Chomsky and Greibach Normal Forms

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Note the difference between grammar cleaning and simplifi cation

Note

Normal forms are useful when more advanced topics in computation theory are approached, as we shall see further

Definition

A context-free grammar G is in Chomsky normal form if every rule is of the form:

$$
\begin{array}{rcl}
A & \longrightarrow & BC \\
A & \longrightarrow & a
\end{array}
$$

where a is a terminal, A, B, C are nonterminals, and B, C may not be the start variable (the axiom)

Note

The rule $S \longrightarrow \epsilon$, where S is the start variable, is not excluded from a CFG in Chomsky normal form.

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- Show that any CFG G can be converted into a CFG G' in Chomsky normal form
- Conversion procedure has several stages where the rules that violate Chomsky normal form conditions are replaced with equivalent rules that satisfy these conditions
- Order of transformations: (1) add a new start variable, (2) eliminate all ϵ-rules, (3) eliminate unit-rules, (4) convert other rules

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- Show that any CFG G can be converted into a CFG G' in Chomsky normal form
- Conversion procedure has several stages where the rules that violate Chomsky normal form conditions are replaced with equivalent rules that satisfy these conditions
- Order of transformations: (1) add a new start variable, (2) eliminate all ϵ-rules, (3) eliminate unit-rules, (4) convert other rules
- Check that the obtained CFG G' defi nes the same language

Proof

Let $G = (N, T, R, S)$ be the original CFG.

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Note: this change guarantees that the start symbol of G' does not occur on the rhs of any rule

Step 2: eliminate ϵ-rules

Repeat

1. Eliminate the ϵ rule $A \longrightarrow \epsilon$ from R where A is not the start symbol
2. For each occurrence of A on the rhs of a rule, add a new rule to R with that occurrence of A deleted
   Example: replace $B \longrightarrow uAv$ by $B \longrightarrow uAv|uv$;
   replace $B \longrightarrow uAvAw$ by $B \longrightarrow uAvAw|uvAw|aAvw|uvw$
3. Replace the rule $B \longrightarrow A$, (if it is present) by $B \longrightarrow A|\epsilon$ unless the rule $B \longrightarrow \epsilon$ has been previously eliminated

until all ϵ rules are eliminated

Step 3: remove unit rules

Repeat

1. Remove a unit rule $A \longrightarrow B \in R$
2. For each rule $B \longrightarrow u \in R$, add the rule $A \longrightarrow u$ to R, unless $B \to u$ was a unit rule previously removed

until all unit rules are eliminated

Note: u is a string of variables and terminals

Convert all remaining rules

Repeat

1. Replace a rule $A \longrightarrow u_1u_2 \ldots u_k$, $k \geq 3$, where each u_i, $1 \leq i \leq k$, is a variable or a terminal, by:
   $A \longrightarrow u_1A_1$, $A_1 \longrightarrow u_2A_2$, ..., $A_{k-2} \longrightarrow u_{k-1}u_k$
   where A_1,A_2, ..., A_{k-2} are new variables
2. If $k \geq 2$ replace any terminal u_i with a new variable U_i and add the rule $U_i \longrightarrow u_i$

until no rules of the form $A \longrightarrow u_1u_2 \ldots u_k$ with $k \geq 3$ remain

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Example CFG conversion

Consider the grammar G_6 whose rules are:

$$\begin{array}{rcl} S & \longrightarrow & ASA|aB \\ A & \longrightarrow & B|S \\ B & \longrightarrow & b|\epsilon \end{array}$$

Notation: symbols removed are green and those added are red.

After fi rst step of transformation we get:

$$\begin{array}{rcl} S_0 & \longrightarrow & S \\ S & \longrightarrow & ASA|aB \\ A & \longrightarrow & B|S \\ B & \longrightarrow & b|\epsilon \end{array}$$

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Removing ϵ rules

Removing $B \to \epsilon$:

$$\begin{array}{rcl} S_0 & \longrightarrow & S \\ S & \longrightarrow & ASA|aB|a \\ A & \longrightarrow & B|S|\epsilon \\ B & \longrightarrow & b|\epsilon \end{array}$$

Removing $A \to \epsilon$:

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More unit rules

Removing $A \rightarrow B$:

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Converting remaining rules

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Note

- The conversion procedure produces several variables U_i along with several rules $U_i \rightarrow a$.
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Greibach Normal Form

A context-free grammar $G = (V, \Sigma, R, S)$ is in Greibach normal form if each rule $r \in R$ has the property: $lhs(r) \in V$, $rhs(r) = a\alpha$, $a \in \Sigma$ and $\alpha \in V^*$.

Note: Greibach normal form provides a justifi cation of operator prefi x-notation usually employed in algebra.

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Greibach Theorem

Every CFL L where $\epsilon \notin L$ can be generated by a CFG in Greibach normal form.

Proof idea: Let $G = (V, \Sigma, R, S)$ be a CFG generating L. Assume that G is in Chomsky normal form

- Let $V = \{A_1, A_2, \ldots, A_m\}$ be an ordering of nonterminals.
- Construct the Greibach normal form from Chomsky normal form

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Construction

1. Modify the rules in R so that if $A_i \to A_j\gamma \in R$ then $j > i$
2. Starting with A_1 and proceeding to A_m this is done as follows:
   (a) Assume that productions have been modifi ed so that for $1 \leq i \leq k$, $A_i \to A_j\gamma \in R$ only if $j > i$
   (b) If $A_k \to A_j\gamma$ is a production with $j < k$, generate a new set of productions substituting for the A_j the rhs of each A_j production
   (c) Repeating (b) at most $k - 1$ times we obtain rules of the form $A_k \to A_p\gamma$, $p \geq k$
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Removing left-recursion

Left-recursion can be eliminated by the following scheme:

- If $A \to A\alpha_1|A\alpha_2 \ldots |A\alpha_r$ are all A left recursive rules, and $A \to \beta_1|\beta_2|\ldots|\beta_s$ are all remaining A-rules then chose a new nonterminal, say B
- Add the new B-rules $B \to \alpha_i|\alpha_i B$, $1 \leq i \leq r$
- Replace the A-rules by $A \to \beta_i|\beta_i B$, $1 \leq i \leq s$

This construction preserve the language L.

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More on Greibach NF

See Introduction to Automata Theory, Languages, and Computation, J.E, Hopcroft and J.D Ullman, Addison-Wesley 1979, p. 94–96

Example

Convert the CFG

$G = (\{A_1, A_2, A_3\}, \{a, b\}, R, A_1)$

where

$R = \{A_1 \rightarrow A_2A_3, A_2 \rightarrow A_3A_1|b, A_3 \rightarrow A_1A_2|a\}$

into Greibach normal form.

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into Greibach normal form.

Solution

1. *Step 1:* ordering the rules: (Only A_3 rules violate ordering conditions, hence only A_3 rules need to be changed).
   Following the procedure we replace A_3 rules by:
   $A_3 \rightarrow A_3A_1A_3A_2|bA_3A_2|a$
2. Eliminating left-recursion we get: $A_3 \rightarrow bA_3A_2B_3|aB_3|bA_3A_2|a$,
   $B_3 \rightarrow A_1A_3A_2|A_1A_3A_2B_3$
3. All A_3 rules start with a terminal. We use them to replace $A_1 \rightarrow A_2A_3$. This introduces the rules $B_3 \rightarrow A_1A_3A_2|A_1A_3A_2B_3$
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