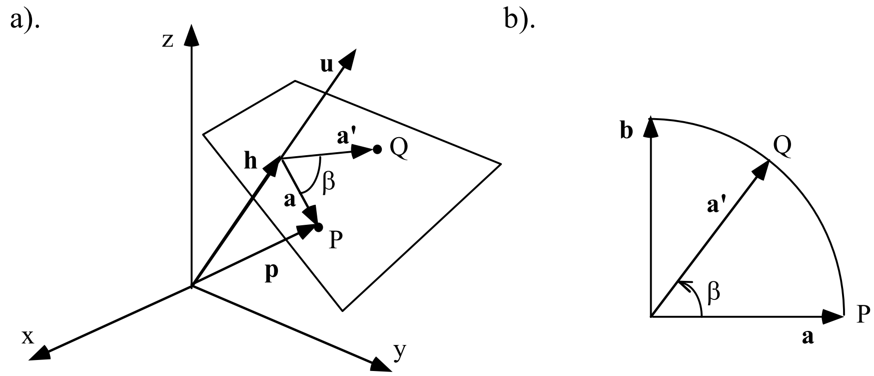
- **The classic way.** Decompose the required rotation into a sequence of known steps:
  - Perform two rotations so that **u** becomes aligned with the *z*-axis.
  - Do a *z*-roll through angle *β*.
  - Undo the two alignment rotations to restore **u** to its original direction.
- $R_{\mathbf{u}}(\beta) = R_z(-\theta)\, R_y(-\Phi)\, R_z(\beta)\, R_y(\Phi)\, R_z(\theta)$ is the desired rotation.



Rotating about an Arbitrary Axis (3)

- **The constructive way.** Using some vector tools we can obtain a matrix $R_{\mathbf{u}}(b)$.
- We wish to express the operation of rotating point *P* through angle *b* into point *Q*.

Rotating about an Arbitrary Axis (5)



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Rotating about an Arbitrary Axis (6)

- c = cos(β), s = sin(β), and u_x, u_y, u_z are the components of u.
- Then

$$R_u(\beta) = \begin{pmatrix} c+(1-c)u_x^2 & (1-c)u_y u_x - s u_z & (1-c)u_z u_x + s u_y & 0 \\ (1-c)u_x u_y + s u_z & c+(1-c)u_y^2 & (1-c)u_z u_y - s u_x & 0 \\ (1-c)u_x u_z - s u_y & (1-c)u_y u_z + s u_x & c+(1-c)u_z^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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Rotating about an Arbitrary Axis (6)

- Find the matrix that produces a rotation through 45° about the axis **u** = (1,1,1) / 3 = (0.577,0.577,0.577) .
- **Solution:** For a 45° rotation, *c* = *s* = 0.707, and filling in the terms in matrix

$$R_{\mathbf{u}}(45^0) = \begin{pmatrix} .8047 & -.31 & .5058 & 0 \\ .5058 & .8047 & -.31 & 0 \\ -.31 & .5058 & .8047 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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Rotating about an Arbitrary Axis (6)

- Open-GL provides a rotation about an arbitrary axis:

  glRotated (beta, ux, uy, uz);
- beta is the angle of rotation.
- ux, uy, uz are the components of a vector u normal to the plane containing P and Q.

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Summary of Properties of 3D Affine Transformations

- **Affine transformations** preserve **affine combinations of points**.
- **Affine transformations preserve lines and planes.**
- **Parallelism of lines and planes is preserved.**
- **The columns of the matrix reveal the transformed coordinate frame.**
- **Relative ratios are preserved.**

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Summary of Properties of 3D Affine Transformations (2)

- **The effect of transformations on the volumes of objects.** If 3D object D has volume V, then its image $T(D)$ has volume $|det\ M\ |\ V$, where $|det\ M|$ is the absolute value of the determinant of M.
- **Every affine transformation is composed of elementary operations.** A 3D affine transformation may be decomposed into a composition of elementary transformations.

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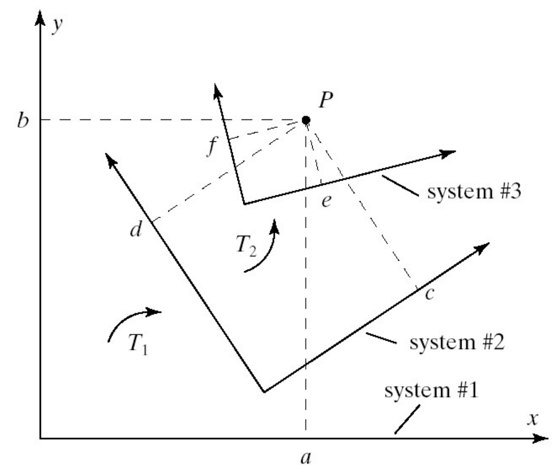
Transforming Coordinate Systems

- We have a 2D coordinate frame #1, with origin $\mathcal{O}$ and axes **i** and **j**.
- We have an affine transformation *T*(.) with matrix *M*, where *T*(.) transforms coordinate frame #1 into coordinate frame #2, with new origin $\mathcal{O}'$ = *T*($\mathcal{O}$), and new axes **i**' = *T*(**i**) and **j**' = *T*(**j**).



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Transforming Coordinate Systems (2)

- Now let *P* be a point with representation $(c, d, 1)^T$ in the new system #2.
- What are the values of *a* and *b* in its representation $(a, b, 1)^T$ in the original system #1?
- The answer: simply *premultiply* $(c, d, 1)^T$ by *M*:

$$(a, b, 1)^T = M\,(c, d, 1)^T$$

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Transforming Coordinate Systems (3)

- We have the following theorem:
- Suppose coordinate system #2 is formed from coordinate system #1 by the affine transformation *M*. Further suppose that point $P = (P_x, P_y, P_z, 1)$ are the coordinates of a point *P* expressed in system #2. Then the coordinates of *P* expressed in system #1 are *MP*.

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Successive Transformations

- Now consider forming a transformation by making two successive changes of the coordinate system. What is the overall effect?
- System #1 is converted to system #2 by transformation $T_1(.)$, and system #2 is then transformed to system #3 by transformation $T_2(.)$. Note that system #3 is transformed *relative* to #2.

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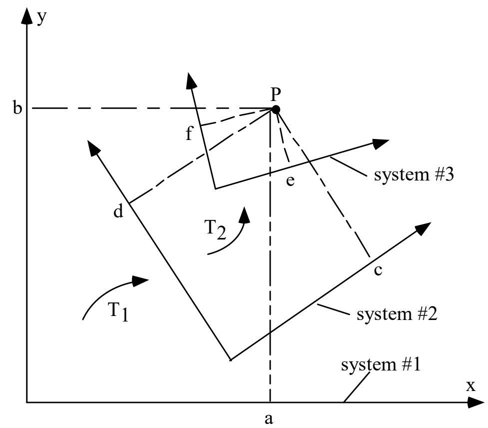
Successive Transformations (2)

- Point P has representation $(e, f, 1)^T$ with respect to system #3. What are its coordinates $(a, b, 1)^T$ with respect to the original system #1?



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Successive Transformations (3)

- To answer this, collect the effects of each transformation:
  - In terms of system #2 the point P has coordinates $(c, d, 1)^T = M_2(e, f, 1)^T$.
  - And in terms of system #1 the point $(c, d, 1)^T$ has coordinates $(a, b, 1)^T = M_1(c, d, 1)^T$.
  - So $(a, b, 1)^T = M_1(d, c, 1)^T = M_1M_2(e, f, 1)^T$
- The essential point is that when determining the desired coordinates $(a, b, 1)^T$ from $(e, f, 1)^T$ we *first* apply M_2 and *then* M_1, just the *opposite* order as when applying transformations to points.

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Successive Transformations (4)

- **To transform points.** To apply a sequence of transformations $T_1()$, $T_2()$, $T_3()$ (in that order) to a point P, form the matrix $M = M_3 \times M_2 \times M_1$.
- Then P is transformed to *MP; pre-multiply* by M_i.
- **To transform the coordinate system.** To apply a sequence of transformations $T_1()$, $T_2()$, $T_3()$ (in that order) to the coordinate system, form the matrix $M = M_1 \times M_2 \times M_3$.
- Then P in the transformed system has coordinates *MP* in the original system. To compose each additional transformation M_i you must *post-multiply* by M_i.

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Open-GL Transformations

- Open-GL actually transforms coordinate systems, so in your programs you will have to apply the transformations in reverse order.
- E.g., if you want to translate the 3 vertices of a triangle and then rotate it, your program will have to do rotate and then translate.

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Using Affine Transformations in Open-GL

- glScaled (sx, sy, sz); // 2-d: sz = 1.0
- glTranslated (tx, ty, tz); //2-d: tz = 0.0
- glRotated (angle, ux, uy, uz); // 2-d: ux = uy = 0.0; uz = 1.0
- The sequence of commands is
  - glLoadIdentity();
  - glMatrixMode (GL_MODELVIEW);
  - // transformations 1, 2, 3, ... (in reverse order)
- This method makes Open-GL do the work of transforming for you.

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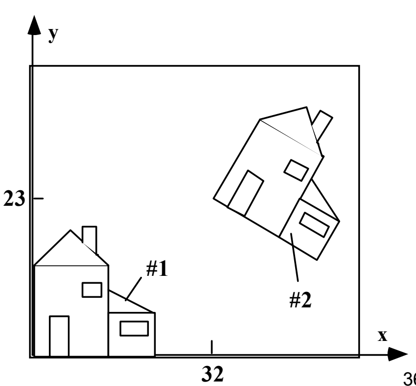
Example

- We have version 1 of the house defined (vertices set), but what we really want to draw is version 2.
- We could write routines to transform the coordinates – this is the hard way.
- The easy way lets GL do the transforming.

a).



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