

NFA - DFA Equivalence

The following theorem proves that for any NFA (without ϵ transitions) there exists an equivalent DFA which accepts the same language. Note that the resulting DFA may have an exponential number of states with respect to the number of states in the original NFA.

Theorem 1

Let $L \subseteq \Sigma^*$, and suppose L is accepted by the NFA $M_a = (Q_a, \Sigma, \delta_a, q_{0-a}, F_a)$. There is a FA $M_b = (Q_b, \Sigma, \delta_b, q_{0-b}, F_b)$ that accepts L (i.e. $L(M_1) = L(M_2)$).

Proof: The trick here is to construct a new FA out of the given NFA where each state in the FA represents a set of states in NFA. We define:

$Q_b = 2^Q$: the set of subsets of Q_a .

$q_{0-b} = \{q_{0-a}\}$.

$F_b = \{q \in Q_b \mid q \cap F_a \neq \emptyset\}$.

$\delta_b(q, a) = \bigcup_{p \in q} \delta_a(p, a)$ for $q \in Q_b$ and $a \in \Sigma$.

We now show: $\widehat{\delta}_b(q_{0-b}, x) = \widehat{\delta}_a(q_{0-a}, x)$ (for every x)
which will imply that FA M_b accepts language L .

Let's use induction (on the length of the string x):

Base Case: $|x| = 0$

$$\begin{aligned} \widehat{\delta}_b(q_{0-b}, x) &= \widehat{\delta}_b(q_{0-b}, \epsilon) \\ &= q_{0-b} \text{ (by definition of } \widehat{\delta}_b) \\ &= \{q_{0-a}\} \\ &= \widehat{\delta}_a(q_{0-a}, \epsilon) \\ &= \widehat{\delta}_a(q_{0-a}, x) \end{aligned}$$

Our Induction Hypothesis is that for $n \geq 0$ and for any y , s.t. $|y| = n$, $\widehat{\delta}_b(q_{0-b}, y) = \widehat{\delta}_a(q_{0-a}, y)$.

Need to show that for string z , s.t. $|z| = n + 1$, $\widehat{\delta}_b(q_{0-b}, z) = \widehat{\delta}_a(q_{0-a}, z)$. Of course we will write z as the concatenation of string y ($|y| = n$) and the character $a \in \Sigma$.

$$\begin{aligned} \widehat{\delta}_b(q_{0-b}, z) &= \widehat{\delta}_b(q_{0-b}, ya) \\ &= \delta_b(\widehat{\delta}_b(q_{0-b}, y), a) \\ &= \delta_b(\widehat{\delta}_a(q_{0-a}, y), a) \\ &= \bigcup_{p \in \widehat{\delta}_a(q_{0-a}, y)} \delta_a(p, a) \text{ (by definition of } \delta_b) \\ &= \widehat{\delta}_a(q_{0-a}, ya) \end{aligned}$$

$$= \widehat{\delta}_a(q_{0-a}, z)$$

A string is accepted by M_b if and only if $\widehat{\delta}_b(q_{0-b}, x) \in F_b$. By our above construction we can state that a string is accepted by M_b if and only if $\widehat{\delta}_a(q_{0-a}, x) \cap A \neq \emptyset$.

A string x is accepted by M_b if and only if it is accepted by M_a . \diamond