# Chapter 3

# Vector Spaces

## 3.1 Vector spaces and subspaces

We discussed how to solve a system $A\mathbf{x} = \mathbf{b}$ of linear equations, and we saw that the basic questions of the existence or uniqueness of the solution were much easier to answer after Gaussian-elimination. In this chapter, we introduce the notion of a vector space, which is an abstraction of the usual algebraic structures of the 3-space $\mathbb{R}^3$ and then elaborate our study of a system of linear equations to this framework.

Usually, many physical quantities, such as length, area, mass, temperature are described by real numbers as magnitudes. Other physical quantities like force or velocity have directions as well as magnitudes. Such quantities with direction are called **vectors**, while the numbers are called **scalars**. For instance, an element (or a point) $\mathbf{x}$ in the 3-space $\mathbb{R}^3$ is usually represented as a triple of real numbers:

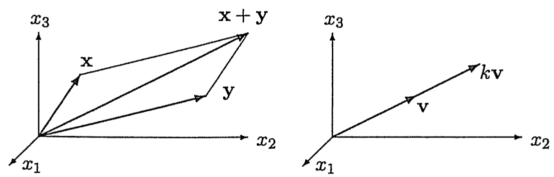
$$\mathbf{x} = (x_1,\ x_2,\ x_3),$$

where $x_i \in \mathbb{R}$, $i = 1,\ 2,\ 3$, are called the **coordinates** of $\mathbf{x}$. This expression provides a rectangular coordinate system in a natural way. On the other hand, pictorially such a point in the 3-space $\mathbb{R}^3$ can also be represented by an arrow from the origin to $\mathbf{x}$. In this way, a point in the 3-space $\mathbb{R}^3$ can be understood as a vector. The direction of the arrow specifies the direction of the vector, and the length of the arrow describes its magnitude.

In order to have a more general definition of vectors, we extract the most basic properties of those arrows in $\mathbb{R}^3$. Note that for all vectors (or points) in $\mathbb{R}^3$, there are two algebraic operations: the addition of any two vectors and scalar multiplication of a vector by a scalar. That is, for two vectors $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$ and $k$ a scalar, we define

$$\begin{aligned} \mathbf{x}+\mathbf{y} &= (x_1+y_1, x_2+y_2, x_3+y_3), \\ k\mathbf{x} &= (kx_1, kx_2, kx_3). \end{aligned}$$

The addition of vectors and scalar multiplication of a vector in the 3-space $\mathbb{R}^3$ are illustrated as follows:




Even though our geometric visualization of vectors does not go beyond the 3-space $\mathbb{R}^3$, it is possible to extend the above algebraic operations of vectors in the 3-space $\mathbb{R}^3$ to the general **n-space** $\mathbb{R}^n$ for any positive integer $n$. It is defined to be the set of all ordered $n$-tuples $(a_1, a_2, \ldots, a_n)$ of real numbers, called *vectors*: *i.e.*,

$$\mathbb{R}^n = \{(a_1, a_2, \ldots, a_n) \,:\, a_i \in \mathbb{R}, i = 1, 2, \ldots, n\}.$$

For any two vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ in the $n$-space $\mathbb{R}^n$, and a scalar $k$, the *sum* $\mathbf{x}+\mathbf{y}$ and the *scalar multiplication* $k\mathbf{x}$ of them are vectors in $\mathbb{R}^n$ defined by

$$\begin{aligned} \mathbf{x}+\mathbf{y} &= (x_1+y_1, x_2+y_2, \ldots, x_n+y_n), \\ k\mathbf{x} &= (kx_1, kx_2, \ldots, kx_n). \end{aligned}$$

It is easy to verify the following list of arithmetical rules of the operations:

**Theorem 3.1** *For any scalars $k$ and $\ell$, and vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, and $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ in the $n$-space $\mathbb{R}^n$, the following rules hold:*

(1) $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$,

(2) $\mathbf{x}+(\mathbf{y}+\mathbf{z}) = (\mathbf{x}+\mathbf{y})+\mathbf{z}$,

(3) $\mathbf{x}+\mathbf{0} = \mathbf{x} = \mathbf{0}+\mathbf{x}$,

(4) $\mathbf{x} + (-1)\mathbf{x} = \mathbf{0}$,

(5) $k(\mathbf{x} + \mathbf{y}) = k\mathbf{x} + k\mathbf{y}$,

(6) $(k + \ell)\mathbf{x} = k\mathbf{x} + \ell\mathbf{x}$,

(7) $k(\ell\mathbf{x}) = (k\ell)\mathbf{x}$,

(8) $1\mathbf{x} = \mathbf{x}$,

*where* $\mathbf{0} = (0,\ 0,\ \ldots,\ 0)$ *is the* **zero** *vector.*

We usually write a vector $(a_1,\ a_2,\ \ldots,\ a_n)$ in the $n$-space $\mathbb{R}^n$ as an $n \times 1$ column matrix

$$(a_1,\ a_2,\ \ldots,\ a_n) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = [a_1\ a_2\ \cdots\ a_n]^T,$$

also called a *column vector*. Then the two operations of the matrix sum and the scalar multiplication of column matrices coincide with those of vectors in $\mathbb{R}^n$, and the above theorem is just Theorem 1.2.

These rules of arithmetic of vectors are the most important ones because they are the only rules that we need to manipulate vectors in the $n$-space $\mathbb{R}^n$. Hence, an (abstract) vector space can be defined with respect to these rules of operations of vectors in the $n$-space $\mathbb{R}^n$ so that $\mathbb{R}^n$ itself becomes a vector space. In general, a *vector space* is defined to be a set with two operations: an addition and a scalar multiplication which satisfy the above rules of operations in $\mathbb{R}^n$.

**Definition 3.1** A (real) **vector space** is a nonempty set $V$ of elements, called **vectors**, with two algebraic operations that satisfy the following rules.

(A) There is an operation called *vector addition* that associates to every pair $\mathbf{x}$ and $\mathbf{y}$ of vectors in $V$ a unique vector $\mathbf{x} + \mathbf{y}$ in $V$, called the the **sum** of $\mathbf{x}$ and $\mathbf{y}$, so that the following rules hold for all vectors $\mathbf{x},\ \mathbf{y},\ \mathbf{z}$ in $V$:

(1) $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ (commutativity in addition),

(2) $\mathbf{x} + (\mathbf{y}+\mathbf{z}) = (\mathbf{x}+\mathbf{y}) + \mathbf{z} (= \mathbf{x} + \mathbf{y} + \mathbf{z})$ (associativity in addition),

(3) there is a unique vector $\mathbf{0}$ in $V$ such that $\mathbf{x} + \mathbf{0} = \mathbf{x} = \mathbf{0} + \mathbf{x}$ for all $\mathbf{x} \in V$ (it is called the **zero vector**),

(4) for any $\mathbf{x} \in V$, there is a vector $-\mathbf{x} \in V$, called the **negative** of $\mathbf{x}$, such that $\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$.

**(B)** There is an operation called **scalar multiplication** that associates to each vector $\mathbf{x}$ in $V$ and each scalar $k$ a unique vector $k\mathbf{x}$ in $V$ called the **multiplication** of $\mathbf{x}$ by a (real) scalar $k$, so that the following rules hold for all vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ in $V$ and all scalars $k$, $\ell$:

**(5)** $k(\mathbf{x}+\mathbf{y}) = k\mathbf{x} + k\mathbf{y}$ (distributivity with respect to vector addition),

**(6)** $(k+\ell)\mathbf{x} = k\mathbf{x} + \ell\mathbf{x}$ (distributivity with respect to scalar addition),

**(7)** $k(\ell\mathbf{x}) = (k\ell)\mathbf{x}$ (associativity in scalar multiplication),

**(8)** $1\mathbf{x} = \mathbf{x}$.

Clearly, the $n$-space $\mathbb{R}^n$ is a vector space by Theorem 3.1. A **complex vector space** is obtained if, instead of real numbers, we take complex numbers for scalars. For example, the set $\mathbb{C}^n$ of all ordered $n$-tuples of complex numbers is a complex vector space. In Chapter 7 we shall discuss complex vector spaces, but until then we will discuss only real vector spaces unless otherwise stated.

**Example 3.1** **(1)** For any positive integer $m$ and $n$, the set $M_{m\times n}(\mathbb{R})$ of all $m \times n$ matrices forms a vector space under the matrix sum and scalar multiplication defined in Section 1.3. The zero vector in this space is the zero matrix $\mathbf{0}_{m\times n}$, and $-A$ is the negative of a matrix $A$.

**(2)** Let $C(\mathbb{R})$ denote the set of real-valued continuous functions defined on the real line $\mathbb{R}$. For two functions $f(x)$ and $g(x)$, and a real number $k$, the sum $f+g$ and the scalar multiplication $kf$ of them are defined by

$$\begin{aligned} (f+g)(x) &= f(x)+g(x), \\ (kf)(x) &= kf(x). \end{aligned}$$

Then one can easily verify, as an exercise, that the set $C(\mathbb{R})$ is a vector space under these operations. The zero vector in this space is the constant function whose value at each point is zero.

**(3)** Let $A$ be an $m \times n$ matrix. Then it is easy to show that the set of solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is a vector space (under the sum and scalar multiplication of matrices).

**Theorem 3.2** *Let $V$ be a vector space and let $\mathbf{x}$, $\mathbf{y}$ be vectors in $V$. Then*

**(1)** $\mathbf{x}+\mathbf{y} = \mathbf{y}$ *implies* $\mathbf{x} = \mathbf{0}$,

**(2)** $0\mathbf{x} = \mathbf{0}$,

**(3)** $k\mathbf{0} = \mathbf{0}$ *for any* $k \in \mathbb{R}$,

(4) $-\mathbf{x}$ *is unique and* $-\mathbf{x} = (-1)\mathbf{x}$,

(5) *if* $k\mathbf{x} = \mathbf{0}$, *then* $k = 0$ *or* $\mathbf{x} = \mathbf{0}$.

**Proof:** (1) By adding $-\mathbf{y}$ to both sides of $\mathbf{x} + \mathbf{y} = \mathbf{y}$, we have

$$\mathbf{x} = \mathbf{x} + \mathbf{0} = \mathbf{x} + \mathbf{y} + (-\mathbf{y}) = \mathbf{y} + (-\mathbf{y}) = \mathbf{0}.$$

(2) $0\mathbf{x} = (0+0)\mathbf{x} = 0\mathbf{x} + 0\mathbf{x}$ implies $0\mathbf{x} = \mathbf{0}$ by (1).

(3) This is an easy exercise.

(4) The uniqueness of the negative $-\mathbf{x}$ of $\mathbf{x}$ can be shown by a simple modification of Lemma 1.6. In fact, if $\bar{\mathbf{x}}$ is another negative of $\mathbf{x}$ such that $\mathbf{x} + \bar{\mathbf{x}} = \mathbf{0}$, then

$$-\mathbf{x} = -\mathbf{x} + \mathbf{0} = -\mathbf{x} + (\mathbf{x} + \bar{\mathbf{x}}) = (-\mathbf{x} + \mathbf{x}) + \bar{\mathbf{x}} = \mathbf{0} + \bar{\mathbf{x}} = \bar{\mathbf{x}}.$$

On the other hand, the equation

$$\mathbf{x} + (-1)\mathbf{x} = 1\mathbf{x} + (-1)\mathbf{x} = (1-1)\mathbf{x} = 0\mathbf{x} = \mathbf{0}$$

shows that $(-1)\mathbf{x}$ is another negative of $\mathbf{x}$, and hence $-\mathbf{x} = (-1)\mathbf{x}$ by the uniqueness of $-\mathbf{x}$.

(5) Suppose $k\mathbf{x} = \mathbf{0}$ and $k \neq 0$. Then $\mathbf{x} = 1\mathbf{x} = \frac{1}{k}(k\mathbf{x}) = \frac{1}{k}\mathbf{0} = \mathbf{0}$. □

*Problem 3.1* Let $V$ be the set of all pairs $(x,\ y)$ of real numbers. Suppose that an addition and scalar multiplication of pairs are defined by

$$(x,\ y) + (u,\ v) = (x + 2u,\ y + 2v), \quad k(x,\ y) = (kx,\ ky).$$

Is the set $V$ a vector space under those operations? Justify your answer.

A subset $W$ of a vector space $V$ is called a **subspace** of $V$ if $W$ is itself a vector space under the addition and scalar multiplication defined in $V$. Usually, in order to show that a subset $W$ is a subspace, it is not necessary to verify all the rules of the definition of a vector space, because certain rules satisfied in the larger space $V$ are automatically satisfied in every subset, if vector addition and scalar multiplication are closed in subset.

**Theorem 3.3** *A nonempty subset $W$ of a vector space $V$ is a subspace if and only if $\mathbf{x} + \mathbf{y}$ and $k\mathbf{x}$ are contained in $W$ (or equivalently, $\mathbf{x} + k\mathbf{y} \in W$) for any vectors $\mathbf{x}$ and $\mathbf{y}$ in $W$ and any scalar $k \in \mathbb{R}$.*

**Proof:** We need only to prove the sufficiency. Assume both conditions hold and let $\mathbf{x}$ be any vector in $W$. Since $W$ is closed under scalar multiplication, $\mathbf{0} = 0\mathbf{x}$ and $-\mathbf{x} = (-1)\mathbf{x}$ are in $W$, so rules (3) and (4) for a vector space hold. All the other rules for a vector space are clear. $\square$

A vector space $V$ itself and the zero vector $\{\mathbf{0}\}$ are trivially subspaces. Some nontrivial subspaces are given in the following examples.

**Example 3.2** Let $W = \{(x,\ y,\ z) \in \mathbb{R}^3\ :\ ax + by + cz = 0\}$, where $a, b, c$ are constants. If $\mathbf{x} = (x_1,\ x_2,\ x_3)$, $\mathbf{y} = (y_1,\ y_2,\ y_3)$ are points in $W$, then clearly $\mathbf{x} + \mathbf{y} = (x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3)$ is also a point in $W$, because it satisfies the equation in $W$. Similarly, $k\mathbf{x}$ also lies in $W$ for any scalar $k$. Hence, $W$ is a subspace of $\mathbb{R}^3$ and is a plane passing through the origin in $\mathbb{R}^3$.

**Example 3.3** Let $A$ be an $m \times n$ matrix. Then, as we have seen in Example 3.1 (3), the set

$$W = \{\mathbf{x} \in \mathbb{R}^n\ :\ A\mathbf{x} = \mathbf{0}\}$$

of solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is a vector space. Moreover, since the operations in $W$ and in $\mathbb{R}^n$ coincide, $W$ is a subspace of $\mathbb{R}^n$.

**Example 3.4** For a nonnegative integer $n$, let $P_n(\mathbb{R})$ denote the set of all real polynomials in $x$ with degree $\leq n$. Then $P_n(\mathbb{R})$ is a subspace of the vector space $C(\mathbb{R})$ of all continuous functions on $\mathbb{R}$.

**Example 3.5** Let $W$ be the set of all $n \times n$ real symmetric matrices. Then $W$ is a subspace of the vector space $M_{n\times n}(\mathbb{R})$ of all $n \times n$ matrices, because the sum of two symmetric matrices is symmetric and a scalar multiplication of a symmetric matrix is also symmetric. Similarly, the set of all $n \times n$ skew-symmetric matrices is also a subspace of $M_{n\times n}(\mathbb{R})$.

*Problem 3.2* Which of the following sets are subspaces of the 3-space $\mathbb{R}^3$? Justify your answer.

(1) $W = \{(x,\ y,\ z) \in \mathbb{R}^3\ :\ xyz = 0\}$,
(2) $W = \{(2t,\ 3t,\ 4t) \in \mathbb{R}^3\ :\ t \in \mathbb{R}\}$,
(3) $W = \{(x,\ y,\ z) \in \mathbb{R}^3\ :\ x^2 + y^2 - z^2 = 0\}$,
(4) $W = \{\mathbf{x} \in \mathbb{R}^3\ :\ \mathbf{x}^T\mathbf{u} = 0 = \mathbf{x}^T\mathbf{v}\}$, where $\mathbf{u}$ and $\mathbf{v}$ are any two fixed nonzero vectors in $\mathbb{R}^3$.

Can you describe all subspaces of the 3-space $\mathbb{R}^3$?

*Problem 3.3* Let $V = C(\mathbb{R})$ be the vector space of all continuous functions on $\mathbb{R}$. Which of the following sets $W$ are subspaces of $V$? Justify your answer.

(1) $W$ is the set of all differentiable functions on $\mathbb{R}$.

(2) $W$ is the set of all bounded continuous functions on $\mathbb{R}$.

(3) $W$ is the set of all continuous nonnegative-valued functions on $\mathbb{R}$, *i.e.*, $f(x) \geq 0$ for any $x \in \mathbb{R}$.

(4) $W$ is the set of all continuous odd functions on $\mathbb{R}$, *i.e.*, $f(-x) = -f(x)$ for any $x \in \mathbb{R}$.

(5) $W$ is the set of all polynomials with integer coefficients.

## 3.2 Bases

Recall that any vector in the 3-space $\mathbb{R}^3$ is of the form $(x_1,\ x_2,\ x_3)$ which can alsobe written as

$$(x_1,\ x_2,\ x_3) = x_1(1,\ 0,\ 0) + x_2(0,\ 1,\ 0) + x_3(0,\ 0,\ 1).$$

That is, any vector in $\mathbb{R}^3$ can be expressed as the sum of scalar multiples of $\mathbf{e}_1 = (1,\ 0,\ 0)$, $\mathbf{e}_2 = (0,\ 1,\ 0)$ and $\mathbf{e}_3 = (0,\ 0,\ 1)$, which are also denoted by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, respectively. The following definition gives a name to such expressions.

**Definition 3.2** Let $V$ be a vector space, and let $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m\}$ be a set of vectors in $V$. Then a vector $\mathbf{y}$ in $V$ of the form

$$\mathbf{y} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_m\mathbf{x}_m,$$

where $a_1,\ \ldots,\ a_m$ are scalars, is called a **linear combination** of the vectors $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m$.

The next theorem shows that the set of all linear combinations of a finite set of vectors in a vector space forms a subspace.

**Theorem 3.4** *Let* $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m$ *be vectors in a vector space* $V$. *Then the set* $W = \{a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_m\mathbf{x}_m\ :\ a_i \in \mathbb{R}\}$ *of all linear combinations of* $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m$ *is a subspace of* $V$ *called the* **subspace of** $V$ **spanned** *by* $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m$.

**Proof:** We want to show that $W$ is closed under addition and scalar multiplication. Let $\mathbf{u}$ and $\mathbf{w}$ be any two vectors in $W$. Then

$$\begin{array}{rcl} \mathbf{u} & = & a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_m\mathbf{x}_m, \\ \mathbf{w} & = & b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + \cdots + b_m\mathbf{x}_m \end{array}$$

for some scalars $a_i$'s and $b_i$'s. Therefore,

$$\mathbf{u}+\mathbf{w} = (a_1+b_1)\mathbf{x}_1 + (a_2+b_2)\mathbf{x}_2 + \cdots + (a_m+b_m)\mathbf{x}_m$$

and, for any scalar $k$,

$$k\mathbf{u} = (ka_1)\mathbf{x}_1 + (ka_2)\mathbf{x}_2 + \cdots + (ka_m)\mathbf{x}_m.$$

Thus, $\mathbf{u}+\mathbf{w}$ and $k\mathbf{u}$ are linear combinations of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ and consequently contained in $W$. Therefore, $W$ is a subspace of $V$. $\square$

Suppose that $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ is any set of $m$ vectors in a vector space $V$. If any vector in $V$ can be written as a linear combination of these vector $\mathbf{x}_i$'s, we say that it is a **spanning set** of $V$.

**Example 3.6** **(1)** For a nonzero vector $\mathbf{v}$ in a vector space $V$, linear combinations of $\mathbf{v}$ are simply scalar multiples of $\mathbf{v}$. Thus the subspace $W$ of $V$ spanned by $\mathbf{v}$ is $W = \{k\mathbf{v} : k \in \mathbb{R}\}$.

**(2)** Consider three vectors $\mathbf{v}_1 = (1,1,1)$, $\mathbf{v}_2 = (1,-1,1)$ and $\mathbf{v}_3 = (1,0,1)$ in $\mathbb{R}^3$. The subspace $W_1$ spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ is written as

$$W_1 = \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = (a_1+a_2, a_1-a_2, a_1+a_2) : a_i \in \mathbb{R}\},$$

and the subspace $W_2$ spanned by $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ is written as

$$W_2 = \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = (a_1+a_2+a_3, a_1-a_2, a_1+a_2+a_3) : a_i \in \mathbb{R}\}.$$

Then $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + 0\mathbf{v}_3$ implies $W_1 \subseteq W_2$. On the other hand, any vector in $W_2$ is of the form $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$. But, since $\mathbf{v}_3 = \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_2)$, this can be rewritten as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. This means that $W_2 \subseteq W_1$, thus $W_1 = W_2$ which is a plane in $\mathbb{R}^3$ containing the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. In general, a subspace in a vector space can have many different spanning sets. $\square$

**Example 3.7** Let

$$\begin{aligned}\mathbf{e}_1 &= (1,\ 0,\ 0,\ \ldots,\ 0),\\ \mathbf{e}_2 &= (0,\ 1,\ 0,\ \ldots,\ 0),\\ &\ \vdots\\ \mathbf{e}_n &= (0,\ 0,\ 0,\ \ldots,\ 1)\end{aligned}$$

be $n$ vectors in the $n$-space $\mathbb{R}^n$ $(n \geq 3)$. Then a linear combination of $\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3$ is of the form

$$a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = (a_1,\ a_2,\ a_3,\ 0,\ \ldots,\ 0).$$

Hence, the set

$$W = \{(a_1,\ a_2,\ a_3,\ 0,\ \ldots,\ 0) \in \mathbb{R}^n\ :\ a_1,\ a_2,\ a_3 \in \mathbb{R}\}$$

is the subspace of the $n$-space $\mathbb{R}^n$ spanned by the vectors $\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3$. Note that the subspace $W$ can be identified with the 3-space $\mathbb{R}^3$ through the identification

$$(a_1,\ a_2,\ a_3,\ 0,\ \ldots,\ 0) \equiv (a_1,\ a_2,\ a_3)$$

with $a_i \in \mathbb{R}$. In general, for $m \leq n$, the $m$-space $\mathbb{R}^m$ can be identified as a subspace of the $n$-space $\mathbb{R}^n$. □

**Example 3.8** Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ be an $m \times n$ matrix. Then the column vectors $\mathbf{a}_i$'s are in $\mathbb{R}^m$, and the matrix product $A\mathbf{x}$ represents the linear combination of the column vector $\mathbf{a}_i$'s whose coefficients are the components of $\mathbf{x} \in \mathbb{R}^n$, *i.e.*, $A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$. Therefore, the set

$$W = \{A\mathbf{x} \in \mathbb{R}^m\ :\ \mathbf{x} \in \mathbb{R}^n\}$$

of all linear combinations of the column vectors of $A$ is a subspace of $\mathbb{R}^m$ called the **column space** of $A$. Therefore, $A\mathbf{x} = \mathbf{b}$ *has a solution* $(x_1,\ \cdots,\ x_n)$ *in* $\mathbb{R}^n$ *if and only if the vector* $\mathbf{b}$ *belongs to the subspace* $W$ *spanned by the column vectors of* $A$. □

*Problem 3.4* Let $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m$ be vectors in a vector space $V$ and let $W$ be the subspace spanned by $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m$. Show that $W$ is the smallest subspace of $V$ containing $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m$, *i.e.*, if $U$ is a subspace of $V$ containing $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m$, then $W \subseteq U$.

*Problem 3.5* Show that the set of all matrices of the form $AB - BA$ does not span the vector space $M_{n\times n}(\mathbb{R})$.

As we saw above, any nonempty subset of a vector space $V$ spans a subspace through the linear combinations of the vectors, and a subspace can have many spanning sets with a different number of vectors. This means that a vector can be written as linear combinations in various ways. If one can find a finite number of vectors in $V$ such that any vector in $V$ can be expressed in a unique way as a linear combination of them, then the study of the vector space $V$ might be easier and the computations of vectors may be simplified. Thus, for a fixed set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ in a vector space $V$, we look at their linear combinations $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_m\mathbf{x}_m$ and see whether any vector in $V$ can be written in this form in exactly one way. This problem can be rephrased as to whether or not a nontrivial linear combination produces the zero vector, while the trivial combination, with all scalars $c_i = 0$, obviously produces the zero vector.

**Definition 3.3** A set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ in a vector space $V$ is said to be **linearly independent** if the vector equation, called the **linear dependence** of $\mathbf{x}_i$'s,

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_m\mathbf{x}_m = \mathbf{0}$$

has only the trivial solution $c_1 = c_2 = \cdots = c_m = 0$. Otherwise, it is said to be **linearly dependent**.

Therefore, a set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ is linearly dependent if and only if there is a linear dependence

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_m\mathbf{x}_m = \mathbf{0}$$

with a nontrivial solution $(c_1, c_2, \ldots, c_m)$. In this case, we may assume that $c_m \neq 0$. Then the equation can be rewritten as

$$\mathbf{x}_m = -\frac{c_1}{c_m}\mathbf{x}_1 - \frac{c_2}{c_m}\mathbf{x}_2 - \cdots - \frac{c_{m-1}}{c_m}\mathbf{x}_{m-1}.$$

That is, *a set of vectors is linearly dependent if and only if at least one of the vectors in the set can be written as a linear combination of the others.*

**Example 3.9** Let $\mathbf{x} = (1,2,3)$ and $\mathbf{y} = (3,2,1)$ be two vectors in the 3-space $\mathbb{R}^3$. Then clearly $\mathbf{y} \neq \lambda\mathbf{x}$ for any $\lambda \in \mathbb{R}$ (or $a\mathbf{x} + b\mathbf{y} = \mathbf{0}$ only when $a = b = 0$). This means that $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent in $\mathbb{R}^3$. If $\mathbf{w} = (3,6,9)$, then $\{\mathbf{x}, \mathbf{w}\}$ is linearly dependent since $\mathbf{w} - 3\mathbf{x} = \mathbf{0}$. In general, if $\mathbf{x}, \mathbf{y}$ are noncollinear vectors in the 3-space $\mathbb{R}^3$, the set of all linear combinations of $\mathbf{x}$ and $\mathbf{y}$ determines a plane $W$ through the origin in $\mathbb{R}^3$, *i.e.*, $W = \{a\mathbf{x} + b\mathbf{y} : a, b \in \mathbb{R}\}$. Let $\mathbf{z}$ be another nonzero vector in the 3-space $\mathbb{R}^3$. If $\mathbf{z} \in W$, then there are some numbers $a$, $b \in \mathbb{R}$, not all of them are zero, such that $\mathbf{z} = a\mathbf{x} + b\mathbf{y}$, that is, the set $\{\mathbf{x},\ \mathbf{y},\ \mathbf{z}\}$ is linearly dependent. If $\mathbf{z} \notin W$, then $a\mathbf{x} + b\mathbf{y} + c\mathbf{z} = \mathbf{0}$ is possible only when $a = b = c = 0$ (prove it). Therefore, the set $\{\mathbf{x},\ \mathbf{y},\ \mathbf{z}\}$ is linearly independent if and only if $\mathbf{z}$ does not lie in $W$. □

By abuse of language, it is sometimes convenient to say that "the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ are linearly independent," although this is really a property of a set.

**Example 3.10** The columns of the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 & 0 \\ 4 & 2 & 6 & 8 \\ 2 & -1 & 1 & 3 \end{bmatrix}$$

are linearly dependent in the 3-space $\mathbb{R}^3$, since the third column is the sum of the first and the second.

As this example shows, the concept of linear dependence can be applied to the row or column vectors of any matrix.

**Example 3.11** Consider an upper triangular matrix

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 4 \end{bmatrix}.$$

The linear dependence of the column vectors of $A$ may be written as

$$c_1 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which, in matrix notation, may be written as a homogeneous system:

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

From the third row, $c_3 = 0$, from the second row $c_2 = 0$, and substitution of them into the first row forces $c_1 = 0$, *i.e.*, the homogeneous system has only the trivial solution, so that the column vectors are linearly independent. □

The following theorem can be proven by the same argument.

**Theorem 3.5** *The nonzero rows of a matrix of a row-echelon form are linearly independent, and so are the columns that contain leading 1's.*

In particular, the rows of any triangular matrix with nonzero diagonals are linearly independent, and so are the columns.

In general, if $V = \mathbb{R}^m$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are $n$ vectors in $\mathbb{R}^m$, then they form an $m \times n$ matrix $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n]$. On the other hand, Example 3.8 shows that the linear dependence $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ of $\mathbf{v}_i$'s is nothing but the homogeneous equation $A\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = (c_1, c_2, \cdots, c_n)$. Thus, *the column vectors $\mathbf{v}_i$'s of $A$ are linearly independent in $\mathbb{R}^m$ if and only if the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution,* and *they are linearly dependent if and only if $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.*

If $U$ is the reduced row-echelon form of $A$, then we know that $A\mathbf{x} = \mathbf{0}$ and $U\mathbf{x} = \mathbf{0}$ have the same set of solutions. Moreover, a homogeneous system $A\mathbf{x} = \mathbf{0}$ with unknowns more than the number of equations always has a nontrivial solution (see the remark on page 11). This proves the following lemma.

**Lemma 3.6** **(1)** *Any set of $n$ vectors in the $m$-space $\mathbb{R}^m$ is linearly dependent if $n > m$.*

**(2)** *If $U$ is the reduced row-echelon form of $A$, then the columns of $U$ are linearly independent if and only if the columns of $A$ are linearly independent.*

**Example 3.12** Consider the vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$ and $\mathbf{e}_3 = (0, 0, 1)$ in the 3-space $\mathbb{R}^3$. The vector equation $c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 = \mathbf{0}$ becomes

$$c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(0, 0, 1) = (0, 0, 0)$$

or, equivalently, $(c_1, c_2, c_3) = (0, 0, 0)$. Thus, $c_1 = c_2 = c_3 = 0$, so the set of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is linearly independent and also spans $\mathbb{R}^3$.

**Example 3.13** In general, it is quite clear that the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ in $\mathbb{R}^n$ are linearly independent. Moreover, they span the $n$-space $\mathbb{R}^n$: In fact, when we write a vector $\mathbf{x} \in \mathbb{R}^n$ as $(x_1, \ldots, x_n)$, it means just the linear combination of the vector $\mathbf{e}_i$'s:

$$\mathbf{x} = (x_1, \ldots, x_n) = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n.$$

However, if any one of the $\mathbf{e}_i$'s is missed, then they cannot span $\mathbb{R}^n$. Thus, this kind of vector plays a special role in the vector space.

**Definition 3.4** Let $V$ be a vector space. A **basis** for $V$ is a set of linearly independent vectors that spans $V$.

For example, as we saw in Example 3.13, the set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ forms a basis, called the **standard basis** for the $n$-space $\mathbb{R}^n$. Of course, there are many other bases for $\mathbb{R}^n$.

**Example 3.14** **(1)** The set of vectors $(1, 1, 0)$, $(0, -1, 1)$, and $(1, 0, 1)$ is not a basis for the 3-space $\mathbb{R}^3$, since this set is linearly dependent (the third is the sum of the first two vectors), and cannot span $\mathbb{R}^3$. (The vector $(1, 0, 0)$ cannot be obtained as a linear combination of them (prove it).) This set does not have enough vectors spanning $\mathbb{R}^3$.

**(2)** The set of vectors $(1, 0, 0)$, $(0, 1, 1)$, $(1, 0, 1)$ and $(0, 1, 0)$ is not a basis either, since they are not linearly independent (the sum of the first two minus the third makes the fourth) even though they span $\mathbb{R}^3$. This set of vectors has some redundant vectors spanning $\mathbb{R}^3$.

**(3)** The set of vectors $(1, 1, 1)$, $(0, 1, 1)$, and $(0, 0, 1)$ is linearly independent and also spans $\mathbb{R}^3$. That is, it is a basis for $\mathbb{R}^3$, different from the standard basis. This set has the proper number of vectors spanning $\mathbb{R}^3$, since the set cannot be reduced to a smaller set nor does it need any additional vector spanning $\mathbb{R}^3$. □

By definition, in order to show that a set of vectors in a vector space is a basis, one needs to show two things: *it is linearly independent, and it spans the whole space.* The following theorem shows that a basis for a vector space represents a coordinate system just like the rectangular coordinate system by the standard basis for $\mathbb{R}^n$.

**Theorem 3.7** *Let $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space $V$. Then each vector $\mathbf{x}$ in $V$ can be uniquely expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, i.e., there are unique scalars $a_i$'s, $i = 1, \ldots, n$, such that*

$$\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n.$$

In this case, the column vector $[a_1\ a_2\ \cdots\ a_n]^T$ is called the **coordinate vector** of $\mathbf{x}$ with respect to the basis $\alpha$, and it is denoted $[\mathbf{x}]_\alpha$.

**Proof:** If $\mathbf{x}$ can be also expressed as $\mathbf{x} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n$, then we have $\mathbf{0} = (a_1 - b_1)\mathbf{v}_1 + (a_2 - b_2)\mathbf{v}_2 + \cdots + (a_n - b_n)\mathbf{v}_n$. By the linear independence of $\mathbf{x}_i$'s, $a_i = b_i$ for all $i = 1, \ldots, n$. □

**Example 3.15** Let $\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbb{R}^3$, and let $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with $\mathbf{v}_1 = (1,1,1) = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, $\mathbf{v}_2 = (0,1,1) = \mathbf{e}_2 + \mathbf{e}_3$, $\mathbf{v}_3 = (0,0,1) = \mathbf{e}_3$. Then

$$[\mathbf{v}_1]_\alpha = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ [\mathbf{v}_2]_\alpha = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \ [\mathbf{v}_3]_\alpha = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

while $[\mathbf{v}_1]_\beta = [1\ 0\ 0]^T$, $[\mathbf{v}_2]_\beta = [0\ 1\ 0]^T$, $[\mathbf{v}_3]_\beta = [0\ 0\ 1]^T$.

*Problem 3.6* Show that the vectors $\mathbf{v}_1 = (1,\ 2,\ 1)$, $\mathbf{v}_2 = (2,\ 9,\ 0)$ and $\mathbf{v}_3 = (3,\ 3,\ 4)$ in the 3-space $\mathbb{R}^3$ form a basis.

*Problem 3.7* Show that the set $\{1,\ x,\ x^2,\ \ldots,\ x^n\}$ is a basis for $P_n(\mathbb{R})$, the vector space of all polynomials of degree $\leq n$ with real coefficients.

*Problem 3.8* In the $n$-space $\mathbb{R}^n$, determine whether or not the set

$$\{\mathbf{e}_1 - \mathbf{e}_2,\ \mathbf{e}_2 - \mathbf{e}_3,\ \ldots,\ \mathbf{e}_{n-1} - \mathbf{e}_n,\ \mathbf{e}_n - \mathbf{e}_1\}$$

is linearly dependent.

*Problem 3.9* Let $\mathbf{x}_k$ denote the vector in $\mathbb{R}^n$ whose first $k-1$ coordinates are zero and whose last $n-k+1$ coordinates are 1. Show that the set $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_n\}$ is a basis for $\mathbb{R}^n$.

## 3.3 Dimensions

We often say that the line $\mathbb{R}^1$ is one-dimensional, the plane $\mathbb{R}^2$ is two-dimensional and the space $\mathbb{R}^3$ is three-dimensional, etc. This is mostly due to the fact that the freedom in choosing coordinates for each element in the space is 1, 2 or 3, respectively. This means that the concept of *dimension* is closely related to the concept of bases. Note that for a vector space in general there is no unique way to choose a basis. However, there is something common to all bases, and this is related to the notion of dimension. We first need the following important lemma from which one can define the dimension of a vector space.

**Lemma 3.8** *Let $V$ be a vector space and let $\alpha = \{\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m\}$ be a set of $m$-vectors in $V$.*

**(1)** *If $\alpha$ spans $V$, then every set of vectors with more than $m$ vectors cannot be linearly independent.*

**(2)** *If $\alpha$ is linearly independent, then any set of vectors with fewer than $m$ vectors cannot span $V$.*

**Proof:** Since (2) follows from (1) directly, we prove only (1). Let $\beta = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ be a set of $n$-vectors in $V$ with $n > m$. We will show that $\beta$ is linearly dependent. Indeed, since each vector $\mathbf{y}_j$ is a linear combination of the vectors in the spanning set $\alpha$, *i.e.*, for $j = 1, \ldots, n$,

$$\mathbf{y}_j = a_{1j}\mathbf{x}_1 + a_{2j}\mathbf{x}_2 + \cdots + a_{mj}\mathbf{x}_m = \sum_{i=1}^{m} a_{ij}\mathbf{x}_i,$$

we have

$$\begin{aligned}
c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n &= c_1(a_{11}\mathbf{x}_1 + a_{21}\mathbf{x}_2 + \cdots + a_{m1}\mathbf{x}_m) \\
&\quad + c_2(a_{12}\mathbf{x}_1 + a_{22}\mathbf{x}_2 + \cdots + a_{m2}\mathbf{x}_m) \\
&\quad\ \vdots \\
&\quad + c_n(a_{1n}\mathbf{x}_1 + a_{2n}\mathbf{x}_2 + \cdots + a_{mn}\mathbf{x}_m) \\
&= (a_{11}c_1 + a_{12}c_2 + \cdots + a_{1n}c_n)\mathbf{x}_1 \\
&\quad + (a_{21}c_1 + a_{22}c_2 + \cdots + a_{2n}c_n)\mathbf{x}_2 \\
&\quad\ \vdots \\
&\quad + (a_{m1}c_1 + a_{m2}c_2 + \cdots + a_{mn}c_n)\mathbf{x}_m.
\end{aligned}$$

Thus, $\beta$ is linearly dependent if and only if the system of linear equations

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n = \mathbf{0}$$

has a nontrivial solution $(c_1, c_2, \ldots, c_n) \neq (0, 0, \cdots, 0)$. This is true if all the coefficients of $\mathbf{x}_i$'s are zero but not all of $c_i$'s are zero. This means that the homogeneous system of linear equations in $c_i$'s,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

has a nontrivial solution. This is guaranteed by Lemma 3.6, since $A$ is an $m \times n$ matrix with $m < n$. $\square$

It is clear by Lemma 3.8 that if a set $\alpha = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ of $n$ vectors is a basis for a vector space $V$, then no other set $\beta = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_r\}$ of $r$ vectors can be a basis for $V$ if $r \neq n$. This means that all bases for a vector space $V$ have the same number of vectors, even if there are many different bases for a vector space. Therefore, we obtain the following important result:

**Theorem 3.9** *If a basis for a vector space $V$ consists of $n$ vectors, then so does every other basis.*

**Definition 3.5** The **dimension** of a vector space $V$ is the number, say $n$, of vectors in a basis for $V$, denoted by $\dim V = n$. When $V$ has a basis of a finite number of vectors, $V$ is said to be **finite dimensional**.

**Example 3.16** The following can be easily verified:

(1) If $V$ has only the zero vector: $V = \{\mathbf{0}\}$, then $\dim V = 0$.

(2) If $V = \mathbb{R}^n$, then $\dim \mathbb{R}^n = n$, since $V$ has the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$.

(3) If $V = P_n(\mathbb{R})$ of all polynomials of degree less than or equal to $n$, then $\dim P_n(\mathbb{R}) = n + 1$ since $\{1, x, x^2, \ldots, x^n\}$ is a basis for $V$.

(4) If $V = M_{m\times n}(\mathbb{R})$ of all $m \times n$ matrices, then $\dim M_{m\times n}(\mathbb{R}) = mn$ since $\{E_{ij} : i = 1, \ldots, m,\ j = 1, \ldots, n\}$ is a basis for $V$, where $E_{ij}$ is the $m \times n$ matrix whose $(i, j)$-th entry is 1 and all others are zero. □

If $V = C(\mathbb{R})$ of all real-valued continuous functions defined on the real line, then one can show that $V$ is not finite dimensional. A vector space $V$ is **infinite dimensional** if it is not finite dimensional. In this book, we are concerned only with finite dimensional vector spaces unless otherwise stated.

**Theorem 3.10** *Let $V$ be a finite dimensional vector space.*

(1) *Any linearly independent set in $V$ can be extended to a basis by adding more vectors if necessary.*

(2) *Any set of vectors that spans $V$ can be reduced to a basis by discarding vectors if necessary.*

**Proof:** We prove **(1)** only and leave **(2)** as an exercise. Let $\alpha = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ be a linearly independent set in $V$. If $\alpha$ spans $V$, then $\alpha$ is a basis. If $\alpha$ does not span $V$, then there exists a vector, say $\mathbf{x}_{k+1}$, in $V$ that is not contained in the subspace spanned by the vectors in $\alpha$. Now $\{\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1}\}$ is linearly independent (check why). If $\{\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1}\}$ spans $V$, then this is a basis for $V$. If it does not span $V$, then the same procedure can be repeated, yielding a linearly independent set that spans $V$, *i.e.*, a basis for $V$. This procedure must stop in a finite step because of Lemma 3.8 for a finite dimensional vector space $V$. □

Theorem 3.10 shows that a basis for a vector space $V$ is a set of vectors in $V$ which is *maximally* independent and *minimally* spanning in the above sense. In particular, if $W$ is a subspace of $V$, then any basis for $W$ is linearly independent also in $V$, and can be extended to a basis for $V$. Thus $\dim W \le \dim V$.

**Corollary 3.11** *Let $V$ be a vector space of dimension $n$. Then*
**(1)** *any set of $n$ vectors that spans $V$ is a basis for $V$, and*
**(2)** *any set of $n$ linearly independent vectors is a basis for $V$.*

**Proof:** Again we prove **(1)** only. If a spanning set of $n$ vectors were not linearly independent, then the set would be reduced to a basis that has a smaller number of vectors than $n$ vectors. □

Corollary 3.11 means that if it is known that $\dim V = n$ and if a set of $n$ vectors either is linearly independent or spans $V$, then it is already a basis for the space $V$.

**Example 3.17** Let $W$ be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{x}_1 = (1,\ -2,\ 5,\ -3),\ \ \mathbf{x}_2 = (0,\ 1,\ 1,\ 4),\ \ \mathbf{x}_3 = (1,\ 0,\ 1,\ 0).$$

Find a basis for $W$ and extend it to a basis for $\mathbb{R}^4$.

Solution: Note that $\dim W \le 3$ since $W$ is spanned by three vectors $\mathbf{x}_i$'s. Let $A$ be the $3 \times 4$ matrix whose rows are $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$:

$$A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Reduce $A$ to a row-echelon form:

$$U = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & \frac{5}{6} \end{bmatrix}.$$

The three nonzero row vectors of $U$ are clearly linearly independent, and they also span $W$ because the vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ can be expressed as a linear combination of these three nonzero row vectors of $U$. Hence, $U$ provides a basis for $W$. (Note that this implies $\dim W = 3$ and hence $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ is also a basis for $W$ by Corollary 3.11. The linear independence of $\mathbf{x}_i$'s is a by-product of this fact).

To extend this basis, just add any nonzero vector of the form $\mathbf{x}_4 = (0,\ 0,\ 0,\ t)$ to the rows of $U$ to get a basis for the space $\mathbb{R}^4$. □

*Problem 3.10* Let $W$ be a subspace of a vector space $V$. Show that if $\dim W = \dim V$, then $W = V$.

*Problem 3.11* Find a basis and the dimension of each of the following subspaces of $M_{n\times n}(\mathbb{R})$ of all $n \times n$ matrices:

(1) the space of all $n \times n$ diagonal matrices whose traces are zero;
(2) the space of all $n \times n$ symmetric matrices;
(3) the space of all $n \times n$ skew-symmetric matrices.

Now consider two subspaces $U$ and $W$ of a vector space $V$. The **sum** of these subspaces $U$ and $W$ is defined by

$$U + W = \{\mathbf{u} + \mathbf{w} \ : \ \mathbf{u} \in U,\ \mathbf{w} \in W\}.$$

It is not hard to see that this is a subspace of $V$.

*Problem 3.12* Let $U$ and $W$ be subspaces of a vector space $V$.

(1) Show that $U + W$ is the smallest subspace of $V$ containing $U$ and $W$.
(2) Prove that $U \cap W$ is also a subspace of $V$. Is $U \cup W$ a subspace of $V$? Justify your answer.

**Definition 3.6** A vector space $V$ is called the **direct sum** of two subspaces $U$ and $W$, written $V = U \oplus W$, if $V = U + W$ and $U \cap W = \{\mathbf{0}\}$.

For example, one can easily show that $\mathbb{R}^3 = \mathbb{R}^1 \oplus \mathbb{R}^2 = \mathbb{R}^1 \oplus \mathbb{R}^1 \oplus \mathbb{R}^1$.

**Theorem 3.12** *A vector space $V$ is the direct sum of subspaces $U$ and $W$, i.e., $V = U \oplus W$, if and only if for any $\mathbf{v} \in V$ there exist unique $\mathbf{u} \in U$ and $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$.*

**Proof:** Suppose that $V = U \oplus W$. Then, for any $\mathbf{v} \in V$, there exist vectors $\mathbf{u} \in U$ and $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$, since $V = U + W$. To show the uniqueness, suppose that $\mathbf{v}$ is also expressed as a sum $\mathbf{u}' + \mathbf{w}'$ for $\mathbf{u}' \in U$ and $\mathbf{w}' \in W$. Then $\mathbf{u} + \mathbf{w} = \mathbf{u}' + \mathbf{w}'$ implies

$$\mathbf{u} - \mathbf{u}' = \mathbf{w}' - \mathbf{w} \in U \cap W = \{\mathbf{0}\}.$$

Hence, $\mathbf{u} = \mathbf{u}'$ and $\mathbf{w} = \mathbf{w}'$.

Conversely, if there exists a nonzero vector $\mathbf{v}$ in $U \cap W$, then $\mathbf{v}$ can be written as sum of vectors in $U$ and $W$ in many different ways:

$$\mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{v} = \frac{1}{3}\mathbf{v} + \frac{2}{3}\mathbf{v} \in U + W. \qquad \square$$

**Example 3.18** Consider the three vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ in $\mathbb{R}^3$. Let $U = \{a_1\mathbf{e}_1 + b_3\mathbf{e}_3 \,:\, a_1, b_3 \in \mathbb{R}\}$ be the subspace spanned by $\mathbf{e}_1$ and $\mathbf{e}_3$ ($xz$-plane), and let $W = \{a_2\mathbf{e}_2 + c_2\mathbf{e}_3 \,:\, a_2, c_3 \in \mathbb{R}\}$ be the subspace of $\mathbb{R}^3$ spanned by $\mathbf{e}_2$ and $\mathbf{e}_3$ ($yz$-plane). Then a vector in $U + W$ is of the form

$$(a_1\mathbf{e}_1 + b_3\mathbf{e}_3) + (a_2\mathbf{e}_2 + c_2\mathbf{e}_3) = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + (b_3 + c_2)\mathbf{e}_3 = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$$

where $a_3 = b_3 + c_3$ and $a_1$, $a_2$, $a_3$ are arbitrary numbers. Thus $U + W = \mathbb{R}^3$. However, $\mathbb{R}^3 \neq U \oplus W$ since clearly $\mathbf{e}_3 \in U \cap W \neq \{\mathbf{0}\}$. In fact, the vector $\mathbf{e}_3 \in \mathbb{R}^3$ can be written as many linear combinations of vectors in $U$ and $W$:

$$\mathbf{e}_3 = \frac{1}{2}\mathbf{e}_3 + \frac{1}{2}\mathbf{e}_3 = \frac{1}{3}\mathbf{e}_3 + \frac{2}{3}\mathbf{e}_3 \in U + W.$$

Note that if we had taken $W$ to be the subspace spanned by $\mathbf{e}_2$ alone, then it would be easy to see that $\mathbb{R}^3 = U \oplus W$. Note also that there are many choices for $W$. $\square$

As a direct consequence of Theorem 3.10 and the definition of the direct sum, one can show the following.

**Corollary 3.13** *If $U$ is a subspace of $V$, then there is a subspace $W$ in $V$ such that $V = U \oplus W$.*

**Proof:** Choose a basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ for $U$, and extend it to a basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ for $V$. Then the subspace $W$ spanned by $\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ satisfies the requirement. $\square$

*Problem 3.13* Let $U$ and $W$ be the subspaces of the vector space $M_{n\times n}(\mathbb{R})$ consisting of all symmetric matrices and all skew-symmetric matrices, respectively. Show that $M_{n\times n}(\mathbb{R}) = U \oplus W$. Therefore, the decomposition of a square matrix $A$ given in (3) of Problem 1.10 is unique.

*Problem 3.14* Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space $V$ and let $W_i = \{r\mathbf{v}_i : r \in \mathbb{R}\}$ be the subspace of $V$ spanned by $\mathbf{v}_i$. Show that $V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$.

## 3.4 Row and column spaces

In this section, we go back to systems of linear equations and study them in terms of the concepts introduced in the previous sections. Note that an $m \times n$ matrix $A$ can be abbreviated by the row vectors or column vectors as follows:

$$
\begin{aligned}
A &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_m \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix},
\end{aligned}
$$

where the $\mathbf{r}_i$'s are the row vectors of $A$ that are in $\mathbb{R}^n$, and the $\mathbf{c}_j$'s are the column vectors of $A$ that are in $\mathbb{R}^m$.

**Definition 3.7** Let $A$ be an $m \times n$ matrix with row vectors $\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ and column vectors $\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$.

**(1)** The **row space** of $A$ is the subspace in $\mathbb{R}^n$ spanned by the row vectors $\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$, denoted by $\mathcal{R}(A)$.

**(2)** The **column space** of $A$ is the subspace in $\mathbb{R}^m$ spanned by the column vectors $\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$, denoted by $\mathcal{C}(A)$.

**(3)** The solution set of the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is called the **null space** of $A$, denoted by $\mathcal{N}(A)$.

Note that the null space $\mathcal{N}(A)$ is a subspace of the $n$-space $\mathbb{R}^n$, whose dimension is called the **nullity** of $A$. Since the row vectors of $A$ are just the column vectors of its transpose $A^T$, and the column vectors of $A$ are the row vectors of $A^T$, the row space of $A$ is the column space of $A^T$; that is,

$$\mathcal{R}(A) = \mathcal{C}(A^T) \quad \text{and} \quad \mathcal{C}(A) = \mathcal{R}(A^T).$$

Since $A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \dots x_n\mathbf{c}_n$ for any vector $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, we get

$$\mathcal{C}(A) = \{A\mathbf{x} \ : \ \mathbf{x} \in \mathbb{R}^n\}.$$

Thus, for a vector $\mathbf{b} \in \mathbb{R}^m$, *the system $A\mathbf{x} = \mathbf{b}$ has a solution if and only if* $\mathbf{b} \in \mathcal{C}(A) \subseteq \mathbb{R}^m$. Thus, the column space $\mathcal{C}(A)$ is the set of vectors $\mathbf{b} \in \mathbb{R}^m$ for which $A\mathbf{x} = \mathbf{b}$ has a solution.

It is quite natural to ask what the dimensions of those subspaces are, and how one can find bases for them. This will help us to understand the structure of all the solutions of the equation $A\mathbf{x} = \mathbf{b}$. Since the set of the row vectors and the set of the column vectors of $A$ are spanning sets for the row space and the column space, respectively, a minimally spanning subset of each of them will be a basis for each of them.

This is not difficult for a matrix of a (reduced) row-echelon form.

**Example 3.19** Let $U$ be in a reduced row-echelon form given as

$$U = \begin{bmatrix} 1 & 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly, the first three nonzero row vectors containing leading 1's are linearly independent and they form a basis for the row space $\mathcal{R}(U)$, so that $\dim \mathcal{R}(U) = 3$. On the other hand, note that the first three columns containing leading 1's are linearly independent (see Theorem 3.5), and that the last two column vectors can be expressed as linear combinations of them. Hence, they form a basis for $\mathcal{C}(U)$, and $\dim \mathcal{C}(U) = 3$. To find a basis for the null space $\mathcal{N}(U)$, we first solve the system $U\mathbf{x} = \mathbf{0}$ with arbitrary values $s$ and $t$ for the free variables $x_4$ and $x_5$, and get the solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s - 2t \\ s - 3t \\ -4s + t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ -4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} = s\mathbf{n}_s + t\mathbf{n}_t,$$

where $\mathbf{n}_s = (-2,\ 1,\ -4,\ 1,\ 0)$, $\mathbf{n}_t = (-2,\ -3,\ 1,\ 0,\ 1)$. It shows that these two vectors $\mathbf{n}_s$ and $\mathbf{n}_t$ span the null space $\mathcal{N}(U)$, and they are clearly linearly independent. Hence, the set $\{\mathbf{n}_s,\ \mathbf{n}_t\}$ is a basis for the null space $\mathcal{N}(U)$. □

In the following, the row, the column or the null space of a matrix $A$ will be discussed in relation to the corresponding space of its (reduced) row-echelon form. We first investigate the row space $\mathcal{R}(A)$ and the null space $\mathcal{N}(A)$ of $A$ by comparing them with those of the reduced row-echelon form $U$ of $A$. Since $A\mathbf{x} = \mathbf{0}$ and $U\mathbf{x} = \mathbf{0}$ have the same solution set by Theorem 1.1, we have $\mathcal{N}(A) = \mathcal{N}(U)$.

Let $A = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_m \end{bmatrix}$ be an $m \times n$ matrix, where $\mathbf{r}_i$'s are the row vectors of $A$. The three elementary row operations change $A$ into the following three types:

$$A_1 = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ k\mathbf{r}_i \\ \vdots \\ \mathbf{r}_m \end{bmatrix} \text{ for } k \neq 0, \; A_2 = \begin{bmatrix} \vdots \\ \mathbf{r}_j \\ \vdots \\ \mathbf{r}_i \\ \vdots \end{bmatrix} \text{ for } i < j, \; A_3 = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_i + k\mathbf{r}_j \\ \vdots \\ \mathbf{r}_m \end{bmatrix}.$$

It is clear that the row vectors of the three matrices $A_1$, $A_2$ and $A_3$ are linear combinations of the row vectors of $A$. On the other hand, by the inverse elementary row operations, these matrices can be changed into $A$. Thus, the row vectors of $A$ can also be written as linear combinations of those of $A_i$'s. This means that if matrices $A$ and $B$ are row equivalent, then their row spaces must be equal, *i.e.*, $\mathcal{R}(A) = \mathcal{R}(B)$.

Now the nonzero row vectors in the reduced row-echelon form $U$ are always linearly independent and span the row space of $U$ (see Theorem 3.5). Thus they form a basis for the row space $\mathcal{R}(A)$ of $A$. We have the following theorem.

**Theorem 3.14** *Let $U$ be a (reduced) row-echelon form of a matrix $A$. Then*

$$\mathcal{R}(A) = \mathcal{R}(U) \quad \textit{and} \quad \mathcal{N}(A) = \mathcal{N}(U).$$

*Moreover, if $U$ has $r$ nonzero row vectors containing leading 1's, then they form a basis for the row space $\mathcal{R}(A)$, so that the dimension of $\mathcal{R}(A)$ is $r$.*

The following example shows how to find bases for the row and the null spaces, and at the same time how to find a basis for the column space $\mathcal{C}(A)$.

**Example 3.20** Let $A$ be a matrix given as

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \mathbf{r}_4 \end{bmatrix}.$$

Find bases for the row space $\mathcal{R}(A)$, the null space $\mathcal{N}(A)$, and the column space $\mathcal{C}(A)$ of $A$.

Solution: (1) Find a basis for $\mathcal{R}(A)$: By Gauss-Jordan elimination on $A$, we get the reduced row-echelon form $U$:

$$U = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the three nonzero row vectors

$$\begin{aligned} \mathbf{v}_1 &= (1, \ 0, \ 2, \ 0, \ 1), \\ \mathbf{v}_2 &= (0, \ 1, \ -1, \ 0, \ 1), \\ \mathbf{v}_3 &= (0, \ 0, \ 0, \ 1, \ 1) \end{aligned}$$

of $U$ are linearly independent, they form a basis for the row space $\mathcal{R}(U) = \mathcal{R}(A)$, so $\dim \mathcal{R}(A) = 3$. (Note that in the process of Gaussian elimination, we did not use a permutation matrix. This means that the three nonzero rows of $U$ were obtained from the first three row vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ of $A$ and the fourth row $\mathbf{r}_4$ of $A$ turned out to be a linear combination of them. Thus the first three row vectors of $A$ also form a basis for the row space.)

(2) Find a basis for $\mathcal{N}(A)$. It is enough to solve the homogeneous system $U\mathbf{x} = \mathbf{0}$, since $\mathcal{N}(A) = \mathcal{N}(U)$. That is, neglecting the fourth zero equation, the equation $U\mathbf{x} = \mathbf{0}$ takes the following system of equations:

$$\begin{cases} x_1 \phantom{{}+x_2} + 2x_3 \phantom{{}+x_4} + x_5 = 0 \\ \phantom{x_1 +{}} x_2 - x_3 \phantom{{}+x_4} + x_5 = 0 \\ \phantom{x_1 + x_2 - x_3 +{}} x_4 + x_5 = 0. \end{cases}$$

Since the first, the second and the fourth columns of $U$ contain the leading 1's, we see that the basic variables are $x_1$, $x_2$, $x_4$, and the free variables are

$x_3$, $x_5$. By assigning arbitrary values $s$ and $t$ to the free variables $x_3$ and $x_5$, we find the solution $\mathbf{x}$ of $U\mathbf{x} = \mathbf{0}$ as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s - t \\ s - t \\ s \\ -t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = s\mathbf{n}_s + t\mathbf{n}_t,$$

where $\mathbf{n}_s = (-2,\ 1,\ 1,\ 0,\ 0)$ and $\mathbf{n}_t = (-1,\ -1,\ 0,\ -1,\ 1)$. In fact, the two vectors $\mathbf{n}_s$ and $\mathbf{n}_t$ are the solutions when the values of $(x_3, x_5) = (s, t)$ are $(1, 0)$ and those of $(x_3, x_5) = (s, t)$ are $(0, 1)$, respectively. They must be linearly independent, since $(1,\ 0)$ and $(0,\ 1)$, as the $(x_3, x_5)$-coordinates of $\mathbf{n}_s$ and $\mathbf{n}_t$ respectively, are clearly linearly independent. Since any solution of $U\mathbf{x} = \mathbf{0}$ is a linear combination of them, the set $\{\mathbf{n}_s,\ \mathbf{n}_t\}$ is a basis for the null space $\mathcal{N}(U) = \mathcal{N}(A)$. Thus $\dim \mathcal{N}(A) = 2 =$ *the number of free variables in* $U\mathbf{x} = \mathbf{0}$.

(3) Find a basis for $\mathcal{C}(A)$. Let $\mathbf{c}_1,\ \mathbf{c}_2,\ \mathbf{c}_3,\ \mathbf{c}_4,\ \mathbf{c}_5$ denote the column vectors of $A$ in the given order. Since these column vectors of $A$ span $\mathcal{C}(A)$, we only need to discard some of the columns that can be expressed as linear combinations of other column vectors. But, the linear dependence

$$x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + x_3\mathbf{c}_3 + x_4\mathbf{c}_4 + x_5\mathbf{c}_5 = \mathbf{0}, \quad \textit{i.e.,}\ A\mathbf{x} = \mathbf{0},$$

holds if and only if $\mathbf{x} = (x_1,\ \cdots,\ x_5) \in \mathcal{N}(A)$. By taking $\mathbf{x} = \mathbf{n}_s = (-2,\ 1,\ 1,\ 0,\ 0)$ or $\mathbf{x} = \mathbf{n}_t = (-1,\ -1,\ 0,\ -1,\ 1)$, the basis vectors of $\mathcal{N}(A)$ given in (2), we obtain two nontrivial linear dependencies of $\mathbf{c}_i$'s:

$$\begin{aligned} -2\mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3 &= \mathbf{0}, \\ -\mathbf{c}_1 - \mathbf{c}_2 - \mathbf{c}_4 + \mathbf{c}_5 &= \mathbf{0}, \end{aligned}$$

respectively. Hence, the column vectors $\mathbf{c}_3$ and $\mathbf{c}_5$ corresponding to the free variables in $A\mathbf{x} = \mathbf{0}$ can be written as

$$\begin{aligned} \mathbf{c}_3 &= 2\mathbf{c}_1 - \mathbf{c}_2, \\ \mathbf{c}_5 &= \mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_4. \end{aligned}$$

That is, the column vectors $\mathbf{c}_3, \mathbf{c}_5$ of $A$ are linear combinations of the column vectors $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_4$, which correspond to the basic variables in $A\mathbf{x} = \mathbf{0}$. Hence, $\{\mathbf{c}_1,\ \mathbf{c}_2,\ \mathbf{c}_4\}$ spans the column space $\mathcal{C}(A)$.

We claim that $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_4\}$ is linearly independent. Let $\tilde{A} = [\mathbf{c}_1\ \mathbf{c}_2\ \mathbf{c}_4]$ and $\tilde{U} = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_4]$ be submatrices of $A$ and $U$, respectively, where $\mathbf{u}_j$ is the $j$-th column vector of the reduced row-echelon form $U$ of $A$ obtained in (1). Then clearly $\tilde{U}$ is the reduced row-echelon form of $\tilde{A}$ so that $\mathcal{N}(\tilde{A}) = \mathcal{N}(\tilde{U})$. Since the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4$ are just the columns of $U$ containing leading 1's, they are linearly independent, by Theorem 3.5, and $\tilde{U}\mathbf{x} = \mathbf{0}$ has only a trivial solution. This means that $\tilde{A}\mathbf{x} = \mathbf{0}$ has also only a trivial solution, so $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_4\}$ is linearly independent. Therefore, it is a basis for the column space $\mathcal{C}(A)$ and $\dim \mathcal{C}(A) = 3 =$ *the number of basic variables.* That is, *the column vectors of $A$ corresponding to the basic variables in $U\mathbf{x} = \mathbf{0}$ form a basis for the column space $\mathcal{C}(A)$.* □

In summary, given a matrix $A$, we first find the (reduced) row-echelon form $U$ of $A$ by Gauss-Jordan elimination. Then a basis for $\mathcal{R}(A) = \mathcal{R}(U)$ is the set of nonzero rows vectors of $U$, and a basis for $\mathcal{N}(A) = \mathcal{N}(U)$ can be found by solving $U\mathbf{x} = \mathbf{0}$, which is easy. On the other hand, one has to be careful for $\mathcal{C}(U) \neq \mathcal{C}(A)$ in general, since the column space of $A$ is not preserved by Gauss-Jordan elimination. However, we have $\dim \mathcal{C}(A) = \dim \mathcal{C}(U)$, and a basis for $\mathcal{C}(A)$ can be selected from the column vectors in $A$, not in $U$, as those corresponding to the basic variables (or the leading 1's in $U$). To show that those column vectors indeed form a basis for $\mathcal{C}(A)$, we used a basis for the null space $\mathcal{N}(A)$ to eliminate the redundant columns.

Note that a basis for the column space $\mathcal{C}(A)$ can be also found with the elementary column operations, which is the same as finding a basis for the row space $\mathcal{R}(A^T)$ of $A^T$.

*Problem 3.15* Let $A$ be the matrix given in Example 3.20. Find a relation of $a, b, c, d$ so that the vector $\mathbf{x} = (a, b, c, d)$ belongs to $\mathcal{C}(A)$.

*Problem 3.16* Find bases for $\mathcal{R}(A)$ and $\mathcal{N}(A)$ of the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}.$$

Also find a basis for $\mathcal{C}(A)$ by finding a basis for $\mathcal{R}(A^T)$.

*Problem 3.17* Let $A$ and $B$ be two $n \times n$ matrices. Show that $AB = \mathbf{0}$ if and only if the column space of $B$ is a subspace of the nullspace of $A$.

*Problem 3.18* Find an example of a matrix $A$ and its row-echelon form $U$ such that $\mathcal{C}(A) \neq \mathcal{C}(U)$.

## 3.5 Rank and nullity

The argument in Example 3.20 is so general that it can be used to prove the following theorem, which is one of the most fundamental results in linear algebra. The proof given here is just a repetition of the argument in Example 3.20 in a general form, and so may be skipped at the reader's discretion.

**Theorem 3.15 (The first fundamental theorem)** *For any $m \times n$ matrix $A$, the row space and the column space of $A$ have the same dimension; that is,* $\dim \mathcal{R}(A) = \dim \mathcal{C}(A)$.

**Proof:** Let $\dim \mathcal{R}(A) = r$ and let $U$ be the reduced row-echelon form of $A$. Then $r$ is the number of the nonzero row (or column) vectors of $U$ containing leading 1's, which is equal to the number of basic variables in $U\mathbf{x} = \mathbf{0}$ or $A\mathbf{x} = \mathbf{0}$. We shall prove that the $r$ columns of $A$ corresponding to the $r$ leading 1's (or basic variables) form a basis for $\mathcal{C}(A)$, so that $\dim \mathcal{C}(A) = r = \dim \mathcal{R}(A)$.

(1) They are linearly independent: Let $\tilde{A}$ denote the submatrix of $A$ whose columns are those of $A$ corresponding to the $r$ basic variables (or leading 1's) in $U$, and let $\tilde{U}$ denote the submatrix of $U$ containing $r$ leading 1's. Then, it is quite clear that $\tilde{U}$ is the reduced row-echelon form of $\tilde{A}$, so that $\tilde{A}\mathbf{x} = \mathbf{0}$ if and only if $\tilde{U}\mathbf{x} = \mathbf{0}$. However, $\tilde{U}\mathbf{x} = \mathbf{0}$ has only a trivial solution since the columns of $U$ containing the leading 1's are linearly independent by Theorem 3.5. Therefore, $\tilde{A}\mathbf{x} = \mathbf{0}$ also has only the trivial solution, so the columns of $\tilde{A}$ are linearly independent.

(2) They span $\mathcal{C}(A)$: Note that the columns $A$ corresponding to the free variables are not contained in $\tilde{A}$, and each of these column vector of $A$ can be written as a linear combination of the column vectors of $\tilde{A}$ (see Example 3.20). In fact, if $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ is the set of free variables whose corresponding columns are not in $\tilde{A}$, then, for an assignment of value 1 to $x_{i_j}$ and 0 to all the other free variables, one can always find a nontrivial solution of

$$A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \ldots + x_n\mathbf{c}_n = \mathbf{0}.$$

When the solution is substituted into this equation, one can see that the column $\mathbf{c}_{i_j}$ of $A$ corresponding to $x_{i_j} = 1$ is written as a linear combination of the columns of $\tilde{A}$. This can be done for each $j = 1, \ldots, k$, so the columns of $A$ corresponding to those free variables are redundant in the spanning set of $\mathcal{C}(A)$. □

**Remark:** In the proof of Theorem 3.15, once we have shown that the columns in $\tilde{A}$ are linearly independent as in (1), we may replace step (2) by the following argument: One can easily see that $\dim \mathcal{C}(A) \geq \dim \mathcal{R}(A)$ by Theorem 3.10. On the other hand, since this inequality holds for arbitrary matrices, in particular for $A^T$, we get $\dim \mathcal{C}(A^T) \geq \dim \mathcal{R}(A^T)$. Moreover, $\mathcal{C}(A^T) = \mathcal{R}(A)$ and $\mathcal{R}(A^T) = \mathcal{C}(A)$ implies $\dim \mathcal{C}(A) \leq \dim \mathcal{R}(A)$, which means $\dim \mathcal{C}(A) = \dim \mathcal{R}(A)$. This also means that the column vectors of $\tilde{A}$ span $\mathcal{C}(A)$, and so form a basis.

In summary, the following equalities are now clear from Theorem 3.14 and 3.15:

$$\begin{aligned}
\dim \mathcal{R}(A) &= \dim \mathcal{R}(U) \\
&= \text{the number of nonzero row vectors of } U \\
&= \text{the maximal number of linearly independent} \\
&\quad\ \text{row vectors of } A \\
&= \text{the number of basic variables in } U\mathbf{x} = \mathbf{0}. \\
&= \text{the maximal number of linearly independent} \\
&\quad\ \text{column vectors of } A \\
&= \dim \mathcal{C}(A).
\end{aligned}$$

$$\begin{aligned}
\dim \mathcal{N}(A) &= \dim \mathcal{N}(U) \\
&= \text{the number of free variables in } U\mathbf{x} = \mathbf{0}.
\end{aligned}$$

**Definition 3.8** For an $m \times n$ matrix $A$, the **rank** of $A$ is defined to be the dimension of the row space (or the column space), denoted by rank $A$.

Clearly, rank $I_n = n$ and rank $A =$ rank $A^T$. And for an $m \times n$ matrix $A$, rank $A = \dim \mathcal{R}(A) = \dim \mathcal{C}(A)$. Since $\dim \mathcal{R}(A) \leq m$ and $\dim \mathcal{C}(A) \leq n$, we have the following corollary:

**Corollary 3.16** *If $A$ is an $m \times n$ matrix, then* rank $A \leq \min\{m,\ n\}$.

Since $\dim \mathcal{R}(A) = \dim \mathcal{C}(A) =$ rank $A$ is the number of basic variables in $A\mathbf{x} = \mathbf{0}$, and $\dim \mathcal{N}(A) =$ nullity of $A$ is the number of free variables $A\mathbf{x} = \mathbf{0}$, we have the following corollary.

**Corollary 3.17** *For any $m \times n$ matrix $A$,*

$$\begin{aligned}
\dim \mathcal{R}(A) + \dim \mathcal{N}(A) &= \text{rank } A + \text{nullity of } A &= n, \\
\dim \mathcal{C}(A) + \dim \mathcal{N}(A^T) &= \text{rank } A + \text{nullity of } A^T &= m.
\end{aligned}$$

If $\dim \mathcal{N}(A) = 0$ (or $\mathcal{N}(A) = \{\mathbf{0}\}$), then $\dim \mathcal{R}(A) = n$ (or $\mathcal{R}(A) = \mathbb{R}^n$), which means that $A$ has exactly $n$ linearly independent rows and $n$ linearly independent columns. In particular, if $A$ is a square matrix of order $n$, then the row vectors are linearly independent if and only if the column vectors are linearly independent. Therefore, by Theorem 1.8, we get the following corollary.

**Corollary 3.18** *Let $A$ be an $n \times n$ square matrix. Then $A$ is invertible if and only if* $\operatorname{rank} A = n$.

**Example 3.21** For a $4 \times 5$ matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ -1 & -2 & 1 & 1 & 0 \\ 1 & 2 & -3 & -7 & 2 \\ 1 & 2 & -2 & -4 & 3 \end{bmatrix},$$

by Gaussian elimination, we get

$$U = \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The first three nonzero rows containing leading 1's in $U$ form a basis for $\mathcal{R}(U) = \mathcal{R}(A)$. Note that $x_1$, $x_3$ and $x_5$ are the basic variables in $U\mathbf{x} = \mathbf{0}$, since the first, third and fifth columns of $U$ contain leading 1's. Thus the three columns $\mathbf{c}_1 = (1, -1, 1, 1)$, $\mathbf{c}_3 = (0, 1, -3, -2)$ and $\mathbf{c}_5 = (1, 0, 2, 3)$ of $A$, not the three columns in $U$, corresponding to those basic variables $x_1$, $x_3$ and $x_5$ form a basis for $\mathcal{C}(A)$. Therefore, $\operatorname{rank} A = \dim \mathcal{R}(A) = \dim \mathcal{C}(A) = 3$, the nullity of $A = \dim \mathcal{N}(A) = 2$, and $\dim \mathcal{N}(A^T) = 1$. □

*Problem 3.19* Find the nullity and the rank of each of the following matrices:

$$(1)\, A = \begin{bmatrix} 1 & 3 & 1 & 7 \\ 2 & 3 & -1 & 9 \\ -1 & -2 & 0 & -5 \end{bmatrix}, \quad (2)\, A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 5 & -2 \end{bmatrix}.$$

For each of the matrices, show that $\dim \mathcal{R}(A) = \dim \mathcal{C}(A)$ directly by finding their bases.

*Problem 3.20* Show that a system of linear equations $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\operatorname{rank} A = \operatorname{rank}\, [A\ \mathbf{b}]$, where $[A\ \mathbf{b}]$ denotes the augmented matrix of $A\mathbf{x} = \mathbf{b}$.

**Theorem 3.19** *For any two matrices $A$ and $B$ for which $AB$ can be defined,*

(1) $\mathcal{N}(AB) \supseteq \mathcal{N}(B)$,

(2) $\mathcal{N}((AB)^T) \supseteq \mathcal{N}(A^T)$,

(3) $\mathcal{C}(AB) \subseteq \mathcal{C}(A)$,

(4) $\mathcal{R}(AB) \subseteq \mathcal{R}(B)$.

**Proof:** (1) and (2) are clear, since $B\mathbf{x} = \mathbf{0}$ implies $(AB)\mathbf{x} = A(B\mathbf{x}) = \mathbf{0}$.

(3) For an $m \times n$ matrix $A$ and an $n \times p$ matrix $B$,

$$\begin{aligned} \mathcal{C}(AB) &= \{AB\mathbf{x} \ : \ \mathbf{x} \in \mathbb{R}^p\} \\ &\subseteq \{A\mathbf{y} \ : \ \mathbf{y} \in \mathbb{R}^n\} = \mathcal{C}(A), \end{aligned}$$

because $B\mathbf{x} \in \mathbb{R}^n$ for any $\mathbf{x} \in \mathbb{R}^p$.

(4) $\mathcal{R}(AB) = \mathcal{C}((AB)^T) = \mathcal{C}(B^T A^T) \subseteq \mathcal{C}(B^T) = \mathcal{R}(B)$. □

**Corollary 3.20** $\text{rank}(AB) \leq \min\{\text{rank } A, \ \text{rank } B\}$.

In some particular cases, the equality holds. In fact, it will be shown later in Theorem 5.23 that for any square matrix $A$, $\text{rank}(A^T A) = \text{rank } A = \text{rank}(AA^T)$. The following problem illustrates another such case.

*Problem 3.21* Let $A$ be an invertible square matrix. Show that, for any matrix $B$, $\text{rank}(AB) = \text{rank } B = \text{rank}(BA)$.

**Theorem 3.21** *Let $A$ be an $m \times n$ matrix of rank $r$. Then*

(1) *for every submatrix $C$ of $A$,* rank $C \leq r$, *and*

(2) *the matrix $A$ has at least one $r \times r$ submatrix of rank $r$, that is, $A$ has an invertible submatrix of order $r$.*

**Proof:** (1) We consider an intermediate matrix $B$ which is obtained from $A$ by removing the rows that are not wanted in $C$. Then clearly $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and hence rank $B \leq$ rank $A$. Moreover, since the columns of $C$ are taken from those of $B$, $\mathcal{C}(C) \subseteq \mathcal{C}(B)$ and rank $C \leq$ rank $B$.

(2) Note that we can find $r$ linearly independent row vectors of $A$, which form a basis for the row space of $A$. Let $B$ be the matrix whose row vectors consist of these vectors. Then rank $B = r$ and the column space of $B$ must be of dimension $r$. By taking $r$ linearly independent column vectors of $B$, one can find an $r \times r$ submatrix $C$ of $A$ with rank $r$. □

*Problem 3.22* Prove that the rank of a matrix is equal to the largest order of its invertible submatrices.

*Problem 3.23* For each of the matrices given in Problem 3.19, find an invertible submatrix of the largest order.

## 3.6 Bases for subspaces

In this section, we discuss how to find bases for $V+W$ and $V\cap W$ of two subspaces $V$ and $W$ of the $n$-space $\mathbb{R}^n$, and then derive an important relationship between the dimensions of those subspaces in terms of the dimensions of $V$ and $W$.

Let $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and $\beta = \{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$ be bases for $V$ and $W$, respectively. Let $Q$ be the $n \times (k+\ell)$ matrix whose columns are those bases vectors:

$$Q = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_k \ \mathbf{w}_1 \ \cdots \ \mathbf{w}_\ell]_{n\times(k+\ell)}.$$

Then it is quite clear that $\mathcal{C}(Q) = V+W$, so that a basis for $\mathcal{C}(Q)$ is a basis for $V+W$. On the other hand, one can show that $\mathcal{N}(Q)$ can be identified with $V\cap W$.

In fact, if $\mathbf{x} = (a_1, \ldots, a_k, b_1, \ldots, b_\ell) \in \mathcal{N}(Q) \subseteq \mathbb{R}^{k+\ell}$, then

$$Q\mathbf{x} = a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k + b_1\mathbf{w}_1 + \cdots + b_\ell\mathbf{w}_\ell = \mathbf{0}.$$

This means that corresponding to $\mathbf{x}$ there is a vector

$$\mathbf{y} = a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k = -(b_1\mathbf{w}_1 + \cdots + b_\ell\mathbf{w}_\ell)$$

that belongs to $V\cap W$, since the middle part is in $V$ as a linear combination of the basis vectors in $\alpha$ and the right side is in $W$ as a linear combination of the basis vectors in $\beta$. On the other hand, if $\mathbf{y} \in V\cap W$, $\mathbf{y}$ can be written as linear combinations of both bases for $V$ and $W$:

$$\begin{aligned}\mathbf{y} &= a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k\\ &= b_1\mathbf{w}_1 + \cdots + b_\ell\mathbf{w}_\ell,\end{aligned}$$

for some $a_1, \ldots, a_k$ and $b_1, \ldots, b_\ell$. Let $\mathbf{x} = (a_1, \ldots, a_k, -b_1, \ldots, -b_\ell)$. Then it is quite clear that $Q\mathbf{x} = \mathbf{0}$, *i.e.*, $\mathbf{x} \in \mathcal{N}(Q)$. That is, for each $\mathbf{x} \in \mathcal{N}(Q)$, there corresponds a vector $\mathbf{y} \in V\cap W$, and *vice versa*. Moreover, if $\mathbf{x}_i$, $i = 1, 2$, correspond to $\mathbf{y}_i$, then one can easily check that $\mathbf{x}_1 + \mathbf{x}_2$ corresponds to $\mathbf{y}_1 + \mathbf{y}_2$, and $k\mathbf{x}_1$ corresponds to $k\mathbf{y}_1$. This means that the two vector spaces $\mathcal{N}(Q)$ and $V \cap W$ can be identified as vector spaces. In particular, for a basis for $\mathcal{N}(Q)$, the corresponding set in $V \cap W$ is also a basis, that is, if the set of vectors

$$\left\{\begin{array}{lcl} \mathbf{x}_1 & = & (a_{11}, \ldots, a_{1k}, b_{11}, \ldots, b_{1\ell}), \\ \vdots & & \\ \mathbf{x}_s & = & (a_{s1}, \ldots, a_{sk}, b_{s1}, \ldots, b_{s\ell}), \end{array}\right.$$

is a basis for $\mathcal{N}(Q)$, then the set

$$\left\{\begin{array}{lcl} \mathbf{y}_1 & = & a_{11}\mathbf{v}_1 + \cdots + a_{1k}\mathbf{v}_k, \\ \vdots & & \\ \mathbf{y}_s & = & a_{s1}\mathbf{v}_1 + \cdots + a_{sk}\mathbf{v}_k, \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{lcl} \mathbf{y}_1 & = & -(b_{11}\mathbf{w}_1 + \cdots + b_{1\ell}\mathbf{w}_\ell), \\ \vdots & & \\ \mathbf{y}_s & = & -(b_{s1}\mathbf{w}_1 + \cdots + b_{s\ell}\mathbf{w}_\ell) \end{array}\right.$$

is also a basis for $V \cap W$, and *vice versa*. This means that

$$\dim \mathcal{N}(Q) = \dim V \cap W.$$

Note that $\dim(V + W) \neq \dim V + \dim W$, in general. The following corollary gives a relation of them.

**Corollary 3.22** *For any subspaces $V$ and $W$ of the $n$-space $\mathbb{R}^n$,*

$$\dim(V + W) + \dim(V \cap W) = \dim V + \dim W.$$

**Proof:** Let $\dim V = k$ and $\dim W = \ell$. Recall that rank $A+$ nullity $A =$ the number of the columns of a matrix $A$. Thus, for the matrix $Q$ above, we have

$$\dim \mathcal{C}(Q) + \dim \mathcal{N}(Q) = k + \ell.$$

However, we have $\dim \mathcal{C}(Q) = \dim(V + W)$, $\dim \mathcal{N}(Q) = \dim(V \cap W)$. $\dim V = k$ and $\dim W = \ell$. □

**Example 3.22** Let $V$ and $W$ be two subspaces of $\mathbb{R}^5$ with bases

$$\left\{\begin{array}{lcl} \mathbf{v}_1 & = & (1,\ 3,\ -2,\ 2,\ 3), \\ \mathbf{v}_2 & = & (1,\ 4,\ -3,\ 4,\ 2), \\ \mathbf{v}_3 & = & (1,\ 3,\ 0,\ 2,\ 3), \end{array}\right. \qquad \left\{\begin{array}{lcl} \mathbf{w}_1 & = & (2,\ 3,\ -1,\ -2,\ 9), \\ \mathbf{w}_2 & = & (1,\ 5,\ -6,\ 6,\ 1), \\ \mathbf{w}_3 & = & (2,\ 4,\ 4,\ 2,\ 8), \end{array}\right.$$

respectively. Then the matrix $Q$ takes the following form:

$$Q = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{w}_1\ \mathbf{w}_2\ \mathbf{w}_3\ ] = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 2 \\ 3 & 4 & 3 & 3 & 5 & 4 \\ -2 & -3 & 0 & -1 & -6 & 4 \\ 2 & 4 & 2 & -2 & 6 & 2 \\ 3 & 2 & 3 & 9 & 1 & 8 \end{bmatrix}.$$

After Gauss-Jordan elimination, we get

$$U = \begin{bmatrix} 1 & 0 & 0 & 5 & 0 & 0 \\ 0 & 1 & 0 & -3 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

From this, one can directly see that $\dim(V + W) = 4$. The columns $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}_3$ corresponding to the basic variables in $Q\mathbf{x} = \mathbf{0}$ form a basis for $\mathcal{C}(Q) = V + W$. Moreover, $\dim \mathcal{N}(Q) = \dim(V \cap W) = 2$, since there are two free variables $x_4$ and $x_5$ in $Q\mathbf{x} = \mathbf{0}$.

To find a basis for $V \cap W$, we solve $U\mathbf{x} = \mathbf{0}$ for $(x_1, x_2, x_3, 1, 0, x_5)$ and $(x_1, x_2, x_3, 0, 1, x_5)$. After a simple computation, we obtain a basis for $\mathcal{N}(Q)$:

$$\mathbf{x}_1 = (-5, 3, 0, 1, 0, 0) \text{ and } \mathbf{x}_2 = (0, -2, 1, 0, 1, 0).$$

From $Q\mathbf{x}_i = \mathbf{0}$, we obtain two equations:

$$\begin{array}{rcrcrcl} -5\mathbf{v}_1 & + & 3\mathbf{v}_2 & + & \mathbf{w}_1 & = & \mathbf{0}, \\ -2\mathbf{v}_2 & + & \mathbf{v}_3 & + & \mathbf{w}_2 & = & \mathbf{0}. \end{array}$$

Therefore, $\{\mathbf{y}_1, \mathbf{y}_2\}$ is a basis for $V \cap W$, where

$$\mathbf{y}_1 = 5\mathbf{v}_1 - 3\mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ -1 \\ -2 \\ 9 \end{bmatrix} = \mathbf{w}_1, \quad \mathbf{y}_2 = 2\mathbf{v}_2 - \mathbf{v}_3 = \begin{bmatrix} 1 \\ 5 \\ -6 \\ 6 \\ 1 \end{bmatrix} = \mathbf{w}_2.$$

Clearly, the equality

$$\dim(V + W) + \dim(V \cap W) = 4 + 2 = 3 + 3 = \dim V + \dim W$$

holds in this example. □

**Remark:** In Example 3.22, we showed a method for finding bases for $V+W$ and $V \cap W$ for given subspaces $V$ and $W$ of $\mathbb{R}^n$ by constructing a matrix $Q$ whose columns are basis vectors for $V$ and basis vectors for $W$. There is another method for finding their bases by constructing a matrix $Q$ whose rows are basis vectors for $V$ and basis vectors for $W$.

If $Q$ is the matrix whose row vectors are basis vectors for $V$ and basis vectors for $W$ in order, then clearly $V + W = \mathcal{R}(Q)$. By finding a basis for the row space $\mathcal{R}(Q)$, we can get a basis for $V + W$.

On the other hand, a basis for $V \cap W$ can be found as follows: Let $A$ be the $k \times n$ matrix whose rows are basis vectors for $V$, and $B$ the $\ell \times n$ matrix whose rows are basis vectors for $W$. Then, $V = \mathcal{R}(A)$ and $W = \mathcal{R}(B)$. Let $\bar{A}$ denote the matrix $A$ with an unknown vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ attached at the bottom row, *i.e.*,

$$\bar{A} = \begin{bmatrix} A \\ \mathbf{x} \end{bmatrix},$$

and the matrix $\bar{B}$ is defined similarly. Then it is clear that $\mathcal{R}(A) = \mathcal{R}(\bar{A})$ and $\mathcal{R}(B) = \mathcal{R}(\bar{B})$ if and only if $\mathbf{x} \in V \cap W = \mathcal{R}(A) \cap \mathcal{R}(B)$. This means that the row-echelon form of $A$ and that of $\bar{A}$ should be the same via the same Gaussian elimination. Thus, by comparing the row vectors of the row-echelon form of $A$ with those of $\bar{A}$, we can obtain a system of linear equations for $\mathbf{x} = (x_1, \ldots, x_n)$. By the same argument applied to $B$ and $\bar{B}$, we get another system of linear equations for the same $\mathbf{x} = (x_1, \ldots, x_n)$. Solutions to these two systems together will provide us with a basis for $V \cap W$.

The following example illustrates how one can apply this argument to find bases for $V + W$ and $V \cap W$.

**Example 3.23** Let $V$ be the subspace of $\mathbb{R}^5$ spanned by

$$\begin{aligned} \mathbf{v}_1 &= (1,\ 3,\ -2,\ \ 2,\ 3), \\ \mathbf{v}_2 &= (1,\ 4,\ -3,\ \ 4,\ 2), \\ \mathbf{v}_3 &= (2,\ 3,\ -1,\ -2,\ 10), \end{aligned}$$

and $W$ the subspace spanned by

$$\begin{aligned} \mathbf{w}_1 &= (1,\ 3,\ \ 0,\ 2,\ 1), \\ \mathbf{w}_2 &= (1,\ 5,\ -6,\ 6,\ 3), \\ \mathbf{w}_3 &= (2,\ 5,\ \ 3,\ 2,\ 1). \end{aligned}$$

Find a basis for $V+W$ and for $V\cap W$.

Solution: Note that the matrix $A$ whose row vectors are $\mathbf{v}_i$'s is reduced to a row-echelon form

$$\begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

so that $\dim V = 3$. Similarly, the matrix $B$ whose row vectors are $\mathbf{w}_j$'s is reduced to a row-echelon form

$$\begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 2 & -6 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

so that $\dim W = 2$.

Now, if $Q$ denotes the $6\times 5$ matrix whose row vectors are $\mathbf{v}_i$'s and $\mathbf{w}_j$'s, then $V+W=\mathcal{R}(Q)$. By Gaussian elimination, $Q$ is reduced to a row-echelon form, excluding zero rows:

$$\begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, the four nonzero row vectors

$$(1,\ 3,\ -2,\ 2,\ 3),\ (0,\ 1,\ -1,\ 2,\ -1),\ (0,\ 0,\ 1,\ 0,\ -1),\ (0,\ 0,\ 0,\ 0,\ 1)$$

form a basis for $V+W$, so that $\dim(V+W)=4$.

We now find a basis for $V\cap W$. A vector $\mathbf{x}=(x_1,\ x_2,\ x_3,\ x_4,\ x_5)\in\mathbb{R}^5$ is contained in $V\cap W$ if and only if $\mathbf{x}$ is contained in both the row space of $A$ and that of $B$.

Let $\bar{A}$ be $A$ with $\mathbf{x}$ attached at the last row:

$$\bar{A}=\begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 1 & 4 & -3 & 4 & 2 \\ 2 & 3 & -1 & -2 & 10 \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}.$$

Then by the same Gaussian elimination $\bar{A}$ is reduced to

$$\begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -x_1+x_2+x_3 & 4x_1-2x_2+x_4 & 0 \end{bmatrix}.$$

Therefore, $\mathbf{x} \in \mathcal{R}(A) = V$ if and only if $\mathcal{R}(A) = \mathcal{R}(\bar{A})$. By comparing the row vectors of the row-echelon form of $A$ with those of $\bar{A}$, it gives that $\mathbf{x} \in \mathcal{R}(A)$ if and only if the last row vector of the row-echelon form of $\bar{A}$ is the zero vector, that is, $\mathbf{x}$ is a solution of the homogeneous system of equations

$$\begin{cases} -x_1 + x_2 + x_3 & = 0 \\ 4x_1 - 2x_2 + x_4 & = 0. \end{cases}$$

We do the same calculation with $\bar{B}$, and obtain another homogeneous system of linear equations for $\mathbf{x}$:

$$\begin{cases} -9x_1 + 3x_2 + x_3 & = 0 \\ 4x_1 - 2x_2 + x_4 & = 0 \\ 2x_1 - x_2 + x_5 & = 0. \end{cases}$$

Solving these two homogeneous systems together yields

$$V \cap W = \{t(1,\ 4,\ -3,\ 4,\ 2) : t \in \mathbb{R}\}.$$

Hence, $\{(1,\ 4,\ -3,\ 4,\ 2)\}$ is a basis for $V \cap W$ and $\dim(V \cap W) = 1$. □

*Problem 3.24* Let $V$ and $W$ be the subspaces of the vector space $P_3(\mathbb{R})$ spanned by

$$\begin{cases} v_1(x) = 3 - x + 4x^2 + x^3, \\ v_2(x) = 5 + 5x^2 + x^3, \\ v_3(x) = 5 - 5x + 10x^2 + 3x^3, \end{cases}$$

and

$$\begin{cases} w_1(x) = 9 - 3x + 3x^2 + 2x^3, \\ w_2(x) = 5 - x + 2x^2 + x^3, \\ w_3(x) = 6 + 4x^2 + x^3, \end{cases}$$

respectively. Find the dimensions and bases for $V + W$ and $V \cap W$.

*Problem 3.25* Let

$$\begin{aligned} V &= \left\{(x,\ y,\ z,\ u) \in \mathbb{R}^4 : y+z+u=0\right\}, \\ W &= \left\{(x,\ y,\ z,\ u) \in \mathbb{R}^4 : x+y=0,\ z=2u\right\} \end{aligned}$$

be two subspaces of $\mathbb{R}^4$. Find bases for $V$, $W$, $V+W$, and $V \cap W$.

## 3.7 Invertibility

We now can have the following existence and uniqueness theorems for a solution of a system of linear equations $A\mathbf{x} = \mathbf{b}$ for an $m\times n$ matrix $A$ and a vector $\mathbf{b} \in \mathbb{R}^m$.

**Theorem 3.23 (Existence)** *Let $A$ be an $m\times n$ matrix. Then the following statements are equivalent.*

**(1)** *For each $\mathbf{b} \in \mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ has* at least *one solution $\mathbf{x}$ in $\mathbb{R}^n$.*

**(2)** *The column vectors of $A$ span $\mathbb{R}^m$, i.e., $\mathcal{C}(A) = \mathbb{R}^m$.*

**(3)** rank $A = m$, *and hence $m \le n$.*

**(4)** *There exists an $n \times m$ right inverse $B$ of $A$ such that $AB = I_m$.*

**Proof:** **(1) $\Leftrightarrow$ (2)**: Note that $\mathcal{C}(A) \subseteq \mathbb{R}^m$ in general. For any $\mathbf{b} \in \mathbb{R}^m$, there is a solution $\mathbf{x} \in \mathbb{R}^n$ of $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{b}$ is a linear combination of the column vectors of $A$. This is equivalent to saying that $\mathbb{R}^m = \mathcal{C}(A)$.

**(2) $\Leftrightarrow$ (3)**: Since $\dim \mathcal{C}(A) = \operatorname{rank} A = \dim \mathcal{R}(A) \le \min\{m, n\}$, $\mathcal{C}(A) = \mathbb{R}^m$ if and only if $\dim \mathcal{C}(A) = m \le n$ (see Problem 3.10).

**(1) $\Rightarrow$ (4)** : Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$ be the standard basis for $\mathbb{R}^m$. Then for each $i = 1, 2, \ldots, m$ we can find at least one solution $\mathbf{x}_i \in \mathbb{R}^n$ such that $A\mathbf{x}_i = \mathbf{e}_i$ by the condition. If $B$ is the $n\times m$ matrix whose columns are these solutions, *i.e.*, $B = [\mathbf{x}_1\ \mathbf{x}_2\ \cdots\ \mathbf{x}_m]$, then it follows by matrix multiplication that

$$AB = A\,[\mathbf{x}_1\ \mathbf{x}_2\ \cdots\ \mathbf{x}_m] = [\mathbf{e}_1\ \mathbf{e}_2\ \cdots\ \mathbf{e}_m] = I_m.$$

Hence, the matrix $B$ is a required right inverse.

**(4) $\Rightarrow$ (1)** : If $B$ is a right inverse of $A$, then for any $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{x} = B\mathbf{b}$ is a solution of $A\mathbf{x} = \mathbf{b}$. $\square$

Condition (2) means that $A$ has $m$ linearly independent column vectors, and condition (3) implies that there exist $m$ linearly independent row vectors of $A$, since $\operatorname{rank} A = m = \dim \mathcal{R}(A)$.

Note that if $\mathcal{C}(A) \subsetneq \mathbb{R}^m$, then $A\mathbf{x} = \mathbf{b}$ has no solution for $\mathbf{b} \notin \mathcal{C}(A)$.

**Theorem 3.24 (Uniqueness)** *Let $A$ be an $m\times n$ matrix. Then the following statements are equivalent.*

**(1)** *For each $\mathbf{b} \in \mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ has* at most *one solution $\mathbf{x}$ in $\mathbb{R}^n$.*

**(2)** *The column vectors of $A$ are linearly independent.*

(3) $\dim \mathcal{C}(A) = \text{rank } A = n$, *and hence* $n \leq m$.

(4) $\mathcal{R}(A) = \mathbb{R}^n$.

(5) $\mathcal{N}(A) = \{\mathbf{0}\}$.

(6) *There exists an* $n \times m$ *left inverse* $C$ *of* $A$ *such that* $CA = I_n$.

**Proof:** **(1) ⇒ (2)** : Note that the column vectors of $A$ are linearly independent if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only a trivial solution. However, $A\mathbf{x} = \mathbf{0}$ has always a trivial solution $\mathbf{x} = \mathbf{0}$ and (1) means that it is the only one.

**(2) ⇔ (3)** : Clear, because all the column vectors are linearly independent if and only if they form a basis for $\mathcal{C}(A)$, or $\dim \mathcal{C}(A) = n \leq m$.

**(3) ⇔ (4)**: Clear, because $\dim \mathcal{R}(A) = \text{rank } A = \dim \mathcal{C}(A) = n$ if and only if $\mathcal{R}(A) = \mathbb{R}^n$ (see Problem 3.10).

**(4) ⇔ (5)**: Clear, since $\dim \mathcal{R}(A) + \dim \mathcal{N}(A) = n$.

**(2) ⇒ (6)** : Suppose that the columns of $A$ are linearly independent so that rank $A = n$. Extend these column vectors of $A$ to a basis for $\mathbb{R}^m$ by adding $m - n$ more independent vectors to them. Construct an $m \times m$ matrix $S$ with those vectors in columns. Then the matrix $S$ has rank $m$ and is hence invertible. Let $C$ be the $n \times m$ matrix obtained from $S^{-1}$ by throwing away the last $m - n$ rows. Since the first $n$ columns of $S$ constitute the matrix $A$, we have $CA = I_n$.

**(6) ⇒ (1)** : Let $C$ be a left inverse of $A$. If $A\mathbf{x} = \mathbf{b}$ has no solution, then we are done. Suppose that $A\mathbf{x} = \mathbf{b}$ has two solutions, say $\mathbf{x}_1$ and $\mathbf{x}_2$. Then

$$\mathbf{x}_1 = CA\mathbf{x}_1 = C\mathbf{b} = CA\mathbf{x}_2 = \mathbf{x}_2.$$

Hence, the system can have at most one solution. □

**Remark:** **(1)** We have proved that an $m \times n$ matrix $A$ has a right inverse if and only if rank $A = m$, and $A$ has a left inverse if and only if rank $A = n$. In the first case $A\mathbf{x} = \mathbf{b}$ always has a solution, and in the second case the solution (if it exists) is unique. Therefore, if $m \neq n$, $A$ cannot have both left and right inverses.

**(2)** For a practical way of finding a right or a left inverse of an $m \times n$ matrix $A$, we will show later (see Corollary 5.24) that if rank $A = m$, then $(AA^T)^{-1}$ exists and $A^T(AA^T)^{-1}$ is a right inverse of $A$, and if rank $A = n$, then $(A^TA)^{-1}$ exists and $(A^TA)^{-1}A^T$ is a left inverse of $A$.

(3) Note that if $m = n$ so that $A$ is a square matrix, then $A$ has a right inverse (and a left inverse) if and only if rank $A = m = n$. Moreover, in this case the inverses are the same (see Theorem 1.8). Therefore, a square matrix $A$ has rank $n$ if and only if $A$ is invertible. This means that for a square matrix "Existence = Uniqueness", and the ten conditions in the above two theorems are all equivalent. In particular, for the invertibility of a square matrix it is enough to show the existence of a one-side inverse.

*Problem 3.26* For each of the following matrices, discuss the number of possible solutions to the system of linear equations $A\mathbf{x} = \mathbf{b}$ for any $\mathbf{b}$:

$$(1)\ A = \begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 2 & 7 & -3 & 6 & 13 \end{bmatrix}, \qquad (2)\ A = \begin{bmatrix} 2 & 3 \\ 3 & -7 \\ -6 & 1 \end{bmatrix},$$

$$(3)\ A = \begin{bmatrix} 1 & 2 & -3 & -2 & -3 \\ 1 & 3 & -2 & 0 & -4 \\ 3 & 8 & -7 & -2 & -11 \\ 2 & 1 & -9 & -10 & -3 \end{bmatrix}, \qquad (4)\ A = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 5 & 5 \\ 1 & 2 & -2 \end{bmatrix}.$$

The following theorem is a collection of the results proved in Theorems 1.8, 3.23, 3.24, and the Remark before Definition 4.3.

**Theorem 3.25** *For a square matrix $A$ of order $n$, the following statements are equivalent.*

(1) *$A$ is invertible.*

(2) $\det A \neq 0$.

(3) *$A$ is row equivalent to $I_n$.*

(4) *$A$ is a product of elementary matrices.*

(5) *Elimination can be completed: $PA = LDU$, with all $d_i \neq 0$.*

(6) *$A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^n$.*

(7) *$A\mathbf{x} = \mathbf{0}$ has only a trivial solution, i.e., $\mathcal{N}(A) = \{\mathbf{0}\}$.*

(8) *The columns of $A$ are linearly independent.*

(9) *The columns of $A$ span $\mathbb{R}^n$, i.e., $\mathcal{C}(A) = \mathbb{R}^n$.*

(10) *$A$ has a left inverse.*

(11) $\operatorname{rank} A = n$.

(12) *The rows of $A$ are linearly independent.*

(13) *The rows of $A$ span $\mathbb{R}^n$,* i.e., $\mathcal{R}(A) = \mathbb{R}^n$.

(14) *$A$ has a right inverse.*

**(15)*** *The linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ via $A(\mathbf{x}) = A\mathbf{x}$ is injective.*

**(16)*** *The linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ is surjective.*

**(17)*** *Zero is not an eigenvalue of $A$.*

**Proof:** Exercise: where have we proved which claim? Prove any not covered. The numbers with asterisks will be explained in the following places: (15) and (16) in the Remark on page 141 and (17) in Theorem 6.1. □

## 3.8 Application: Interpolation

In many scientific experiments, a scientist wants to find the precise functional relationship between input data and output data. That is, in his experiment, he puts various input values into his experimental device and obtains output values corresponding to those input values. After his experiment, what he has is a table of inputs and outputs. The precise functional relationship might be very complicated, and sometimes it might be very hard or almost impossible to find the precise function. In this case, one thing he can do is to find a polynomial whose graph passes through each of the data points and comes very close to the function he wanted to find. That is, he is looking for a polynomial that approximates the precise function. Such a polynomial is called an **interpolating polynomial**. This problem is closely related to systems of linear equations.

Let us begin with a set of given data: Suppose that for $n+1$ distinct experimental input values $x_0, x_1, \ldots, x_n$, we obtained $n+1$ output values $y_0 = f(x_0), y_1 = f(x_1), \ldots, y_n = f(x_n)$. The output values are supposed to be related to the inputs by a certain function $f$. We wish to construct a polynomial $p(x)$ of degree less than or equal to $n$ which interpolates $f(x)$ at $x_0, x_1, \ldots, x_n$: *i.e.*, $p(x_i) = y_i = f(x_i)$ for $i = 0, 1, \ldots, n$.

Note that if there is such a polynomial, it must be unique. Indeed, if $q(x)$ is another such polynomial, then $h(x) = p(x) - q(x)$ is also a polynomial of degree less than or equal to $n$ vanishing at $n+1$ distinct points $x_0, x_1, \ldots, x_n$. Hence $h(x)$ must be the identically zero polynomial so that $p(x) = q(x)$ for all $x \in \mathbb{R}$.

In fact, the unique polynomial $p(x)$ can be found by solving a system of linear equations: If we write $p(x) = a_0 + a_1x + \cdots + a_nx^n$, then we are supposed to determine the coefficients $a_i$'s. The set of equations

$$p(x_i) = a_0 + a_1 x_i + \cdots + a_n x_i^n = y_i = f(x_i),$$

for $i = 0,\ 1,\ \ldots,\ n$, constitutes a system of $n+1$ linear equations in $n+1$ unknowns $a_i$'s:

$$\begin{bmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

The coefficient matrix $A$ is a square matrix of order $n+1$, known as **Vandermonde's matrix** (see Problem 2.10), whose determinant is

$$\det A = \prod_{0 \le i < j \le n} (x_j - x_i).$$

Since the $x_i$'s are all distinct, $\det A \neq 0$. It follows that $A$ is nonsingular, and hence $A\mathbf{x} = \mathbf{b}$ always has a unique solution, which determines the unique polynomial $p(x)$ of degree $\le n$ passing through the given $n+1$ points $(x_0,\ y_0)$, $(x_1,\ y_1)$, $\cdots$, $(x_n,\ y_n)$ in the plane $\mathbb{R}^2$.

**Example 3.24** Given four points

$$(0,\ 3),\ (1,\ 0),\ (-1,\ 2),\ (3,\ 6)$$

in the plane $\mathbb{R}^2$, let $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ be the polynomial passing through the given four points. Then, we have a system of equations

$$\left\{ \begin{array}{rcrcrcrcl} a_0 & & & & & & & = & 3 \\ a_0 & + & a_1 & + & a_2 & + & a_3 & = & 0 \\ a_0 & - & a_1 & + & a_2 & - & a_3 & = & 2 \\ a_0 & + & 3a_1 & + & 9a_2 & + & 27a_3 & = & 6. \end{array} \right.$$

Solving this system, we find that $a_0 = 3$, $a_1 = -2$, $a_2 = -2$, $a_3 = 1$ is the unique solution, and the unique polynomial is $p(x) = 3 - 2x - 2x^2 + x^3$. $\square$

*Problem 3.27* Let $f(x) = \sin x$. Then at $x = 0,\ \frac{\pi}{4},\ \frac{\pi}{3},\ \frac{3\pi}{4},\ \pi$, the values of $f$ are $y = 0,\ \frac{1}{\sqrt{2}},\ \frac{\sqrt{3}}{2},\ \frac{1}{\sqrt{2}},\ 0$. Find the polynomial $p(x)$ of degree $\le 4$ that passes through these five points. (One may need to use a computer due to messy computation).

*Problem 3.28* Find a polynomial $p(x) = a + bx + cx^2 + dx^3$ that satisfies $p(0) = 1$, $p'(0) = 2$, $p(1) = 4$, $p'(1) = 4$.

*Problem 3.29* Find the equation of a circle that passes through the three points $(2, -2)$, $(3, 5)$, and $(-4, 6)$ in the plane $\mathbb{R}^2$.

**Remark: (1)** It is suggested that the readers think about the differences between this interpolation and the Taylor polynomial approximation to a differentiable function.

**(2)** Note again that the interpolating polynomial $p(x)$ of degree $\leq n$ is uniquely determined when we have the correct data, *i.e.*, when we are given precisely $n+1$ values of $y$ at precisely $n+1$ distinct points $x_0, x_1, \ldots, x_n$.

However, if we are given fewer data, then the polynomial is under-determined: *i.e.*, if we have $m$ values of $y$ with $m < n+1$ at $m$ distinct points $x_1, x_2, \ldots, x_m$, then there are as many interpolating polynomials as the null space of $A$ since in this case $A$ is an $m \times (n+1)$ matrix with $m < n+1$.

On the other hand, if we are given more than $n+1$ data, then the polynomial is over-determined: *i.e.*, if we have $m$ values of $y$ with $m > n+1$ at $m$ distinct points $x_1, x_2, \ldots, x_m$, then there need not be any interpolating polynomial since the system could be inconsistent. In this case, the best we can do is to find a polynomial of degree $\leq n$ to which the data is closest. We will review this statement again in Section 5.8.

## 3.9 Application: The Wronskian

Let $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ be $n$ vectors in an $m$-dimensional vector space $V$. To check the independence of the vectors $\mathbf{y}_i$'s, consider its linear dependence:

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n = \mathbf{0}.$$

Let $\alpha = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ be a basis for $V$. By expressing each $\mathbf{y}_i$ as a linear combination of the basis vectors $\mathbf{x}_i$'s, the linear dependence of $\mathbf{y}_i$'s can be written as a linear combination of the basis vectors $\mathbf{x}_i$'s, so that all of the coefficients (which are also linear combinations of $c_i$'s) must be zero. It gives a homogeneous system of linear equations in $c_i$'s, say $A\mathbf{c} = \mathbf{0}$ with an $m \times n$ matrix $A$, as in the proof of Lemma 3.8. Recall that the vectors $\mathbf{y}_i$'s are linearly independent if and only if the system $A\mathbf{c} = \mathbf{0}$ has only a trivial solution. Hence, the linear independence of a set of vectors in a finite dimensional vector space can be tested by solving a homogeneous system of linear equations. But, if $V$ is not finite dimensional, this test for the linear independence of a set of vectors cannot be applied.