Examples of CFL

Example 1. $L=\{0^n 1^n \mid n \geq 0\}$ is a CFL.

To show that L is a CFL, we need to construct a CFG G generating L.

Let $G = (V, \Sigma, R, S)$ where

$V = \{S\}$

$\Sigma = \{0, 1\}$

$R = \{S \rightarrow \varepsilon | 0S1 \}$

Example 2. Construct CFG to generate $L=\{xx^R \mid x \text{ is in } (0+1)^*\}$.

$G = (V, \Sigma, R, S)$ where

$V = \{S\}$

$\Sigma = \{0, 1\}$

$R = \{ S \rightarrow \varepsilon \mid 0S0 \mid 1S1\}$

Example 3 $L=\{ x \in (0+1)^* \mid \#_0(x) = \#_1(x)\}$

Idea: what relations exist between longer strings and shorter strings in this language?

Consider four cases:

Case 1. $x = 0w1$. Then x is in L iff w is in L.
If we use S to represent a string in L, then
the relation in Case 1 can be represented
As a rule

$$S \rightarrow 0S1$$

Case 2. $x=1w0$. Similar to case 1. This case gives rule

$$S \to 1S0$$

Case 3. $x=0w0$. Suppose $w = w_1 w_2 \cdots w_n$.
Consider the following sequence:

$$\#_0(0) - \#_1(0) > 0,$$
$$\#_0(0w_1) - \#_1(0w_1),$$
$$\ldots,$$
$$\#_0(0w_1 \cdots w_n) - \#_1(0w_1 \cdots w_n) < 0.$$

Note that in this sequence, two adjacent numbers Have difference 1. Therefore, there exists I such that

$$\#_0(0w_1 \cdots w_i) - \#_1(0w_1 \cdots w_i) = 0$$

This means that x is the concatenation of two shorter strings in L. So, we have a rule

$$S \to SS$$

Case 4. $x=1w1$. Similar to Case 3.

Based on the above analysis, we have CFG

$G=(V, \Sigma, R, S)$ where

$V = \{S\}$

$\Sigma = \{0, 1\}$

$R = \{S \rightarrow \varepsilon \mid 0S1 \mid 1S0 \mid SS\}$

Each nonterminal symbol represents a language.

Each rule represents a relationship between languages represented by nonterminal symbols.

CFL is closed under union.

Proof. Suppose $A = L(G_A)$ and $B = L(G_B)$ where

$G_A = (V_A, \Sigma_A, R_A, S_A)$

$G_B = (V_B, \Sigma_B, R_B, S_B)$

Without loss of generality, assume $V_A \cap V_B = \emptyset$.
(Otherwise, we may change some nonterminal symbols.)

Then $A \cup B = L(G)$ for $G=(V, \Sigma, R, S)$ where

$V = V_A \cup V_B \cup \{S\}$

$\Sigma = \Sigma_A \cup \Sigma_B$

$R = R_A \cup R_B \cup \{S \to S_A \mid S_B\}$

Example 4 $L = \{ 0^m 1^n \mid m \neq n, m, n \geq 0\}$

$L = \{0^m 1^n \mid m > n \geq 0\} \cup \{0^m 1^n \mid n > m \geq 0\}$

Hence, $L = L(G)$ for $G = (\{S, S_A, S_B\}, \{0,1\}, R, S)$
where

$R = \{S \rightarrow S_A \mid S_B,$

$S_A \rightarrow 0 \mid 0S_A \mid 0S_A1,$

$S_B \rightarrow 1 \mid S_B1 \mid 0S_B1\}$

CFL is closed under concatenation.

Proof. Suppose $A = L(G_A)$ and $B = L(G_B)$ where

$G_A = (V_A, \Sigma_A, R_A, S_A)$

$G_B = (V_B, \Sigma_B, R_B, S_B)$

Without loss of generality, assume $V_A \cap V_B = \emptyset$.
(Otherwise, we may change some nonterminal symbols.)

Then $AB = L(G)$ for $G=(V, \Sigma, R, S)$ where

$V = V_A \cup V_B \cup \{S\}$

$\Sigma = \Sigma_A \cup \Sigma_B$

$R = R_A \cup R_B \cup \{S \rightarrow S_A S_B\}$

Example 5 $L = \{xx^R w \mid x \in (0+1)^+,\ w \in (0+1)^*\}$

$L = \{xx^R \mid x \in (0+1)^+\}\{0,1\}^*$

$L=L(G)$ for $G = (\{S, S_A, S_B\}, \{0, 1\}, R, S)$
where
$R = \{\ S \rightarrow S_A S_B,$
$S_A \rightarrow 00 \mid 11 \mid 0S_A0 \mid 1S_A1,$
$S_B \rightarrow \varepsilon \mid 0S_B \mid 1S_B\ \}$

CFL is closed under star-closure.

Proof. Suppose L = (G) for G=(V, Σ, R, S).

Then L* = L(G*) for G* =(V, Σ, R*, S)

where R* = R U { S → ε | SS}.

S represents L and S^* represents L^*.

Then $S^* \to \varepsilon \mid S^* S$.

So, $S^* \Rightarrow S$.

Example 6 $L=(0+1)^*00$

$L=L(G)$ for $G=(\{S, A\}, \{0,1\}, R, S)$
where
$R=\{S \rightarrow A00,\ A \rightarrow \varepsilon \mid AA \mid 0 \mid 1\}$

The role of nonterminat symbol.

Every nonterminal symbol A represents a language which can be generated by using A as start symbol.

Example 7.

$L=\{a^m b^n c^p d^q \mid m+n = p+q,\ m, n, p, q \geq 0\}$

Let S represent L,

A represent $\{b^n c^n \mid n \geq 0\}$

B represent $\{a^m b^n c^p \mid m+n = p,\ m, n, p \geq 0\}$

C represent $\{b^n c^p d^q \mid n = p+q,\ n, p, q \geq 0\}$

Then we can find relations

$S \rightarrow aSd \mid B \mid C,$

$B \rightarrow aBc \mid A,$

$C \rightarrow bCd \mid A,$

$A \rightarrow bAc \mid \varepsilon.$

Example 8
L ={x $\in$ (0+1)*| x $\neq$ ww for any w $\in$ (0+1)*}

Let us analyze what would happens for x in L.

Case 1. |x| = odd.
In this case, x is either in language
A = {u0v | |u|=|v|, u, v (0+1)*}
or
B = {u1v | |u|=|v|, u, v (0+1)*}.

Case 2. |x| = even. Write
$x = x_1\ x_2\ \ldots\ x_m\ y_1\ y_2\ \ldots y_m$.
There exists i such that $x_i \neq y_i$.

Subcase 2.1 $x = x_1 \ldots x_{i-1} 0 x_{i+1} \ldots x_m y_1 \ldots y_{i-1} 1 y_{i+1} \ldots y_m$.
x is in AB

Subcase 2.2 $x = x_1 \ldots x_{i-1} 1 x_{i+1} \ldots x_m y_1 \ldots y_{i-1} 0 y_{i+1} \ldots y_m$.
x is in BA

Thus, $L = A+B+AB+BA$.

So, $L=L(G)$ for $G=(\{L,A,B\}, \{0,1\}, R, L)$
where

$$
\begin{aligned}
R = \{ & L \to A \mid B \mid AB \mid BA, \\
& A \to 0 \mid 0A0 \mid 0A1 \mid 1A0 \mid 1A1, \\
& B \to 1 \mid 0B0 \mid 0B1 \mid 1B0 \mid 1B1 \}.
\end{aligned}
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