Linear algebra

Find combination $\mathbf{x}$ of columns of matrix $\mathbf{A}$ that gives vector $\mathbf{b}$. In matrix notation:

$$\mathbf{Ax = b}$$

There are different ways to solve this problem:

1. Direct: Forward elimination and back substitution
2. Matrix: $\mathbf{x = A^{-1} b}$
3. Using column and null space of A.
4. The solution may not exist.

Example:

$$2x_1 + 3x_2 + 2x_3 = 0$$
$$x_1 - 2x_2 + 3x_3 = 11$$
$$-3x_1 + x_2 + x_3 = -3$$

Find x_1, x_2, and x_3.

Equivalent matrix form representation:

$$\underbrace{\begin{bmatrix} 2 & 3 & 2 \\ 1 & -2 & 3 \\ -3 & 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 11 \\ -3 \end{bmatrix}}_{b}$$

$Ax = b$. Find x.

1. Direct: First find x_3. Substitute. Then find x_2. Substitute. Then find x_1.

2. Matrix: $x = A^{-1} b$

Inverse of A (will learn later).

3. Vector space solution.
4. Solution may not exist:

Example:

$$x_1 + x_2 = 3$$
$$x_1 + x_2 = 4$$

There exists no x_1, x_2 pair that satisfies both equations.

Understand when $Ax=b$ is solvable.

Important: Ax is a linear operation:

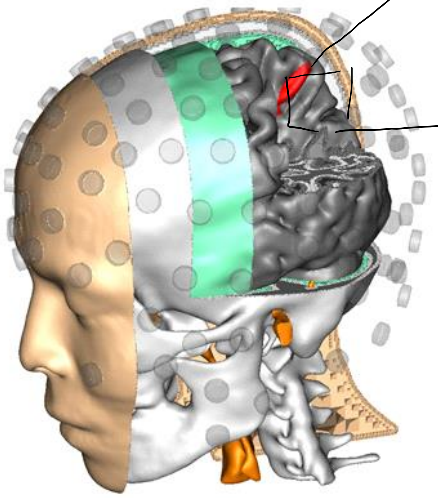
$$A(x+y) = Ax + Ay$$
$$A(kx) = kAx, \quad k \text{ constant.}$$

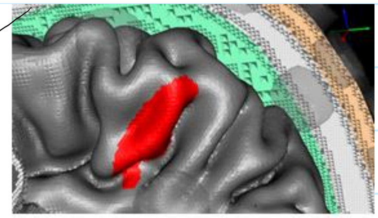
GOAL of first half of DS 5020:

Better understanding of $Ax=b$, or a system of linear equations.

- Large problems.
- Non-square problems. # equations ≠ # unknowns
- Breaking-up the problem into smaller pieces.
- Algebraic manipulations to turn harder problems into simpler ones.

Example: Numerical solutions to partial differential eqn's.





Brain Stimulation:

Where does the electrical currents go in the brain?

Solution: Convert general Poisson's equation that describes the electromagnetic properties of tissue into a system of linear equations. Then solve! (with the help of a computer and human guidance.)

A continuous partial differential equation:

$$\nabla \cdot \sigma \nabla u = 0$$

$\xrightarrow{\text{Discretization}}$

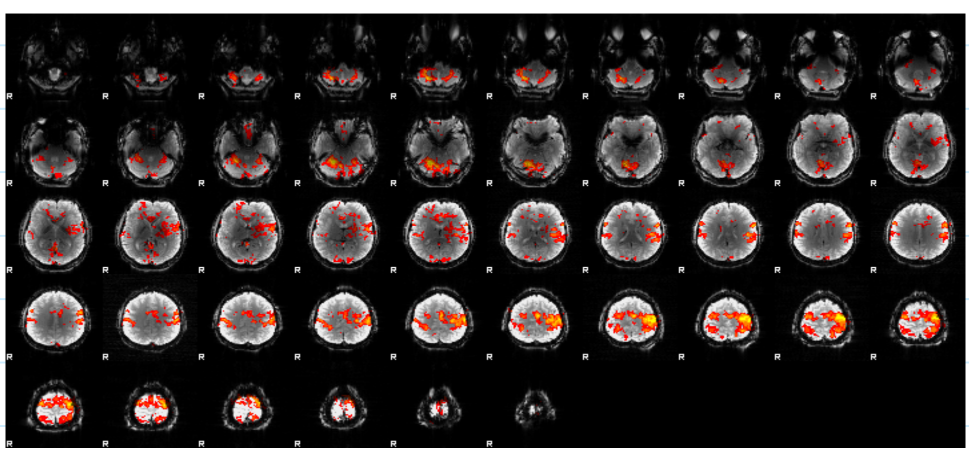
$$Mu = b$$

Find u.

Need to understand!

Example:

Neuroimaging: What linear analysis can we perform to better understand the brain?



Vectors

A vector is a collection of numbers or variables.

Ex:

$$x = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad x_1, x_2, x_3 \in \mathbb{R}$$

Ex: your position on Earth.
30° North, 50° West.

Generalization:

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$ is a <u>column</u> vector with n components.
(components of x: $x_1, x_2, \dots, x_n$)

Vectors help us summarize, only one letter x represent a collection of numbers.

Vector addition:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad x + y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$$

$$x - y = \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \end{bmatrix}$$

Scalar multiplication: c is a scalar;

$$cx = c \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}$$

Scalar value c is multiplied with each component separately.

Linear combination:

Given that c and d are scalars,

Sum of cx and dy is a <u>linear combination</u> of x and y.

Four special linear combinations: sum, difference, zero vector, and vector cx:

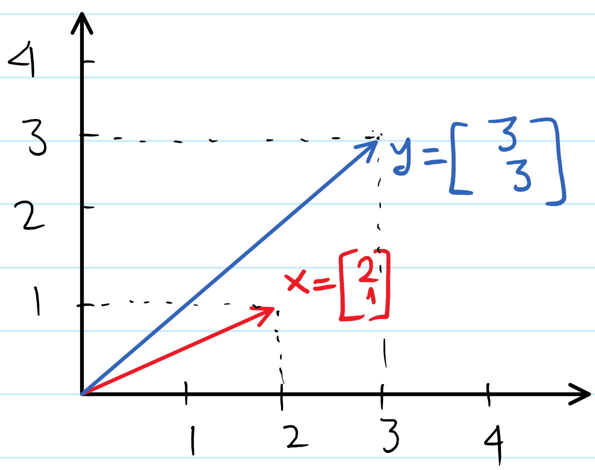
$x+y$: sum of x and y

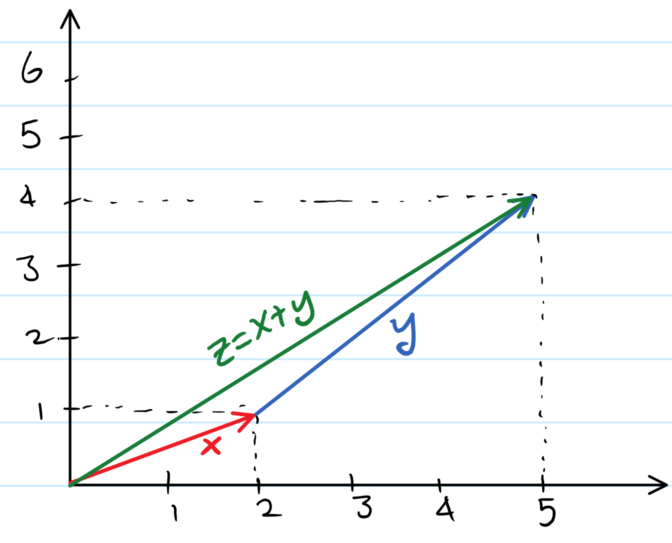
$x-y$: difference of x and y

$0x + 0y =$ zero vector.

$cx + 0y = cx$

Geometric interpretation of vector summation:





$$z = x+y = \begin{bmatrix}2\\1\end{bmatrix} + \begin{bmatrix}3\\3\end{bmatrix} = \begin{bmatrix}2+3\\1+3\end{bmatrix} = \begin{bmatrix}5\\4\end{bmatrix}$$

* We placed second vector arrow starting at the point where first vector arrow ended, given that first vector arrow starts from origin.

Example: $u = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ $v = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$

Find the following linear combinations:

- $u + v = ?$
- $2u - v = ?$
- $0u + 0v = ?$
- $3u + 0v = ?$

Solution:

$$u + v = \begin{bmatrix} 1 - 2 \\ 3+3 \\ 4+1 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \\ 5 \end{bmatrix}$$

$$2u - v = 2\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \times 1 \\ 2 \times 3 \\ 2 \times 4 \end{bmatrix} + \begin{bmatrix} -1 \times -2 \\ -1 \times 3 \\ -1 \times 1 \end{bmatrix} = \begin{bmatrix} 2 + 2 \\ 6 - 3 \\ 8 - 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$$

$$0u + 0v = \begin{bmatrix} 0 \times 1 \\ 0 \times 3 \\ 0 \times 4 \end{bmatrix} + \begin{bmatrix} 0 \times -2 \\ 0 \times 3 \\ 0 \times 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$3u + 0v = \begin{bmatrix} 3 \times 1 \\ 3 \times 3 \\ 3 \times 4 \end{bmatrix} + \begin{bmatrix} 0 \times -2 \\ 0 \times 3 \\ 0 \times 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 12 \end{bmatrix}$$

Example: All linear combinations of a vector, two vectors, or three vectors:

All linear combinations of vector u: cu

All linear combinations of vectors u and v: $cu + dv$

All linear combinations of vectors u, v, and w: $cu + dv + ew$

where $c, d, e \in \mathbb{R}$ are scalars.

If u, v, w are typical non-zero vectors, then:

- cu fills a line.
- $cu + dv$ fills a plane.
- $cu + dv + ew$ fills a three-dimensional space

Dot product

Dot product of vectors $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is:

$$x \cdot y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1 y_1 + x_2 y_2$$

a "dot" between two vectors represents dot (inner) product of these two vectors.

Observation: Dot product is a number! True for vectors with higher number of components than 2.

Example:

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 2 \end{bmatrix} = 2 \times -3 + 1 \times 2 = -4$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 2 \times -1 + 1 \times 2 = 0$$

Note: $x \cdot y = y \cdot x$

Example: We have 3 books, each \$35 worth.
2 pencils, each \$5 worth.
1 eraser, \$1 worth.

How much money would we make if we sell these school supplies?

Solution: total earnings = Dot product of price vector and quantities vector

$$= \begin{bmatrix} 35 \\ 5 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = 35 \times 3 + 5 \times 2 + 1 \times 1 = 116 \,\$$$

Length of a vector:

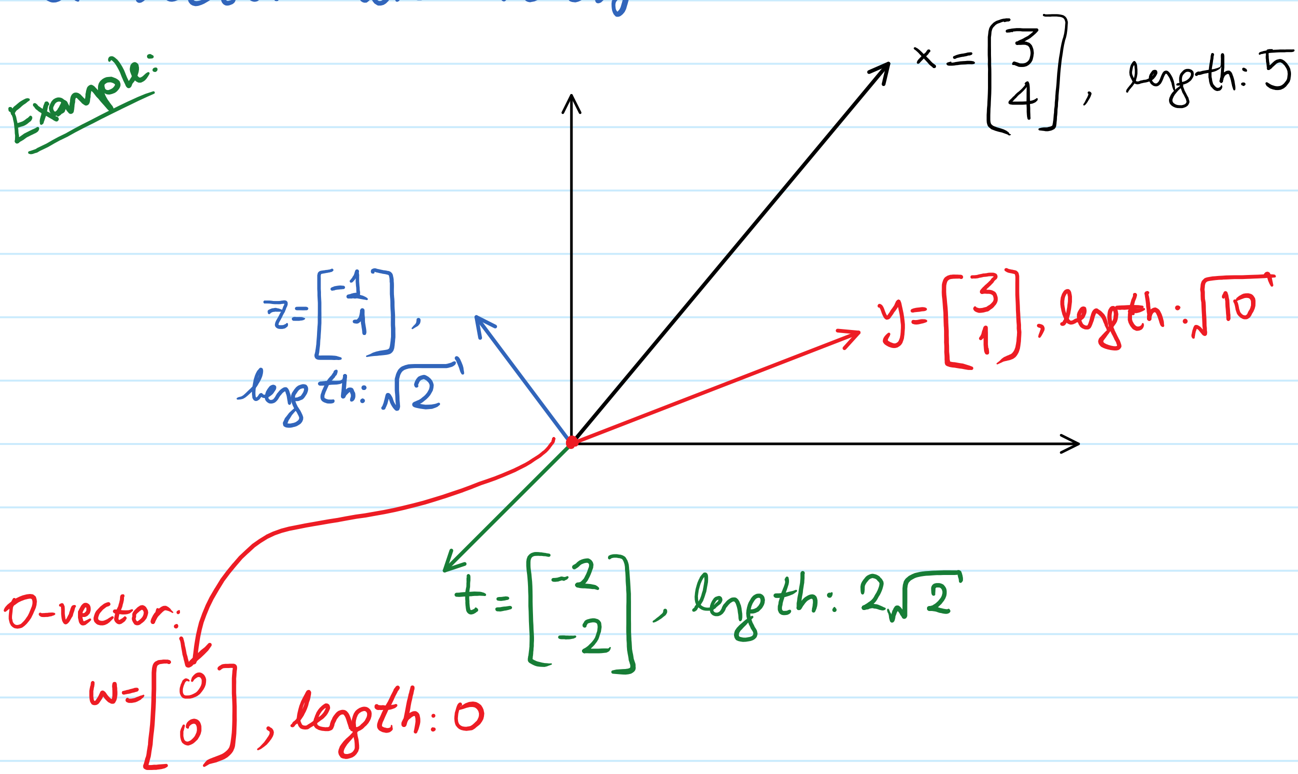
Length of vector $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is square root of $x.x$:

$$\text{length of } x = \sqrt{x.x} = \sqrt{x_1 \times x_1 + x_2 \times x_2 + \dots + x_n \times x_n}$$

$$= \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

- Length of x is usually denoted by $\|x\|$.
- Observation: Length of a vector is always non-negative. So is the dot product of a vector with itself.

Example:



Example: Find the lengths of following vectors:

$$u = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad v = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}, \quad w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Unit vectors:

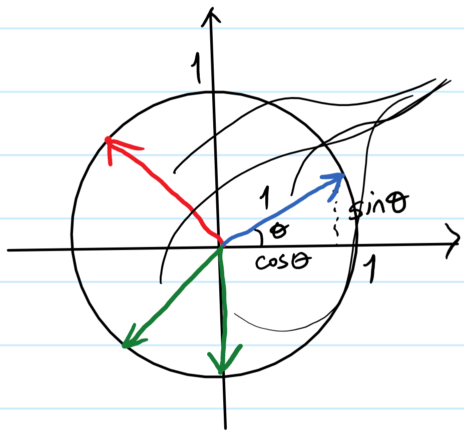
A unit vector u is a vector whose length is equal to 1.

Ex: $i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad u = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

If v is a non-zero vector (that is it has at least one non-zero component), then

$$u = \frac{v}{\|v\|}$$ is a unit vector.

Ex:



Unit vectors in two-dimensinal (i.e. vectors have two components) space.

Perpendicular vectors:

If two vectors are perpendicular, their inner product is zero: $u \cdot v = 0$

Cosine formula:

u, v are non-zero vectors.

Then, $$\frac{u \cdot v}{\|u\| \, \|v\|} = \cos\theta,$$ θ being the angle between these two vectors.

Schwarz Inequality:

$$|v \cdot w| \leq \|v\| \, \|w\|$$

absolute value of dot product of two vectors is smaller than or equal to product of their lengths.