Probability

Uncertainty

- Let action A_t = leave for airport t minutes before flight
  - Will A_t get me there on time?
- Problems:
  - Partial observability (road state, other drivers' plans, etc.)
  - Noisy sensors (traffic reports)
  - Uncertainty in action outcomes (flat tire, etc.)
  - Complexity of modeling and predicting traffic
- Hence a purely logical approach either
  - Risks falsehood: "A_{25} will get me there on time," or
  - Leads to conclusions that are too weak for decision making:
    - A_{25} will get me there on time if there's no accident on the bridge and it doesn't rain and my tires remain intact, etc., etc.
    - A_{1440} might reasonably be said to get me there on time but I'd have to stay overnight in the airport

Probability

Probabilistic assertions summarize effects of

- Laziness: failure to enumerate exceptions, qualifications, etc.
- Ignorance: lack of explicit theories, relevant facts, initial conditions, etc.
- Intrinsically random behavior

Making decisions under uncertainty

- Suppose the agent believes the following:

  $P(A_{25}$ gets me there on time$) = 0.04$

  $P(A_{90}$ gets me there on time$) = 0.70$

  $P(A_{120}$ gets me there on time$) = 0.95$

  $P(A_{1440}$ gets me there on time$) = 0.9999$

- Which action should the agent choose?
  - Depends on preferences for missing flight vs. time spent waiting
  - Encapsulated by a *utility function*
- The agent should choose the action that maximizes the *expected utility*:

  $P(A_t$ succeeds$) * U(A_t$ succeeds$) + P(A_t$ fails$) * U(A_t$ fails$)$

- Utility theory is used to represent and infer preferences
- Decision theory = probability theory + utility theory

Where do probabilities come from?

- **Frequentism**
  - Probabilities are relative frequencies
  - For example, if we toss a coin many times, P(heads) is the proportion of the time the coin will come up heads
  - But what if we're dealing with events that only happen once?
    - E.g., what is the probability that Republicans will take over Congress in 2010?
  - "Reference class" problem
- **Subjectivism**
  - Probabilities are degrees of belief
  - But then, how do we assign belief values to statements?
  - What would constrain agents to hold consistent beliefs?

Random variables

- We describe the (uncertain) state of the world using *random variables*
  - Denoted by capital letters
  - **R**: *Is it raining?*
  - **W**: *What's the weather?*
  - **D**: *What is the outcome of rolling two dice?*
  - **S**: *What is the speed of my car (in MPH)?*
- Just like variables in CSP's, random variables take on values in a *domain*
  - Domain values must be mutually exclusive and exhaustive
  - **R** in {True, False}
  - **W** in {Sunny, Cloudy, Rainy, Snow}
  - **D** in {(1,1), (1,2), … (6,6)}
  - **S** in [0, 200]

Events

- Probabilistic statements are defined over *events*, or sets of world states
  - *"It is raining"*
  - *"The weather is either cloudy or snowy"*
  - *"The sum of the two dice rolls is 11"*
  - *"My car is going between 30 and 50 miles per hour"*
- Events are described using propositions:
  - R = True
  - W = "Cloudy" $\vee$ W = "Snowy"
  - D $\in$ {(5,6), (6,5)}
  - $30 \leq S \leq 50$
- Notation: P(A) is the probability of the set of world states in which proposition A holds
  - P(X = x), or P(x) for short, is the probability that random variable X has taken on the value x

Kolmogorov’s axioms of probability

- For any propositions (events) A, B
  - $0 \leq P(A) \leq 1$
  - $P(True) = 1$ and $P(False) = 0$
  - $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$
    - Subtraction accounts for double-counting

- Based on these axioms, what is $P(\neg A)$?

- These axioms are sufficient to completely specify probability theory for *discrete* random variables
  - For continuous variables, need *density functions*

Probabilities and rationality

- Why should a rational agent hold beliefs that are consistent with axioms of probability?

- De Finetti (1931): If an agent has some degree of belief in proposition A, he/she should be able to decide whether or not to accept a bet for/against A
  - E.g., if the agent believes that $P(A) = 0.4$, should he/she agree to bet \$6 that A will occur against \$4 that A will not occur?

- **Theorem:** An agent who holds beliefs inconsistent with axioms of probability can be tricked into accepting a combination of bets that are guaranteed to lose them money

Atomic events

- Atomic event: a complete specification of the state of the world, or a complete assignment of domain values to all random variables
  - Atomic events are mutually exclusive and exhaustive

- E.g., if the world consists of only two Boolean variables *Cavity* and *Toothache*, then there are 4 distinct atomic events:

$$Cavity = false \wedge Toothache = false$$
$$Cavity = false \wedge Toothache = true$$
$$Cavity = true \wedge Toothache = false$$
$$Cavity = true \wedge Toothache = true$$

Joint probability distributions

- A **joint distribution** is an assignment of probabilities to every possible atomic event

Atomic event	**P**
Cavity = false $\wedge$ *Toothache = false*	0.8
Cavity = false $\wedge$ *Toothache = true*	0.1
Cavity = true $\wedge$ *Toothache = false*	0.05
Cavity = true $\wedge$ *Toothache = true*	0.05

  – Why does it follow from the axioms of probability that the probabilities of all possible atomic events must sum to 1?

Joint probability distributions

- Suppose we have a joint *distribution* $P(X_1, X_2, \ldots, X_n)$ of n random variables with domain sizes d
  - What is the size of the probability table?
  - Impossible to write out completely for all but the smallest distributions
- Notation:
  - $P(X = x)$ is the probability that random variable X takes on value x
  - $P(X)$ is the *distribution* of probabilities for all possible values of X

Marginal probability distributions

- Suppose we have the joint distribution P(X,Y) and we want to find the *marginal distribution* P(Y)

P(Cavity, Toothache)	
Cavity = false $\wedge$ *Toothache = false*	0.8
Cavity = false $\wedge$ *Toothache = true*	0.1
Cavity = true $\wedge$ *Toothache = false*	0.05
Cavity = true $\wedge$ *Toothache = true*	0.05

P(Cavity)	
Cavity = false	?
Cavity = true	?

P(Toothache)	
Toothache = false	?
Toochache = true	?

Marginal probability distributions

- Suppose we have the joint distribution P(X,Y) and we want to find the *marginal distribution* P(Y)

$$P(X = x) = P\big((X = x \wedge Y = y_1) \vee \ldots \vee (X = x \wedge Y = y_n)\big)$$

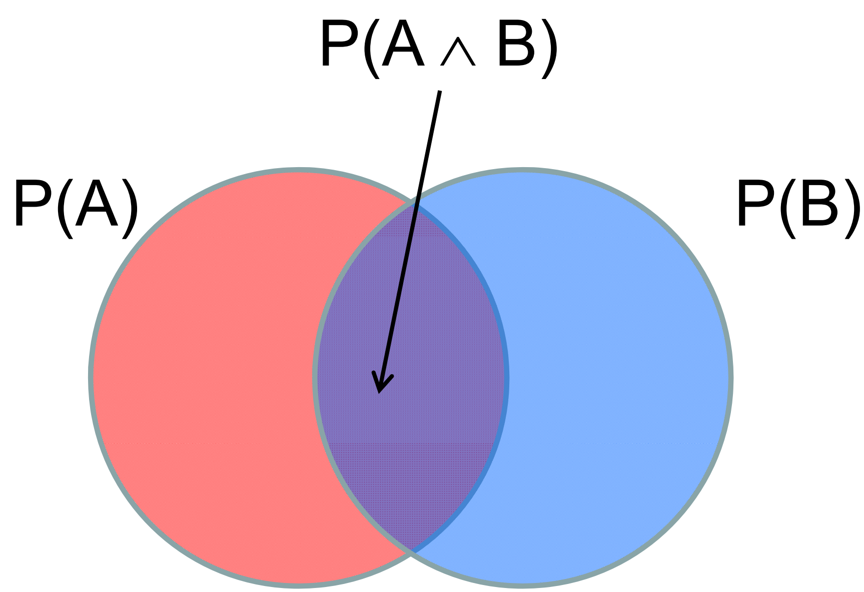
$$= P\big((x, y_1) \vee \ldots \vee (x, y_n)\big) = \sum_{i=1}^{n} P(x, y_i)$$

- General rule: to find P(X = x), sum the probabilities of all atomic events where X = x.

Conditional probability

- Probability of cavity given toothache:
  P(*Cavity* = *true* | *Toothache* = *true*)

- For any two events A and B, $P(A \mid B) = \dfrac{P(A \wedge B)}{P(B)} = \dfrac{P(A, B)}{P(B)}$



Conditional probability

P(Cavity, Toothache)	
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P(Cavity)	
Cavity = false	0.9
Cavity = true	0.1

P(Toothache)	
Toothache = false	0.85
Toothache = true	0.15

- What is P(*Cavity = true* | *Toothache = false*)?

  0.05 / 0.85 = 0.059
- What is P(*Cavity = false* | *Toothache = true*)?

  0.1 / 0.15 = 0.667

Conditional distributions

- A conditional distribution is a distribution over the values of one variable given fixed values of other variables

P(Cavity, Toothache)	
Cavity = false $\wedge$ *Toothache = false*	0.8
Cavity = false $\wedge$ *Toothache = true*	0.1
Cavity = true $\wedge$ *Toothache = false*	0.05
Cavity = true $\wedge$ *Toothache = true*	0.05

P(Cavity | Toothache = true)	
Cavity = false	0.667
Cavity = true	0.333

P(Cavity|Toothache = false)	
Cavity = false	0.941
Cavity = true	0.059

P(Toothache | Cavity = true)	
Toothache= false	0.5
Toothache = true	0.5

P(Toothache | Cavity = false)	
Toothache= false	0.889
Toothache = true	0.111

Normalization trick

- To get the whole conditional distribution P(X | y) at once, select all entries in the joint distribution matching Y = y and renormalize them to sum to one

P(Cavity, Toothache)	
Cavity = false $\wedge$ *Toothache = false*	0.8
Cavity = false $\wedge$ *Toothache = true*	0.1
Cavity = true $\wedge$ *Toothache = false*	0.05
Cavity = true $\wedge$ *Toothache = true*	0.05

Select

Toothache, Cavity = false	
Toothache= false	0.8
Toothache = true	0.1

Renormalize

P(Toothache | Cavity = false)	
Toothache= false	0.889
Toothache = true	0.111

Normalization trick

- To get the whole conditional distribution P(X | y) at once, select all entries in the joint distribution matching Y = y and renormalize them to sum to one
- Why does it work?

$$\frac{P(x,y)}{\sum_{a'} P(x',y)} = \frac{P(x,y)}{P(y)} \quad \text{by marginalization}$$

Product rule

- Definition of conditional probability: $P(A \mid B) = \dfrac{P(A,B)}{P(B)}$

- Sometimes we have the conditional probability and want to obtain the joint:

$$P(A,B) = P(A \mid B)P(B) = P(B \mid A)P(A)$$

Product rule

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$$P(A, B) = P(A \mid B)P(B) = P(B \mid A)P(A)$$

- The chain rule:

$$P(A_1, \ldots, A_n) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1, A_2)\ldots P(A_n \mid A_1, \ldots, A_{n-1})$$

$$= \prod_{i=1}^{n} P(A_i \mid A_1, \ldots, A_{i-1})$$

Bayes Rule

Rev. Thomas Bayes
(1702-1761)

- The product rule gives us two ways to factor a joint distribution:

$$P(A, B) = P(A \mid B)P(B) = P(B \mid A)P(A)$$

- Therefore, $P(A \mid B) = \dfrac{P(B \mid A)P(A)}{P(B)}$

- Why is this useful?
  - Can get *diagnostic probability* P(cavity | toothache) from *causal probability* P(toothache | cavity)
  - Can update our beliefs based on evidence
  - Important tool for probabilistic inference

Independence

- Two events A and B are independent if and only if $P(A \wedge B) = P(A)\,P(B)$
  - In other words, $P(A \mid B) = P(A)$ and $P(B \mid A) = P(B)$
  - This is an important simplifying assumption for modeling, e.g., *Toothache* and *Weather* can be assumed to be independent
- Are two *mutually exclusive* events independent?
  - No, but for mutually exclusive events we have $P(A \vee B) = P(A) + P(B)$
- **Conditional independence**: A and B are *conditionally independent* given C iff $P(A \wedge B \mid C) = P(A \mid C)\,P(B \mid C)$

Conditional independence: Example

- *Toothache*: boolean variable indicating whether the patient has a toothache
- *Cavity*: boolean variable indicating whether the patient has a cavity
- *Catch*: whether the dentist's probe catches in the cavity

- If the patient has a cavity, the probability that the probe catches in it doesn't depend on whether he/she has a toothache

  $P(Catch \mid Toothache, Cavity) = P(Catch \mid Cavity)$
- Therefore, *Catch* is conditionally independent of *Toothache* given *Cavity*
- Likewise, *Toothache* is conditionally independent of *Catch* given *Cavity*

  $P(Toothache \mid Catch, Cavity) = P(Toothache \mid Cavity)$
- Equivalent statement:

  $P(Toothache, Catch \mid Cavity) = P(Toothache \mid Cavity)\, P(Catch \mid Cavity)$

Conditional independence: Example

- How many numbers do we need to represent the joint probability table P(*Toothache, Cavity, Catch*)?

  $2^3 - 1 = 7$ independent entries

- Write out the joint distribution using chain rule:

  P(*Toothache, Catch, Cavity*)

  = P(*Cavity*) P(*Catch* | *Cavity*) P(*Toothache* | *Catch, Cavity*)

  = P(*Cavity*) P(*Catch* | *Cavity*) P(*Toothache* | *Cavity*)

- How many numbers do we need to represent these distributions?

  1 + 2 + 2 = 5 independent numbers

- In most cases, the use of conditional independence reduces the size of the representation of the joint distribution from exponential in *n* to linear in *n*

Naïve Bayes model

- Suppose we have many different types of observations (symptoms, features) that we want to use to diagnose the underlying cause
- It is usually impractical to directly estimate or store the joint distribution $P(Cause, Effect_1, \ldots, Effect_n)$.
- To simplify things, we can assume that the different effects are conditionally independent *given the underlying cause*

Naïve Bayes model

- Suppose we have many different types of observations (symptoms, features) that we want to use to diagnose the underlying cause
- It is usually impractical to directly estimate or store the joint distribution $P(Cause, Effect_1, \ldots, Effect_n)$.
- To simplify things, we can assume that the different effects are conditionally independent *given the underlying cause*
- Then we can estimate the joint distribution as

$$P(Cause, Effect_1, \ldots, Effect_n) = P(Cause)\prod_i P(Effect_i \mid Cause)$$

- This is usually not accurate, but very useful in practice

Example: Naïve Bayes Spam Filter

- **Bayesian decision theory:** to minimize the probability of error, we should classify a message as spam if P(spam | message) > P(¬spam | message)
  - ***Maximum a posteriori* (MAP)** decision
- We have

$$P(spam \mid message) = \frac{P(message \mid spam)P(spam)}{P(message)} \quad \text{and}$$

$$P(\neg spam \mid message) = \frac{P(message \mid \neg spam)P(\neg spam)}{P(message)}$$

- Notice that P(message) is just a constant normalizing factor and doesn't affect the decision
- Therefore, all we need is to find P(message | spam) P(spam) and P(message | ¬spam) P(¬spam)

Example: Naïve Bayes Spam Filter

- We need to find P(message | spam) P(spam) and P(message | ¬spam) P(¬spam)
- The message is a sequence of words $(w_1, \ldots, w_n)$
- ***Bag of words*** representation
  - The order of the words in the message is not important
  - Each word is conditionally independent of the others given message class (spam or not spam)

$$P(message \mid spam) = P(w_1, \ldots, w_n \mid spam) = \prod_{i=1}^{n} P(w_i \mid spam)$$

- Our filter will classify the message as spam if

$$P(spam) \prod_{i=1}^{n} P(w_i \mid spam) \quad > \quad P(\neg spam) \prod_{i=1}^{n} P(w_i \mid \neg spam)$$

Example: Naïve Bayes Spam Filter

$$\underbrace{P(spam \mid w_1, \ldots, w_n)}_{\text{posterior}} = \underbrace{P(spam)}_{\text{prior}} \underbrace{\prod_{i=1}^{n} P(w_i \mid spam)}_{\text{likelihood}}$$

Probabilistic inference

- In general, the agent observes the values of some random variables $X_1, X_2, \ldots, X_n$ and needs to reason about the values of some other *unobserved* random variables $Y_1, Y_2, \ldots, Y_m$
  – Figuring out a diagnosis based on symptoms and test results
  – Classifying the content type of an image or a document based on some features
- This will be the subject of the next few lectures