Physics-PHY101-Lecture #01

INTRODUCTION TO PHYSICS & THIS COURSE

1.1. Introduction

Welcome to Physics. We will embark on a long journey that will consist of 45 lectures. Nevertheless, I hope will find it interesting and enjoyable. As I continued to study and research in this subject my interest in this subject grew. But before we come to physics, I want to remind all students that physics is the branch of science and must be aware of the fact that our modern world is based on concepts from science, modern machines/equipment rely on science and inventions such as telephones, satellites, etc. are based on science. But in reality, science is a way of thinking that only accepts the rule of reasoning. In which the judgment of truth and falsehood is based on the sets of results from an experiment. So as mentioned earlier physics is the branch of science that is often termed as the "queen" of science and is rightly said to be the greatest science.

The history of science is as old as the story of mankind. When did it start? It is probably tens of thousands of years old. Every civilization has contributed to it. It may have started from the time of Babylon, but after that, the Greeks created great perfection in it, and then Chinese, Hindu and Islamic civilizations advanced it. Physics in its present form is not that much old. It started about three and a half hundred years ago, and it was a time when a great scientific revolution took place in Europe, which is called the Scientific Revolution, and this was the time when great scientists like Newton revolutionized it which is the reason why our present-day world is so different from previous eras.

Physics is related to every worldly thing (actually everything in the whole universe) with its main purpose being to fully understand the whole physical world/universe and it means that everything, no matter how big or small it is. For example, if we look at our solar system (as shown in Figure 1.1), the sun is at the center and planets revolve around it. Human beings have thought a lot over many fundamental questions like, how heat is produced in the sun and then how it reaches the Earth. Why and how does the Earth rotate around the sun? Contrary to this, in general, if anything moves it moves in a straight line but the earth continues to move around the sun in a circular path.

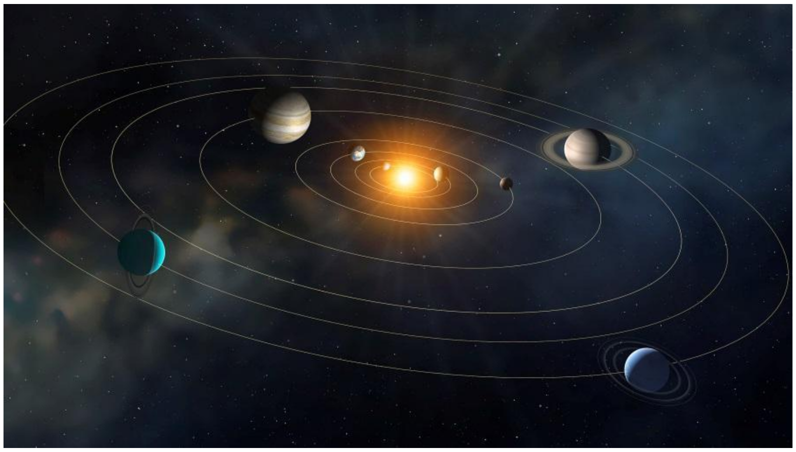

Figure 1.1. Schematic of Solar System. Sun at the center and planets revolving around the Sun.

Atom was considered to be the smallest thing in the world making up all the matter. The atom was considered to be a basic building block but now we know that it can be further broken down as it has a center called the nucleus and electrons revolve around the nucleus as shown in Figure 1.2(a). Upon closer inspection of the nucleus Protons and neutrons can be observed. Interestingly protons and neutrons can be further subdivided into particles called quarks as shown in Figure 1.2(b).

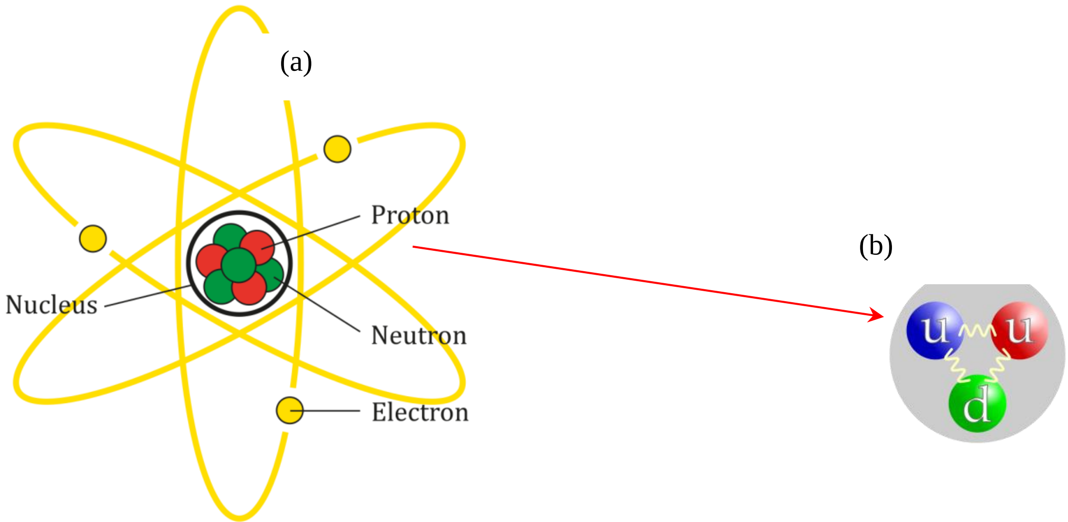

Figure 1.2. (a) Atomic Structure showing nucleus at the center having Proton and Neutron. Electrons revolve around the nucleus. (b) Protons can be subdivided into quarks.

So, on the one hand physics is related to the sun, the solar system, galaxies, and the universe itself, and on the other hand, it is also related to atoms and subatomic particles, and lastly all kinds of entities that come in between these two extremes. These entities can be classified as forms of matter which are as follows

1. Solids
2. Liquids
3. Gases

The first form of matter is called solids in which atoms can't move very far from each other for example table salt. The sodium atom in the salt cannot move away from the chlorine atoms. Atoms are fixed to each other and immobile as shown in Figure 1.3.

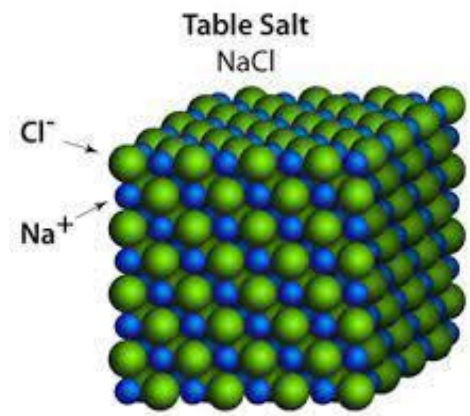

Figure 1.3. Structure of common table Salt (Sodium Chloride, NaCl)

Matter can be in a liquid state and when the substance is in the liquid state, the atoms or molecules that are there can move quite far from each other but still there is some kind of attraction among them confining them. For example, if we put water in a vessel, then this water takes the shape of the vessel and this is a characteristic of the matter when it is in a liquid state. Lastly, there is another situation in which matter can be in a gas state. Well, if the matter is in a gaseous state, then its atoms and molecules can move around freely anywhere and far away from each other. Gases like oxygen, hydrogen, and nitrogen make up our atmosphere and without oxygen, we cannot even breathe.

The purpose of this course is to introduce students to the vast subject of physics. Although even experts in the field only know a fraction of it, we will do our best to learn and gain knowledge related to physics in all 45 lectures. The main goal of this course is to teach problem-solving to students. Science is not just about solving existing problems, but also about understanding and solving new and challenging problems.

To achieve this goal, students must listen to all video lectures, read all materials (such as handouts and referred books) given for this course, and complete the assignments (samples) provided after going through the materials. It's essential to pay attention to the assignments since they will help students to apply the knowledge they have gained.

Studying physics has multiple purposes, one of which is to gain knowledge for future courses. In every field of engineering, ranging from bridge construction to designing electric components, fundamental physics knowledge is required. However, it is crucial to develop the habit of critical thinking and evaluation based on intellect.

Science relies on reason, logic, and experience to determine the truth or falsehood of a claim. Physics is often considered a difficult subject because it involves mathematics. Calculus is required, but we will develop as much calculus as we need. Knowledge of algebra, trigonometry, linear equations, or quadratic equations is necessary prerequisite knowledge.

1.2. Main Areas of Physics:

Following are the main areas of physics

1. Classical Mechanics
2. Electricity and Magnetism
3. Thermal Physics
4. Quantum Mechanics

First of all, classical mechanics is the field of physics, which is attributed to Isaac Newton. It is related to objects, the movement of objects, momentum, force, and energy, these are all concepts that come under classical mechanics. After this, we will talk about electricity and magnetism. Now, electricity and magnetism are not so different from each other and this is a discovery of James Clerk Maxwell from about 200 years ago. And now all modern technologies in the world (telephone, radio, television etc.) are based on these discoveries. So, we will study electricity and magnetism after classical mechanics. When we have this much background, then we will come to thermal physics, i.e. the physics of heat. Concepts like temperature and entropy will be discussed.

In these three areas, classical mechanics, electricity and magnetism and thermal physics, we will spend a lot of time on them. But there is another field, which is more related to the physics of atoms. We will not be able to spend much time on this because to understand it and to study it correctly, you need special mathematics. So, my advice to those people who want to study physics further is that you should study mathematics separately.

1.3. Dimension:

Just like a house is built with bricks, in every field of physics, everything is built with three types of bricks. These are called dimensions. Dimension is a concept which is about 200 years old and it was proposed by a French scientist named Joseph Fourier. Now, let us focus on the three most important fundamental dimensions[1].

1. Time T
2. Length L
3. Mass M

First of all, there is a dimension of time. What is the reality of time? Now, philosophers have been discussing this for centuries, but scientists and physicists are not very interested in these discussions. In physics, we want to know how to measure a thing or a dimension. There is a method to measure time which is called a clock. Now, clocks can be of different types. For example, my pulse is also a clock although not an accurate one. When I stand here, my pulse moves at one speed but when I run, my pulse increases. But there is a better clock, a pendulum. Now, the pendulum moves to and fro. And every time it moves to and fro, the time duration is the same. The earth rotates around its axis and when it rotates, it comes back exactly to the same position after 24 hours. Similarly, when the earth rotates around the sun, we say that one year has passed. The best clock is called the atomic clock which is based on the regular vibration of an atom. With this, we can measure time with better accuracy than one part in a billion. To measure length, normally we need a ruler but its accuracy is not very good. In modern ways, we measure with the wavelength of atoms. So, accuracy is much better as compared to the ruler. The third fundamental dimension is mass, which tells us how much matter is present in a body. For example, if I have an apple and I add another apple to it, the mass will double (assuming both apples are identical). The more mass

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[1] Fundamental and Derived Dimensions

there is, the more difficult it will be to move. For example, if I put equal force on a rickshaw and a taxi, the rickshaw will move faster and the taxi will move slower. So, mass tells us how much resistance to motion there is.

Now, let us look at dimensional quantities, which are made up of mass, length and time. For example, the dimension of the area is LxL or L^2. For a volume, it will be L^3. For example, for a box, its length multiplied by width multiplied by height will give its volume having the dimension of L^3. We can make another thing from this, which is called density. The dimensions of density will be mass per unit volume or M / L^3. In the same way, we find out the dimensions of frequency. What is frequency? For example, there is a pendulum that completes 2 cycles in a second. So, its frequency is 2. Or, if we consider alternating current (AC) of electricity, which changes its polarity 50 times every second. So, its frequency is 50. Frequency has a dimension of 1/T or T^{-1}. The dimensions of speed are L/T or LT^{-1}. But there are some quantities which have no dimensions such as angle. The angle between two adjacent fingers is dimensionless. Degrees are units not the dimension of the angle[2]. In short, there are seven fundamental dimensions as given in the table. And everything in physics, every other quantity can be constructed in terms of these. Fundamental set dimension is given below in form of table.

Fundamental Quantity	**Dimension**	**SI Unit**
Length	[L]	Meter (m)
Time	[T]	Second (s)
Mass	[M]	Kilogram (kg)
Temperature	[θ]	Kelvin (K)
Electric Current	[I]	Ampere (A)

[2] Dimensions refer to the fundamental physical properties that describe a quantity. They are abstract and universal, not tied to any specific measurement system. In physics, there are several fundamental dimensions, such as length (L), mass (M), and time (T). Units, on the other hand, are specific measurements that quantify the magnitude of a physical quantity. Units are used to express how much of a particular dimension is present in a given quantity. Dimensions represent the abstract properties of a quantity, while units are specific measurements that combine a numerical value with dimensions to quantify that property. The combination of dimensions and units provides a complete description of a physical quantity in a specific measurement system.

Amount of Substance	[N]	Mole (mol)
Luminous Intensity	[J]	Candela (cd)

1.4. Units

As per the previous example, there is an angle between my two adjacent fingers, and measure this angle in terms of degrees or radians. So now we will need a system of units. There are two different unit systems. The first one is called the MKS system. MKS means meter, kilogram, second system. where the length is measured in meters. The mass is measured in kilograms. And the time is measured in seconds. Then there is another system called the CGS system. CGS means centimeter, gram, second system. Here, the length is measured in centimeters. The mass is measured in grams. And the time is measured in seconds, like MKS.

Approximate lengths in meters (m) of a few entities are given below in the table

Distance	**Length (m)**
Tall Person	2×10^{0}
Cricket Ground	3×10^{2}
Radius of Earth	6.4×10^{6}
Earth to Sun	1.5×10^{11}
Radius of Universe	1×10^{26}
Thickness of Paper	1×10^{-4}
Diameter of Hydrogen atom	1×10^{-10}
Diameter of Proton	1×10^{-15}

For very large lengths, scientific notation is required and a power of 10 is used. So, for example, the size of the earth is 6.4×10^6 meters and the distance of the sun from the Earth is about 1.5×10^{11} meters. Similarly, the universe is very vast, but today we know that it is not unlimited, it is limited and its radius is about 1×10^{26} meters. Now, on the other hand, there are many small things in the world. For example, the thickness of a piece of paper is about 10^{-4} meters and atom, for example, a hydrogen atom, its radius is about 10^{-10} meters. The proton in it is 100,000 times smaller than that and its size is 10^{-15} meters. So, these are different scales of length.

Similarly, there are different scales of time and mass. The approximate time for particular events is given below.

Event	**Time (s)**
Light travels from Earth to the moon	1.3×10^0
One hour	3.6×10^3
One year	3.2×10^7
Age of universe	5×10^{17}
Open/close eyelid	1×10^0
One cycle of radio wave	1×10^{-8}

Light moves at a very fast speed, but light also needs time to go from Earth to the moon and this is about 1.3 seconds, i.e. 1.3×10^0 s. In an hour, there are 3600 seconds, i.e. 3.6×10^3 s. In a year, there are 3.2×10^7 s. Today, we also know that the universe was formed about 15 billion years ago and if written in seconds, then it becomes about 5×10^{17} s. On the other hand, there are such events in which the duration is very short. For example, if you blink your eyes, you can do this twice in a second, i.e. the frequency is 2 per second. For a radio wave (electromagnetic wave) in which the descent and ascent happens 10^8 in a second, i.e. its time duration is 1×10^{-8} seconds.

There are also different scales of mass as listed in the table below.

Object	Mass (kg)
Student	7×10^{1}
Car	1×10^{3}
Ship	1×10^{6}
Erath	6×10^{24}
Sun	2×10^{30}
Milky Way (Galaxy)	4×10^{41}
Dust Particle	1×10^{-9}
Oxygen Atom	3×10^{-25}
Electron	9×10^{-31}

For example, the weight of an ordinary person is about 70 kilograms, the weight of a car is about 1000 kilograms, while the weight of a ship is 1000 times more, i.e. 1×10^{6} kg. It is worth noting that powers of 10 are used to write very large as well as very small quantities. Earth's weight is 6×10^{24} kg and the sun are about 1 million times heavier; its weight is 2×10^{30} kg and if we calculate the weight of our galaxy, then that is about 4×10^{41} kg. Dust particle barely weighs 1×10^{-9} kg. An atom of oxygen is much lighter than this as its weight is about 3×10^{-25} kg and an electron is 1000 times lighter than that, and its weight is 9×10^{-31} kg.

1.5. <u>Conversion of Units</u>

Going from a one-unit system (like MKS or CGS) to another system is called conversion of units. There are some useful conversion factors i.e. 1 inch is equal to 2.54 centimeters and 1 meter is equal to 3.28 feet. Sometimes it is required to convert other quantities as well. For example, converting miles per hour (mi/hr) into meters per second (m/s). 1 mile per hour (mi/hr) can be easily converted to meter per second (m/s) using only two following conversions and cross multiplications:

$$1\ mi = 5280\ ft\ \rightarrow 1 = \frac{5280\ ft}{1\ mi}$$

$$1\ m = 3.28\ ft\ \rightarrow 1 = \frac{1\ m}{3.28\ ft}$$

$$1\ hr = 3600\ s \rightarrow 1 = \frac{1\ hr}{3600\ s}$$

So, multiplying all the above unities by $1\frac{mi}{hr}$

$$1\ \frac{mi}{hr}.1\ .1.1 = \left(1\ \frac{mi}{hr}\right).\left(\frac{5280\ ft}{1\ mi}\right).\left(\frac{1\ m}{3.28\ ft}\right).\left(\frac{1\ hr}{3600\ s}\right)$$

By cancelling units, we get

$$1\ \frac{mi}{hr} = \frac{5280}{3.28 \times 3600}\ \frac{m}{s} = 0.477\frac{m}{s}$$

This was also an example of dimensional analysis. Even though units were of miles per hour on the left side and meters per second on the right side, both were length over time [L/T]. So, the dimensions were equal in both cases. So, it becomes very important that the dimensions on the left side should be on the right side in every equation. If this is not the case, then it means that the equation cannot be correct.

For example, let's assume the following equation

$$d = vt^2$$

Where d is distance, v is velocity and t is time. Dimension on the left side of the equation is [L] and on the right side, it is $[LT^{-1}]\ [T^2]$ = [LT] which is not equal to the left side so this equation is incorrect. Explain whether the following equation(s) could possibly be right or wrong based on Dimensional analysis.

$$v^2 = u^2 + at$$

$$v^2 = u^2 + 3a^2t^2$$

Here *v* and *u* are velocities, *a* is acceleration, and *t* is time.

1.6. <u>Rules of Dimensions</u>

Dimensions (M, L, and T) can be considered as algebraic quantities. Likewise, symbols and dimensions can be added and subtracted but rules differ from conventional algebraic rules. Multiplication and division are applicable to dimensions. For example, in the case of length divided by length [L/L] dimension will cancel and the result will be a dimensionless quantity. Dimension for speed can be simplified to $[LT^{-1}]$ based on division. So [M], [L] and [T] can be divided and multiplied but cannot be added or subtracted from each other. For example, [L] + [T] is physically not realizable. Even $[T^2]$ cannot be added to [T]. So, addition and subtraction of the same dimension is only possible.

1.7. Accuracy

Whenever we measure a quantity its accuracy cannot be 100%. For example, if a certain length is measured using a conventional ruler to be 2.95 centimeters (cm), it cannot be measured as 2.952467 cm. This is because of the fact that measuring equipment always has inaccuracy associated with it so significant figures become relevant. It is not advised to do calculations using certain equipment that misrepresents their accuracy.

For example, if you are asked to measure the weight of 3 apples and then calculate the average weight. Its average weight cannot be measured at 550.234 grams. This is not possible because the equipment being used to measure weight is not 100% accurate. For this reason, we need to be aware of the fact that what level of accuracy is required for the measurement of an algebraic or an arithmetic quantity. Only an appropriate and reliable number of digits should be used for the measurement process.

If length is being measured by a ruler, then there can be at most 1% accuracy, not more than that. Some general rules should be considered for calculations. For example, when two numbers are added together if one number has an accuracy of 95.9 and the other has an accuracy of 39.32, then you add them together and it becomes 135.22. But it would be more appropriate to call its accuracy 135. So, there is no point in adding the decimal figures to it. It would not be of any benefit. When subtracting when multiplying and when dividing, for example, multiplying, one number has an accuracy of one figure of decimal i.e. 105.8 and the other has an accuracy of 31.4. If we multiply both, then we have six figures. But we should round off this. This means that after the decimal figure, the 0.12 does not have any meaning. We should leave it. So, if we divide 105.8 by 31.4, then we will get many decimal figures. But it would be appropriate to cut it off at 3.37 because the two numbers are known only to one decimal figure of accuracy. Therefore, 5 decimal figures after 3 do not give much information.

1.8. Orders of Magnitude:

Order of magnitudes means that we want to approximately calculate a quantity. In physics, it is often not possible to do exact calculations. Either we do not have enough information or the problem is very difficult or whatever. So, it is extremely important that we can estimate. Now, we would like to discuss the order of magnitude. Order of magnitude means that when we write in powers of 10, then if it is written near the nearest power of 10. Let us give you an example. A man's weight is approximately 100 kilograms. Now, I agree that 100 kilograms is of very few people. Usually, it is half of that. But writing it as 10 to the power of 2 kilograms can be useful in some situations. Similarly, if we take the weight of a child, then it can be approximately 10 kilograms. And if you take a cricket ball, then its weight is approximately 1 kilogram. Now, here we have only talked about the closest powers of 10. So, we are not saying that this is accurate, but as mentioned earlier, whenever an estimate is required, we can use this method.

Let's have a look at an example. How many seconds are there in a person's life? Let's calculate it. If a person is 80 years old, then you can calculate it from 365 days per year. Next, there are 24 hours in a day. And there are 60 minutes in an hour. And there are 60 seconds in a minute. So, calculate by multiplying these conversion factors (i.e. 80 x 365 x 24 x 60 x 60). As a result, we will get 2.5 x 10^9 seconds. On the other hand, if we estimate it, then assume a person's life is 100 years for the purpose of approximation. There are 365 days in a year but we assume 100 days per year which is the closest power of 10 and so on as shown in the table below. Upon multiplication of all conversion factors, we get 10^9 seconds. On one hand, we did a very accurate calculation to get 80 years which is equivalent to 2.5 x 10^9 seconds. And on the other hand, if we make these coarse estimates, then there is not much difference. In both cases, we get 10^9 seconds only. Although in one calculation we get 2.5 and in the other we get unity. But this difference is very small. Approximate reasoning sometimes gives us very useful results.

Conversion Factor	**Closest Power of 10**	**Closest Power of 10 (scientific notation)**

80 yr	100 yr	10^2 yr
365 days/yr	100 days/yr	10^2 days/yr
24 hr/day	10 hr/day	10^1 hr/day
60 min/hr	100 min/hr	10^2 min/hr
60 s/min	100 s/min	10^2 s/min

There once a news that armed men robbed 10 crore rupees (100 million rupees) from shopkeepers and ran away on a motorcycle. Now the question arises, can such a large amount be put on a motorcycle or not? To find out, let us do a small experiment. A balancing scale (as shown in the video lecture) is used to measure the weight of 10 currency notes each having a value of 1000 rupees. As seen in the video lecture, the weight of 10,000 rupees (10 notes x 1000 rupees/note) is 20 grams. This means that the weight of 100,000 rupees is 200 grams, which is equal to 0.2 kilograms. And the weight of 1 crore rupees is 20 kilograms. The weight of 10 crores is 200 kilograms. Now 200 kilograms is equal to the weight of 3 people. And it is not possible that such a large amount can be put on a motorcycle other than two people riding it. Therefore, there is something doubtful about the whole incident. This was a very ordinary example of scientific methodology. However, we will see in the upcoming lectures that this methodology is used in every field of physics, and it is called the scientific method.

Observations are made based on which an initial thought is created. After that, the hypothesis is born in the mind. For now, we do not know whether this hypothesis is right or wrong. Now, to test it, its predictable outcome has to be compared with more observations. It has to be experimentally verified. When a law is tested again and again and it is seen again and again that it is successful, means the hypothesis combined with the observations and they connect, then it gets the status of scientific theory. We will give you an example of Newton's theory of gravitation and Newton's theory of motion. Now, we will see in the next lectures, those lectures which are related to classical mechanics and which are associated with Newton's theory, what phenomena are achieved by those. But as we know, Einstein who came after Newton proved that Newton's theory cannot be completely correct. So, does this mean that Newton's theory is wrong? No, this does not mean at all. Now, if you apply Newton's theory to any particle, as long as that particle is not moving close to the speed of light, that is, it is not moving at this extreme speed, then Newton's theory can be used very easily and effectively. For example, when we send a spacecraft to a planet far away from the Earth, then we use Newton's laws. We do not need to use Einstein's laws. The theory of relativity created by Einstein is used only in those cases where particles' speed is close to the speed of light. For example, in research of particle physics particle accelerators are used and electrons move at a speed close to the speed of light.

Finally, I will say something which may seem a little strange to you. In this lecture, I emphasized that physics is related to the material world and that its purpose is to find out the rules and laws that are the basis of our universe. So, we are talking about materialistic things. But on the other hand, the concepts of physics are abstract and only exist in our minds. These concepts are the creation of our minds. Now, we would like to clarify this a little. In the upcoming lectures, you will hear again and again that this is a free particle, a free body, that this is a point, that this is a straight line. But there is no such thing as a free body. We say that there is nothing free on Earth because gravity pulls that body towards the Earth. So, take it a little farther from Earth, take it to the outer atmosphere. How far should we take it? Take it 1 crore miles away from the earth. Take it 100 crore miles away. But that will not be enough because there will always be a minute force that pulls that body towards the earth.

And if we take it too far, then there are galaxies and other planets, therefore, a completely free body does not exist. It exists only in one place and that is in our brains. And similarly, if we talk about a dot. We assume that a dot is a very tiny thing and we can make it by putting a pencil on paper. But if we look at it under a microscope, the higher the magnification bigger the dot we will observe. Now a pencil's tip cannot be made so fine because a pencil's tip itself is made up of atoms and atoms themselves are not dots. Therefore, there is no dot in the world and there is no straight line either. The point is that the concepts of physics are in one's mind. Therefore, these concepts are created and evolve, which is why physics is a progressive discipline. In which there is progress all the time. And this is the most interesting thing about physics. So, I hope that in the next lectures, you will feel that this is an interesting topic. That there is progress in it and that it invokes thinking. To learn this subject, you will have to work a little hard. As long as you do not solve problems by yourself. You have to believe that you can solve every problem by yourself. And the more problems you solve, your physics skills will increase accordingly. Secondly, do not assume that you will be able to solve all your problems with a single course be it this course or any other course. There is a vast ocean of knowledge in this world and we can pick up a drop of it at one time. It is a lifelong effort to understand physics and science. And we hope that you will also be a part of this effort. The journey of physics will continue in the next lecture. We will meet again at that time. With your permission, goodbye.

Physics-PHY101-Lecture #02

Kinematics

"Kinematic" is derived from the Greek word "kinesis," which means to move or motion. Our world is full of motion. For example, cars move, birds can fly, and fish can move underwater. It is necessary to understand motion and its causes.

The main topics of this lecture are:

- Displacement
- Velocity
- Acceleration

Whenever we discuss the motion of a body, it is essential to locate its position. To achieve this, we need an origin point. For example, a point x has a value of x = 0 when stationary, but as it starts moving, its values begin to increase.

If the position of the vector x depends on "t," then we can express its position as x(t), where x is referred to as a function of "t." (Think of a function as a factory where you input raw material and receive products, like inputting a number into a function and obtaining a result as output.)

In kinematics, we require a specific function denoted as x (t). This function allows us to input a value of t and obtain the position of a moving particle.

Now, let's delve into the concept of "Displacement."

Displacement:

As we know, the position at time t is denoted as x(t). Here, "t" can be any number, such as 9, 7, 6, etc. Similarly, when a particle moves at different times "t", it may be denoted with different notations, such as $x(t_1)$ and $x(t_2)$, and its function may be defined as x(t).

Now, when we subtract these two quantities, we will obtain a value $\triangle x$. The displacement $\triangle x$ in time interval $\triangle t = t_2 - t_1$ is:

$$\triangle x = x(t_2) - x(t_1)$$

Sometimes we don't want to write equation as $x(t_1)$ and $x(t_2)$ so we do write the above equation as:

$$\triangle x = x_2 - x_1$$

This is what we call displacement. Now consider that a particle is moving in one dimension. Imagine a graph of particle's position along the x-axis as a function of time. From the Figure. 2.1 we can see that when the particle is at position x_1 the time is t_1 and when at x_2 the time is t_2.

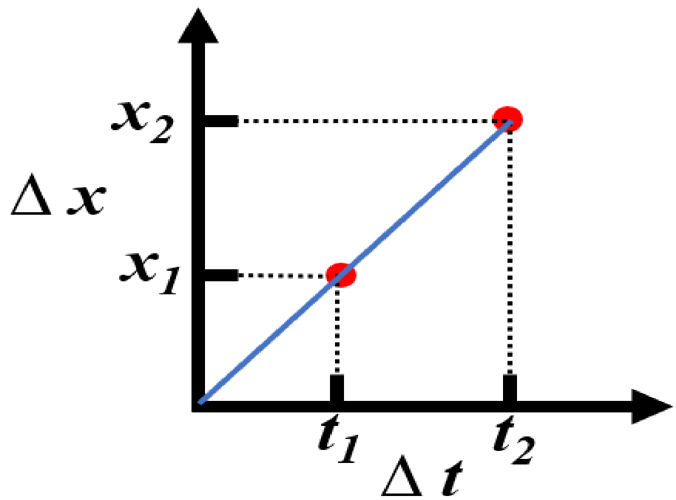


Figure 2.1. Displacement vs. time graph of an object.

If an object as shown in figure 1. is at 10 m from origin at t_1 and reach at 30 m at time t_2, then, we can define displacement $\triangle x$ as:

$$\triangle x = x_2 - x_1$$

$$=30 - 10$$

$$\triangle x = 20 \text{ m}$$

But if the object at $t_1$is at 30 m and reach at 10 m at t_2, then the magnitude will remain same but negative sign appears due to net displacement in negative direction.

$$\triangle x = x_2 - x_1$$

$$=10 - 30$$

$$\triangle x = - 20 \text{ m}$$

This is what we call displacement, which can be both positive and negative.

Speed and Velocity:

Speed and velocity measure how position changes with time. There are two major concepts to consider. First, let's look at average speed:

$$\text{Average speed} = \frac{\text{Total distance}}{\text{Total time}}$$

If we divide the particle's total distance travelled by the total time taken, we can determine the average speed of that particle. Average speed can vary, being either maximum or minimum. Meanwhile, average velocity can be defined as:

$$\text{Average velocity} = \frac{\text{Total displacement}}{\text{Total time}}$$

$$= \frac{x_2 - x_1}{t_2 - t_1}$$

$$v = \frac{\Delta x}{\Delta t}$$

Note that distance is always positive, while displacement can be positive or negative. It's important to clarify that this is average velocity because we are dividing displacement by time, not distance by time. This is also known as slope or gradient. For example, consider a car moving on a slope – the steeper the slope, the greater the gradient.

Now, consider the Figure. 2.2 (a) where the distance covered by two points creates a slope from which average velocity is calculated.

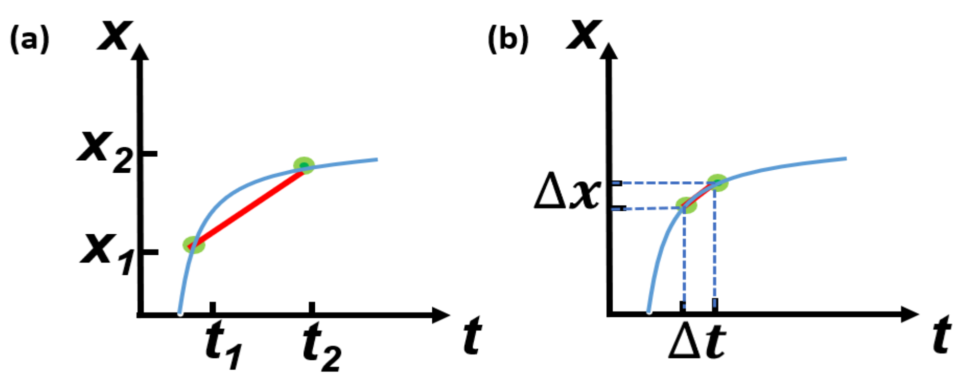


Figure 2.2. Displacement vs. time graph of an object presenting (a) constant velocity, (b) Instantaneous velocity.

In contrast**, instantaneous velocity** is determined by taking two times very close to each other, with the interval between them approaching zero. In Figure 2.2 (b). if we take two values so close to each other that they approach zero, we can call it instantaneous velocity. Now, let's discuss what is meant by "being too close." Being too close implies approaching a distance of zero. (It's important to note that approaching zero doesn’t mean you are making the distance zero.)

Acceleration

Acceleration measures how the velocity changes with time. As we have defined average velocity, we can also define average acceleration as:

$$\text{Average acceleration} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\triangle v}{\triangle t}$$

We can say that average acceleration is change in velocity divided by time taken to undergo that change. It is also known as slope of graph of velocity against time.

$$a = \lim_{\triangle t \to 0} \frac{\triangle v}{\triangle t}$$

Now to understand definition better we will again construct a graph as shown in Figure. 2.3:

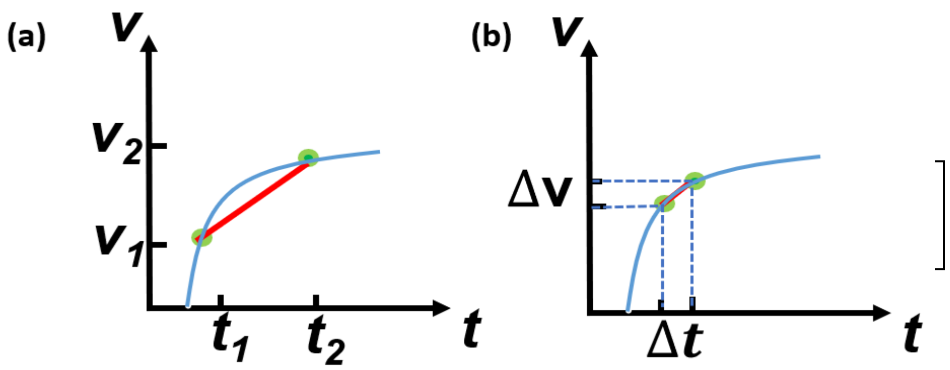


Figure 2.3. Velocity vs. time graph of an object presenting (a) constant acceleration, (b) Instantaneous acceleration.

When t_1 and t_2 come closer, v_1 and v_2 also come closer and we draw a tangent line at that point which is known as slope of tangent acceleration as shown in Figure 2.3 (b). Remember that we are approaching t to zero, not making it zero.

The acceleration can be positive and negative. Moreover, it is not necessary for acceleration to be in the direction of velocity. They can have different direction as shown in Figure 2.4, in which velocity of train is decreasing as time passes which presents deacceleration. By convention we take positive x-axis in right direction and negative x-axis in left direction. If the velocity of an object is increasing in positive x-axis direction than its **acceleration is positive.** If the velocity of the object is increasing in negative x-axis, then its **acceleration will be negative**, but it never be deacceleration or retardation. The SI unit of acceleration is m s^{-2}.

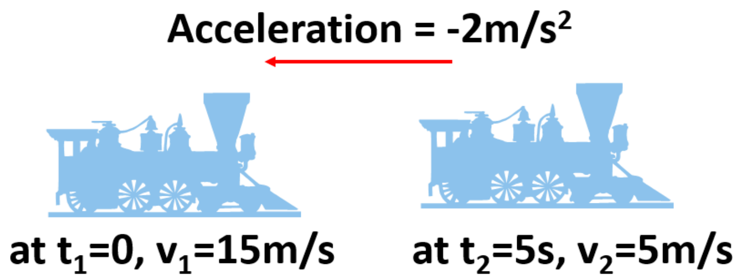


Figure 2.4. The negative acceleration of a train.

Constant Acceleration

When an object's acceleration stays constant over time, it's referred to as having constant acceleration. Stated differently, the object's velocity changes at a constant rate.

Mathematically:

Let us talk about constant acceleration in more detail. For this purpose, let's do simple calculations.

For convenience, take:

$$t_1 = 0,\ t_2 = \mathrm{t}$$

Then: $$x_1 = x_o \text{ and } x_2 = x$$

$$v_1 = v_o \text{ and } v_2 = v$$

At time t_1, the velocity of particle is v_1 which is v_o, and at time t_2 its velocity becomes $v_2 = v$. If acceleration is constant, then we can write the average acceleration as:

$$a_{vg} = \frac{v_2 - v_1}{t_2 - t_1}$$

And as acceleration is constant, so, v is equals to,

$$v = v_o + at \text{ ………. (2)}$$

$$\therefore a = v/t, \text{ so } v = at$$

The average velocity as "it is the average of v and v_o divided by 2".

$$v_{av} = \frac{1}{2}(v_o + v)$$

If we want to write in previous notation, then:

$$v_{av} = \frac{x_2 - x_1}{t_2 - t_1}$$

If we substitute values from equation given above, then:

$$\frac{x - x_o}{t - 0} = \frac{v + v_o}{2}$$

$$\frac{x - x_o}{t} = \frac{v_o + at + v_0}{2}$$

$$\frac{x - x_o}{t} = v_o + \frac{1}{2}at$$

$$x = x_0 + v_o t + \frac{1}{2}at^2$$

As we have discussed earlier that x is function of t. So, by putting t = 0 in above equation, we will have value of x as x_o, which is the initial point from, we started.

From all that procedure we have got two main equations:

$$x = x_0 + v_o t + \frac{1}{2} a t^2 .... \text{(A)}$$

$$v = v_o + at \text{ .... (B)}$$

Now let's plot again them on a graph for better understanding:

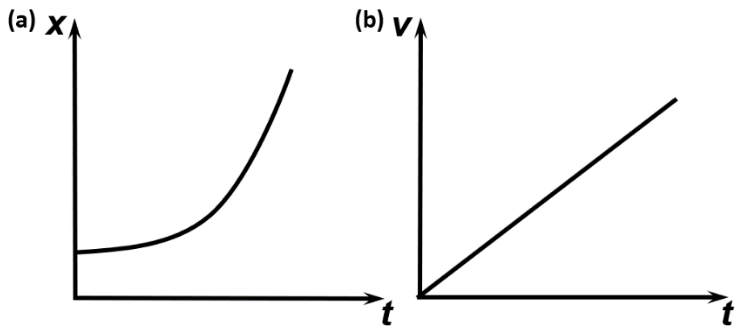


Figure 2.5. (a) Displacement vs. time, (b) velocity vs. time graph of an object with constant acceleration.

From the graphs in Figure. 2.5 it can be observed that as the time increases particle start moving away from the origin and its velocity increases linearly, while its displacement increases quadratically.

From equation (A) and (B),

$$t = \left[\frac{v - v_o}{a}\right]$$

$$x = x_o + v_o \left[\frac{v - v_o}{a}\right] + \frac{1}{2} a \left[\frac{v - v_o}{a}\right]^2$$

$$x - x_o = \frac{v_o v - v_o^2}{a} + \frac{v^2 + v_o^2 - 2 v v_o}{2a}$$

$$x - x_o = \frac{2 v_o v - 2 v_o^2 + v^2 + v_o^2 - 2 v v_o}{2a}$$

$$2a(x - x_o) = v^2 - v_o^2$$

$$v^2 = v_0^2 + 2a(x - x_0)$$

Where v is function of x. This equation tells us when the value of x changes, the value of v also changes.

Example:

For better understanding, let's consider an example with cars. In the context of cars equipped with a speedometer ranging from 0 to 160 km/h, the initial state of the car is at rest, denoted as $v = 0$. When the car is started, its speed, let's consider it as = 70 km/h, increases to this value and moves at this speed for some time. Afterward, it changes its velocity to 60 km/h, indicating negative acceleration. If we bring the car to a stop, its velocity will gradually decrease and eventually become zero, signifying that the car is at rest.

Introduction to Vectors

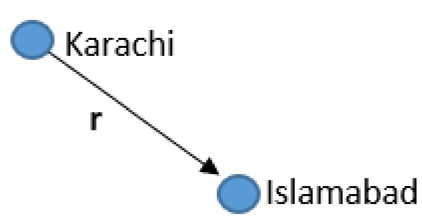
Up until now, the discussion has focused on motion in one dimension. Now, the exploration will extend to motion in two and three dimensions. Vectors exhibit two primary properties:

- Magnitude
- Direction

Consider the position vector r in two dimensions. For example, envision drawing a vector on a map of Pakistan, where one end (origin) is in Karachi, and the other end is in Islamabad. To achieve this, two essential considerations are needed:

- Set the origin at Karachi.
- Choose coordinates for distance (in kilometres) and direction (North, West, East, South)

Two sets of numbers or coordinates are necessary. Subsequently, two arrows will be drawn, representing vectors originating from Islamabad and Karachi.



Consider a car moving on a non-uniform surface. We will notice that’s its velocity is changing its direction. Although it is moving with constant speed, but its direction keep changing. The components of vector in two dimensions can be expressed as,

$$r_X = x = rcos\theta$$

$$r_y = y = rsin\theta$$

The position vector is written as,

$$r = (r_X, r_y) = (x, y)$$

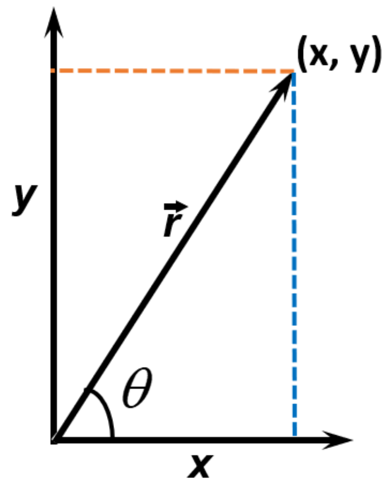


Figure 2.6. The point (x, y) is in two-dimensional plane with origin (0,0). So, we write this vector as r or in two coordinates (x, y). We call these coordinates as components.

For x = rcos(θ):

From the first trigonometric identity, we have:

$$\cos(\theta) = x/r$$

Multiplying both sides by r, we get:

$$r\cos(\theta) = x$$

Therefore, x = rcos(θ)

For y = rsin(θ):

Similarly, from the second trigonometric identity, we have:

$\sin(\theta) = y/r$

Multiplying both sides by r, we get:

$r\sin(\theta) = y$

Therefore, $y = r\sin(\theta)$.

Mathematically we write them as x and y as:

$$r_X = x = r cos\theta$$
$$r_y = y = r sin\theta$$

We are already familiar from these basic trigonometric formulas.

$$x^2 + y^2 = r^2(\sin^2\theta + \cos^2\theta)$$
$$x^2 + y^2 = r^2(1) = r^2$$

Which proves that if we square two rectangular components and then add them, we get a constant number.

Although the length of the vector doesn't depend on the direction of vectors. We can write it mathematically as:

Where $\mathbf{r} = |\mathbf{r}|$

$$\theta = \tan^{-1}\frac{y}{x}$$

Magnitude of r can be found by Pythagorean theorem.

$$|\mathbf{r}| = r = \sqrt{x^2 + y^2}$$

Where r is independent of value of angle.

Vectors can be of different types. We have discussed position vector till now. For instance, we have also discussed velocity vector with example of moving car. Acceleration is a vector itself although it made up from velocity but still it is independent of it.

Vector addition:

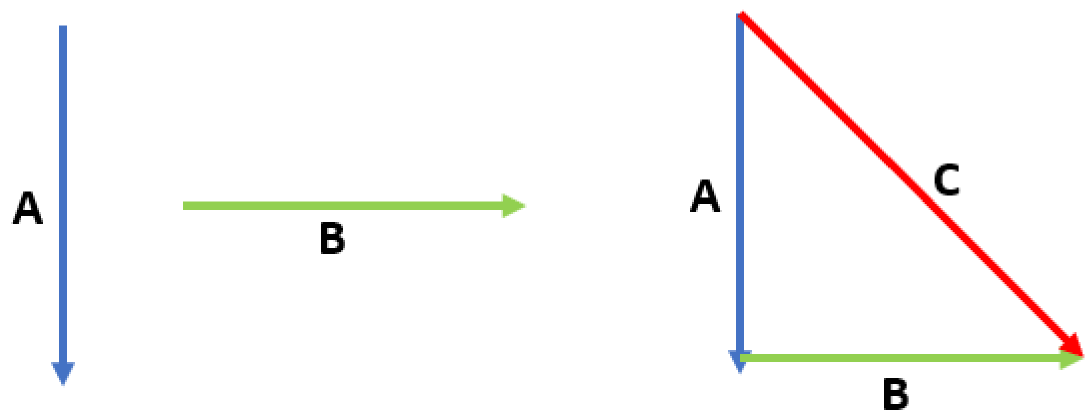
We can add vectors in one dimension. For example, consider a vector in one direction with a length of 10 m and another in the opposite direction with a length of 20 m.

20 m −10 m = 10 m

So, adding vectors in one dimension is an easy task. Now, let’s discuss two dimensions.
We want to add two vectors **A and B and** resultant as **C.** As,

$$\mathbf{C} = \mathbf{A}+\mathbf{B}$$

Graphically if we add these two vectors, they form a triangle with a resultant vector **C.**



We can arrange the vectors any way we want if we maintain their length and direction.

Parallelogram method for vector addition:

We can also add two vectors by parallelogram method, Continuing from previous example,

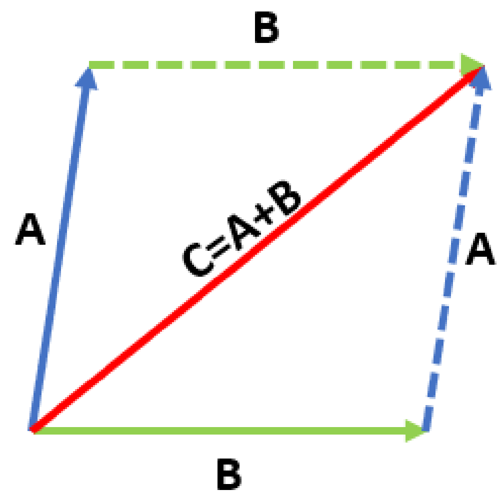


Figure 2.7. We translate the vectors A and B and form the sides of a parallelogram. The diagonal between them represents their resultant vector, denoted as C. This method is known as the parallelogram method.

In the parallelogram method for vector addition, the vectors are translated, (i.e., moved) to a common origin and the parallelogram constructed as follows:

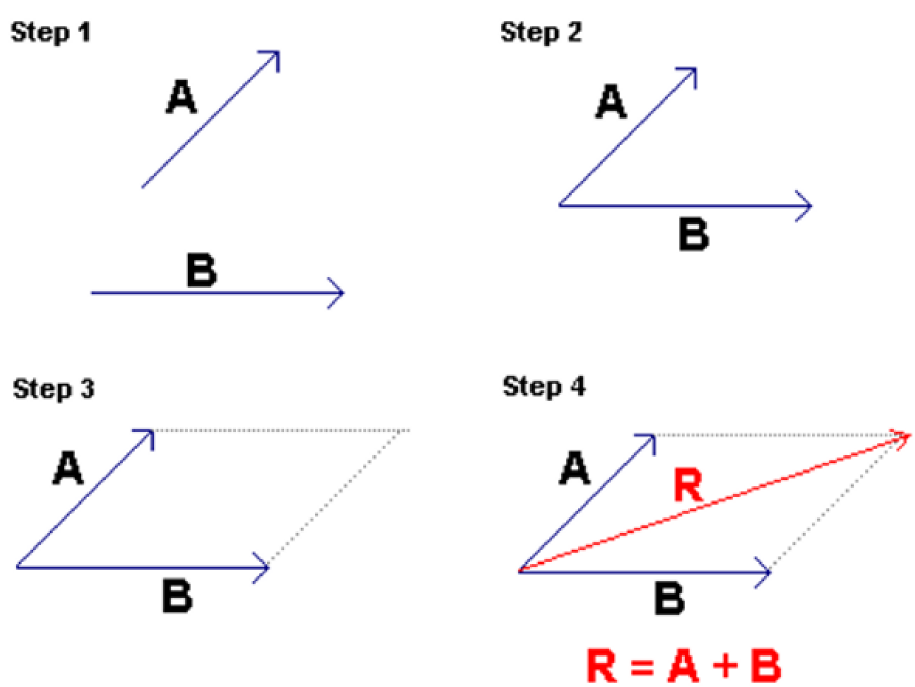


Figure 2.8. The resultant R is the diagonal of the parallelogram drawn from the common origin.

Component Method:

The components of a vector are those vectors which, when added together, give the original vector. The sum of the components of two vectors is equal to the sum of these two vectors. When we add two vectors their components do get added.

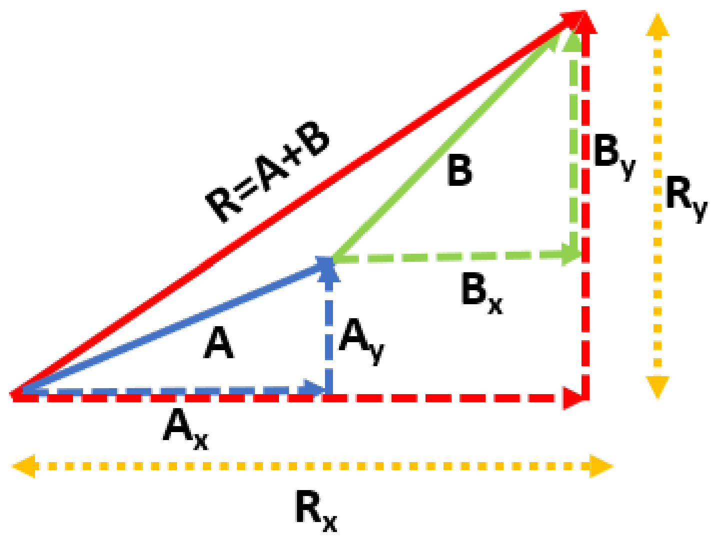


Figure 2.9. Figure presents the concept of vector addition by component method.

Here vector **A** and **B** have two components along x-axis and along y-axis we get a resultant.

vector **R** by adding these vectors components in x and y direction.

Summarizing it as,

- Add components.

$$\boldsymbol{R_x} = \boldsymbol{A_x} + \boldsymbol{B_x}$$

$$\boldsymbol{R_y} = \boldsymbol{A_y} + \boldsymbol{B_y}$$

Bold letters present the vector nature.

- Then calculate the magnitude from following formula,

$$R = \sqrt{{R_x}^2 + {R_y}^2}$$

- Calculate the angle by using formula.

$$\theta = tan^{-1}\frac{R_y}{R_X}$$

| Physics | PHY101_Lecture#03

Kinematics-II

Student Learning Outcomes

After listening the lecture, students will be able to,

- Understanding Position Functions
- Analyzing Constants and Dimensions
- Exploring Derivatives and Velocity
- Motion in Two Dimensions
- Vector Operations and Applications

Here are some questions for the students,

1. Is it possible for a car to have acceleration while at rest?
2. Are velocity and acceleration always in the same direction? If a car is accelerating, does it acceleration with constant acceleration?

To comprehend these questions, one must understand the term 'motion.' In this lecture, we will delve into the concept of motion and explore some functions that elucidate this term.

Function

The function represents a machine where an input is inserted, producing an output. The emphasis is specifically on the position function, denoted as X, which is a function of time, t.

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \ldots..$$

Where $c_0, c_{1,} c_{2,} c_3$ are constants and remain fixed. 'Constant' refers to values that do not change over time. It is essential to focus initially on the dimensions on both the left and right side of the equation, ensuring their equality.

Upon observing the dimension on the left side, it is identified as length, even if t is not explicitly stated. This implies that the function describes the movement of a particle in a straight line, indicating the distance covered over time, t. So, this is a function that says the particle is moving in a straight line and the distance it covers in time t. This is what we called $x(t)$.

The dimensions must match,

$$Dim[c_0] = L$$
$$Dim[c_1] = L/T$$
$$Dim[c_2] = L/T^2$$
$$Dim[c_3] = L/T^3$$

And obviously, when the value of t is zero, then x is C_0. So, the dimension of C_0 is equal to L. Now, focus on the dimension of C_1. The dimension of C_1 is equal to L/T. Similarly, the dimension of C_2 is L/T^2, and the dimension of C_3 is L/T^3. In short, if the values of C_0, C_1, C_2, etc., are unknown, then at any point, the value of X(t) can be calculated. If dimensions of constants, and function $x(t)$ are known, the dimensions of "t" can easily be determined.

Derivatives

Derivative is the invention of Newton. In differential calculus we take the differences and then dividing them by the difference the ratio of both becomes finite. Let's try to understand the concept of derivatives, which represents the change. This change might be in position, velocity, acceleration etc. The change in position "x" w.r.t time is written in form of derivative is,

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

Never make the mistake of cancelling the "d" in numerator and denominator; this is not possible and doesn't hold any meaning.

The meaning of Δt is clear in this definition. On one value of "t", you take a value of x, and then on the second value of "t", you take another value of x. Then, take the difference, which we call Δx, and divide by Δt. Here we emphasize that Δt should be very small. You might ask how small?

0.1 is not enough? no. Then 0.01 is not enough either. Even if you take 0.0001, that is still not enough. But if we make it small enough so that Δt approaches 0."

We will discuss a little detail about it now,

$$x(t) = t$$

$$\Delta x = x(t + \Delta t) - x(t)$$

$$= (t + \Delta t) - t = \Delta t$$

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = 1$$

From this it is clear that the function whose value is 1 is called a linear function and the derivative of linear function is constant.

Let's calculate the derivative of "t^2",

$$x(t) = t^2$$

$$\Delta x = x(t + \Delta t) - x(t)$$

$$\Delta x = (t + \Delta t)^2 - t^2$$

$$= t^2 + (\Delta t)^2 + 2t\Delta t - t^2$$

$$\frac{\Delta x}{\Delta t} = \Delta t + 2t$$

$$\Rightarrow \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = 2t$$

This case is different from the previous case. Now dx/dt the derivative of x of with respect to "t" is not constant, but it depends on function. And we know that dx/dt is known as speed or velocity. As in this case dx/dt is not constant but proportional to t.

Similarly,

$$x(t) = t^3$$

$$\Delta x = (t + \Delta t)^3 - t^3$$

$$= t^3 + 3t^2\Delta t + 3t\Delta t^2 + \Delta t^3 - t^3$$

$$\frac{\Delta x}{\Delta t} = (\Delta t)^2 + 3t^2 + 3t\Delta t$$

$$\Rightarrow \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = 3t^2$$

Now generalize it for t^4, t^5…. etc.

consider, the function with power “n”, where n=integer

$$x(t) = t^n$$

$$then:$$

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = nt^{n-1}$$

Then value of dx/dt is nt^{n-1}. Look for previous cases when n = 1 value of dx/dt is 1. And for n = 2 value of dx/dt is 2t. And for n = 0 the value of dx/dt should be 0. The derivative of a constant is always zero.

Geometrical interpretation of derivative

When value of t is increased to Δt, the value of x increases as well as Δx. As if dx/dt is like a gradient and in gradients, when you go a bit horizontal, you also go a bit vertical, resulting in a slope.

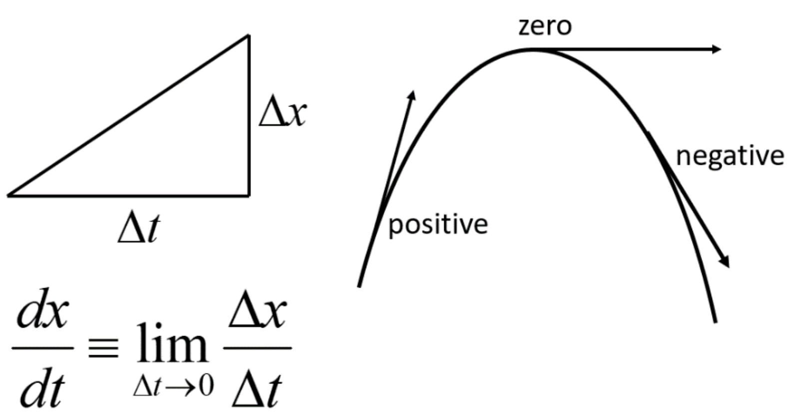


Figure 3.6. present the concept of derivatives.

In Figure 3.1. Three arrows represent the change w.r.t time. The arrow pointing upward indicates a positive gradient, meaning its derivative (dx/dt) is positive or the rate of change with respect to time (t) is positive. As we progress to the right, the value of t increases, and the arrow's direction changes. It transitions from an upward orientation to a horizontal one and eventually moves downward. Consequently, the sign of the derivative changes from positive to zero and then from zero to negative. This shift in sign signifies the geometrical significance of the derivative.

The second eq. of motion is,

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

Now let’s see how the derivative formulas we have drawn apply to this.

Now differentiate x with respect to t

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$
$$\frac{dx}{dt} = 0 + v_0 + \frac{1}{2} a(2t)$$
$$\Rightarrow v = \frac{dx}{dt} = v_0 + at$$
$$\frac{dv}{dt} = 0 + a = a$$

x_0, v_0 is the initial displacement and velocity, respectively.

Now answer the question asked by a student about how the car will accelerate when it is parked. A car at rest can be accelerating very fast

$$v = at$$
$$\frac{dv}{dt} = a \neq 0$$

The car is not moving at t = 0, then it starts to move with some acceleration. So, from here it is clear that speed is a separate thing and acceleration is separate thing. It is possible that the object is moving in a positive direction with a constant velocity, while its acceleration is directed in the negative direction. Both speed and acceleration are always considered relative to a specific origin.

Now take the example of a stone. Stone always falls downward. A stone can be at rest yet accelerating.

$$v = -gt$$

$$\frac{dv}{dt} = -g \neq 0$$

The value of "g" does not remain constant at 9.8 m/s^2. If an object goes upward, it will decrease, and as they go further up, it will decrease even more. If an object goes out into space, the value of "g" will be zero. However, it's important to note that the units of "g" are in meters per second squared. Since we live on Earth, it would be beneficial to remember this value as 9.8 m/s^2.

A useful notation:

$$\frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$$
$$= \frac{d^2x}{dt^2}$$

When we take second derivative of x then we write it as: $\frac{d^2x}{dt^2}$

Remember that speed is $\frac{dx}{dt}$ and acceleration is $\frac{d^2x}{dt^2}$.

Motion in 2-dimension

Let's discuss some characteristics of vectors. Each vector has a specific direction and a certain length. Now, there are special vectors known as unit vectors. Their characteristic is that their length is equal to one, and they only indicate directions. For example, one unit vector may point in a certain direction, while another may be perpendicular to it. A unit vector is a vector with a magnitude of 1 (no units), and it is obtained by dividing a vector by its length or magnitude.

$$\hat{A} = \bar{A}/A$$

Example of unit vector are $\hat{\imath}$ and $\hat{\jmath}$ in 2 dimensions. The vector i and j are perpendicular to each other, and their magnitude is 1. The resultant vector is written as,

$$\vec{A} = A_x\hat{\imath} + A_Y\hat{\jmath}$$

Resolution of vectors into its components

The vector component along the x direction is called x-component and the component along the y direction is called y-component.

$$\vec{A} = A_x\hat{\imath} + A_Y\hat{\jmath}$$

In this way, we can resolve a vector into its components as shown in Figure 3.2. Resolution of vectors means breaking down a vector into its components. We can do this in 3D too, except that it requires three vectors, î, $\hat{\jmath}$ and $\hat{k}$.

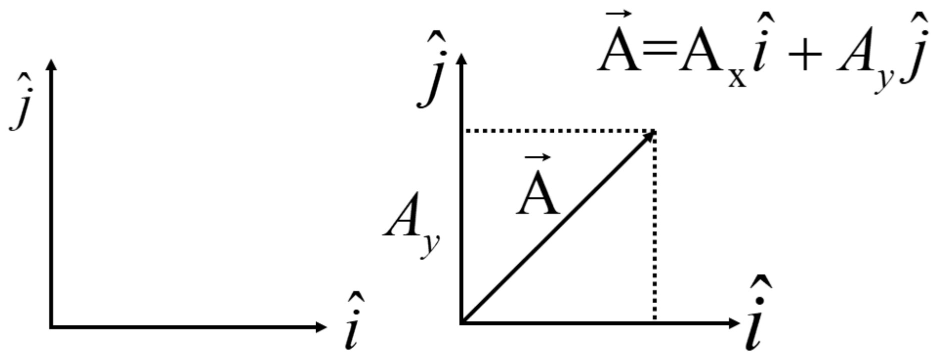


Figure 3.7. Illustrate the concept of addition of vectors.

Velocity in 2 dimensions

We have already discussed velocity in 1 dimension in detail. Velocity in 2 dimensions come from differentiating a displacement vector in 2 dimensions.

$$\vec{r} = x(t)\hat{\imath} + y(t)\hat{\jmath}$$
$$\vec{v} = \frac{\overrightarrow{dr}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath}$$
$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$$

Acceleration in 2 dimensions

Acceleration is the rate of change of velocity. When the velocity in 2-dimension is differentiated with respect to time, the acceleration is

$$\vec{a} = \frac{d\vec{v}}{dt}$$
$$= \frac{dv_x}{dt}\hat{\imath} + \frac{dv_y}{dt}\hat{\jmath}$$
$$= a_x\hat{\imath} + a_y\hat{\jmath}$$

Addition of vectors

Two vectors **A** and **B** can be added by head to tail rule. Mathematically,

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$$

$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath}$$

$$\vec{R} = \vec{A} + \vec{B}$$

$$= \left(A_x\hat{\imath} + A_y\hat{\jmath}\right) + \left(B_x\hat{\imath} + B_y\hat{\jmath}\right)$$

$$= (A_x + B_x)\hat{\imath} + \left(A_y + B_y\right)\hat{\jmath}$$

$$= R_x\hat{\imath} + R_y\hat{\jmath}$$

Example

$$\vec{A} = 6\hat{\imath} + 5\hat{\jmath}$$

$$\vec{B} = 8\hat{\imath} + 7\hat{\jmath}$$

What is the magnitude of of $2\vec{A} - \vec{B}$?

Letting

$$\vec{R} = 2\vec{A} - \vec{B}$$

$$= 2\ (\ 6\hat{\imath} + 5\hat{\jmath}) - (\ 8\hat{\imath} + 7\hat{\jmath})$$

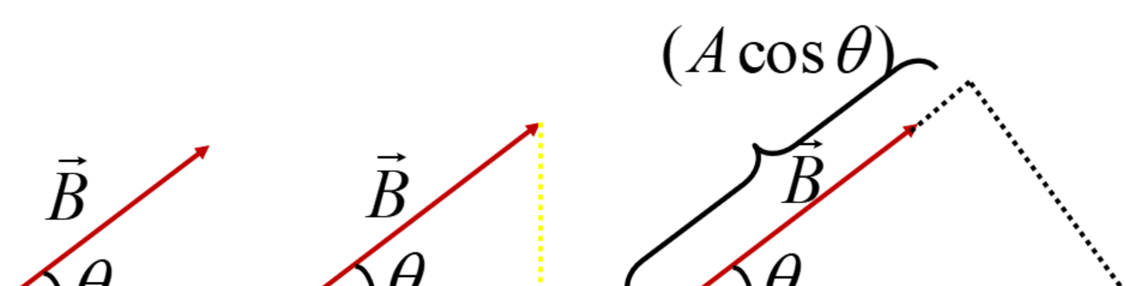
$$= (\ 12 - 8\ )\hat{\imath} + (\ 10 - 7\ )\ \hat{\jmath}$$

$$\vec{R}\ = 4\hat{\imath} + 3\hat{\jmath}$$

The magnitude is,

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{4^2 + 3^2} = 5$$

Consider two vectors $\vec{A}$ and $\vec{B}$ making an angle θ with each other as shown in Figure 3.3,



The scalar product of $\vec{A}$ and $\vec{B}$ is defined as:

$$\vec{A}.\vec{B} = AB\cos\theta$$

We can also write it as:

$$\vec{A}.\vec{B} = (A)(B\cos\theta)$$

= (length of A) × (projection of B on A)

Scalar product of unit vectors

The dot product of vector with itself is always 1, and dot product of two mutually perpendicular vectors is always zero.

$$\hat{i}.\hat{i} = \hat{j}.\hat{j} = (1)(1)\cos(0) = 1$$
$$\hat{i}.\hat{j} = (1)(1)\cos(90^{\circ}) = 0$$

We can take dot product easily between any two vectors.

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$$
$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath}$$
$$\vec{A}.\vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath}).(B_x\hat{\imath} + B_y\hat{\jmath})$$
$$= A_xB_x\hat{\imath}.\hat{\imath} + A_xB_y\hat{\imath}.\hat{\jmath} + A_tB_x\hat{\jmath}.\hat{\imath} + A_yB_y\hat{\jmath}.\hat{\jmath}$$
$$= A_xB_x + A_yB_y$$

Now you can do all these things in the same way in 3 dimensions. There would be only one more unit vector needed which is $\hat{k}$.

Generalization in 3 dimensions

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$

$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

$$\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

Application:

Projectile Motion:

Consider the example of a ball being thrown. When the ball is thrown, it follows a trajectory until it reaches its destination. The ball's velocity is composed of two distinct components: one in the y-direction and another in the x-direction. Notably, the x and y components of velocity are independent of each other. Acceleration is a factor affecting the ball's motion. Upon release, the ball descends in the negative y-direction, experiencing acceleration in the y-direction. However, there is no acceleration acting in the x-direction. For the sake of clarity, let's denote the acceleration of the ball in the y-direction as "a_y."

- Acceleration along y-axis is $a_y = -g$
- Velocity along x is constant
- Acceleration along x-axis is $a_x = 0$

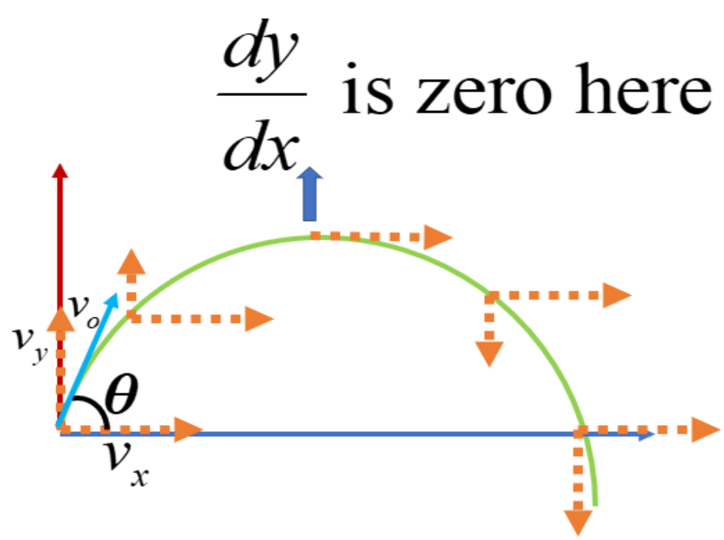


Figure 3.9. Shows the projectile motion of an object at various instants.

$$V_{0x} = V_0 \cos\theta$$
$$V_{0y} = V_0 \sin\theta$$

As presented in Figure 3.4 the velocity V of the ball is represented by two components: horizontal V_x and vertical V_y. As the ball traverses its trajectory, its velocity undergoes changes. Notably, the value of V_x remains constant because the acceleration a_x in the horizontal x direction is equal to zero, indicating no acceleration along the x-direction. In contrast, the value of V_y changes as the ball moves. At its highest point in the trajectory, V_y becomes zero. As discussed earlier, the gradient dy/dx is also zero at this point. As x increases beyond this point, dy/dx continues to be zero. Subsequently, the ball descends, causing its vertical component V_y to become negative. V_y eventually reaches zero again at a lower point in the trajectory, and further descent results in negative values for V_y. A more detailed examination using mathematical formulas will provide further insight into this phenomenon.

Along x-axis

As the variable "x" undergoes change, it is important to note that when "x" is constant, it equals the product of velocity (v) and time (t). The initial velocity "v" is the same as the velocity with which the ball was initially thrown. When the ball is thrown, the component along the horizontal direction remains constant when time is equal to zero. At this point, the value of "x" is zero, and this choice of coordinates is convenient, as it aligns with the starting point of the ball. While alternative coordinate choices are valid, setting "x" to zero at "t" not equal to zero proves to be a practical choice.

Examining this concept in mathematical terms, the relationship is described through the following formulas.

X direction

$$V_x = V_{0x}$$
$$x = x_0 + V_{0x}t$$
$$a_x = 0$$

Y-direction

$$a_y = -g$$
$$V_y = V_{0y} - gt$$
$$y = y_0 + V_{0y}t - \frac{1}{2}gt^2$$

How much its velocity changes depend on the value of “t”. In the equation, $V_y = V_{0y} - gt$, if “t” is equal to zero, then the values of V_y will be equal to the initial vertical velocity, but with the increase of “t”, they are decreasing and finally when they reach their maximum, where V_y will be zero.

Consider the following scenario: an inquiry into the maximum height a ball can attain when thrown lightly versus when thrown with greater force. The analysis suggests that a faster throw results in a higher ascent. To quantify this, calculations are performed with the objective of determining the optimal height. Subsequently, attention is turned to calculating the maximum distance the ball can cover in the horizontal (x) direction—the maximum range. To address these questions, algebraic methods are employed to derive the necessary equations and relationships that govern the ball's trajectory.

$$V_{0x} = V_0 \cos\theta$$
$$V_{0y} = V_0 \sin\theta$$

$$V_y = V_{0y} - gt$$
$$v_y = 0$$
$$v_0 \sin\theta - gt = 0$$
$$t = \frac{v_0 \sin\theta}{g}$$
$$y = (v_0 \sin\theta)t - \frac{1}{2}gt^2$$
$$H = (v_0 \sin\theta)\left(\frac{v_0 \sin\theta}{g}\right) - \frac{1}{2}g\left(\frac{v_0 \sin\theta}{g}\right)^2$$

$$\boldsymbol{H = \frac{(v_0 \sin\theta)^2}{2g}}$$

In the quest to achieve the maximum height for a thrown ball, it is essential to set the angle 90, as the sine function attains its maximum value at 90 degrees. Thus, to propel the ball to its highest point, aligning it straight upward is optimal. On the other hand, if the objective is to maximize the horizontal distance covered by the ball, the strategy involves achieving the maximum range "R". Given that the speed of the ball remains constant in the horizontal direction, the distance covered "x" in time "t" is crucial. At two distinct points in time, "y" attains the value of zero: firstly, at the release point (initial point) and secondly, at the landing point (final point). Consequently, there are two unique values of "x" where "y" equals zero. The first is evident at "x = 0", and the second, denoted as x = R, represents the distance from the starting to the ending point. Notably, the choice of θ is pivotal for achieving the maximum range. To optimize the horizontal distance covered, it is recommended to throw the ball at an angle of $\theta = 45^{o}$.

$$x = (v_0 \cos\theta)t$$

$$t = \frac{x}{(v_0 \cos\theta)}$$

$$y = (v_0 \sin\theta)t - \frac{1}{2}gt^2$$

$$= (v_0 \sin\theta)\frac{x}{(v_0 \cos\theta)} - \frac{1}{2}g\left(\frac{x}{(v_0 \cos\theta)}\right)^2$$

$$= x\tan\theta - x^2\left(\frac{g\sec^2\theta}{2{v_0}^2}\right)$$

$$y = x\left[\tan\theta - x\left(\frac{g}{2{v_0}^2\cos^2\theta}\right)\right] = 0$$

This equation has two solutions for x,

$$x = 0, \qquad and \qquad x = R\ (range)$$

$$\left[\tan\theta - R\left(\frac{g}{2{v_0}^2\cos^2\theta}\right)\right] = 0$$

$$R\left(\frac{g}{2{v_0}^2\cos^2\theta}\right) = \tan\theta$$

$$R = \frac{2{v_0}^2\cos^2\theta}{g}.\tan\theta$$

$$R = \frac{2{v_0}^2 \cos^2\theta}{g} \cdot \frac{\sin\theta}{\cos\theta}$$

$$R = \frac{2{v_0}^2 \sin\theta \cos\theta}{g}$$

$$\boldsymbol{R = \frac{{v_0}^2 \sin 2\theta}{g}}$$

Since $-1 \le \sin 2\theta \le 1$

therefore $(\sin 2\theta)_{\max} = 1$

$$\Rightarrow R_{\max} = \frac{{v_0}^2}{g}(\sin 2\theta)_{\max} = \frac{{v_0}^2}{g}$$

Physics-PHY101-Lecture#04

FORCE AND NEWTON'S LAWS

Till now in this course of physics, we have looked at kinematics which includes basic concepts such as displacement, velocity, and acceleration. We talked about one dimension that if the body can move only on one dimension i.e., on a straight line then how do we define velocity and acceleration? We talked about derivatives, then we generalized it into two and three dimensions. Then some interesting problems were also solved. But this question remains to be raised why do bodies/objects move like this, where do they get this acceleration from? We will look at these issues in today's lecture.

The theme of today's lecture is dynamics. By dynamics, we mean the force which acts on the body and gives it acceleration. In this context, we will talk about Newton's three laws of Motion. So, the real question arises where does acceleration come from? For this, we will need a new concept which we will call Force. Dynamics means it is the study of forces and the resulting motion. So how does a body get motion and what is the effect of force on it will be discussed in this lecture? Before Newton's time, it was believed that the natural state of everything is to come to a standstill, which means that everything wants to come to a standstill or stop, so for example: if we roll a ball then the ball stops after some time or if you throw any other thing then it moves forward for a while and then stops as if it is a natural state that is the of rest and if something moves then there must be some force acting on it, hence this idea was common before Newton's time that some force behind moving body.

For this reason, it was believed that the sun, the moon and all the other planets are moving then there must be some force behind them. For example: if mars rotates in its orbit like this then there must be something pushing it. This was an old idea but after Newton a great revolution took place. Newton said that the natural state of everything is that it wants to continue its movement. The modern view is that objects tend to remain in their initial state, that is they want to remain in the same state in which they were and unless some force acts on it will maintain their condition. Isaac Newton was probably the world's greatest scientist and thinker. He wrote a book called Principia Mathematica about 350 years ago in which he proposed three laws of motion.

Newton’s First Law of Motion:

It states that everything either remains at rest or moves with constant speed unless some force acts on it. This means that the body moving will keep moving on its own without the application of external force and can be considered as a free body.

Frame of reference:

If we measure the movement of a body and someone else measures the motion of the same body then there can be differences between the two, so here we have to talk to you about reference frames as shown in figure 4.1. These are two frames of reference, one we can call S, and the other can be called S'. Imagine that you are at rest and standing on land and exist in frame S. There is another person who exists in frame S' and it is moving away from us at velocity V. Let’s assume a point P having distance x from our frame of reference. Now the person who is moving will say that I measured this distance as x'.

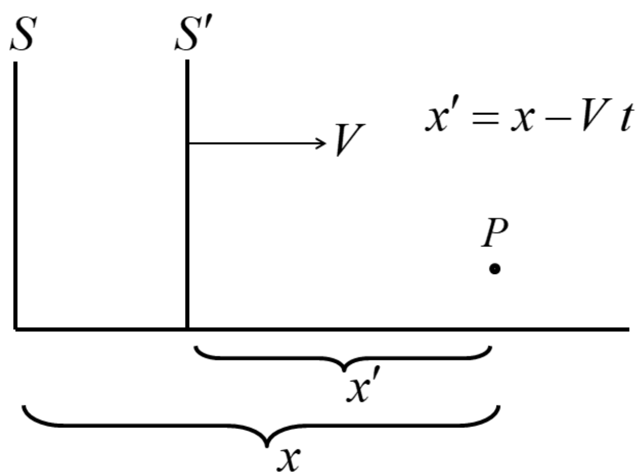


Figure 4.1: Frames of references S and S'.

The relation between these two distances is given by

$$x' = x - V t$$

and at $t = 0$

$$x' = x$$

So initially ($t = 0$) both distances are measured to be the same, but as the second observer moved forward, there was a difference between x and x' which is called a frame of reference. According to this, two different observers will have different sets of coordinates for the same point and whatever measurements we make will depend on these coordinates. Now the question arises of whose measurements are more accurate.

Inertial and non- Inertial Frame of Reference:

Newton's first law applies in every frame which is moving with constant velocity and such frames are called **inertial frames.** Newton's law is valid in all inertial frames and all non-accelerating frames are called inertial frames.

For example: Imagine that you are sitting in the car and there is a hydrogen or helium balloon in that car, and it is suspended inside the car. Now when the car accelerates, we will observe that the balloon starts moving backwards. It seems like some force is acting on the balloon. But in reality there is no force acting on it, but we feel it as if it’s moving backwards. And that is because you are not in the same inertial frame. The frame which accelerates is a **non-inertial frame** and apparently, a force acts in it sometimes called a fictitious force. It appears to be force but in reality, it is only observable because we have chosen the wrong frame of reference. So, Newton’s first law is true only in inertial frames.

Difference between inertial and non-inertial frames of reference:

Inertial frame	• The frame of reference which is moving with uniform velocity and does not accelerate (a=0). • Obeys Newton’s law of motion. • Example, a train moving with uniform velocity is an inertial frame of reference.
Non-inertial frame	• The frame of reference which is accelerating ($a \neq 0$) is called non-inertial frame of reference. • Does not obey Newton’s law of motion. • Example, a freely falling elevator is taken as non-inertial frame of reference.

Law of inertia:

Now the question arises when you want to change the state of motion of a body, you feel some resistance. How much does a body resist when you try to change its state of motion? It depends on its mass. The heavier the body, the more it will resist change in its state of motion. There are many examples of this.

For example: If you push a light body (like a shopping cart), it moves easily. If you push a heavy body (like a car or bus) it is possible that you may not even be able to move it. We define this resistance as **"inertia"**. So, inertia is the resistance to change in motion in other words resistance to acceleration and mass quantifies it. The greater the mass, inertia will be more accordingly. Now the question arises whether mass means size? to which the answer is no. Sometimes smaller bodies offer more resistance. It is related to density. To which the answer is also no. Is this related to weight? Is mass the same as weight? This is also not the answer. Mass and weight are not equal.

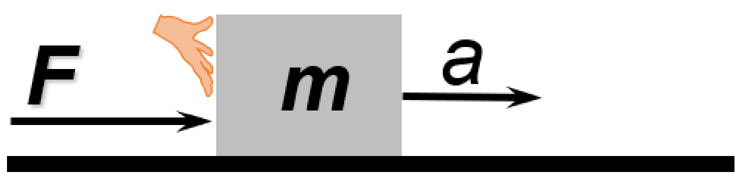


Figure 4.2: Force *F* acting on a body having Mass *m* and causing acceleration *a*.

Now look at Figure 4.2. Force F is acting on a mass whose value is m. Now the more force applied to it the acceleration 'a' will be greater accordingly. So mathematically it is written as

$$a \propto F$$

More force leads to more acceleration. But on the other hand, if we consider a body having more mass then the acceleration will be less.

$$a \propto \frac{1}{m}$$

Hence

$$a \propto \frac{F}{m}$$

$$F = kma \ (k=1)$$

Increasing the force increases the acceleration but increasing mass reduces acceleration.

Newton's second law of motion:

Newton's second law of motion states that the product of mass and acceleration is equal to the total external force acting upon mass. Now there can be many forces (shown by subscripts) applied simultaneously in different directions. The total net force will produce acceleration.

$$F = ma \rightarrow a = \frac{F}{m}$$

Where

$$F = F_1 + F +_2 + F_3 + ...$$

By this, we can also define mass as:

$$m = \frac{F}{a}$$

So, how to rightly write this equation? There is an equation and there are three variables. Here you can measure the acceleration separately because acceleration is the rate of change of velocity and velocity is the rate of change of position. As position can be measured as a function of time so, velocity and consequently time-dependent acceleration can be measured. There is a need to measure force and how do we do this? We can give you many simple examples of this.

Example: We can consider a spring balance. In spring balance, we can place some mass and there is a spring. The more mass we place, the more spring will extend. Similarly, if we pull something with a spring balance, there is a scale on the spring balance showing the magnitude pulling force. If we pull a rubber band, this rubber band exerts a force on our hand and the more it is pulled, the greater the force. So, a rubber band can measure force.

Introduction to force:

Force is the vector and hence it has a magnitude and also a direction. The direction of force is also the direction of acceleration. There are many types of forces.

Let us consider **contact forces**. When two bodies come in contact with each other, one body exerts a force on another body that is a force acts on it, i.e., when we push a box, a force acts on the box, air exerts pressure on a moving car which is called **air resistance**. Consider a rope with a weight tied at one end that results in **tension** which is also a type of force.

Just like other physical quantities force also has dimensions given by,

$$[Force] = [Mass] \times [Acceleration] = [M] \times \left[\frac{L}{T^2}\right] = \left[\frac{ML}{T^2}\right] = \left[MLT^{-2}\right]$$

Units for force in the MKS system is Newton. Newton is defined as, if we apply 1 Newton force on 1 kilogram of mass then its acceleration becomes 1 m/s^2.

$$1N = 1kg.1\frac{m}{s^2}$$

Force F means external forces only. Consider a body made up of atoms (as all matter is made up of atoms), every atom attracts or repels the other atom. These interactions are internal forces so they cancel out each other. When we say that the acceleration of a body is *F/m*, then that F refers to external force only.

External force actually is a total force so if various forces are acting such as F_1, F_2 and F_3 till F_N then we have to add all these, given by:

$$F = F_1 + F_2 + F_3 + ... + F_N = \sum_{i=1}^{N} F_i$$

here we see a new symbol which is called the Greek symbol "Sigma $\sum$" or symbol of summation. So, this summation means we add F_1, and F_2 till F_N and this makes the total force.

As force is a vector quantity it can be added by two methods. One method is already discussed (the Parallelogram method) earlier. This can also be done and the other way is through vectors addition by components. The following sample problem (figure 4.3) can help us better understand the addition of two forces by components method. We are given two forces (F_1 whose value is 4

N along the negative y-axis) and F_2 whose value is 5 N at an angle of 36.9° with the x-axis (angle not mentioned on PPT slide).

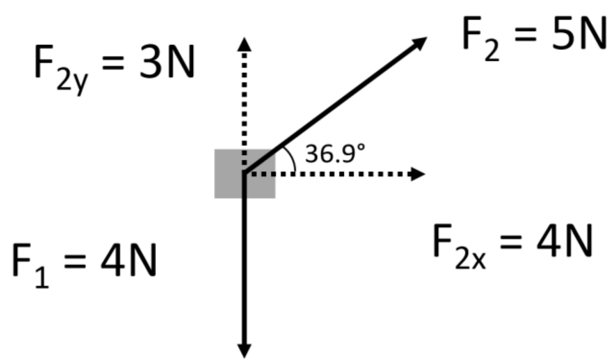


Figure 4.3: Sample problem.

Components of F_1:

$F_{1x} = 0$ (as no component along x-axis)

$F_{1y} = -4N$ (as it is along negative y-axis)

Components of F_2:

$$F_{2x} = F_2 \cos(36.9) \simeq 5*0.8 = 4N$$

$$F_{2y} = F_2 \sin(36.9) \simeq 5*0.6 = 3N$$

The sum of components along the x-axis,

$$\sum F_x = F_{1x} + F_{2x} = 0 + 4 = 4N$$

$$\sum F_y = F_{1y} + F_{2y} = -4 + 3 = -1N$$

So, the **magnitude of resultant force F** can be calculated by,

$$|F| = \sqrt{\sum F_x^2 + \sum F_y^2} = \sqrt{(4)^2 + (-1)^2} = \sqrt{17}N = 4.12N$$

Exercise for student: Determine the direction of resultant force F.

Mass and Weight:

Now we will discuss the difference between weight W and mass m. Weight actually is a force, the force due to gravity. If I have a kilogram of matter it will have a specific weight on earth's surface, but if I take it to the moon's surface, its weight will be different. The mass will be the same but its weight is different on both (earth and moon). Let's look at it in the formula, the weight is the force of gravity and if you apply the following equation.

$$F = ma$$

Where F becomes the weight W, mass m remains tha same, and acceleration a is replaced by acceleration due to gravity g.

$$W = mg$$

Like forces, weight is also a vector and is measured in Newton. If we measure the weight of the same body on earths and moon's surface, we will observe that the weight on earth will be seven times as compared to weight on moon although mass is same. If we go out into space where there is no celestial body, then our weight will be zero there as there will be no acceleration due to gravity, although mass will be non-zero and remain the same.

Difference between mass and weight:

Mass	**Weight**
Mass is a property of matter. The mass of an object is the same everywhere.	Weight depends on the effect of gravity. Weight increases or decreases with higher or lower gravity.
Mass can never be zero.	Weight can be zero if no gravity acts upon an object, as in space.
Mass does not change according to location.	Weight varies according to location.

Mass is a scalar quantity. It has magnitude.	Weight is a vector quantity. It has magnitude and is directed toward the center of the Earth or other gravity well.
Mass may be measured using an ordinary balance.	Weight is measured using a spring balance.
Mass usually is measured in grams and kilograms.	Weight often is measured in newtons, a unit of force.

<u>Newton's third law of motion:</u>

Newtown's third law says that the action of every force produces a negative force. The magnitude of these two forces is the same and direction is opposites (antiparallel).

Example: Two boxing gloves A and B collide and come in contact with each other while punching. A Pushes on B and B pushes on A. So, these are action and reaction forces are in opposite direction but have same magnitude (negative sign on the right-hand side of equation in figure 4.4 shows opposite direction)

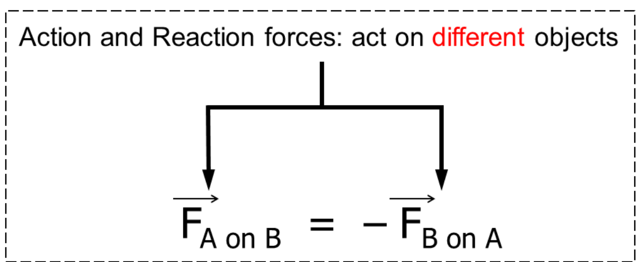


Figure 4.4: Action reaction forces.

Note that $F_{A\ on\ B}$ is equal to negative $F_{B\ on\ A}$, so the effect of A is on B & B is on A. A is not affecting C, D or anything else, nor is B on anything else. It is so that A is on B & B is on A.

Example: An apple is lying on the table, so this is the force of the earth on the apple. We will call it action, and this is due to gravity. Its reaction is not the force of the table on the earth, but the force of the apple on the earth. Earth pulls this apple down, and on the other hand this apple pulls the earth to itself. We assume earth is not pulled to apple which is wrong. Apple is light and very small in comparison to the earth, but the apple also pulls the earth toward itself. Now because the earth is so heavy its acceleration is very less. But apple definitely exerts a force on earth. Table also exerts as a force on apple and its reaction is that apple also exerts a force on table. Now you can ask how the table exerts a force on the apple. How does the force exerted by the table affect it? It's obvious that if there was no apple there would be no force. But if you keep the apple on the table there is a slight bend produced in the table as a result of which a force is produced in upward direction.

To make this discussion interesting we will narrate a story of a very educated horse. He studied Newton's law in his spare time and especially Newton's third law. One day, while he was studying, his owner got angry. His owner attached a cart to him and ordered horse to pull it. The horse said that there is no use of it because he has just read Newton's third law and according to that if he pulls the cart then the cart will also pull him back with same force. He will not move forward hence it is useless to pull the cart. The owner got angry, he gave a pat at the back and what happened? The cart started moving forward. Why did it happen? If the horse is right then the cart would not have moved but there is a mistake in it. So, let's see where the mistake is.

If we look at this student in figure 4.5 walking forward on the ground. Similar to the student depicted in figure we also push ground backwards with our feet. The ground pushes us forward as a reaction. Newton's third law says that if we push something back, it pushes us forward. In this way we move forward. So whenever one object exerts a force on a second object, the second object exerts an equal and opposite force on the first.

Figure 4.5: Person ‘P’ walking on ground ‘G’. F_{GP} is a backward force exerted on ground by a person. F_{PG} is forward force exerted on a person by ground.

Now look at Figure 4.6 of assistant boy ‘A’ boy is moving forward on ground ‘G’ and at the same time he is also pulling a sledge ‘S’ forward. When he pushes ground backward (F_{GA}), the ground pushes him forward (F_{AG}). The tension in this rope (F_{AS}) also pulls this boy backwards. But if the total force produced in the forward direction is more than the backward pull, then this sledge will move forward.

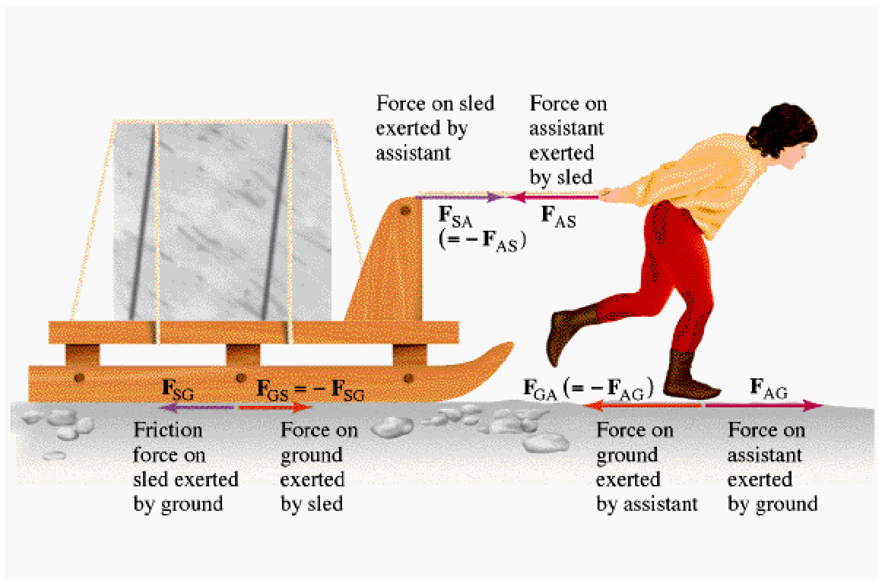


Figure 4.6. Assistant boy ‘A’ moving forward on ground ‘G’ and also pulling sledge ‘S’ forward.

Now one more **example** is depicted in figure 4.7. Your hand is on the table in front of you, and with your hand, you are applying pressure on the table and pushing it downwards but table is pushing your hand upward in opposite direction.

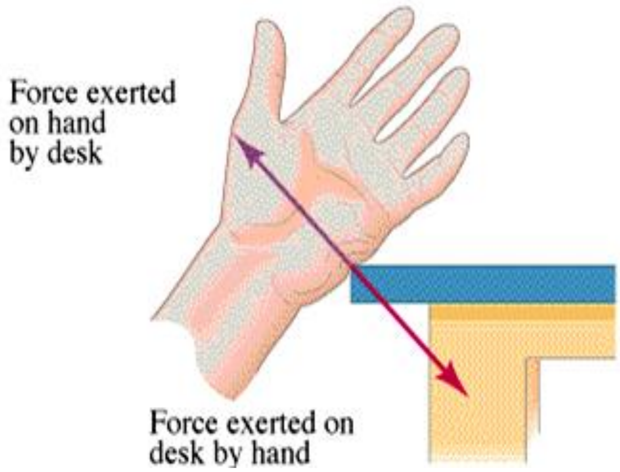


Figure 4.7: Action-reaction pair for hand and table.

Let's consider another **example** shown in figure 4.8. The person is standing and pushing against the wall and the wall pushes the person back with same force. If the action-reaction forces are equal and opposite in direction, why they do not cancel out each other?

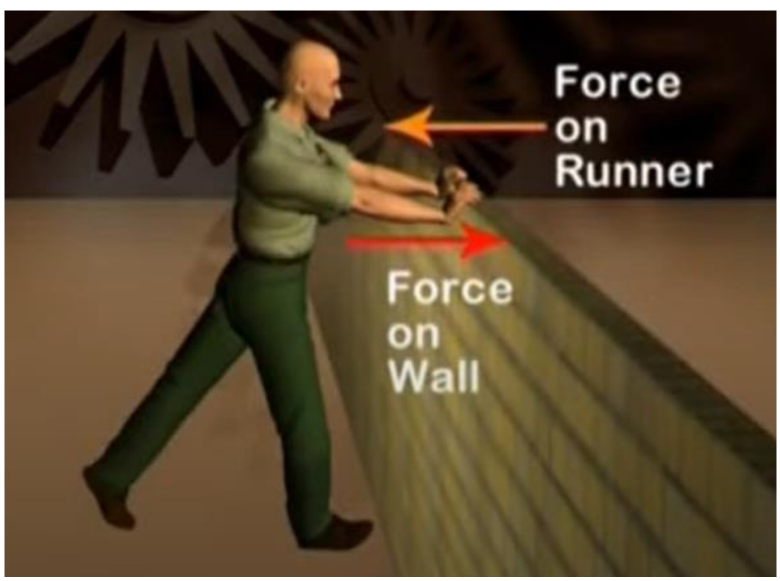


Figure 4.8: Action-reaction pair for person pushing wall.

Figure 4.9 elaborates forces involved with more details. A person is leaning against a wall. Two forces are acting on this person. One is his weight and written it as $F_{m\ on\ f}$ meaning the force of mass 'm' on the floor 'f'. This force act in downward direction. Reaction to this force i.e., the force of floor on the man $F_{f\ on\ m}$ acts upward and cancel out each other. The force exerted on the left side is force of man on the wall $F_{m\ on\ w}$ and the force of wall on the man $F_{m\ on\ w}$ is on opposite direction. Both these forces are equal in magnitude and in opposite direction.

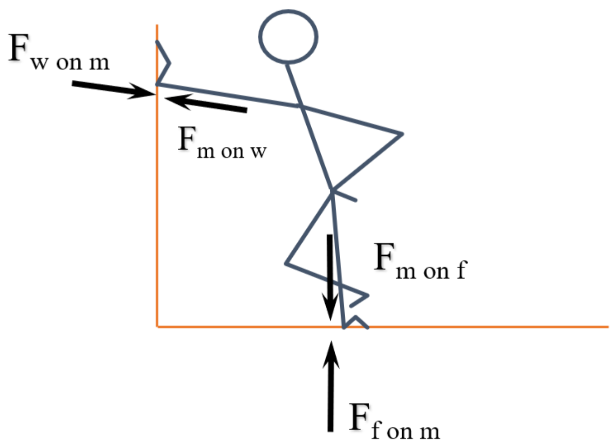


Figure 4.9: Forces involved for physical scenario of man 'm' leaning against wall 'w'.

Now if we push a body with much greater force then there is possibility that it will start moving in the direction of applied force. Such **an example** is shown in figure 4.10. A block/box is on frictionless (ice) surface. Man pushes block/box to the left (negative x-axis) by a force $F_{m\ on\ b}$ (force of man on block). Reaction to this is force of block on man ($F_{b\ on\ m}$) is towards right side (positive x-axis). Although both forces ($F_{m\ on\ b}$ and $F_{b\ on\ m}$) are equal and opposite in direction but block moves to left side. So how is it possible? Why are these forces not cancelling each other?

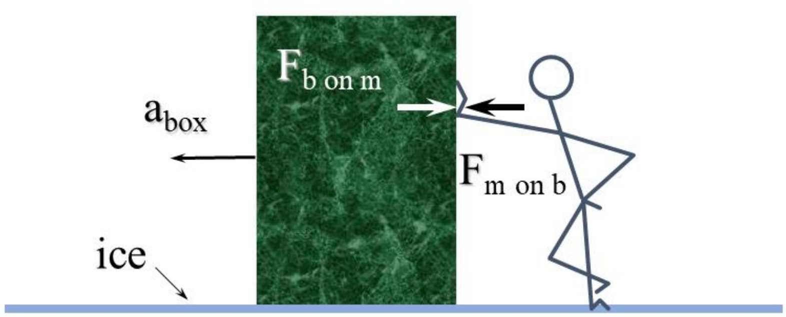


Figure 4.10: A block/box on frictionless (ice) surface being pushed by man.

The answer to why these forces is not cancelling each other out is that a body moves depending upon the total external force acting on that body (having mass m). The force that will act on a body will create acceleration in it. By considering ONLY the block/box as the whole system we can answer above mentioned questions. The force on block/box ($F_{\text{on box}}$) is equal to force of man on block/box ($F_{\text{m on b}}$) which by newton's second law is equal to product of mass m and acceleration of block/box (a_{box}). Mathematically it can be written as

$$F_{\text{on box}} = F_{\text{m on b}} = ma_{\text{box}}$$

Acceleration of block/box (a_{box}) will be,

$$a_{\text{box}} = \frac{F_{\text{m on b}}}{m}$$

<u>Misconception:</u>

Table 4.1: This table enlist few misconceptions related newton's laws

Sr. No.	Claim	True or False. Explanation
1.	If something is moving, there must be a net force on it.	This is a false claim. A body moving at constant velocity has no net force on it. An accelerating body must have a net force on it.
2.	All equal and opposite forces are action-reaction pairs.	This claim is false. The weight of a book sitting on a tabletop and the normal force of the table acting on the book are equal and opposite, but they are not an action-reaction pair!
3.	If there is a force on an object, it must be accelerating.	This claim is false. Only a net force on the object leads to acceleration.

In our daily lives, we observe the validation of Newton's law on many occasions, especially the first and second laws. There is a concept about these which we call inertia that a body tends to maintain its state of motion or rest and its speed doesn't change. This resistance to change of state is proportional to its mass. An example involving glass and paper is discussed in the video lecture followed by the stretching of rubber band as a function of applied force (amount of water).

We will pay close attention to the applications of Newton’s law during the next few lectures. The application of Newton’s laws is not limited or localized. These laws are applicable in the whole universe (also termed as universal set of laws). Interestingly concepts at the base of Newton’s laws do not exist in the world. They only exist in our minds. For example, if we talk about free particles (as mentioned earlier that free particle is the one on which no force acts on it) but is there any free particle in reality? The answer is NO. There is no such thing as a free particle. If we even take a particle/body million and millions of miles away from Earth, there will be a minute force acting on it due to gravity. We have also talked about this earlier there is such a thing as a point mass or point body. Despite these facts, we are related to the abstract concept that Newton's laws related to things that do not exist yet this is the achievement of human thinking. We will discuss the application of these laws and how phenomenon are explained.

Physics-PHY101-Lecture#05

APPLICATIONS OF NEWTON’S LAWS–I

In today's lecture, we will be focusing on Newton's laws of motion and how they can be applied to different situations and circumstances. We will be solving various examples to gain a better understanding of how important Newton Laws are in understanding the world around us. Living in the 21^{st} century, we can sometimes forget the significant difference between the latest and old concepts/laws. Before Newton's laws, it was generally believed that every physical body or particle naturally wanted to remain at rest. However, Newton's first two laws of motion state that every physical body or particle tends to maintain its state of motion, whether it's at rest or in motion. If the state of motion of a body changes, such as a change in speed, it's because some force has acted upon it.

Equilibrium

If a body or a particle is in a state of rest, then we refer to it as an equilibrium state. Equilibrium means that in the state of a system, there is no force acting upon it, i.e., also called the state of rest (no motion). Newton’s law means that if the sum of all the forces is equal to zero on that body or combination of bodies, then we get a state of rest. As per the definition, "State of a system when the sum of all forces acting upon it vanishes" is called 'Equilibrium'.

What is meant by the system here? The system here refers to many bodies which are connected to each other or isolated from each other. When you consider all these together then it is called a system.

Examples of equilibrium:

1) The gravity acting on the apple continuously pulls it downwards but when placed on the table as shown in Figure 5.1, the apple remains at rest. It doesn't move further down. Which forces are acting on this apple? Gravity is one as mentioned earlier but there must be at least one more force.

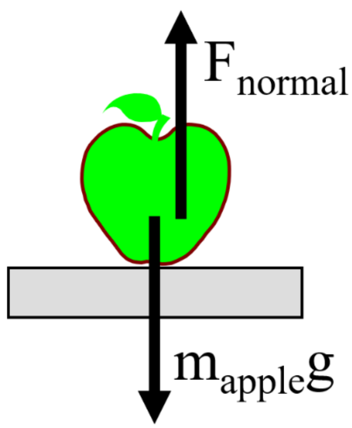


Figure 5.1: Apple placed on the horizontal top surface of the table. Normal force F_{normal} and weight of apple ($W_{apple} = m_{apple}g$).

Newton's law governs that whenever a force is applied to a body, the body starts accelerating. The table applies a force opposite to the gravitational force which stops this apple from falling. This force is called the Normal Force (F_{normal}). The word "normal" has two meanings in English, one is "as per usual" and the other is "perpendicular" as in geometry. In Figure 5.1, both these forces are indicated, the normal force (F_{normal}) and the $m_{apple}g$ which is the force due to gravity (also called weight). Mathematically Newton's second law (for y-axis or vertical direction) can be written as,

$$\sum F = m\,a_y = F_{Normal} - m_{apple}\,g$$

Normal force (F_{Normal}) is written as positive and gravitational force ($m_{apple}g$) is written with the negative sign as it is opposite to the direction of F_{Normal}. All forces along the x-axis (horizontal direction) are zero as gravity does not act along this direction and also table is not applying any force sideways. As the apple is at rest, acceleration (a_y) is also zero. The above equation can be simplified to

$$F_{Normal} - m_{apple}\, g = 0 \text{ (as } a_y = 0)$$
$$F_{Normal} = m_{apple}\, g$$

So normal force applied to the apple by the table is equal to its weight under the equilibrium (at rest/no motion). It is normally supposed that equilibrium means there is no force acting on the body which is a wrong presumption. Equilibrium or a state of rest arises when the total net force is zero (sum of all forces).

2) <u>Equilibrium Example: Rubber band as slingshot</u>

A rubber band can be considered a system and when pulled tension is created in it. This tension increases as it is pulled more and more. If a piece of paper in the form of a pellet is attached to a rubber band and stretched only and kept in place (at rest, equilibrium) like a slingshot, the tension in the rubber pulls the paper pellet forward. If released, the paper pellet will go forward like a projectile. So, it remains in equilibrium only under the force applied by us which is equal and opposite to tension in the rubber band. Now the question is whether this paper pellet was in the state of equilibrium when released or when it reached its maximum height considering that it is projectile motion. The answer is no, it was not in the equilibrium state. Although its velocity along the y-direction was zero momentarily at the highest point, its acceleration at this point was not zero as acceleration due to gravity was continuously acting on it. There was no force acting other than force due to gravity so it was not in the state of equilibrium. The total net force acting on it was force due to gravity (weight) which is nonzero.

3) <u>Equilibrium Example: Aircraft flying</u>

Consider an aircraft craft flying at a constant speed and same elevation. This is an example of equilibrium motion as there is no acceleration (constant speed) indicating net force (sum of all forces) acting on the aircraft is zero.

Now we will solve another problem considering another aircraft of mass m has a position vector given by $\vec{r} = (\mathrm{at} + \mathrm{bt}^3)\hat{i} + (\mathrm{ct}^2 + d\,\mathrm{t}^4)\hat{j}$. What force is acting upon it? Is it in equilibrium?

As velocity $\vec{v}$ is the time derivative of position $\vec{r}$ by

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left((\mathrm{at} + \mathrm{bt}^3)\hat{i} + (\mathrm{ct}^2 + d\,\mathrm{t}^4)\hat{j}\right)$$
$$\vec{v} = (\mathrm{a} + 3\mathrm{bt}^2)\hat{i} + (2\mathrm{ct} + 4d\,\mathrm{t}^3)\hat{j}$$

Now as acceleration $\vec{a}$ itself is time derivative of $\vec{v}$, so

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left((\mathrm{a} + 3\mathrm{bt}^2)\hat{i} + (2\mathrm{ct} + 4d\,\mathrm{t}^3)\hat{j}\right)$$
$$\vec{a} = (6\mathrm{bt})\hat{i} + (2c + 12d\,\mathrm{t}^2)\hat{j}$$

Now force $\vec{F}$ is a product of mass m and acceleration $\vec{a}$, so

$$\vec{F} = m\vec{a} = m((6\mathrm{bt})\hat{i} + (2c + 12d\,\mathrm{t}^2)\hat{j}) = 6m\mathrm{bt}\,\hat{i} + m(2c + 12d\,\mathrm{t}^2)\hat{j}$$

The above equation shows the force acting on the aircraft. Force along with acceleration is nonzero so the aircraft is not in equilibrium.

Tension

Now consider a string tied to a body as shown in Figure 5.2(a). The body has a certain weight (downward force) due to gravity which causes tension in the string. This tension force in the string is in an upward direction as it balances the weight. If we cut the string which is horizontally stretched, we will perceive tension towards the right as well as towards the left as shown in Figure 5.2 (b).

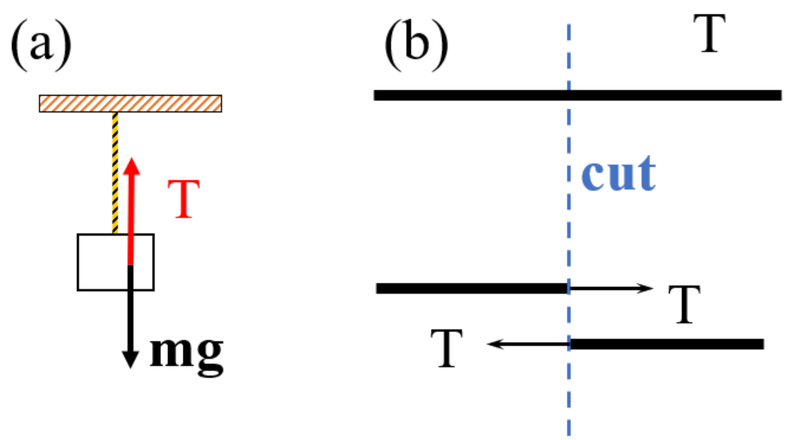


Figure 5.2: (a) A body having mass m tied to a string. The other end of the string is tied to a fixed support. Mass hanging vertically. (b) Tension T is considered in horizontal string. Blue dashed line indicating cut-line.

Now the big advantage of ropes is that we can pull a body from a distance and the direction of the force can be changed. Consider a horizontal string under tension as shown in figure 5.3. the yellow part represents a small part of the string. T_2 acts to the right-side direction and T_1 to the left-side direction of that small part of the string. According to Newton's law, total net force action on that small part of the rope is given by

$$\sum F = T_2 - T_1 = ma \text{ …………5.1}$$

Where m is the mass and a is the acceleration of that small part of the string. Considering that string is very light in terms of mass, the mass of the small part of the string can be neglected and assumed to be zero.

$$m = 0 \text{ (small part of string whose mass is very less)}$$

Eq 5.1 becomes

$$T_2 - T_1 = 0 * a$$

$$T_2 - T_1 = 0$$

$$T_2 = T_1$$

As T_2 and T_1 are the same, we can equate both quantities to be T. This means that tension in a string has very little mass and is constant throughout the string.

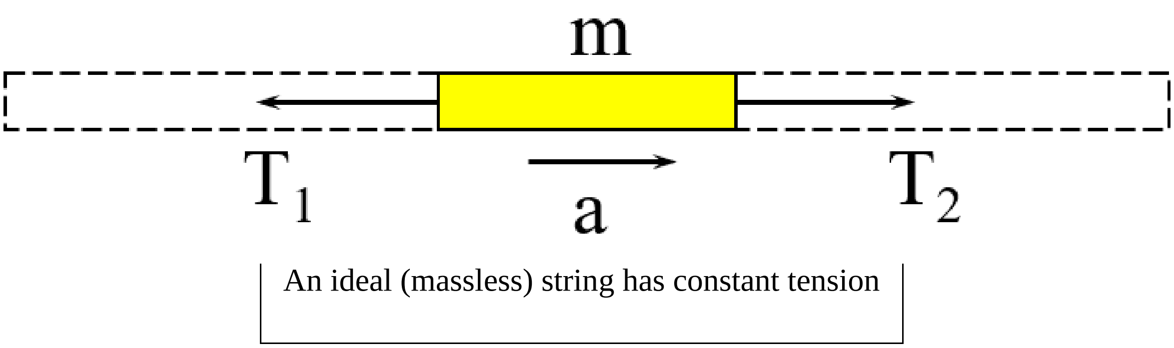


Figure 5.3: A string having horizontal tension. The central yellow part shows a small part of the string having mass m and accelerating by a being pulled to the left and right by tensions T_1 and T_2, respectively.

Newton's third law applies as forces act on all segments of a string, but these action and reaction forces are equal to each other. As shown in Figure 5.2(a), the body having mass is under equilibrium and is at rest. Newton's second law can be as applied as

$$\sum F = T - mg = ma_y$$

Where a_y is acceleration along the y-axis and is zero ($a_y = 0$).

$$T - mg = 0$$

$$T = mg$$

So, for this mass system, the tension is equal to the weight and the value of this tension will be equal everywhere inside the string. Now for string having as considerable mass such as rope, will the tension be equal in all parts of this rope if it is allowed to hang vertically without any additional body? The answer is NO, because the lower end of the rope has to support less mass as compared to the higher part of the rope. So, tension at the higher end is greater as compared to tension at the lower end.

A massive string (rope) does not have constant tension

Changing the direction of Tension

We can change the direction of tension as per requirement. As mentioned earlier, if the string is considered massless tension will be constant throughout the string even for the physical configurations shown in Figure 5.4. Figure 5.4(a) shows mass-II m_{II} hanging vertically from an ideal string and the direction of tension is changed using a pulley. Now tension is acting horizontally on mass-I m_I.

Figure 5.4 (b) shows another example of the change of tension direction. The weight ($W = mg$) is being lifted. A worker is pulling a rope and with it, the weight is rising upwards. So, F_T is shown on two sides and both of them are adding up. Under equilibrium, the net force upward is two times F_T and it is equal to weight ($W = mg$).

$$\sum F = F_T + F_T - mg = ma_y \quad \ldots\ldots\ldots\ldots\ldots 5.2$$

Under equilibrium $a_y = 0$

$$F_T + F_T - mg = 0$$

$$2F_T = mg$$

If $2F_T$ is greater than weight ($W=mg$) such as $2F_T > mg$ then eq. 5.2 can be written as

$$2F_T - mg = ma_y$$

$$ma_y > 0$$

$a_y > 0$ (Upward direction or lifting up)

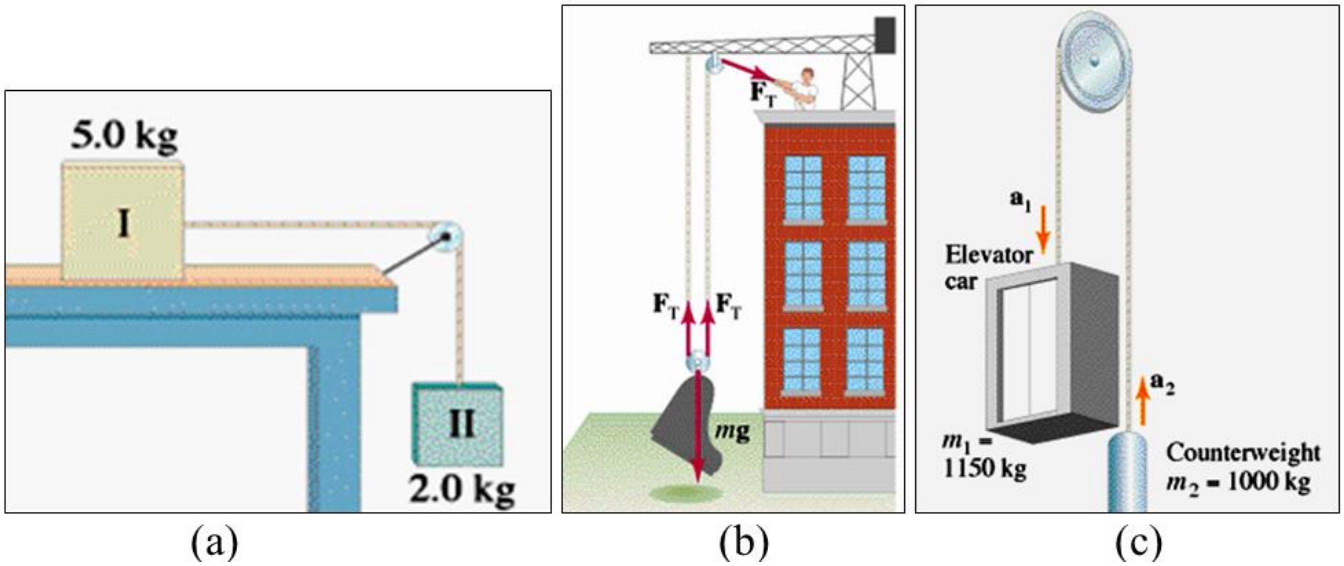
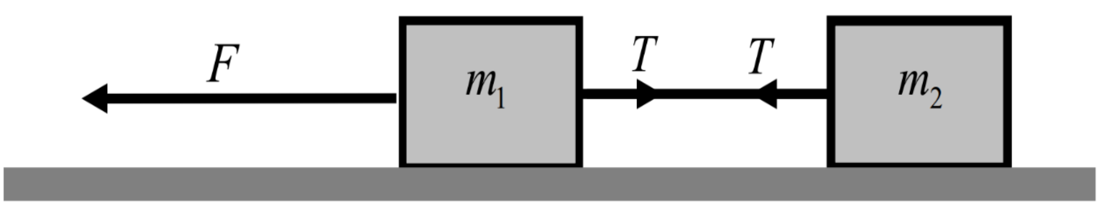


Figure 5.4: Examples of change of tension direction by pulley.

Figure 5.3(c) shows another example elevator car. The tension direction has been changed through the pulley. The mass of the elevator car is 1150 kg and on the other side, there is a counterweight which is equal to 1000 kg. Now the acceleration a_1 is downward as the elevator is more massive as compared to counterweight. The acceleration of the counterweight a_2 is downward.

Problem:

Question: Consider the two blocks shown below on a frictionless surface. Determine the tension and acceleration.



Solution:

F on m_1 :

$$\sum F = m_1 a$$

$$F - T = m_1 a \text{.........1)}$$

F on m_2 :

$$\sum F = T$$

$$T = m_2 a \text{...............2)}$$

To find "a"

Plugging eq. 2) in eq. 1)

$$F - m_2 a = m_1 a \Rightarrow F = m_1 a + m_2 a$$

$$a = \frac{F}{m_1 + m_2} \text{...............3)}$$

To find "T"

Plugging eq. 3) in eq. 2)

$$T = m_2 a = \frac{m_2 F}{m_1 + m_2} \text{.......4)}$$

Friction

This is another important type of force that we encounter most frequently in our daily lives and it is called force of friction. The force of friction opposes every kind of motion. For example, if you place a body having a certain weight on the table and pull that body, then the force which is produced due to applied force and contact between the body and table is called frictional force as shown in Figure 5.5. There are countless examples of this in our everyday life. Suppose there is a plate lying on a book on this table, and it will not move on its own unless some force is applied to it. Now when we pull it (as per demonstration in video lecture), we will observe that it moves forward, because we have offset the frictional force. We pulled it with more force and it started moving. The normal force N is upwards and equal to its weight *mg*. Simultaneously, the greater the weight, the greater will force that opposes its motion. If we double the weight of this plate, then the force required to move the plate is also doubled. As depicted in Figure 5.5, friction results in a force parallel to the surface in the direction opposite to the direction of motion. So whichever direction we pull the body ($F_{Applied}$), the force of friction will be in the opposite direction. Additionally, the frictional force is perpendicular to the normal force N which is always upward.

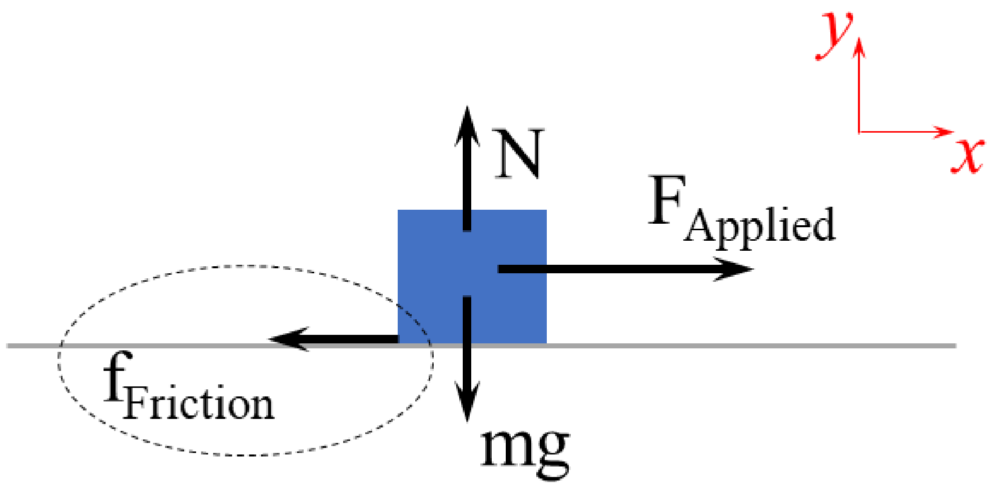


Figure 5.5: Mass m placed on a horizontal surface. Normal force N and weigh mg balanced along the y-axis. Frictional force $f_{Friction}$ produced along the negative x-axis as response to external force $F_{Applied}$ which is along the positive x-axis.

Now the pertinent question that why does the friction occur? We observe friction in various situations, for example, when we rub our hands, then we can feel opposing force due to contact of hands. To understand the underlying reason for friction, we need to observe the two surfaces in contact under a microscope. Any surface if seen under the microscope has very fine ups and downs in it and due to this unevenness, friction is created. The more the ups and downs, the more the roughness, then the friction will be higher accordingly. If we look at contact between surfaces of bodies at a microscopic level as shown in Figure 5.6, we will be able to observe the roughness caused by ups and downs which fundamentally causes the frictional force. If we can reduce this roughness then the frictional force or simply friction will reduce, and the fun will reduce.

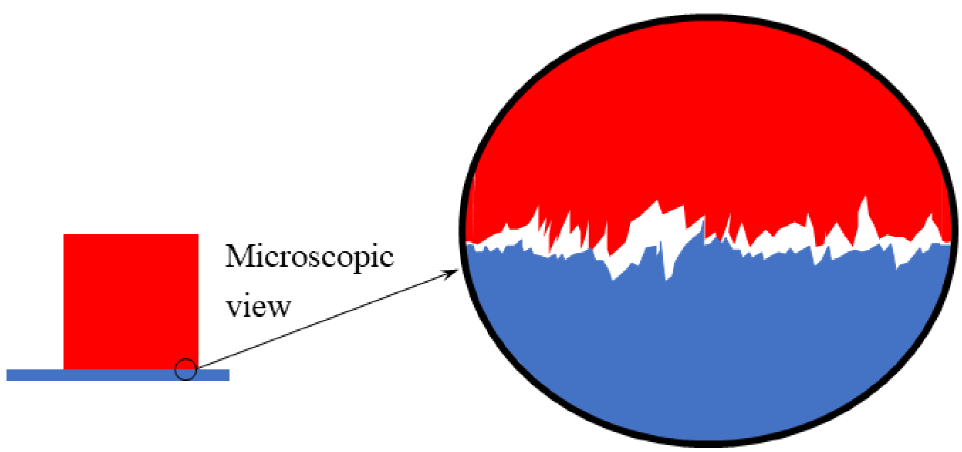


Figure 5.6: Microscopic view of two surfaces in contact which causes friction.

Characteristics of Frictional force f_F:

There are some special characteristics of frictional force. When the body is in motion, the friction that is produced is called kinetic friction. Friction force is proportional to the normal force N.

$$f_F \propto N$$

This sign of proportionality can be removed by the introduction of constant "μ_k" of proportionality and is called the "coefficient of kinetic friction".

$$f_F = \mu_k N$$

$$f_F = \mu_k mg \text{ (as } N = W = mg)$$

This equation shows that the "heavier" the object or a body, the greater the friction will be. This is an example of empirical law where $f_F = \mu_k N = \mu_k mg$. Can it be compared to fundamental laws such as Newton's second law? This is an example of an empirical law, it means that it is based on the observation that if the weight of the body/object is doubled then you have to apply double the force, if it is three times then, three times more force is required and so on. These statements are correct to some extent but it is obvious that if we keep increasing the weight of this body/object, then eventually a situation will arise when this law will not be applicable. If the body/object becomes too heavy then the surface (table) under the body/object might break and friction will not arise. So, this is an example of an Empirical Law which means that that law is based on observations and their scope is limited. Unlike empirical laws, the scope of Newton's law is very wide as we can apply it everywhere under every situation such as on the earth, on the moon, on small bodies and heavy bodies etc. Frictional law is not a such fundamental law. The frictional force is independent of the contact area. It does not matter whether this contact area is more or less. The force is just proportional to the Normal force, not to the area. We can say that it is correct to a great extent but we cannot claim that that is completely true. If the contact area is too large or too small, it can make a difference. Imagine that the entire weight of the plate is concentrated to something finer than a needle then it is quite possible that it will pierce the surface below and frictional force law would not be applicable. These things are very important to keep in mind that there is a difference between empirical laws and fundamental laws.

Frictional Dynamics

Now we will discuss the frictional dynamics, that is, how a body moves in the presence of friction. Now consider the forces along the x-axis (as shown in Figure 5.5).

$$\sum F_x = F_{Applied} - f_{Friction} = ma_x$$

$$\sum F_x = F_{Applied} - \mu_k N = ma_x \text{ ....5.3}$$

Where a_x is the acceleration body having mass m along the x direction. Similarly, for forces along the y-axis

$$\sum F_y = N - mg = ma_y = 0 \text{ (as } a_y = 0)$$

$$N = mg$$

So eq. 5.3 can be written as

$$F_{Applied} - \mu_k mg = ma_x$$

By this equation we can determine the acceleration for any given force and any given mass provided we are given the coefficient of Friction μ_k. Friction acts whenever the body is in motion but if a body is at rest then friction is also present. There is a difference between friction at rest and kinetic friction. Kinetic friction means the friction that exists when the body is in motion. Its value ($\mu_k N = \mu_k mg$) is a constant. But when the body is at rest, the value of friction force is not fixed. As we start pulling this plate it does not move (as per demonstration in the video lecture). We pulled it by applying a small force and its reaction is also a small frictional force accordingly. Only when we pull it with more than a certain limit force then it will move by balancing and overcoming frictional force. This means that static friction adjusts its value depending upon applied force, so static friction is different from kinetic friction in this regard. Now as the body is at rest, a_x will also be zero. Forces along the x-axis will be

$$\sum F_x = F_{Applied} - f_{Friction} = 0 \text{ (as } a_x = 0\text{)}$$

$$F_{Applied} = f_{Friction}$$

and for the y-axis, it will be similar to the previous (as in both kinetic and static cases, there is no motion along the y-axis)

$$\sum F_y = N - mg = ma_y = 0 \text{ (as } a_y = 0\text{)}$$

$$N = mg$$

The maximum possible force that can be produced by static frictions is f_{MAX} and is given by

$$f_{MAX} = \mu_s N$$

Here μ_s is the coefficient of static friction. The higher its value, the greater will be the static frictional force accordingly. There is a difference between kinetic friction and static friction. **Kinetic friction** is a constant force that means whether you go fast or slow, you have to counter that friction but static friction is different as it is not constant. For example, in the case of pulling a plate on the surface (as demonstrated in the video lecture), static friction adjusts depending on how hard we pull (apply the force) in that direction. The values μ_s may vary. If it is a frictionless surface, μ_s will be zero. But if there is a lot of friction, the value of μ_s would probably be one.

However, it is dependent on materials as different materials have different friction with each other. We can increase or decrease the friction between two materials, for example, if there are two materials and we smoothen them then friction gets reduced. If we lubricate any surface by using oil, then a layer of oil is formed which reduces the friction. When a body starts moving, the coefficient of friction is relatively less. If a body is at rest, the force required to initiate the motion is higher as compared to the force required to counter the friction when the body is in motion.

In simple words, the kinetic frictional force is less than the static frictional force. This is also depicted in Figure 5.7. We have to apply more force to make the body move and less force to keep it in motion. We talked about static and kinetic/sliding friction but there is a third one. There is also a type of friction which is called sliding friction. For example, when a wheel passes over a railway track, there is friction there too, but it is the rolling friction. The value of rolling friction is very low in comparison to static and kinetic friction.

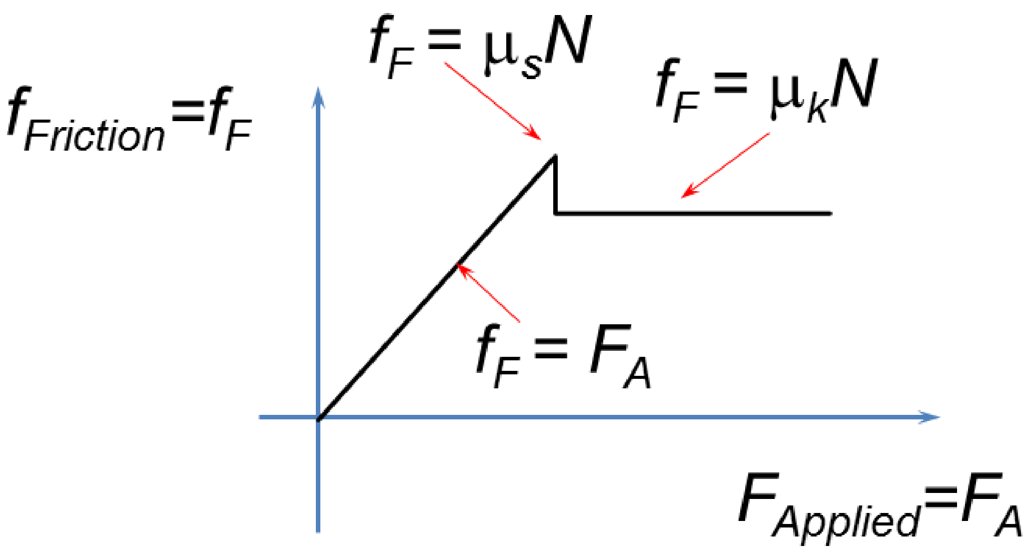


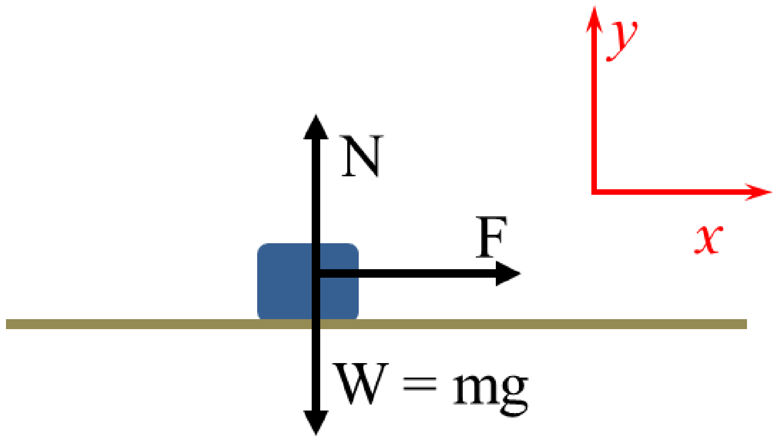
Figure 5.7:

Frictional force ($f_{Friction} = f_F$) against applied force ($F_{Applied} = F_A$) showing that initially magnitude of frictional force is equal to the applied force but in the opposite direction thus body remains at rest. Once f_{MAX} is achieved the body starts moving and frictional force becomes constant i.e. $\mu_k N = \mu_k mg$ which is mostly less than f_{MAX}.

<u>Problem:</u>

Question: A box of mass m = 2 kg slides on a frictionless floor. A force F_x = 10N pushes on it in the x direction. What is the acceleration of the box? What forces acting on the box?

Solution:



We will solve it by applying Newton's second law along the x-axis and y-axis as shown in the figure below.

$$\sum F_x = F = ma_x$$

$$a_x = \frac{F}{m} = \frac{10N}{2kg} = 5\frac{m}{s^2}$$

$$\sum F_y = N - mg = ma_x$$

$a_y = 0$ (as no motion in y-direction) and $N = mg$

Body on an inclined plane and Angle of Repose:

Now we will consider a physical configuration where the body (plate shown in the video lecture) is on a surface/book which can be tilted. As we tilt the surface, the normal component acting on the body, because of the surface below will decrease. If we continue tilting the surface (raising one end of the surface while keeping the second end fixed), the body will start moving/sliding at a certain angle. This angle is called the Angle of Repose as shown by 'α' in Figure 5.8(a).

Now we will determine the value of the angle of repose for the physical configuration shown in Figure 5.8(a), where the body having mass M is placed on an inclined plane having angle α with horizontal which can be increased. The coefficient of static friction μ_s is also provided. The angle of repose can be calculated by determining the point till where the equilibrium condition persists. Figure 5.8(a) shows force *Mg* which is the weight of the body. Its components are also shown.

An unconventional xy-plane coordinate system (shown in by red inset of Figure 5.8(a)) is utilized for the sake of convenience, where weight *Mg* (which is in the vertical direction) is now considered making an angle α with the x-axis. Mg is resolved into its x and y components as per elaboration in Figure 5.8(b). Normal force N is perpendicular to the inclined surface and static frictional force f_s is parallel to the inclined surface. Under equilibrium, forces parallel to the plane (f_s and Mgsinα) are balanced. At an angle of repose (the angle at which the body starts sliding), acceleration due to weight (Mgsinα) will barely overcome static frictional force. Forces perpendicular to the plane (N and Mgcosα) will always be balanced.

$$f_s = Mg\sin\alpha \quad \text{.......5.4}$$

$$N = Mg\cos\alpha \quad \text{.......5.5}$$

Dividing eq. 5.4 by eq. 5.5 will give

$$\frac{f_s}{N} = \frac{Mg\sin\alpha}{Mg\cos\alpha} = \tan\alpha$$

As $f_s = \mu_s N$, above equation can be written as

$$\tan\alpha = \frac{\mu_s N}{N} = \mu_s$$

$$\alpha = \tan^{-1}(\mu_s)$$

Conclusion: The result we have derived is very reasonable and is also very easy to understand. If the friction is eliminated then the angle of repose will become zero.

$$\alpha = \tan^{-1}(\mu_s) \qquad \therefore \mu_s = 0$$
$$\alpha = \tan^{-1}(0) = 0$$

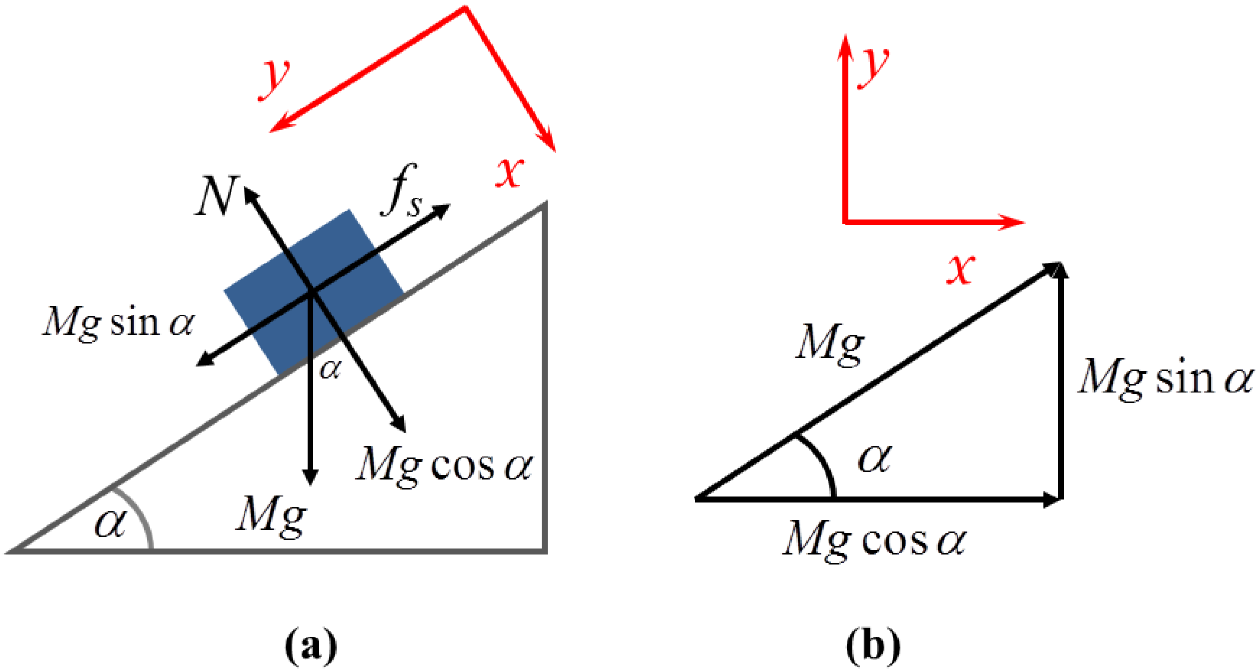


Figure 5.8: (a) Body having mass M on an inclined plane having an angle of inclination α. Note: xy-plane orientation differs from conventional representation where x and y correspond to horizontal and vertical, respectively. (b) Components of weight Mg are shown as per normal xy-plane orientation.

Question: Why are the brakes of a car not so effective on an inclined road as compared to a levelled road?

The reason is that the normal force inclined road gets reduced. Remember that that component is Mgcosα. If α is zero then it is Mg and if α is 90° then it becomes zero. The brakes of a car do not work so well on an inclined road.

Question: Why a four-wheeler can use the brakes better and why does it grip better on the road?

The normal force is the same for that given coefficient of friction but instead of multiplying by two, if we multiply by four then obviously the frictional force increases. And still many concepts related to friction are still to discussed, especially that if we move through a fluid or liquid then why does friction occur? How does it occur? Can we include all effects in equations? We will discuss all these scenarios in our next lecture.

Physics-PHY101-Lecture#06

APPLICATIONS OF NEWTON'S LAWS – II

Introduction:

In previous lecture, we have applied Newton's law on frictional problems for solid objects. As we know, frictional force is produced because of the rubbing of two objects together. But this is not necessary for only solid objects, fluid friction is also possible.

Example:

1. If you wave your hand in the air, apparently you will not feel any frictional force. But if you take an umbrella in your hand and then wave your hand you will feel a frictional force because of air resistance with umbrella.
2. If you pass your hand from water, then you will feel a frictional force easily.

Importance of fluid friction:

Newton's law is very important in fluid friction also because when an object is passing through a fluid, object has to make some space to move in fluid, where the use of pair of forces (action-reaction force) is important. For Example, as a body moves through a body it displaces the fluid. It has to exert a force on the fluid to push it out of the way. By Newton's third law, the fluid pushes back on the body with an equal and opposite force.

The two main conclusions of fluid friction are.

- The direction of the fluid resistance force on a body is always opposite to the direction of the body's velocity relative to the fluid.
- The magnitude of the fluid resistance force usually increases with the speed of the body through the fluid

Q. Think, what will happen without fluid friction in sea water?

Ans: Off course fishes would die there. Fishes would not swim in water and eventually leads to death without fluid friction.

Dependence of fluid power:

The power of fluid to resist the object is depends on,

- The speed of object moving in fluid
- The higher the speed of object usually, maximizes to resist the motion of object.

Fluid frictional force and velocity of object:

According to a linear law, we can say that "The fluid resistance force is proportional to the velocity but if the velocity of the object has increased then fluid resistance force becomes proportional to the square or cube of the velocity".

Limitations of this law:

This law would not be applicable to the case where the speed of the object becomes faster. For solid objects, we use empirical laws (which can be calculated experimentally) but with some limitations.

Example of equilibrium under two forces:

To see how friction/resistance affects different bodies, scientist Galileo performed an experiment at the leaning tower of Pisa. He dropped two balls of different masses (M and m, where M >m) from the top of Pisa tower. Both balls hit the ground at the same time regardless of their masses. These results shocked him for a while because before his experiment, it was thought that the heavier ball hit the ground before the lighter ball, but the conclusion was different. Then Galileo again performed the experiment with a ball and a feather. This time he finds that the ball hit the ground before feather even though both have same gravitational force act on them. This experiment leads to the conclusion that there is some other resistive force (frictional force) which restricts their motion. Now in order to see, how resistance affects acceleration and velocity of feather, we have the following,

Let's, the magnitude of fluid resistance 'f' is almost proportional to the speed of the body through the fluid given as,

$$f \propto v$$
$$f = kv$$

Where, "k" is called the proportionality constant. As, the upward resistive force on feather when it's falling downward is always in opposite direction to the motion of object. Hence,

Upward resistive force= $F_r = -\ kv$

Downward weight force = $F_w = mg$

The net force acting on the object is,

$$F_w + F_r = F$$

$$mg + (-kv) = ma$$

After certain interval, acceleration 'a' of the feather becomes constant (a=0) and attains a velocity called as terminal velocity (V_T) which is given as,

$$mg - kv_T = 0$$

$$v_T = \frac{mg}{k}$$

This was a simple example of equilibrium between two forces.

Problem solving steps:

In general, while solving actual problems one should do the following steps,

- Draw a free body diagram.
- Define an origin for a system of coordinates.
- Define x, y coordinate system.
- Identify all forces (tension, normal, friction, weight, etc.) and their x & y-components.
- Apply Newton's 2nd law separately on x-axis and y-axis.

$$\sum F_x = ma_x$$

$$\sum F_y = ma_y$$

- Find the unknown quantities i.e.,
- Acceleration, velocity, and displacement:

$$F = ma_x$$
$$V = V_o x + a_x t$$
$$X = X_o + V_o x + \frac{1}{2} a_x t^2$$

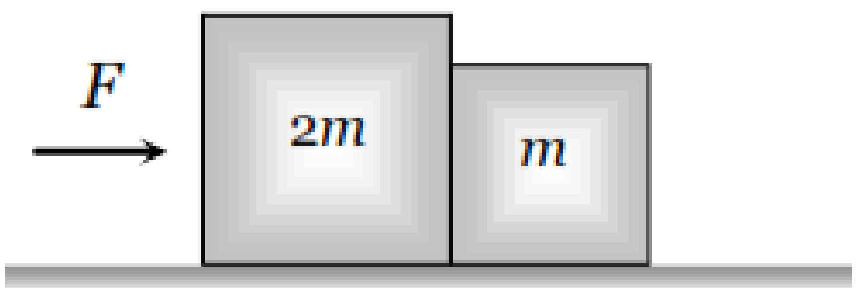
- Forces:

$$F_w = mg \text{ (gravitational force)}$$
$$f = \mu N \text{ (frictional force)}$$

Applications to solve frictional problems:

Problem 1:

Consider two blocks of mass 2m and m on a frictionless surface and a force F is applied on 2m block from left side. As the two blocks acts as one body, hence, begins to move towards right side. We have to calculate force on mass ‘m’ due to mass ‘2m’. As shown in figure.



Solution:

Acceleration of the blocks can be calculated as,

$F = ma$

$Total\ mass = 2m + m$

$$F = (2m+m)a \Rightarrow F = 3ma \Rightarrow\Rightarrow a = \frac{F}{3m}$$

Now isolate block 'm' and then apply newtons 2nd law,

$$\sum F = ma \qquad \therefore a = \frac{F}{3m}$$

$$F_{2on1} = m\left(\frac{F}{3m}\right) \Rightarrow F_{2on1} = \frac{F}{3}$$

This is the force exerted on mass m from mass 2m. Similarly, the force on mass 2m from mass m is,

$$F_{1on2} = 2m\left(\frac{F}{3m}\right) = \frac{2F}{3}$$

Conclusion: Results show that the force acts on m block because the force of 2m block is just one third ($1/3^{rd}$) of the total force.

Limitation of newtons law:

Newtons laws are only applicable to inertial frame of reference.

Difference between inertial and non-inertial frame of reference:

Inertial frame	▪ The frame of reference which is moving with uniform velocity and does not accelerate (a=0). ▪ Obeys Newton’s law of motion. ▪ **For example**, a train moving with uniform velocity is an inertial frame of reference.
Non-inertial frame	▪ The frame of reference which is accelerating ($a \neq 0$) is called non-inertial frame of reference. ▪ Does not obey Newton’s law of motion. ▪ **For example**, a freely falling elevator is taken as non-inertial frame of reference.

Problems in non-inertial frame of reference:

Problem 2:

Suppose you are in a lift. Here we have to deal with different cases:

- **Case-1:** where, you are at rest or moving at constant velocity (a=0) then normal force N (i.e., the force with which the floor of the lift is pushing on you) and the force due to gravity are exactly equal,

$$N - Mg = 0$$
$$N = Mg$$

The relation shows the apparent weight is equal to the true weight just for the condition that the acceleration must be equal to zero.

- **Case-2:** if the lift is accelerating upwards then,

$$N - Mg = Ma$$
$$N = M(g + a)$$

These relations show that the apparent weight is more than the true weight (you feel heavier) as there is an added gravitational (g) force of 9.8 m/s^2. Acceleration would add an extra gravitational force, making you feel twice as heavy.

- **Case-3:** if the lift is accelerating downwards then,

$$N - Mg = Ma$$
$$N = M(g - a)$$

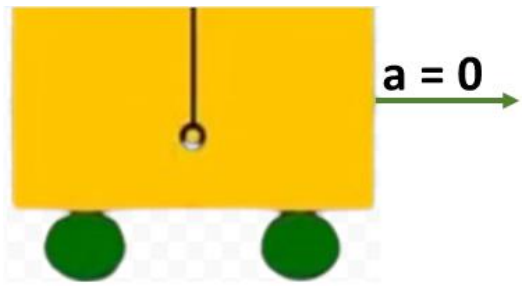
This relation shows that the apparent weight is less than the true weight (you feel lighter).

- **Case-4:** if the cable, supporting the lift is breaks, the lift falls downward with a= g. then

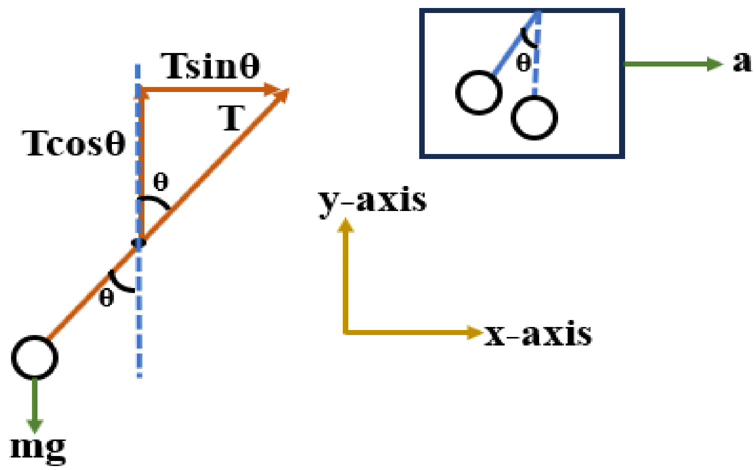
$$N = M(g - g) = 0$$

This shows the apparent weight under the free fall is zero and you will experience weightlessness just like astronauts in space do.

Problem 3: Imagine that you are in a railway wagon and want to know how much you are accelerating. You are not able to look out of the windows. A mass is hung from the roof. Find the acceleration of the car from the angle made by the mass.



Solution:



According to the balancing of forces,

Forces acting horizontally,

$\sum F_x = T\sin\theta$

$ma = T\sin\theta \rightarrow 1)$

Forces acting vertically,

$\sum F_y = T\cos\theta - mg$

$0 = T\cos\theta - mg$

$T\cos\theta = mg \rightarrow 2)$

On dividing equation 1) and 2), we get

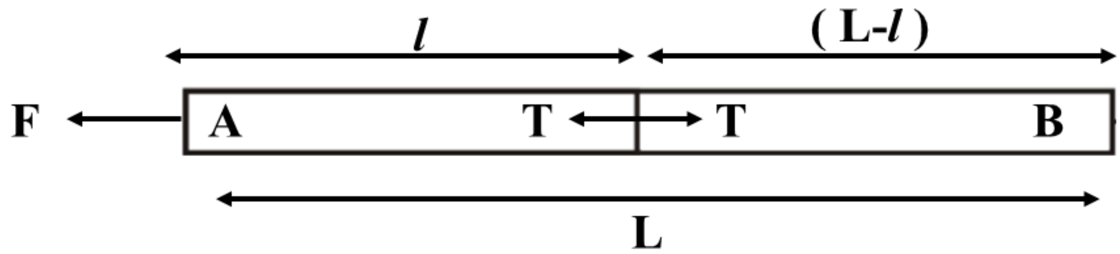
$$\frac{T\sin\theta}{T\cos\theta} = \frac{ma}{mg}$$

$$\tan\theta = \frac{a}{g}$$

Conclusion: Result shows that the mass doesn't matter during the motion.

Problems in inertial frame of reference:

Problem 4: A uniform rope of length L, lying on a horizontal smooth floor, is pulled by a horizontal force F. What is the tension in the rope at a distance l from the end where the force is applied?



Solution:

Massive rope have different value of tention at different position, $\sum F_x = F - T$

$\left(\text{mass of rope upto length '}l\text{'}\right) a = F - T$

$\left(ml\right) a = F - T \Rightarrow\Rightarrow T = F - mla \rightarrow 1)$

$\because m = \frac{M}{L} = \frac{mass}{length} = \text{linear mass density}$

$\because F = Ma, \because F = mLa \rightarrow 2)$

plug in equation 2) in 1), we get,

$T = mLa - mla = ma\left(L - l\right)$

$T = maL\left(1 - \frac{l}{L}\right) = F\left(1 - \frac{l}{L}\right) \rightarrow 3$

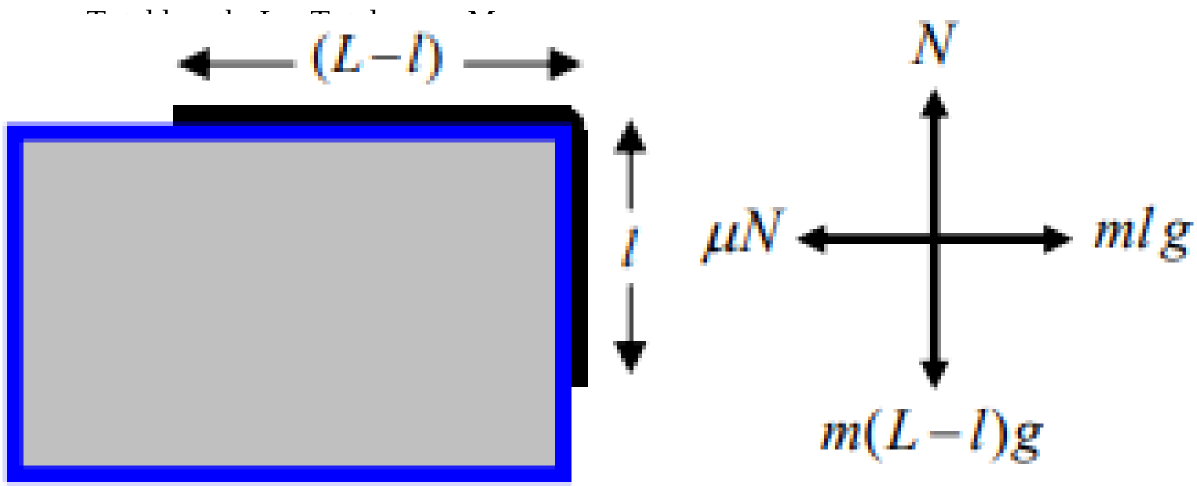
Checking:

a. If $l = 0$ then, T=F(as, this is the point where we have applied force)

b. If $l = L$ then, T=0(as, at the end of the rope, there is nothing (no force) to act upon)

<u>Problem 5:</u> A rope of total length L and mass per unit length m is put on a table with a length l hanging from one edge. What should be length l such that the rope just begins to slip?

<u>Solution:</u> To solve this, look at the balance of forces in the diagram.



$$\sum F_y = N - mg\left(L-l\right)$$

$$0 = N - mg\left(L-l\right) \Rightarrow N = mg\left(L-l\right) \rightarrow 2$$

Putting eq.2 in eq.1,we get

$$\mu N = mgl$$

$$\mu\left(mg\left(L-l\right)\right) = mgl$$

$$\mu mgL - \mu mgl = mgl$$

$$\mu mgL = \mu mgl + mgl$$

$$\mu mgL = mg\left(l + \mu l\right) \Rightarrow \mu L = \left(\mu + 1\right)l$$

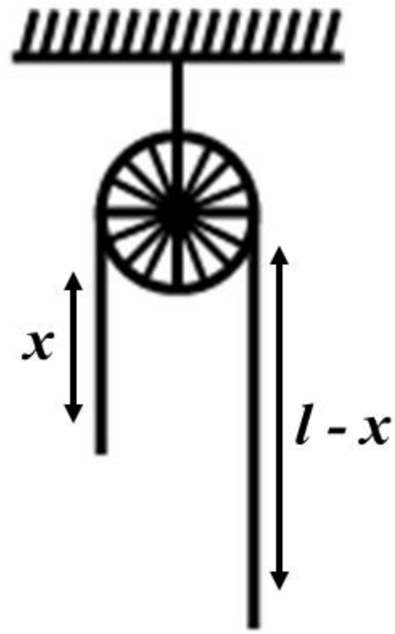
$$l = \frac{\mu L}{\left(\mu + 1\right)}$$

<u>Conclusion:</u>

- Note that if μ is very small (μ=0) then, even a small piece of string that hangs over the edge will cause the entire string to slip down.
- And if you take μ to be a very large number then, a very small portion of the length of rope is enough to stay on table.

Problem 6: A massive rope of total length l is passing through a pulley as shown in figure. The larger length of rope is moving downward with some acceleration which we have to calculate here.

Solution:



Total length $= l$

mass per unit length= m

on the right side of fig.

$$\sum F_y = m(l-x)g - mgx$$

$$(\text{total mass of the rope})a = m\,\text{lg} - mxg - mxg$$

$$mla = ml\,\text{g} - 2mxg$$

$$mla = m(\text{lg} - 2xg) \Rightarrow la = (\text{lg} - 2xg)$$

$$a = \frac{\text{lg}}{l} - \frac{2xg}{l} \Rightarrow g - \frac{2xg}{l}$$

$$a = g\left(1 - \frac{2x}{l}\right)$$

Checking; if,

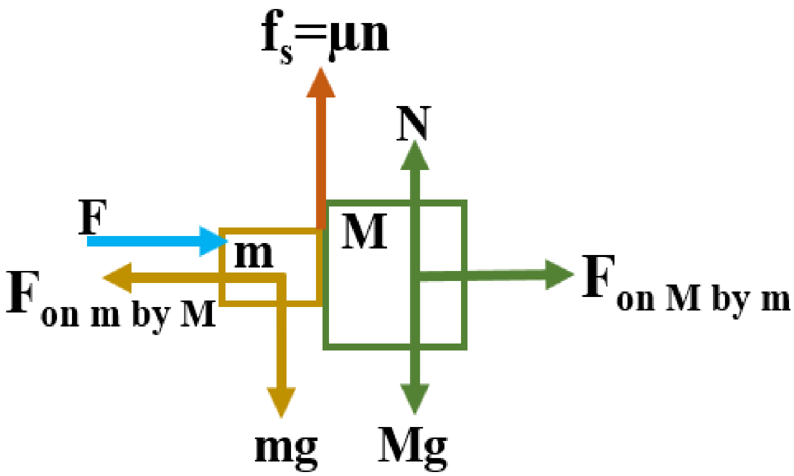
$$x = \frac{l}{2} \;\Rightarrow a = g\left(1 - \frac{2\frac{l}{2}}{l}\right)$$

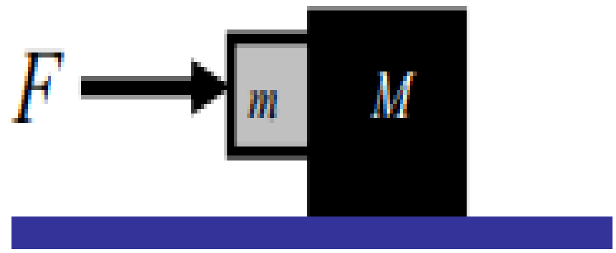
$$a = g$$

Both sides of the rope will have equal acceleration.

Problem 7: In this problem, two masses M and m attached together. We would like to calculate the minimum force such that the small block does not slip down.

Solution:





Since masses moves together so, $\sum F_x = F$

$(m+M)a = F \Rightarrow a = \dfrac{F}{m+M} \rightarrow 1$

At contact surface, $\sum F_y = f_s - mg$

$0 = f_s - mg \Rightarrow f_s = mg \quad \Rightarrow\Rightarrow mg \le f_s \rightarrow 2$

Where, $f_s = \mu n \Rightarrow mg \le \mu n \rightarrow 3$

$n =$ force by M on m $\Rightarrow n = ma$

$n = (\text{mass of box on which force act})(\text{acceleration of the system})$

$n = m\left(\dfrac{F}{m+M}\right) \rightarrow 4 \qquad \because a = \dfrac{F}{m+M}$

Using eq. 4 in eq.3, we get,

$mg \le \mu m\left(\dfrac{F}{m+M}\right) \Rightarrow g \le \mu\left(\dfrac{F}{m+M}\right)$

$\mu \ge \dfrac{g(m+M)}{F} \rightarrow 5$

Where, μ is a dimensionless quantity as it's the ratio of two forces.

Conclusion:

- We want the friction 'µn' to be at least as large as the downwards force 'mg' so that, it minimizes the horizontal force to prevent it from slipping.

Problem 8: Suppose two bodies m_1 and m_2 attached to a string and pass over the pulley. Mass m_1 is on the frictionless table and the forces acted on the masses are shown in figure. We have to calculate the tension and acceleration produced in the string.

Solution: If the string is massless then the tension in rope would be same at all points. So according to the balancing of forces, one can calculate as following:

Lets calculate the forces on m_1 & $m_2$first,

Forces on m_1;

$$\sum F_x = T \Rightarrow m_1 a = T \rightarrow 1$$

$$\sum F_y = N - m_1 g$$

$$0 = N - m_1 g \Rightarrow N = m_1 g$$

Forces on m_2;

$$\sum F_y = m_2 g - T \Rightarrow m_2 a = m_2 g - T \rightarrow 2$$

To find "a", putting eq. 1 in eq. 2,

$$m_2 a = m_2 g - m_1 a$$

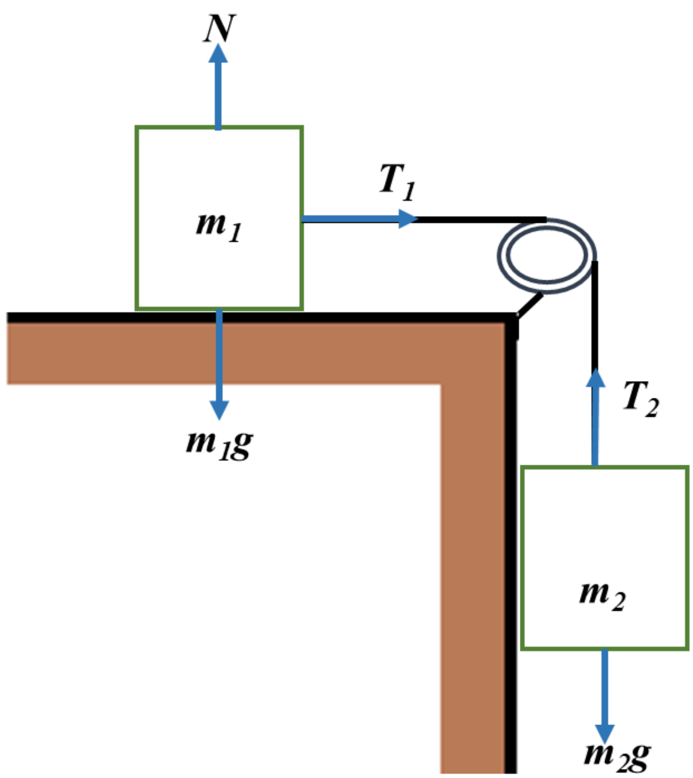
$$(m_1 + m_2)a = m_2 g \Rightarrow a = \left(\frac{m_2}{m_1 + m_2}\right) g \rightarrow 3$$

Now, to calculate "T", using eq.1 in eq.2,

$$m_2 \frac{T}{m_1} = m_2 g - T \Rightarrow m_2 T = m_1 m_2 g - m_1 T$$

$$(m_1 + m_2)T = m_1 m_2 g$$

$$T = \left(\frac{m_1 m_2}{(m_1 + m_2)}\right) g \rightarrow 4$$

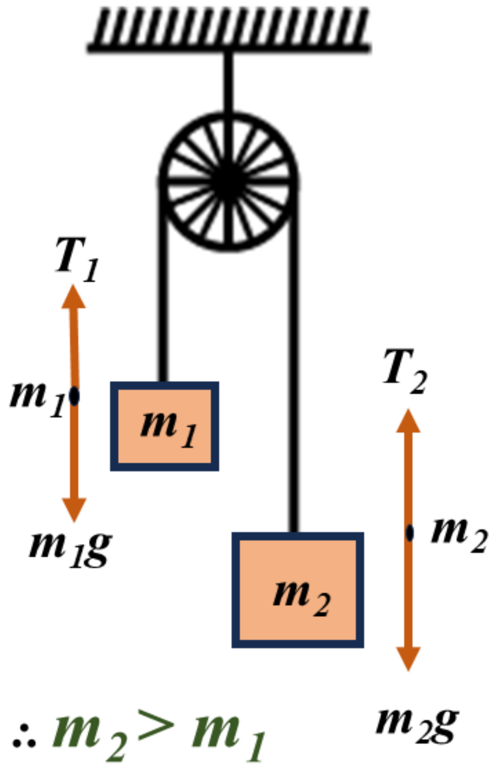


Checking:

- If we put $m_1 = m_2$ in eq.3 and consider both masses as one mass, then it concludes that half of the mass is accelerated downward, and rest of the half is not. We can say that half of the mass is responsible for this acceleration.
- If m_1 or $m_2 = 0$ in eq.4, then T (tension in string) becomes zero.

Problem 9: Consider two bodies of unequal masses m_1 and m_2 connected by the ends of a string, which passes over a frictionless pulley as shown in the diagram. We have to calculate the tension in the string and acceleration of the masses.

Solution:



Forces on m_1;

$$\sum F_y = T - m_1 g$$

$$m_1 a = T - m_1 g \rightarrow 1$$

Forces on m_2;

$$\sum F_y = m_2 g - T$$

$$m_2 a = m_2 g - T \rightarrow 2$$

On subtracting, eq.1 & eq.2, we will get acceleration as,

$$m_1 a = T - m_1 g$$

$$\underline{m_2 a = m_2 g - T}$$

$$(m_1 + m_2) a = (m_2 - m_1) g$$

$$a = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) g \rightarrow 3$$

To calculate 'T', multiply eq.1 to m_2

and multiply eq.2 to m_1 and then on subtracting, we get

$$m_1 m_2 a = m_2 T - m_1 m_2 g$$

$$\underline{m_1 m_2 a = m_1 m_2 g - m_1 T}$$

$$0 = (m_1 + m_2) T - 2 m_1 m_2 g$$

$$(m_1 + m_2) T = 2 m_1 m_2 g$$

$$T = \left(\frac{2 m_1 m_2}{m_1 + m_2}\right) g \rightarrow 4$$

Conclusion: If $m_1 = m_2$ then, the body would not be able to accelerate because at both sides, masses are equal so, $a = 0$.

WORK AND ENERGY

Definition of Work:

“Force applied in direction of displacement x displacement”.

This means that the force F acts at an angle θ with respect to the direction of motion as shown in figure 7.1.

$$W = \vec{F}.\,\vec{d} = Fdcos\theta$$

a) Work is a scalar - it has magnitude but no direction.

b) Work has dimensions: (MLT^{-2}. $L = ML^2T^{-2}$)

c) Work has units: Newton · Meter ≡ Joule (J)

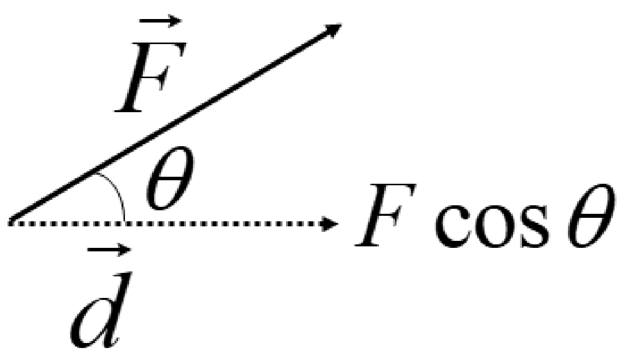


Figure 7.1: Work done by a constant force F.

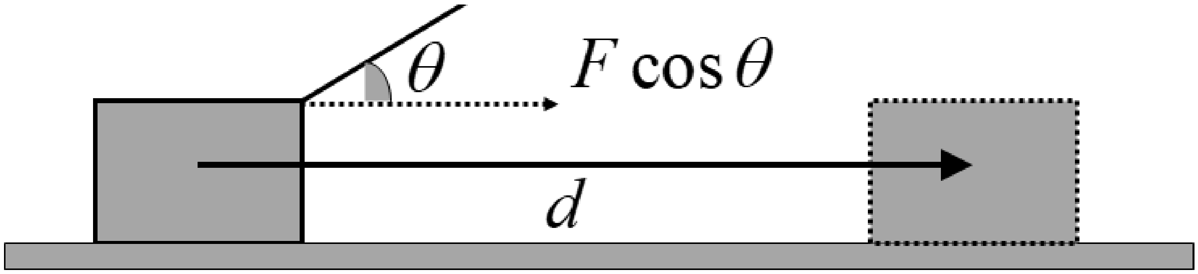
Nature of work:

- **Positive work:** if the applied force displaces the object in its direction, then the work done is known as positive work. θ = 0°
- **Negative work:** if the force and displacement work in the opposite direction, then the work done is known as negative work. θ = 180°
- **Zero work;** If the force and displacement act perpendicular to each other, then the work done is known as zero work. θ = 90°

Forces do work on objects:

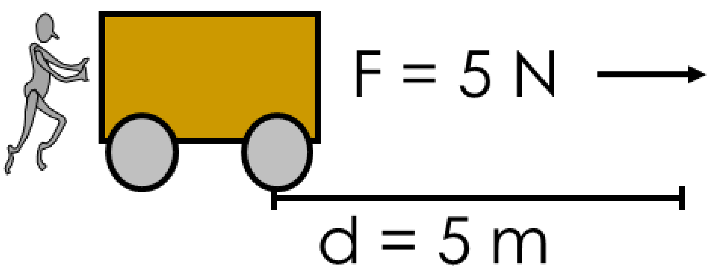
If a crate is pulled along the floor, only the force component parallel to the displacement d contributes to the work as shown in figure 7.2.

Figure 7.2: Work done by a force F in displacing object through displacement d.



We will justify this by following problem.

<u>Problem 1:</u> Suppose a man pushing a cart. A 5N force acts on it and displaces the cart through a displacement of 5m. As the force and displacement both are in same direction (parallel) to each other, calculate the work done on cart.



Solution: As we know,

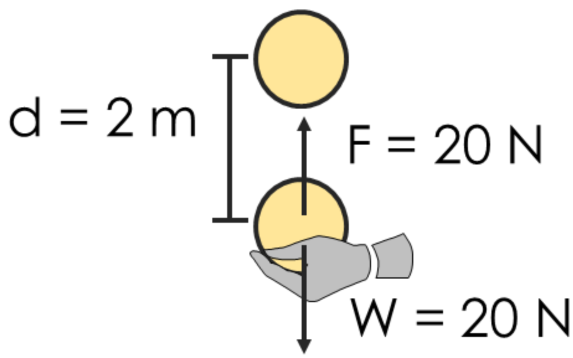
$$W = F.d\cos\theta = 5.5\cos 0 = 25\ (1)\ Nm = 25\ J$$

<u>Problem 2:</u> Suppose an apple falling from a tree. Due to its weight (1N), a force of gravity acts on it. Let suppose if it covers a displacement of 3m, then calculate the work done on apple due to gravity.

Solution: As we know,

$$W = F.d\cos\theta = 1N \times 3m = 3\ Nm = 3J$$

Problem 3: A heavy weight of 20 N is lifted from a height of 2m. calculate the work done on the heavy weight.



Solution: As we know,

$$W = F.d\cos\theta = 20N \times 2m\cos\theta$$

Here, we have to deal with two cases:

Work on ball by F_{hand} = 20 N x 2 m cos0 = 40 J (+ve work done)

Work on ball by $F_{gravity}$ = 20 N x 2 m cos180 = – 40 J (because force and displacement are anti-parallel to each other i.e., -ve work done)

Conclusion:

- If a mass is attached to a string and is moving in a circular motion, tension T is produced in a string. But this tension is not responsible for work as, T is perpendicular to the d (displacement).
- If the angle between two quantities is 90°, then it contributes towards no work.

Work done by a variable force:

Let the force F (x) act in the x direction, and let it vary in magnitude with x according to the function F(x) as shown in figure 7.3. What is the work done when the body moves from some initial position to some final position?

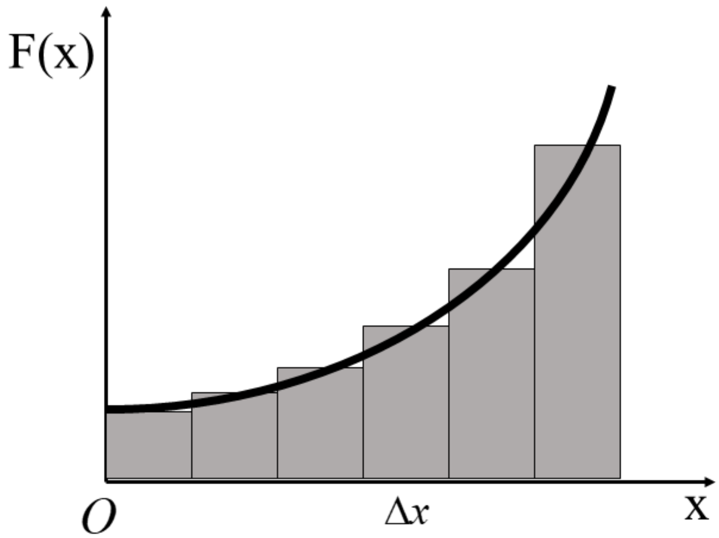


Figure 7.3. Variation of Force with Respect to x. The graph illustrates the dynamic nature of the force applied, showing fluctuations in response to changes in x.

What if the force varies with distance (say, a spring pulls harder as it becomes longer). In that case, we should break up the distance over which the force acts into small pieces so that the force is approximately constant over each bit. As we make the pieces smaller and smaller, we will approach the exact result.

$$\begin{aligned}\Delta W_1 &= F_1 \Delta x \\ \Delta W_2 &= F_2 \Delta x \\ \Delta W_3 &= F_3 \Delta x\end{aligned}$$

Now add up all the little pieces of work:

$$\begin{aligned}W &= \Delta W_1 + \Delta W_2 + \cdots + \Delta W_N \\ &= F_1 \Delta x + F_2 \Delta x + \cdots + F_N \Delta x\end{aligned}$$

To get the exact result let $\Delta x \to 0$ and the number of intervals N $\to \infty$:

$$W = \sum_{n=1}^{N} F_n \Delta x \qquad\qquad W = \lim_{\Delta x \to 0} \sum_{n=1}^{\infty} F_n \Delta x$$

Definition: $\lim_{\Delta x \to 0} \sum_{n=1}^{\infty} F_n \; \Delta x \equiv \int_{x_i}^{x_f} \mathrm{F(x)dx}$

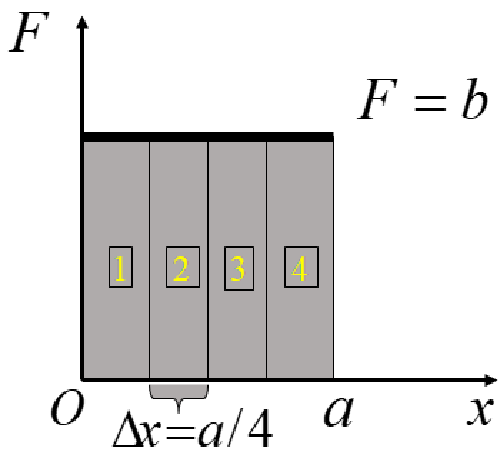
is the integral of F with respect to x from x_i to x_f.

This quantity is the work done by a force, constant or non-constant. So, if the force is known as a function of position, we can always find the work done by calculating the definite integral.

Example (for constant force):

Let suppose we have a rectangle, which is divided into 4 parts.

Area =



$$W = \int_0^a F dx$$

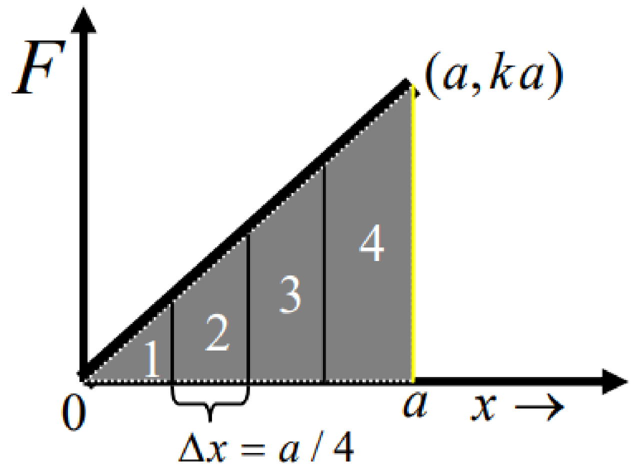
$\frac{1}{4}a(b)+\frac{1}{4}a(b)+\frac{1}{4}a(b)+\frac{1}{4}a(b)=ab$ For a constant force, $W = F\int_0^a dx = F\left|x\right|_0^a$

$$\therefore F = b$$

$$W = b\left(a-0\right) = ab$$

Example (for variable force):

Let suppose we have a triangle, which is divided into 4 parts. Here, F = kx, force is proportional to x i.e., the force increases linearly with x.



$$\text{Area} = \frac{1}{2} * base * height$$

From the fig. base is "a" and height is "F", $\therefore$ F = ka

$$\text{Area of shaded region} = \frac{1}{2}(a)(ka) = k\frac{a^2}{2}$$

Mathematically, work done is given as, W=$\int Fdx$

$$W = \int_0^a Fdx = \int_0^a kxdx = \frac{k}{2}\left|x^2\right|_0^a = \frac{k}{2}\left(a^2 - 0\right) = k\frac{a^2}{2}$$

Example: Calculation of area under the curve:

Suppose we have a parabolic function y = x^2 as shown in figure 7.4. The parabola is divided into 4 parts so, Δx = 1/n = 1/4 (as maximum value on x and y-axis is "1") and evaluated at four different points as shown in figure.

Area under the curve between 0 and 1 is approximately, $\sum_{i=1}^{n} y_i \Delta x$

As, number of rectangles, n=4, hence, $\Delta x = \dfrac{1.00}{n} \to 1$

$x_i = \left(i - \dfrac{1}{2}\right)\Delta x \Rightarrow$ putting the value of $\Delta x = \dfrac{1.00}{n}$

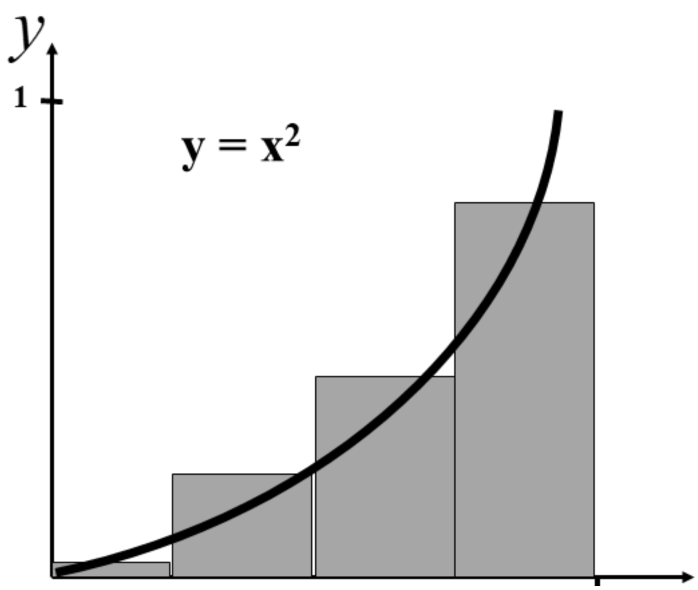


Figure 7.4: Graph of the Parabolic Function $y = x^2$. The curve represents the quadratic relationship between the independent variable x and the dependent variable y, forming a symmetric parabola centered at the origin.

$$x_i = \left(i - \frac{1}{2}\right)\frac{1.00}{n} = \frac{\left(i - \frac{1}{2}\right)}{n}$$

As, the function is, $y_i = x_i^{\,2}$, hence, $y_i = \dfrac{\left(i - \frac{1}{2}\right)^2}{n^2} \rightarrow 2$

Since, area under the curve is, $\sum_{i=1}^{n} y_i \Delta x, so$

$$\sum_{i=1}^{n} y_i \Delta x = \sum_{i=1}^{n} \frac{\left(i - \frac{1}{2}\right)^2}{n^2} . \frac{1.00}{n} = \sum_{i=1}^{n} \frac{\left(i - \frac{1}{2}\right)^2}{n^3}$$

Now lets solve R.H.S of the equation,

$$\sum_{i=1}^{n} \frac{\left(i - \frac{1}{2}\right)^2}{n^3} = \sum_{i=1}^{n} \frac{1}{n^3}\left(i^2 + \frac{1}{4} - i\right) \Rightarrow \ = \frac{1}{n^3}\left[\sum_{i=1}^{n} i^2 + \frac{1}{4}\sum_{i=1}^{n} 1 - \sum_{i=1}^{n} i\right] \rightarrow 3$$

Using properties of summation,

$$\because \sum_{i=1}^{n} 1 = n, \because \sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \because \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Now 3 equation becomes,

$$= \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6} + \frac{1}{4}n - \frac{n(n+1)}{2}\right] \Rightarrow = \frac{1}{n^3}\left[\frac{(n^2+n)(2n+1)}{6} + \frac{n}{4} - \frac{(n^2+n)}{2}\right]$$

$$= \frac{1}{n^3}\left[\frac{2n^3 + n^2 + 2n^2 + n}{6} + \frac{n}{4} - \frac{(n^2+n)}{2}\right] \Rightarrow = \frac{1}{n^3}\left[\frac{4n^3 + 2n^2 + 4n^2 + 2n + 3n - 6n^2 - 6n}{12}\right]$$

$$= \frac{1}{n^3}\left[\frac{4n^3 - n}{12}\right] = \frac{n(4n^2-1)}{12n^3} = \frac{(4n^2-1)}{12n^2} = \frac{4n^2}{12n^2} - \frac{1}{12n^2} = \frac{1}{3} - \frac{1}{12n^2}$$

$$\sum_{i=1}^{n} \frac{\left(i - \frac{1}{2}\right)^2}{n^3} = \frac{1}{3} - \frac{1}{12n^2}$$

putting $n = 4$, $\sum_{i=1}^{n} \dfrac{\left(i - \frac{1}{2}\right)^2}{n^3} = 0.32812$

Now if we divide the parabola into 8 parts (n = 8) then the result will be: $\sum_{i=1}^{n} \dfrac{\left(i - \frac{1}{2}\right)^2}{n^3} = 0.33203$

And if n = 16, then, $\sum_{i=1}^{n} \frac{\left(i - \frac{1}{2}\right)^2}{n^3} = 0.33301$

And if n = 32, then, $\sum_{i=1}^{n} \frac{\left(i - \frac{1}{2}\right)^2}{n^3} = 0.33325$

Now if we divide the parabola into infinite parts (n = ∞) then the result will be :

$$\sum_{i=1}^{n} y_i \Delta x = \lim_{\Delta x \to 0} \sum_{i=1}^{n} y_i \Delta x = \int_0^1 x^2 dx = \lim_{n \to \infty} \left[\frac{1}{3} - \frac{1}{12n^2} \right] = \frac{1}{3} = 0.3333$$

In general, if we integrate a function $\int_0^1 x^n dx$ then the result would be $= \frac{1}{n+1}$. In this way we can calculate the total work done by a variable force.

Introduction to Energy:

In physics work and energy is of great importance. Energy is the capacity of a physical system to do work.

- It comes in many forms – mechanical, electrical, chemical, nuclear, etc.
- It can be stored
- It can be converted into different forms
- It can never be created or destroyed

Types of Energy:

1) **Elastic energy:**

   Elastic energy is stored in an elastic object - such as a coiled spring or a stretched elastic band. Elastic objects store elastic energy when a force causes them to be stretched.

2) **Gravitational energy:**

   Gravitational energy associated with gravity or gravitational force. In other words, the energy held by an object when it is in a high position compared to a lower position.

3) **Electrical energy:**

   Electrical energy is the movement of electrons. Lightning is an example of electrical energy.

4) **Chemical energy:**

Chemical energy is stored in the bonds of atoms and molecules. It is the energy that holds these particles together. Stored chemical energy is found in food, biomass, petroleum, and natural gas.

5) **Thermal energy:**

Thermal energy is created from the vibration of atoms and molecules within substances. The faster they move, the more energy they possess and the hotter they become. Thermal energy is also called heat energy.

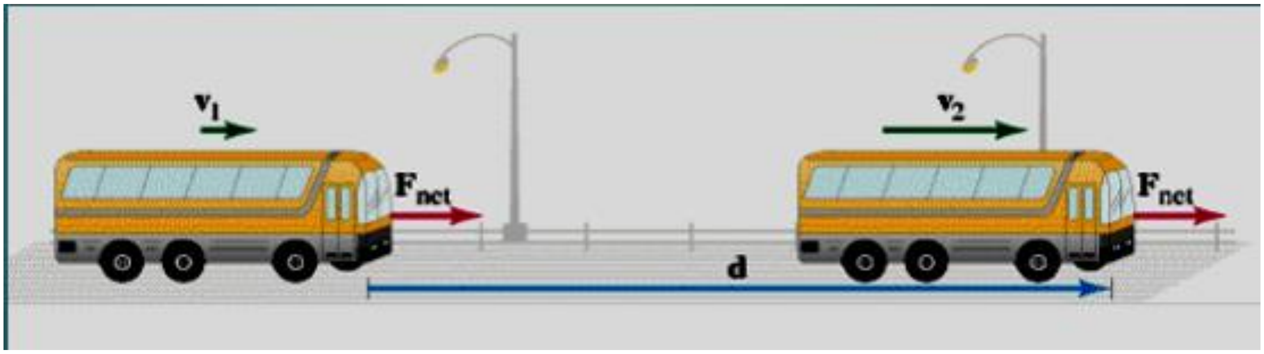
6) **Nuclear energy:**

Nuclear energy is stored in the nucleus of atoms. This energy is released when the nuclei are combined (fusion) or split apart (fission). Nuclear power plants split the nuclei of uranium atoms to produce electricity.

7) **Sound energy:**

Sound energy is the energy produced due to vibrations. There is usually much less energy in sound than in other forms of energy.

Problem 4: A constant force accelerates a bus (mass m) from speed v_1 to speed v_2 over a distance d. What work is done by the engine?



Solution:

Recall: $v_2^2 - v_1^2 = 2a(x_2 - x_1)$

where: v_2 = final velocity, x_2 = final position

v_1 = initial velocity, x_1 = initial position

$$\therefore \quad a = \frac{v_2^2 - v_1^2}{2d}$$

To calculate work:

$W = F\,d$

$= ma\,d$

$= m \dfrac{v_2^2 - v_1^2}{2d}\, d$

$= \frac{1}{2} m{v_2}^2 - \frac{1}{2} m{v_1}^2$

Define Kinetic Energy:

$KE = \frac{1}{2} mv^2$

so, work done is equal to the change in kinetic energy.

$W = \Delta K.E$

<u>Problem 5:</u> A truck weighs 20 times more than a rickshaw but moves 5 times slower. Which has more kinetic energy?

Solution:

$$KE\ (\text{rickshaw}) = \tfrac{1}{2} m v^2$$

$$KE\ (\text{truck}) = \tfrac{1}{2}(20m)(v/5)^2$$

$$= \tfrac{20}{25} \cdot \tfrac{1}{2} m v^2$$

As, the speed of truck is 5 times less than the rickshaw, hence, the kinetic energy of truck is less than the rickshaw.

<u>Work Energy Principle:</u>

Net work done on object = Change in $K.E$ of object

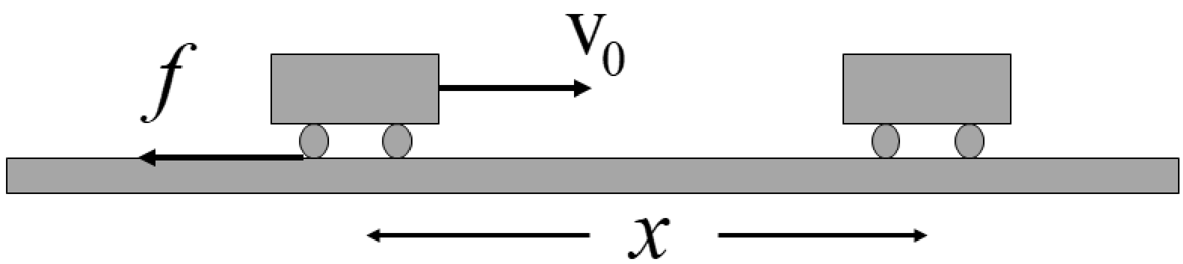
Work can be:

- Positive (K.E increases)
- Negative (K.E decreases)

<u>Units of energy:</u> Energy has the same units as work:

Joule = Newton. Meter = Nm

Problem 6: A car is moving on a road, and a frictional force acts on it. We want to know how far the car travel before it comes to rest.



Solution: According to Work Energy Principle,

$$W_{net} = K_f - K_i = 0 - K_i$$

$$-f\,x = -\frac{1}{2}m\text{v}_0^{\ 2} \quad \text{but } f = \mu N = \mu mg$$

k.E consumed by the frictional work is, $f\,x = \frac{1}{2}m\text{v}_0^{\ 2}$

$$\mu mgx = \frac{1}{2}m\text{v}_0^{\ 2} \quad \Rightarrow \quad x = \frac{\text{v}_0^{\ 2}}{2\mu \text{g}}$$

Checking:

- If $\mu = 0$, then x = ∞, which means, in the absence of friction, car does not stop and move through an infinite distance.
- If $\mu = \infty$, then x = 0, which means, car would not move or cover a distance because of infinite friction.

Introduction to power:

The work done by a force is just the force multiplied by the distance – it does not depend upon time. But suppose that the same amount of work is done in half the time. We then say that the power is twice as much.

Power is the "rate of doing work"

$$\text{Power} = \frac{\text{Work done}}{\text{Time taken}}$$

If the force does not depend on time:

$$\frac{\text{Work}}{\text{Time}} = \frac{F\,\Delta x}{\Delta \text{t}} = F\ \text{v}$$

$\therefore$ Power = F v

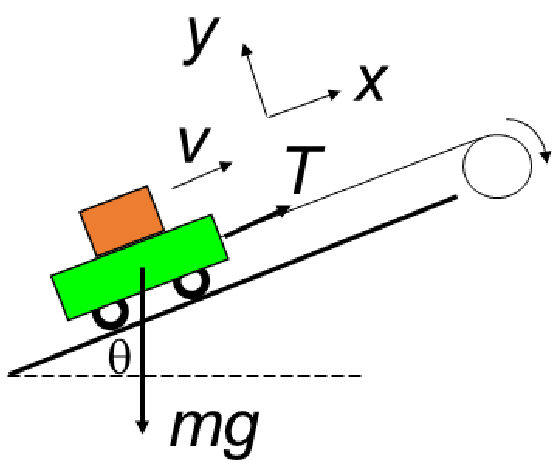
Units of power: J/sec = Watts

Old units: horsepower (hp)

1 hp = 746 W = 0.746 kW

<u>Problem 7:</u> A 2000 kg trolley is pulled up a 30° hill at 20 mi/hr by a rope. How much power is the machine providing?

Solution:



- The power is $P = F\,v = T\,v$
- No acceleration $\Rightarrow$ no net force
- Balance forces along and normal to plane

In the x direction: $T = mg\sin\theta$

v = 20 mi/hr = 8.93 m/s

g = 9.8 m/s^2

m = 2000 kg

Sinθ = sin(30°) = 0.5

P = (2000 kg) . (9.8 m/s^2) (8.93 m/s) (0.5)

P = 88,000 W (power of machine) = 88 kw

Physics-PHY101-Lecture 8

Conservation of Energy

Potential energy:

Potential energy is, as the word suggests, the energy "locked up" up somewhere and which can do work. It can be defined as, "**energy possessed by a body due to its position**".

It has the ability or capacity to do work like other forms of energies. Potential energy can be converted into kinetic energy (1/2 mv^2) and vice versa.

Formula: Potential energy can be calculated using the following formula

$$P.E = m.g.h$$

Where,

- m is the mass in kilograms
- g is the acceleration due to gravity (10 m/s^2)
- h is the height in meters

SI unit:

$$P.E = m.g.h = kg.ms^{-2}.m = kgm^2s^{-2} = \text{Joule (J)}$$

Types of potential energy:

The types of potential energy are,

1. Elastic potential energy
2. Gravitational potential energy

Elastic potential energy:

When an object is compressed or stretched, the energy stored in object is called elastic potential energy. The more the object stretch or compress, the more elastic potential energy is.

Examples:

- A twisted rubber band that powers a toy plane
- An archer's stretched bow
- A bent diver's board just before a diver dive in

Formula: Elastic potential energy can be calculated using the following formula,

$$U = \frac{1}{2}kx^2$$

Where,

- U is the elastic potential energy
- k is the spring force constant
- x is the string stretch length in m

Problem 1:

Suppose you pull on a spring and stretch it by an amount away from its normal (equilibrium) position. How much energy is stored in the spring?

Solution:

Obviously, the spring gets harder and harder to pull as it becomes longer. When it is extended by length x and you pull it a further distance dx, the small amount of work done is dW = Fdx = kxdx. Adding up all the small bits of work gives the total work:

$$W = \int_0^x F dx = \int_0^x kx dx = \frac{1}{2}kx^2$$

Gravitational potential energy:

The gravitational potential energy of an object is defined as "the energy possessed by an object that raise to a certain height against gravity. As the object is raised against the force of gravity, some amount of work (W) is done on it.

Work done on the object = force × displacement

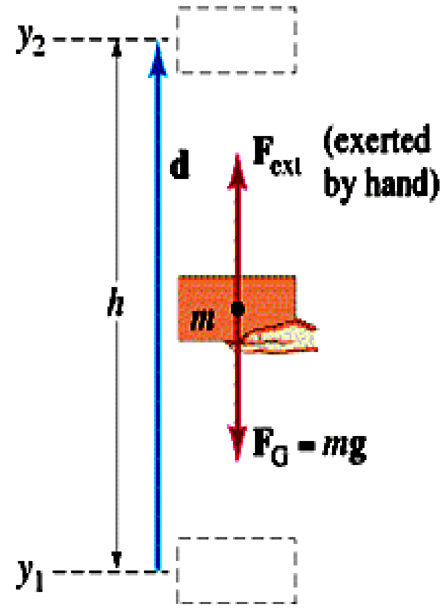
W = F.h

W = mg.h

(a = g = gravitational acceleration)

$$W = P.E = m.g.h$$

Example: If you lift a stone of mass from the ground up a distance, you have to do work against gravity. By conservation of energy, the work done by you was transformed into gravitational potential energy whose values is exactly equal to mgx. Where is the energy stored?



Answer: it is stored neither in the mass nor in the earth. It is stored in the gravitational field of the combined system of stone + earth.

1. How much work does it take to lift a mass m to height h, as shown in figure?

Answer:

$$W_{ext} = F_{ext}\, d$$
$$W_{ext} = mg\, h$$
$$W_{ext} = m\, g\, (y_2 - y_1)$$

You did work on the object. Therefore, its energy increased.

Key points:

- PE is measured with respect to some reference level.
- Only changes in PE actually have physical meaning.
- Changes in PE do not depend on path.
- Energy is a shared property.
- Work is force x distance.
- Energy is the capacity to do work.
- Power is the "rate of doing work"
- $\text{Power} = \dfrac{\text{Work done}}{\text{Time taken}}$

Problem 2:

Your heart is working as a pump and the volume of blood lifted daily is 8000 liters and the blood is lifted up to a height of 1.5 m. calculate its potential energy and power.

Solution:

Volume of blood lifted daily = 8000 litres

Average height lifted = 1.5 m

$\therefore$ Density of blood $\approx$ 1 kg/litre

$P.E = mgh$ $\quad\because$ density = mass/volume $\Rightarrow$ mass = density * volume

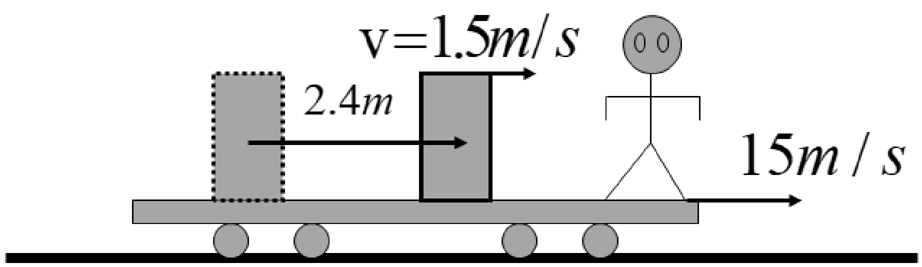
$Work\ done = P.E$ = 8000 x 1 x 9.8 x 1.5

$\approx$ 120,000 J in 24 hours

$\therefore \quad \text{Power} = \dfrac{Work\ done}{time} = \dfrac{120000}{24 \text{ x } 60 \text{ x } 60} \approx 1.4 \text{ W}$

Problem 3:

A box of mass 12kg is pushed with a constant force so that its speed goes from zero to 1.5m/sec (as measured by the person at rest on the cart) and it covers a distance of 2.4m. Assume there is no friction.



Solution:

mass of box $= 12\ kg$

$$\Delta K = K_f - K_i = \frac{1}{2}(12kg)(1.5m/s)^2 - 0 = 13.5J$$

$$a = \frac{v_f^2 - v_i^2}{2(x_f - x_i)} = \frac{(1.5m/s)^2 - 0}{2(2 \cdot 4m)} = 0.469m/s^2$$

This acceleration results from a constant net force given by:

$F = ma = (12kg)(0.469\,m/s^2) = 5.63N$

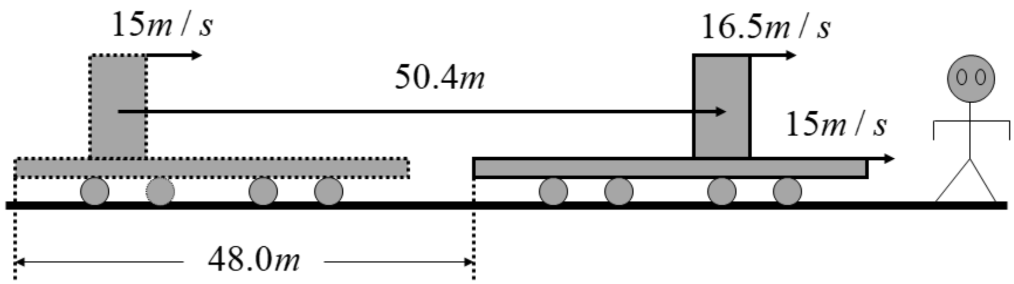
Work done on the crate is:

$W = F\Delta x = (5.63N)(2.4m) = 13.5J$

(same as $\Delta K = 13.5J$!)

Problem 4:

Suppose there is somebody standing on the ground, and that the trolley moves at 15 m/sec relative to the ground.



Solution:

Let us repeat the same calculation:

$$\begin{aligned}\Delta K' &= K'_f - K'_i = \frac{1}{2} m\mathrm{v}_{\mathrm{f}}^{\,2} - \frac{1}{2} m\mathrm{v}_{\mathrm{i}}^{\,2} \\ &= \frac{1}{2}(12kg)(16.5m/s)^2 - \frac{1}{2}(12kg)(15.0m/s)^2 \\ &= 284J\end{aligned}$$

(This is not equal to $\Delta K = 13.5J$)

This example clearly shows that work and energy have different values in different frames. Now, if this person calculate the force to be 5.63 N, then calculate work done by the person.

$$\because\ a' = a\ \ \therefore\ F' = F = 5.63N$$

$$t = \frac{\mathrm{v}_f - \mathrm{v}_i}{a} = \frac{1.5m/s}{0.469m/s^2} = 3.2s$$

and train moves (15m/s)(3.2s) $= 48m$

$$\begin{aligned}&\text{Total displacement} = \Delta x' \\ &= 48m + 2.4m = 50.4m\end{aligned}$$

The ground-based observer also concludes that the work is:

$$W' = F'\Delta x' = (5.63N)(50.4m) = 284J$$

Conclusion:

- If you calculate work or kinetic energy in different frame of reference, the results would be different in both frames.

- Potential energy and kinetic energy depend on the frame you choose to measure it in. If you are running with a ball, it has zero kinetic energy with respect to you. But someone who is standing will see that it has kinetic energy.
- In the absence of friction, the total energy of the system is conserved.
- Total mechanical energy is:

  E_{mech} = KE + PE
- If no friction then E_{mech} is conserved:

  $\Delta(E_{mech}) = \Delta(KE) + \Delta(PE) = 0$
- E_{mech} = KE + PE is constant

Problem 5:

A ball is thrown upwards at speed v_o. How high will it go before it stops?

Solution:

The loss of potential energy is equal to the gain of potential energy,

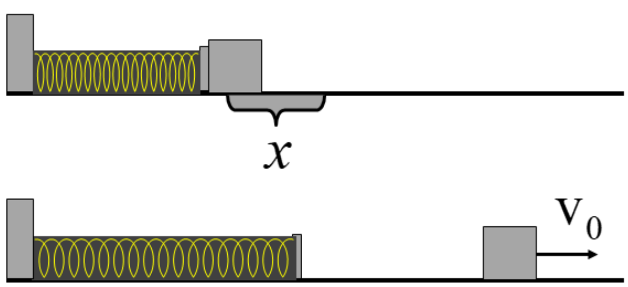
$$K.E = P.E$$

$$\frac{1}{2} m v_0^2 = mgh$$

$$h = \frac{v_0^2}{2g}$$

Problem 6:

A mass is attached to a spring, this spring is compressed, and elastic potential energy is stored in it. The extension created in this string is x. When mass m is released from the spring, it moves with velocity V_o. Here, we want to calculate that velocity.



Solution:

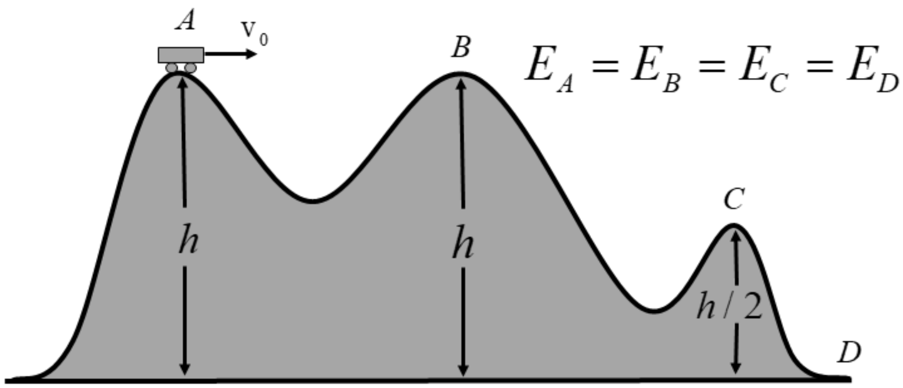
As there is no friction, the kinetic energy of the spring becomes equal to the elastic potential energy.

$$\frac{1}{2}m\mathrm{v}_0^2 = \frac{1}{2}kx^2$$

$$\mathrm{v}_0 = \sqrt{\frac{k}{m}}x$$

Problem 7:

Now look at the smooth, frictionless motion of a car over the hills as shown below:



What will be the speed of the car,

a) at point B,

b) at point C,

c) at point D?

Solution: As, in the absence of friction, energy at every point would be the same.

$$E_A = E_B = E_C = E_D$$

According to the law of conservation of energy, From A to B;

$$\frac{1}{2}m\mathrm{v}_A{}^2 + mgh = \frac{1}{2}m\mathrm{v}_B{}^2 + mgh$$

$$\mathrm{v}_A = \mathrm{v}_B = \mathrm{v}_0 \rightarrow 1$$

From B to C;

$$\frac{1}{2}m\mathrm{v}_B{}^2 + mgh = \frac{1}{2}m\mathrm{v}_C{}^2 + mg\frac{h}{2}$$

$$\frac{1}{2}m\mathrm{v}_B{}^2 + mgh = \frac{1}{2}\left(m\mathrm{v}_C{}^2 + mgh\right)$$

$$m\mathrm{v}_B{}^2 + mgh = \left(m\mathrm{v}_C{}^2 + mgh\right)$$

$$\mathrm{v}_B{}^2 + gh = \mathrm{v}_C{}^2$$

$$\mathrm{v}_\mathrm{C} = \sqrt{\mathrm{v}_0{}^2 + gh} \rightarrow 2$$

From C to D;

$$\frac{1}{2}m\mathrm{v}_C{}^2 + mg\frac{h}{2} = \frac{1}{2}m\mathrm{v}_D{}^2$$

$$\mathrm{v}_D{}^2 = \mathrm{v}_C{}^2 + gh \rightarrow 3$$

putting eq.2 in eq.3

$$\mathrm{v}_D{}^2 = \mathrm{v}_o{}^2 + gh + gh$$

$$\mathrm{v}_\mathrm{D} = \sqrt{\mathrm{v}_0{}^2 + 2gh} \rightarrow 4$$

Remember that potential energy has meaning only for a force that is conservative. A conservative force is that for which the work done in going from point A to point B is independent of the path chosen. Friction is an example of a non-conservative force, and a potential energy cannot be defined. For a conservative force,

$$F = -\frac{dV}{dx}$$

$$as, V = \frac{1}{2}kx^2$$

$$F = -\frac{d}{dx}\left[\frac{1}{2}kx^2\right] = -\frac{1}{2}k.\frac{d}{dx}\left[x^2\right]$$

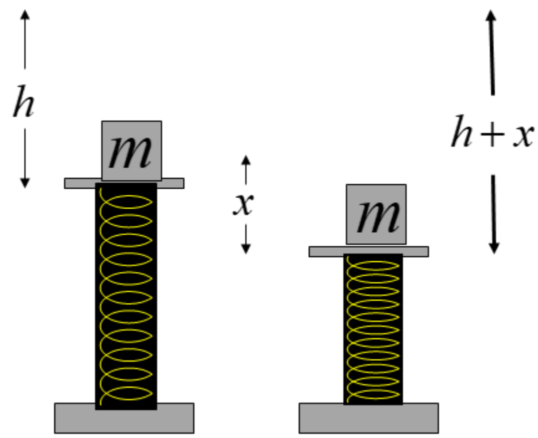
$$F = -\frac{1}{2}k.2x = -kx$$

Problem 8:

A mass m is taken on height h above spring. The mass compresses the spring upon falling on it as shown in the figure. The extension in the spring is x. We have to calculate that x.

Solution:

According to the conservation of energy, initial K.E becomes equal to the P.E stored in the spring,



$$\frac{1}{2}kx^2 = mg(x+h)$$

$$\frac{1}{2}kx^2 = mgx + mgh \Rightarrow \frac{1}{2}kx^2 - mgx - mgh = 0$$

$$a = \frac{1}{2}k, \quad \text{b}= -mg, \quad \text{c}= -mgh$$

On applying quadratic formula, we get,

$$\text{x}=\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\text{x}=\frac{mg \pm \sqrt{(mg)^2 - 4\left(\frac{1}{2}k\right)(-mgh)}}{k} = \frac{mg \pm \sqrt{m^2 g^2 + (2k)(mgh)}}{k}$$

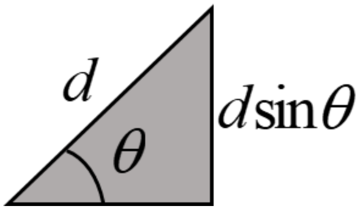
$$\text{x}=\frac{mg \pm mg\sqrt{1+\frac{2kh}{mg}}}{k}$$

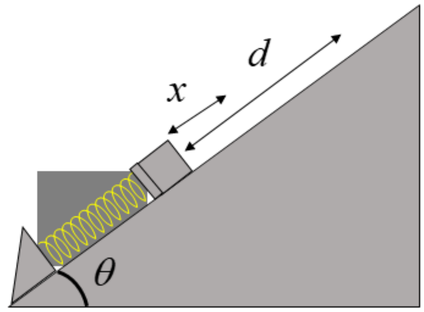
$$\Rightarrow x = \frac{mg}{k}(1 \pm \sqrt{1+2hk/mg})$$

Here, we find the two solutions for this problem. The two solutions prove the oscillation of the spring (up & down) when mass falls on it.

Problem 9:

Consider an inclined plane. A spring with attached mass is taken on it. The mass is stretched through a distance of d, which compresses the spring through x. We want to calculate the value of x.





Solution: According to the conservation of energy, K.E becomes equal to the P.E stored in the spring,

$$\frac{1}{2}kx^2 = mgd\sin\theta$$

$$\Rightarrow \quad d = \frac{kx^2}{2mg\sin\theta}$$

$$mgd\sin\theta = \frac{1}{2}m\text{v}^2$$

$$\Rightarrow \quad \text{v} = \sqrt{2gd\sin\theta}$$

To find d, we have,

$$mg(x+d)\sin\theta = \frac{1}{2}kx^2 \Rightarrow mgx\sin\theta + mgd\sin\theta = \frac{1}{2}kx^2$$

$$mgd\sin\theta = \frac{1}{2}kx^2 - mgx\sin\theta$$

$$d = \frac{kx^2}{2mg\sin\theta} - x$$

To find x, we have,

$$\frac{1}{2}kx^2 - mgx\sin\theta - mgd\sin\theta = 0$$

$$a = \frac{1}{2}k, \quad \text{b}= -\text{mg}\sin\theta, \quad \text{c}= -mgd\sin\theta$$

on applying quadratic formula, we get,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{mg\sin\theta\pm\sqrt{(mg\sin\theta)^2-4\left(\frac{1}{2}k\right)(-mgd\sin\theta)}}{k}$$

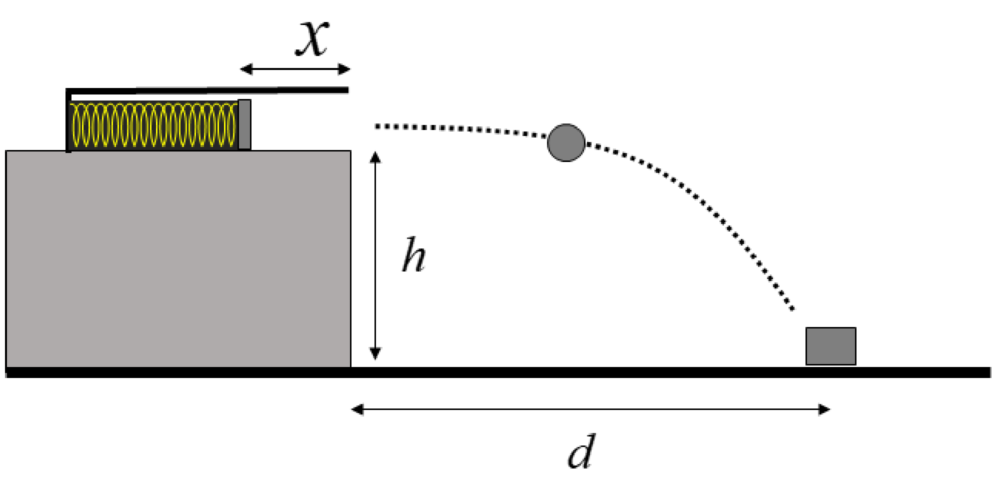
$$x=\frac{mg\sin\theta\pm\sqrt{(mg\sin\theta)^2+2k\,(mgd\sin\theta)}}{k}$$

$$x=\frac{mg\sin\theta\pm mg\sin\theta\sqrt{1+\left(\frac{2kd}{mg\sin\theta}\right)}}{k}$$

$$x=\frac{mg\sin\theta}{k}\left(1\pm\sqrt{1+\frac{2kd}{mg\sin\theta}}\right)$$

Problem 10:

A spring is taken at height h from ground level and a ball is attached to it. When the spring is compressed, the ball falls down to earth and make a parabolic path and touches the mass which is on ground at distance d. We want to know the value of x (extension of the spring) as shown in figure.



Solution:

According to the conservation of energy, K.E becomes equal to the P.E stored in the spring,

$$\frac{1}{2}m\mathrm{v}_0^2 = \frac{1}{2}kx^2$$

$$\Rightarrow \quad x = \sqrt{\frac{m}{k}}\mathrm{v}_0$$

$$\mathrm{v}_0 = \mathrm{v}_{ix} = \sqrt{\frac{kx}{m}}$$

$$y - component;$$

$$S_y = \mathrm{v}_{iy}t + \frac{1}{2}a_y t^2$$

$$S_y = h, \quad a_y = 9.8$$

$$h = o - \frac{1}{2}gt^2 = \frac{1}{2}gt^2$$

$$\Rightarrow \quad t = \sqrt{\frac{2h}{g}}$$

$$x - component;$$

$$S_x = \mathrm{v}_{ix}t + \frac{1}{2}a_x t^2$$

$$d = \sqrt{\frac{kx}{m}}t + 0$$

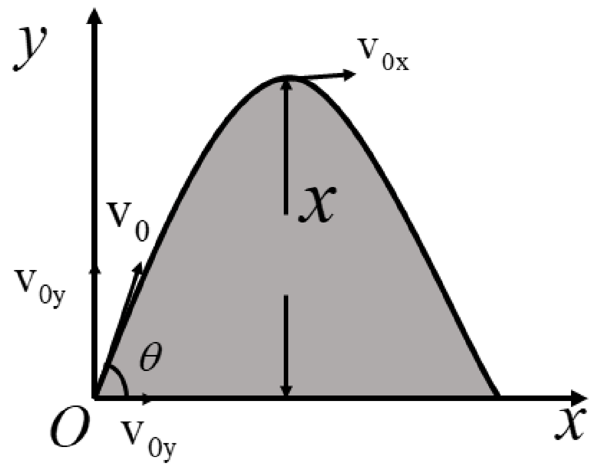
$$d = \sqrt{\frac{kx}{m}}.\sqrt{\frac{2h}{g}} = \sqrt{\frac{2kh}{mg}}x$$

$$\Rightarrow \quad x = d\sqrt{\frac{mg}{2hk}}$$

Problem 11:

A ball of mass m is falling vertically upward which performs a projectile motion under constant acceleration. We want to calculate the maximum height of the ball reaches during its motion.

Solution: According to the conservation of energy, initial K.E and P.E of the ball becomes equal to the final K.E and P.E,



$$K_0 + U_0 = K + U$$

$$\frac{1}{2}mv_{0x}^2 + \frac{1}{2}mv_{0y}^2 = \frac{1}{2}mv_{0x}^2 + mgh$$

$$\frac{1}{2}mv_{0y}^2 = mgh$$

$$\because v_{0y} = v_0 \sin\theta$$

$$v_{0y}^2 = 2gh$$

$$v_0{}^2 \sin^2\theta = 2gh$$

$$h = \frac{v_0^2 \sin^2\theta}{2g}$$

At different angles, the ball reaches to different heigh but it reaches its maximum height at 90 degree.

$\because \sin 90 = 1$

$$h_{max} = \frac{v_0^2}{2g}$$

Conservative force:

Conservative force means, work does not depend on the path taken.

- Gravitational force
- Electrostatic force
- Elastic force

Note: Remember that potential energy has meaning only for a force that is conservative. A conservative force is that for which the work done in going from point A to point B is independent of the path chosen.

Non-Conservative force:

Non-Conservative force means work is dependent on the path taken.

For Example:

- Frictional force
- air resistance force
- velocity dependent forces

Potential energy cannot be defined for non-conservative forces.

Problem 12:

Suppose a particle has potential energy as V(x), and the particle followed path as shown in figure.

Solution:

If the particle moves Δx

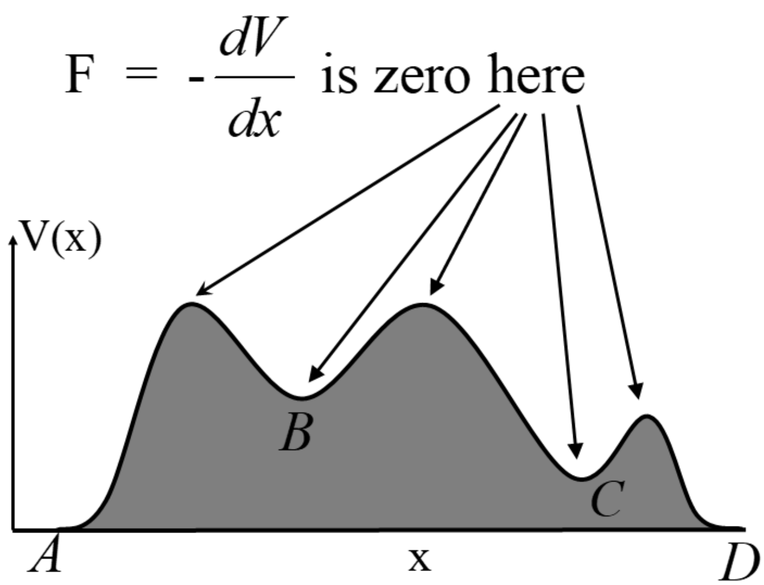
Then change in PE is ΔV

Where: $\Delta V = - F\, \Delta x$

$$\Rightarrow \qquad F = -\frac{\Delta V}{\Delta x}$$

Now let $\Delta x \rightarrow 0$

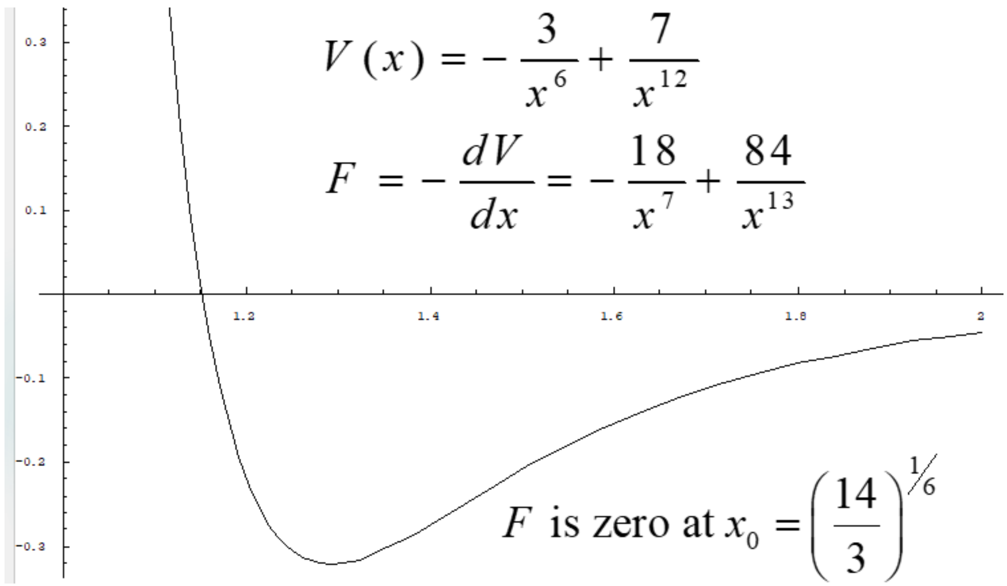
$$F = -\lim_{\Delta x \to 0} \frac{\Delta V}{\Delta x} = -\frac{dV}{dx}$$



The tangent shows that the derivative of potential is zero at the top point of the curve, where the tangent is flat to curve, force is also zero there.

Example:

Suppose we have a potential function as shown in the figure.



The potential function is given as,

$$V(x) = -\frac{3}{x^6} + \frac{7}{x^{12}}$$

$$As, \Rightarrow F = -\frac{dV}{dx} = -\frac{d}{dx}\left[-\frac{3}{x^6} + \frac{7}{x^{12}}\right],$$

Using power rule nx^{n-1}, and taking differential w.r.t "x",

$$F = 3\left(6.x^{-6-1}\right) + 7\left(12.x^{-12-1}\right) = \left[-\frac{18}{x^7} + \frac{84}{x^{13}}\right]$$

F is zero at x_0, *Hence,*

$$0 = \left[-\frac{18}{x_0^{\ 7}} + \frac{84}{x_0^{\ 13}}\right]$$

$$\frac{x_0^{\ 13}}{x_0^{\ 7}} = \frac{84}{18} \Rightarrow x_0^{\ 13-7} = \frac{28}{6}$$

$$x_0^{\ 6} = \frac{14}{3} \Rightarrow x_0 = \left(\frac{14}{3}\right)^{\frac{1}{6}}$$

Conclusion: We can conclude that potential can be defined only for the conservative forces. We cannot define potential for non-conservative forces.

Because non-conservative forces do not satisfy the path-independence property. The work done by non-conservative forces depends on the specific path taken, and as a result, it is not possible to define a potential energy associated with non-conservative forces in the same way. Examples of non-conservative forces include friction and air resistance.

Physics-PHY101-Lecture 9

MOMENTUM

Momentum is the "quantity of motion" possessed by a body. More precisely, it is defined as,

"The product of mass and velocity of a body".

$$p = mv$$

Dimensions of momentum:

The dimensions of momentum are MLT^{-1}.

Unit of momentum:

The units of momentum are kg-m/s.

Momentum is a vector quantity and has both magnitude and direction.

Newton's Second Law and Momentum:

Newton's Second Law can be expressed in terms of momentum. It is defined as,

"The rate of change of momentum of a body is equal to the resultant force acting on the body and is in the direction of that force".

$$m\vec{\mathbf{a}} = \vec{\mathbf{F}} \text{ (old form)}$$

$$\frac{d\vec{\mathbf{p}}}{dt} = \vec{\mathbf{F}} \quad \text{(new form of Newton's Law)}$$

They are the same:

$$\frac{d\vec{\mathbf{p}}}{dt} = \frac{d(m\vec{\mathbf{v}})}{dt} = m\frac{d\vec{\mathbf{v}}}{dt} = m\vec{\mathbf{a}} = \vec{\mathbf{F}}$$

Newton's 2nd law for several particles:

When there are many particles, then total momentum is,

$$\vec{\mathbf{P}} = \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 + \cdots \vec{\mathbf{p}}_N$$

$$\frac{d}{dt}\vec{\mathbf{P}} = \frac{d}{dt}\vec{\mathbf{p}}_1 + \frac{d}{dt}\vec{\mathbf{p}}_2 + \cdots \frac{d}{dt}\vec{\mathbf{p}}_N$$

$$\frac{d}{dt}\vec{\mathbf{P}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \cdots \vec{\mathbf{F}}_N$$

$$\frac{d}{dt}\vec{\mathbf{P}} = \sum_{i=1}^{i=N} \vec{\mathbf{F}}_i = \text{total external force}$$

This shows that when there are several particles, the rate at which the total momentum changes is equal to the total force.

Conservation of Linear Momentum:

If the sum of the total external forces vanishes, then the total momentum is conserved.

$$\sum \vec{\mathbf{F}}_{ext} = 0 \text{ Then,}$$

$$\frac{d\vec{\mathbf{P}}}{dt} = 0$$

$$\vec{\mathbf{P}} = \text{constant}$$

Momentum is conserved for an isolated system. This is quite independent of what sort of forces act between the bodies i.e., electric force, gravitational force, etc. - or how complicated these are. We shall see why this is so important from the following examples.

Problem 1:

Two balls, which can only move along a straight line, collide with each other as shown in figure 9.1. The initial momentum is $p_i = m_1u_1 + m_2u_2$ and the final momentum is $p_f = m_1v_1 + m_2v_2$. Obviously one ball exerts a force on the other when they collide, so their momentum changes. But, from the fact that there is no external force acting on the balls,

$$p_i = p_f$$

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

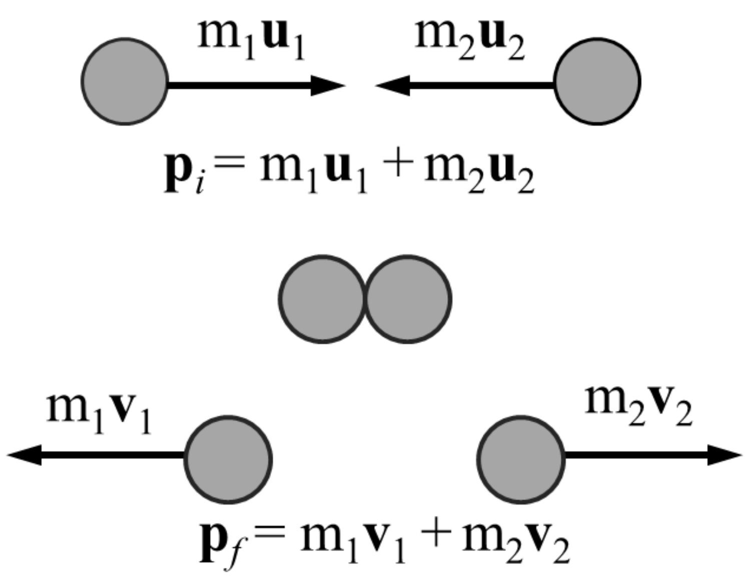


Figure 9.1. Momentum Transfer in a one-dimensional collision between the two balls along a Straight Path.

Problem 2:

A bomb at rest explodes into two fragments as shown in figure 9.2. Before the explosion the total momentum was zero. So obviously it is zero after the explosion as well.

$$\mathbf{P}_i = \mathbf{0} \quad \text{and}$$
$$\mathbf{P}_f = \mathbf{0} \quad \text{but } \mathbf{P}_f = m_1\mathrm{v}_1 + m_2\mathrm{v}_2$$
$$\therefore\ m_1\mathrm{v}_1 = -m_2\mathrm{v}_2$$

During the time when the explosion happens, the forces acting upon the pieces are very complicated and change rapidly with time. But when all is said and done, there are two pieces flying away with a total zero final momentum. In other words, the fragments fly apart with equal momentum but in opposite directions. The center-of-mass stays at rest. So, knowing the velocity of one fragment permits knowing the velocity of the other fragment.

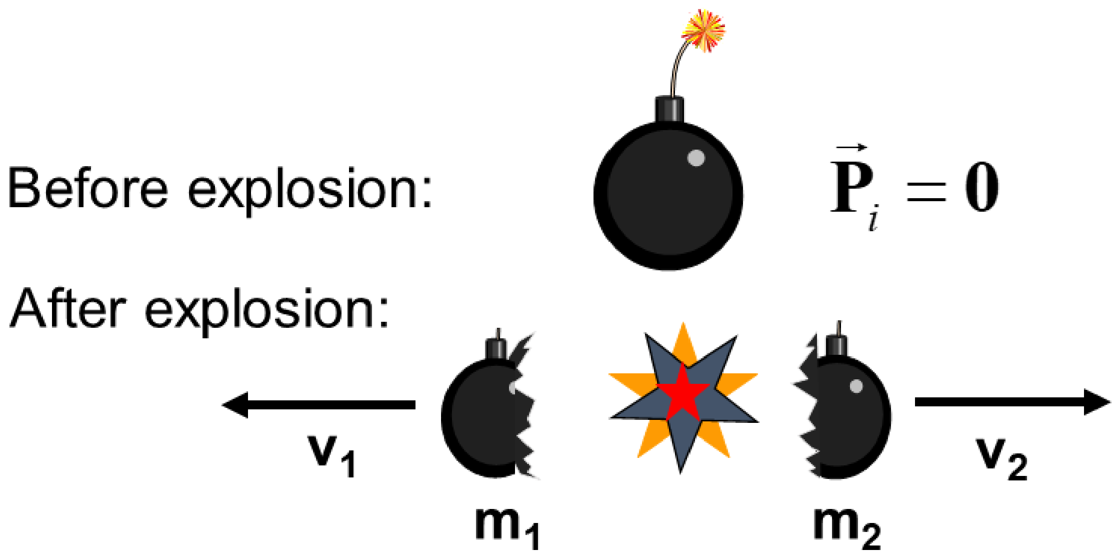


Figure 9.2. The physics of explosions as a bomb at rest divides into two fragments, shows momentum conservation.

Problem 3:

A bomb of mass 10 kg, initially at rest, explodes into two pieces of masses 4 kg and 6 kg. If the speed of the 4 kg piece is 12 m/s, find the speed of the 6 kg piece.

Solution:

Mass of bomb=$M = 10kg$

After explosion,

$m_1 = 4kg$

$V_1 = 12m/s$

$m_2 = 6kg$

$V_2 = ?$

Using formula,

$m_1V_1 = m_2V_2$

$$V_2 = \frac{m_1V_1}{m_2} = \frac{4\times12}{6} = 8m/s$$

$V_2 = 8m/s$

Problem 4:

A rocket conserves momentum through the principle of action and reaction, as described by Newton's third law of motion: "For every action, there is an equal and opposite reaction." In the context of a rocket, this law explains how momentum is conserved during the propulsion process as shown in figure 9.3.

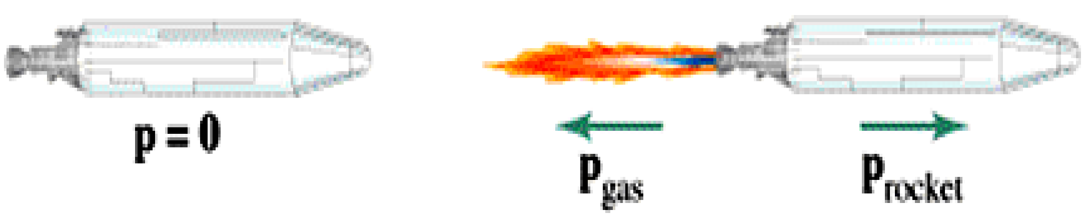


Figure 9.3. Momentum Conservation in the Propulsion Process.

When a rocket engine expels exhaust gases at high velocity in one direction, it generates a force in the opposite direction according to Newton's third law. This force is what propels the rocket forward. The expelled gases have mass and velocity, contributing to a significant momentum in the opposite direction. Since momentum is conserved in an isolated system, the rocket gains an equal and opposite momentum. This results in the rocket moving forward. Mathematically, this can be expressed using the equation for momentum:

$$p_i = p_f$$
$$p_{\text{gas}} + p_{\text{rocket}} = 0$$
$$p_{\text{gas}} = -p_{\text{rocket}}$$

Since the initial momentum is small or zero, the final momentum of the rocket is primarily determined by the momentum of the expelled exhaust gases. The larger the momentum of the gases, the larger the momentum gained by the rocket.

Problem 5:

When a gun is fired, there are two main components involved: the gun itself and the bullet (projectile) being expelled from the gun as shown in figure 9.4. The gun and the bullet are initially at rest, so their initial momentum is zero. When the gun is fired, the bullet accelerates out of gun and gains momentum in one direction (forward), the gun and the shooter gain an equal and opposite momentum in the opposite direction (backward).

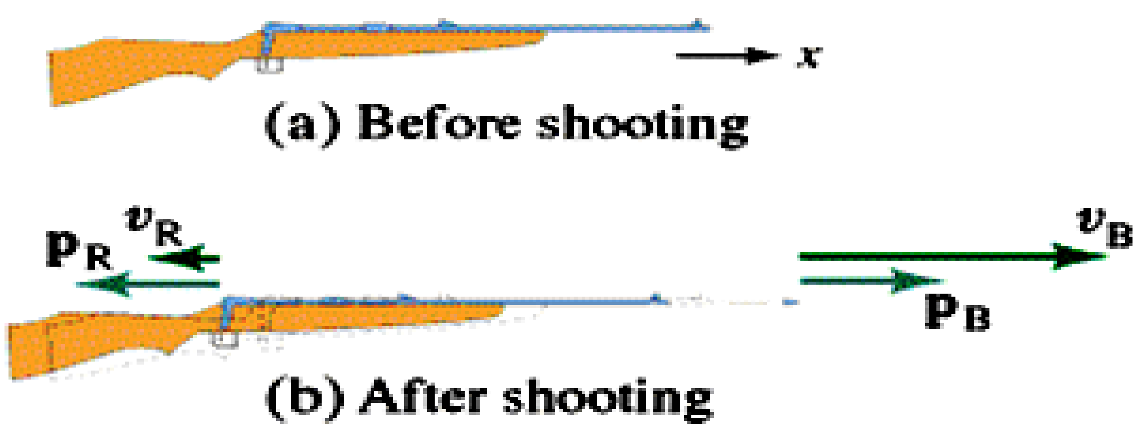


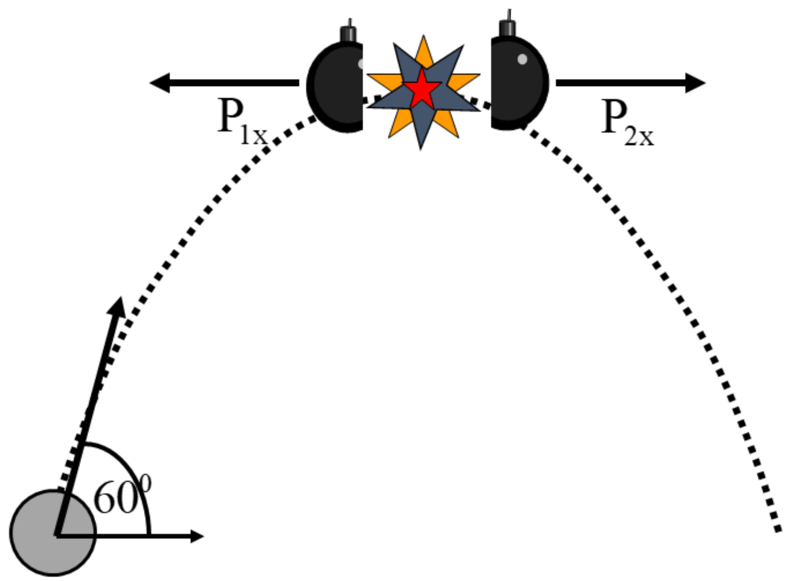
Figure 9.4. The dynamics of gun firing, where bullet acceleration and recoil impart equal and opposite momentum to maintain conservation.

$$p_i = p_f$$
$$p_{\text{rifle}} + p_{\text{bullet}} = 0$$
$$p_{\text{rifle}} = -p_{\text{bullet}}$$

This conservation of momentum ensures that the total momentum before and after the gunshot remains zero, as no external forces are acting on the system.

Problem 6:

A shell is fired from a cannon with a speed of 10 m/s at an angle 60^0 with the horizontal. At the highest point in its path, it explodes into two pieces of equal masses. One of the pieces retraces its path to the cannon. Find the velocity of the other piece immediately after the explosion.



Solution:

Before explosion:

$V_{ix} = V_i \cos\theta = 10\cos 60 = 5\,m/s$

$P_x = M.V_x = 5$ M kg m/s

After explosion:

$$MV_{ix} = -\frac{M}{2}V_{1x} + \frac{M}{2}V_2$$

$$\therefore V_1 = V_{ix}, \text{ and } V_2 = V_{2x}$$

$$V_{ix} = -\frac{1}{2}V_{ix} + \frac{1}{2}V_{2x}$$

$$2V_{ix} = -V_{ix} + V_{2x}$$

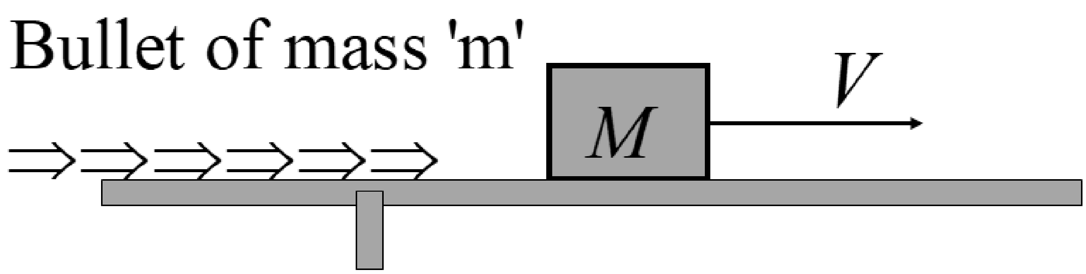
$$2V_{ix} + V_{ix} = V_{2x}$$

$$V_{2x} = 3V_{ix}$$

$$V_{2x} = 3 \times 5 = 15\,m/s$$

Problem 7:

A stream of bullets, each of mass m, is fired horizontally with a speed v into a large wooden block of mass M that is initially at rest on a horizontal table. If the block is free to slide without friction, what speed will it acquire after it has absorbed N bullets?



Solution:

Linear momentum is conserved:

$$P_f = P_i$$

$$(M + Nm)V = N(m\text{v})$$

$$V = \frac{Nm}{(M + Nm)}\text{v}$$

Problem 7:

A cannon with mass M equal to 1300 kg fires a 72 kg ball in horizontal direction with a muzzle speed v of 55 m/s. The cannon is mounted so that it can recoil freely.

a) What is the velocity V of the recoiling cannon with respect to the earth?
b) What is the initial velocity V_E of the ball with respect to the earth?

Solution:

a) Linear momentum is conserved:

$$P_f = P_i$$

$$MV + m(\mathrm{v} + V) = 0$$

$$V = -\frac{m\mathrm{v}}{m+M} = -\frac{(72kg)(55m/s)}{1300kg + 72kg}$$

$$V = -2.9\ m/s$$

b) $$\mathrm{V_E} = \mathrm{v} + V$$

$$\mathrm{V_E} = 55m/s + (-2.9\ m/s)$$

$$\mathrm{V_E} = 52m/s$$

Problem 8:

Consider a body at point A which is at height h from the ground. The body has no friction, when it is at point A, but when it comes down straight to earth it has friction which renders the body to move again to the curve as shown in fig. For that case, we want to know,

a) Is the momentum conserved?
b) Where does the particle finally come to rest?

Solution:

Suppose the total distance moved on the flat part before it comes to rest is ‘x’,

$$mgh = \frac{1}{2}mv^2$$
$$\therefore\ \frac{1}{2}mv^2 = f.x$$
$$\frac{1}{2}mv^2 = \mu Nx = \mu mgx$$
$$mgh = \mu mgx$$
$$x = \frac{h}{\mu}$$

Impulse and Momentum:

When you hit your thumb with a hammer it hurts, doesn't it? Why? Because a large amount of momentum has been destroyed in a short amount of time. If you wrap your thumb with foam, it will hurt less. To understand this better, remember that force is the rate of change of momentum,

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$
$$d\mathbf{p} = \mathbf{F}dt = I$$

Impulse:

It is defined as, “force acting over time to change the momentum of an object”.

If the force changes with time between the limits, then one should define impulse as,

$$\mathrm{I} \equiv \int_{t_1}^{t_2} \mathrm{F}dt$$
$$\text{Since } \int_{t_1}^{t_2} \mathrm{F}dt = \int_{p_i}^{p_f} dp$$
$$\mathrm{I} = \mathbf{p}_f - \mathbf{p}_i$$

In words, the change of momentum equals the impulse, which is equal to the area under the curve of force versus time. Even if you wrap your thumb in foam, the impulse is the same. But the force is definitely not.

Sometimes we only know the force numerically (i.e., there is no expression like F=something). But we still know what the integral means: it is the area under the curve of force versus time. The curve here is that of a hammer striking a table. Before the hammer strikes, the force is zero, reaches a peak, and goes back to zero.

Graphical representation:

1. Let us draw a graph between force F(t) and time (Δt) as shown in figure 9.5. From this graph, one can see the area under the curve gives impulse and the value of force is maximum at t_2 but minimum at t_1 and t_3 (also, F(t) is same at t_1 and t_3). Below t_1 and above t_3, F(t) begins to zero.

$$I = F_{av}\Delta t$$

Area under the curve = impulse

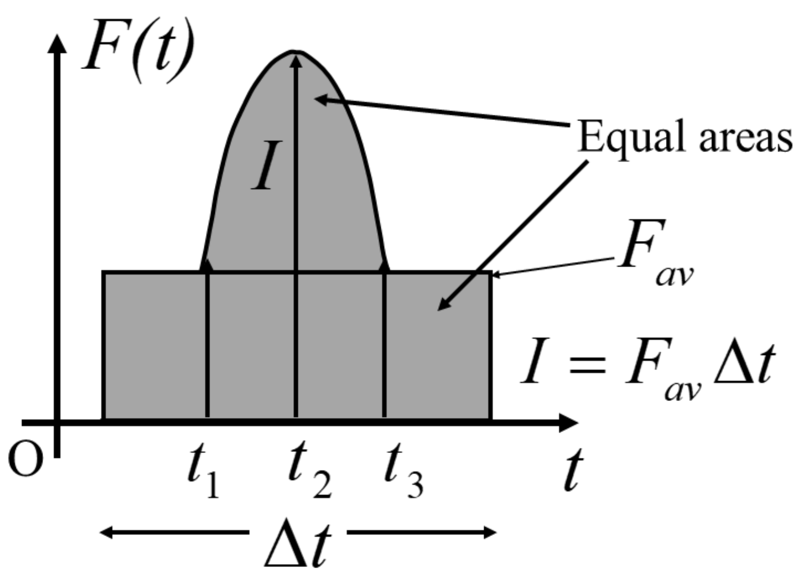


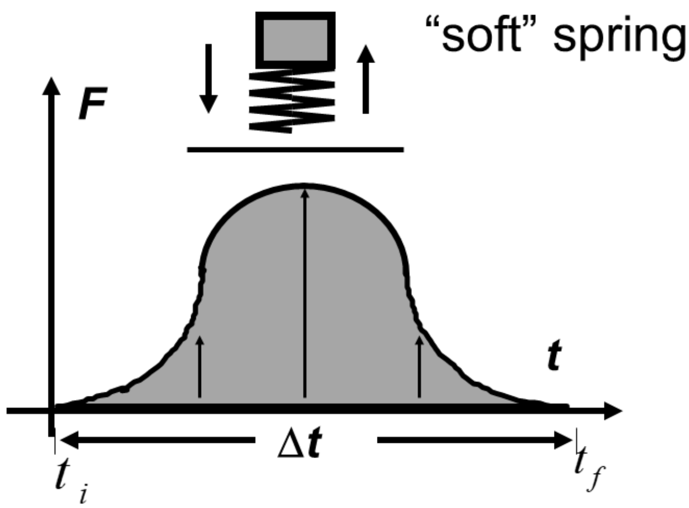
Figure 9.5. Graphical representation of F(t) and Δt, unraveling impulse as the area under the curve, with maximal force at t_2 and minimum force at t_1 and t_3.

2. Force may be of any type. Let us now consider a case where a soft spring is taken upon which when force is applied, it shows less resistance as being soft in nature. When a weight is taken over a soft spring it shows,

$$\Delta t \text{ is big, } F \text{ is small}$$

But the product of force and time gives us a constant value which is called impulse.

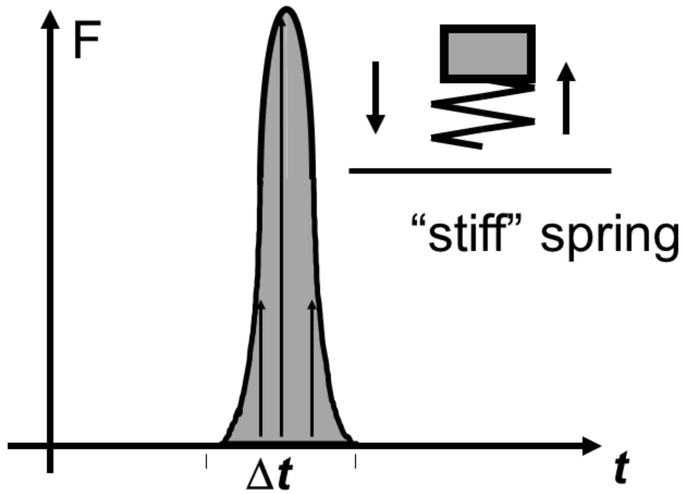
$$I = F\Delta t$$



3. Now consider a case of a stiff spring. When a small force is applied to that spring, it may create a significant resistance.

$$\Delta t \text{ is small, } F \text{ is big}$$

But the product of force and time again gives us a constant value which is called impulse.



Conclusion:

Momentum is a concept that is useful because of Newton’s 2nd law. Physics is a quantitative field. It turns out that momentum conservation still holds even when we go beyond Newton’s laws. But momentum is not just mv.

Practice Questions:

Q: Would you rather land with your legs bending or stiff?

Q: Why do cricket fielders move their hands backwards when catching a fast ball?

Q: Why do railway carriages have dampers at the front and back?

Physics-PHY101-Lecture 10

COLLISIONS

Collisions are extremely important to understand because they happen all the time - electrons collide with atoms, a bat with a ball, cars with trucks, star galaxies with other galaxies. In every case, the sum of the initial momenta equals the sum of the final momenta. This follows directly from Newton's Second Law, as we have already seen.

Elastic collision in one dimension:

Let us take the simplest collision. Consider two bodies of mass m_1 and m_2 moving with velocities u_1 and u_2. After the collision they are moving with velocities v_1 and v_2 as shown in figure 10.1. For elastic collision, the total linear momentum, and kinetic energies of the two bodies before and after collision must remain the same.

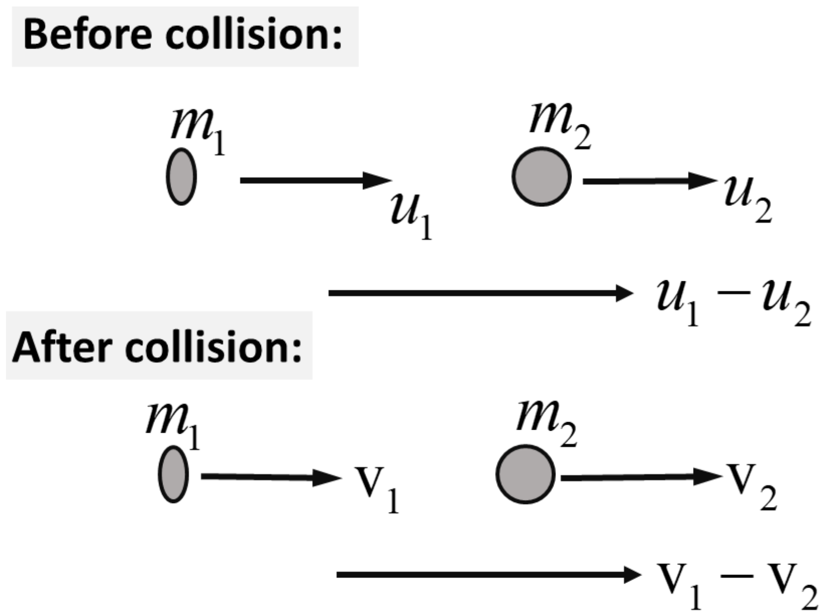


Figure 10.1. Illustrating one-dimensional elastic collision dynamics, where masses m_1 and m_2 exchange velocities u_1, u_2 to v_1, and v_2 after collision while preserving total linear momentum and kinetic energies.

From the law of conservation of linear momentum,

Total momentum before collision (p_i) = Total momentum after collision (p_f)

$m_1u_1 + m_2u_2 = m_1v_1 + m_2\mathrm{v}_2$

$m_1(u_1 - \mathrm{v}_1) = m_2(\mathrm{v}_2 - u_2) \rightarrow 1$

For elastic collision,

Total kinetic energy before collision $(\mathrm{K.E_i})$ = Total kinetic energy after collision $(\mathrm{K.E_f})$

$\frac{1}{2}m_1u_1^{\,2} + \frac{1}{2}m_2u_2^{\,2} = \frac{1}{2}m_1v_1^{\,2} + \frac{1}{2}m_2v_2^{\,2}$

$\frac{1}{2}m_1(u_1^{\,2} - \mathrm{v}_1^{\,2}) = \frac{1}{2}m_2(\mathrm{v}_2^{\,2} - u_2^{\,2})$

$m_1(u_1^{\,2} - \mathrm{v}_1^{\,2}) = m_2(\mathrm{v}_2^{\,2} - u_2^{\,2}) \rightarrow 2$

Using the formula, $\mathrm{a}^2 - \mathrm{b}^2 = (\mathrm{a}+\mathrm{b})(\mathrm{a}-\mathrm{b})$, we can rewrite the above equation as

$m_1(u_1 + \mathrm{v}_1)(u_1 - \mathrm{v}_1) = m_2(\mathrm{v}_2 + u_2)(\mathrm{v}_2 - u_2) \rightarrow 3$

Dividing equation (3) by (1) gives,

$$\frac{m_1(u_1 + \mathrm{v}_1)(u_1 - \mathrm{v}_1)}{m_1(u_1 - \mathrm{v}_1)} = \frac{m_2(\mathrm{v}_2 + u_2)(\mathrm{v}_2 - u_2)}{m_2(\mathrm{v}_2 - u_2)}$$

$u_1 + \mathrm{v}_1 = \mathrm{v}_2 + u_2$

$u_1 - u_2 = \mathrm{v}_2 - \mathrm{v}_1$

$u_1 - u_2 = -(\mathrm{v}_1 - \mathrm{v}_2) \rightarrow 4$

This means that for any elastic head on collision, the relative speed of the two elastic bodies after the collision has the same magnitude as before collision but in opposite direction. Further note that this result is independent of mass.

Rewriting the equation 4 for v_1 and v_2,

$\mathrm{v}_1 = \mathrm{v}_2 + u_2 - u_1 \rightarrow 5$

$\mathrm{v}_2 = u_1 + \mathrm{v}_1 - u_2 \rightarrow 6$

<u>To find the final velocities v_1 and v_2:</u>

Substituting equation (5) in equation (1) gives the velocity of m1 as,

$m_1(u_1 - \mathrm{v}_1) = m_2(\mathrm{v}_2 - u_2)$

$m_1(u_1 - \mathrm{v}_1) = m_2(u_1 + \mathrm{v}_1 - 2u_2)$

$m_1u_1 - m_1\mathrm{v}_1) = m_2u_1 + m_2\mathrm{v}_1 - 2m_2u_2$

$(m_1 - m_2)u_1 + 2m_2u_2 = (m_1 + m_2)v_1$

$$v_1 = \frac{(m_1 - m_2)}{(m_1 + m_2)}u_1 + \frac{2m_2}{(m_1 + m_2)}u_2 \rightarrow 7$$

Similarly, by substituting equation (7) in equation (6), we get the final velocity of m_2 as,

$$v_2 = u_1 + \frac{(m_1 - m_2)}{(m_1 + m_2)} u_1 + \frac{2m_2}{(m_1 + m_2)} u_2 - u_2$$

$$v_2 = \frac{2m_1}{(m_1 + m_2)} u_1 + \frac{(m_2 - m_1)}{(m_1 + m_2)} u_2 \rightarrow 8$$

These equations holds true in all inertial frames.

<u>Case-I:</u>

When bodies have same mass i.e., $m_1 = m_2$,

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) u_1 + \left(\frac{2m_2}{m_1 + m_2}\right) u_2$$

$$v_1 = (0) u_1 + \frac{2m_2}{2m_2} u_2$$

$$v_1 = u_2$$

$$v_2 = \left(\frac{2m_1}{m_1 + m_2}\right) u_1 + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) u_2$$

$$v_2 = \frac{2m_1}{2m_1} u_1 + (0) u_2$$

$$v_2 = u_1$$

The equations show that in one dimensional elastic collision, when two bodies of equal mass collide after the collision their velocities are exchanged.

<u>Case-II:</u>

When bodies have the same mass i.e., $m_1 = m_2$ and second body (usually called target) is at rest ($u_2 = 0$), By substituting $m_1 = m_2$ and $u_2 = 0$ in equations we get,

$$v_1 = 0$$

$$v_2 = u_1$$

Equations show that when the first body comes to rest the second body moves with the initial velocity of the first body.

Case-III:

$$m_1 \ll m_2 \text{ and } u_2 = 0$$

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \left(\frac{2m_2}{m_1 + m_2}\right)u_2$$

$$v_1 = -u_1$$

$$v_2 = \left(\frac{2m_1}{m_1 + m_2}\right)u_1 + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)u_2$$

$$v_2 = 0$$

The equations implies that the first body which is lighter, returns back in the opposite direction with the same initial velocity as it has a negative sign. The second body which is heavier in mass continues to remain at rest even after collision.

For example, if a ball is thrown at a fixed wall, the ball will bounce back from the wall with the same velocity with which it was thrown but in opposite direction.

Case-IV:

$$m_2 << m_1 \quad u_2 = 0$$

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \left(\frac{2m_2}{m_1 + m_2}\right)u_2$$

$$v_1 = u_1$$

$$v_2 = \left(\frac{2m_1}{m_1 + m_2}\right)u_1 + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)u_2$$

$$v_2 = 2u_1$$

The equations implies that the first body which is heavier continues to move with the same initial velocity. The second body, which is lighter, will move with twice the initial velocity of the first body. It means that the lighter body is thrown away from the point of collision.

Elastic and inelastic collision:

Elastic Collision	Inelastic Collision

The total kinetic energy is conserved	The total kinetic energy of the bodies at the beginning and the end of the collision is different.
Momentum is conserved	Momentum is conserved
No conversion of energy takes place	Kinetic energy is changed into other energy such as sound or heat energy.
Highly unlikely in the real world as there is almost always a change in energy.	This is the normal form of collision in the real world.
An example of this can be swinging balls or a spacecraft flying near a planet but not getting affected by its gravity in the end.	An example of an inelastic collision can be the collision of two cars.

Sometimes we wish to slow down particles by making them collide with other particles. In a nuclear reactor, neutrons can be slowed down in this way.

A completely inelastic collision:

A collision that is completely inelastic in physics is one in which the two colliding objects stick together to form a single mass as shown in figure 10.2. The total momentum is conserved in such collisions but kinetic energy is not conserved; instead, part or all of it is converted into deformation energy or internal kinetic energy.

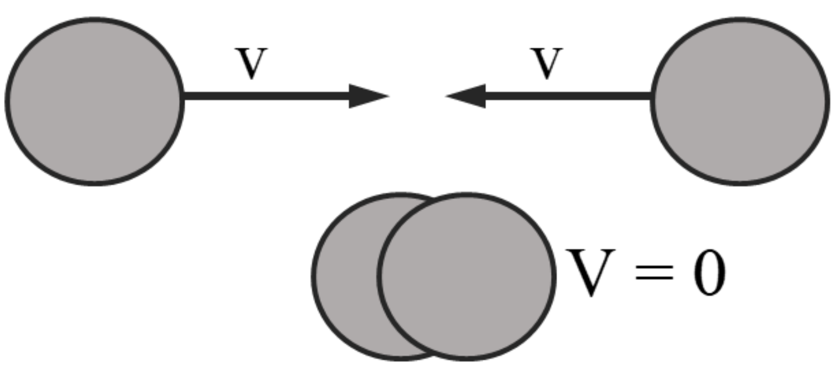


Figure 10.2. Illustrating one-dimensional in-elastic collision dynamics, where two masses moves with the same velocity V, conserving only the total linear momentum but the kinetic energies are not conserved.

Mathematically,

$$\text{Initial momentum} = \text{Final momentum} = 0$$

$$\text{Initial kinetic energy} = 2\ \frac{1}{2}\text{mv}^2\left(\text{for two massess}\right)$$

$$\text{Final kinetic energy} = 0\ \left(\text{as V} = 0\right)$$

A common example of a totally inelastic collision is the result of two balls colliding and sticking to one another. After the collision, the combined mass that results travels at the same speed.

Problem 1: By what fraction is the kinetic energy of a neutron (mass m_1) decreased in a head-on collision with an atomic nucleus (mass m_2) initially at rest?

Solution:

$$m_1 \ll m_2, \quad v_{2i} = 0$$

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right)u_{2i}$$

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_{1i} \Rightarrow \frac{v_f}{u_i} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)$$

$$\left(\frac{v_f}{u_i}\right)^2 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2$$

Fractional decrease in neutron K.E :

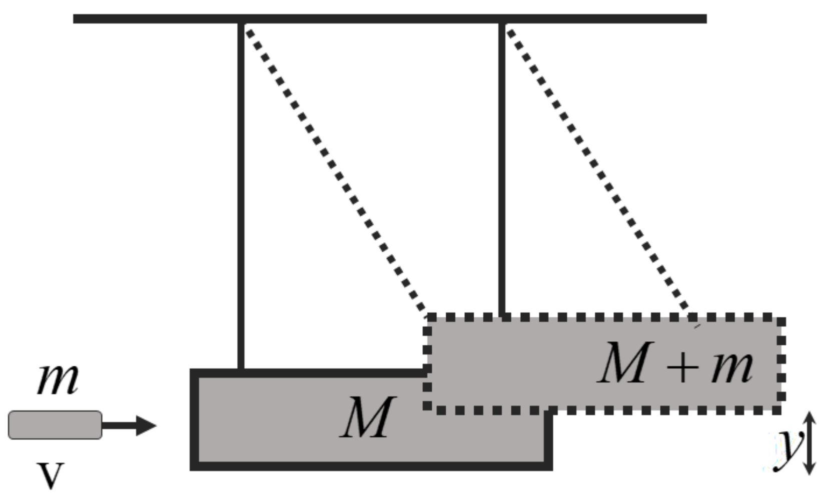
$$\frac{K_i - K_f}{K_i} = 1 - \frac{K_f}{K_i} = 1 - \frac{v_f^{\,2}}{v_i^{\,2}}$$

$$\frac{K_i - K_f}{K_i} = 1 - \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2$$

$$\frac{K_i - K_f}{K_i} = \frac{(m_1 + m_2)^2 - (m_1 - m_2)^2}{(m_1 + m_2)^2}$$

$$\frac{K_i - K_f}{K_i} = \frac{4m_1 m_2}{(m_1 + m_2)^2}$$

Problem 2: A bullet with mass m is fired into a block of wood with mass M, suspended like a pendulum and makes a completely inelastic collision with it. After the impact, the block swings up to a maximum height y. What is the initial speed of the bullet?



Solution:

$$mv + 0 = (m + M)V$$

$$m\mathrm{v} = (m + M)V$$

$$\mathrm{v} = \frac{(m + M)}{m}V \rightarrow 1$$

Conservation of energy gives,

$$\frac{1}{2}(m + M)V^2 = (m + M)gy$$

$$y = \frac{(m + M)V^2}{2(m + M)g}$$

$$y = \frac{V^2}{2g}$$

$$V = \sqrt{2gy} \rightarrow 2$$

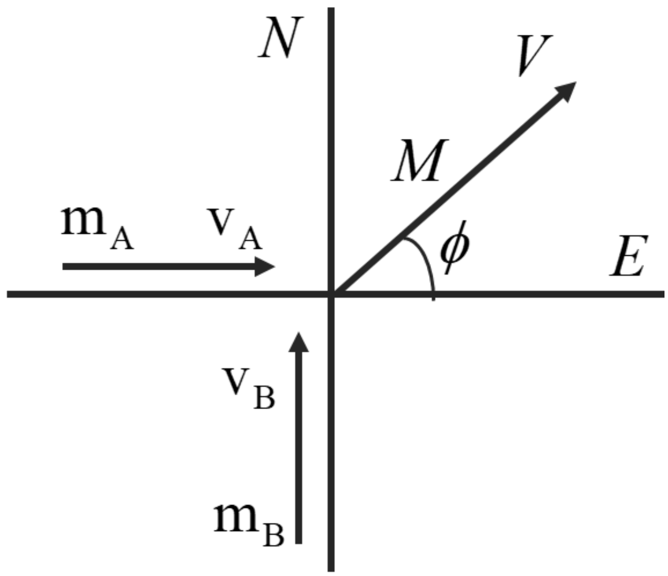
On putting equation (2) in equation (1),

$$\mathrm{v} = \frac{(m + M)}{m}\sqrt{2gy}$$

Conclusion: Equation shows the direct relation between the initial speed of bullet and the wooden block's height 'y'. The wooden block gains height as much as speedily the bullet enters the wooden block.

Problem 3: A car 'A' of mass 1000 kg is traveling north at 15 m/s collides with another car B of mass 2000 kg traveling east at 10 m/s. After a collision they move as one mass. Find the total momentum just after the collision.

Solution:

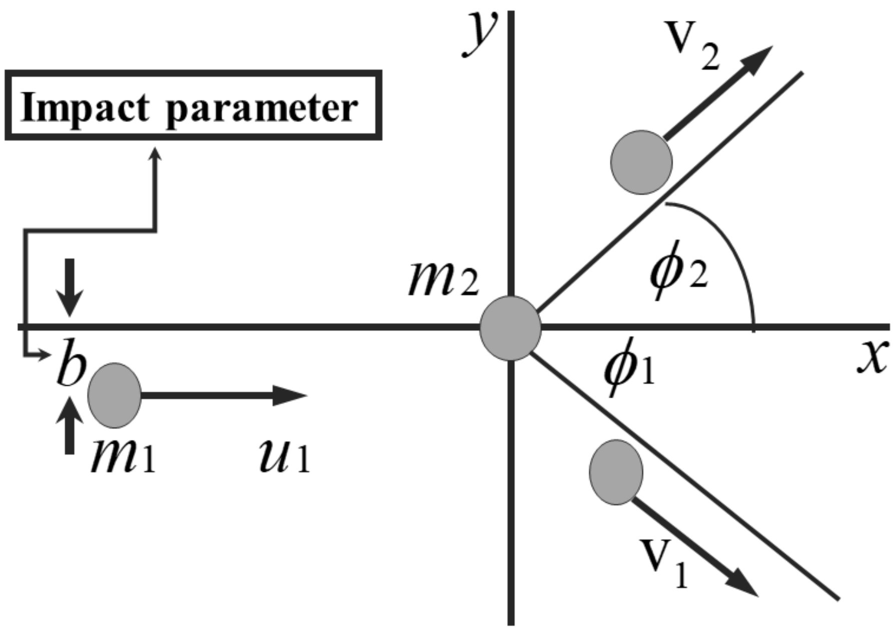


$$P_x = p_{Ax} + p_{Bx} = \mathrm{m_A v}_{Ax} + \mathrm{m_B v}_{Bx}$$
$$P_x = 0 + 2000 \times 10 = 20,000\ kg\,m/s$$
$$P_y = p_{Ay} + p_{By} = \mathrm{m_A v}_{Ay} + \mathrm{m_B v}_{By}$$
$$P_y = 1000 \times 15 + 0 = 15000\ kg\,m/s$$
$$P = \sqrt{P_x{}^2 + P_y{}^2} = \sqrt{20,000^2 + 15,000^2} = 25000\ kg\,m/s$$
$$\tan\theta = \frac{P_y}{P_x} = \frac{15,000}{20,000} = 0.75 \quad \Rightarrow \quad \theta = 37^0$$

Problem 4:

Consider two masses m_1 moving with velocity u_1 and m_2 at rest (u_2=0) as shown in figure. This is not a central collision (i.e., m_1 directly collide to m_2), instead m_1 slightly touches m_2 by one of its sides. Apply the law of conservation of momentum to this case.

Solution:



$$1)\ p_{ix} = p_{fx}$$
$$\Rightarrow m_1 u_1 = m_1 v_1 \cos\phi_1 + m_2 v_2 \cos\phi_2$$
$$2)\ p_{iy} = p_{fy}$$
$$\Rightarrow 0 = m_1 v_1 \sin\phi_1 - m_2 v_2 \sin\phi_2$$
$$3)\ KE_i = KE_f$$
$$\Rightarrow \tfrac{1}{2} m_1 u_1^2 = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2$$

As there are 4-unknowns and 3-equations, they cannot be solved.

Problem 5: Consider two masses m_1 moving with velocity v_1 and m_2 moving with velocity v_2. Both masses collied with each other. After collision, they become a one body with mass M moving with velocity v_f. Apply the law of conservation of momentum to this case.

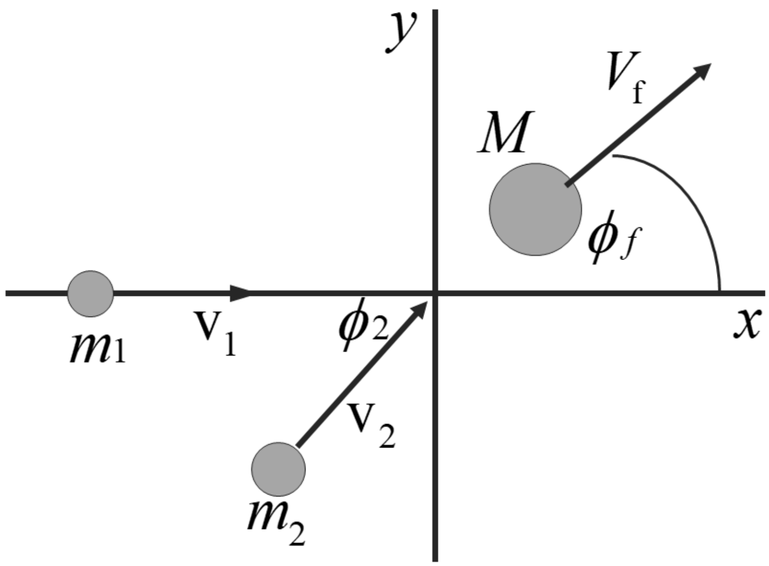
Solution:

1) $p_{ix} = p_{fx}$

$$\Rightarrow m_1 v_1 + m_2 v_2 \cos\phi_2$$
$$= (m_1 + m_2) V_f \cos\phi_f$$

2) $p_{iy} = p_{fy}$

$$\Rightarrow m_2 v_2 \sin\phi_2 = M V_f \sin\phi_f$$



Now there are 2-unknowns and 2-equations, hence, can be solved.

Conclusion:

- Momentum is always conserved in collisions, but energy may or may not be.
- We have come to trust momentum conservation very much: discovery of the neutrino, hints of black holes, discovery of dark matter.

PHY101-Lecture#11

Rotational Kinematics

When an object undergoes rotation or revolution, it means that the distance from a specific point, known as the center, remains constant. This signifies that during rotation, regardless of the object's position, this distance remains unchanged. We refer to this fixed distance as the radius. As the radius vector changes its position, an associated angle ϕ (phi), also changes accordingly. Every rotation entails an angle, ϕ, which defines the rotational position. Unlike Cartesian coordinates (X and Y) used to pinpoint a location, in rotational motion, we simplify this to a single value, ϕ. Furthermore,

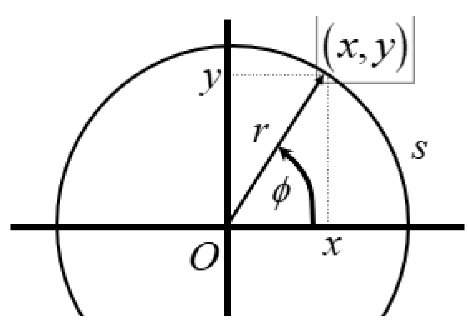


Fig. 11. 1: For a circle of radius r, one radian is the angle subtended by an arc length equal to r.

during rotation, a fixed distance ‘s’ known as arc length is maintained. This arc length corresponds to the radius vector, and as ϕ increases, so does the arc length, denoted by 's'. It's noted that 's' is directly proportional to ϕ, indicating that as ϕ increases, the arc length also increases.

$$\text{arc length} = \text{radius} \times \text{angular displacement}$$

$$s = r\phi$$

$$\begin{aligned} \text{one revolution} &= 2\pi \text{ radians} \\ &= 360 \text{ degrees} \\ 1 \text{ radian} &= 57.3^0 \\ 1 \text{ radian} &= 0.159 \text{ revolution} \end{aligned}$$

$$s = 2\pi r = \text{total circumference}$$

Angular Speed

Consider an object at this moment, it occupies a certain position, let's denote the fixed distance from the center as 'A'. Initially positioned at ϕ_1, it then rotates to another position, which we'll refer to as ϕ_2. Consequently, it maintains a fixed angular difference of $\phi_2 - \phi$. Observe its depiction: 'r' has a complex value. Initially at ϕ_1 and later at ϕ_2. Suppose it's at ϕ_1 at time t_1 and at ϕ_2 at time t_2. From this, we derive the concept of angular speed.

$$\bar{\omega} = \frac{\phi_2 - \phi_1}{t_2 - t_1} = \frac{\Delta\phi}{\Delta t}$$

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\phi}{\Delta t}$$

$$\omega = \frac{d\phi}{dt}$$

as $\Delta\phi$ decreases, we assume that $\Delta\phi$ and Δt are very small. This difference occurs over a brief period, causing Δt to

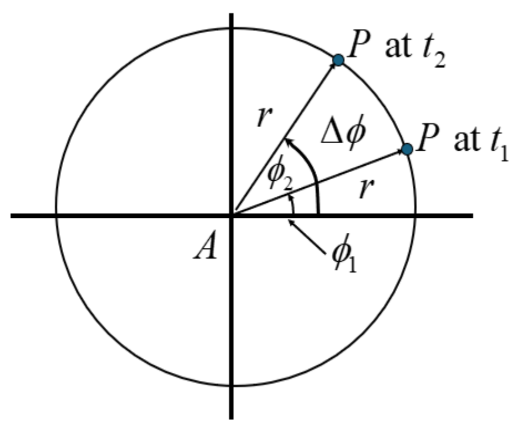


Fig. 11. 2: A point on a rotating circular path for t_1 and t_2 displaced through the angle $\Delta\phi = \phi_2 - \phi_1$.

decrease. Consequently, ω (Omega), our angular speed, approaches the value of $\frac{d\phi}{dt}$. We denote this angular speed as 'ω'.

Consider the example of a clock. You'll notice that the second hand moves rapidly, followed by the minute hand, and then the hour hand. Now, let's explore their respective angular speeds. To calculate angular speed, we must remember that each hand covers a fixed angle during every revolution. For the second hand, this angle is traversed within 60 seconds. Therefore, its angular speed is 2π divided by 60 seconds. For the minute hand, this distance is covered within one hour, which means 60 minutes times 60 seconds equals 3600 seconds. For the hour hand, spanning 12 hours, you'd divide this time by 12, then calculate its respective values.

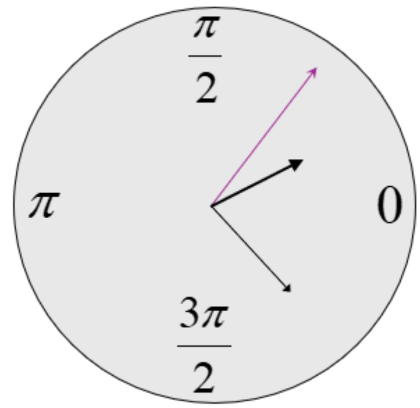


Fig. 11. 3: Three different angular speeds in a wall clock are connected to the radian angles.

$$\omega = \frac{2\pi}{T}$$

$$\omega_{\text{second}} = \frac{2\pi}{60} = 0.105 \; rad / s$$

$$\omega_{\text{minute}} = \frac{2\pi}{60 \times 60} = 1.75 \times 10^{-3} \; rad / s$$

$$\omega_{\text{hour}} = \frac{2\pi}{60 \times 60 \times 12} = 1.45 \times 10^{-4} \; rad / s$$

Problem 11.1: Our sun is 2.3 x 10^4 light years away from the center of our Milky Way galaxy. It moves in a circle around this center at 250 km/s.

(a) How long does it take the sun to make one revolution about the galactic center?

(b) How many revolutions has the sun completed since it was formed about 4.5 x 10^9 years ago?

Solution:

$$\text{a) 1 Light Year} = 9.46\times10^{15}\,m$$

$$\text{v} = R\omega = R\frac{\theta}{t} = R\frac{2\pi}{T}$$
$$\therefore \text{ for one revolution } T = \frac{2\pi R}{\text{v}}$$
$$T = 5.5\times10^{15}\,s = 1.74\times10^{8}\text{ years}$$

$$b)\ \frac{4.5\times10^{9}}{1.74\times10^{8}} = 26\text{ revolutions}$$

Angular acceleration

Angular Acceleration is defined as the time rate of change of angular velocity. It is usually expressed in radians per second per second. Thus,

$$a = \frac{d\omega}{dt}$$

The angular acceleration is also known as rotational acceleration. It is a quantitative expression of the change in angular velocity per unit time.

The average angular acceleration ($\bar{\alpha}$) over a specific time interval. Here, ω represents angular velocity, and t represents time. ω_1 and ω_2 are the initial and final angular velocities respectively, and t_1 and t_2 are the initial and final times respectively.

$$\bar{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$
$$\alpha = \lim_{\Delta t\to 0}\frac{\Delta\omega}{\Delta t}$$
$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\frac{d\phi}{dt} = \frac{d^2\phi}{dt^2}$$

Here, ϕ represents angular displacement. This equation shows that angular acceleration (α) can also be expressed as the second derivative of angular displacement with respect to time. It represents how rapidly the angular velocity is changing, which in turn indicates how quickly an object is rotating.

When the angle changes, the distance also changes accordingly. This implies that the length of the arc also undergoes alteration. As I rotate my hand in a certain direction, the angle increases, and there's a corresponding change in the distance, referred to as *'s'*, which equals *'r'* multiplied by *'* ϕ *'*, the angle through which it has moved. This *'* ϕ *'* varies with time, so *'s'* equals *'r'* multiplied by ' ϕ '. Now, let's differentiate it with respect to time. Since *'r'* is constant, it becomes *'r'* times the derivative of *'* ϕ *'* with respect to *'t'*, denoted as *d/dt* of *'r'* times ' ϕ ', which is *'* ω *'*. The formula for the speed at which this object is moving is *'r'* times *'* ω *'*. If *'* ω *'* is constant, then differentiating it again is straightforward.

$$s = r\phi$$

$$\frac{ds}{dt} = r\frac{d\phi}{dt}$$

$$\mathrm{v} = r\omega$$

$$\frac{d\mathrm{v}}{dt} = r\frac{d\omega}{dt}$$

$$\mathrm{a_T} = r\alpha$$

the acceleration '$\mathrm{a_T}$' in the tangent direction equals *'r'* times *'* α *'*. The tangent direction refers to the direction tangent to the motion path. I'll illustrate this with a diagram shortly in Fig.11.4. There exists a close relationship between linear motion and angular motion, despite the differences: one involves straight-line movement while the other involves rotation with constant distance. However, their mathematical principles differ, although to a similar extent. Let's revisit the formulas we derived earlier for linear motion.

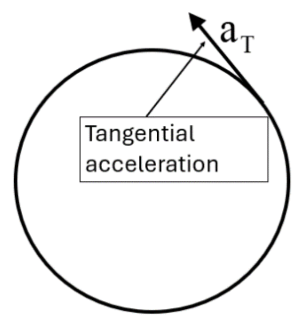


Fig. 11. 4: Tangential acceleration is tangent of a circle.

Relationship between linear and angular variables:

Translational Motion	**Rotational Motion**
$\mathrm{v} = \mathrm{v}_0 + \mathrm{a}t$ $x = x_0 + \mathrm{v}_0 t + \frac{1}{2}\mathrm{a}t^2$ $\mathrm{v}^2 = \mathrm{v}_0^2 + 2\mathrm{a}(x - x_0)$	$\omega = \omega_0 + \alpha t$ $\phi = \phi_0 + \omega_0 t + \frac{1}{2}\alpha t^2$ $\omega^2 = \omega_0^2 + 2\alpha(\phi - \phi_0)$

Problem 11.2: A point on the rim of a 0.75 m diameter grinding wheel changes speed from 12 m/s to 25 m/s in 6.2 s. What is the angular acceleration during this interval?

Solution:

$$\mathrm{a} = \frac{\mathrm{v}_f - \mathrm{v}_i}{t} = 2.1\,\mathrm{m/s^2}$$

$$\because \quad \mathrm{a} = r\alpha$$

$$r = d/2 = 0.75/2 = 0.375$$

$$\alpha = \frac{\mathrm{a}}{r} = \frac{2.1}{0.375} = 5.6\,\mathrm{rad/s^2}$$

Problem 11.3: The angular speed of a car engine is increased from 1170 rev/min to 2880 rev/min in 12.6 s.

(a) Find the average angular acceleration in rev/ min^2.

(b) How many revolutions does the engine make during this time?

Solution:

$$\omega_i = 1170\ \mathrm{rev/min}$$

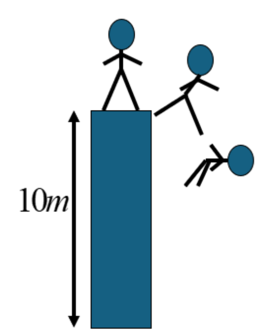
$$\omega_f = 2880\ rev/\min$$

$$t = 12.6\ s = 12.6/60 = 0.21\,\mathrm{min}$$

$$\alpha = \frac{\omega_f - \omega_i}{t} = \frac{(2880 - 1170)\ \mathrm{rev/min}}{0.21\,\mathrm{min}} = 8140\,\mathrm{rev/min^2}$$

$$\phi = \omega_i t + \frac{1}{2}\alpha t^2 = 425\,\mathrm{rev}$$

Problem 11.4: A diver makes 2.5 complete revolutions on the way from a 10 m platform to the water below shown in Fig.11.5. Assuming zero initial vertical velocity, calculate the average angular velocity.



$$h = v_i t + \frac{1}{2} g t^2$$

$$\therefore v_i = 0$$

$$h = \frac{1}{2} g t^2$$

$$h = 10\ m\ ,\ \text{g} = 10\ \text{m/s}^2$$

$$t = \sqrt{\frac{2h}{g}} = 1.43\ s$$

$$\therefore \omega = \frac{\phi}{t} = \frac{2\pi(\frac{rad}{rev})n}{t}$$

$$n = 2.5\ (rev)$$

$$\omega = \frac{2\pi(\frac{rad}{rev}) \times 2.5\ (rev)}{1.43\ s} = 11\,\text{rad/s}$$

Fig. 11. 5: Path of a diver from a 10 m Platform.

Rotation is a common phenomenon in our surroundings, not limited to just the wheels of a car. If you observe any machinery, whether it's a car, motorcycle, or in a lathe shop, you'll notice the interaction between small and large wheels. Understanding the concepts of angular speed and angular acceleration allows us to comprehend many aspects of our surroundings.

Consider a bicycle as an example. In a bicycle, there are two gears: a large one driving a smaller one. As the large gear rotates once, the smaller gear rotates several times. We can calculate this rotation ratio using their respective radii. If the radius of the larger gear is represented as 'r_c' and that of the smaller gear as 'r_a' as shown in Fig.11.6, then the smaller gear will rotate ' r_c / r_a ' times for every rotation of the larger gear. If there's a threefold difference between their radii, the smaller gear will rotate three times for every rotation of the larger gear. Now, let's focus on the chain mechanism. The chain's purpose is to transmit force effectively. Despite its motion, the chain maintains a constant speed throughout its length. This uniformity ensures that as one part of the chain moves forward, the other part moves backward with the same speed, ensuring effective force transmission. When acceleration occurs, it is also transmitted backward through the chain. Now, let's work through a problem related to this concept.

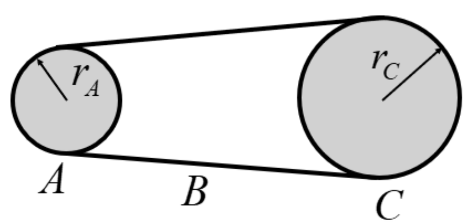


Fig. 11. 6: Angular speed between gears of a bicycle.

Problem 11.5: Wheel A of radius r_A = 10.0 cm is coupled by a chain B to wheel C of radius r_C = 25.0 cm as shown in Fig.11.7. Wheel A increases its angular speed from rest at a uniform rate of 1.60 rad/s^2.

Determine the time for wheel C to reach a rotational speed of 100 rev/min.

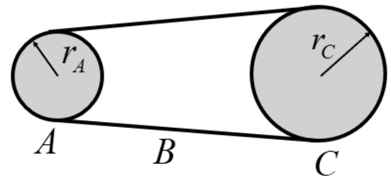


Fig. 11. 7: Angular speed between gears of a bicycle provides certain rotational speed for a specific time.

$$\begin{aligned}
&Given:\\
&r_A = 10.0\ cm\\
&r_C = 25.0\ cm\\
&Angular\ acceleration\ of\ wheel\ A,\ \alpha = 1.60\ rad/s^2\\
&Desired\ rotational\ speed\ of\ wheel\ C,\ \omega_C = 100\ rev/\min\\
&First,\ we\ need\ to\ convert\ the\ desired\ rotational\ speed\ of\ wheel\ C\\
&from\ revolutions\ per\ \min ute\ to\ radians\ per\ \sec ond:\\
&Angular\ speed\ of\ wheel\ C = \omega_C = \frac{100\times 2\pi}{60} = \frac{100\times 2(3.14)}{60} = 10.5\ rad/s\\
&Now,\ we\ can\ use\ the\ given\ equations:\\
&v_A = v_C \Rightarrow\ r_A\omega_A = r_C\omega_C\\
&1.\ \omega_A = \frac{r_C\omega_C}{r_A}
\end{aligned}$$

$$\omega_A = \frac{25.0\times 10.5}{10.0} = 26.3\ rad/s$$

$$2.\ \alpha = \frac{\omega_A - 0}{t}$$

$$3.\ t = \frac{\omega_A}{\alpha} = \frac{26.3}{1.60} = 16.4\ s$$

Uniform circular motion

Consider an object rotating in a circle at a constant speed. For instance, imagine I've tied it to a thread and I'm rotating it. Now, the length of the thread remains constant, ensuring that the distance from the center, or the radius, remains unchanged, and the speed of the object also remains constant. However, does this imply that the speed remains constant without any acceleration? We need to delve deeper into this. It's essential to consider the direction of velocity. Initially, as mentioned before, the speed remains constant, denoted as v. However, there's a difference in direction: if v_1 is upward, then v_2 is slightly sideways, creating an angle between them. Moreover, v_2 is directed towards the center, giving rise to what

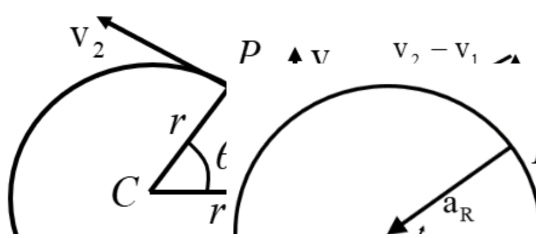


Fig. 11. 8: As object moves from P_1 to P_2 the direction of its velocity changes from v_1 to v_2. The resultant vector of velocity $v_2 - v_1$ is toward the center of the circle.

we call centripetal acceleration. Now, let's determine its value. Understanding its direction is crucial; it's directed towards the center, thus termed centripetal acceleration.

To find out the difference, we need to calculate v_2 - v_1. Consider this scenario: if the body is initially positioned at P_1 and after some time it moves to the point P_2, forming an angle θ (theta), Let's denote its length a_R, as the radius, and θ as the angle through which it has rotated. By drawing a triangle here, we can notice a minor discrepancy between the triangle and the actual path. This occurs when theta is very small. However, if I adjust it slightly, ensuring theta is small, there will be no discernible difference between the triangle and the actual path. Thus, Δr is approximately equal to $r\theta$.

Fig. 11. 9: As the object moves along the cicular path the direction of its velocity changes so the object undergoes a radial acceleration.

$$\Delta r = \mathrm{v}\Delta t \approx r\theta$$

This same concept directly applies to velocity. Notice that another triangle is formed, where delta and velocity are both influenced by theta.

$$\Delta \mathrm{v} \approx \mathrm{v}\theta$$

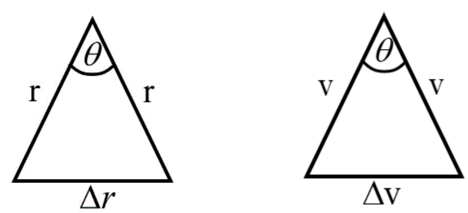


Fig. 11. 10: Triangular approximation for radius r and velocity v.

Applying this concept again, we find that alpha, denoted by 'a', equals v^2 divided by 'r', representing the average acceleration. However, you'll learn that there's minimal disparity between average and instantaneous values, given negligible differences in distances and times. Consequently, as we let Δt approach zero, we obtain v^2/r, known as centripetal acceleration.

$$\bar{a} = \frac{\Delta v}{\Delta t} \approx \frac{v\theta}{r\theta / v} = \frac{v^2}{r}$$

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{v^2}{r}$$

$$\vec{a}_R = -\frac{v^2}{r}\hat{r}$$

Negative sign shows the acceleration is readily inward.

If there's no other acceleration, meaning the object maintains a constant speed, then only radial acceleration exists. However, if you gradually increase its speed, acceleration occurs; likewise, when decelerating. This introduces another acceleration component: tangential acceleration as shown in Fig 11.11. The total acceleration becomes a vector with two components: one being centripetal and the other is tangential. The resultant acceleration of these two components equals the total acceleration ‘a’.

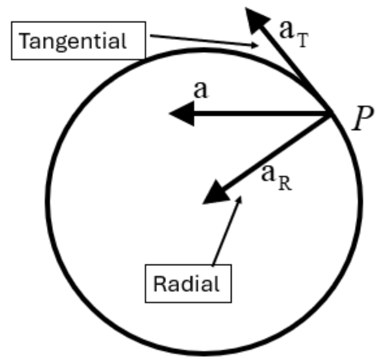


Fig. 11. 11: Total acceleration of the rotating object at a point P.

$$a_T = r\alpha \ , \ a_R = \frac{v^2}{r} = r\omega^2$$

$$a = \sqrt{{a_T}^2 + {a_R}^2}$$

Problem 11.6: The Moon revolves about the Earth as shown in Fig.11.12, making a complete revolution in 27.3 days. Assume that the orbit is circular and has a radius of 238,000 miles. What is the magnitude of the acceleration of the Moon towards the Earth?

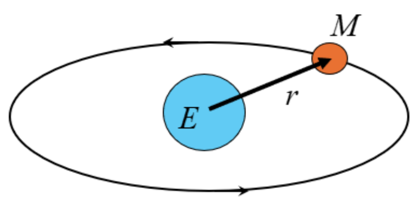


Fig. 11. 12: Illustration of the Moon and Earth during orbital motion.

Given Data:

$Radius\ of\ the\ Moon's\ orbit: r = 3.81952 \times 10^8\ meters$

$Period\ of\ the\ Moon's\ orbit: T\ =\ 27.3\ days$

$Calculate\ Orbital\ Speed\ (v):$

$Convert\ the\ period\ of\ the\ Moon's\ orbit\ from\ days\ to\ seconds:$

$$T = 27.3 \times 24 \times 60 \times 60 \text{seconds}$$

$$T = 2.36 \times 10^6 \text{seconds}$$

$Use\ the\ formula\ for\ the\ circumference\ of\ a\ circle\ to\ find\ the\ orbital\ speed:$

$$v = \frac{2\pi r}{T} = \frac{2 \times 3.14159 \times 3.81952 \times 10^8}{2.36 \times 10^6} \text{m/s}$$

$$v = \frac{23.95546 \times 10^8}{2.36 \times 10^6} = 1015.421 \text{m/s}$$

$Calculate\ Acceleration\ (\mathrm{a}):$

$Use\ the\ centripetal\ acceleration\ formula:$

$$a = \frac{v^2}{r}$$

$Substitute\ the\ calculated\ values:$

$$a = \frac{(1015.421 \text{m/s})^2}{3.81952 \times 10^8} \text{m/s}^2 = \frac{1030885.961}{3.81952 \times 10^8} \text{m/s}^2$$

$$a \approx 2.719 \times 10^{-3} \text{m/s}^2$$

Problem 11.7: Calculate the speed of an Earth satellite as shown in Fig.11.13 that it is traveling at an altitude h of 210 km where g = 9.2 m/s^2. The radius R of the Earth is 6370 km.

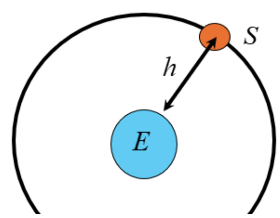


Fig. 11. 13: Artificial satellite ‘S’ orbiting around the Earth at a height h.

Solution:

$$\mathrm{a} = \frac{\mathrm{v}^2}{\mathrm{r}}$$

$$\mathrm{a} = g \text{ and } r = R + h$$

$$g = \frac{\mathrm{v}^2}{R + h}$$

$$\mathrm{v} = \sqrt{(R+h)g} = 7780\, m/s$$

Vector Cross Products

Let's explore another method of defining the product of two vectors, known as the cross product. Imagine drawing lines on a piece of paper. First, we'll create vector A and then position the second vector, 'B', anywhere within the same plane on this paper. Between them, there exists an angle θ. Now, while the dot product involves the sine of theta multiplied by the magnitude of A times B, the cross product takes a different approach. It's perpendicular to both A and B as shown in Fig.11.14, extending in a direction we'll call the 'n' direction. This perpendicularity is similar to the tip of a pencil rooted on the paper. The magnitude of this vector, denoted as 'A cross B', equals the magnitude of A times B times sine theta, with this being its limit.

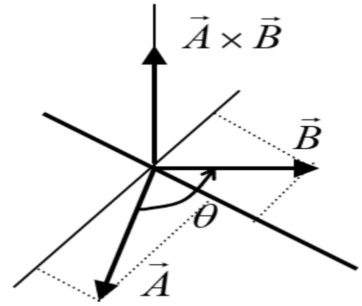


Fig. 11. 14: Vectors A and B and their cross-product A × B are perpendicular to each other.

$$\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$$

$\hat{n}$ is perpendicular to AB-plane

Now, let's consider unit vectors in three dimensions, represented by $\hat{i}$, $\hat{j}$, and $\hat{k}$ cross products of these unit vectors are well defined. As the angle between any two different unit vectors is always 90 degrees, if we take the cross product of $\hat{i}$ and $\hat{j}$, it results in the third direction, $\hat{k}$, and similarly for other combinations.

$$\hat{i}\times\hat{j} = \hat{k}$$
$$\hat{k}\times\hat{i} = \hat{j}$$
$$\hat{j}\times\hat{k} = \hat{i}$$

Important properties

Several important properties characterize the cross product. For instance, if two vectors are parallel, their cross product is zero. When the angle between them is zero, the cross product of a vector with itself is also zero.

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$$
$$\vec{A}\times\vec{A} = 0$$

There are other properties as well, such as the distributive property.

$$\left(\vec{A}+\vec{B}\right)\times\vec{C} = \left(\vec{A}\times\vec{B}\right)+\left(\vec{A}\times\vec{C}\right)$$

When expressed using unit vectors, A cross B can be represented as a determinant.

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$$

$$=(A_yB_z-A_zB_y)\hat{i}+(A_zB_x-A_xB_z)\hat{j}+(A_xB_y-A_yB_x)\hat{k}$$

Summary

In this chapter, we explained the fundamental concepts of rotational kinematics, specifically focusing on angular speed, angular velocity, and angular acceleration. These concepts come into play when a body undergoes rotational motion, moving along a circular path. We questioned why objects tend to move in circular paths when, inherently, they tend to move in straight lines. Exploring these principles helps us understand the forces at play that compel objects to rotate. Overall, this chapter provided an overview of rotational kinematics, shedding light on the dynamics of circular motion**.**

Physics-PHY101-Lecture 12

PHYSICS OF MANY PARTICLES

Introduction to center of mass:

Everybody is made up of particles and every particle has a direction with every other particle, that it either pulls it towards itself or pushes it. There are some quantities about which if we have the information then we can say something about the system as a whole which means they include the center of mass. So, where the center of the mass will be, there will be a kind of concentration of mass.

Let's say there is an elephant. An ant moves on top of the elephant. It will not make much difference if the ant moves here or there. The mass of the elephant is so much that the center of mass will be the center of elephant. We would like to define the **center of mass** as,

A system's or object's center of mass is the location where the mass is considered to be concentrated. Simply, it's the average position of the mass distribution.

Mathematically for two masses the center of mass r_{cm} is:

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

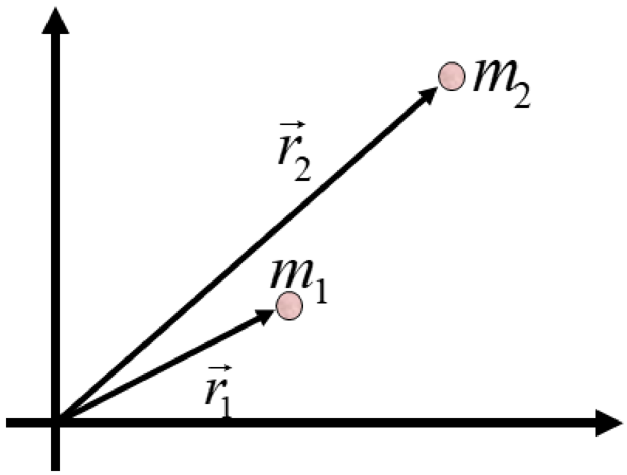


Figure 12.1. Distribution of masses 'm_1 and m_2' with their position vectors 'r_1 and r_2'.

The center of mass equation can be written in two or three dimensions, depending on the given situation in question. In the case of two dimensions, its coordinates will be x and y; in the case of three dimensions, there will be x, y, and z. Equation of center of mass for two dimension is:

$$x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

$$y_{cm} = \frac{m_1y_1 + m_2y_2}{m_1 + m_2}$$

The center of mass will lie between two masses if their masses are equal ($m_1 = m_2 = m$), and closer to the heavier mass if one mass is significantly greater than the other. If M is heavier than m, the center of mass is near mass M as can be seen from figure 12.2.

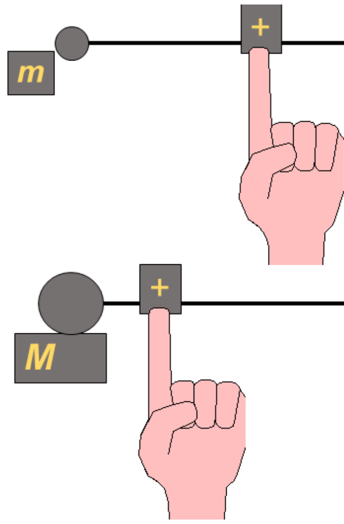


Figure 12.2. Center of mass alignment for unequal masses. Equal masses ($m_1 = m_2 = m$) result in a center of mass equidistant between them. For unequal masses ($M > m$), the center of mass is closer to the heavier mass.

Let's take another example. Consider two bodies of equal masses, one is placed at x = 2 m and other is placed at x = 6 m as shown in figure 12.3, then center of mass will be ,

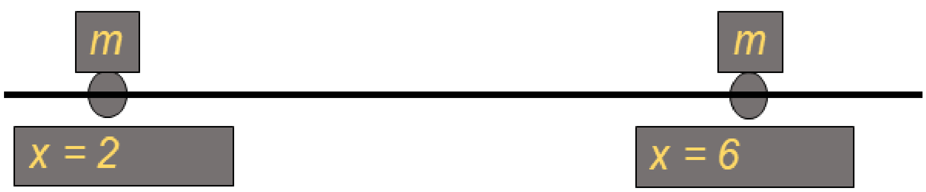


Figure 12.3. Two bodies of equal masses are positioned at x = 2 m and x = 6 m. The center of mass location is determined by their equilibrium.

$$x_{cm} = \frac{mx_1 + mx_2}{m + m}$$

$$x_{cm} = \frac{m(2\ meter) + m(6\ meters)}{2m}$$

$$x_{cm} = \frac{8m}{2m} = 4m \text{ (4 meters)}$$

Now consider two bodies of unequal masses. One is a lighter 'mass m' placed at x = 6 m and other is a heavier 'mass 3m' placed at x = 2 m as shown in figure 12.4.

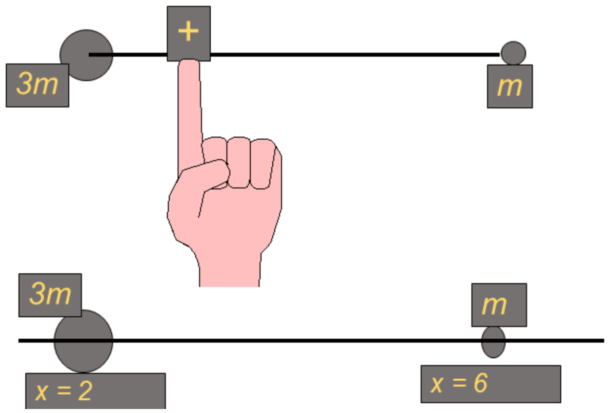


Figure 12.4. A lighter mass 'm' is positioned at x = 6 m and a heavier mass '3m' at x = 2 m. The center of mass is influenced by the masses and their respective positions.

$$x_{cm} = \frac{(3m)x_1 + mx_2}{3m + m} = \frac{(3m)2 + 6m}{4m} = \frac{12m}{4m} = 3\ meters$$

For N masses, the center of mass is:

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N}{m_1 + m_{2+} + \cdots + m_N}$$

$$\vec{r}_{cm} = \frac{1}{M}\left(\sum m_n\vec{r}_n\right)$$

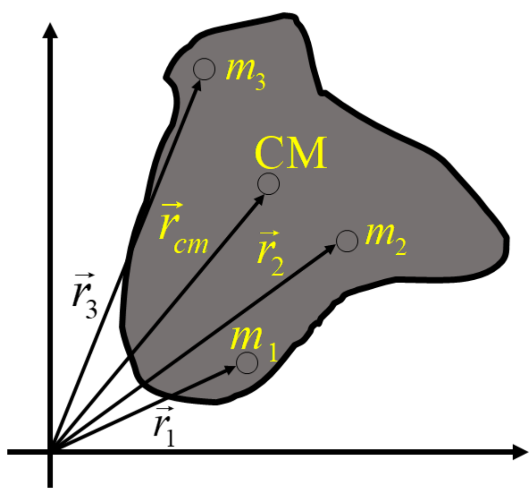
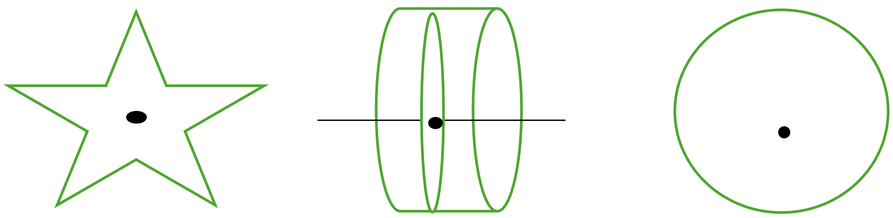


Figure 12.5: There is a point CM which we call center of mass, from the origin O, vector r be the position vector.

For symmetrical object, the center of mass is easy to guess as can be seen from the figures below.



(a) (b) (c)

In figure (a), the center of mass is in the center of star and in (b) the center of mass is in the center of cylinder if the material of cylinder will be uniform and in figure (c) if it's a uniform sphere then its center of mass is also in the center of sphere.

In order to understand that why the center of mass is useful concept, we have to apply newton's law. The velocity and acceleration of the center of mass can be calculated by the definition of the center of mass and differentiating it w.r.t time.

$$\vec{v}_{cm} = \frac{d\vec{r}_{cm}}{dt} = \frac{1}{M}\left(\sum m_n \vec{v}_n\right) \quad \therefore \vec{r}_{cm} = \frac{1}{M}\left(\sum m_n \vec{r}_n\right)$$

$$\vec{a}_{cm} = \frac{d\vec{v}_{cm}}{dt} = \frac{1}{M}\left(\sum m_n \vec{a}_n\right)$$

It is known as the acceleration of a single point, and we know that a particle possesses acceleration only when a force act on it. Here, the force can act for two reasons.

- One reason is that the particle is in external field. For example, the particle is inside the gravity and the gravity is pulling it down or up or in some direction.
- The second reason is that the other particles exert force and push or pull the particle in different directions.

The two types of forces that act on every particle are divided into external and internal forces. Therefore, the acceleration of each particle is due to the internal and external forces, but the internal force gets cancelled (because action and reactions are equal and opposite) and only external force remains.

$$M\vec{a}_{cm} = \sum \vec{F}_n = \sum \left(\vec{F}_{ext} + \vec{F}_{\text{int}} \right)$$

$$\sum \vec{F}_{ext} = M\vec{a}_{cm}$$

In conclusion, the center of mass moves in such a way to concentrate the whole mass on it.

A center of mass is a point but it’s not necessarily for a mass to be actually there. It's a common misconception that there's actually something. It is only appearing that all the masses concentrated there but when Newton's Law is applied, this imaginary point moves just like a mass that is concentrated at one point.

The center of mass for a combination of objects is the average center of mass location of objects. The center of mass can be outside the body; it does not have to be inside for all the structures as can be seen from figure 12.6.

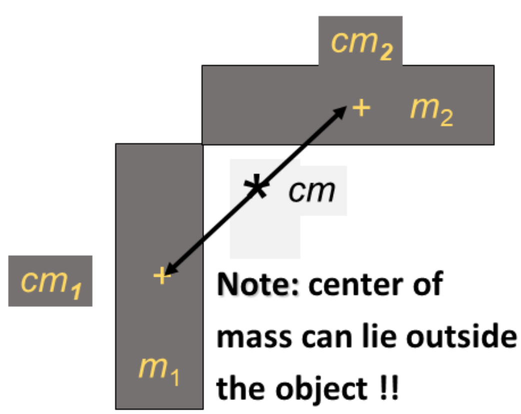


Figure 12.6. The center of mass represents the average location, and it can be located outside the physical boundaries of the objects.

Rotational Energy of Rigid Bodies:

Where there is movement there will be energy called kinetic energy. Motion is not only in one direction it can also be rotational motion. Consider a rigid body rotating about a fixed axis as shown in figure 12.7.

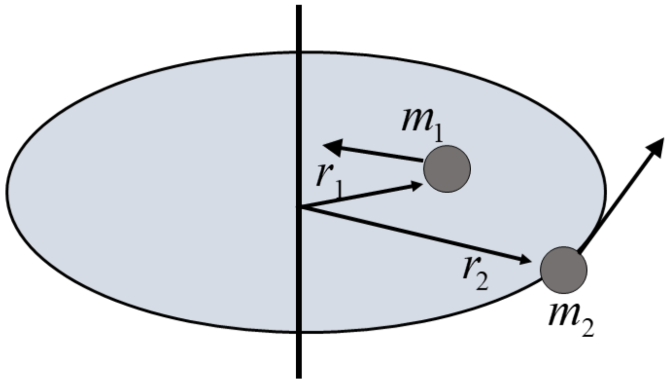


Figure 12.7. Rigid body rotation about a fixed axis.

Now let’s know what a rigid body is. A body which has no elasticity, in which all its particles move together, such as wheel. If you calculate the total kinetic energy of a rigid body, then it is the sum of all the kinetic energies of each body.

Total kinetic energy is:

$$K = \frac{1}{2}m_1{v_1}^2 + \frac{1}{2}m_2{v_2}^2 + \frac{1}{2}m_3{v_3}^2 + \cdots$$
$$\because v = r\omega$$

For a rigid body or an inelastic body, all the particles are moving together, which means that each particle has the same angular velocity ‘ω’ as the other particles, when we add all the K.E we get,

$$K = \frac{1}{2}m_1{r_1}^2\omega^2 + \frac{1}{2}m_2{r_2}^2\omega^2 + \frac{1}{2}m_3{r_3}^2\omega^2 + \cdots$$

$$K = \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2$$

Where $\sum m_i r_i^2$ is called as ‘moment of inertia I’.

Rotational inertia:

$$K = \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2$$

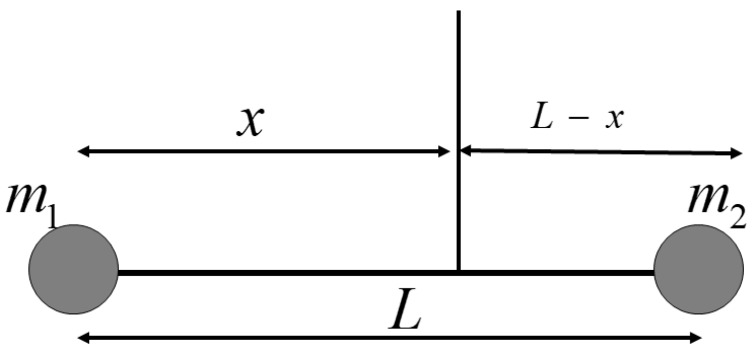
$$K = \frac{1}{2} I\omega^2 \qquad \therefore I = \sum m_i r_i^2$$

This implies that we will take the mass of each body and use the square of the distance to calculate each body's moment of inertia. Then, we will add up all of the body's moments of inertia to get the total moment of inertia. When a body moves on straight line than its KE is:

$$K = \frac{1}{2} Mv^2 \qquad \therefore \text{ v is the linear velocity.}$$

Problem:

Two particles m_1 and m_2 are connected by a light rigid rod of length L. neglect the mass of rod, find the rotational inertia I of this system about an axis perpendicular to the rod and at a distance x from m_1.



Solution:

$$I = I_1 + I_2$$

$$I = m_1 x^2 + m_2 (L - x)^2$$

For what x, is I the largest?

$$\frac{dI}{dx} = \frac{d}{dx}\left(m_1 x^2 + m_2 (L-x)^2\right)$$

$$\frac{dI}{dx} = m_1(2x) + m_2 2(L-x)(-1)$$

For a maxima, $\frac{dI}{dx} = 0$

$$0 = 2m_1x - 2m_2\left(L - x\right)$$
$$0 = 2m_1x - 2m_2L + 2m_2x$$
$$\left(2m_1 + 2m_2\right)x = 2m_2L$$
$$x = \frac{2m_2L}{2(m_1 + m_2)} = \frac{m_2L}{(m_1 + m_2)}$$

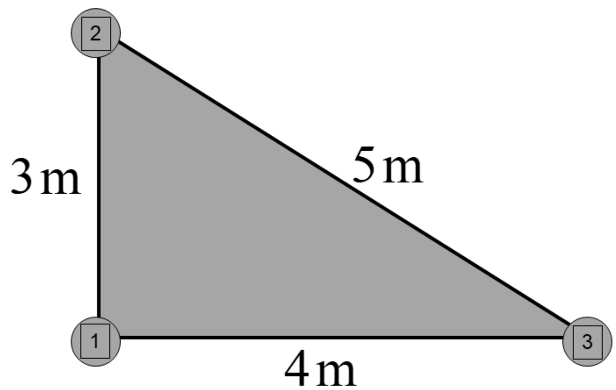
Conclusion: At this distance, the system of particles possesses the highest value of moment of inertia. If $m_1 = m_2$ then x = L/2 for maxima.

Problem:

Three particles of masses m_1 (2.3 kg), m_2 (3.2 kg) and m_3 (1.5 kg) are at the vertices of a triangle.

Part-I: Find the rotational inertia about axes perpendicular to the xy plane and passing through each of the particles.

Part-II: What is the moment of inertia about the center of mass?



Solution:

Moment of inertia about each axis of rotation =?

When passing through particle 1:

$$I = I_1 + I_2 + I_3 = mr_1^2 + mr_2^2 + mr_3^2$$

$$I = 2.3kg(0) + 3.2kg(3)^2 + 1.5kg(4)^2 = 52.8kg$$

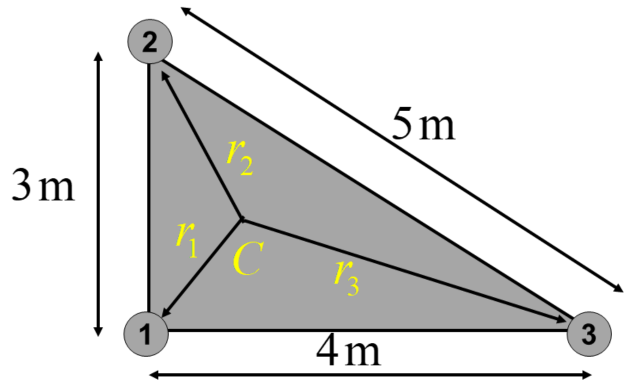
When passing through particle 2:

$$I = 2.3kg(3)^2 + 3.2kg(0) + 1.5kg(5)^2 = 58.2kg$$

When passing through particle 3:

$$I = 2.3kg(4)^2 + 3.2kg(5)^2 + 1.5kg(0) = 116.8kg$$

Part-II: What is the moment of inertia about the center of mass?



Lets find the center of mass first.

Since, masses are in xy-plane so,

$$\overrightarrow{r_{cm}} = x_{cm}\hat{i} + y_{cm}\,j$$

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} = \frac{m_1(0) + m_2(0) + m_3(4)}{2.3 + 3.2 + 1.5} = \frac{6}{7} = 0.86m$$

$$y_{cm} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3} = \frac{m_1(0) + m_2(3) + m_3(0)}{2.3 + 3.2 + 1.5} = \frac{9.6}{7} = 1.37m$$

Using pythagoras theorem,

$$r_1^2 = x_{cm}^2 + y_{cm}^2 = (0.86)^2 + (1.37)^2 = 2.62\text{m}^2$$

$$r_2^2 = x_{cm}^2 + (y_2 - y_{cm})^2 = (0.86)^2 + (3 - 1.37)^2 = 3.40\,\text{m}^2$$

$$r_3^2 = (x_3 - x_{cm})^2 + y_{cm}^2 = (4 - 0.857)^2 + (1.37)^2 = 11.74\text{m}^2$$

$$I_{cm} = \sum m_ir_i^2 = m_1r_1^2 + m_2r_2^2 + m_3r_3^2$$

$$I_{cm} = 2.3 * 2.62 + 3.2 * 3.40 + 1.5 * 11.74$$

$$I_{cm} = 34.5\,\text{kg}\,\text{m}^2$$

This problem is for three masses, you may solve it for infinite masses. We use integration if we have infinite masses. If we divide a body into small parts, if mass of a small parts is dm. Take the distance r from the center and then take square of distance and then multiply with dm and then sum up over all the parts of the body that make up. This is called integration, and the total moment of inertia is the integration or the integral of the entire body of r square times dm.

For solid bodies:

$$I = \int r^2 dm$$

Hoop about cylinder axes:

$$I = \int r^2 dm \qquad \because \text{r = R= Fix}$$
$$I = R^2 \int dm$$
$$I = MR^2$$

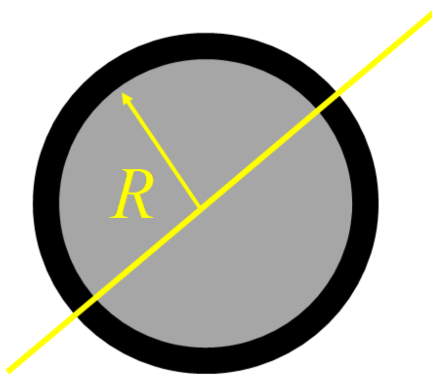


Figure 12.8. Hoop rotation about cylinder axis.

Solid plate about cylinder axes:

$$I = \int r^2 dm$$

$$\therefore \rho_O = \frac{M}{Area} = \frac{M}{\pi r^2}$$

$$if\, \rho_O = \frac{dm}{dA} = \frac{dm}{2\pi r dr}$$

$$dm = 2\pi r dr \rho_O$$

$$I = \int_0^R r^2 .2\pi r dr \rho_O$$

$$I = \int_0^R 2\pi r^3 dr \rho_O = 2\pi\rho_O \int_0^R r^3 dr$$

$$I = 2\pi\rho_O \left. \frac{r^4}{4} \right|_0^R$$

$$I = \frac{1}{2}\left(\pi R^4 \rho_O\right) = \frac{1}{2}\left(\pi R^4\right)\frac{M}{\pi R^2}$$

$$I = \frac{1}{2} MR^2$$

Solid sphere about diameter:

$$I = \frac{2}{5} MR^2$$

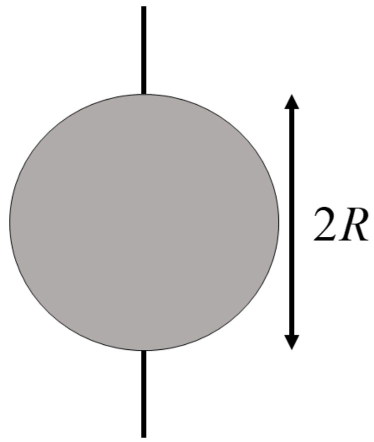


Figure 12.9. Solid sphere rotation about diameter.

For a hollow sphere, the mass is concentrated towards the outside, and you would get the result that a hollow sphere of mass m will have a greater moment of inertia than a sphere of mass m.

The moment of inertia measures how much energy is generated inside an object when you spin it. For example as shown in figure 12.10 (a) and (b).

(a) Solid cylinder or disk about cylinder axis

$$I = \frac{1}{2} MR^2$$

(b) Solid cylinder or disk about central diameter:

$$I = \frac{1}{4} MR^2 + \frac{1}{12} ML^2$$

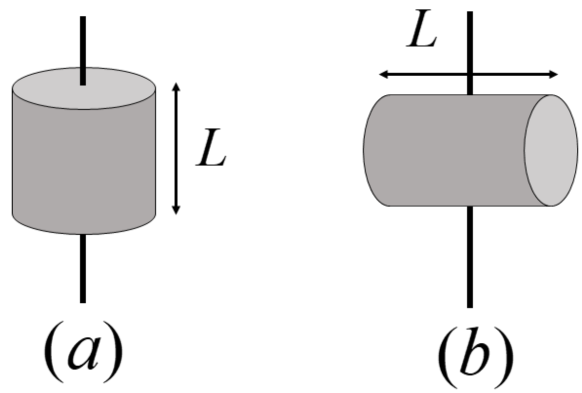


Figure 12.10. (a) solid cylinder or disk about cylinder axis, (b) solid cylinder or disk about central diameter .

Rectangular plate about central axis:

$$I = \frac{1}{12} M \left(a^2 + b^2\right)$$

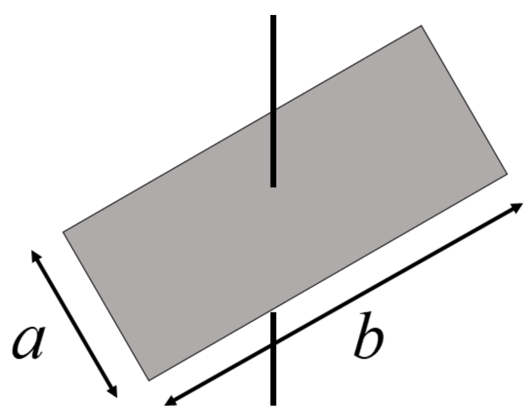


Figure 12.11. Rectangular plate about central axis.

So in conclusion, if an object rotates, its kinetic energy becomes equal to the ½ I ω^2, but the question arises here is that, why the object rotates? Of course, force helps to rotate the object, but it depends on where the force is applied. For example, we have wrenches of various sizes. Because of torque, when we use a small-length wrench, it is difficult to tighten the nut; however, when we use a large-length wrench, the nut is easily tightened. A wrench with a long length has higher torque than one with a little length.

Now we define **torque** mathematically,

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Where 'r' is the distance at which force acts. If we take its magnitude, we have

$$\tau = rF \sin\theta$$

And θ is the angle between r and F as shown in figure 12.12.

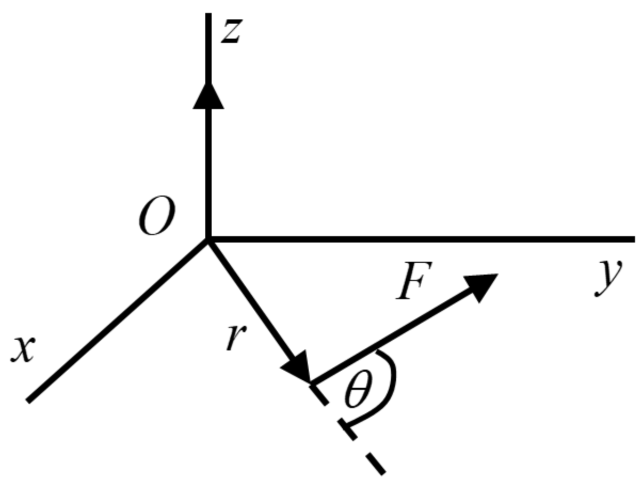


Figure 12.12. Illustration of torque, where 'r' is the distance at which force (F) acts, and θ is the angle between r and F.

Work can be done wherever a force occurs and when it acts and moves a distance S, the force into a distance is equal to work as shown in figure 12.13. So F.ds is the amount of work done.

$$dW = \vec{F}.\vec{ds}$$

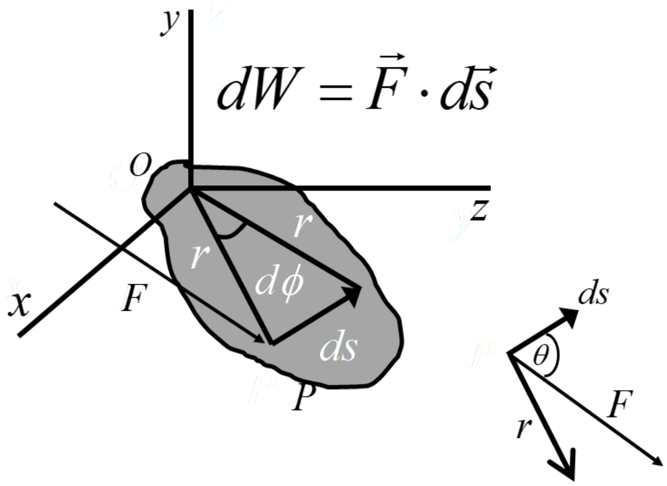


Figure 12.13. Work (W) is the product of force (F) and displacement (ds), where the force acting over a distance results in the performance of work.

So suppose there are several bodies and forces acting on them. Now the work done is,

$$\begin{aligned} dW &= \vec{F}.d\vec{s} = F\cos\theta ds \\ &= (F\cos\theta)(rd\phi) \\ &= \tau d\phi \end{aligned}$$

For all particles,

$$\begin{aligned} dW_{net} &= (F_1\cos\theta_1)r_1 d\phi + (F_2\cos\theta_2)r_2 d\phi + \cdots + (F_n\cos\theta_n)r_n d\phi \\ &= (\tau_1 + \tau_2 + \cdots + \tau_n)d\phi \end{aligned}$$

See, there's only one angle here, and that's because all the particles move together because it's a rigid body.

$$W_{net} = \left(\sum \tau_{ext}\right)d\phi = \left(\sum \tau_{ext}\right)\omega dt$$

When we differentiate the K.E with respect to ω,

$$K = \left(\frac{1}{2} I\omega^2 \right)$$

$$\frac{dK}{d\omega} = \frac{d}{d\omega}\left(\frac{1}{2} I\omega^2 \right) = \frac{1}{2} I(2\omega) = I\omega$$

$$dK = I\omega d\omega \qquad \therefore \alpha = \frac{d\omega}{dt} \Rightarrow d\omega = \alpha dt$$

$$dK = \left(I\alpha \right)\omega dt$$

Remember that α is the angular acceleration.

$$dW_{net} = dK$$

Work has been done by you in rotating it so that work is converted into kinetic energy. Then the result is the total external torque and is equal to moment of inertia multiplied by angular acceleration,

$$\sum \tau_{ext} = I\alpha$$

This equation is analogous to the Newton's law F = ma.

Before continuing we should know where translational and rotational motions are similar.

'x' is displacement at the one side and Ø is angular displacement on the other side. As x increases with the time and we take the derivative dx/dt, likewise Ø increases with time and we take the derivative from there we get the angular speed.

Translational motion	**Angular motion**
x, M	ϕ, I
$v = \frac{dx}{dt}$	$\omega = \frac{d\phi}{dt}$
$a = \frac{dv}{dt}$	$\alpha = \frac{d\omega}{dt}$

In term of dynamics, the moment of inertia has exactly the same role for rotational motion as that of mass in translational motion.

Translational	**Rotational motion**
$F = Ma$	$\tau = I\alpha$
$W = \int F dx$	$W = \int \tau d\phi$
$K = \frac{1}{2}Mv^2$	$K = \frac{1}{2}I\omega^2$

Combined Rotational and Translational Motion:

In order to understand this type of motion, let's take the example of a car. Take the wheel of a car, put a mark on the wheel, and put a mark on its rim. The car goes in a straight line, but at the same time, its wheel rotates. So, this is an example of combined rotational and translational motion.

Now we study it in detail, there is a body with two vectors on it. One vector is of center of mass and the second is at point p, point P is any random point on a body and a vector which goes from the center of mass to P is called r_i' as shown in figure 12.14.

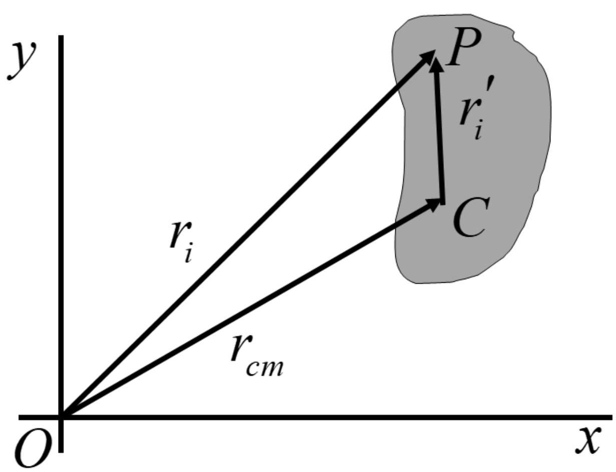


Figure 12.14. Body with center of mass vector and position vector at point P. The vector from the center of mass to a random point p on the body is defined as the position vector r_i'.

If we take out the total kinetic energy of it, then it will have two parts, one part is the kinetic energy of the center of the mass and the other part is due to its rotation.

$$K = \frac{1}{2}M{v_{cm}}^2 + \frac{1}{2}I_{cm}\omega^2$$

Remember, we didn't take a regular shape here. The body is rolling along its path in addition to traveling in a straight line. We now aim to find out if it is indeed the case that this body possesses two different kinds of kinetic energy. Let's prove it.

First of all, we will extract the kinetic energies of all its different particles, then we divide the velocity of every particle into two parts. One is the part of center of mass and other is due to rotation.

$$K = \sum \frac{1}{2} m_i v_i^2 = \sum \frac{1}{2} m_i \vec{v}_i \,.\vec{v}_i$$
$$\sum \frac{1}{2} m_i \left(\vec{v}_{cm} + \vec{v}_i'\right).\left(\vec{v}_{cm} + \vec{v}'\right)$$
$$\sum \frac{1}{2} m_i \left(v^2{}_{cm} + 2\vec{v}_{cm}.\vec{v}_i' + v_i'^2\right)$$
$$\sum m_i \vec{v}_{cm}.\vec{v}_i' = \vec{v}_{cm}.\sum m_i \vec{v}_i'$$
$$\sum \vec{p}_i' = \sum m_i \vec{v}_i' = M\vec{v}_{cm}'$$
$$\vec{v}_{cm}' = 0$$

$\vec{v}_{cm}' = 0$ in the center of mass frame.

$$K = \sum \frac{1}{2} m_i v^2{}_{cm} + \sum \frac{1}{2} m_i v'^2$$
$$= \frac{1}{2} M v^2{}_{cm} + \sum \frac{1}{2} m_i r_i'^2 \omega^2$$
$$K = \frac{1}{2} M v^2{}_{cm} + \frac{1}{2} I_{cm} \omega^2$$

Here, we emphasized that it is a rigid body that moves along. However, we cannot apply this to a non-rigid body in such a way that, for example, rotating a bucket of water will cause the water inside to rotate at one speed while the water attached to the bucket rotates at a different speed, indicating that the bucket is not a rigid body.

Rolling without slipping:

Take a rigid body with a center of the mass speed (v_{cm}) moving toward the right side and this is just translational motion. And on the other hand, it is just rotational motion that is rotating along the fix center and its speed is Rω on their edges as shown in figure 12.15.

Translational motion **Rotational motion**

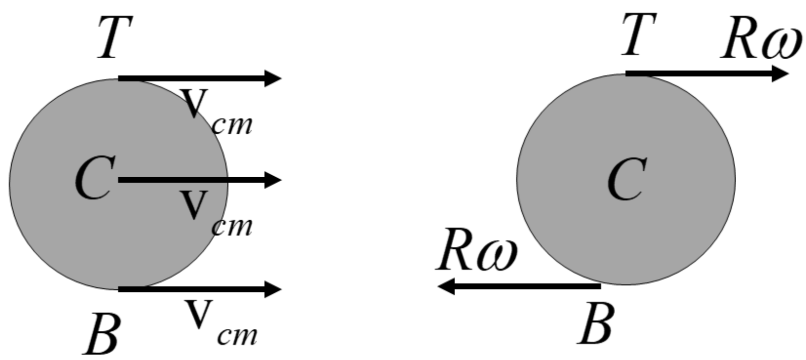


Figure 12.15. The body exhibits translational motion with center of mass speed (v_{cm}) to the right and rotational motion about a fixed center with speed Rω.

It is possible that it is rolling and rotating together, so it is a combination of translational plus rotational motion. Here point B is attached to the ground so it is at rest. It is moving at the speed of V_{cm}, and the top point is moving at two times the velocity of center of mass as shown in figure 12.16.

Translational + Rotational motion

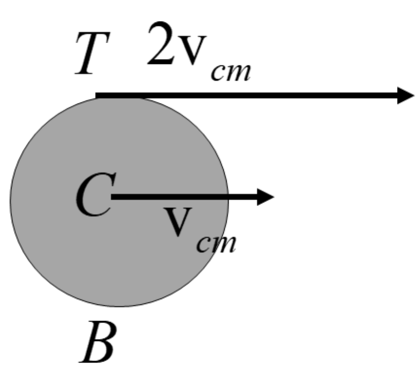


Figure 12.16. Rigid body with combined translational (vcm) and rotational (Rω) motion about a fixed axis.

If we find the total energy then we see the result of rolling without slipping is,

$$v_{cm} = R\omega$$
$$K = \frac{1}{2}Mv^2_{cm} + \frac{1}{2}I_{cm}\left(\frac{v^2_{cm}}{R^2}\right)$$
$$K = \frac{1}{2}MR^2\omega^2 + \frac{1}{2}I_{cm}\omega^2$$

We didn't say that this is a sphere or a hoop or the other shape we only discuss it as a rigid body.

Example:

In the absence of friction, energy is conserved, meaning that the same amount of energy at the beginning remains at the end and since we know that energy is divided into two parts; kinetics energy of center of mass plus rotational motion so from this we can solve many types of questions.

Now we take the example of a body rolling down a slope as shown in figure 12.17. For example a hoop which has the moment of inertia ½ mR^2. Now we want to know how fast it will be when it leaves the slope.

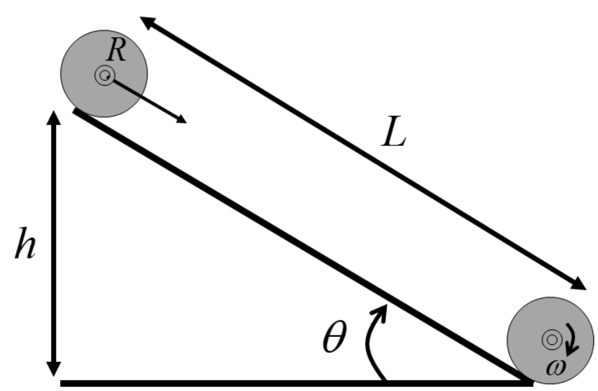


Figure 12.17. Rolling Motion of a Hoop on a Slope.

Solution:

Here we use the conversation of energy, in the start it has potential energy but later its loss the potential energy and only kinetic energy remains. When body leaves the slop its K.E is, At starting point, object possess gravitational P.E = Mgh

As it moves down, it possess rotational and translational K.E, which is:

$$K=\frac{1}{2}Mv^2{}_{cm}+\frac{1}{2}I_{cm}\omega^2$$

$$Mgh=\frac{1}{2}Mv_{cm}^2+\frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v_{cm}}{R}\right)^2 \qquad \therefore \mathrm{v}_{cm}=\mathrm{r}\omega$$

$$gh=\frac{v_{cm}^2}{2}+\frac{v_{cm}^2}{4}=\frac{3v_{cm}^2}{4}$$

$$v_{cm}=\sqrt{\frac{4}{3}gh}$$

Summary: Kinetic energy consists of two parts, one related to the center of mass and one related to its rotation. Center of mass and moment of inertia are two properties associated with a body and which define the body. What we've discussed is very important for mechanical engineers, people who design machines that have rotating objects inside, whether they're aircraft engineers or automobile engineers or others. We will go far with these concepts, which are essential in every other subject that we have learnt.

<u>Lecture_13</u>

Angular momentum

In physics, momentum is a fundamental concept. Momentum is determined by multiplying the velocity of a particle by its mass. Its significance lies in that when a force acts on a particle, its momentum changes. Conversely, in the absence of force, the momentum remains constant. When dealing with multiple particles, the total momentum equals the sum of the momenta of all individual particles. It is crucial to note that momentum is a vector quantity.

Today's lecture focuses on angular momentum, which is equally fundamental. This lecture will explore how angular momentum shares properties with regular momentum and possesses additional characteristics. This distinguishes it from a general definition of angular momentum.

Angular momentum of a single particle:

Angular momentum describes the rotational motion of an object. It is a vector quantity defined as the cross product of the object's position vector and its linear momentum vector, for a chosen origin, as shown in figure 13.1.

Mathematically,

$$\vec{L} = \vec{r} \times \vec{p}$$

There are different ways to write angular momentum.

$$L = r p \sin\theta$$
$$L = (r \sin\theta)\, p = r_\perp p$$
$$L = r(p \sin\theta) = r\, p_\perp$$

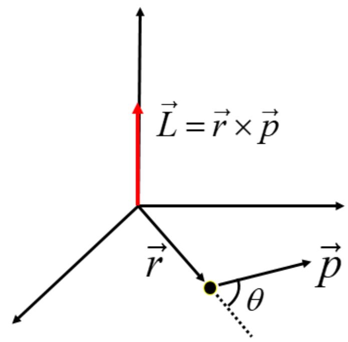


Figure13.1: Relation between angular and linear momentum.

Angular momentum of projectile:

A projectile is launched, following the trajectory of a parabola, originating from a specified point. Our objective is to determine its angular momentum for point O, as shown in figure 13.2.

It is crucial to clearly define point O, as the calculation of angular momentum depends on the reference point from which the projectile is thrown.

Let's find angular momentum ***L*** about origin O after some time t:

$$x = (v_0 \cos\theta)t \qquad \text{eq(1)}$$
$$y = (v_0 \sin\theta)t - \frac{1}{2}gt^2 \qquad \text{eq(2)}$$
$$v_x = v_0 \cos\theta \qquad \text{eq(3)}$$
$$v_y = v_0 \sin\theta - gt \qquad \text{eq(4)}$$

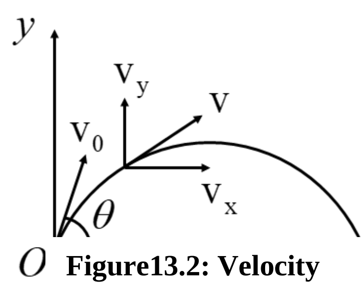


Figure13.2: Velocity components of a projectile.

$$\vec{L} = \vec{r} \times \vec{p} = \left(x\hat{i} + y\,\hat{j}\right) \times \left(\mathrm{v}_x\hat{i} + \mathrm{v}_y\,\hat{j}\right)m$$
$$\vec{L} = m\left(x\mathrm{v}_x\,\hat{i} \times \hat{i} + x\mathrm{v}_y\,\hat{i} \times \hat{j} + y\mathrm{v}_x\,\hat{j} \times \hat{i} + y\mathrm{v}_y\,\hat{j} \times \hat{j}\right)$$
$$\vec{L} = m\left(x\mathrm{v}_x\,(0) + x\mathrm{v}_y\,(\hat{k}) + y\mathrm{v}_x\,(-\hat{k}) + y\mathrm{v}_y\,(0)\right) \qquad \because \hat{i} \times \hat{i} = \hat{k} \times \hat{k} = \hat{j} \times \hat{j} = 0$$
$$\vec{L} = m\left(x\,\mathrm{v}_y - y\,\mathrm{v}_x\right)\hat{k} \qquad \because\ \hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{i} = -\hat{k}$$

putting values from eq(1-4), we get

$$\vec{L} = m\left(\left(t\mathrm{v}_0 \cos\theta\right)(\mathrm{v}_0 \sin\theta - gt) - \{(t\mathrm{v}_0 \sin\theta - \frac{1}{2}gt^2)\mathrm{v}_0 \cos\theta\}\right)\hat{k}$$
$$\vec{L} = m\left((t\mathrm{v}_0 \cos\theta \mathrm{v}_0 \sin\theta - gt^2\mathrm{v}_0 \cos\theta) - \{t\mathrm{v}_0 \sin\theta \mathrm{v}_0 \cos\theta - \frac{1}{2}gt^2\,\mathrm{v}_0 \cos\theta\}\right)\hat{k}$$
$$\vec{L} = m\left(t\mathrm{v}_0 \cos\theta \mathrm{v}_0 \sin\theta - gt^2\mathrm{v}_0 \cos\theta - t\mathrm{v}_0 \sin\theta \mathrm{v}_0 \cos\theta + \frac{1}{2}gt^2\,\mathrm{v}_0 \cos\theta\right)\hat{k}$$
$$\vec{L} = m\left(\frac{1}{2}gt^2\mathrm{v}_0 \cos\theta - gt^2\mathrm{v}_0 \cos\theta\right)\hat{k}$$
$$\vec{L} = mgt^2\mathrm{v}_0 \cos\theta\left(\frac{1}{2} - 1\right)\hat{k}$$
$$\vec{L} = mgt^2\mathrm{v}_0 \cos\theta\left(-\frac{1}{2}\right)\hat{k}$$
$$\vec{L} = -\frac{m}{2}gt^2\mathrm{v}_0 \cos\theta\,\hat{k}$$

Here we are seeing that the angular momentum is proportional to time. It changes with t-square and is increasing in the negative k-direction because it has a minus angular momentum.

Torque (reminder):

As we know torque is given as,

$$\vec{\tau} = \vec{r} \times \vec{F}$$
$$\tau = rF\sin\theta$$

Relation between torque and angular momentum

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{L} + \Delta\vec{L} = \left(\vec{r} + \Delta\vec{r}\right) \times \left(\vec{p} + \Delta\vec{p}\right)$$

$$\vec{L}+\Delta\vec{L}=\vec{r}\times\vec{p}+\vec{r}\times\Delta\vec{p}+\Delta\vec{r}\times\vec{p}+\Delta\vec{r}\times\Delta\vec{p}$$
$$\vec{L}+\Delta\vec{L}=\vec{L}+\vec{r}\times\Delta\vec{p}+\Delta\vec{r}\times\vec{p}$$
$$\because \Delta\vec{r}\times\Delta\vec{p}=\text{ignored, very small term}$$

$$\Rightarrow \Delta\vec{L}=\vec{r}\times\Delta\vec{p}+\Delta\vec{r}\times\vec{p}$$

$$\frac{\Delta\vec{L}}{\Delta t}=\frac{\vec{r}\times\Delta\vec{p}+\Delta\vec{r}\times\vec{p}}{\Delta t}=\frac{\vec{r}\times\Delta\vec{p}}{\Delta t}+\frac{\Delta\vec{r}\times\vec{p}}{\Delta t}$$
$$=\vec{r}\times\frac{\Delta\vec{p}}{\Delta t}+\frac{\Delta\vec{r}}{\Delta t}\times\vec{p}$$

Take limit as $\Delta t\to 0$:

$$\because \quad \lim_{\Delta t\to 0}\frac{\Delta\vec{L}}{\Delta t}=\frac{d\vec{L}}{dt}$$

$$\Rightarrow \quad \frac{d\vec{L}}{dt}=\vec{r}\times\frac{d\vec{p}}{dt}+\frac{d\vec{r}}{dt}\times\vec{p}$$

But $\frac{d\vec{r}}{dt}$ is $\vec{\mathrm{v}}$ and $\vec{p}=m\vec{\mathrm{v}}$!

$$\because \quad \frac{d\vec{r}}{dt}\times\vec{p}=\vec{\mathrm{v}}\times m\vec{\mathrm{v}}=m\left(\vec{\mathrm{v}}\times\vec{\mathrm{v}}\right)=0$$

And we are left with only,

$$\frac{d\vec{L}}{dt}=\vec{r}\times\frac{d\vec{p}}{dt}$$

Now use Newton's second law:

$$\vec{F}=\frac{d\vec{p}}{dt}$$
$$\Rightarrow\frac{d\vec{L}}{dt}=\vec{r}\times\vec{F}$$

$$\therefore \quad \frac{d\vec{L}}{dt}=\vec{\tau}$$

$$\boxed{\frac{d\vec{L}}{dt} = \sum \vec{\tau}}$$

The net torque acting on a particle is equal to the time rate of change of its angular momentum.

The Spinning Top:

As we observed from experiment, the torque changes the direction but not the magnitude of the angular momentum and the motion is called precession, as shown in figure 13.3.

Let's do it mathematically, starting from torque.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

where $\vec{F} = m\vec{g}$, so

$$\tau = Mgr\sin\theta \quad \text{eq(1)}$$

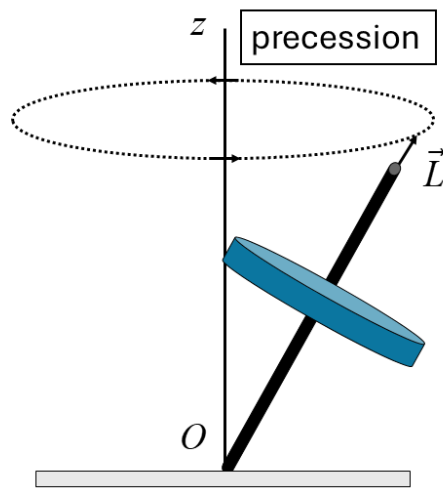


Figure 13.3: The precession about the z-axis.

$\vec{\tau}$ is perpendicular to $\vec{L}$ as shown in figure 13.4

$\therefore$ it cannot change the magnitude of $\vec{L}$!!

we know form figure 13.5

$$\Delta\vec{L} = \vec{\tau}\ \Delta t$$

using $\theta = \dfrac{s}{r}$, we get

$$\Delta\phi = \frac{\Delta L}{L\sin\theta}$$

or,

$$\Delta\phi = \frac{\tau\ \Delta t}{L\sin\theta} \quad \text{eq(2)}$$

Precession speed ω_P is:

$$\omega_P = \frac{\Delta\phi}{\Delta t}$$

using $\Delta\phi$ from eq(2)

$$= \frac{\tau\ \Delta t}{L\sin\theta}\frac{1}{\Delta t}$$

$$= \frac{\tau}{L\sin\theta}$$

using τ from eq(1)

$$= \frac{Mgr\sin\theta}{L\sin\theta}$$

$$\omega_P = \frac{Mgr}{L}$$

Here, precession is proportional to 1/L. So, as angular momentum decreses with time the precession increase.

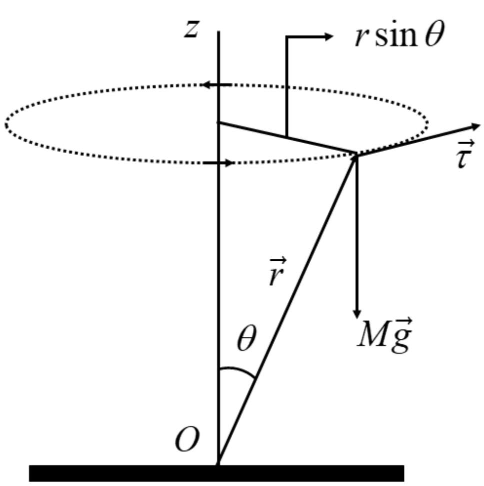


Figure 13.4: Torque, weight, and component of radius is being drawn.

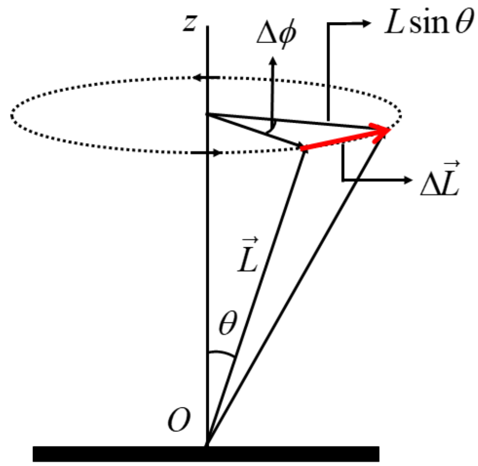


Figure 13.5: The incremental change in angular momentum and angular displacement is presented.

Example:

A mass m is released from a distance b along x-axis. A Position vector is r from origin O, as shown in the figure 13.6. Calculate torque and angular momentum.

Solution:

Let's calculate torque,

$$\vec{\tau} = \vec{r} \times \vec{F}$$
$$\tau = rF\sin\theta$$

putting $r\sin\theta = b$ and $F = mg$ from figure 13.6 we get

$$\tau = mgb$$

Right hand rule shows that $\vec{\tau}$ is directed inwards, as shown in figure 13.6.

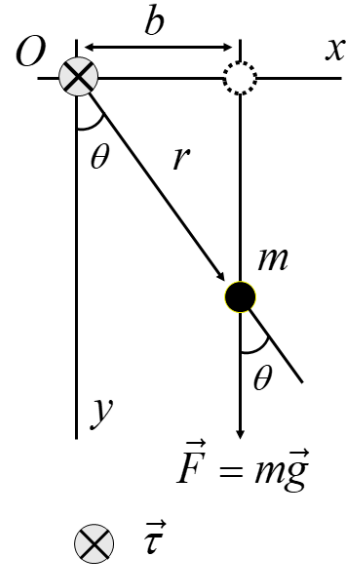


Figure 13.6: Direction of torque is into the paper.

Let's calculate angular momentum,

$$\vec{L} = \vec{r} \times \vec{p}$$
$$L = rp\sin\theta$$

putting $r\sin\theta = b$ and $p = m\mathrm{v} = mgt$ form figure 13.7, we get

$$L = mgbt$$

$\vec{L}$ is directed inwards, as shown in figure 13.7.

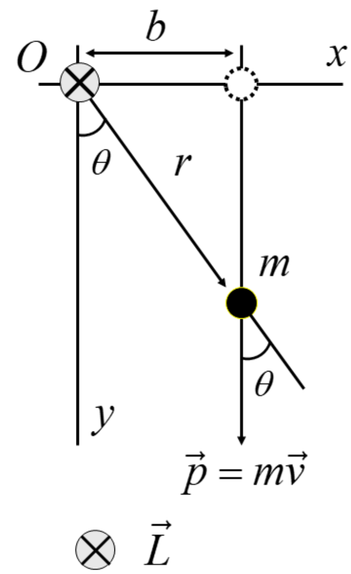


Figure 13.7: The direction of angular momentum is into the paper.

We know angular momentum and torque are connected, so let's calculate torque from angular momentum and check our expression.

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(mgtb)\hat{k}$$
$$= mgb\,\hat{k}$$
$$\tau = mgb\,\hat{k}$$

As we expected, it is the same.

Angular momentum for a system of particles:

Suppose we have large number of particles in a system, as shown in figure 13.8, to calculate its total angular momentum we do vector addition of angular momentum for individual particles.

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \cdots + \vec{L}_N = \sum_{n=1}^{N} \vec{L}_n$$

$$\frac{d\vec{L}}{dt} = \frac{d\vec{L}_1}{dt} + \frac{d\vec{L}_2}{dt} + \cdots + \frac{d\vec{L}_N}{dt} = \sum_{n=1}^{N} \frac{d\vec{L}_n}{dt}$$

$$\text{Since } \frac{d\vec{L}_n}{dt} = \vec{\tau}_n$$

$$\frac{d\vec{L}}{dt} = \sum_{n=1}^{N} \vec{\tau}_n$$

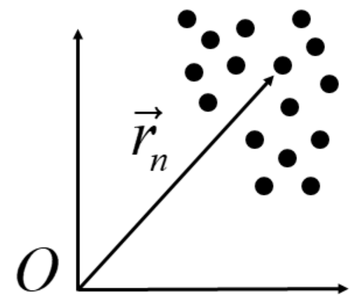


Figure 13.8: A general position vector for a system of particle.

Thus, the time rate of change of the total angular momentum of a system of particles equals the net torque acting on the system.

There are two sources of the torque acting on the system:

1) The torque exerted on the particles of the system by internal forces between the particles.

2) The torque exerted on the particles of the system by external forces.

$$\sum \vec{\tau} = \sum \vec{\tau}_{\text{int}} + \sum \vec{\tau}_{ext}$$

If the forces between two particles, as shown in figure 13.8 not only are equal and opposite but are also directed along the line joining the two particles, then the total internal torque is zero.

$$\sum \vec{\tau}_{\text{int}} = 0$$

$$\sum \vec{\tau}_{\text{int}} = \vec{\tau}_1 + \vec{\tau}_2$$

$$= \vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{21}$$

but

$$\vec{F}_{12} = -\vec{F}_{21} = F\hat{r}_{12}$$

$$\therefore \sum \vec{\tau}_{\text{int}} = \left(\vec{r}_1 - \vec{r}_2\right) \times \vec{F}_{12} = \vec{r}_{12} \times \left(F\hat{r}_{12}\right)$$
$$= F\left(\vec{r}_{12} \times \hat{r}_{12}\right) = 0$$

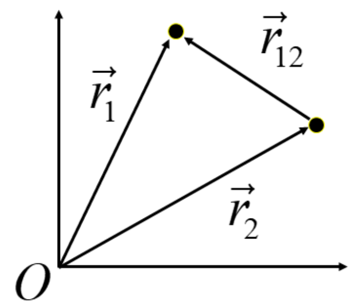


Figure 13.9: Two particles at position vector r_1 and r_2 respectively.

Hence

$$\sum \vec{\tau} = \sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$$

The net external torque acting on a system of particles is equal to the time rate of change of the total angular momentum of the system.

$$\sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$$

$$\sum \vec{F}_{ext} = \frac{d\vec{L}}{dt}$$

Conservation of Angular Momentum:

If no net external torque acts on the system, then the angular momentum of the system does not change with time.

$$\frac{d\vec{L}}{dt} = 0 \quad \Rightarrow \quad \vec{L} = \text{a constant}$$

Mathematical formulism of linear motion and angular motion shows some resemblance. As clear from eq(1) now go for angular momentum L since it is origin dependent while linear momentum p is not.

$$\vec{F} = \frac{d\vec{p}}{dt} \Leftrightarrow \vec{\tau} = \frac{d\vec{L}}{dt} \quad \text{eq(1)}$$

$$\vec{p} = m\vec{\mathrm{v}} \Leftrightarrow \vec{\mathrm{L}} = \; ??$$

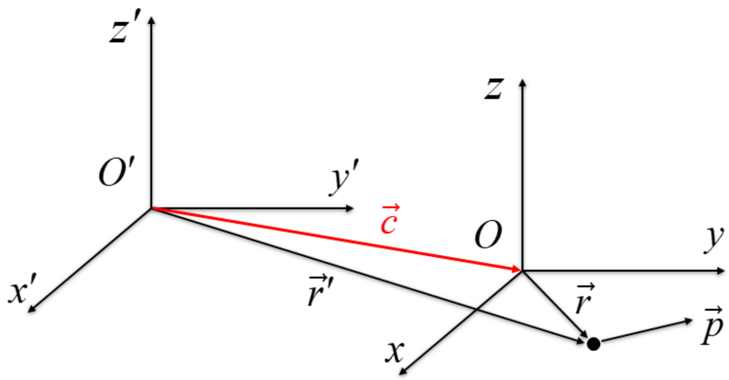


Figure 13.10: Angular momentum for two different origins is different.

$\vec{L}$ depends on the choice of the origin, from figure 13.10 we get

$\vec{L}' = \vec{r}' \times \vec{p} = (\vec{c} + \vec{r}) \times \vec{p} = \vec{c} \times \vec{p} + \vec{L}$

Rotation of Rigid Bodies:

A rigid body is an object whose shape doesn't deform under the influence of external forces. In other words, the distances between points on the object remain constant. Consider a rigid body that rotates about z-axis, as shown in figure 13.11. As reference line AP rotates through an angle all the points on it move with the same angular speed.

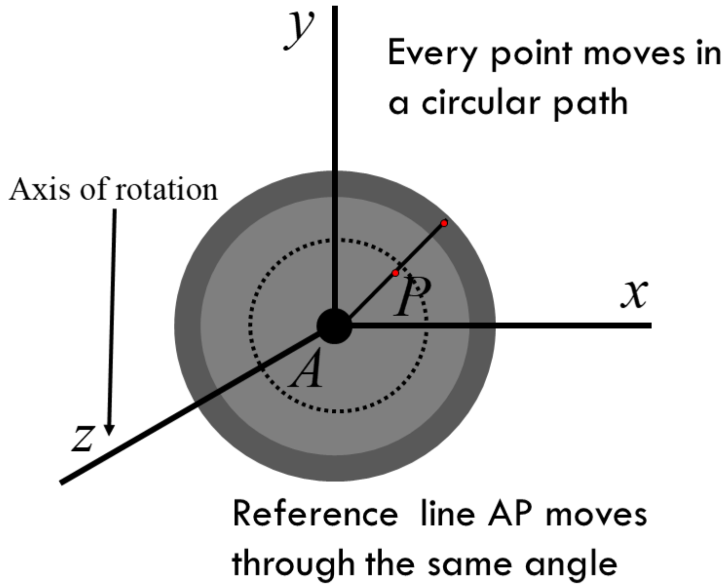


Figure 13.11: Rigid body rotates about z-axis.

Kinematics of a rigid body can be described by the motion of point P. There are two ways to understand one is by observing the motion of the point P as shown in figure 13.12 and the other is by observing the motion of cross-sectional slice of the rigid body, as shown in figure 13.13.

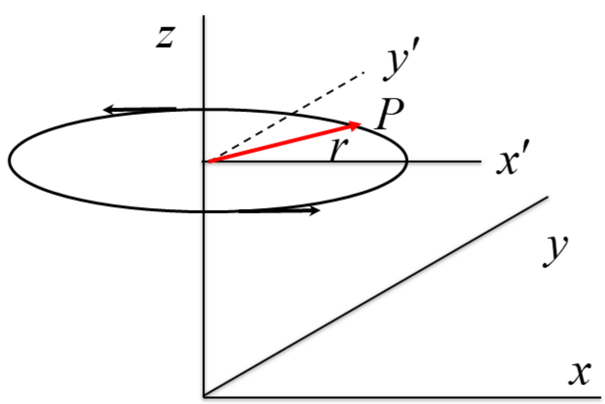


Figure 13.12: Motion of point P with translated coordinates.

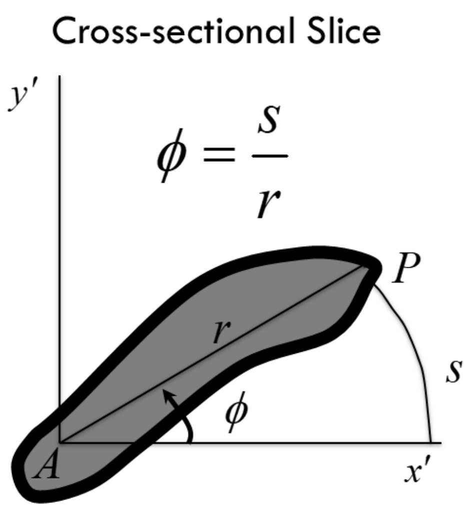
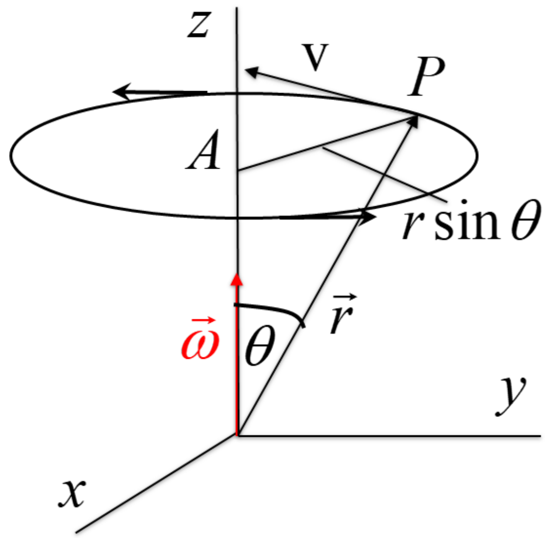


Figure 13.13: Motion of a cross-section of rigid body.

Linear and angular velocity:

Let's look at linear and angular velocity of point P. Their vector representation along with components of position vector **r** is shown in figure 13.14.

$$\vec{\mathrm{v}} = \vec{\omega} \times \vec{r} \quad \text{eq(1)}$$
$$\mathrm{v} = \omega r \sin\theta$$



angular velocities of point P.

Linear and angular acceleration:

As we know acceleration is rate change of velocity so using eq(1) we can get,

$$\vec{\mathrm{a}} = \frac{d\vec{\mathrm{v}}}{dt} = \frac{d}{dt}\left(\vec{\omega} \times \vec{r}\right)$$
$$= \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$
$$= \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{\mathrm{v}} \quad \because \vec{\alpha} = \frac{d\vec{\omega}}{dt} \text{ and } \vec{\mathrm{v}} = \frac{d\vec{r}}{dt}$$
$$\vec{\mathrm{a}} = \vec{\mathrm{a}}_{\mathrm{T}} + \vec{\mathrm{a}}_{\mathrm{R}}$$

Where $\vec{\mathrm{a}}_{\mathrm{T}}$ and $\vec{\mathrm{a}}_{\mathrm{R}}$ are tangential and radial (towards the point A) components of acceleration, respectively, as shown in the figure 13.15.

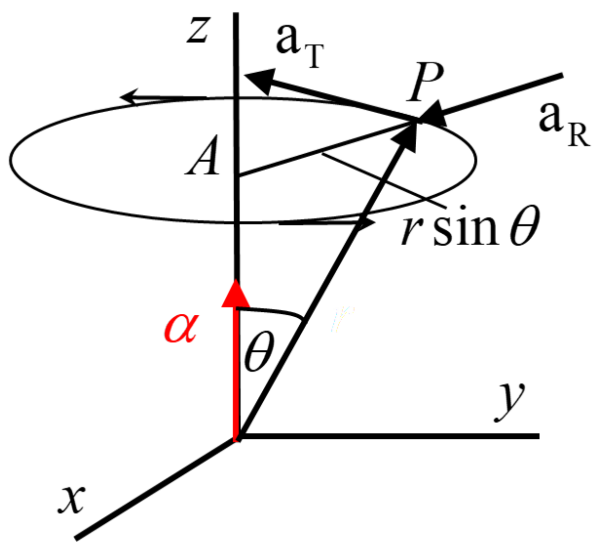


Figure 13.15: linear and angular acceleration of point P.

Now let's discuss a special case in which there are two equal masses. In this way, the center of mass will be in between the two. Axis of rotation passes through the middle, as shown in figure 13.16. Here, the direction of angular momentum is along the z-axis same as of angular momentum.

$$\vec{L} = \left(2mr^2\right)\vec{\omega}$$
$$\vec{L} = I\vec{\omega}$$

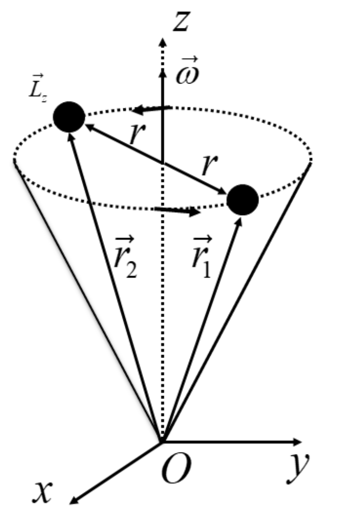


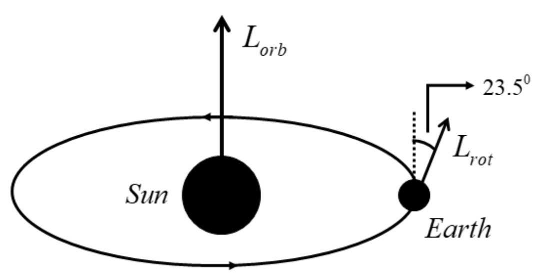
Figure 13.16: Two equal masses rotating about the z-axis.

<u>Problem # 1:</u>

Which is greater?
(a) The angular momentum of the Earth due to rotation on its axis.
(b) The angular momentum of the Earth due to its orbital motion around the sun.

<u>Solution:</u>

L_{rot} is the angular momentum of earth due to rotational motion about its own axis and L_{orb} is the angular momentum of earth due to the orbital motion about the sun. R_E is the radius of the earth and R_{orb} is the distance between the sun and the earth (clearly $R_{orb} \gg R_E$).



$$L_{rot} = I\omega = \left(\frac{2}{5}MR_E^2\right)\omega \;\; \text{eq(1)} \quad \because I = \frac{2}{5}MR_E^2 \text{ for solid sphere}$$

$$L_{orb} = R_{orb}p = R_{orb}M\text{v} = R_{orb}M\left(R_{orb}\omega\right) = MR_{orb}^2\omega \;\; \text{eq(2)}$$

By dividing eq(2) by eq(1), we get

$$\frac{L_{orb}}{L_{rot}} = \frac{MR_{orb}^2\omega}{\left(\frac{2}{5}MR_E^2\right)\omega}$$

$$L_{orb}/L_{rot} = \frac{5}{2}\left(\frac{R_{orb}}{R_E}\right)^2$$

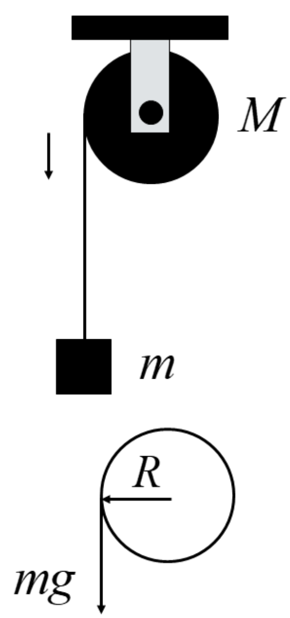
we know, $R_{orb} \gg R_E$ or $\frac{R_{orb}}{R_E} \gg 1$ so,

$$L_{orb}/L_{rot} = \frac{5}{2}\left(\frac{R_{orb}}{R_E}\right)^2 \gg 1 \Rightarrow L_{orb} \gg L_{rot}$$

The angular momentum of the earth due to its orbital motion aroud the sun is much greater than the angular momentum of the earth due to rotation about its axis because of the fact that the distance between the sun and the earth is much larger than the radius of the earth.

Problem # 2:

A mass m is tied to a pulley of mass M with a massless rope, as shown. As mass m moves down, the pully starts rotating. What is angular acceleration?



Solution:

Total angular momentum of system consists of two sections, one is due to pully and other is due to mass m. Origin is at center of the pully so,

$$L = I\omega + m\text{v}R$$

We also know,

$$\tau = \frac{dL}{dt} \quad \because \tau = (mg)R$$

We get,

$$(mg)R = \frac{d}{dt}(I\omega + m\text{v}R)$$

$$(mg)R = I\left(\frac{d\omega}{dt}\right) + mR\left(\frac{d\text{v}}{dt}\right)$$

$$(mg)R = I\alpha + mRa \quad \because \alpha = \frac{d\omega}{dt} \text{ and } a = \frac{d\text{v}}{dt}$$

$a = \alpha R$ or $\dfrac{a}{R} = \alpha$ $\because$ relation between linear and angular accleration

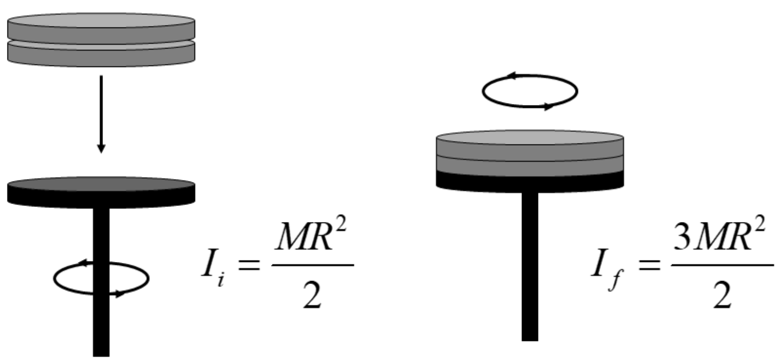
$I = \dfrac{1}{2}MR^2$ $\because$ moment of inertia of pully

$$\Rightarrow \ mgR = \left(\frac{1}{2}MR^2\right)(a/R) + mRa$$

$$a = \frac{2mg}{M + 2m}$$

Problem # 3:

A disc which comes rotating and gets attached to another disc below. This upper disc which is twice as heavy as the lower disc and twice as big, when it gets attached to the lower disc, then these three discs start rotating together. What is the angular velocity?



Solution:

Here, angular momentum is conserved because there is no external torque. So,

$$I_i\omega_i = I_f\omega_f$$
$$\Rightarrow \ \omega_f = \omega_i\left(\frac{I_i}{I_f}\right)$$

$\because$ For one disc $I_i = I = \frac{1}{2}MR^2$ and for three discs $I_f = I = \frac{3}{2}MR^2$

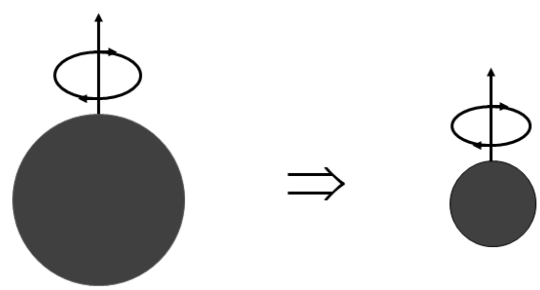
$$\omega_f = \omega_i\left(\frac{MR^2}{2}\times\frac{2}{3MR^2}\right)$$
$$\omega_f = \frac{1}{3}\omega_i$$

So, the final angular velocity is one third of the initial angular velocity.

Additional Problems:

Problem # 1:

If the radius of the earth suddenly shrinks to half its present value, while the mass of the Earth remains unchanged, what will be the duration of one day where earth assumed to be a perfect sphere?



Solution:

Applying the law of conservation of angular momentum

$$I_i\omega_i = I_f\omega_f$$

$\because$ Moment of inertia of a solid sphere $I = \frac{2}{5}MR^2$

$$\because \omega = \frac{\text{angular displacment}}{\text{time}} = \frac{2\pi}{T}$$

$$\left(\frac{2}{5}MR_i^2\right)\left(\frac{2\pi}{T_i}\right) = \left(\frac{2}{5}MR_f^2\right)\left(\frac{2\pi}{T_f}\right)$$

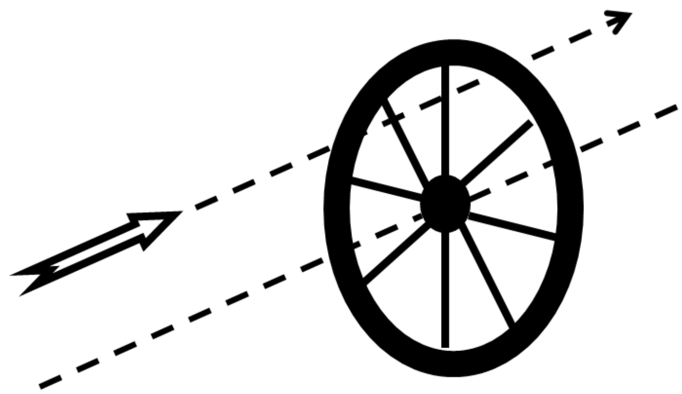
$$\frac{R_i^2}{T_i} = \frac{R_f^2}{T_f} \quad or \quad T_f = \left(\frac{R_f}{R_i}\right)^2 T_i = \left(\frac{1}{2}\right)^2 (24\text{hours})$$

$$T_f = \left(\frac{1}{4}\right)(24\text{hours}) = 6 \text{ hours}$$

So, duration of one day will be 6 hours.

Problem # 2:

A wheel has eight spokes and a radius of 30 cm. It is mounted on a fixed axle and is spinning at 2.5 rev/s. You want to shoot a 24 cm arrow parallel to this axle and through the wheel without hitting any of the spokes. Assume that the arrow and the spokes are very thin. (a) What minimum speed must the arrow have? (b) Does it matter where between the axle and the rim of the wheel you aim? If so, where is the best location?



Solution:

$$\text{minimum speed} = \frac{\text{length of the arrow}}{\text{time to pass one spoke}}$$

$$\text{s = distance traveled by one spoke} = \frac{2\pi \text{r}}{8}$$

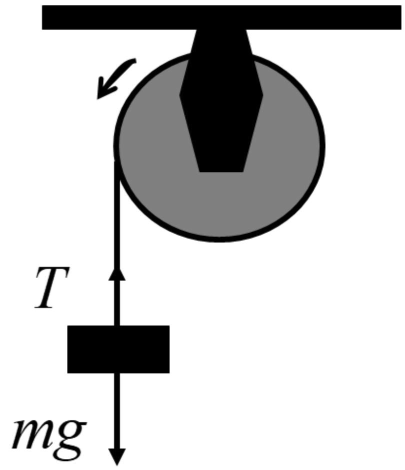
$$\text{time to pass one spoke} = \frac{\text{distance traveled by one spoke}}{\text{speed of spoke}}$$

$$\text{time to pass one spoke} = \frac{\text{s}}{\text{v}} = \frac{2\pi \text{r}}{8\text{r}\omega}$$

$$\text{So minimum speed} = \frac{\ell \times 8\text{r}\omega}{2\pi \text{r}} = 4.8\,\text{m/s}$$

Does not matter where we aim!!

Problem # 3:



A disk of mass M = 2.5 kg and radius R = 20 cm is mounted on a fixed horizontal axle. A block of mass m = 1.2 kg hangs from a light cord that is wrapped around the rim of the disk. Find the acceleration of the falling block, the tension in the cord, and the angular acceleration of the disk.

Solution:

$$\sum F = mg - T = m\text{a}$$

$$m\text{a} = mg - T \quad \text{eq(1)}$$

$$\because \tau = \text{r F also } \tau = \text{I}\alpha$$

$$\sum \tau = TR \quad \text{eq(2)}$$

$$\sum \tau = \frac{1}{2}MR^2\left(\frac{\text{a}}{R}\right) \quad \text{eq(3)} \quad \because \text{a} = R\alpha$$

from eq(2) and eq(3) we get

$$TR = \frac{1}{2}MR^2\left(\frac{\text{a}}{R}\right)$$

$$T = \frac{1}{2}M\text{a} \quad \text{eq(4)}$$

substituting eq(4) into eq(1), we get

$$m\text{a} = mg - \frac{1}{2}M\text{a}$$

$$m\text{a} + \frac{1}{2}M\text{a} = mg$$

$$\text{a}\,\frac{M+2m}{2} = mg$$

$$\text{a} = g\,\frac{2m}{M+2m} \quad \text{eq(5)}$$

$$\text{a} = 4.8\,\text{m/s}^2$$

putting eq(5) into eq(4), we get

$$T = mg\,\frac{M}{M+2m}$$

$$T = 6.0N$$

$$\alpha = \frac{\text{a}}{R} = 3.8\,\text{rev/s}^2$$

Physics-PHY101-Lecture

14 Equilibrium of rigid bodies:

Countless examples of stillness and motionlessness are found in our environment, for example, a ball or a chair lying on the floor or a ship sailing at a steady speed or the car is moving at a slow speed, these are some examples which are related to the ground but if you look from the sky then you will also see all the solar system is at a state of peace i.e. it is in equilibrium, so this lecture is about equilibrium of rigid/solid body.

During the last few lectures, we had talked to you in detail about momentum and angular momentum and before that we had also discussed centre of mass and moment of inertia**.** A body which is composed of many smaller objects, no matter how complex the interactions between them may be, but there is a point which is called the centre of mass and as this centre of mass moves, it carries momentum, depends on how much force acts on it.

"The rate of change of this momentum is equal to the net external force acting on the body".

Therefore, if no external force acts on this body, then this is the state of equilibrium. This is a necessary condition for equilibrium, as if the total external force has to vanish. It certainly does not mean that all the forces vanish. If forces are in different directions and they cancel each other out, only then you will say that the net external force is zero.

For equilibrium, it is not enough that we say that the net external force should be zero.

Conditions for Equilibrium:

A rigid body is in mechanical equilibrium if both the linear momentum $\vec{P}$ and angular momentum $\vec{L}$ have a constant value.

$$\text{i.e.,} \quad \frac{d\vec{P}}{dt} = 0 \quad \text{and} \quad \frac{d\vec{L}}{dt} = 0$$

$$\vec{P} = 0 \quad \text{and} \quad \vec{L} = 0 \quad \Rightarrow \text{ static equilibrium}$$

Examples of Equilibrium:

Here are some practical examples of equilibrium.

1) *How does a lever work?*

Solution:

Torques balance about an axis through the fulcrum, as shown in the figure 14.1:

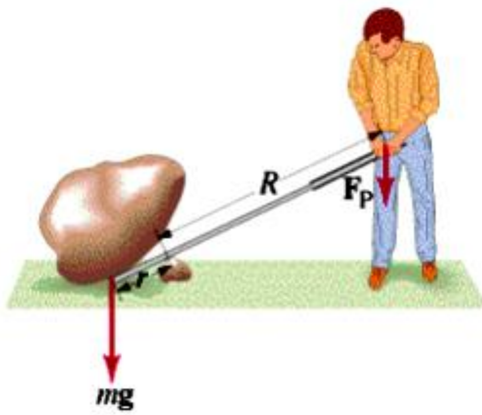


Figure 14.1: Torques by applied force and by force of gravity balance about an axis through the fulcrum.

Tourqe produced by person = Tourqe produced by stone

$$\overrightarrow{R} \times \overrightarrow{F_P} = \overrightarrow{\mathrm{r}} \times \overrightarrow{F_g} \qquad \because F_g = mg$$

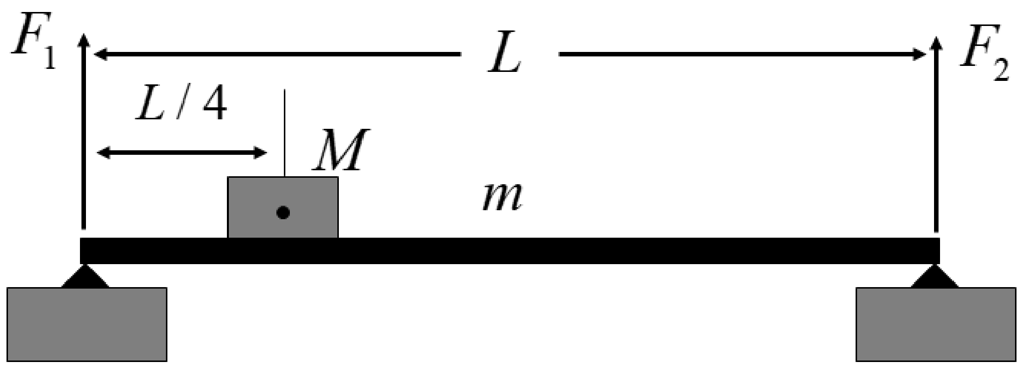
$$F_P \cdot R = mg \cdot r \qquad \because \theta = 90 \; so \; \sin\theta = 1$$

Solve for the applied force (F_P):

$$F_P = mg\frac{r}{R}$$

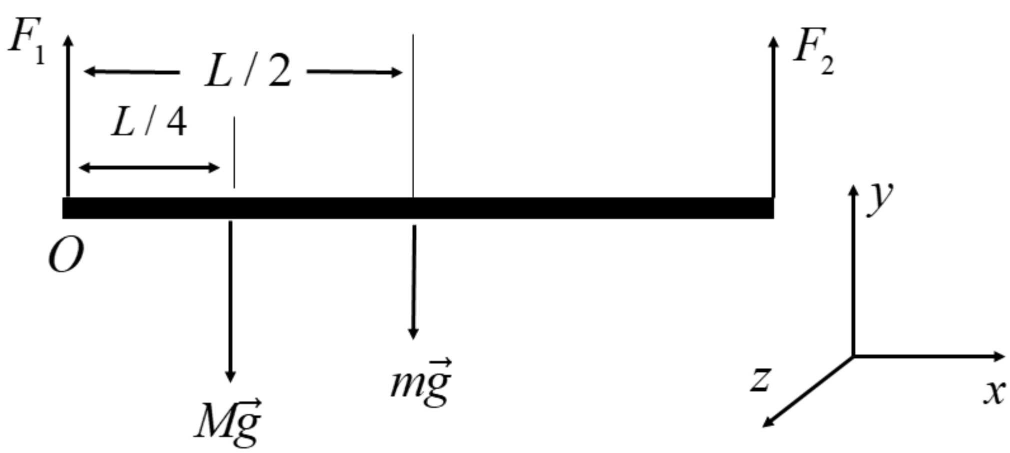
F_p (force applied by person) can be rather small depending on r and R. In the given figure R is bigger than r so a small F_p is required to lift the stone.

2) Consider a uniform rod of mass m (lies at center of mass) and length L. A mass M is placed on it as shown in figure. Find forces F_1 and F_2?



Solution:

For equilibrium, the sum of all forces and sum of all torque must be equal to zero.



Sum of all forces must be equal to zero

$$\sum F_y = F_1 + F_2 - Mg - mg = 0$$

$$\Rightarrow F_1 + F_2 - Mg - mg = 0$$

$$\Rightarrow F_1 + F_2 = Mg + mg \;\; or \;\; F_1 = \; Mg + mg - F_2 \; ....(eq\;\; 14.1)$$

Sum of all torques must be equal to zero

$$\sum \tau_y = (F_1)(0) + (F_2)(L) - (Mg)(L/4) - (mg)(L/2) = 0$$

$$\Rightarrow F_2 L - Mg\frac{L}{4} \; - mg\frac{L}{2} = 0$$

$$or \; F_2 L = Mg\frac{L}{4} \; - mg\frac{L}{2} \; or \; L(Mg\frac{1}{4} \; + mg\frac{1}{2})$$

$$F_2 L = L(Mg\frac{1}{4} \; + mg\frac{1}{2})$$

$$Cancelling \; L \; on \; both \; sides$$

$$F_2 = (Mg\frac{1}{4} \; + mg\frac{1}{2})$$

$$F_2 = \frac{Mg + 2mg}{4} = \frac{(M + 2m)g}{4}$$

$$F_2 = \frac{(M + 2m)g}{4}$$

$$Now \; put \; this \; valuse \; of \; F_2 \; in \; above \; eq 14.1,$$

$$we \; get \;\; F_1 = \frac{(3M + 2m)g}{4}$$

<u>For a body in equilibrium, the choice of origin for calculating torques is unimportant. Torque must vanish everywhere irrespective of position of O.</u>

Proof:

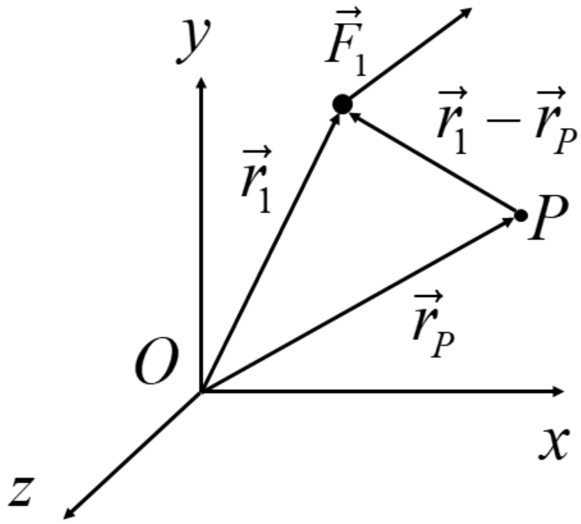


Figure 14.2: Torque about a general point P is being described.

Total torque is the sum of torques

$$\vec{\tau}_O = \vec{\tau}_1 + \vec{\tau}_2 + \cdots + \vec{\tau}_N$$
$$= \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \cdots\cdots + \vec{r}_N \times \vec{F}_N$$

Torque about point P as shown in figure 14.2, written as

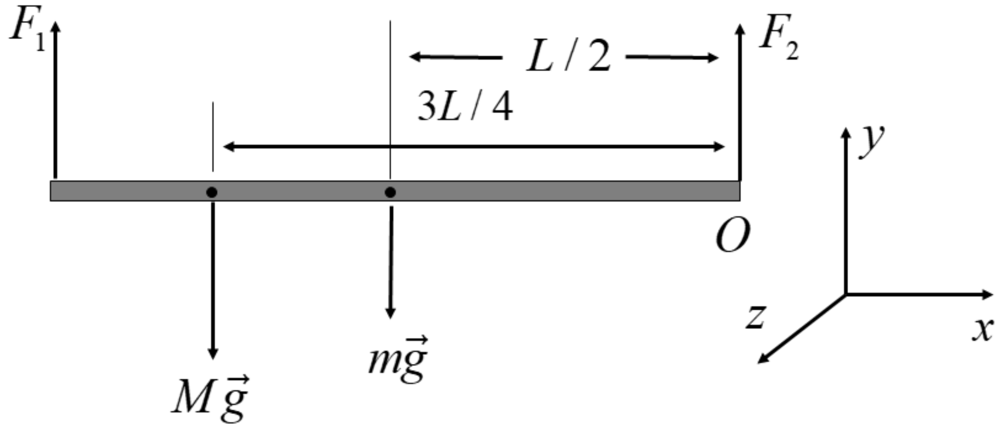
$$\vec{\tau}_P = \left(\vec{r}_1 - \vec{r}_P\right) \times \vec{F}_1 + \left(\vec{r}_2 - \vec{r}_P\right) \times \vec{F}_1 + \cdots\cdots + \left(\vec{r}_N - \vec{r}_P\right) \times \vec{F}_N$$
$$= \left[\vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \cdots + \vec{r}_N \times \vec{F}_N\right] - \left[\vec{r}_P \times \vec{F}_1 + \vec{r}_P \times \vec{F}_2 + \cdots + \vec{r}_P \times \vec{F}_N\right]$$
$$= \vec{\tau}_O - \left[\vec{r}_P \times \left(\vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_N\right)\right]$$
$$= \vec{\tau}_O - \left[\vec{r}_P \times \left(\sum \vec{F}_{ext}\right)\right]$$

but $\sum \vec{F}_{ext} = 0,$ for a body in translational equilibrium

$$\therefore \quad \vec{\tau}_P = \vec{\tau}_O$$

Hence the torque about any two points has the same value when the body is in translational equilibrium.

Now, origin O is at the right side (as shown in figure) lets recalculate forces to see the effect of different origin.



$$\sum F_y = F_1 + F_2 - Mg - mg = 0$$
$$\Rightarrow F_1 + F_2 - Mg - mg = 0$$
$$\Rightarrow F_1 + F_2 = Mg + mg \;\; or \;\; F_2 = \; Mg + mg - F_1 \;\; (eq\, ii)$$
$$\sum \tau_y = -(F_1)(L) + (F_2)(0) + (M\,g)(3L/4) + (mg)(L/2) = 0$$
$$\Rightarrow -F_1 L + Mg\frac{3L}{4} \; + mg\frac{L}{2} = 0$$
$$or \; F_1 L = Mg\frac{3L}{4} \; + mg\frac{L}{2} \; or \; L(Mg\frac{3}{4} \; + mg\frac{1}{2})$$
$$F_1 L = L(Mg\frac{3}{4} \; + mg\frac{1}{2})$$

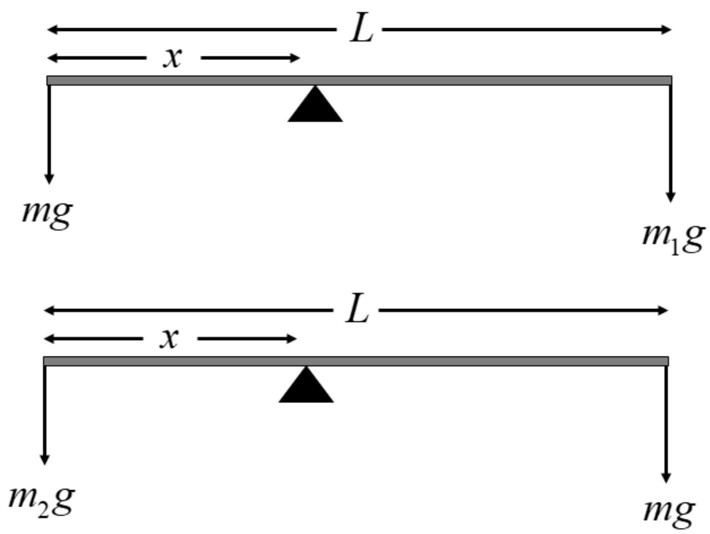
Cancelling L on both sides

$$F_1 = (Mg\frac{3}{4} \; + mg\frac{1}{2})$$
$$F_1 = \frac{3Mg + 2\,mg}{4} = \frac{(3M + 2\,m)g}{4}$$
$$F_1 = \frac{(3M + 2\,m)g}{4} \text{ Now put this valuse of } F_1 \text{ in eq ii above}$$
$$\text{we get } \; F_2 = \frac{(M + 2\,m)g}{4}$$

Conclusion: As we expected, change of origin does not affect the forces.

3) A massless rod of length L is placed on a fulcrum at a distance x from the left side. An unknown mass m is placed first on the left side and then on the right side of the rod (as shown in figure). If rod is balanced (by m_1 and m_2) in both situations, then calculate the value of m?

Solution:



$$m = ?$$

Taking the torques about the knife edge in the two cases (as shown in figure), we have,

$$mgx = m_1 g(L-x)$$
$$m_2 gx = mg(L-x)$$

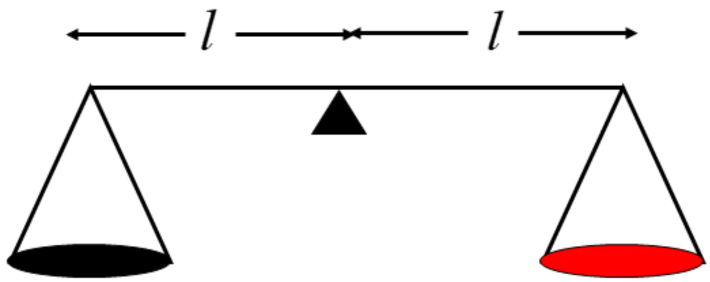
by dividing these equations we get

$$\frac{mgx}{m_2 gx} = \frac{m_1 g(L-x)}{mg(L-x)}$$

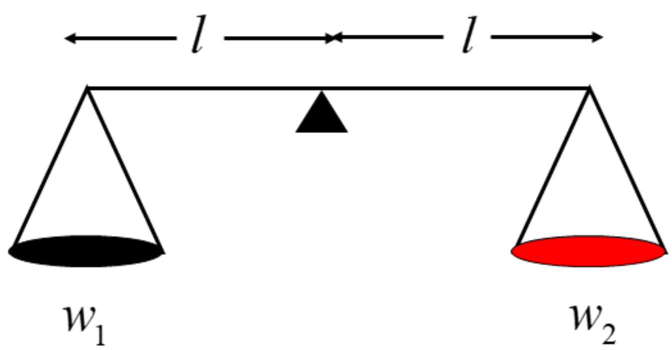
$$\Rightarrow \frac{m}{m_2} = \frac{m_1}{m} \text{ or } m = \sqrt{m_1 m_2}$$

So, we can find out the unknown mass with the help of two known masses using this methodology.

4) A false balance (weight of both pans are not equal) has equal arms. An object weights x when placed in one pan and y when placed in the other pan. What is the true weight of the object?



Solution:



Let the weights of the pans be w_1 and w_2 and the true weight be *w*.

Taking torques about the knife edge gives:

$$(w+w_1)l=(x+w_2)l$$
$$\Rightarrow (w+w_1)=(x+w_2) \quad \text{eq(1)}$$

$$\left(w+w_2\right)l=\left(y+w_1\right)l$$

$$\Rightarrow\left(w+w_2\right)=\left(y+w_1\right)\quad \text{eq(2)}$$

Adding eq(1) and eq(2)

$$\left(w+w_2\right)+\left(w+w_1\right)=\left(x+w_2\right)+\left(y+w_1\right)$$

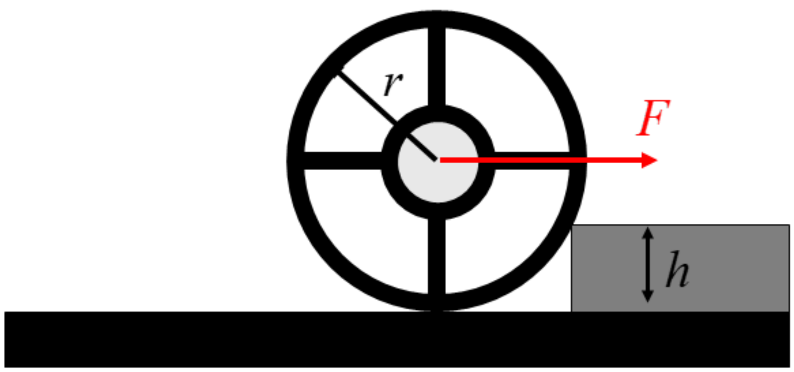
$$2w+(w_1+w_2)=x+y+(w_1+w_2)$$

$$2w=x+y$$

$$\Rightarrow w=\frac{x+y}{2}$$

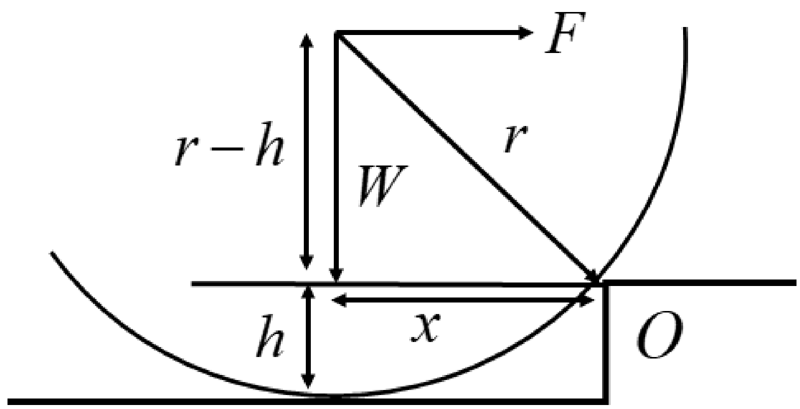
Conclusion: The actual weight is the average of both weights.

5) What minimum force F applied horizontally at the axle of the wheel (of the radius r) is necessary to raise the wheel over an obstacle of height h, as shown in the figure?



Solution:

The normal force vanishes as the wheel leaves the ground. The remaining two forces (applied F and weight w) participate in torque.



Take torques about O (the point of contact):

$$Wx = F\left(r - h\right)$$

$$x = \frac{F\left(r - h\right)}{W} \qquad \text{eq(1)}$$

Using Pythagoras Theorem on right-angle triangle:

$$x^2 + \left(r - h\right)^2 = r^2$$

$$x^2 = r^2 - \left(r - h\right)^2$$

$$x^2 = r^2 - \left(r^2 + h^2 - 2rh\right)$$

$$x^2 = r^2 - r^2 - h^2 + 2rh$$

$$x^2 = 2rh - h^2$$

using equation (1)

$$\left\{\frac{F\left(r - h\right)}{W}\right\}^2 = 2rh - h^2$$

$$\Rightarrow \frac{F\left(r - h\right)}{W} = \sqrt{2rh - h^2}$$

$$\Rightarrow \; F = W\,\frac{\sqrt{2rh - h^2}}{r - h}.$$

So, this F will raise the wheel over an obstacle of height h.

Review: Center of Mass (CM)

The center of mass (CM) is a theoretical point in a system of particles where, on average, the mass of the system is concentrated.

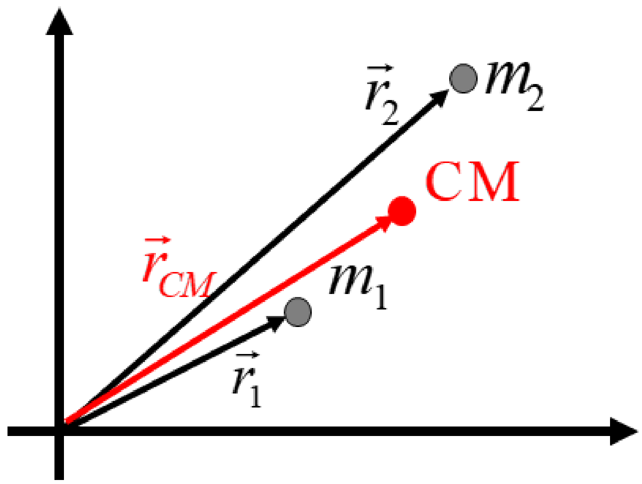


Figure 14.3: CM and r_{CM} for system of two masses.

Mathematical expression of figure 14.3 is:

$$\vec{r}_{cm} \equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

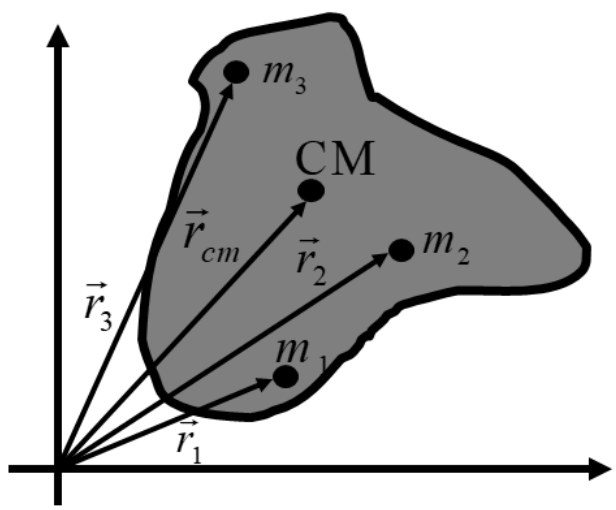


Figure 14.4: CM and r_{CM} for system of N masses is presented.

For N number of masses, as shown in figure 14.4 it can be written as

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N}{m_1 + m_2 + \cdots + m_N}$$

We also know

$$\vec{F} = M\,\vec{a}_{cm}$$

where,

$$\vec{F} = \sum \vec{F}_{ext}$$

The specialty of the center of mass is that if we throw a body of any shape, then the trajectory of CM, the curve on which that body will move will be a parabola, its shape may be complicated, but its one point, which is the center of mass, moves on a parabola. The parabolic path (dotted line) of the center of mass (red dot) on the trajectory of the object is shown in figure 14.5.

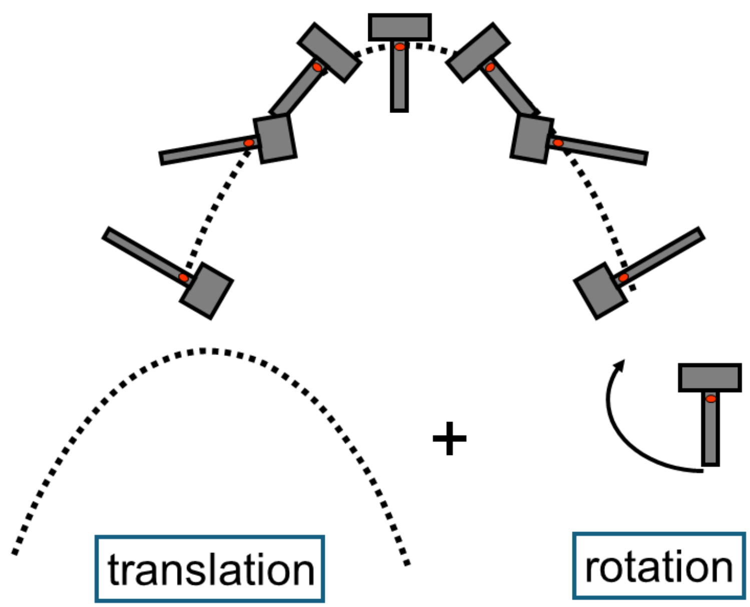


Figure 14.5: An object of arbitrary shape has translational and rotational motion while CM (red dot) has a parabolic path.

Centre of gravity:

The center of gravity is the average location of the weight of an object. As it is shown for different shapes in figure 14.6.

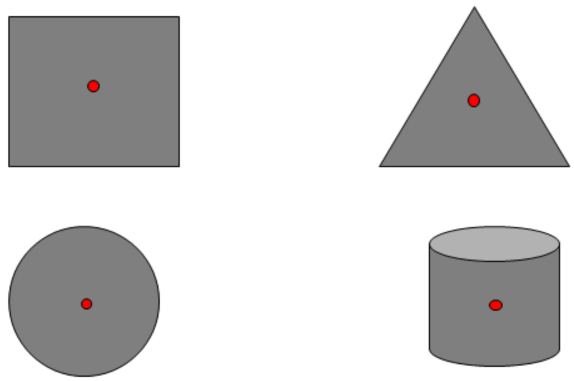


Figure 14.6: Center of gravity (as a red dot) is shown for different shapes.

Suppose the gravitational acceleration g has the same value at all points of a body. Then

- The weight is equal to mg
- And the center of gravity "coincides" with the center of mass.

Lets take look on equilibrium conditions, here
The net force on the whole = sum over all individual particles

$$\sum \vec{F} = \sum m_i \vec{g}$$

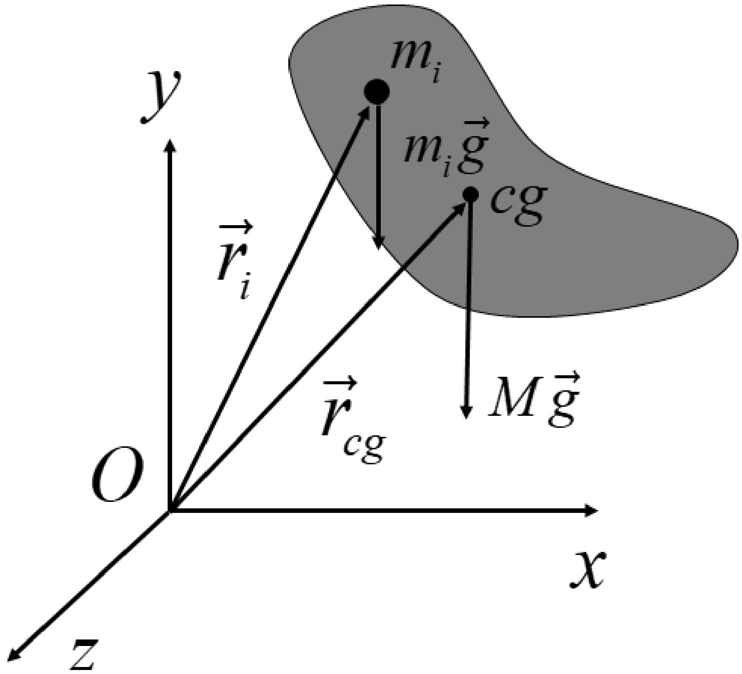


Figure 14.7: Position vector $\mathbf{r_{cg}}$ and cg is center of gravity for an arbitrary shape is depicted.

Since $\vec{g}$ has the same value for every particle of the body

$\therefore \sum \vec{F} = \vec{g} \sum m_i = M\vec{g}$

The net torque about the origin O, as shown in figure 14.7:

$$\sum \vec{\tau} = \sum \left(\vec{r}_i \times m_i \vec{g} \right)$$

$$= \sum \left(m_i \vec{r}_i \times \vec{g} \right) \qquad \because \vec{r}_{CM} = \frac{m_i \vec{r}_i}{M}$$

$$\therefore \sum \vec{\tau} = M\vec{r}_{cm} \times \vec{g} = \vec{r}_{cm} \times M\vec{g}$$

Figure 14.8 shows the torque due to gravity about the centre of mass of a body is zero, and object is balanced at CM/CG.

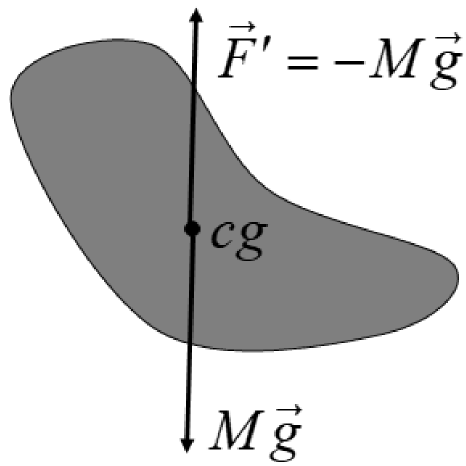


Figure 14.8: The object is balanced at CM/CG.

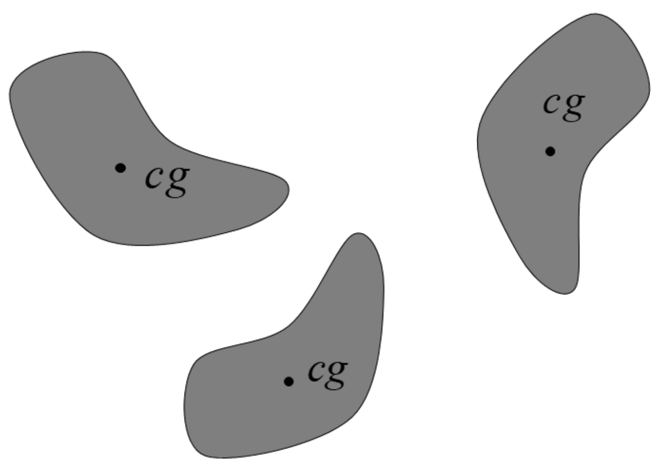
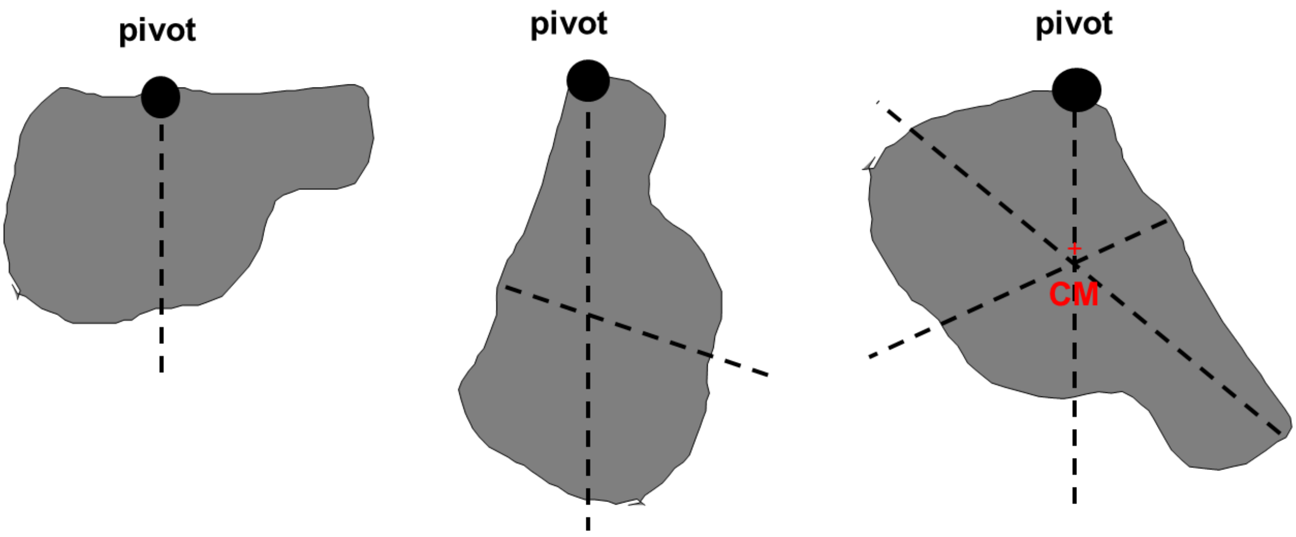


Figure 14.9: Center of gravity for different shapes is shown.

The object will be in equilibrium no matter what its orientation.

Figure 14.10: The arbitrary shaped object is being pivoted at different edges to find out CM.



We find that the center of mass is at the "center" of the object.

Difference Between Center of Mass and Center of Gravity:

If g is not constant over the body, then the center of gravity and the center of mass do not coincide, as shown in figure 14.11.

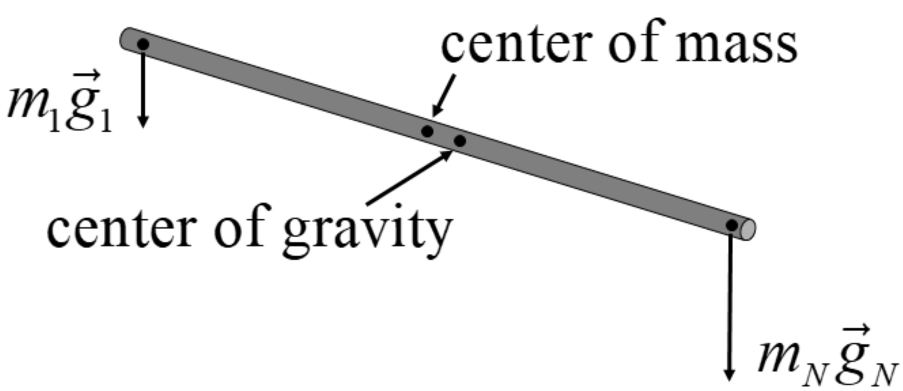


Figure 14.11: For different values of g over a body can led to different location of CM and CG of the body.

Types of Equilibrium:

- **Stable equilibrium:** Object returns to its original position if displaced slightly, as shown in figure 14.12.

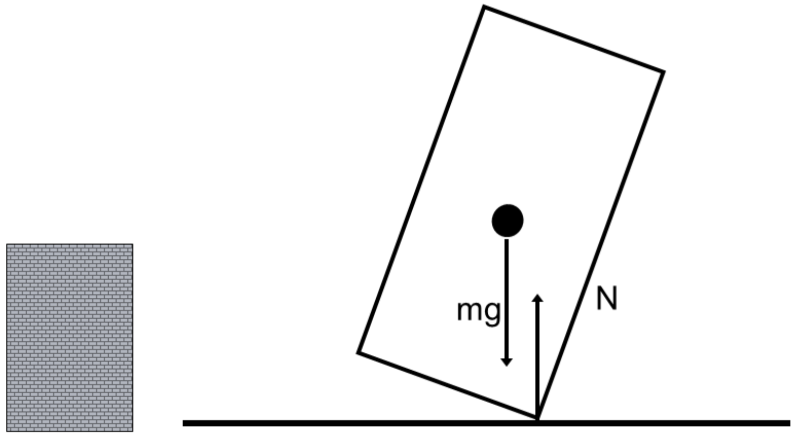


Figure 14.12: The object possesses stable equilibrium if slightly tilted.

- **Unstable equilibrium:** Object moves farther away from its original position if displaced slightly, as shown in figure 14.13.

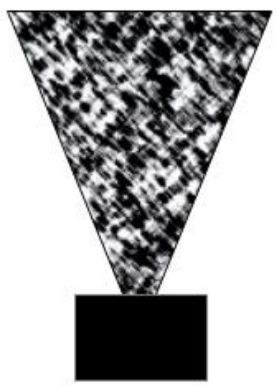


Figure 14.13: The object possesses unstable equilibrium if slightly tilted.

- **Neutral equilibrium:** Object stays in its new position if displaced slightly, as shown in figure 14.14.

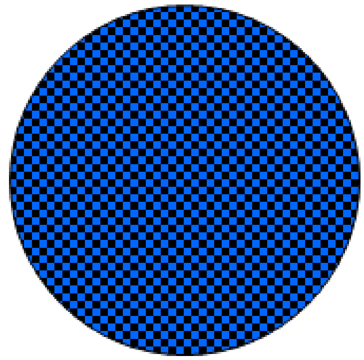


Figure 14.14: The object possesses neutral equilibrium if slightly tilted.

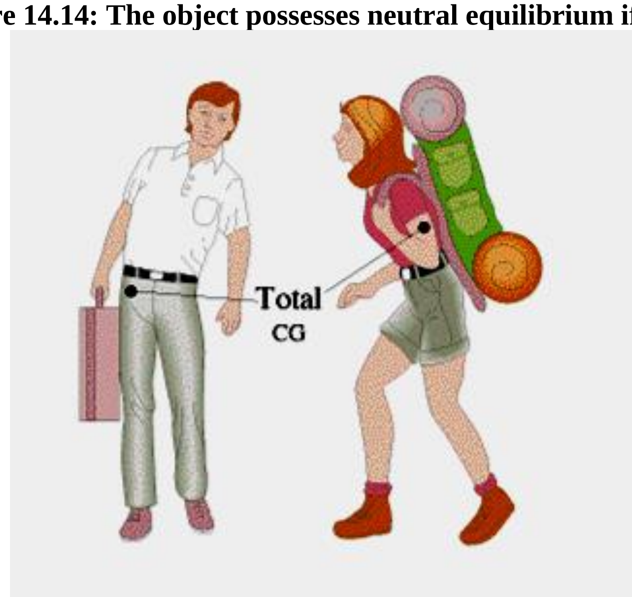


Figure 14.15: To achieve stable equilibrium while carrying a weight people tend to keep CG over their feet.

- We can observe that people try to keep the CG over their feet, to feel stable while carrying weight, as shown in figure 14.15.

6) A thin bar AB of negligible weight is pinned to a vertical wall at A and supported by a thin wire BC, a weight w can be moved along the bar.

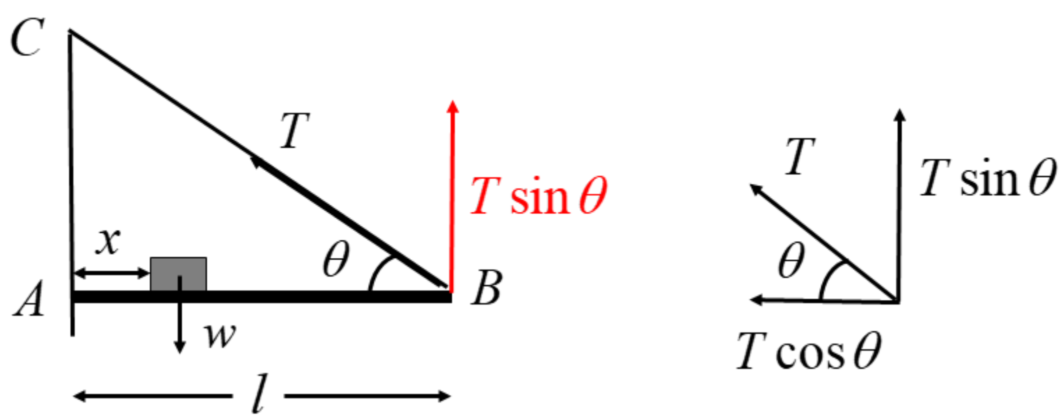
(a) Find T as a function of x. (b) Find the horizontal and vertical components of the force exerted on the bar by the pin at A.

Solution:

Since the system is in rotational equilibrium, the net torque about A is zero.

$$\therefore\ w\cdot x-(T\sin\theta)l=0$$

$$\text{or}\quad T=\frac{w\cdot x}{l\sin\theta}\quad \text{eq(1)}$$



Let F_H and F_V be the horizontal and verticalcomponents of the force exerted by the pin at A. Then since there is translational equilibrium we have

$$F_H=T\cos\theta$$

Using $eq(1)$

$$F_H=\frac{w\cdot x}{l\sin\theta}\cos\theta=\frac{wx\cot\theta}{l}\qquad \because \cot\theta=\frac{\cos\theta}{\sin\theta}$$

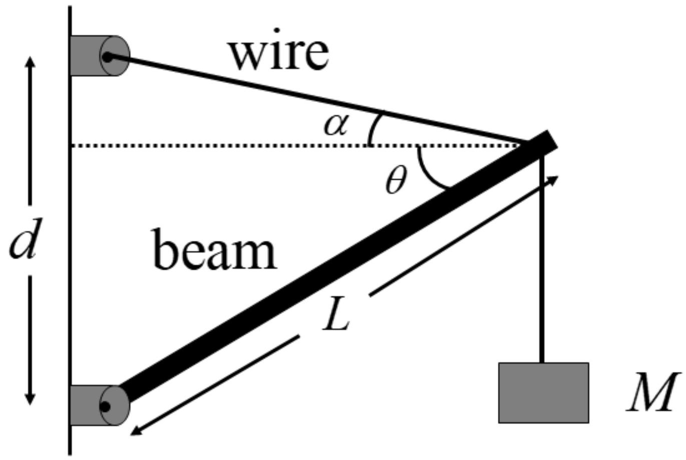
and

$$F_V=w-T\sin\theta$$

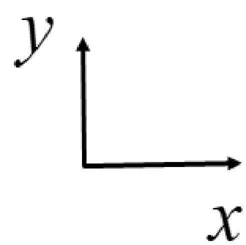
Using $eq(1)$

$$F_V=w-\frac{wx}{l\sin\theta}\sin\theta=w\left(1-\frac{x}{l}\right)$$

7) A mass M is hanged over a pivoted beam of length L, making an angle α with the horizontal. Find the tension in the wire and the force exerted by the hinge on the beam.



Solution:



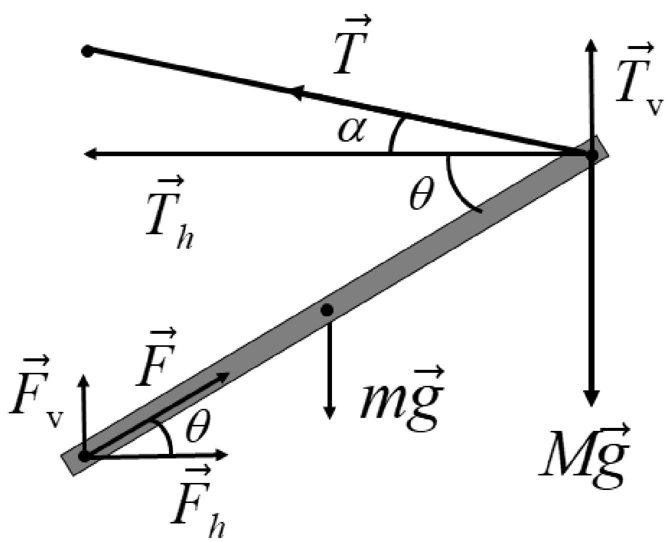
From the translational equilibrium

$$\sum F_x = F_h - T_h = 0$$

$$\sum F_y = F_v + T_v - mg - Mg = 0$$

$$T_v = T_h \tan\alpha$$

$$\tan\alpha = \frac{(d - Lsin\theta)}{L\cos\theta}$$



Applying the rotational equilibrium around the upper end of the beam

$$\sum \tau_z = F_v(L\cos\theta) + F_h(L\sin\theta)_h$$

$$+mg\left(\frac{L}{2}\cos\theta\right) = 0 \Rightarrow F_v = F_h \tan\theta + \frac{mg}{2}$$

Solving these equations simultaneously, we get

$$F_v = \frac{(dm + L(m + 2M)\sin\theta)}{2d} g$$

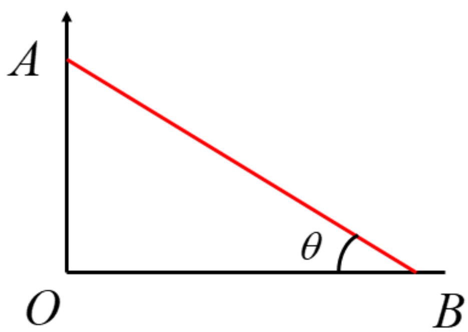
$$F_h = \frac{L(2M+m)\cos\theta}{2d}g$$

$$T_v = \frac{(m+2M)(d-L\cos\theta)}{2d}g$$

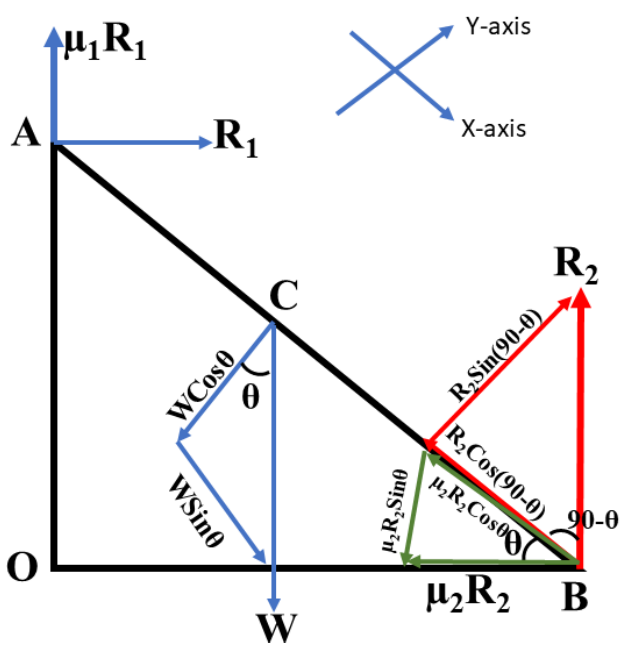
$$T_h = \frac{(m+2M)L\cos\theta}{2d}g$$

8) Show that the least angle θ at which the rod can lean to the horizontal without slipping is given by

$$\theta = \tan^{-1}\left(\frac{1-\mu_1\mu_2}{2\mu_2}\right)$$



Solution:



W = weight of the rod

R_1 *and* R_2 = *normal force perpendicular to the surface.*

Acoording to the first condition of equilibrium,

$\sum F_x = 0, \ \sum F_y = 0$

Along x-axis:

$R_1 - \mu_2 R_2 = 0 \Rightarrow R_1 = \mu_2 R_2 \ \rightarrow$ eq(1)

Along y-axis:

$R_2 + \mu_1 R_1 - W = 0 \Rightarrow \ R_2 + \mu_1 R_1 = W \ \rightarrow$ eq(2)

substituting R_1 from eq(1) into eq(2), we get

$$R_2 + \mu_1 \mu_2 R_2 = W$$

$$R_2 (1 + \mu_1 \mu_2) = W$$

$$\Rightarrow \quad R_2 = \frac{W}{(1 + \mu_1 \mu_2)} \quad \text{eq(3)}$$

According to the second condition of equilibrium,

$\sum \tau = 0$

Since A is our center of rotation which means, there is no torque contribution from R_1 and $\mu_1 R_1$ (*i.e.*, τ = r*F where, r = 0)

Now Considering the rotational equilibrium about A,

$$R_2 \times OB = W \times OD + \mu_2 R_2 \times OA$$

$$R_2 \times AB \cos\theta = W \times \frac{AB \cos\theta}{2} + \mu_2 R_2 \times AB \sin\theta$$

$$R_2 \cos\theta = W \frac{\cos\theta}{2} + \mu_2 R_2 \sin\theta$$

$$R_2 \cos\theta - W \frac{\cos\theta}{2} = \mu_2 R_2 \sin\theta$$

$$\text{or } \cos\theta \left(R_2 - \frac{W}{2} \right) = \mu_2 R_2 \sin\theta$$

$$\Rightarrow \quad \tan\theta = \frac{R_2 - \frac{W}{2}}{\mu_2 R_2} \qquad \because \tan\theta = \frac{\sin\theta}{\cos\theta}$$

using the value of R_2 from eq(3), we get

$$\tan\theta = \frac{\dfrac{W}{(1+\mu_1\mu_2)} - \dfrac{W}{2}}{\mu_2 \dfrac{W}{(1+\mu_1\mu_2)}}$$

$$\tan\theta = \frac{\dfrac{2W - W(1+\mu_1\mu_2)}{2(1+\mu_1\mu_2)}}{\mu_2 \dfrac{W}{(1+\mu_1\mu_2)}}$$

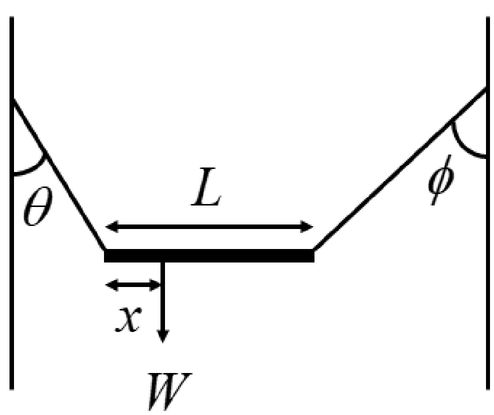
$$\tan\theta = \frac{2W - W(1+\mu_1\mu_2)}{2(1+\mu_1\mu_2)} \frac{(1+\mu_1\mu_2)}{W\mu_2}$$

$$\tan\theta = \frac{2W - W(1+\mu_1\mu_2)}{2W\mu_2}$$

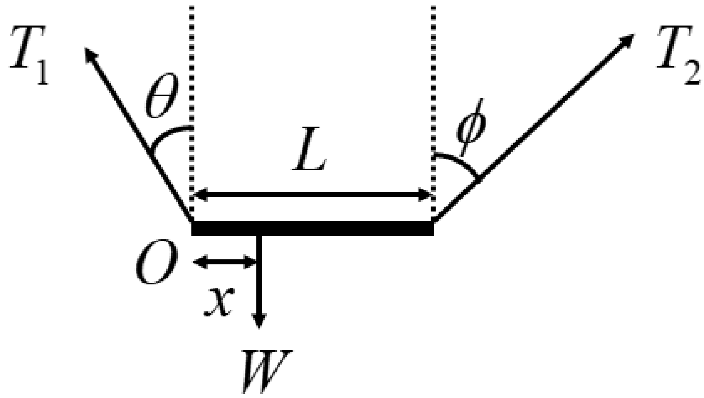
$$\tan\theta = \frac{W(2-1-\mu_1\mu_2)}{2W\mu_2}$$

$$\Rightarrow \quad \tan\theta = \frac{1\text{-}\mu_1\mu_2}{2\mu_2}$$

9) A non-uniform bar of weight W is suspended at rest in a horizontal position by two light cords. Find the distance x from the left-hand end to the center of gravity.



Solution:



Summition of horizontal components

$T_2 \sin\phi - T_1 \sin\theta = 0$

multiplying it with $\cos\theta,$ we get

$T_2 \sin\phi\cos\theta - T_1 \sin\theta\cos\theta = 0$ eq(1)

Summition of vertical components

$T_2 \cos\phi + T_1 \cos\theta - W = 0$

multiplying it with $\sin\theta,$ we get

$T_2 \cos\phi\sin\theta + T_1 \cos\theta\sin\theta = W\sin\theta$ eq(2)

adding equation 1 and 2

$T_2 \sin\phi\cos\theta - T_1 \sin\theta\cos\theta + T_2 \cos\phi\sin\theta + T_1 \cos\theta\sin\theta = W\sin\theta$

$T_2 \sin\phi\cos\theta + T_2 \cos\phi\sin\theta = W\sin\theta$

$T_2(\sin\phi\cos\theta + \cos\phi\sin\theta) = W\sin\theta$

$\because \sin\left(\theta+\phi\right) = \sin\phi\cos\theta + \cos\phi\sin\theta$

$$\Rightarrow T_2 = \frac{W\sin\theta}{\sin\left(\theta+\phi\right)}$$

Now for rotational equilibrium about O:

$$-Wx + \left(T_2\cos\phi\right)L = 0$$

$$\Rightarrow x = \frac{\left(T_2\cos\phi\right)L}{W} = \frac{W\sin\theta}{\sin\left(\theta+\phi\right)}\frac{\cos\phi L}{W}$$

$$x = \frac{L\sin\theta\cos\phi}{s\,\text{in}\left(\theta+\phi\right)}$$

Equilibrium means no net force and no net torque but, in the several physics books, it is called static equilibrium when a body is stationary, for example, a ladder is attached to a wall, then it is called static equilibrium or if a box is lying on a table, it is also static. However, if a plane is flying in the air and is moving at a constant speed if it is doing so, then it is called dynamic equilibrium. However, the difference between this and that is not in the sense that if you were to move along with that ship, then it would appear static to you, hence whether something is static or dynamic depends on your frame of reference so, I have not differentiated between these two. As you have seen during this lecture how many different types of systems, we can apply the concepts of equilibrium and how much we can learn about the subject of mechanics.

Lecture#15 PHY101

Oscillations

Oscillation and vibration are fundamental physical phenomena. Oscillation is a back-and-forth movement, like stirring a pan or playing cards. It's all around us, seen in a swinging pendulum or a plucked guitar string. Even a floating object on water demonstrates oscillatory motion. This repetitive pattern continues unless acted upon by outside forces. For example, a spring with added weight will oscillate continuously. While some oscillations are visible, others need special tools like an oscilloscope to detect. Interestingly, the quartz crystal in a watch uses oscillations for accurate timekeeping.

We're exploring the idea of oscillation.

The **time period** is the amount of time it takes to reach a certain point again, often referred to as "T". It's the time it takes to complete one full cycle. The time period is measured in seconds.

Now, let's talk about **frequency** - it's closely related to the time period. Frequency is how often a cycle happens in one second. It's the inverse of the time period and is measured in hertz (Hz).

Finally, there's **amplitude** - this is the extent or size of the oscillation. Amplitude reflects how much the movement varies. A larger amplitude means a greater range of motion.

Let's understand these ideas using **a simple pendulum**. Picture a weight hanging from a string. When you move it sideways and it swings back, that's its time period - the time it takes to complete one full swing. Now, how many times does it swing back and forth in a second? That's the frequency. Imagine it takes 3 seconds to complete a full swing; that means the frequency is one-third swing per second. Finally, consider the amplitude - how far the pendulum swings. Notice how changing the amplitude changes the motion. In an ideal situation without air resistance, the pendulum would keep swinging forever. But air resistance gradually drains its energy over time, causing it to eventually stop.

Now, let's e-xplore the concept of **equilibrium position** - the point where a body naturally stays. Any force- acting on it near this position causes it to oscillate. Consider a pendulum: when displaced, gravity pulls it towards equilibrium, accelerating its motion. As it moves away, its velocity decreases. Notice, the pivotal force always directs towards equilibrium, governing the body's movement dynamics.

In the pendulum example, we observed that the force is directly proportional to the height it moves. When this force aligns horizontally, it's called a **linear restoring force**. However, if it's proportional to x^2, it pulls in the opposite direction. Whether the force is linear depends on not over stretching the spring. A mass attached to a spring as shown in Fig 15.1 will cause the system to oscillate, as the mass experiences the force of gravity. Let's see this in action by adding a mass to the spring.

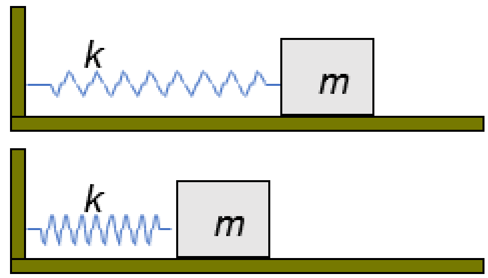


Fig 15. 1: Mass spring system

Now, observe that the system is at a balanced position when still. When released, it starts moving up and down in a repeating cycle. We call this the equilibrium point, which we'll mark as x = 0. As we change the mass attached to the spring, we notice the frequency of the oscillations changing. A heavier mass requires more force to stretch the spring, affecting the balance between gravity and spring force. This causes the oscillation frequency to decrease. This demonstrates how adjusting the mass can alter the oscillation frequency, an important concept to understand in oscillatory systems.

What describes oscillations?

- *period* T = time for completing one cycle
- *frequency* f = 1 / T
- *amplitude* A = maximum displacement from *equilibrium* position

Restoring force

- Suppose a force is always directed towards a central *equilibrium* position
- Force always acts to return the object to its equilibrium position
- The object will oscillate around the equilibrium position
- This “back-and-forth” motion around an equilibrium position is called: Periodic motion.

Periodic motion

- Simple Harmonic Motion (SHM)
- The restoring force depends on the displacement Δx

$$F_{\text{restore}} = -k\,\Delta x$$

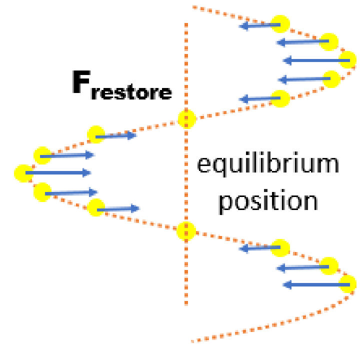


Fig 15. 2: Restoring force of a body at its equilibrium position**.**

Derivation of Equation of Motion for a Mass-Spring System

1. Hooke's Law

Hooke's Law describes the behaviour of an ideal spring:

$$F(x) = -kx$$

where:

- $F(x)$ is the force exerted by the spring
- k is the spring constant (a measure of the stiffness of the spring)
- x is the displacement from the equilibrium position.

Hooke's Law tells us that the force exerted by the spring is directly proportional to the displacement from its equilibrium position, with the negative sign indicating that the force is opposite to the direction of displacement.

2. Stored Energy in the Spring

When a spring is stretched or compressed, it stores potential energy. The amount of energy stored in the spring is given by:

$$U(x) = \frac{1}{2}kx^2$$

where:

- $U(x)$ is the potential energy stored in the spring

- k is the spring constant, and
- x is the displacement from the equilibrium position.

This equation shows that the potential energy stored in the spring is directly proportional to the square of the displacement from its equilibrium position.

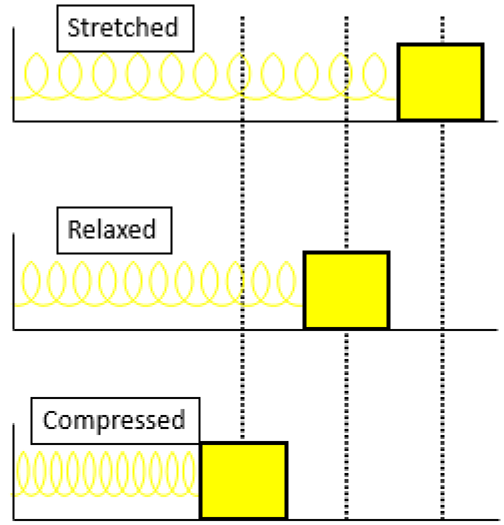


Fig 15. 3: A spring at equilibrium, neither compressed nor stretched. A block of mass m on a frictionless surface is pushed against the spring. If x is the compression in the spring, the potential energy stored in the spring is ½ kx^2 . When the block is released, this energy is transferred to the block in the form of kinetic energy.

3. Equating Force and Acceleration

According to Newton's Second Law, the force acting on an object is equal to the product of its mass and acceleration:

$$F = ma$$

Substituting $F = -kx$ (from Hooke's Law), we have:

$$ma = -kx$$

This equation relates the acceleration *(a)* of the mass to the force *(F)* exerted by the spring.

4. Differential Equation of Motion

By expressing acceleration as the second derivative of displacement with respect to time, we obtain:

$$a = \frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} = -kx$$

This equation represents the differential equation of motion for the mass-spring system, where:

- m is the mass of the object,
- $\frac{d^2x}{dt^2}$ is the acceleration of the mass, and
- $-kx$ is the restoring force exerted by the spring.

5. Solving the Differential Equation

Dividing both sides by m, we get:

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

This is a second-order linear homogeneous ordinary differential equation with constant coefficients. Its solution provides the equation of motion for the mass-spring system, describing how the displacement (x) of the mass varies with time during simple harmonic motion.

<u>Simple Harmonic Motion</u>

1. Equation of Motion for Simple Harmonic Oscillator:

We start with the general equation of motion for a simple harmonic oscillator:

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

This equation represents the second-order linear ordinary differential equation governing the motion of the oscillator.

2. Rearrangement:

We can rearrange the equation by moving the term involving x to one side:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

This equation shows that the acceleration of the oscillator ($\frac{d^2x}{dt^2}$) is directly proportional to its displacement (x) from the equilibrium position, with a negative sign indicating that the acceleration is opposite to the displacement.

3. Introduction of Angular Frequency:

We introduce the concept of angular frequency (ω) for the oscillator:

$$\omega^2 = \frac{k}{m}$$

The angular frequency (ω) represents the rate at which the oscillator oscillates back and forth. It is related to the stiffness of the oscillator (determined by k) and its mass (determined by m). Squaring both sides simplifies the equation.

How to calculate $\frac{\mathrm{d}}{\mathrm{dt}}\cos\omega t$?

$x(t) = \cos\omega t$

$x(t+\Delta t) = \cos\omega(t+\Delta t)$

Differece between $x(t+\Delta t)$ and x(t)

Apply Trigonometric Identity (Angle Addition Formula for Cosine):

$$\cos(A+B)-\cos(A) = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

Applying this formula with $A = \omega t + \omega\Delta t$ and $B = -\omega t$

$$\cos(\omega t+\omega\Delta t)-\cos(\omega t) = -2\sin\left(\frac{\omega t+\omega\Delta t-(-\omega t)}{2}\right)\sin\left(\frac{\omega t+\omega\Delta t-\omega t}{2}\right)$$

Simplifying the expression inside the sine functions:

$$\sin\left(\frac{\omega t+\omega\Delta t+\omega t}{2}\right)\sin\left(\frac{\omega\Delta t}{2}\right) = \sin(\omega t+\omega\Delta t/2)\sin(\omega\Delta t/2)$$

Approximation:

Since Δt is very small, $\omega\Delta t$ is slso very small.

For small angles, $\sin(\theta) \approx \theta$

So, $\sin(\omega\Delta t/2) \approx \omega\Delta t/2$

Therefore, $\sin(\omega t+\omega\Delta t/2)\sin(\omega\Delta t/2)$ is approximated as $(\omega\Delta t/2)^2$,

and the negative sign from the trigonometric identity is retained.

Result:

Substituting the approximation into the equation:

$$x(t+\Delta t)-x(t) \approx -\sin\omega\Delta t\cdot\sin(\omega t+\omega\Delta t/2) \approx -\omega\Delta t\cdot\sin\omega t$$

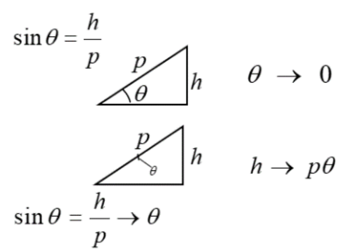
$$\therefore \frac{\mathrm{d}}{\mathrm{dt}}\cos\omega t = -\omega\sin\omega t$$

Remember two important results:

$$\frac{d}{dt}(\sin\omega t) = \omega\cos\omega t$$

$$\frac{d}{dt}(\cos\omega t) = -\omega\sin\omega t$$

Why is $\sin\theta \approx \theta$ for small θ?



What happens if you differentiate twice?

$$\frac{d^2}{dt^2}(\sin \omega t) = \omega \frac{d}{dt}\cos \omega t$$
$$= - \omega^2 \sin \omega t$$

$$\frac{d^2}{dt^2}(\cos \omega t) = -\omega \frac{d}{dt}\sin \omega t$$
$$= - \omega^2 \cos \omega t$$

Any function of the form:

$x = a\cos \omega t + b\sin \omega t$

is a solution of $\frac{d^2x}{dt^2} = -\omega^2 x$

Physical significance of constant ω

$$x = x_m \cos \omega\left(t + \frac{2\pi}{\omega}\right)$$
$$x = x_m \cos(\omega t + 2\pi)$$
$$x = x_m \cos \omega t$$

That is, the function merely repeats itself after a time $2\pi / \omega$

So $2\pi / \omega$ is the period of the motion T

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

The frequency ν of the oscillator is the number of complete vibrations per unit time:

$$\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$\omega = 2\pi\nu = \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

ω is called the angular frequency

Fig 15. 4: For small angles, $sin\theta$ is approximately equal to θ due to the linear behaviour near the origin."

$$\dim[\omega] = T^{-1}$$

Unit of ω is radian/second

$$x(t) = a\cos\omega t + b\sin\omega t$$
$$\text{x(0)=a}$$
$$\frac{\mathrm{d}}{\mathrm{dt}}x(t) = -\omega a\sin\omega t + \omega b\cos\omega t$$
$$= \omega b \text{ (at t=0)}$$

This solution can also be written as:

$$x(t) = x_m\cos\left(\omega t + \phi\right)$$

Physical significance of constant x_m

$$x = x_m\cos(\omega t + \phi)$$

$$\Rightarrow \; -x_m \le x \le +x_m$$

x_m is called the amplitude of the motion

The frequency of the simple harmonic motion is independent of the amplitude of the motion

Phase

We're exploring the concept of "phi" (a Greek letter) in sinusoidal waves. Phi represents a phase shift, which changes where the wave begins on the vertical axis. When we assign different constant

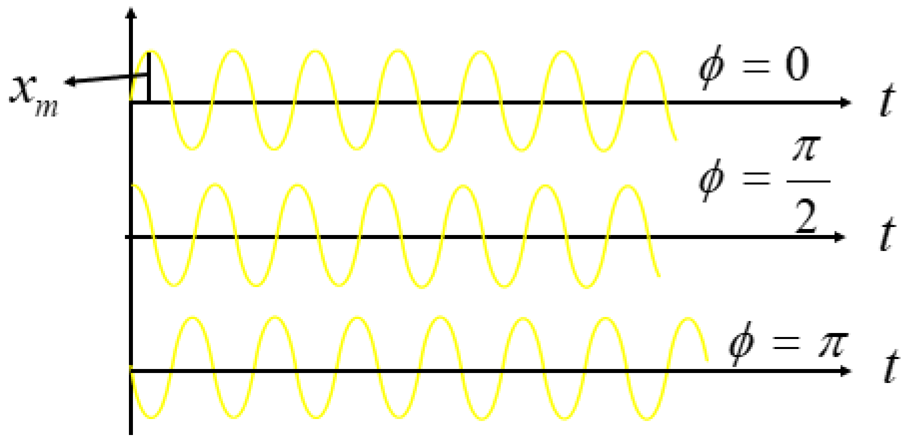


Fig 15. 5: Impact of Phase Shift (Phi) on Sinusoidal Waves: Different constant values of phi shift waveforms along the vertical axis."

values to phi, we see distinct changes in the waveform. For example, setting phi to zero gives us a standard sine wave with a frequency denoted by omega as shown in Fig.15.5. Increasing phi shifts the wave along the axis, moving its starting point from a peak to a trough. We can visually see this shift through plots. Additionally, differentiating the wave reveals its acceleration, which is linked to omega and the displacement amplitude. By exploring phi, we grasp its significant role in shaping sinusoidal waves and its importance in wave analysis.

$$x = x_m \sin(\omega t + \phi)$$

The quantity $\theta = \omega t + \phi$ is called the phase of the motion. The constant ϕ is called the phase constant.

__**Energy of simple harmonic motion**__

1. Expressing Displacement (*x*):

In simple harmonic motion, the object oscillates back and forth around a central point as shown in Fig 15.6. We express this oscillation as $x = x_m \cos(\omega t)$

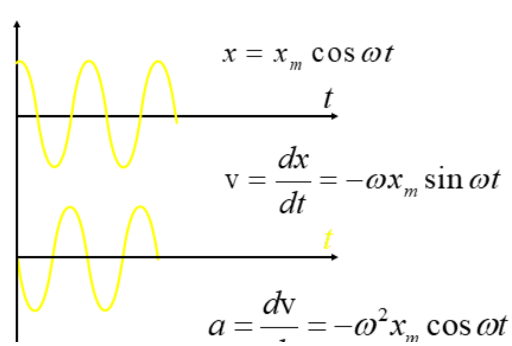


Fig 15. 6: Representation of Displacement in Simple Harmonic Motion: Explaining the components of displacement (x) in the oscillatory motion.

Where:

x is the displacement from the central point at time t.

x_m is the amplitude, representing the maximum displacement from the central point.

ω is the angular frequency, determining the rate of oscillation.

2. Calculating Potential Energy (*U*):

The potential energy of the oscillator, *U*, arises from the restoring force exerted by the spring or restoring element. We use Hooke's Law to calculate it:

$$U = \frac{1}{2}kx^2$$

Substituting $x = x_m \cos(\omega t)$, we find $U = \frac{1}{2}kx_m^2 \cos^2(\omega t)$

This equation shows how the potential energy of the oscillator varies with its displacement from the central point.

3. Calculating Kinetic Energy (*K*):

The kinetic energy of the oscillator, *K*, arises from its motion. We calculate it using the formula:

$$K = \frac{1}{2}mv^2$$

Since velocity $v = \frac{dx}{dt}$, and $x = x_m \cos(\omega t)$, we find $v = -x_m \omega \sin(\omega t)$

Substituting this into the equation for kinetic energy, we find

$$K = \frac{1}{2}m\omega^2 x_m^2 \sin^2(\omega t)$$

This equation illustrates how the kinetic energy of the oscillator varies with its velocity and displacement.

4. Calculating Total Mechanical Energy (*E*):

The total mechanical energy (*E*) of the oscillator is the sum of its potential and kinetic energies:

$$E = K + U$$

Substituting the expressions for *K* and *U*, we find

$$E = \frac{1}{2}kx_m^2 \cos^2(\omega t) + \frac{1}{2}kx_m^2 \sin^2(\omega t)$$

Since $\cos^2(\omega t) + \sin^2(\omega t) = 1$, we find $E = \frac{1}{2}kx_m^2$

This equation shows that the total mechanical energy of the oscillator remains constant throughout its motion, illustrating the principle of energy conservation as shown in Fig 15.7.

$$E=\frac{1}{2}m\text{v}^2+\frac{1}{2}kx^2=\frac{1}{2}kx_m^2$$

$$\text{v}=\frac{dx}{dt}=\pm\sqrt{\frac{k}{m}\left(x_m^2-x^2\right)}$$

speed is maximum at x=0

speed is zero at $x=\pm x_m$

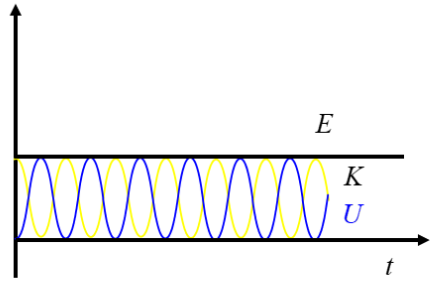


Fig 15. 7: Total mechanical energy of the oscillator remains constant.

Problem: 15.1: Two spring systems, illustrated in Fig 15.8, require determination of their respective time period.

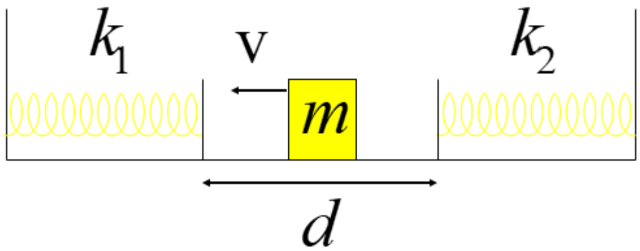


Fig 15. 8: Illustration of Two Spring System.

Solution:

$T=t_1+t_2+t_3$

$t_1=$ period of oscillation of spring k_1

$t_2=$ period of oscillation of spring k_2

$t_3=$ time to cover the distance d

$$T=2\pi\sqrt{\frac{m}{k_1}}+2\pi\sqrt{\frac{m}{k_2}}+\frac{2d}{\text{v}}$$

Springs Coupled in Series

Consider a system where two springs, each with spring constants k_1 and k_2, are connected end to end, such that they act in series.

1. Total Displacement (y)

When a force (F) is applied to the system, it causes a total displacement (y) in the system, which can be expressed as the sum of the displacements (y_1 and y_2) caused by each spring as shown in Fig 15.9:

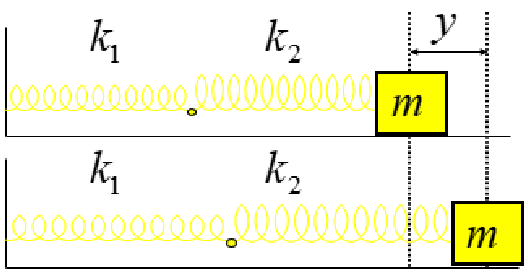


Fig 15. 9: Illustration of Springs Coupled in Series.

$$y = y_1 + y_2$$

2. Hooke's Law for Each Spring

According to Hooke's law, the force exerted by each spring (*F*) is proportional to its displacement (y_1 and y_2). Therefore:

$$F = -k_1 y_1 = -k_2 y_2$$

where k_1 and k_2 are the spring constants of the respective springs.

3. Displacement of Each Spring

Solving for the displacements y_1 and y_2 in terms of the force *F*:

$$y_1 = -\frac{F}{k_1}\ , y_2 = -\frac{F}{k_2}$$

4. Total Displacement and Effective Spring Constant

Substituting the expressions for y_1 and y_2 into the total displacement equation, we get:

$$y = -F\left(\frac{1}{k_1} + \frac{1}{k_2}\right) = -F\left(\frac{k_1 + k_2}{k_1 k_2}\right)$$

This expression shows the total displacement *y* in terms of the force *F* and the inverse of the sum of the reciprocals of the spring constants k_1 and k_2.

5. Effective Spring Constant (k_{eff})

The effective spring constant (k_{eff}) of the coupled springs can be obtained by equating the force *F* to - $k_{eff}\, y$, which yields:

$$F = -\left(\frac{k_1 k_2}{k_1 + k_2}\right) y\ ,\ k_{eff} = \left(\frac{k_1 k_2}{k_1 + k_2}\right)$$

This equation provides a way to calculate the effective spring constant of the series-coupled springs based on their individual spring constants.

6. Period of Oscillation (*T*)

Finally, the period of oscillation (T) of the system can be determined using the effective spring constant (k_{eff}) and the mass (m) connected to the springs:

$$\frac{1}{k_{eff}} = \frac{1}{k_1} + \frac{1}{k_2} \ , T = 2\pi\sqrt{\frac{m}{k_{eff}}}$$

This equation relates the period of oscillation to the effective spring constant and the mass of the system, providing insight into the dynamics of the coupled springs in series.

<u>Springs in parallel</u>

1. Introduction to Displacement (x):

We begin by defining x as the displacement of the mass from its equilibrium position.

2. Force Exerted by Spring 1 (F_1)

Spring 1 exerts a force (F_1) on the mass, given by Hooke's Law:

$F_1 = -k_1 x$

Here,

k_1 represents the spring constant of spring 1.

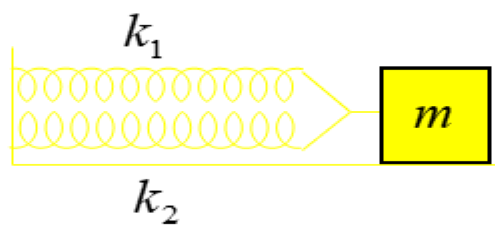


Fig 15. 10: Illustration of Springs in parallel.

3. Force Exerted by Spring 2 (F_2)

Similarly, spring 2 exerts a force (F_2) on the mass, also given by Hooke's Law:

$$F_2 = -k_2 x$$

k_2 is the spring constant of spring 2.

4. Total Force (F):

The total force (F) exerted on the mass is the sum of the forces exerted by both springs:

$$F = F_1 + F_2 = -(k_1 + k_2)x$$

This equation shows that the total force is the sum of the forces from each spring acting in the same direction.

5. Effective Spring Constant (k_{eff})

We define an effective spring constant (k_{eff}), representing the combined stiffness of the two springs:

$$k_{eff} = k_1 + k_2$$

This effective spring constant accounts for the total restoring force acting on the mass due to both springs.

6. Angular Frequency (ω):

The angular frequency (ω) of the system is given by:

$$\omega = \sqrt{\frac{k_1 + k_2}{m}} = \sqrt{\frac{k_{\text{eff}}}{m}}$$

It represents the rate at which the mass oscillates back and forth under the influence of the combined springs.

Torsional Oscillator

Consider a system where a wire or a shaft is twisted, exerting a restoring torque (τ) proportional to the angle of twist (θ) as shown in Fig 15. The constant of proportionality (κ), known as the torsional constant, depends on the properties of the wire or shaft.

1. Expression for Torque (τ)

The torque (τ) acting on the system can be expressed as:

$$\tau = -\kappa\theta$$

Here, θ represents the angular displacement from the equilibrium position.

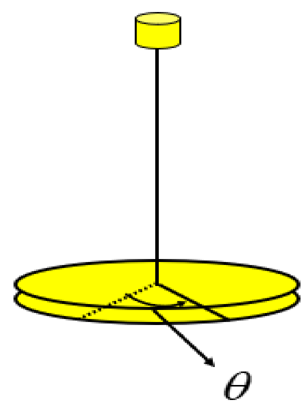


Fig 15. 11: Illustration of a Torsional System with Expression for Torque.

2. Newton's Second Law for Rotation

Applying Newton's second law for rotation, we equate the torque (τ) to the moment of inertia (I) times the angular acceleration (α):

$$\tau = I\alpha = I\frac{d^2\theta}{dt^2}$$

This equation relates the torque to the angular acceleration, where I is the moment of inertia of the system.

3. Equating Torque and Angular Acceleration

Equating the expressions for torque and angular acceleration, we get:

$$-\kappa\theta = I\frac{d^2\theta}{dt^2}$$

4. Differential Equation for Angular Displacement

Rearranging terms, we obtain the differential equation for angular displacement:

$$\frac{d^2\theta}{dt^2} = -\left(\frac{\kappa}{I}\right)\theta$$

This equation governs the motion of the torsional oscillator, relating the angular displacement (θ) to its second derivative with respect to time.

5. Solution for Angular Displacement

Assuming harmonic motion, we propose a solution of the form:

$$\theta = \theta_m \cos\omega t$$

Here,

θ_m represents the maximum angular displacement, and ω is the angular frequency of oscillation.

6. Angular Frequency (ω)

Substituting the proposed solution into the differential equation, we find the angular frequency (ω) of the oscillation:

$$\omega = \sqrt{\frac{\kappa}{I}}$$

This equation gives the rate at which the angular displacement oscillates back and forth, determined by the torsional constant (κ) and the moment of inertia (I) of the system.

Physics-PHY101-Lecture 16

OSCILLATIONS: II

Oscillations are a fundamental concept in physics that describes the repetitive motion of a system around a central equilibrium point. In this chapter, we will explore various types of oscillatory motion and their properties. We begin with the simple pendulum, which is a classic example of a harmonic oscillator. Next, we will discuss the physical pendulum, which is a more complex system than the simple pendulum. We will also discuss the composition of two harmonic oscillators with the same period, which leads to interesting phenomena like beats and resonance. Additionally, we will look into damped oscillators, which describe systems where the amplitude of oscillations decreases over time due to damping forces. We will analyze the behavior of damped oscillators and derive their equations of motion. Finally, we will examine forced and free oscillations. **Forced oscillations** occur when an external force is applied to a system, while **free oscillations** occur without any external force. By the end of this chapter, you will have a deep understanding of the fundamental concepts of oscillatory motion and be able to analyze and describe a wide range of oscillatory systems.

Simple Pendulum:

A simple pendulum is an ideal body consisting of a body of mass m and suspended by a light in-extensible string. The pendulum oscillates periodically under the action of gravity when it is pulled to one side of its equilibrium point and then released.

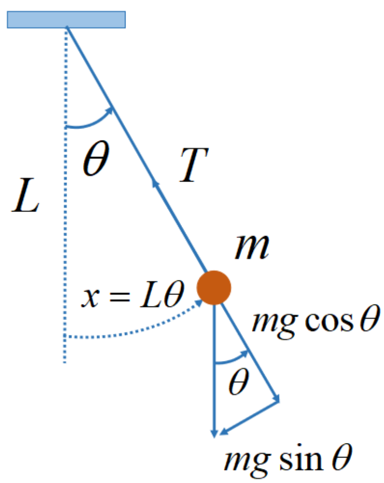


Figure 16. 1: The simple pendulum. A pendulum of mass m with a length L and making an angle with the vertical. We choose the x axis to be in the tangential direction and the y axis to be in the radial direction at this particular time.

The forces acting on mass are the weight mg and the tension T in the string. The weight mg is resolved into a radial component of magnitude mg cos θ and a tangential component of magnitude mg sin θ. The centripetal acceleration required to keep the particle travelling in a circular path is provided by the radial components of the forces. The restoring force on m that pushes it towards equilibrium is represented by the tangential component. The restoring force is thus

$$F = -mg\,sin\,\theta \qquad\qquad (1)$$

The negative sign shows that F is opposite to the direction of increasing L= x / θ.

It should be noted that the restoring force is proportional to sin θ rather θ. As a result, the motion is not simply harmonic. However, for small angles, the sine function can be approximated using the small-angle approximation:

$$sin\,\theta \approx \theta$$

Thus

$$x = L\theta \rightarrow \theta = \frac{x}{L}$$

Putting the value in eq. (1) we get

$$F = -mg\frac{x}{L}$$

where $$a = -g\frac{x}{L} \qquad\qquad (2)$$

$$F = -\left(\frac{mg}{L}\right)x$$

By Newton's second law , F = ma where $a = \frac{d^2x}{dt^2}$. Hence

$$F = m\frac{d^2x}{dt^2}$$

Also, $F = -\left(\frac{mg}{L}\right)x$

On comparing, we get,

$$\frac{d^2x}{dt^2} = -\left(\frac{g}{L}\right)x$$

$$\frac{d^2x}{dt^2} + \left(\frac{g}{L}\right)x = 0$$

The general solution to this differential equation is of the form:

$$x(t) = Acos(\omega t) + Bsin(\omega t)$$

Where A and B are constants, and ω is the angular frequency, related to g and L by the equation ω = √(L/g) . For the specific, where the pendulum is released from its maximum displacement (x_m), the initial conditions are $x(0) = x_m$ and $\frac{dx}{dt}(0) = 0$. Solving for A and B based on these conditions yields:

$$x(0) = Acos(0) + Bsin(0) \rightarrow x_m = A$$

So, A = x_m and B = 0. Therefore, the solution for the given initial condition is:

$$x = x_m \cos \omega t$$

In SHM acceleration is given by

$$a = -\omega^2 x \qquad (3)$$

After comparing eq. 2 and 3, we will get the expression of angular frequency defined as,

$$\omega^2 = \frac{g}{L}$$

$$\omega = \sqrt{\frac{g}{L}}$$

The period of the oscillation of physical pendulum is given by,

$$T = \frac{2\pi}{\omega}\,.$$

Substituting ω into the expression, we get

$$T = 2\pi\sqrt{\frac{m}{mg/L}} = 2\pi\sqrt{\frac{L}{g}}$$

Physical Pendulum

A physical pendulum is a rigid body suspended from a pivot point that oscillates back and forth in a vertical plane under the influence of gravity. Unlike a simple pendulum, which consist of a mass suspended from weightless string, a physical pendulum has a non-negligible mass and size. The motion of a physical pendulum can be described by using the principle of rotational motion and the conservation of energy.

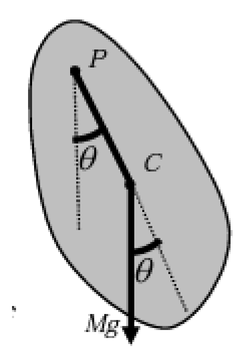


Figure 16. 2: Physical pendulum consisting of a rigid body of mass M.

Let us consider a physical pendulum consisting of a rigid body of mass M and length L, suspended from a pivot point P.

The pivot point P is fixed, and the pendulum is free to oscillate about it. Let the pendulum make a small angular displacement θ from its equilibrium position. The torque acting on the pendulum on the pendulum due to gravity is given by,

$$\tau = -Mgdsin\,\theta \qquad (4)$$

Where g is the acceleration due to gravity and L is the distance between the pivot point P and the center of mass of the pendulum. The negative sign indicates that the torque acts in the opposite direction to the displacement.

By Newton's second law for rotational motion, the torque is defined as,

$$\tau = I\,\alpha \qquad (5)$$

where I is the moment of inertia of the pendulum and α is the angular acceleration, $\alpha = \frac{d^2\theta}{dt^2}$.

Using the small angle approximation $sin\,\theta \approx \theta,$ and the definition of angular acceleration, we can write the above equation as,

$$I\frac{d^2\theta}{dt^2} = -Mgdsin\,\theta,$$

Dividing both sides by I and rearranging, we get

$$\frac{d^2\theta}{dt^2} + \left(\frac{Mgd}{I}\right)\theta = 0. \qquad (6)$$

The solution to this differential equation is a sinusoidal function of the form,

$$\theta(t) = Acos(\omega t + \varphi),$$

where A is the amplitude of oscillation, ω is the angular frequency, and φ is the phase angle. To find ω, take the second derivative of $\theta(t)$.

$$\theta''(t) = -A\omega^2\cos\,(\omega t + \varphi)$$

Substituting this into the differential equation, we get

$$-A\omega^2\cos(\omega t + \varphi) + \left(\frac{Mgd}{I}\right)Acos(\omega t + \varphi) = 0$$

Dividing both sides by $Acos(\omega t + \varphi)$ and after simplification we get

$$\omega^2 = \left(\frac{Mgd}{I}\right)$$

After substituting this into the expression for the period, we get

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{Mgd}}$$

Simple Harmonic Motion and Uniform Circular Motion

Simple harmonic motion (SHM) and uniform circular motion (UCM) are both types of periodic motion, but they differ in their nature and characteristics. Although SHM and UCM have different characteristics, they are related in some ways.

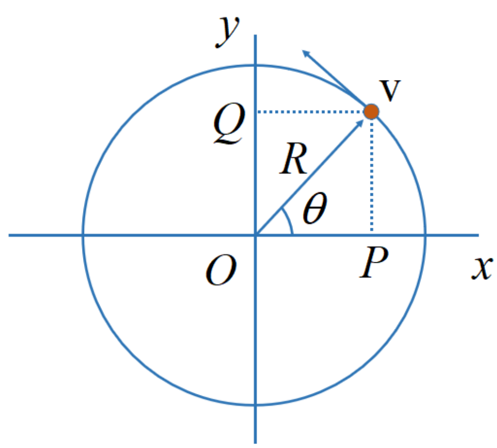


Figure 16. 3: Particle P executing uniform circular motion.

For example, the projection of UCM onto any diameter of the circle is SHM, with the amplitude of the SHM equal to the radius of the circle and the period of the SHM equal to the period of the UCM. Similarly, the motion of a particle in SHM can be considered as the projection of UCM onto a straight line. Furthermore, the equation for SHM can be derived from the equation for UCM by considering the projection of UCM onto a diameter of the circle.

Figure shows a particle P in uniform circular motion where **v** is the linear velocity and Ris radius of circle.

At a time t, the vector R which locates point P relative to the origin O, makes an angle θ with the x axis and the x component of vector R is,

$$x = R\cos\theta$$

Differentiating above equation two times w.r.t time we will get

$$\frac{dx}{dt} = R(\sin\theta)\frac{d\theta}{dt}$$

$$\frac{d^2x}{dt^2} = -R\cos\theta\frac{d^2\theta}{dt^2}$$

$$\frac{d^2x}{dt^2} = -x\frac{d^2\theta}{dt^2}$$

As we knew that the left side of above equation is the definition of acceleration and right side is the definition of angular displacement so above equation simplified as,

$$a = -\omega^2 x$$

Which is the expression of the body executing simple harmonic motion. Therefore, we can conclude that uniform circular motion can be considered a form of simple harmonic motion. When referring to "circular motion," the acceleration connected to this concept is referred to as centripetal acceleration, which is defined as:

$$\vec{a} = -\frac{v^2}{R}\hat{r} = -R\omega^2\hat{r}$$

where $v = R\omega$. The acceleration along x direction is

$$a_x = -R\omega^2 cos\ \theta$$

By substituting $R cos\theta = x$ (i.e. $\cos\theta = \frac{x}{R}$) in above equation we will get

$$a_x = -\omega^2 x$$

which shows that point P executes simple harmonic motion.

Similarly, the y component of acceleration is

$$a = -R\omega^2 sin\theta = -\omega^2 y \qquad \because\ y = Rsin\theta.$$

Composition of Two Simple Harmonic Oscillator along Same Line

When two simple harmonic motions with the same period are added together, the resulting motion is also a simple harmonic motion with the same period. The amplitude and phase of the resulting motion depend on the amplitudes and phases of the two individual motions.

Let's consider two simple harmonic motions with the same period T,

$$x_1(t) = A_1 sin\omega t,$$

$$x_2(t) = A_2 sin(\omega t + \varphi),$$

where A_1 and A_2 are the amplitudes and φ is any fixed angle and ω is the angular frequency. The sum of two motions is,

$$x(t) = x_1(t) + x_2(t),$$

$$x(t) = A_1 sin\omega t + A_2 sin(\omega t + \varphi).$$

To simplify this expression, we can use the identity

$$sin\ (\alpha+\beta) = sin\alpha cos\beta + cos\alpha sin\beta$$

Using this identity we get,

$$x(t) = A_1 sin\omega t + A_2 sin\omega t \cos\varphi + \ A_2 cos\omega t sin\varphi,$$

$$x(t) = sin\omega t(A_1 + A_2\cos\varphi) + cos\omega t(A_2 sin\varphi).$$

Let $A_1 + A_2\cos\varphi = R\cos\varphi$ and $A_2 sin\varphi = Rsin\theta$. After using some simple trigonometry we can write x in the form, $x = sin(\omega t\ + \theta)$. Thus, the resultant motion is also simple harmonic motion along the same line and has the same time period. Its **amplitude is R** which is defined as:

$$R = \sqrt{\left(R_x^2 + R_y^2\right)}$$

$$R = \sqrt{(A_1 + A_2 \cos\phi)^2 + (A_2 \sin\phi)^2}$$

$$R = \sqrt{(A_1^2 + A_2^2 \cos^2\phi + 2A_1A_2 \cos\phi + A_2^2 \sin^2\phi)}$$

$$R = \sqrt{(A_1^2 + A_2^2(\cos^2\phi + \sin^2\phi) + 2A_1A_2 \cos\phi)}$$

$$R = \sqrt{(A_1^2 + A_2^2(1) + 2A_1A_2 \cos\phi)}$$

$$R = \sqrt{(A_1^2 + A_2^2 + 2A_1A_2 \cos\phi)}$$

And its direction is,

$$\tan\theta = \frac{A_2 sin\varphi}{A_1 + A_2\cos\varphi}.$$

<u>Special Cases</u>

1. Constructive Interference

If $\boldsymbol{\phi} = \boldsymbol{0}$**,** the waves are in phase, resulting in a rise in the total amplitude to a certain degree from the combination of wave amplitudes.

$$\mathrm{R} = \sqrt{{A_1}^2 + {A_2}^2 + 2A_1A_2} = \sqrt{(A_1 + A_2)^2} = (A_1 + A_2),$$

And $\tan\theta = 0$,

which means that if displacements of two harmonic oscillators is equal then the resultant displacement is sum of two individual displacements.

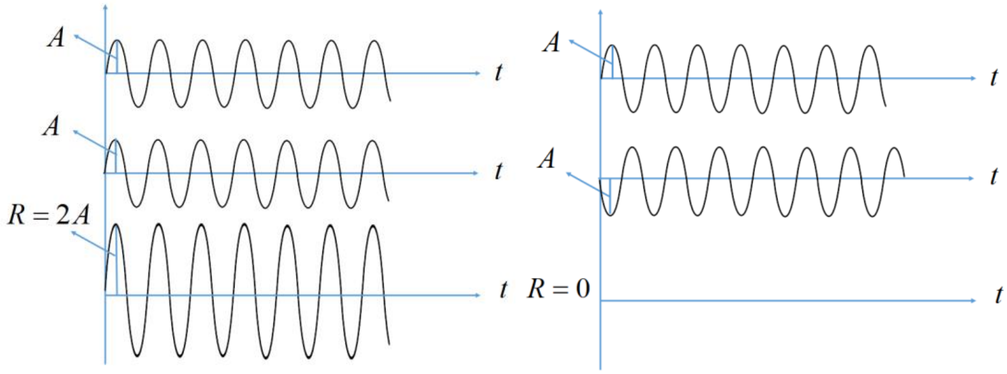


Figure 16. 4: Figure illustrates both constructive and destructive interference: the left side shows waves synchronizing in phase, resulting in reinforcement (constructive), while the right side displays waves being out of phase, causing them to nullify each other (destructive).

2. Destructive Interference

If $\boldsymbol{\phi = \pi}$, the waves are out of phase, their combined amplitudes result in a reduction of the overall amplitude to a certain extent resultant is given by

$$\mathrm{R} = \sqrt{{A_1}^2 + {A_2}^2 - 2A_1A_2} = \sqrt{(A_1 - A_2)^2} = (A_1 - A_2),$$

and $\tan\theta = 0$,

and this is the case of destructive interference. The resultant will be zero, if the amplitude of both waves are equal.

<u>Composition of Two Simple Harmonic Oscillator at Right angles to each other</u>

Until this point, we have only discussed the one-dimensional simple harmonic oscillator, which implies motion along a single direction. However, a particle can also move in two or three dimensions. Therefore, let's now turn our attention to the two-dimensional simple harmonic oscillator.

Let's consider two simple harmonic motions with different amplitudes i.e.

$$\mathrm{x} = \mathrm{A}\sin\omega\mathrm{t} \quad \Longrightarrow \quad \frac{\mathrm{x}}{A} = \sin\omega\mathrm{t} \tag{7}$$

and

$$\mathrm{y} = \mathrm{B}\sin(\omega\mathrm{t} + \phi) \;\Longrightarrow\; \frac{\mathrm{y}}{\mathrm{B}} = \sin(\omega\mathrm{t} + \phi) \tag{8}$$

Using eq. 7 in the trigonometric relation $\cos\theta = \sqrt{1 - sin^2\theta}$, we can write

$$\cos\omega t = \sqrt{1 - \frac{x}{A^2}^2}. \qquad (9)$$

Similarly, by using another trigonometric identity $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ in eq. 8 we can write

$$\frac{y}{B} = \sin\omega t\cos\phi + \cos\omega t\sin\phi \qquad (10)$$

Substituting the values of $\cos\omega t$ and $\sin\omega t$ from eq. 9 and from eq. 7 into eq. 10 we get

$$\frac{y}{B} = \frac{x}{A}\cos\phi + \sqrt{1 - \frac{x}{A^2}^2}\sin\phi$$

$$\frac{y}{B} - \frac{x}{A}\cos\phi = \sqrt{1 - \frac{x}{A^2}^2}\sin\phi$$

Squaring both sides $\therefore (a-b)^2 = a^2 + b^2 - 2ab$

$$\left(\frac{y}{B} - \frac{x}{A}\cos\phi\right)^2 = \left(\sqrt{1 - \frac{x}{A^2}^2}\sin\phi\right)^2$$

$$\frac{y^2}{B^2} + \frac{x^2}{A^2}\cos^2\phi - 2\frac{y}{B}\frac{x}{A}\cos\phi = \left(1 - \frac{x^2}{A^2}\right)\sin^2\phi$$

$$\therefore \cos^2\phi = 1 - \sin^2\phi$$

$$\frac{y^2}{B^2} + \frac{x^2}{A^2}(1 - \sin^2\phi) - 2\frac{y}{B}\frac{x}{A}\cos\phi = \left(1 - \frac{x^2}{A^2}\right)\sin^2\phi$$

$$\frac{y^2}{B^2} + \frac{x^2}{A^2} - \frac{x^2}{A^2}\sin^2\phi - 2\frac{y}{B}\frac{x}{A}\cos\phi = \left(1 - \frac{x^2}{A^2}\right)\sin^2\phi$$

$$\frac{y^2}{B^2} + \frac{x^2}{A^2} - 2\frac{y}{B}\frac{x}{A}\cos\phi = \left(1 - \frac{x^2}{A^2}\right)\sin^2\phi + \frac{x^2}{A^2}\sin^2\phi$$

$$\frac{y^2}{B^2} + \frac{x^2}{A^2} - 2\frac{y}{B}\frac{x}{A}\cos\phi = \left(1 - \frac{x^2}{A^2} + \frac{x^2}{A^2}\right)\sin^2\phi$$

Thus, combined displacement of both harmonic motions is given by the relation

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - 2\frac{x}{A}\frac{y}{B}\cos\phi = \sin^2\phi \qquad (11)$$

Special Cases

Case I

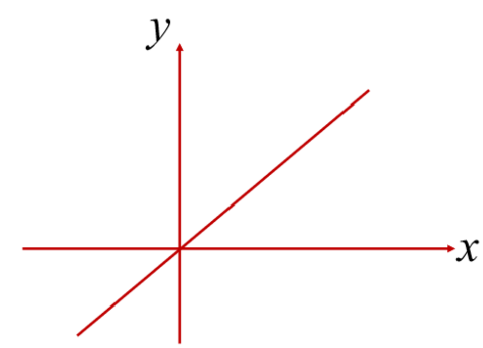
If $\phi = 0$, eq (11) becomes $\therefore Sin\ 0^o = 0,\ Cos\ 0^o = 1$

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - 2\frac{x}{A}\frac{y}{B} = 0$$

It is a special condition in which the ellipse degenerates (i.e. width of one axis becomes zero).

$$\left(\frac{x}{A} - \frac{y}{B}\right)^2 = 0$$

$$\frac{x}{A} - \frac{y}{B} = 0$$



Rearranging the above equation

$$y = \left(\frac{B}{A}\right)x$$

This is the equation of a straight line (y = mx + c) with slope c $= \frac{B}{A}$. Thus, the resultant motion is a S. H. M. along a straight line passing through the origin.

Case II

If $\phi = \pi$, then eq. 11 becomes

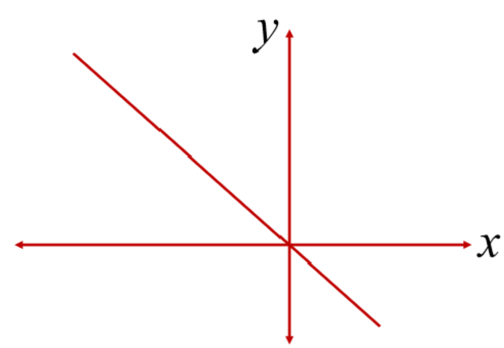
$\therefore$ *Sin 180^o = 0, Cos 180^o = -1*

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} + 2\frac{x}{A}\frac{y}{B} = 0$$

$$\left(\frac{x}{A} + \frac{y}{B}\right)^2 = 0$$

$$\frac{x}{A} + \frac{y}{B} = 0$$

$$y = -\left(\frac{B}{A}\right)x$$



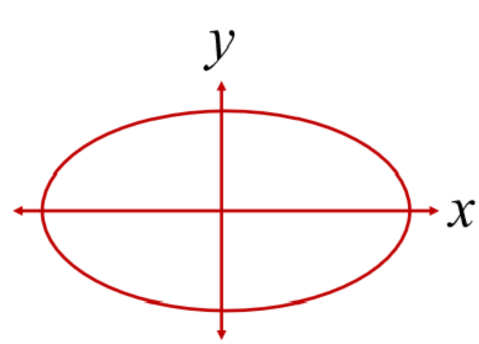
This is also the equation of straight line with negative slope c $= -\frac{B}{A}$.

Case III

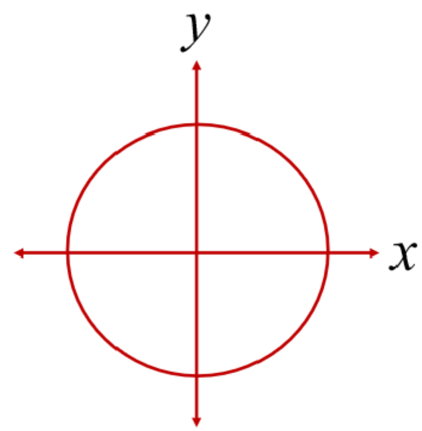
If $\phi = \frac{\pi}{2}$ (for A > B), we get

$\therefore$ *Sin 90^o = 1, Cos 90^o = 0*

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1,$$



which is the equation of an ellipse.



Case IV

If $\phi = \frac{\pi}{2}$ (for A = B), we get

$\because$ *Sin 90º = 1, Cos 90º = 0*

$$x^2 + y^2 = A^2$$

which is a circle with radius A.

Lissajous Figures

If two oscillations of different frequencies at right angles are combined, the resulting motion is more complicated. It is not even periodic unless the two frequencies are in the ratio of integers. This resulting curve is called Lissajous figures.

$$\frac{\omega_x}{\omega_y} = \text{integers} \Rightarrow \text{periodic motion}$$

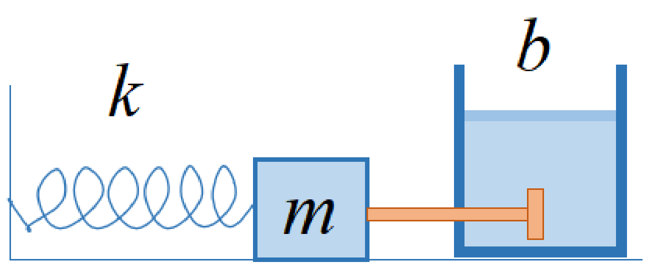
Such a curve can be represented by the following pair of equations.

$$x = A \sin \omega_x t$$
$$y = B \sin \left(\omega_y t + \phi \right)$$

Damped Harmonic Motion

Damped harmonic motion refers to the motion of a system that is subjected to both a restoring force and a damping force. The restoring force tends to return the system to its equilibrium position, while the damping force tends to reduce the amplitude of the oscillation over time. This reduction of amplitude is due to the damping force. There are many cases of damping force including friction, air resistance, and internal forces.

Let us consider a simple model of damped oscillator in which the block slides on a friction-less surface.



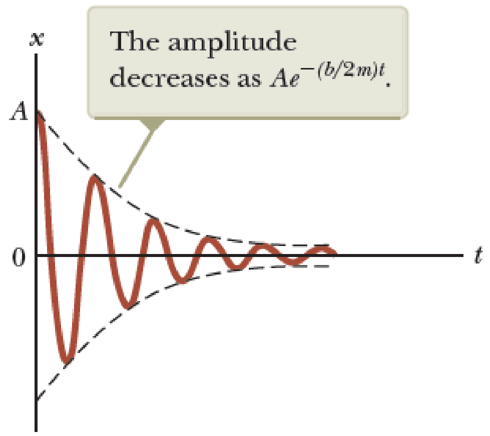


Figure 16. 5: A representation of damped harmonic oscillator.

The damping force is represented in terms of a (mass-less) vane that moves in a viscous fluid. This damping force due to fluid is $-bv_x$, where b is a positive constant called the damping constant that depends on the properties of the fluid and the size and shape of the object. With $\sum F_x = -kx - bv_x$, Newtons second law gives,

$$-kx - bv_x = ma_x$$

with $v_x = \frac{dx}{dt}$ and $a_x = \frac{d^2x}{dt^2}$ the above equation becomes,

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0. \qquad (12)$$

The solution of above equation is,

$$x(t) = x_m e^{\frac{-bt}{2m}} \cos\left(\omega' t + \varphi\right) \qquad (13)$$

Use the above equation to calculate $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$. Then by putting the values of x, $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ in eq. 12 to calculate the value of ω as

$$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)}.$$

Figure 16. 6: Graph of displacement versus time showing decrease in amplitude with time for a damped oscillator.

The above solution is same as we obtain in the case of simple and physical pendulum except the extra term $e^{\frac{-bt}{2m}}$ which is due to damping force. This solution assumes that the damping constant is small so that the quantity under the square root in eq. 16.40 cannot be negative.

Figure shows the position as a function of time for an object oscillating in the presence of a retarding force. When the retarding force is small, the oscillatory character of the motion is preserved but the amplitude decreases exponentially in time, with the result that the motion ultimately becomes undetectable. Any system that behaves in this way is known as a **damped oscillator.** The dashed black lines in Figure, which define the *envelope* of the oscillatory curve, represent the exponential factor in Equation 13. This envelope shows that the amplitude decays exponentially with time.

When the magnitude of the retarding force is small such that $\frac{b}{2m} < \frac{k}{m}$, the system is said to be under-damped**.**

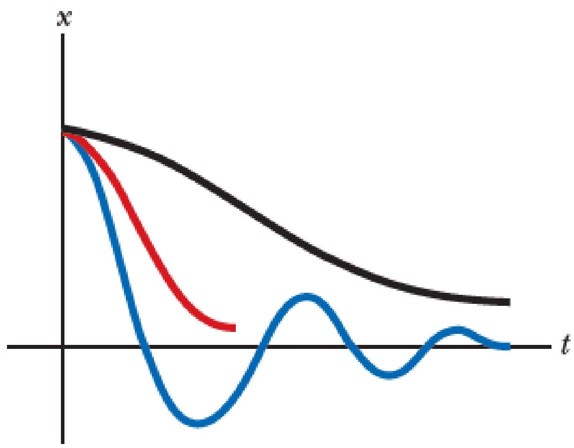


Figure 16. 7: Displacement versus time graphs of an underdamped oscillator (blue line), a critically damped oscillator (red), and an overdamped oscillator (black).

As the value of b increases, the amplitude of the oscillations decreases more and more rapidly. When b reaches a critical value b_c such that $^{b_c}/_{2m} = \frac{k}{m}$,the system does not oscillate and is said to be **critically damped.** In this case, the system, once released from rest at some non-equilibrium position, approaches but does not pass through the equilibrium position. The graph of position versus time for this case is the red curve in Figure.

If the medium is so viscous that the retarding force is large compared with the restoring force—that is, if $\frac{b}{2m} > \frac{k}{m}$ the system is over-damped**.** Again, the displaced system, when free to move, does not oscillate but rather simply returns to its equilibrium position. As the damping increases, the time interval required for the system to approach equilibrium also increases as indicated by the black curve in Figure.

<u>Forced Oscillations and Resonance</u>

An oscillating system undergoes **forced oscillations** whenever it is pushed periodically by an external force. The frequency at which a system would oscillate if neither a driving nor a damping force are present is known as the **natural frequency.**

Resonance is the phenomenon that occurs when the driving frequency ω is equal to the natural frequencyω_o and the forced oscillations reach their maximum displacement amplitude, when the damping is negligible.

$$\omega = \omega_o$$

Where $$\omega_o = \sqrt{\frac{k}{m}}$$

The corresponding frequency is called the **resonant angular frequency.**

We take the driving force to be $F_o cos\boldsymbol{\omega}t$. Newtons second law gives,

$$\sum F_x = -kx + F_o \cos \omega t$$

$$-kx + F_o \cos \omega t = ma_x$$

with $v_x = \frac{dx}{dt}$ and $a_x = \frac{d^2x}{dt^2}$ the above equation becomes,

$$m\frac{d^2x}{dt^2} + kx = F_o \tag{14}$$

The solution of above equation is,

$$x(t) = \frac{F_o}{m(\omega_o^2-\omega^2)} cos\ \ \omega' t \tag{15}$$

Use eq. 15 to calculate $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$. Then by putting the values of x and $\frac{d^2x}{dt^2}$ in eq. 14 we can verify the solution of eq. 14 i.e.

$$\frac{dx(t)}{dt} = \frac{(-\omega)F_o}{m(\omega_o^2-\omega^2)} sin\ \ \omega t \tag{16}$$

$$\frac{d^2x(t)}{dt^2} = -\frac{(-\omega^2)F_o}{m(\omega_o^2-\omega^2)} cos\ \ \omega t \tag{17}$$

After substituting eq. 16 and eq. 17 in eq. 14 we get

$$m\left(\frac{(-\omega^2)F_o}{m(\omega_o^2-\omega^2)} cos\ \ \omega t\right) + k\frac{F_o}{m(\omega_o^2-\omega^2)} cos\ \ \omega t = F_o \cos \omega t$$

$$\frac{(-\omega^2)mF_o + kF_o}{m(\omega_o^2-\omega^2)} cos\ \ \omega t = F_o \cos \omega t \tag{18}$$

Note that the amplitude becomes infinitely large when ω approaches ω_o. This occurs due to the absence of a damping term in the equation. However, when damping is present, the amplitude remains large as ω approaches ω_o but remains finite. All mechanical structures such as buildings, bridges, and airplanes have one or more natural frequencies of oscillation. If the structure is subject to a driving frequency that matches one of the natural frequencies, the resulting large amplitude of oscillation can have disastrous consequences. The collapse of roadways and bridges in earthquakes is a more serious outcome.

Physics-PHY101-Lecture 17

PHYSICS OF MATERIALS

Solids, liquids, gases, and plasma are the four basic states of matter. In a **solid**, particles are packed tightly together with strong intermolecular forces, which give them a definite shape and prevent their flow. The particles in a **liquid** are more loosely packed than they are in a solid and can flow around one another, giving the liquid an arbitrary shape. As a result, the liquid will take the shape of its container. The particles in a **gas** have large spaces between them and possess high kinetic energy. A gas lacks a fixed volume or shape. Highly charged particles with a very high kinetic energy make up **plasma**. Although it is not a common state of matter on Earth, it could be the most typical state of matter across the universe. In short, stars like the sun are hot balls of plasma.

Despite the fact that a solid's shape and volume might be believed to be fixed, these characteristics can be altered by external forces. The term ***deformation*** is used for the shape change caused by the application of force.

Elasticity

The property by virtue of which a body tends to regain its original shape and size when external forces are removed. If a body completely recovers its original shape and size , it is called perfectly elastic. Quartz, steel, and glass are very nearly elastic.

Plasticity

If a body has no tendency to regain its original shape and size , it is called perfectly plastic. Common plastics, kneaded dough, solid honey, etc. are plastics.

Stress

Stress characterizes the strength of the forces causing the stretch, squeeze, or twist. It is defined usually as force per unit area but may have different definitions to suit different situations.

$$stress = \frac{F}{A}$$

The SI unit of stress is $Nm^{-1.}$

Types of Stresses

We distinguish between three types of stresses:

a) **Longitudinal stress**

If the deforming force is applied along some linear dimension of a body, the stress is called ***longitudinal stress*** **or** ***tensile stress*** **or** ***compressive stress***.

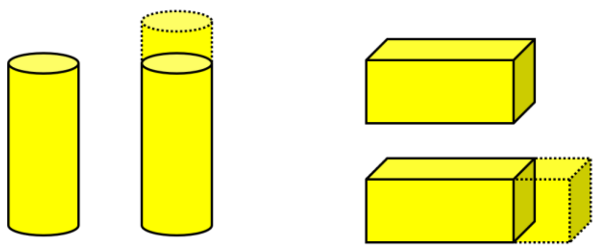


Figure 17. 1: Longitudinal stress causing deformation along the length of a material.

b) **Volume stress**

If the force acts normally and uniformly from all sides of a body, the stress is called ***volume stress.***

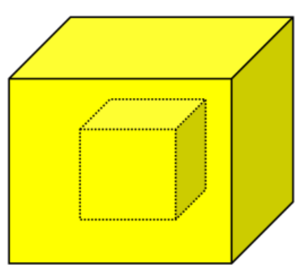


Figure 17. 2: Figure depicting volume stress, highlighting the uniform compression or expansion of a material in response to external forces acting in all directions within its volume.

c) **Shearing stress**

If the force is applied tangentially to one face of a rectangular body, keeping the other face fixed, the stress is called ***tangential or shearing stress***.

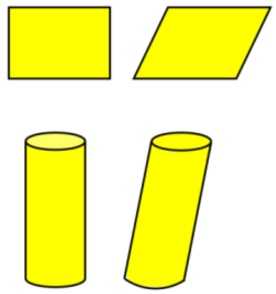


Figure 17. 3: Parallel force component that acts tangentially to a material's surface, causing deformation or slippage between adjacent layers illustrating shear stress.

<u>Strain</u>:

When deforming forces are applied on a body, it undergoes a change in shape or size. The fractional (or relative) change in shape or size is called the strain.

$$\boldsymbol{Strain = \frac{change\ in\ dimension}{original\ dimension}}$$

Strain is a ratio of similar quantities, so it has no units.

<u>Types of Strain</u>

There are 3 different kinds of strain:

i. ***Longitudinal (linear) strain*** is the ratio of the change in length (Δl) to original length (l), i.e.,

$$\boldsymbol{Linear\ strain = \frac{\Delta l}{l}}$$

ii. ***Volume strain*** is the ratio of the change in volume (ΔV) to original volume (V)

$$\boldsymbol{Volume\ strain = \frac{\Delta V}{V}}$$

iii. ***Shearing strain*** the angular deformation (θ) in radians is called shearing stress.

For a smaller angle θ,

$$\boldsymbol{Shearing\ strain \equiv \theta \approx tan\theta = \frac{\Delta x}{l}}$$

The above equation shows that the sheared face moves horizontally across a distance of Δx, while the object's height is given by *l*.

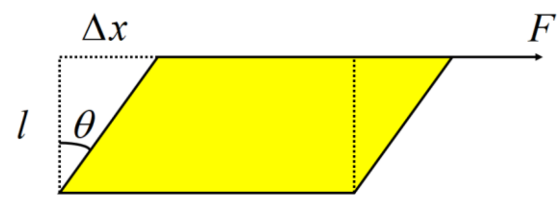


Figure 17. 4: Angular distortion or deformation experienced by a material in response to shear stress, typically represented by the change in angle between initially perpendicular lines within the material.

Hooke's Law

For small deformations, stress is proportional to strain.

$$\textbf{Stress} \propto \textbf{Strain}$$

$$\text{Stress} = \text{E} \times \text{Strain}$$

$$\text{E} = \frac{\text{stress}}{\text{strain}}$$

The constant E is called the modulus of elasticity. It can be interpreted as a measure of a material's stiffness: A material with a high elastic modulus is highly rigid and challenging to deform. Thus, the material being deformed, and the type of deformation affect the constant of proportionality. E has the same units as stress because strain is dimensionless.

There are three moduli of elasticity.

(a) Young's modulus

It is the measure of a solid's resistance to change in length. Consider a rod of original length 'l' and cross-sectional area 'A'. Under the influence of external force F, the internal forces in the rod that resist deformation are balanced by the applied force, resulting in a change in length of the rod by the factor Δl.

Young's modulus (Y) for linear strain is given by the following relation.

$$Y = \frac{longitudinal\ stress}{longitudinal\ strain} = \frac{F/_A}{\Delta l/_l}$$

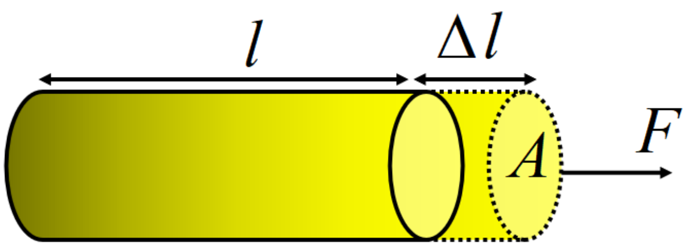
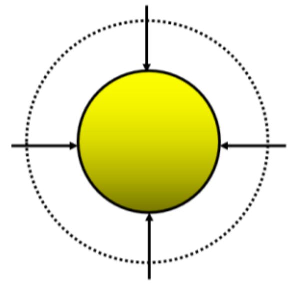


Figure 17. 5: Elastic deformation under tensile or compressive force F.

Bulk Modulus

It determines how resistant solids or liquids are to volume changes. Let a body of volume V be subjected to a uniform pressure on its entire surface, causing a change in volume ΔV and pressure ΔP. Then, bulk modulus (B) for volume strain is given by

$$B = \frac{Volume\ Stress}{Volume\ Strain} = \frac{\Delta F/_A}{\Delta V/_V} = -\frac{\Delta P}{\Delta V/_V}$$



In the above equation, a negative sign is added to make 'B' a positive number. The need for this action arises from the fact that an increase in pressure (positive ΔP) results in a decrease in volume (negative ΔV), and vice versa. 1/B is called **compressibility**. A material having a small value of B can be compressed easily.

(c) Shear Modulus

It evaluates how resistant a solid's sliding planes are to motion. Let a force F produce a strain as in the figure 17.4. It is also known as modulus of rigidity. Then, shear modulus (η) for shearing strain is

$$\eta \equiv \frac{Shering\ stress}{Shearing\ strain} = \frac{F/_A}{\theta} = \frac{F}{A\tan\theta} = \frac{Fl}{A\Delta x}$$

Poisson's Ratio (σ)

When a wire is stretched, its length increases and radius decreases. The ratio of the lateral strain (the ratio of the change in diameter of a circular bar caused by longitudinal deformation to the original diameter of the material) to the longitudinal strain is called Poisson's ratio,

$$\boldsymbol{\sigma = \frac{\Delta r/r}{\Delta l/l}}$$

Its value lies between 0 and 0.5.

Work done in Stretching a Wire

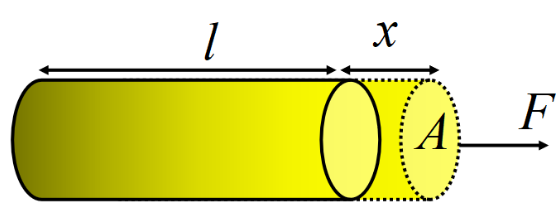
We can calculate the work done by stretching a wire. Obviously, we must do work against the internal restoring force (see fig 17.5). If *x* is the extension (i. e Δl = x) produced by the force F in a wire of length *l*, then

$$\boldsymbol{Y = \frac{longitudinal\ stress}{longitudinal\ strain} = \frac{F/A}{x/l}}$$

$$\boldsymbol{F = \frac{YA}{l}x \qquad (1)}$$

The work done in extending the wire through *Δl* is given by,

$$\boldsymbol{W = \int_0^{\Delta l} F dx}$$



Putting the value of F from eq 1

$$\boldsymbol{W = \frac{YA}{l}\int_0^{\Delta l} x dx = \frac{YA}{l}\left|\frac{x^2}{2}\right|_0^{\Delta l}}$$

Applying limits

$$\boldsymbol{W = \frac{YA}{l}\frac{(\Delta l)^2}{2}}$$

Rearrange the above equation after multiplying and dividing by '*l*'

$$\boldsymbol{W = \frac{1}{2}(Al)\left(\frac{Y\Delta l}{l}\right)\frac{\Delta l}{l} = \frac{1}{2} \times volume \times stress \times strain}$$

Now, we can calculate the amount of energy stored in a unit volume of wire by dividing both sides by that volume.

$$\boldsymbol{Work\ per\ unit\ volume = \frac{1}{2} \times stress \times strain}$$

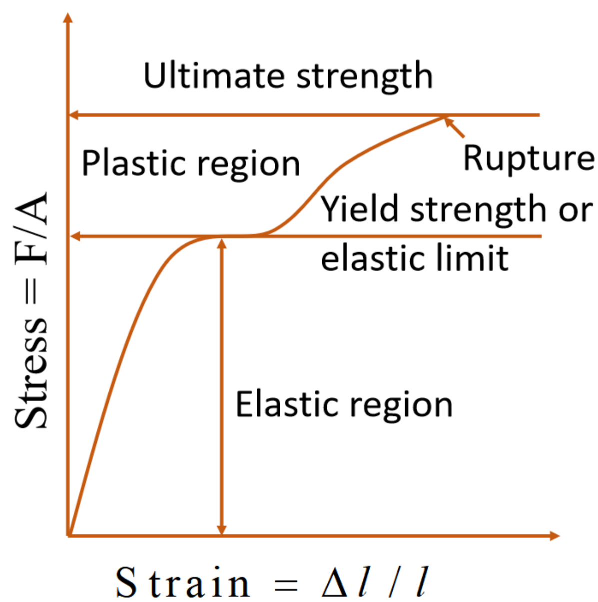
We can also write this as,

$$\boldsymbol{W = \frac{1}{2}\left(\frac{YA\Delta l}{l}\right)\Delta l = \frac{1}{2} \times load \times extension}$$

Behavior of wire under Stress

Figure illustrates how a sufficiently high stress might cause a material to deviate from its elastic limit. The stress-strain curve begins as a straight line in the elastic region till it reaches the ***proportional limit*** (maximum stress endured by the material without losing straight line proportionality between stress and strain). However, as the stress rises, the curve becomes curved.

Figure 17. 6: Graph illustrating the stress-strain curve for a wire, representing the relationship



between strain, and resulting stress, depicting the wire's behavior under increasing loads until it reaches its elastic limit or point of failure.

The highest stress that may be applied to a material before it undergoes permanent deformation is known as the ***elastic limit*** of that material. The object is permanently deformed and does not recover to its original shape when the stress has been removed when the stress surpasses the elastic limit. As a result, the object's shape changes permanently. ***Ultimate strength*** (the maximum stress that a substance can withstand before breaking) is exceeded as the stress is increased more.

Brittle materials have a ***breaking point*** that lies just above their maximum strength. After reaching their maximum strength, ductile metals like copper and gold begin to thin and stretch at lower stress levels before breaking.

Fluid Statics / Fluids at rest

A fluid is a substance that can flow and does not have a shape of its own. Thus all liquids and gases are fluids. Solids possess all the three moduli of elasticity whereas a

fluid possess only the bulk modulus (B). A fluid at rest cannot sustain a tangential force. If such force is applied to a fluid, the different layers simply slide over one another. Therefore the forces acting on a fluid at rest have to be normal to the surface. This implies that the free surface of a liquid at rest, under gravity, in a container, is horizontal.

The normal force per unit area is called **pressure**

$$\boldsymbol{P} = \frac{\Delta \boldsymbol{F}}{\Delta \boldsymbol{A}}$$

Pressure is a scalar quantity. Its **unit** is Newtons/metre^2 , or Pascal (Pa).

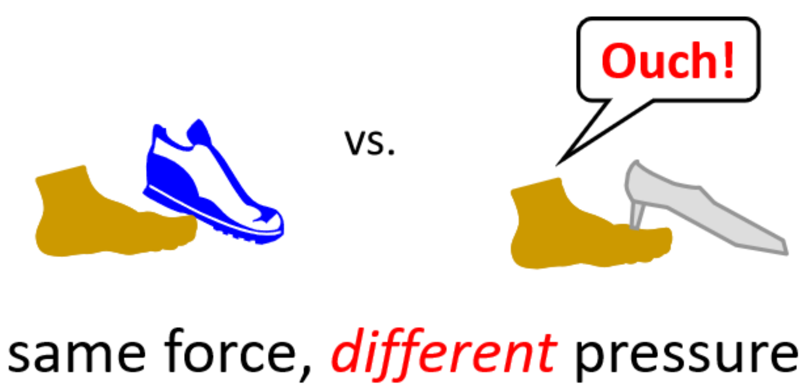


Figure 17. 7: Variation in pressure distribution across areas of contact due to differences in size or shape.

The **density** of a substance is the ratio of its mass to that of its volume.

$$\rho = \frac{\Delta m}{\Delta V}$$

Where Δm is the mass of a small piece of the material and ΔV is the volume it occupies. Density has no directional property and is a scalar quantity. Units for density are kgm^{-3}. If the density is uniform then

$$\rho = \frac{m}{V}$$

The density of water is 1000 kg/m^3.

Variation of pressure in fluid at rest

To calculate how the pressure in a fluid changes with depth, Let's take a fluid of density ρ at rest that is exposed to the environment. Consider a small element dy in the form of a disc of fluid volume submerged within the body of the fluid. The pressure on the bottom face of the cylinder caused by the external liquid is p, while the pressure on the top face of the cylinder is caused by air pressure p+dp. Therefore, the upward force applied to the bottom of the cylinder by the external fluid is

$$F = pA$$

and the downward force applied to the top by the atmosphere.

$$F = (p + dp)A$$

The mass of liquid in disc is

$$dm = \rho dV = \rho A dy$$

and its weight is

$$(dm)g = \rho A dy g$$

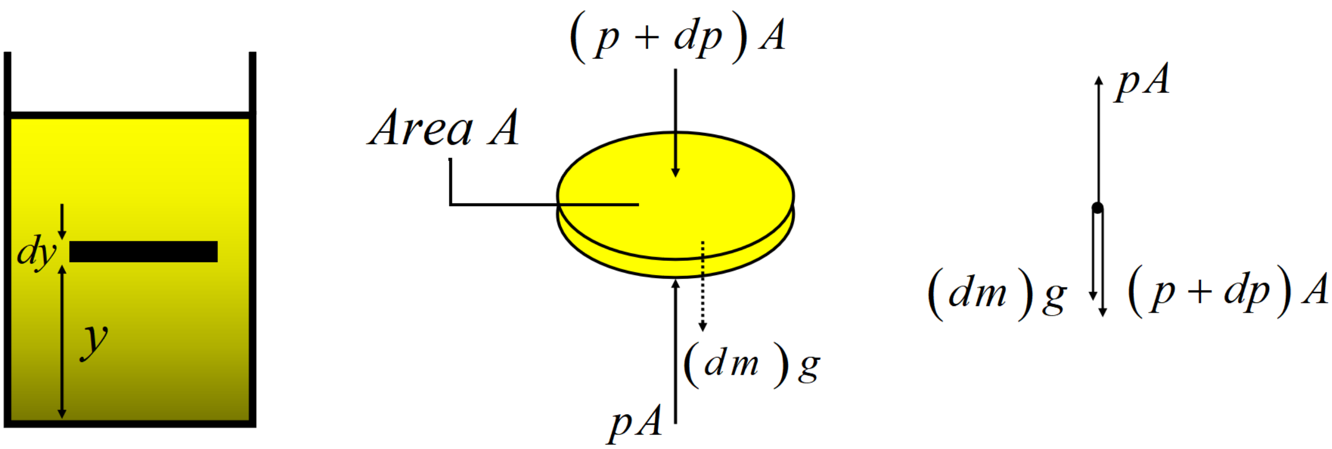


Figure 17. 8: Figure illustrating the variation of pressure on a small fluid volume segment at rest, the forces acting upon the segment and a free body diagram of the segment, highlighting how pressure changes with depth.

The resultant horizontal force is zero, because the element has no horizontal acceleration. The horizontal forces are due only to the pressure of the fluid, and by symmetry, the pressure must be the same at all points within a horizontal plane at y.

The fluid element is in equilibrium, so the resultant vertical force on it must be zero:

$$\sum \boldsymbol{F_y} = \mathbf{0}$$

Now, let us require that the sum of the forces on the fluid element is zero

$$\mathrm{pA} - (\mathrm{p} + \mathrm{dp})\mathrm{A} - \rho\mathrm{gAdy} = 0$$

$$\mathrm{pA} - \mathrm{pA} - \mathrm{dpA} - \rho\mathrm{gAdy} = 0$$

$$-\mathrm{dpA} - \rho\mathrm{gAdy} = 0$$

$$-\mathrm{dpA} = \rho\mathrm{gAdy}$$

$$\frac{\mathrm{dp}}{\mathrm{dy}} = -\rho\mathrm{g}$$

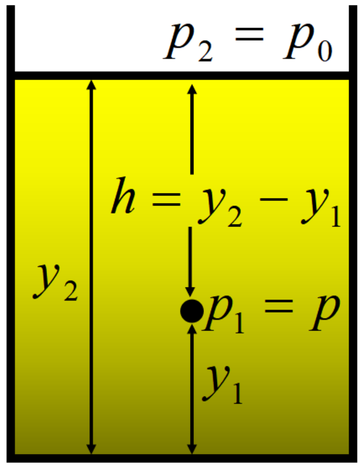


Figure 17. 9: A volume of a liquid with an exposed top surface is contained in a container. The depth h determines the pressure in the liquid at any given time.

This is the equation of hydro-static equilibrium. Note that we are taking the origin (0) at the bottom of the liquid. Therefore, as the elevation increases (*dy positive*), the pressure decreases (*dy negative*). The quantity ρg is the *weight per unit volume* of the fluid. For liquids, which are nearly in-compressible, ρ is practically constant.

$$\therefore \boldsymbol{\rho g = constant}$$

$$\frac{\boldsymbol{dp}}{\boldsymbol{dy}} = \frac{\boldsymbol{\Delta p}}{\boldsymbol{\Delta y}} = \frac{\boldsymbol{p_2 - p_1}}{\boldsymbol{y_2 - y_1}} = \boldsymbol{-\rho g}$$

$$\boldsymbol{p_2 - p_1 = -\rho g(y_2 - y_1)}$$

This equation is for homogeneous liquid (constant density).

If liquid has a free surface, then this becomes the natural level from where we can measure the distances. Therefore

$$\boldsymbol{p_2 - p_1 = -\rho g(y_2 - y_1)}$$

The above equation becomes

$$\boldsymbol{p_0 - p = -\rho g(y_2 - y_1)}$$

But y_2 - y_1 = h, therefore

$$p = p_0 + \rho g h$$

This shows at the bottom of container is greater than at surface. Also the pressure is independent of the container's shape. The depth below the surface is the only factor affecting the pressure.

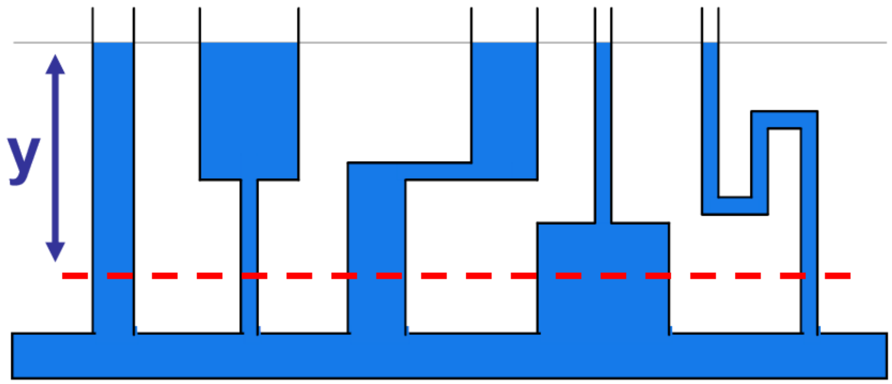
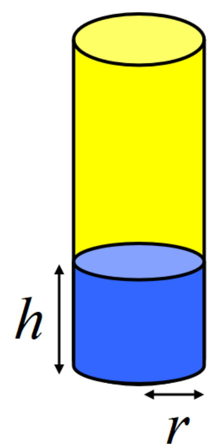


Figure 17. 10: The fluid level is the same everywhere in a connected container. As pressure only depends upon y. As the level of liquid is the same, the pressure will also be same throughout the container.

<u>Example</u>

To what height should a cylindrical vessel be filled with a homogeneous liquid to make the force with which the liquid presses on the sides of the wall equal to the force exerted by the liquid on the bottom of the vessel?



$$\text{Pressure at the bottom of the vessel} = \rho g h$$

$$\text{Area of the bottom face} = \pi r^2$$

By the definition of pressure,

$$\text{pressure } = \frac{F}{A}$$

$$\text{Force exerted by the liquid on the bottom} = (hg\rho)\pi r^2$$

As the pressure varies from top to bottom, we take an average value of pressure on the walls of the vessel.

$$\text{Mean pressure on the wall} = \frac{1}{2}(hg\rho)$$

$$\text{Area of the wall in contact with the liquid} = 2\pi rh$$

$$\text{Force exerted by the liquid on the wall} = \left(\frac{1}{2}hg\rho\right)2\pi rh$$

Since the two forces are equal, hence

$$\left(\frac{1}{2}\boldsymbol{hg\rho}\right)\boldsymbol{2\pi rh} = (\boldsymbol{hg\rho})\boldsymbol{\pi r^2}$$

$$\rightarrow \boldsymbol{h = r}$$

Thus the liquid should be filled up to the height equal to the radius of the cylinder.

Pascal's Principal

Pressure applied to an enclosed fluid is transmitted to every portion of the fluid and to the walls of the containing vessel.

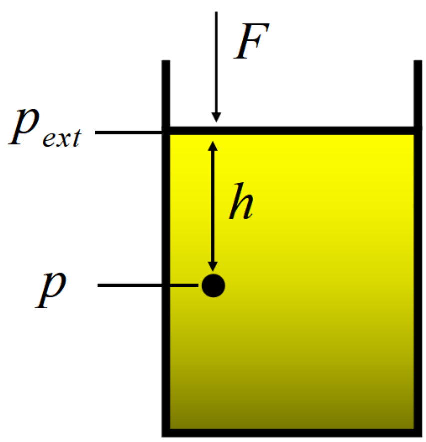


Figure 17. 11: Fluid contained in a cylinder with a moving piston. Every point P experiences pressure because of the force applied by the piston in addition to the weight of the fluid above P.

In other words, if you apply an amount Δp of external pressure to a fluid at one spot, the fluid will experience the same amount of rise in pressure throughout. For an in-compressible liquid, we will demonstrate Pascal's principle. Figure depicts the fluid within a cylinder with a piston. For example, the weight of various objects put on top of the piston can exert an external force on it. An external pressure 'p_{ext}' is exerted on the liquid directly beneath the piston as a result of the external force. The pressure at any location 'P' at a depth 'h' below the surface can be written as follows if the liquid has a density 'ρ':

$$\boldsymbol{p = p_{ext} + \rho g h}$$

Assume that the external pressure has now been increased by an amount 'p_{ext}', by giving the piston greater weight. The fluid's pressure changes as a result of the change in external pressure.

$$\Delta \boldsymbol{p} = \Delta \boldsymbol{p_{ext}} + \Delta(\boldsymbol{\rho g h})$$

Since the liquid is incompressible, the density is constant i.e. $\Delta(\rho gh) = 0$. So,

$$\Delta \boldsymbol{p} = \Delta \boldsymbol{p_{ext}}$$

The change in pressure at any point in the fluid is simply equal to the change in the externally applied pressure. Although we derived the above result for incompressible liquids, Pascal's principle is true for all real (compressible) fluids, gases as well as liquids.

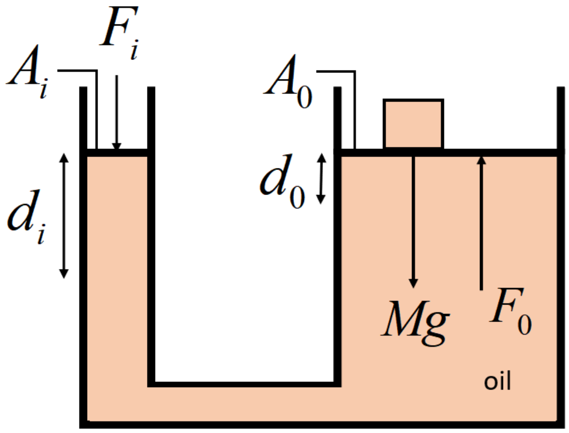
The Hydraulic Lever

An arrangement that is frequently used to lift a heavy object, such as an automobile, is shown in Figure 17.11. It works upon Pascal's principal. A piston with area A_i is subjected to an external force F_i. The larger piston's area A_o experiences a force Mg from the object that needs to be lifted. The larger piston must experience an upward force F_o equal to the weight of the object's weight Mg in order for the system to be in equilibrium. We are trying to determine how the "output force" F_o that the system is capable of applying to the larger piston relates to the applied force F_i.

The pressure on the Liquid at the smaller piston due to our externally applied force is

$$\mathbf{p_i} = \frac{\boldsymbol{F_i}}{\boldsymbol{A_i}}$$

Figure 17. 12: The hydraulics lever. A force exerted on the smaller piston can result in a



considerably greater force acting on the bigger piston, lifting a mass of Mg.

Pascal's principle states that this "input" pressure must equal the "output" pressure that the fluid applies to the larger piston.

$$p_0 = \frac{F_0}{A_0}$$

Therefore,

$$p_i = p_0$$

$$\frac{F_i}{A_i} = \frac{F_0}{A_0}$$

$$F_i = F_0\left(\frac{A_i}{A_0}\right) = \mathrm{Mg}\left(\frac{A_i}{A_0}\right) \ldots\ldots\ldots(1)$$

$$\because\ A_i \ll A_0 \Rightarrow \frac{A_i}{A_0} \ll 1$$

$$\because\ F_i \ll F_0 = Mg$$

The downward movement of the smaller piston through the distance 'd_i' displaces a volume of fluid (volume = area *height).

$$\boldsymbol{V = d_i A_i}$$

If the fluid is incompressible, then this volume must be equal to the volume displaced by the upward motion of the larger piston

$$V = d_i A_i = d_0 A_0$$

$$d_0 = d_i \left(\frac{A_i}{A_0}\right) \dots \dots \dots (2)$$

Now

$$\because \frac{A_i}{A_0} \ll 1 \Rightarrow \ d_o \ll d_i$$

Losing the ability to move anything far is the price we pay for being able to lift a heavy object. Using eq 1 and 2, we can get

$$F_i = F_0 \left(\frac{A_i}{A_0}\right) = F_0 \left(\frac{d_0}{d_i}\right)$$

$$F_i d_i = F_0 d_0$$

Therefore, work done by the external force on the smaller piston equals the work done by the fluid on the larger fluid.

Hydraulic Brakes

Pascal's law is also used by braking systems in cars, buses, and other vehicles. Equal pressure can be transmitted throughout the liquid due to the hydraulic brakes shown in figure. The master cylinder experiences force when the brake pedal is pressed, increasing the liquid pressure inside.

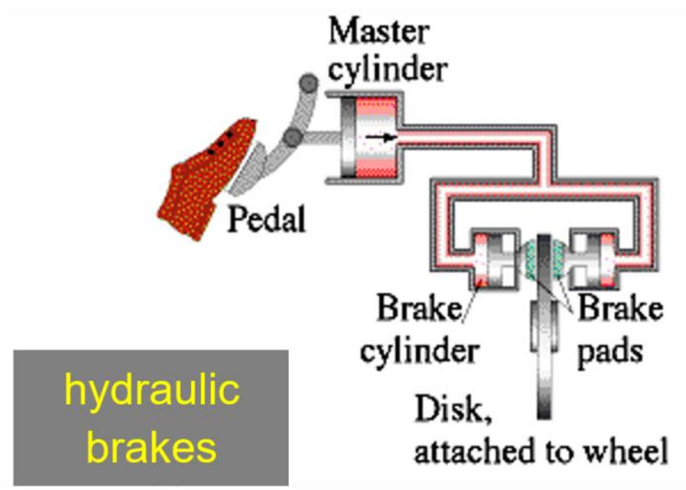


Figure 17. 13: A Schematics of hydraulic breaks

All of the pistons in other cylinders receive the same amount of liquid pressure via the liquid in the metal pipes. The pistons in the cylinders move outward, forcing the brake pads against the braking drums as a result of the rise in liquid pressure. The wheels are stopped by the force of friction between the brake pads and brake drums.

Measuring Pressure

Gauge pressure

The pressure relative to atmospheric pressure is known as gauge pressure. Gauge pressure is positive for pressures above atmospheric pressure and negative for pressures below it.

Gauge pressure = actual pressure – atmospheric pressure

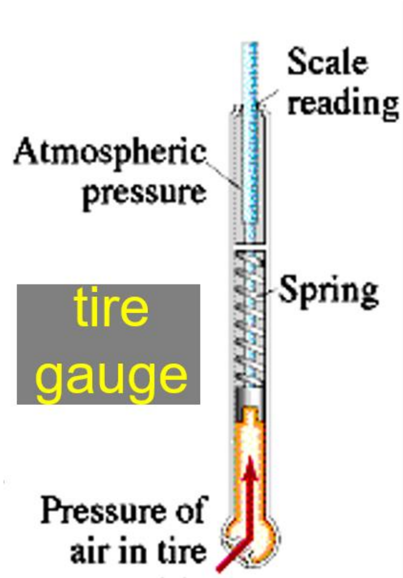


Figure 17. 14: Schematic illustrating gauge pressure, highlighting the pressure measurement relative to atmospheric pressure.

If the pressure inside a car tire is equal to the atmospheric pressure, the tire is flat. The inside pressure has to be greater than atmospheric pressure to support the car.

Manometer

The open tube manometer measures gauge pressure. The U-shaped tube often contains mercury or water. One end of the tube is exposed to the atmosphere, and the other is attached to the system whose pressure is to be determined.

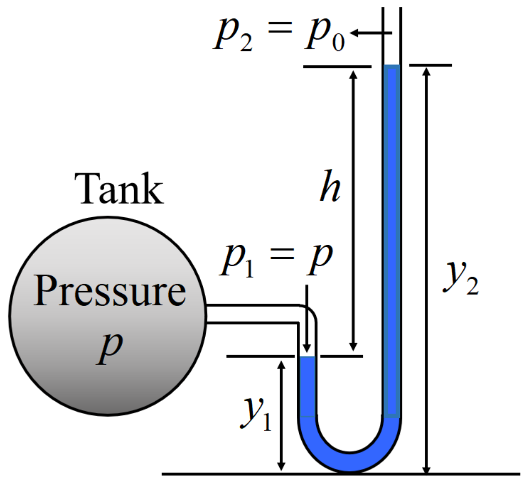


Figure 17. 15: A manometer, device used to measure fluid pressure differences, showing the column height variation between the two arms, indicating the pressure differential between the connected systems or points."

The pressure at the bottom of the tube due to the fluid in the left column is $p_o + \rho g y_2$. And the pressure at the bottom of the tube due to the fluid in the right column is $p + \rho g y_1$. These pressures are measured at the same point so they must be equal

$$\boldsymbol{p + \rho g y_1 = p_0 + \rho g y_2}$$

$$\boldsymbol{p - p_0 = +\rho g (y_2 - y_1) = \rho g h}$$

Thus the gauge pressure $p - p_o$ is proportional to the difference in the height of the liquid column in the U-tube.

Mercury barometer

Evangelista Torricelli (1608–1647) created the barometer, a device used to measure pressure. The mercury barometer is made of a long glass tube that has been filled with mercury and placed upside down into a dish of mercury, as shown in Fig.

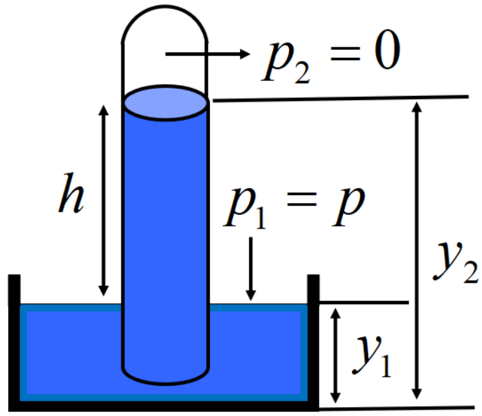


Figure 17. 16: A mercury barometer

Since only mercury vapor exists in the area above the mercury column at normal temperatures, it is effectively a vacuum that may be ignored. The pressure p_1 on the surface of mercury dish corresponds to the unmeasured pressure p.

i. e., $p_2 = 0$

$p_1 = p$ is the unknown pressure.

Therefore

$$p_2 - p_1 = -\rho g(y_2 - y_1)$$

$\because h = y_2 - y_1$

$$0 - p = -\rho g h$$

$$p = \rho g h$$

The pressure is then determined by measuring the height of the column above the dish's surface. The mercury barometer is most often used for measuring the pressure of atmosphere. The pressure of 1 atmosphere (1 atm) is equivalent to that exerted by a column of mercury of height 760 mm at 0^o C under standard gravity (g = 9.90665 m/s^2). The density of mercury at this temperature is 1.35955×10^4 kg/m^3.

$$p_0 = \rho g h$$

$$1\ atm = \left(1.35955 \times 10^4\ \frac{kg}{m^3}\right)\left(9.90665\frac{m}{s^2}\right)(0.76m)$$

$$\mathbf{1\ atm = 1.013 \times 10^5 \frac{N}{m^2} = 1.013 \times 10^5\ Pa}$$

Sometimes, barometer values are expressed in torr, where 1 torr equals the pressure produced by a mercury column 1 mm high. Thus

$$\mathbf{1\ torr = \left(1.35955 \times 10^4 \frac{kg}{m^3}\right)\left(9.90665 \frac{m}{s^2}\right)(0.001m)}$$

$$\mathbf{= 133.326Pa}$$

<u>Archimedes' principle</u>

A body wholly or partially immersed in a fluid is buoyed up by a force (or upthrust) equal in Magnitude to the weight of the fluid displaced by the body.

This net upward force is called the *buoyant force* or *buoyancy*. It is possible to show that the weight of the displaced fluid is equal to the pressure differential between the object's upper and lower sides, which is the physical cause of the buoyant force.

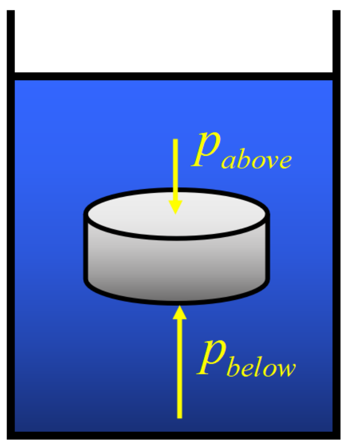


Figure 17. 17: Pressure is applied to the object's sides by the fluid around it. Because pressure increases with depth, the arrows on the underside are bigger than those on the top.

In Figure, the fluid around the object exerts pressure on all sides. The forces generated by the pressure are indicated by arrows. The arrows on the underside are larger than those on top because pressure rises with depth. The horizontal components cancel when they are all added together, but there is a net force upwards. This force, due to differences in pressure, is buoyant force B.

$$\because\ \boldsymbol{p_{below} > p_{above}}$$

$$\therefore \frac{F_{below}}{A} > \frac{F_{above}}{A}$$

$$F_{below} > F_{above}$$

$$F_b = F_{below} - F_{above} > 0$$

This net upward force F_b is called the buoyant force or buoyancy.

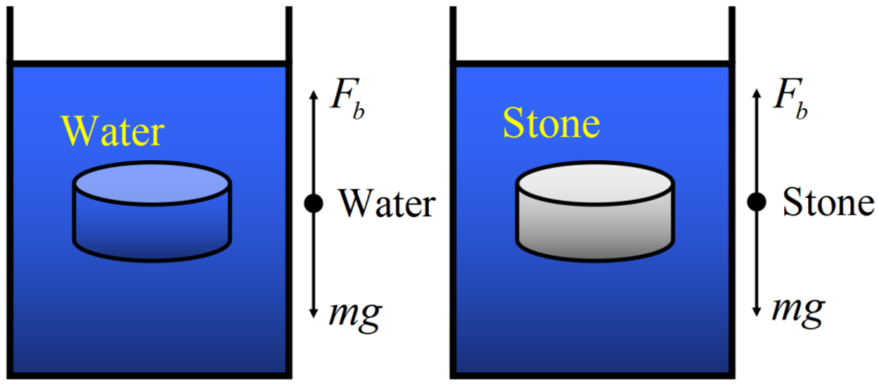


Figure 17. 18: Figure shows the submerged water segment and stone of same volume and the magnitudes of the corresponding forces (F_b and mg) acting on them.

Pressure exerted on a submerged object by the surrounding fluid certainly cannot depend on the material of which the object is made. Therefore, this force acts through the center of gravity of the displaced fluid, called the center of buoyancy. Thus a body appears to weigh less when immersed in a fluid

Apparent Weight = True Weight - Upthrust

Suppose a body of the volume V and density ρ is fully immersed in a liquid of density ρ', Then

$$Weight\ of\ the\ body = W = \rho g V$$

$$Weight\ of\ the\ liquid\ displaced = W' = \rho' g V$$

Net downward force or the apparent weight is

$$W_a = (\rho - \rho')gV = \rho g V\left(1 - \frac{\rho'}{\rho}\right)$$

$$W_a = (\rho - \rho')gV = W\left(1 - \frac{\rho'}{\rho}\right)$$

The following possibilities may occur:

1. If $\rho' < \rho$, $W_a > 0$. Therefore, the body will sink to the bottom
2. If $\rho' = \rho$, $W_a = 0$. Therefore, the body will just float or remain hanging at whatever height it is left inside the liquid.
3. If $\rho' > \rho$, the upthrust will be greater than the weight of the body. Therefore, the body will move partly out of the free surface of the liquid until the upthrust becomes equal to W. The body will then float. Thus the principle of floatation is:

"For a body to float in a liquid, the weight of the liquid displaced by the immersed portion of the body must be equal to its own weight".

Physics-PHY101-Lecture#18

Physics Of Fluids

Introduction to fluids

The field of fluid dynamics is a fascinating branch of physics and engineering. Fluid refers to a substance that flows and takes the shape of its container. In physics, fluid is a state of matter that includes both liquids and gases. Unlike solids, fluids do not have a fixed shape and can easily deform under the influence of external forces. Fluids are characterized by their ability to flow and their relatively low resistance to shear stress. This property is known as viscosity. Liquids, such as water, have a higher viscosity compared to gases like air. The behavior of fluids is governed by principles of fluid dynamics, which study the motion and properties of fluids.

Surface Tension

Liquids exhibit surface tension. Surface tension is a physical property of liquids that describes the force acting on the surface of the liquid due to the cohesive forces between its molecules. It is the reason why liquid droplets form spherical shapes and why some insects can walk on water. At the molecular level, the molecules within a liquid are attracted to each other. In the bulk of the liquid, these attractive forces are balanced in all directions. However, at the surface, the attractive forces are unbalanced, resulting in a net inward force that causes the surface to behave like a stretched elastic membrane. The molecules in the bulk of the liquid are attracted equally in all directions, but at the surface, they experience a stronger attractive force from the molecules below them than from the molecules above them. As a result, the molecules at the surface are pulled inward, creating a sort of "skin" or tension on the surface.

Surface tension can be quantified by measuring the amount of force required to break or distort the surface of a liquid. This force of contraction is at right angles to an imaginary line of unit length, tangential to the surface of a liquid, and is called its surface tension defined as:

$$\gamma = \frac{F}{L}$$

Here F is the force exerted by the "skin" of the liquid. The SI unit of surface tension is $\frac{N}{m}$.

Quantitative measurement of surface tension

In order to quantify the amount of force required to break or distort the surface of liquid can be measured with the help of the following experiment. Let w be the weight of the sliding wire and T is the force with which we pull the wire downward.

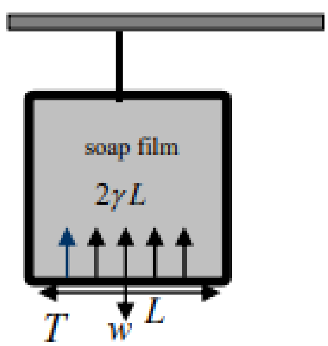


Figure 18.1: Weight of the sliding wire acting downward

From above figure it is obvious that, $T + w = F =$ net downward force. Since film has both front and back surfaces, the force acts along a total length of 2L. By using the equation of the surface tension of the film becomes,

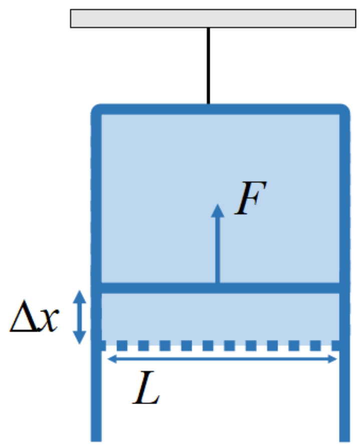
$$\gamma = \frac{F}{2L} \Longrightarrow F = 2\gamma L\ .$$

Hence the surface tension is

$$\gamma = \frac{T+w}{2L}.$$

The γ in above equation is surface tension coefficient.

Surface tension in term of energy



We can comprehend the above phenomenon in terms of energy. When the area of the film increases, the surface energy also increases proportionally. Additionally, this concept can be understood by considering the molecules within the film and the forces acting upon them, which depend on the distance between the molecules. If the molecules within the film are larger, the corresponding forces will be greater.

Let's ask how much work is done when we stretch the skin of liquid. If we move the sliding wire through a displacement Δx, than the work done by the force of surface tension is FΔx. Here F is a conservative force, so there is potential energy defined as:

$$\Delta U = F\Delta x$$

By using, $\gamma = \frac{F}{L}$ in above equation we get

$$\Delta U = \gamma L \Delta x$$

Where L is the length of the surface layer. As $L\Delta x = \Delta A =$change in area of the surface, so above equation becomes,

$$\gamma = \frac{\Delta U}{\Delta A}$$

So we see that surface tension is the ratio of the change in potential energy to the change in the surface area of the liquid when it is stretched.

Example: Soap bubble formation

Soap bubble formation is a fascinating phenomenon that occurs due to the surface tension of soapy water. Surface tension is a property of liquids that arises from the cohesive forces between their molecules. It is the force that causes the surface of a liquid to behave like a stretched elastic sheet.

When we blow air into a soap film or a soapy solution, the air pressure inside the film increases, stretching the film and causing it to form a bubble. This causes a pressure difference between the inside and outside of a soap bubble or a liquid drop. A soap bubble consists of two spherical surface films with a thin layer of liquid between them .

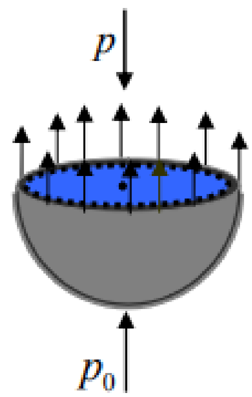


Figure 18. 2: A soap bubble consists of two spherical surface films

Let p = pressure exerted by the upper half, and p_o = external pressure. The force exerted due to surface tension is

$$F = \gamma L$$

and for the above example, the equation becomes,

$$F = 2(2\pi r\gamma)$$

where we have used $L = 2\pi r$ (circumference of soap bubble) and the "2" is for two surfaces. In equilibrium forces must be equal i.e.

$$\text{Total force} = \text{Surface tension force} + \text{Force due to outside}$$

and force can be defined as

$$F = pA = \pi r^2 p$$

So by using the expression of total force, we get

$$\pi r^2 p = 2(2\pi r\gamma) + \pi r^2 p_o$$

$$rp = 2(2\gamma) + rp_o$$

$$r(p - p_o) = 2(2\gamma)$$

∴ As, bubble has two surfaces

$$(p - p_o) = \frac{4\gamma}{r}$$

After solving the above equation for a liquid having only one surface, we get the expression of excess pressure as,

$$\text{Excess pressure} = p - p_o = \frac{2\gamma}{r}$$

Based on the equation above, it is evident that the pressure difference is inversely proportional to the radius of the bubble. This implies that it is significantly easier to form bubbles with a larger radius compared to those with smaller radii.

Fluid Motion

So far, we have only discussed fluid statics, which pertains to fluids that are not in motion. However, understanding fluid dynamics becomes more crucial when fluids are in motion. In this section, we will discuss the topic in detail due to its high significance and numerous real-life examples such as blood flow in our bodies, water flowing from a tank, and the air passing over an airplane, enabling it to fly. This is why the discussion now shifts to fluid motion.

To begin with, fluid motion can be classified as either steady or unsteady. **Steady flow** refers to a constant-speed flow of liquids, while the flow becomes unsteady when the fluid descends from a waterfall, for instance. Additionally, the fluid can either be compressible or in-compressible (constant density). Furthermore, fluid flow can also be categorized as viscous or friction-free.

Types of fluid flow

When discussing fluid dynamics, we encounter two types: laminar flow and turbulent flow. Now, let's proceed to explore the various types of fluid flow.

Laminar flow

Laminar flow is characterized by smooth and orderly movement of fluid particles in parallel layers or streamlines, with minimal mixing between adjacent layers.

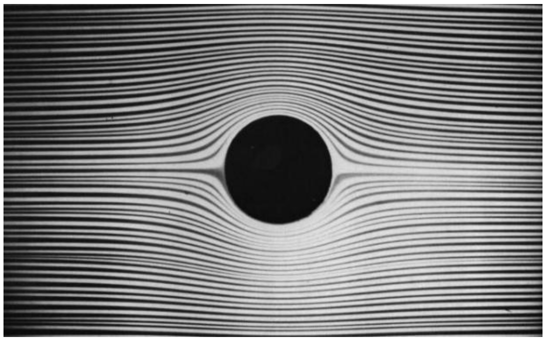


Figure 18. 3: Laminar flow of fluid

Turbulent flow

Turbulent flow, on the other hand, is characterized by irregular and chaotic motion of fluid particles, involving vigorous mixing.

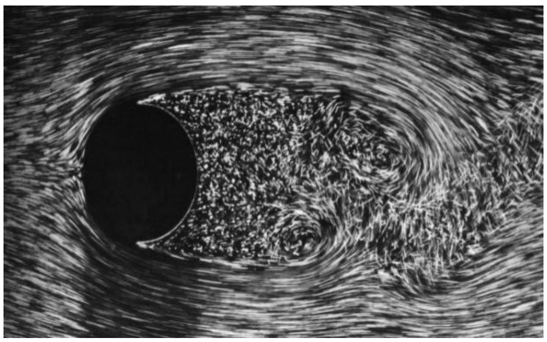


Figure 18. 4: Turbulent flow of fluid

Equation of Continuity

The equation of continuity is a fundamental principle in fluid dynamics that describes the conservation of mass in a fluid flow.

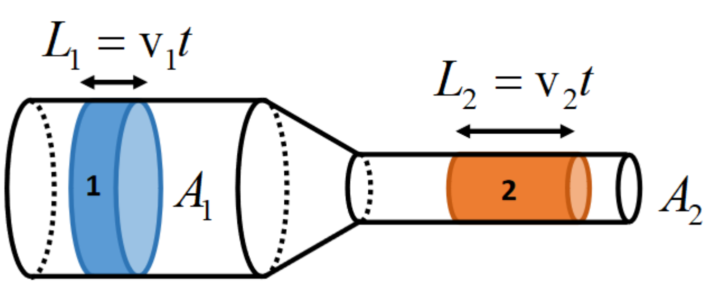


Figure 18. 5: Fluid flowing through a non-uniform pipe.

Consider water flowing in a pipe with a changing diameter. Fluid enters from left where the cross-sectional area is A_1 (volume V_1) and leaves from right where the area is A_2 (volume V_2) . Let v_1 and v_2 are the speeds of fluid particles at both ends of the pipe. If we consider the volume flow rate (V/t), which is the amount of fluid passing through a section of the pipe per unit of time, and since water is incompressible, the volumes at two different points must be equal.

$$\frac{V_1}{t} = \frac{V_2}{t}$$

$$V_1 = V_2$$

The volume of water passing through a section of the pipe can be expressed as the product of cross-sectional area (A) and length (L):

$$A_1 L_1 = A_2 L_2$$

$$\because \text{displacement (L)} = \text{velocity (v)} \times \text{time(t)}$$

$$A_1 v_1 t = A_2 v_2 t$$

$$A_1 v_1 = A_2 v_2$$

This equation represents the inverse relation between the area and speed i.e., smaller the area, greater will be the speed.

This is the key equation of continuity. It states that the product of the cross-sectional area and the velocity at one point in the pipe is equal to the product at another point. This holds true as long as the flow is steady, there are no leaks or ruptures, and the fluid is incompressible. The product of cross-sectional area and velocity is equal to the derivative of volume with respect to time

$$\frac{dV}{dt} = Av = \text{constsnt}$$

This expression emphasizes that the rate of change of volume with respect to time is constant along the pipe.

In summary, the equation of continuity states that for an in-compressible fluid, the mass flow rate remains constant along a streamline. It is based on the principle of conservation of mass, where the volume of fluid passing through a particular section of a pipe must be constant for a steady flow. Thus, as the cross-sectional area changes, the fluid velocity adjusts to ensure that the mass flow rate remains constant.

Example: Blood flow

The heart, functioning as a pump, circulates blood throughout our body, delivering oxygen and nutrients. On average, the heart pumps 60 to 90 ml of blood per heartbeat. If the heart stops, death occurs within four to five minutes.

The heart is divided into left and right sides, each with specific roles. One side pumps blood, connecting to the lungs, while the other directs blood to the body. As blood circulates, it loses oxygen and gains carbon dioxide. Like a tire valve, the heart has four valves allowing one-way blood flow, opening when blood moves and closing under pressure.

Understanding blood flow is vital for general knowledge and heart issue diagnosis. Engineers designing an artificial heart need to grasp concepts like viscous and non-viscous flow, turbulent and streamlined flow, crucial for creating a heart-replacing machine.

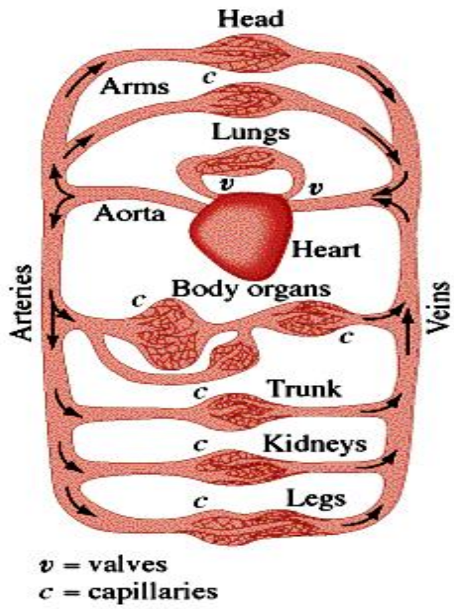


Figure 18. 6: Blood flow within a circulatory system, highlighting the movement of blood through arteries, veins, and capillaries, showcasing the direction and pathways within the vascular network.

Bernoulli's Equation

Bernoulli's Equation is the fundamental equation in fluid dynamics which tells you how fast a fluid will flow when there is also a gravitational field acting upon it. The analysis is based on applying conservation of energy,

$$\Delta K + \Delta U = W$$

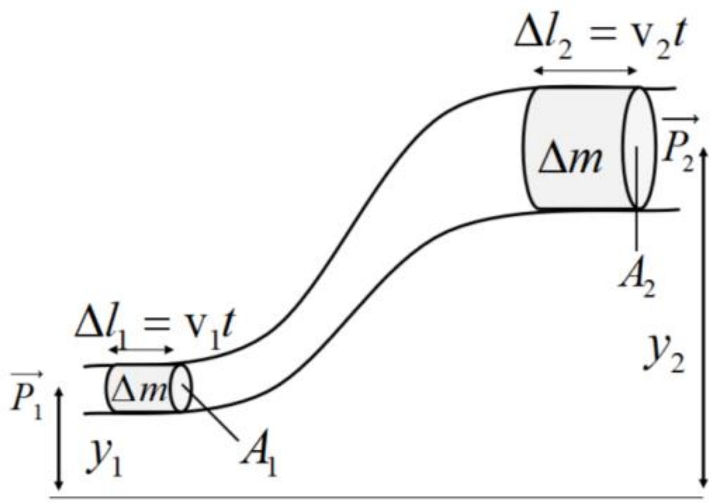


Figure 18. 7: A steady flow of fluid through the pipe in time t+Δt. Transferring the fluid element shown by the dark coloring from the tube's intake end to its output end is the overall result of the flow.

The fluid at the inlet end is subjected to a pressure p_1 (by the fluid behind it) which causes a force F_1 to push the system to the right.

$$F_1 = p_1 A_1$$

At the outlet end, a force F_2 acts on our system to the left due to pressure p_2 (by the fluid in the pipe to the right of the system)

$$F_2 = p_2 A_2$$

The fluid element Δm, moves through the pipe under the combined influence of the two pressure forces and gravity. At the inlet end, the pressure force does work in moving fluid through distance Δl,

$$W_1 = F_1 \Delta l_1 = p_1 A_1 \Delta l_1$$

At the outlet end, the pressure force does work

$$W_2 = -F_2 \Delta l_2 = -p_2 A_2 \Delta l_2$$

The work done by gravity, as Δm moves through the vertical displacement is

$$W_g = -\Delta m g (y_2 - y_1)$$

Where y_1 and y_2 are the heights above some reference level. The negative sign in the above equations indicates that the force and displacement are in opposite directions.

Total work done on the system is

$$W = W_1 + W_2 + W_g$$

$$W = p_1 A_1 \Delta l_1 - p_2 A_2 \Delta l_2 - \Delta m g (y_2 - y_1) \qquad (1)$$

In terms of the (uniform and constant) fluid density , the volume element is

$$A_1 \Delta l_1 = A_2 \Delta l_2 = \Delta V = \frac{\Delta m}{\rho},$$

Thus equation 1 becomes

$$W = (p_1 - p_2) \frac{\Delta m}{\rho} - \Delta m g (y_2 - y_1)$$

The change in kinetic energy of the fluid element Δm is

$$\Delta K = \frac{1}{2} \Delta m v_2^2 - \frac{1}{2} \Delta m v_1^2$$

Where v_1 and v_2 are the velocities at the lower and upper ends respectively. ΔU represents the potential energy due to conservative forces that act among objects within the system. Here we assume that no such forces act within the fluid, so

$$\Delta U = 0$$

Applying conservation of energy,

$$W = \Delta K$$

$$(p_1 - p_2) \frac{\Delta m}{\rho} - \Delta m g (y_2 - y_1) = \frac{1}{2} \Delta m \mathrm{v}_2^2 - \frac{1}{2} \Delta m \mathrm{v}_1^2$$

$$\frac{\Delta m}{\rho}[(p_1 - p_2) - \rho g(y_2 - y_1)] = \Delta m\left[\frac{1}{2}\mathrm{v}_2^2 - \frac{1}{2}\mathrm{v}_1^2\right]$$

$$[(p_1 - p_2) - \rho g(y_2 - y_1)] = \rho\left[\frac{1}{2}\mathrm{v}_2^2 - \frac{1}{2}\mathrm{v}_1^2\right]$$

$$p_1 - p_2 - \rho g y_2 + \rho g y_1 = \frac{1}{2}\rho \mathrm{v}_2^2 - \frac{1}{2}\rho \mathrm{v}_1^2$$

$$p_1 + \frac{1}{2}\rho \mathrm{v}_1^2 + \rho g y_1 = p_2 + \frac{1}{2}\rho \mathrm{v}_2^2 + \rho g y_2$$

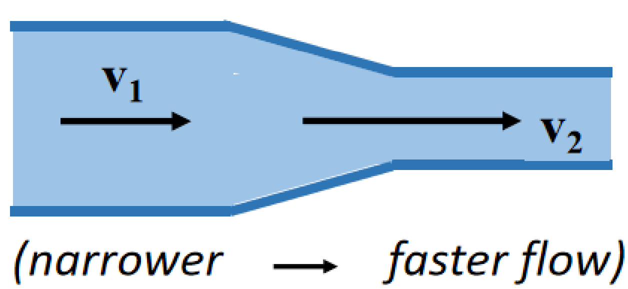
Since the subscripts 1 and 2 refer to two arbitrary locations in the pipe, we can drop the subscripts and write

$$P + \frac{1}{2}\rho \mathrm{v}^2 + \rho g y = constant$$

The above equation is called **Bernoulli equation** which states that "for a steady, non-viscous, in-compressible and irrotational flow fluid the sum of pressure, kinetic energy per unit volume and potential energy per unit volume is always constant".

In summary, Bernoulli's equation indicates that the total energy of a fluid remains constant along a streamline. As the fluid moves faster (increasing its kinetic energy) or climbs to a higher elevation (increasing its potential energy), the pressure exerted by the fluid decreases. Similarly, if the fluid slows down (reducing its kinetic energy) or descends to a lower elevation (decreasing its potential energy), the pressure increases.

This principle has numerous practical applications, including understanding the lift generated by an airplane wing, the flow through pipes and nozzles, and the behavior of fluids in various engineering and natural systems.



Example 1: Let us apply **Bernoulli equation** to water flowing in a pipe whose cross-sectional area decreases along its length as shown in figure (A_1 is area of the wide part, etc.).

The fluid is flowing horizontally, so there is no change in the height ($y_1 = y_2$) hence, the gravitational potential energies get cancelled out, and we get

$$p_1 + \frac{1}{2}\rho v_1^2 + \cancel{\rho g y_1} = p_2 + \frac{1}{2}\rho v_2^2 + \cancel{\rho g y_2}$$

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$

Now, since the liquid is in-compressible, according to equation of continuity it flows faster in the narrower part as given below

$$A_1 v_1 = A_2 v_2 \qquad (2)$$

$$v_2 = \left(\frac{A_1}{A_2}\right) v_1$$

As $A_1 > A_2$, thus $v_2 > v_1$, equation 2 becomes

$$p_2 = p_1 - \frac{1}{2}\rho(v_2^2 - v_1^2)$$

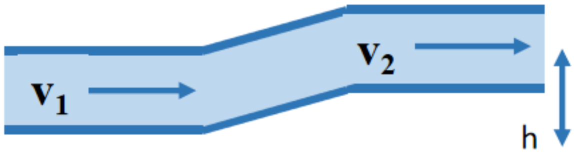
This means that the pressure is smaller in the narrow pipe where the streamlines are closer together than in the wider pipe ($p_2 < p_1$). Hence, **where the speed of fluid is high, the pressure will be low.**

Example 2: Consider a uniform pipe ($A_1 = A_2$) in which an ideal fluid is moving with constant velocity ($v_1 = v_2$). The height of pipe changes from y_1 to y_2 ($y_2 - y_1 = h$) as shown in figure,

Applying Bernoulli's equation

$$p_1 + \cancel{\frac{1}{2}\rho v_1^2} + \rho g y_1 = p_2 + \cancel{\frac{1}{2}\rho v_2^2} + \rho g y_2$$

$$p_2 = p_1 - \rho g(y_2 - y_1) = p_1 - \rho g h$$



Hence, $p_2 < p_1$, greater the height lower will be the pressure and vice versa.

Applications of Bernoulli's equation

Bernoulli's equation, a fundamental principle in fluid dynamics, has wide-ranging applications that span various fields of science and engineering. From aerodynamics and hydraulics to blood flow dynamics and sports, the equation provides valuable insights into the behavior of fluids in different contexts. Understanding and applying Bernoulli's equation enables engineers, researchers, and practitioners to design efficient systems, analyze fluid flow patterns, and even explain phenomena such as lift generation and ball trajectories. In this section, we will explore some of the fascinating applications of Bernoulli's equation and its significance in understanding fluid behavior across diverse disciplines.

Dynamic lift

Dynamic lift is the force that a body experiences as a result of moving through a fluid, like an aircraft wing.

Airplanes

Dynamic lift is exactly the reason why an aircraft flies? The wing shape is curved so that when the aircraft moves through the air, the streamlines are more closely spaced above the wing than below it. The air moves faster on the top part of the wing than on the lower part. Thus, the air pressure is lower on the top compared to the bottom and there is a net pressure upwards. The difference in pressure between the wing's upper and lower surfaces is what produces lift. This is called lift.

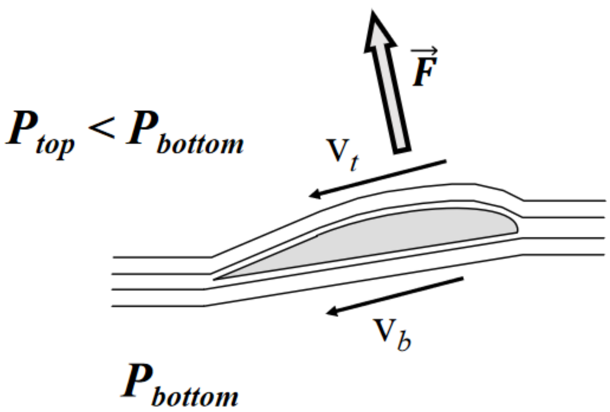


Figure 18. 8: Dynamic lift in airplanes, demonstrating the principle of airflow over the wing creating a pressure difference, resulting in upward lift force during flight.

Another explanation for the lift, however, is that the wing is made to deflect the air passing past it downward. In accordance with Newton's Third Law, this lift is therefore a reaction force $\vec{F}$ (exerted by air on the wing) to the downward force (the wing exerts on the air).

Atomizer

An atomizer is a tool that emits liquid droplets as a thin mist. In this context, the term "atomize" refers to breaking a large body into several, distinct particles. Its work is based on Bernoulli's principle. It provides a low pressure and pulls the liquid and air inside the vertical tube upward when fast horizontal air passes over it. At the end of the horizontal tube, there is a nozzle that disperses the liquid into tiny drops and mixes them with the air.

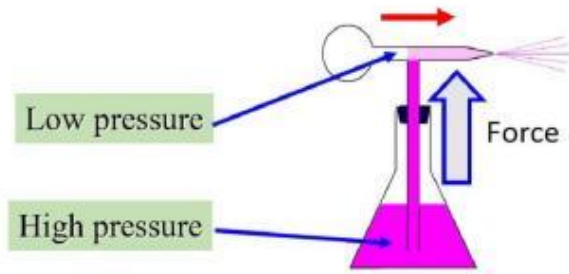


Figure 18. 9: An atomizer, a device used to break down liquids into a fine mist, illustrating the mechanism of liquid dispersion through fine nozzles."

Venturi Meter

A Venturi meter is a device used to measure the flow rate of a fluid in a pipe. It operates based on the principle of Bernoulli's equation and the Venturi effect.

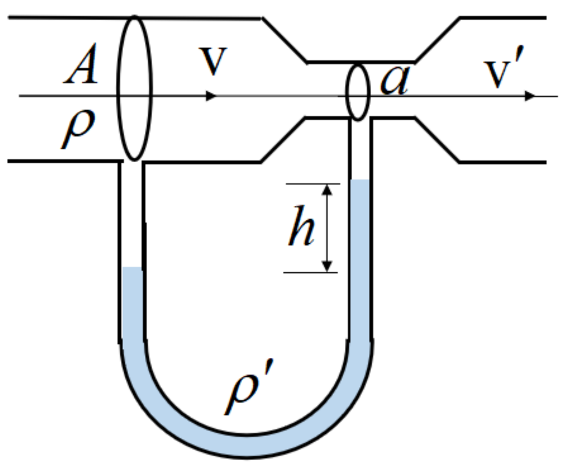


Figure 18. 10: A Venturi meter consists of a pipe with cross-sectional area A_1 is filled with a fluid of density ρ. A manometer tube is inserted at the throat, where the area is reduced to A_2. The manometer liquid, such as mercury, have a density ρ'.

There is no difference in the heights as the pipes are horizontal. We formulate Bernoulli's equation in a more comprehensible manner.

$$p+\frac{1}{2}\rho v^2=p'+\frac{1}{2}\rho v'^2$$

$$p-p'=\tfrac{1}{2}\rho v'^2-\tfrac{1}{2}\rho v^2 \qquad (3)$$

By using equation of continuity

$$\mathrm{Av}=\mathrm{av}'$$

$$\mathrm{v}'=\frac{\mathrm{A}}{\mathrm{a}}\mathrm{v}$$

By using the above equation into eq. 3 we will get

$$p-p'=\frac{1}{2}\rho\left(\frac{\mathrm{A}}{\mathrm{a}}\mathrm{v}\right)^2-\frac{1}{2}\rho v^2$$

$$p-p'=\tfrac{1}{2}\rho v^2\left[\left(\tfrac{\mathrm{A}}{a}\right)^2-1\right] \qquad (4)$$

Also,

$$\mathrm{p}-\mathrm{p}'=\mathrm{h}(\rho'-\rho)\mathrm{g} \qquad (5)$$

So by making the comparison of eq. 4 with eq. 5 we will get

$$\mathrm{h}(\rho'-\rho)\mathrm{g}=\frac{1}{2}\rho v^2\left[\left(\frac{\mathrm{A}}{a}\right)^2-1\right]$$

$$\mathrm{h}(\rho'-\rho)\mathrm{g}=\frac{1}{2}\rho v^2\left[\frac{A^2-a^2}{a^2}\right]$$

Re arranging the above equation, we get

$$\mathrm{v}=\mathrm{a}\sqrt{\frac{2(\rho'-\rho)\mathrm{gh}}{\rho(\mathrm{A}^2-\mathrm{a}^2)}}$$

This equation gives the mathematical expression for the velocity of the fluid at the Venturi throat in terms of the velocities at the inlet v and the pressure difference between the two points, along with the cross-sectional areas of the Venturi meter and the density of the fluid (ρ). The physical explanation of the above equation lies in the Venturi effect, where the increase in fluid velocity in the constricted section leads to a decrease in pressure. This pressure difference can be measured and correlated to the flow rate through the Venturi meter.

The Pitot Tube

The Pitot tube is a velocity measuring device of a fluid (specifically gas), named after the inventor Henri Pitot.

Applying Bernoulli's equation, $v_1=0$, $h_1=h_2$

$$p_2 + \frac{1}{2}\rho v_2^2 = p_1$$

$$p_1 - p_2 = \frac{1}{2}\rho v_2^2 \qquad (6)$$

Also, by the measurement of manometer,

$$p_1 - p_2 = \rho' g h \qquad (7)$$

Comparing equation 6 and 7 we get

$$\frac{1}{2}\rho v_2^2 = \rho' g h$$

$$v_2 = \sqrt{\frac{2gh\rho\prime}{\rho}}$$

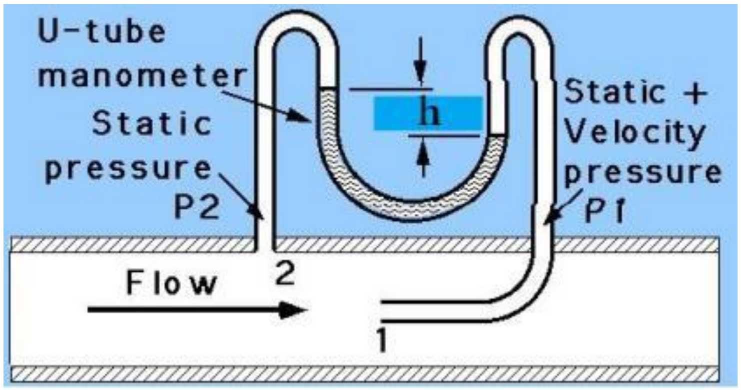


Figure 18. 11: Pitot tube, a device used in aviation to measure fluid flow velocity, illustrating its design and function in determining airspeed on aircraft.

Rotational or Irrotational Fluid Flow

Now, let's explore another perspective involving irrotational and rotational dynamics. If an element of the moving fluid does not rotate about an axis passing through the center of mass of element, the flow is said to be irrotational. If the wheel moves without rotating the motion is irrotational; otherwise it is rotational.

Previously, we covered streamline and turbulent flows, but there's another aspect to consider. When a small object rotates, introducing rotational energy into the flow, it creates a complicated situation, and the basic equation doesn't apply directly to this scenario.

Tennis Ball

Figure (a) depicts streamlines for the constant flow of air passing by a non-rotating, stationary ball at low speeds sufficient to prevent turbulence. Figure (b) displays the streamlines of air being transported by a ball that is rotating quickly. The spinning ball could not move the air in this way without viscosity and the boundary layer.

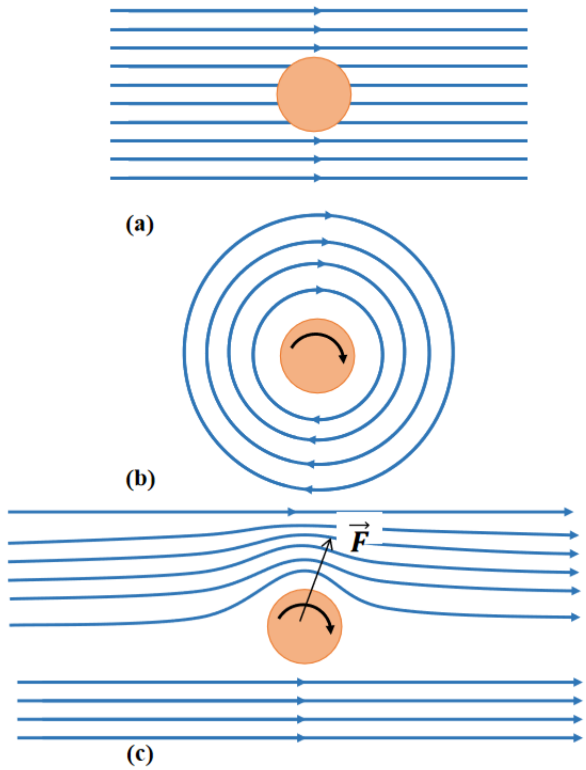


Figure 18. 12: (a) constant flow of air passing by a non-rotating, stationary ball, (b) streamlines of air being transported by a ball that is rotating quickly, (c) combined effect of circulation—caused by the ball's rotation—and steady flow—caused by the ball's translation through the air.

Figure (c) illustrates the outcome of combining circulation—caused by the ball's rotation—and steady flow—caused by the ball's translation through the air. The two velocities in the picture are added above the ball and subtracted below in the situation shown. We can observe that the space between the resulting streamlines indicates that the air velocity below the ball is lower than that above it. Since Bernoulli's equation requires that the air pressure beneath the ball be higher than the air pressure above it, the ball experiences a dynamic lift force.

Super-fluids

Most fluids exhibit viscosity, causing streamlined flow, but turbulence arises at high speeds. However, certain fluids, like liquid helium at extremely low temperatures, defy viscosity norms. At such temperatures, liquid helium undergoes quantum effects, resulting in superfluidity, where viscosity becomes negligible. This phenomenon, observed as helium spontaneously defying gravity by rising from a container's sides, has led to significant discoveries. For instance, a German team in 1988 demonstrated that combining and cooling helium atoms induce superfluidity, eliminating viscosity and friction.

Physics-PHY101-Lecture 19

PHYSICS OF SOUND

Introduction to waves

Waves, including sound waves, are disturbances that transfer energy from one place to another without transferring matter. In the context of sound waves, the medium through which the disturbance travels are typically air, but it can also be other materials like water or solids. Sound waves are produced by vibrations. When an object vibrates, it creates compressions and rarefactions in the surrounding medium. In the case of your voice, the vibrations originate in your vocal cords. As you speak, these vibrations are transmitted to the air molecules, creating areas of high pressure (compressions) and low pressure (rarefactions). These pressure differences are subsequently transmitted as sound waves via the air. Mechanical waves, specifically water waves, are produced by tossing a stone into the water. These waves move water molecules in circular or elliptical patterns by transferring energy over the surface of the water.

For sound waves, the medium is essential. In the absence of a medium (like in outer space), sound cannot travel because there are no molecules to transmit the vibrations. It's important to note that while waves transmit energy, the particles in the medium generally oscillate around a fixed position. At the end of 2004, the biggest tsunami occurred near Indonesia. The energy from the earthquake caused water particles to move in an up-and-down motion, transmitting the energy across vast distances.

In this chapter we'll understand the principles of waves are crucial in various fields, from physics and engineering to telecommunications and environmental science.

Sound waves generated in a tube:

Moving the piston from left to right in a tube with a piston attached to one end causes a motion in the layers of air within the tube. Energy is transferred as one layer imparts energy to the next. These layers of air are constantly vibrating, and their forward movement becomes visible over time. While this is simple in one dimension, it becomes more complicated when onlookers intervene by clapping their hands, forcing the layer of air to scatter in different directions. The difficulty lies in finding and comprehending these dispersed levels.

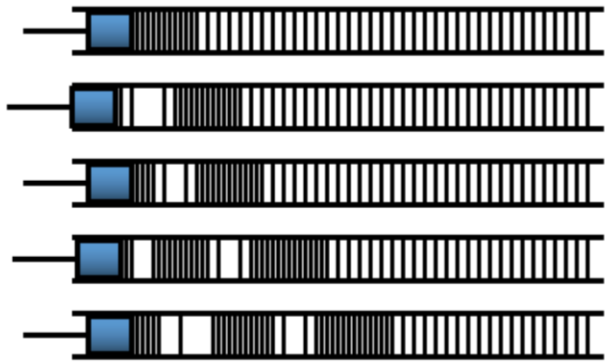
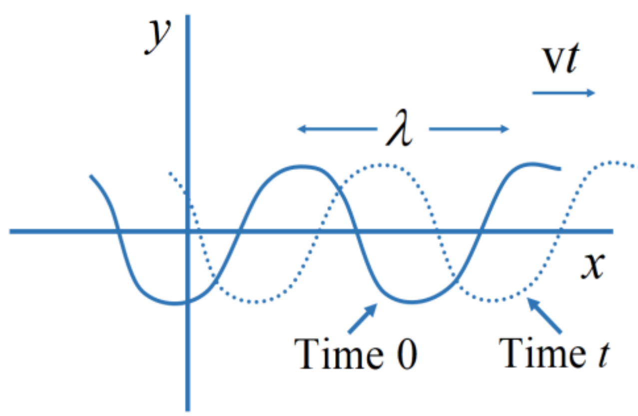


Figure 19. 1: Sound wave propagation within a tube by a moving piston, highlighting the compression and rarefaction patterns as the waves travel through the confined space. The compressible material in the tube is divided into layers with equal masses by the vertical lines.

In a previous lecture, we highlighted the significance of identifying vibrations using a variety of parameters. Firstly, the frequency of the layer determines the number of cycles per second. Secondly, the time taken to complete one cycle is referred to as the time period. Lastly, the amplitude of the vibration signifies its intensity — lower amplitude corresponds to lower sound, and in the context of earthquakes, low-amplitude seismic activity may go unnoticed.



Generation of sound

Working of a loudspeaker

A signal generator produces an electrical signal with a specific frequency and amplitude. To amplify the weak signal, we use a radio amplifier. Connecting the waves from the amplifier to a loudspeaker which consist of basic two parts: an outer diaphragm made of hard paper and a permanent magnet. The coil, when carrying current, experiences a magnetic force, leading to the diaphragm's vibration. By controlling amplitude and frequency, we can control the sound quality. High frequencies, above 15 kHz, become inaudible, demonstrating our control over amplitude and frequency.

Now by placing pallets on the diaphragm, we observe their response to changes in volume and frequency. Resonance occurs when the pallets and diaphragm frequencies align, resulting in increased movement of pallets. This experiment confirms that sound is indeed a vibration.

Intensity of sound

Sound operates on various energy levels, and here it is measured in watts. This quantifies the energy reaching us at any given time. Hence watt is equal to the joule per second (J/s). To gauge/measure the energy in space, sound intensity is measured in watts per centimeter squared.

There are various levels of sound which differ from each other, and here we employ a logarithmic scale for precision. Logarithms, such as $\log_{10}10 = 1$, $\log_{10}10^2 = 2$ and $\log_{10}10^n = n$. Hence, it helps in managing larger values to smaller ones efficiently.

Logarithmic scales are crucial for very low-intensity sounds, around 10^{-12} W/cm^2. This is expressed in **decibels (dB) scale**, "a relative measure for comparing the intensity of different sounds with one another".

$$I_o = \text{Threshold of hearing} = 10^{-12}\ \text{W/ cm}^2$$

$$\text{R = Relative intensity of sound } I = 10log\frac{I}{I_o}$$

Let's now examine the intensity of various sounds by using 10^{-12} as a reference value. Following this, we will explore how sound levels are expressed on a logarithmic scale.

Waves create a displacement between one peak and another, known as wavelength. Mathematically, it is described as follows:

At time t=0

$$y(x,0) = y_m \sin\frac{2\pi}{\lambda}x$$

y_m is the amplitude. At time t in x direction,

$$y(x,t) = y_m \sin\frac{2\pi}{\lambda}(x - \text{v}t)$$

Now, as the waves progress, they travel a distance vt in one second, where v is the velocity of sound. This clarifies that within one oscillation time, the wave moves forward by vt. The **period** T of a wave is the time to undergo one complete cycle of motion.

$$\lambda = vT \to T = \frac{\lambda}{v}$$

The **frequency (**ν) is the number of waves crossing a particular point every second.

Sound	Intensity (W / m^2)	Relative Intensity (I / I$_o$)	Sound Level (dB)
Threshold of hearing	10^{-12}	10^0	0
Rustle of leaves	10^{-11}	10^1	10
Whisper (at 1 atm)	10^{-10}	10^2	20
City street, no traffic	10^{-9}	10^3	30
Conversation (at 1 atm)	10^{-6}	10^6	60
Pressure horn (at 1 atm)	10^{-3}	10^9	90
Ear damage	1	10^{12}	120
Jet engine (at 1 atm)	10	10^{13}	130

$$v = \frac{1}{T}$$

$$y(x,t) = y_m \sin 2\pi(\frac{x}{\lambda} - \frac{t}{T})$$

The value of y at any given position remains the same at t, t+T, t+2T,….

To further describe the wave, we introduce two additional quantities: wave number (k) and angular frequency (ω).

$$k = \frac{2\pi}{\lambda}$$

$$\omega = \frac{2\pi}{T} = 2\pi v$$

In the previous discussion on oscillations, resonance was explored in detail. It was highlighted that when an oscillator is driven by an external force, it exhibits maximum response when the driving frequency aligns closely with the resonance frequency. This principle is evident in musical instruments.

Waves in Guitar Strings

Consider a guitar with various strings, each possessing different widths and tensions. Tensions can be adjusted, creating distinct frequencies for each string. Nodes, points where the string doesn't move, are present at both ends, where the oscillation amplitude is zero. In between these nodes, the amplitude may vary depending on the way of excitation of the strings.

The shape of the oscillation depends on the way of exciting string. Two notable features of the guitar include its unique body shape. There is an air column in the middle of the guitar where resonance occurs when the strings set the body in vibration. By plucking or exciting a string, different tones are generated. The frequency of each tone is proportional to the square root of the tension divided by the mass per unit length, meaning thicker strings have more mass per unit length.

Adjusting tension in the strings of guitar influences frequency. Increasing tension increases frequency. So, the frequency of oscillation does depend upon the square root of tension.

This is because the speed of sound in all strings is equal to the square root of the tension divided by the mass per unit length.

Nodes are present at both ends. When exciting the middle string, a unique sound is produced. The guitarist controls frequency by altering the length, generating multiple frequencies from each string. Unlike pure tones, which consist of a single frequency, the guitar produces a variety of frequencies.

Mathematics of Complex Waveforms

When different frequencies coexist, we can comprehend this phenomenon through mathematics. As mentioned earlier, combining sine waves with varying amplitudes results in a complex waveform that, despite being periodic, appears unconventional compared to simple sine or cosine functions.

Altering the coefficients produces different shapes.

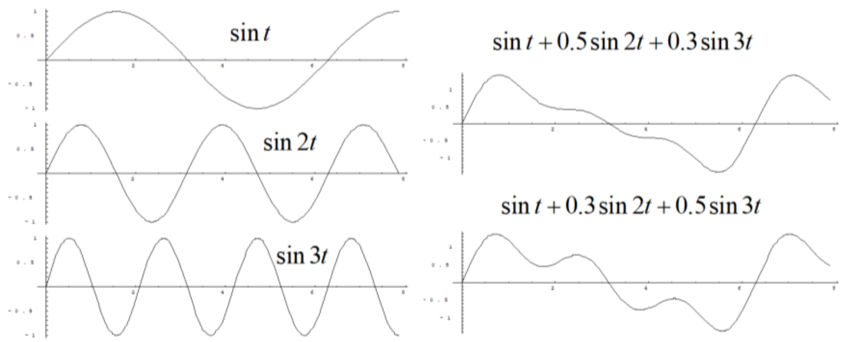


Figure 19. 2: Addition of sin waves i.e., sin t, sin 2t, and sin 3t, each wave with different amplitudes (0.5 and 0.3) is obtained.

You can see cost with various frequencies and amplitudes as well.

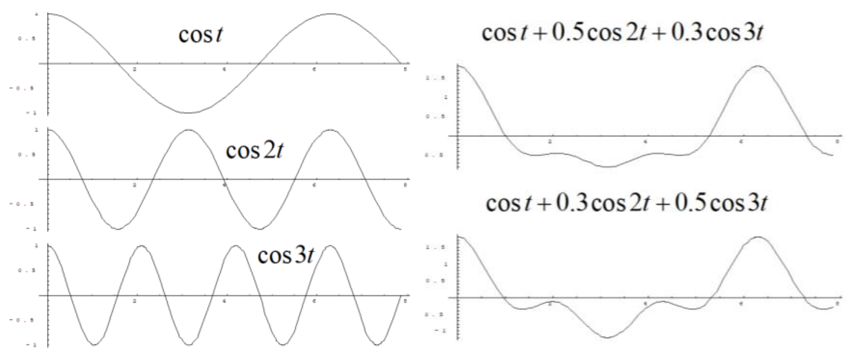


Figure 19. 3: Addition of cosine waves i.e., cos t, cos 2t, and cos 3t.

A remarkable discovery attributed to Joseph Fourier two centuries ago says that any wave, regardless of its shape, can be represented as a sum of sine and cosine functions. This concept is widely recognized as Fourier analysis.

Instrument Tuning Software

There is software designed for tuning various musical instruments, ensuring accurate node generation by comparing with the instrument's natural nodes. Unlike a guitar, which produces multiple notes, the low-frequency note is known as the **fundamental frequency**, and the other frequency notes which appear after the low-frequency notes are termed as **overtones**.

It's important to note the distinction between music and pure tones. While pure tones lack appeal, music incorporates a fundamental frequency and overtones. Comparing the tones generated by the software with recorded guitar sounds reveals that the guitar produces a mix of fundamental frequencies and overtones, around 84 Hz.

To further illustrate this, harmonic analysis quantifies the amplitude of each harmonic, providing information about the coefficients of sine frequencies and others in decibels. The software takes an interesting step by performing Fourier analysis at each moment. As a person speaks, a graph on the screen displays the analysis in real-time.

Doppler Effect

The Doppler Effect is the phenomena in which relative motion between source and observer causes the observer to receive a frequency that is different from that emitted by the source. The phenomenon was discovered by Austrian physicist Christian Johann Doppler in 1842.

Case 1: Observer is moving towards a stationary source

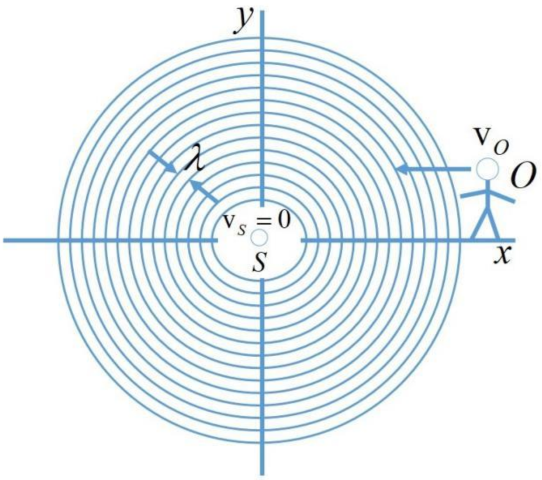


Figure 19. 4: Doppler effect with a stationary sound source (S) and a moving observer (O) travelling at a speed (v_o) towards the source. The circles show wavefronts moving across the medium separated by one wavelength.

If the observer was at rest, the number of waves received in time t would be t/T or vt /λ (1/T = ν = v/λ). Where v is the speed of sound in the medium and λ and ν are the wavelength frequency of sound wave. But if O is moving towards the source with speed v_o, the additional number of waves received is obviously $v_o t$ /λ . By definition,

$$\nu' = \text{frequency actually heard} = \frac{\text{number of waves received}}{\text{unit time}}$$

$$\nu' = \frac{\frac{vt}{\lambda} + \frac{v_o t}{\lambda}}{t} = \frac{v + v_o}{\lambda}$$

$$= \frac{v + v_o}{v/\nu} = \nu\left(\frac{v + v_o}{v}\right)$$

Frequency actually heard is

$$\nu' = \nu\left(1 + \frac{v_o}{v}\right)$$

The increase in frequency ν (v_o/v), brought on by the observer's motion, is added to the frequency (ν) heard at rest to create the frequency (ν') heard by the observer. We finally conclude that the observer hears a higher frequency (higher pitch) while moving towards the source.

Case 2: Observer is moving away from a stationary source

Waves are received at a reduced rate by an observer receding from the source, . Hence, number of waves received per unit time in this case is

$$\nu' = \frac{\frac{vt}{\lambda} - \frac{v_o t}{\lambda}}{t} = \frac{v - v_o}{\lambda} = \frac{v - v_o}{v/\nu} = \nu\left(\frac{v - v_o}{v}\right) = \nu\left(1 - \frac{v_o}{v}\right)$$

The above expression shows a *decrease* in frequency by an amount ν(*vo*/*v*) corresponding to the waves that do not reach the observer in each unit of time because of the receding motion.

Since the observer is moving through the medium while the source is at rest with respect to it, the general relation existing in this situation is

$$\nu' = \nu\left(\frac{v \pm v_o}{v}\right)$$

where motion towards the source is represented by a plus sign, and motion away from the source is represented by a negative sign.

Case 3: Source is moving towards the observer at rest

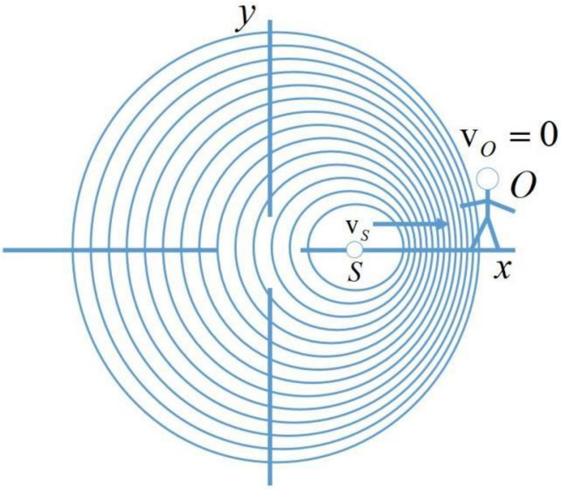


Figure 19. 5: Doppler effect with a source (S) moving at a speed (v_s) towards a stationary observer (O).

The waves are compressed by a quantity known as "doppler shift" ($\Delta\lambda = v_s/\nu$) as the source comes closer to the observer because same number of waves are contained in a smaller region depending upon the velocity of source v_s. So, the wavelength seen by the observer is

$$\lambda' = \lambda - \Delta\lambda = \frac{v}{\nu} - \frac{v_s}{\nu} = \frac{v - v_s}{\nu}$$

From this frequency received by the observer is

$$\nu' = \frac{v}{\lambda'} = \frac{v}{(v - v_s)/\nu}$$

$$\nu' = \nu\frac{v}{(v - v_s)}$$

Consequently, whenever the source is travelling in the direction of the observer, the observed frequency rises.

Case 4: Source is moving away from the observer at rest

The wavelength emitted by the source increases as it gets farther away from the observer, causing the observer to hear a lower frequency.

$$\lambda' = \lambda + \Delta\lambda = \frac{v}{\nu} + \frac{v_s}{\nu} = \frac{v + v_s}{\nu}$$

$$\nu' = \frac{v}{\lambda'} = \frac{v}{(v + v_s)/\nu} = \nu\frac{v}{(v + v_s)}$$

In light of this, the general relationship that holds when the source is moving through a medium and the observer is at rest is

$$\nu' = \nu\frac{v}{(v \pm v_s)}$$

where the minus sign holds for motion toward the observer and the plus sign holds for motion away from the observer.

Case 5: Both source and observer are moving

Let us consider that the source and observer are moving towards each other. Using eq. Doppler shift due to the motion of source is toward observer is

$$\lambda' = \frac{v - v_s}{\nu}$$

While observed frequency due to motion of observer towards source is

$$\nu' = \frac{v + v_o}{\lambda'}$$

Putting the value of λ’ in above equation

$$\nu' = \nu\left(\frac{v + v_o}{v - v_s}\right)$$

Similarly, we can find the expression for the source and observer moving away from each other

$$\nu' = \nu\left(\frac{v - v_o}{v + v_s}\right)$$

Combining the above expressions, we can write in general,

$$\nu' = \nu\frac{v \pm v_o}{(v \mp v_s)}$$

In this expression, the upper signs ($+v_o$ and $-v_s$) refer to motion of one toward the other, and the lower signs ($-v_o$ and $+v_s$) refer to motion of one away from the other.

For all Doppler-effect problems we can conveniently say that:

The word *toward* is related to an *increase* in observed frequency while the words *away from* are related to a *decrease* in observed frequency.

Both Doppler formulas that we derived here can approximate electromagnetic waves when their relative speeds are significantly slower than the speed of light.

Applications of Doppler Effect

Remember, the Doppler shift is commonly used in sound emission, with applications extending to ultrasound in medical diagnoses. **Ultrasound** operates at several MHz, allowing for precise visualization of small distances within the body due to variations in tissue density affecting sound frequency.

In ultrasound, a source emits sound, and reflections are captured by a microphone or probe for computer analysis. In terms of the Doppler effect, it's employed to estimate **blood flow speed** in veins.

Marine applications utilize sound waves to determine the speed and direction of submarines underwater, a technique known as sonar. During World War II, sonar proved valuable for submarine detection, showcasing the practicality of sound waves in underwater scenarios where radio waves fall short.

Physics-PHY101-Lecture 20

WAVE MOTION

Introduction to waves:

Wave motion is a fundamental concept in physics, describing the propagation of disturbances or oscillations through a medium or space. Waves manifest in various forms, encompassing mechanical waves like sound and ocean waves, electromagnetic waves such as light and radio waves, and even quantum waves. Wavelength, amplitude, and frequency are fundamental concepts in the study of wave motion. Whether observing the rhythmic undulations of water, the transmission of sound through air, or the propagation of light through space, understanding wave motion is crucial in analyzing the dynamics of the natural world across a diverse range of phenomena.

Waves, including ocean waves, light waves, X-rays, and radio waves, exhibit common characteristics. When two frequencies combine, they generate additional frequencies, known as **beats**. Additionally, when waves from different sources interact, they can either cancel each other out (destructive interference) or reinforce each other (constructive interference). These principles apply universally across various types of waves.

Difference between constructive and destructive interference:

Constructive Interference	Destructive Interference
1) When the crest of one wave meets with the crest of another wave (or trough to trough) then constructive interference takes place. 2) In the constructive interference when two waves meet each other the two waves are added up each other. 3) The condition for the constructive interference is that the phase difference between two waves must be zero. 4) The intensity of waves is increased in the constructive interference. 5) The waves in the constructive interference are in phase.	1) When the crest of the one wave is meet with the trough of another wave then destructive interference is takes place 2) In destructive interference when two waves meet each other they cancel out each other. 3) The condition for destructive interference is that the phase difference between superimposing waves is 180°. 4) The intensity of light wave decreases in destructive interference. 5) The waves in the destructive interference are in out of phase.

Longitudinal Waves:

In longitudinal waves, medium vibrations align with the wave's direction of motion. If one end of a spring is pushed, the resulting wave travels through the entire spring. An earthquake generates longitudinal waves in the ground, where one-part pushes another forward, causing the waves to propagate.

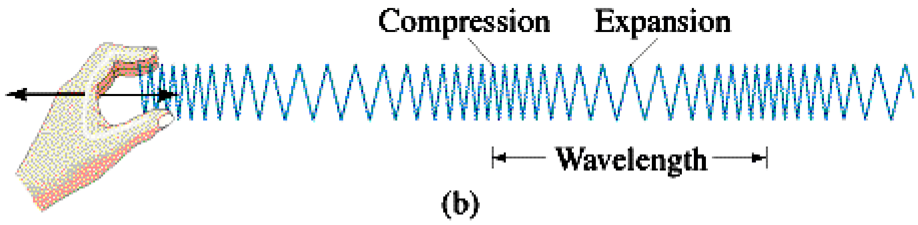


Figure 20. 1: Propagation of a longitudinal wave, showing compressions (high-pressure regions) and rarefactions (low-pressure regions) as the wave moves through a medium.

Transverse Waves:

In transverse waves, medium vibrations are perpendicular to the direction of motion. Transverse waves are evident in phenomena like water ripples when an object is placed on the water's surface, demonstrating upward and downward motion. In earthquake, both longitudinal and transverse waves are present. Light waves are another example of transverse waves.

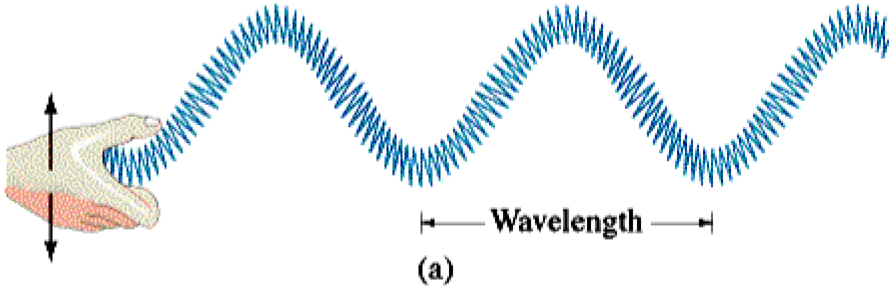


Figure 20. 2: Motion of a transverse wave, highlighting the perpendicular oscillation of particles relative to the direction of wave propagation within a medium.

Difference between Longitudinal and Transverse Waves:

Longitudinal	Transverse
The particles of the medium move in the same direction of the wave.	The particles of the medium are moving perpendicular to the direction of wave.
It acts in one dimension	It acts in two dimensions
The wave cannot be polarized or aligned.	The wave can be polarized or aligned
This wave can be produced in any medium such as gas, liquid or solid.	This wave can be produced in solid and liquid’s surface.
Example: The earthquake P wave, spring waves and sound waves.	Example: Earthquake S wave, and light waves.
It is made of rarefactions and compressions.	It is made of troughs and crests.

Amplitude and Power in Waves:

The relationship between amplitude and power varies in different cases, making it challenging to provide a specific explanation. However, a general point can be made: "**The height of a wave is its amplitude,**" and "**the energy in a wave is proportional to the square of the amplitude.**" This implies that the power or energy of a wave is determined by squaring its amplitude. To illustrate, considering a harmonic oscillator with an extension x expressed as:

$$x = x_m \sin \omega\, t$$

The symbol "x_m" represents the amplitude, which is the maximum value a wave can attain. The potential energy $\left(\frac{1}{2}kx^2\right)$ and kinetic energy $\left(\frac{1}{2}mv^2 = \frac{1}{2}k\left(\frac{dx}{dt}\right)^2\right)$, when summed, are proportional to x_m^2 or the square of the amplitude. This relationship holds true for any wave, where the power or energy is directly proportional to the square of the amplitude.

<u>Plane Waves</u>

Let's consider generating sound waves using a piston, where the area of the piston is assumed to be infinite. By moving the piston back and forth, it produces planar waves that propagate uniformly in one direction. These waves exhibit infinite extent, with wavefronts spaced at a wavelength (λ). In this pressure wave, if the pressure is at a maximum at one wavefront, it will reach maximum again after one wavelength, creating an infinite train of plane waves moving in one direction.

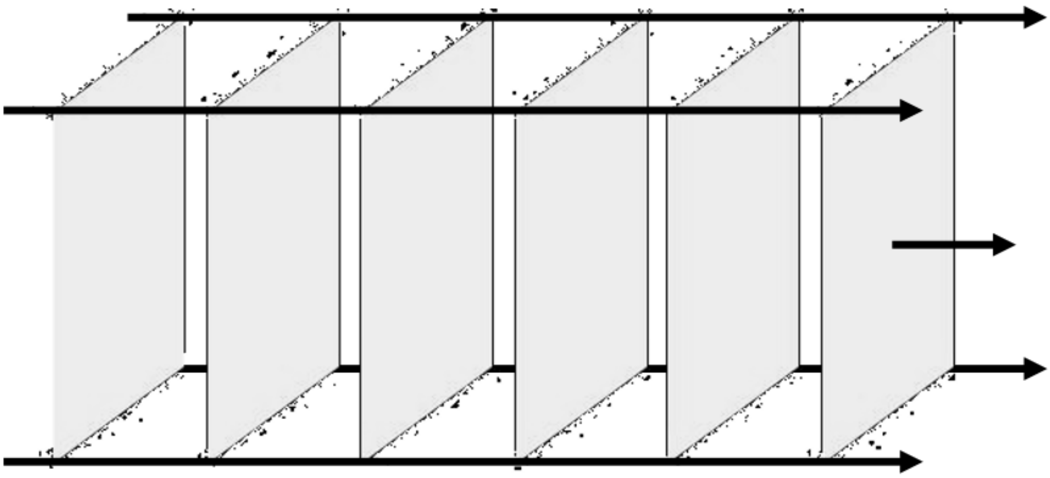


Figure 20. 3: Representation of plane waves. The planes represent wavefronts spaced one wavelength apart, and the arrows represent rays.

<u>Spherical Waves</u>

If sound is emitted spherically, as seen with a chirping sound, spherical waves are formed. Spherical waves decrease in intensity as they propagate, eventually dissipating to zero after traveling a certain distance. The rate of intensity decrease depends on the medium's energy absorption capacity. If the medium does not absorb energy, the produced energy remains within the sphere, creating different conditions for spheres with varying radii.

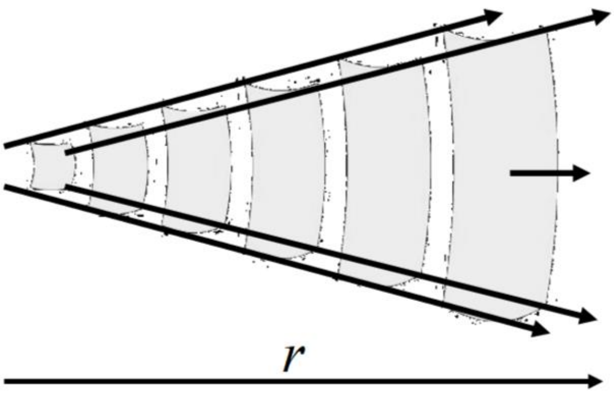
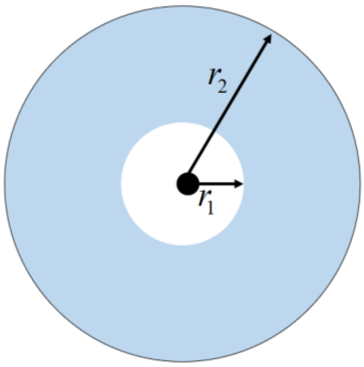


Figure 20. 4: Spherical wavefronts emanating from a point source, illustrating the outward propagation of energy in all directions uniformly.

$$Power \propto \frac{1}{r^2}$$

Let's consider two spheres with radii r_1 and r_2, where the total power passing through both spheres is same. This implies that if power P pass through the first sphere, an equal amount passes through the second sphere. If there is no object in the middle absorbing sound, and only air is present, there is no decrease in intensity. The radiated power (P) can be expressed as,



$$P = 4\pi r^2 I \quad \textbf{(}\text{Area of sphere} = 4\pi r^2\textbf{)}$$

where "I" is the intensity. For the smaller sphere, the surface area is $4\pi r_1^2$, and the power (P_1) is:

$$P_1 = 4\pi r_1^2 I_1$$

Similarly, for the larger sphere, with surface area $4\pi r_2^2$, the power (P_2) is:

$$P_2 = 4\pi r_2^2 I_2$$

$$P_1 = P_2 = P$$

$$\cancel{4\pi} r_1^2 I_1 = \cancel{4\pi} r_2^2 I_2$$

$$\frac{I_1}{I_2} = \frac{r_2^2}{r_1^2}$$

This is evidence that when r^2 increases, intensity—which is measured in W/m^2—will also decrease.

<u>Problem</u>: Spherically sound waves are emitted uniformly in all directions from a point source, the radiated power P being 25 W. What is the intensity, and the sound level of the sound waves at distance r = 2.5 m from the source?

$$P = 25\ W, \quad r = 2.5\ m$$

$$P = 4\pi r^2 I$$

$$I = \frac{P}{4\pi r^2} = \frac{25}{4(3.14)(2.5)^2} = .32\ W/m^2$$

$$SL = 10\log\frac{I}{I_0} = 10\log\frac{.32}{10^{-12}} = 115dB \qquad \therefore\ I_o = 10^{-12}\,W/m^2$$

Phase in Wave Cycles

Phase refers to the specific position in the cycle of a wave at a given point in time. Certainly, the sine wave follows a characteristic pattern, starting at zero, reaching its maximum at π/2, returning to zero at π, descending to negative values, hitting -1 at 3π/2, and rising again to zero at 2π. The cosine wave is closely related to the sine wave, differing by a phase shift of π/2.

The concept of phase introduces a new dimension, representing the argument of a sine or cosine function. In a general wave like sin (kx - ωt), adding a fixed angle to it introduces a constant phase difference, crucial in understanding wave behavior. The interaction of phases, whether added or subtracted, is significant, especially in the context of pure tones where they can either reinforce or cancel each other. Geometrically, phase can be visualized by examining the displacement (y) when x is zero.

$$y(x,t) = y_m \sin\ (kx - \omega\, t - \phi)$$

In the expression (kx - ωt – ϕ) is termed as phase, with ϕ called the phase constant. The phase constant either moves forward or backward in space or time, and the displacement (y) can be expressed in two distinct manners.

$$y(x,t) = y_m \sin\left[k\left(x - \frac{\phi}{k}\right) - \omega t\right]$$

$$y(x,t) = y_m \sin\left[kx - \omega\left(t + \frac{\phi}{\omega}\right)\right]$$

Principle of superposition:

According to principle of superposition, if there are two separate sources producing waves with amplitudes y_1 and y_2, the total amplitude is given by y_1+y_2. This holds true in a linear system. The power obtained from this superimposed amplitude is proportional to $(y_1+y_2)^2$.

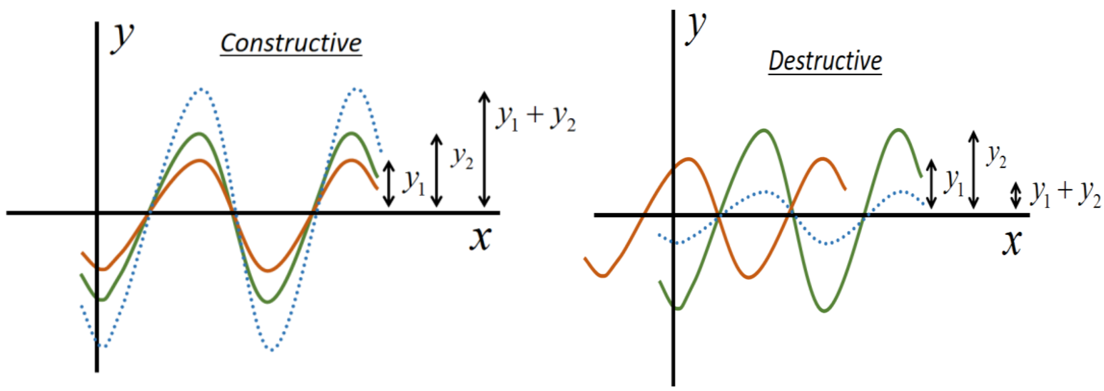
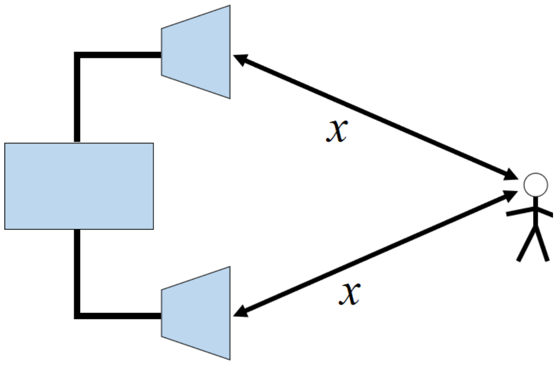


Figure 20. 5: Demonstration of constructive and destructive interference patterns: on the left, waves aligning in phase to produce amplification (constructive), and on the right, waves aligning out of phase leading to cancellation (destructive).

In the case of waves colliding, considering one-dimensional motion, the total amplitude is y_1+y_2. However, it's important to note that at certain locations, this amplitude increases, while at others, it decreases when the waves overlap. This behavior remains consistent as long as the waves are separate and pass through each other while maintaining their original forms. The principle of superposition is applicable to all types of waves.

Example:



When two loudspeakers emit sine waves of a specific frequency or pure tone, and a person stands in front of them, the waves can interfere with each other as the person moves back and forth. Two waves ($y_1(x, t)$ and $y_2(x, t)$) having same magnitude y_m, same wave number k, same angular frequency ω but different phases (ϕ_1 and ϕ_2) are given by,

$$y_1(x,t) = y_m\, sin(kx - \omega t - \phi_1)$$

And

$$y_2(x,t) = y_m\, sin(kx - \omega t - \phi_2)$$

Resultant wave $y(x, t)$ by superposition principal is sum of two waves and can be expressed as

$$y(x,t) = y_1(x,t) + y_2(x,t)$$

$$y(x,t) = y_m\, sin(kx - \omega t - \phi_1) + y_m\, sin(kx - \omega t - \phi_2)$$

$$y(x,t) = y_m[sin(kx - \omega t - \phi_1) + sin(kx - \omega t - \phi_2)]$$

By using trigonometric product identity given below,

$$\sin B + \sin C = 2\cos\left(\frac{B-C}{2}\right)\sin\left(\frac{B+C}{2}\right)$$

$y(x, t)$ becomes,

$$y(x,t) = y_m\left[2\,cos\left(\frac{(kx-\omega t-\phi_1)-(kx-\omega t-\phi_2)}{2}\right).sin\left(\frac{(kx-\omega t-\phi_1)+(kx-\omega t-\phi_2)}{2}\right)\right]$$

$$y(x,t) = y_m\left[2\,cos\left(\frac{\phi_2-\phi_1}{2}\right).sin\left(\frac{2kx-2\omega t-\phi_1-\phi_1}{2}\right)\right]$$

$$y(x,t) = 2y_m\left[cos\left(\frac{\phi_2-\phi_1}{2}\right).sin\left(kx-\omega t-\left(\frac{\phi_1+\phi_1}{2}\right)\right)\right]$$

$$y(x,t) = \left[2y_m\,cos\left(\frac{\Delta\phi}{2}\right)\right].sin(kx-\omega t-\phi')$$

Where $\Delta\phi = \phi_2 - \phi_1$

$$\phi' = \frac{(\phi_1+\phi_2)}{2}$$

The first term can be considered as amplitude (as it constant for particular/fixed values of ϕ_1 and ϕ_2 and is independent of time t).

$$y(x,t) = \underbrace{\left[2y_m \cos\left(\frac{\Delta\phi}{2}\right)\right]}_{\text{amplitude}} . \sin(kx - \omega t - \phi')$$

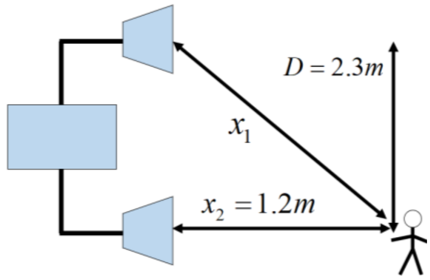
There are two distinct aspects of sine and cosine functions. For constructive interference, when the cos 0° = 1, the amplitude of sine is at its maximum. This occurs when two different frequencies combine, resulting in constructive interference. For constructive interference,

$$\frac{\Delta\emptyset}{2} = 0, \pi, 2\pi\ldots\ldots$$

On the contrary, for destructive interference, when the cosine is zero (as seen with the cos 90° =0), adding π again to phase, results in zero(cos 270° = 0), signifying destructive interference. If two waves cancel each other out, both amplitudes are equal, and this occurs when at the same distance from the loudspeaker. This principle applies more broadly without any specific limitations. For destructive interference,

$$\frac{\Delta\phi}{2} = \frac{\pi}{2}, \frac{3\pi}{2} \ldots\ldots$$

Problem: Two speakers, which are separated by a distance D of 2.3 m, emit a pure tone. The waves are in phase when they leave the speakers. For what wavelengths will the listener hear a minimum in the sound intensity, if the person is standing at the distance of 1.2 m from speaker 2?



To achieve destructive or constructive interference, we need to determine the required phase difference. Using Pythagoras theorem,

Hypotenuse^2 = base^2 + perpendicular^2

$$x_1^2 = x_2^2 + D^2$$
$$x_1 = \sqrt{x_2^2 + D^2} = 2.6m$$
$$x_1 - x_2 = (2.6 - 1.2)m = 1.4m$$

For destructive interference

$$Phase\ difference = \frac{\lambda}{2}, 3\frac{\lambda}{2}, 5\frac{\lambda}{2}, \cdots\cdots$$
$$1.4m = \frac{\lambda}{2}, 3\frac{\lambda}{2}, 5\frac{\lambda}{2}, \cdots\cdots$$
$$\lambda = 2.8m, 0.93m, 0.56m, \cdots$$

Stationary Waves

Stationary waves are formed by the superposition of two waves with the same frequency and amplitude traveling in opposite directions.

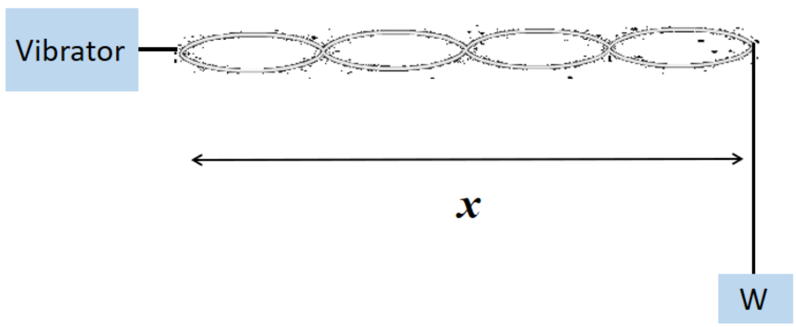


Figure 20. 6: A rope with a weight oscillating over a pulley, producing a standing wave pattern due to interference between waves traveling in opposite directions.

In a standing wave, some sections remain stationary (nodes), while adjacent sections experience maximum displacement (antinodes), forming a distinctive pattern. The resultant amplitude of stationary waves depends on the interference of the individual waves. The wave equations can be expressed as:

$$y_1(x,t) = y_m \sin\left(kx - \omega t\right)$$

$$y_2(x,t) = y_m \sin\left(kx + \omega t\right)$$

The difference in sign here shows that y_1 is the wave which is going in the positive x-direction while the other is in the negative x-direction.

The stationary wave is formed by the superposition of these two waves. So, the resultant displacement, y(x,t), is given by:

$$y(x,t) = y_1(x,t) + y_2(x,t)$$

$$= y_m \sin\left(kx - \omega t\right) + y_m \sin\left(kx + \omega t\right) = y_m \left[\sin\left(kx - \omega t\right) + \sin\left(kx + \omega t\right)\right]$$

Use the trigonometric identity sin (A + B) + sin (A − B) = 2 sin (A) cos (B) to simplify the expression:

$$= \underbrace{\left[2 y_m \sin kx\right]}_{amplitude} \cos \omega t$$

This is the equation for a stationary wave. It shows that the displacement (y)of the particles in the medium depends on both the spatial part (sin(kx)) and the temporal part (cos(ωt)). The amplitude of the stationary wave is $2y_m$, which means it is twice the amplitude of the individual waves, leading to regions of constructive and destructive interference in the resulting wave pattern.

The locations of nodes occur at kx = 0,π,2π,3π,…, where the amplitude is zero. Conversely, at kx = π/2, 3π/2,…, there are antinodes, resulting in maximum amplitude. This proves that the standing waves formed exhibit both nodes and antinodes, and the distance between them can be calculated based on the points where the amplitude is at its maximum and minimum.

So far, we have only considered interference in terms of frequency. Now, let's explore what happens when two separate sources that are considered to be interfering generate pressure waves (sound waves) having same amplitude p_0 but different angular frequencies (ω_1 and ω_2).

$$\Delta p_1(t) = p_0 \sin \omega_1 t$$
$$\Delta p_2(t) = p_0 \sin \omega_2 t$$
$$\Delta p(t) = \Delta p_1(t) + \Delta p_2(t)$$
$$= p_0(\sin \omega_1 t + \sin \omega_2 t) \qquad \therefore \sin\alpha + \sin\beta = 2\cos\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{\alpha+\beta}{2}\right)$$
$$= \left[2p_0 \cos\left(\frac{\omega_1-\omega_2}{2}\right)t\right]\sin\left(\frac{\omega_1+\omega_2}{2}\right)t$$

The argument of sin is (ω_1+ω_2)/2 representing the average of frequency, while the argument of cosine is (ω_1-ω_2)/2. This difference in frequencies is called **beat frequency.**

$$\bar{\omega} = \frac{\omega_1 + \omega_2}{2}$$
$$\omega_{diff} = \frac{\omega_1 - \omega_2}{2} \qquad \therefore p_o = p_m$$
$$\Delta p(t) = \left[2\Delta p_m \cos \omega_{diff}\, t\right]\sin \bar{\omega} t$$
$$\omega_{beat} = 2\omega_{diff} = \omega_1 - \omega_2$$

Consider two audio generators fixed to separate amplifiers. Turning on one generator produces a sound, and when switched to the second generator, a different sound is heard. Our ears cannot easily discern the frequency difference between them. However, when both generators are turned on simultaneously, two frequencies are generated: the beat frequency and the sum of frequencies. As we observe, when the frequencies of the generators are nearly equal, a beat frequency is noticeable, characterized by a slow modulation.

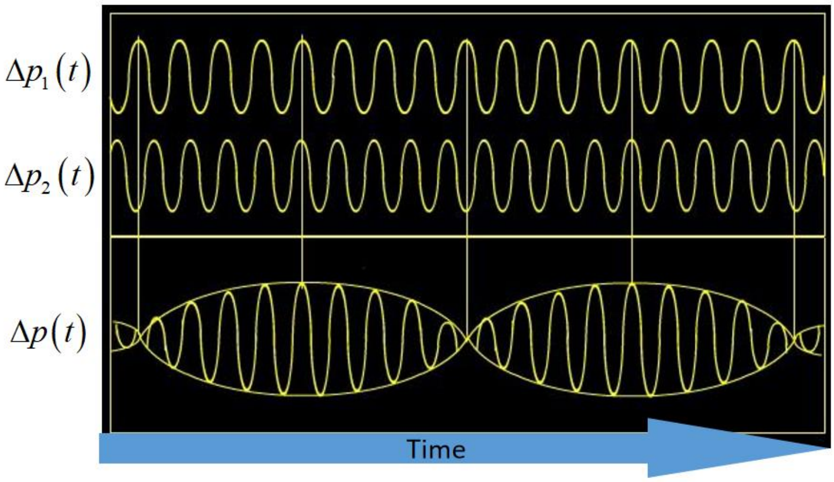


Figure 20. 7: Phenomenon of beats formation, showing the periodic variation in amplitude resulting from the interference of two waves of slightly different frequencies.

Applications of Beats Phenomenon:

The beat phenomenon helps in tuning musical instruments where the frequencies of two tones become nearly equal, producing an audible beat. Another application is in radio modulation. Since audio frequencies range from 20 cycles/s to 20,000 cycles/s, and radio waves operate in kHz and MHz, the beat phenomenon is used for modulation. By adding an audio frequency to a carrier wave, a new frequency is formed, such as combining a 20 kHz frequency with a

20 mega cycle radio frequency. This creates a high-frequency wave suitable for efficient radiation, and upon reaching the radio, it undergoes demodulation.

Speed of sound

A mass attached to a spring exhibits oscillations when pulled and released. The frequency (f) of oscillation is determined by $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$, where k is the spring constant (representing the force with which the spring tries to return to its original position based on its stiffness) and m is the mass. Sound waves obey a similar principle when a particle is displaced from its position; it strives to return to its original position. The force exerted in this process is calculated using bulk modulus (B). The velocity of sound (v) is related to bulk modulus and density (ρ) through the formula $v = \sqrt{\frac{B}{\rho}}$. Sound speed varies in different mediums, being faster in liquids and even more so in solids.

Medium	**Sound speed (ms^{-1})**
Gases	
Air (0^oC)	330
Helium	965
Hydrogen	1274
Liquids	
Water (0^oC)	1402
Solids	
Aluminium	6420
Steel	5941

Pulse Propagation:

Sound typically propagates in the form of pulses (a short burst of a wave or a localized disturbance) rather than pure tones. Pulse shapes can vary, gradually diminishing or ending at different points. At time t = 0, pressure distribution is observed as a function of x. The shape of the pulse is represented by f(x). Moving with the pulse at the speed of sound (v) without changing f(x) is crucial.

At time, the original shape of the function f(x) remains the same.

$$y(x,0) = f(x)$$

At time t it is displaced by an amount vt

$$y(x,t) = f(x') = f(x - vt)$$
$$where, x' = x - vt$$

$$x - vt = \text{constant}$$

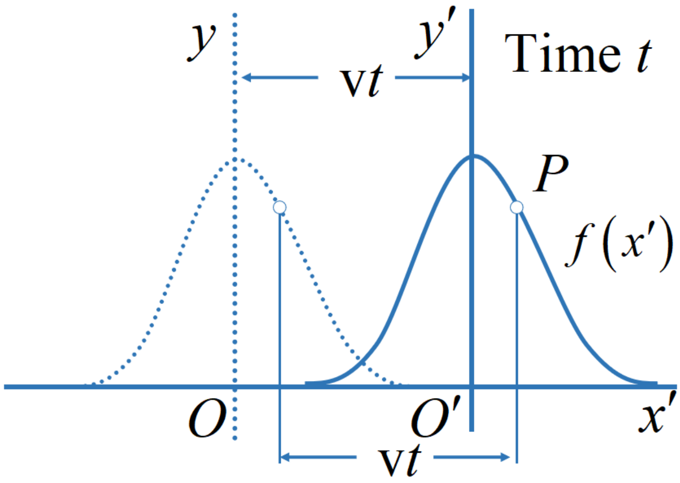


Figure 20. 8: Propagation of a single pulse without change in shape along a medium.

It implies that if you move with speed v, the pulse will continue to move with you, preserving its shape.

$$x - vt = \text{constant}$$

Differentiate w.r.t time

$$\frac{dx}{dt} - v = 0$$

$$\frac{dx}{dt} = v$$

The speed at which the pulse maintains its shape is called **phase velocity**. Phase velocity is independent of frequency, meaning it remains constant regardless of the frequency of the pure tone. Uniform speed for each frequency is crucial to avoid variations in arrival times.

Dispersion in Medium:

Dispersion occurs when there is a correlation between frequency and speed, leading to different frequencies arriving at varying times. **Dispersive medium** is one where different waves move at different speeds. In such a medium, a pulse loses its shape over time, undergoing alterations in its shape. Short peak pulses will spread and flatten over time in a dispersive medium. In cold air, the speed of sound is approximately 330 ms^{-1}. If a moving car emits a sound wave that reaches you, you can deduce that the car is approaching if its speed is slower than the speed of sound (330 m/s). But what happens if it's moving faster? In such a scenario, it is termed supersonic motion.

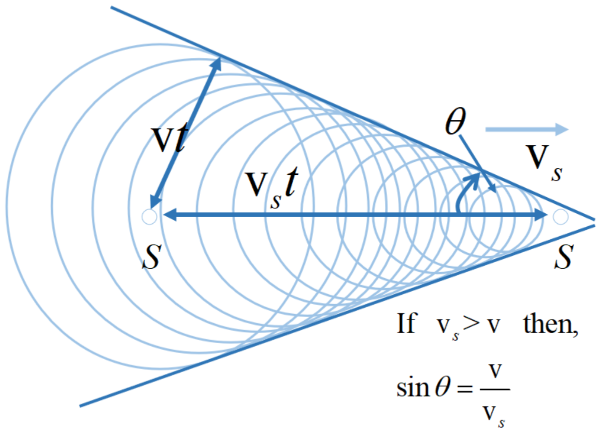


Figure 20. 9: Formation and propagation of shock waves.

Shock Waves:

Supersonic motion occurs when the source moves at a speed surpassing that of sound. This leads to the generation of shock waves or shock fronts, resulting in explosion-like noises. The associated angles (denoted as θ) involve sine θ and cosine θ. The quantification of these angles and their trigonometric functions is expressed through what are known as Mach numbers. **Mach numbers** serve as a numerical representation of the ratio between the speed of a moving object and the speed of sound, providing a standardized measure for describing the intensity and characteristics of supersonic motion. For example, if an object can fly two or three times faster than the speed of sound, it is a Mach Two or Mach Three aircraft.

Applications of Shock Waves:

Shock waves carry substantial energy and find applications in various fields. One practical application is the use of ultrasound shock waves to break down kidney stones, offering a non-surgical solution. The example of using ultrasound shock waves illustrates the potential benefits derived from a profound understanding of wave dynamics.

LECTURE 21

Gravity

Introduction:

Two events were considered very important during early times:

1. Why every thing fall towards the earth.
2. Why stars revolve.

It was assumed that these two occurrences had no connection.

Aristotle proposed that because Earth is the mother of everything, it attracts all other objects to itself. Approximately 2000 years ago, Ptolemy coined the idea of Earth being the centre of the universe and that all celestial objects rotate around it. In contrast, in the 16th century, Nicholas Copernicus proposed that

the Sun is the universe's centre and that the Earth and other celestial bodies orbit around it. In 17th century, Galileo Galilei took his observations andverified the theory of Nicholas Copernicus using his own telescope.
In this lecture we will study the concept of gravity and it’s effect on the objects.

Newton's Law of Universal Gravitation:

It states that the force of attraction between two masses m_1 and m_2 is:

$$F \propto \frac{m_1 m_2}{r^2} \tag{21.1}$$

and is directed along the line joining the two bodies. Putting in a constant of proportionality,

$$F = G\frac{m_1 m_2}{r^2} \tag{21.2}$$

Now let's be a bit careful of the direction of the force. Looking at the **fig. (21.1)** below,

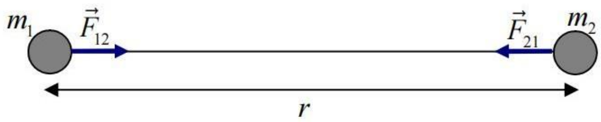
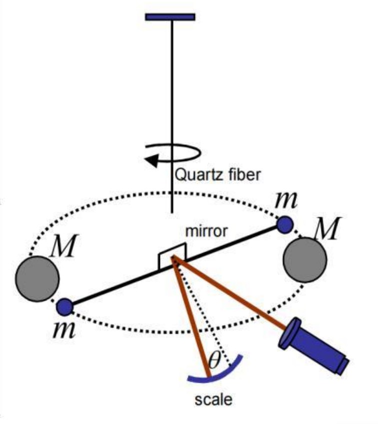


Fig. (21.1): Two masses m_1 and m_2 Exerting force on each other

$\overrightarrow{F}_{21} =$ Force on m_1 by m_2, $\overrightarrow{F}_{12} =$ Force on m_1 by m_2, Both forces are equal in magnitude $|\overrightarrow{F}_{12}| = |\overrightarrow{F}_{21}| = F$, but opposite in direction which can be represented by Newton's Third Law, $\overrightarrow{F}_{12} = -\overrightarrow{F}_{21}$.



The Gravitational Constant:

The gravitational constant G is a very small quantity and needs a very sensitive experiment. An early experiment to find G involved suspending two masses and measuring the attractive force. Suppose the distance between these two masses is L. From the **fig. (21.2)** we can work out an expression for the gravitational torque:

$$\begin{aligned}\tau &= r \times F \\ &= rF\sin\theta\end{aligned} \tag{21.3}$$

Where F is the gravitational force between two masses, $r = \frac{L}{2}$ and $\theta = 90$. We will multiply this expression by 2 because there are two masses m_1 and m_2.

$$\tau = 2\left(\frac{GmM}{r^2}\right)\frac{L}{2} \tag{21.4}$$

A thread provides the restoring torque $\kappa\theta$. κ is the torsion in a thread. The deflection θ can be measured by observing the beam of the light reflected from the small mirror. In equilibrium the torques balance.

$$\frac{GmML}{r^2} = \kappa\theta \tag{21.5}$$

Hence,

$$G = \frac{\kappa\theta r^2}{mML} \tag{21.6}$$

How to find κ?

In Lecture No. 16, we learned that a mass suspended on a string will oscillate and it's oscillating frequency depends on torque. Suppose the mass is suspended, it will oscillate and κ can be found from observing the period of free oscillations,

$$T = 2\pi\sqrt{\frac{I}{k}} \implies \kappa = \frac{2\pi^2 I}{T^2} \tag{21.7}$$

with $I = \frac{mL^2}{2}$. The modern value is $G = 6.67259 \times 10^{-11}\mathrm{Nm^2/kg^2}$.

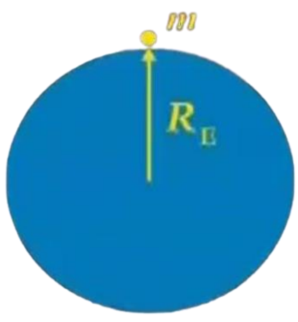
Mass, Volume and Density of the Earth:

The magnitude of the force with which the Earth attracts a body of mass m towards its centre is,

$$F = G\frac{mM_E}{R_E^2} \tag{21.8}$$

where R_E= 6400 km is the radius of the Earth and M_E is the mass. It does not depends in the quality of matter, i.e iron, wood, leather, etc. Objects feel the force in proportion to their masses. According to Newton's 2nd law of motion, if a body fall freely, then it will accelerate with gravitational acceleration. So,

$$F = mg = \frac{GmM_E}{R_E^2} \tag{21.9}$$



We measure g, the acceleration due to gravity, as $9.8\mathrm{m/s^2}$. From this we can immediately deduce the Earth's mass by substituting R_E= 6400 km and $G = 6.67259 \times 10^{-11}\mathrm{Nm^2/kg^2}$:

$$\begin{aligned}M_E &= \frac{gR_E^2}{G} \\ &= \boxed{5.97 \times 10^{24}\mathrm{kg}}\end{aligned} \tag{21.10}$$

What a remarkable achievement!

We can still do more. We can find the volume of the Earth as:

$$\begin{aligned}V_E &= \frac{4}{3}\pi R_E^2 \\ &= \boxed{1.08 \times 10^{21}\mathrm{m^3}}\end{aligned} \tag{21.11}$$

Hence the density of the Earth is:

$$\rho_E = \frac{V_E}{M_E}$$
$$= \boxed{5462\ \mathrm{kgm}^{-3}}$$

So this is 5.462 times greater than the density of water and tells us that theearth must be quite dense inside.

The Gravitational Potential:

The gravitational potential is an important quantity. It is the work done inmoving a unit mass from infinity to a given point R, and equals:

$$V(r) = \frac{U(r)}{m} = -\frac{GM}{R} \qquad (21.12)$$

Proof:

We need to find the work done in bringing a unit mass from the surface of the Earth to a point at infinity (A point where net force is zero). According to conservation of energy,

$$dV = -Fdr \qquad (21.13)$$

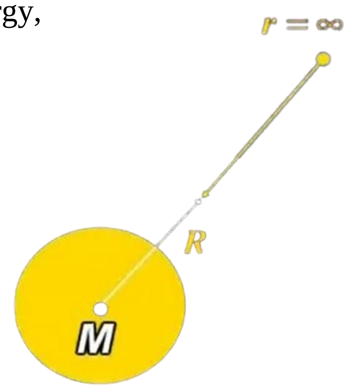
Integrate both sides:

$$\int_{V(R)}^{0} dV = -\int_{R}^{\infty} F(r)dr \qquad .4)$$

$$0 - V(R) = GM\int_{R}^{\infty} \frac{dr}{r^2} \qquad 5)$$

$$V(R) = -GM\left[\frac{1}{r}\right]_R^{\infty} \qquad 6)$$

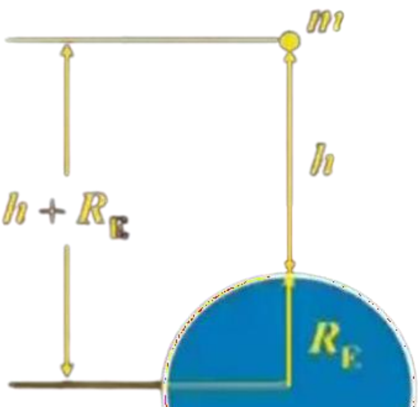
$$V(R) = -\frac{GM}{R} \qquad 7)$$



Negative indicates that the work is done on the object. This work is nowstored in the object in the form of gravitational potential energy.

Change in Potential Energy:

Using Eq. (21.17), let us calculate the change in potential energy ΔU when we raise a body of mass to a height above the Earth's surface. Let R_E be the radius of the earth and $h+R_E$ is the distance from the center of the Earth as shown in the figure.



Hence potential energy ΔU at height h from the surface of the earth will be:

$$\Delta U = U(R+h) - U(R)$$

$$\Delta U = GMm\left(\frac{1}{R_E} - \frac{1}{R_E}\right.$$

$$\Delta U = \frac{GMm}{R_E}\left(1 - \frac{1}{1+\frac{h}{R_E}}\right.$$

$$\Delta U = \frac{GMm}{R_E}\left(1 - \left(1 + \frac{h}{R_E}\right.\right.$$

For $h \ll R_E$, $\left(1 + \frac{h}{R_E}\right)$

Now suppose that the

So, for, we can use **bi**

$$\therefore \Delta U = \frac{GMm}{R_E}\left(1 - \left(1 - \right.\right.$$

$$\Delta U = m\left(\frac{GM}{R^2{}_E}\right)h$$

$$\therefore g = \left(\frac{GM}{R^2{}_E}\right)$$

$$\Delta U = mgh$$

Escape Velocity:

We can use the expression for potential energy and the law of conservation of energy to find the minimum velocity needed for a body to escape the Earth's gravity. Far away from the Earth, the potential energy is zero, and the smallest value for the kinetic energy is zero. Requiring that:

$$(K.E+P.E)_{r=R}=(K.E+P.E)_{r=\infty} \qquad (21.25)$$

Gives,

$$\frac{1}{2}mv_e^2-\frac{GMm}{R_E}=0+0 \qquad (21.26)$$

From this,

$$v_e=\sqrt{\frac{2GM}{R_E}} \qquad (21.27)$$

$$=\sqrt{2gR_E} \qquad (21.28)$$

Putting in some numbers we find that for the Earth, v_e=11.2km/s and for the Sun, v_e = 618km/s.

For a Black Hole, the escape velocity is so high that nothing can escape, even if it could move with the speed of light! (Nevertheless, Black Holes can be observed because when matter falls into them, a certain kind of radiation is emitted.)

$$v_e=\sqrt{\frac{2GM}{R}}$$

Put v_e=c

$$c=\sqrt{\frac{2GM}{R}}$$

$$c^2=\frac{2GM}{R}$$

$$R=\frac{2GM}{c^2}$$

This value of R is known as Schwarzschild radius of a black hole. If a photon(particle of light) is put into the Schwarzschild radius, it will not be able to escape and ultimately it will fall into the black hole. It is interesting to note that the Schwarzschild radius is outside the black hole.

Revolution of a Satellite:

A satellite is in circular orbit over the Earth's surface. The condition for equilibrium,

$$\frac{mv_0^2}{r}=\frac{GMm}{r^2} \qquad (21.29)$$

$$=\sqrt{\frac{GM}{r}} \qquad (21.30)$$

If R_E is the Earth's radius, and h is the height of the satellite above the ground, then $r = R_E + h$. Hence,

$$v_0 = \sqrt{\frac{GM}{R_E + h}} \tag{21.31}$$

For $h << R_E$, we can approximate,

$$v_0 = \sqrt{\frac{GM}{R_E}} \tag{21.32}$$

$$= \sqrt{gR_E} \qquad \because (g = \frac{GM}{R_E^2}) \tag{21.33}$$

We can easily calculate the time for one complete revolution,

$$T = \frac{2\pi}{\omega} \tag{21.34}$$

$$= \frac{2\pi r}{v_o} \tag{21.35}$$

$$= 2\pi r\sqrt{\frac{r}{GM}} \tag{21.36}$$

$$= \frac{2\pi}{\sqrt{GM}} r^{\frac{3}{2}} \tag{21.37}$$

This gives the important result, observed by Kepler nearly 3 centuries ago that,

$$T^2 = \frac{4\pi^2}{GM} r^3 \tag{21.38}$$

Or $T^2 \propto r^3$

Total Energy of a Satellite:

What is the total energy of a satellite moving in a circular orbit around the earth? Clearly, it has two parts, kinetic and potential. Remember that the potential energy is negative. So,

$$E = K.E + P.E \tag{21.39}$$

$$= \frac{1}{2} m v_0^2 - \frac{GM_E m}{r} \tag{21.40}$$

But, $v_0^2 = \frac{GM}{r}$ as we saw earlier and therefore,

$$E = \frac{1}{2}\frac{GMm}{r} - \frac{GMm}{r} \tag{21.41}$$

$$= -\frac{1}{2}\frac{GMm}{r} \tag{21.42}$$

Note that the magnitude of the potential energy is larger than the kinetic energy. If it wasn't, the satellite would not be bound to the Earth!
The negative sign indicates that the work has to be done on the satellite when taken from a point on the Earth to a point at infinity.

Geostationary Satellite:

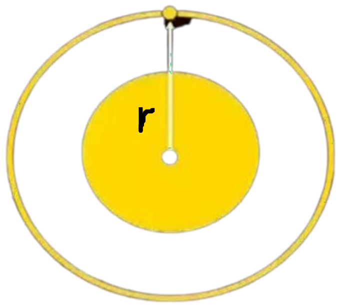
Geo means “Earth” and stationary means “at rest”. A satellite which revolve around the earth with rotation frequency equal to that of the Earth, (1 rotation = 24 hr). It appears to be stationary with respect to the Earth.
Radius (the distance from center of the Earth) of this satellite can be found:

$$T^2 = \frac{4\pi^2}{GM} r^3 \tag{21.43}$$

Rearranging to find r. Multiply and divide by R

$$r^3 = \left(\frac{GM}{R}\right)\frac{RT^2}{4\pi^2} \tag{21.44}$$

$$= g\frac{RT^2}{4\pi^2} \tag{21.45}$$



Taking cube root on both sides: (21.46)

$$r = \sqrt[3]{g\frac{RT^2}{4\pi^2}}$$

$$= \boxed{42.33 \times 10^6 \mathrm{m}}$$

Height of the satellite from the surface of earth:

$$r - R = 35.93 \times 10^6 \mathrm{m}$$

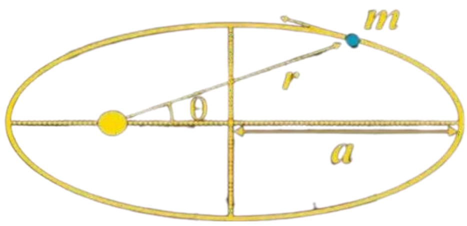
Fig. (21.3): Satellite ata distance r from the centre of the Earth

Kepler's Laws:

German astronomer Johannes Kepler deduce a mathematical model for the motion of the planets. Kepler's complete analysis of planetary motion is summarized in three laws known as Kepler's laws.

• Kepler's 1st Law:

It states that: All planets move in elliptical orbits with the Sun at one focus.



It is also know as Law of Orbits.

• Kepler's 2nd Law:

A famous discovery of the astronomer Johann Kepler some 300 years ago says that, the line joining a planet to the Sun sweeps out equal areas in equal intervals of time.

We can easily see this from the conservation of angular momentum. Call ΔA the area swept out in time . Then from the figure we can see that,

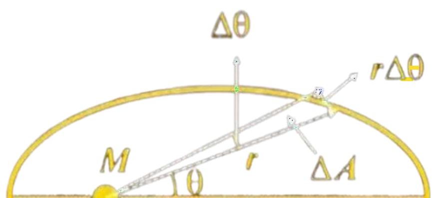
$$\Delta A = \frac{1}{2} r(r\Delta\theta) \tag{21.47}$$

Divide this by Δt and then take the limit where it becomes very small.

$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} \tag{21.48}$$

$$= \frac{1}{2} r^2 \left(\lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t}\right) \tag{21.49}$$

$$= \frac{1}{2} r^2 \omega$$



(21.50)

Recall from Lecture No. 13, Angular

(21.51)

momentum[2] is the momentum multiplied by the perpendicular distance.

$$\begin{aligned}\frac{dA}{dt} &= \frac{1}{2}r^2\omega \\ &= \frac{1}{2}r^2\left(\frac{v}{r}\right) && \because (v = r\omega) \\ &= \frac{1}{2}\frac{\mathrm{r}}{m}mv && \because (\text{Multiply and Divide by } m) \\ &= \frac{1}{2}\frac{\mathrm{r}}{m}p && \because (p = mv) \\ &= \frac{L}{2m} && \because (L = rp)\end{aligned}$$

Since L is a constant, let's see why. Angular momentum changes only when torque acts on it. Remember from previous lectures that:

$$\frac{dL}{dt} = \tau$$

If $\tau = 0$ then $\frac{dL}{dt} = 0$ and L is a constant. $\tau = 0$ because the force acting on the Sun due to the planet, is central means it is acting along the radial direction towards the center of the Sun, hence:

$$\begin{aligned}\vec{r} \times \vec{F} &= \left(\frac{GMm}{r^2}\right)\vec{r} \times \hat{r} \\ &= 0\end{aligned}$$

We have proved one of Kepler's laws (with so little effort)!

- ## Kepler's 3rd Law:

Also known as Law of Periods. It states that: The square of the period of revolution of any planet is proportional to the cube of the planet distance from the Sun. Lets prove this result for circular orbits. (21.52)

Proof:

Suppose a planet orbits around the sun. The gravitational force acting towards the center of the sun will be equal to the centripetal force[3] acting on the planet. Hence:

$$\begin{aligned}F_g &= F_c \\ \frac{GMm}{r^2} &= m\omega r^2\end{aligned}$$

Where $\omega = \frac{2\pi}{T}$.

$$\frac{GM}{r^2} = \left(\frac{2\pi}{T}\right)^2 r, \text{ or} \tag{21.53}$$

$$T^2 = \left(\frac{4\pi^2}{GM}\right)r^3 \tag{21.54}$$

Example:

Suppose a planet revolves around a sun in elliptical orbit. If it has a speed v_1 at a distance d_1 from the Sun. What will be its speed at distance d_2 ?

Solution:

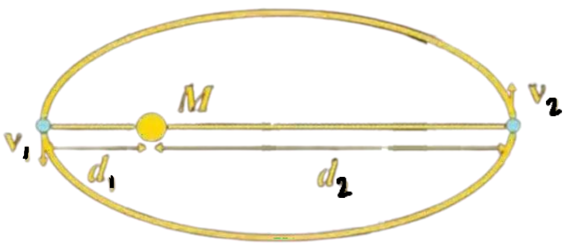
Using the law of conservation of angular momentum:

[2] Angular momentum: $L = r \times p$

[3] Centripetal force: $F = \frac{mv^2}{r} = m\omega r^2$. More detail in Lecture No. 11

$$mv_1d_1 = mv_2d_2 \quad (21.55)$$

$$v_1 = \frac{v_2d_2}{d_1} \quad (21.56)$$



According to conservation of energy:

$$-\frac{GMm}{d_1} + \frac{1}{2}mv_1^2 = -\frac{GMm}{d_2} + \frac{1}{2}mv_2^2 \quad (21.57)$$

Substituting v_1 in above equation to find v_2,

$$-\frac{GMm}{d_1} + \frac{1}{2}m\left(\frac{v_2d_2}{d_1}\right)^2 = -\frac{GMm}{d_2} + \frac{1}{2}mv_2^2 \quad (21.58)$$

$$\frac{1}{2}m\frac{v_2^2d_2^2}{d_1^2} - \frac{1}{2}mv_2^2 = \frac{GMm}{d_1} - \frac{GMm}{d_2}$$

$$\frac{1}{2}mv_2^2\left(\frac{d_2^2}{d_1^2} - 1\right) = GMm\left(\frac{1}{d_1} - \frac{1}{d_2}\right)$$

$$\frac{1}{2}mv_2^2\left(\frac{d_2^2 - d_1^2}{d_1^2}\right) = GMm\left(\frac{d_2 - d_1}{d_1d_2}\right)$$

$$v_2^2 = \frac{2GMm}{m}\left(\frac{d_1^2(d_2 - d_1)}{d_1d_2(d_2^2 - d_1^2)}\right)$$

Using $a^2 - b^2 = (a+b)(a-b)$ in

$$v_2^2 = 2GM\left(\frac{d_1(d_2 - d_1)}{d_2(d_2 - d_1)(d_2 + d_1)}\right)$$

$$v_2^2 = 2GM\left(\frac{d_1}{d_2(d_2 + d_1)}\right)$$

$$v_2 = \sqrt{\frac{2GMd_1}{d_2(d_1 + d_2)}}$$

Substituting value in Eq. (21.55)

$$mv_1d_1 = mv_2d_2$$

$$= m\sqrt{\frac{2GMd_1}{d_2(d_1 + d_2)}}d_2$$

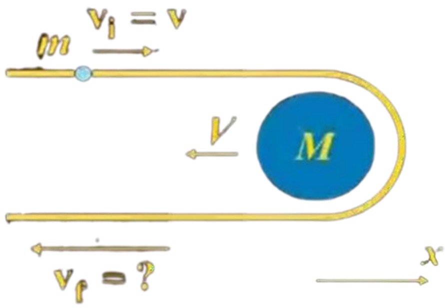
$$= m\sqrt{\frac{2GMd_1d_2^2}{d_2(d_1 + d_2)}}$$

$$= m\sqrt{\frac{2GMd_1d_2}{(d_1 + d_2)}}$$

The Sling Shot Effect:

Sling shot effect is a non-impact collision used to give an extra boost to spacecraft. This collision is one dimensional elastic collision.

Figure shows a planet with mass M and velocity V. A stallite of mass m moving with velocity v_i is directed to pass near the Planet to increase it's velocity to v_f. Such that $M >> m$ and V remains constant.

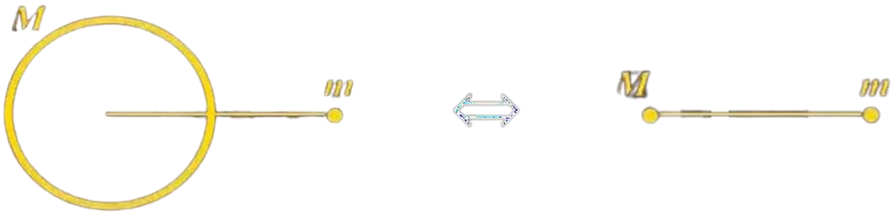


Energy and momentum remainconserved during elastic collision in onedimension. In Lecture No. 10, we already proved that the relative velocity before collision is equal to the relative velocity after collision.

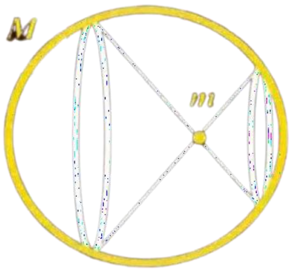
$$v_i + V = -(v_f + V) \tag{21.59}$$

$$v_f = -(v + 2V) \tag{21.60}$$

It is important to note that "A uniformly dense spherical shell attracts an external point mass as if the mass of the shell were concentrated at its center", as show in the figure below.



- ## Gravitational effect of spherical distribution of matter:



Another important point to note is, "A uniform spherical shell of matter exerts no gravitational force on a particle located inside it." In other words, the sum of all the forces acting on a massm at any point inside a uniform spherical shell is, zero.

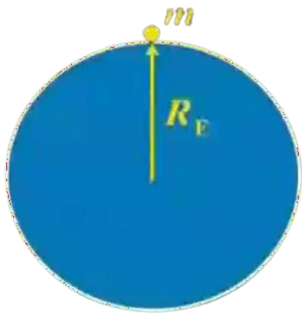
Variation in Gravitational Acceleration (g):

- **At the Surface of the Earth:**

The value of g at the surface of the earth can be found using Newton's law of universal gravitation. The magnitude of force with which Earth attracts the body towards its centre is:

$$F = mg = \frac{GM_E m}{R_E^2} \tag{21.61}$$

$$g = \frac{GM_E}{R_E^2} \tag{21.62}$$

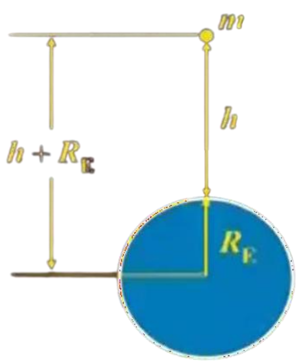


Note that the value of g will be constant all over the surface for a perfectly spherical Earth because it is inversely proportional to R_E^2.

- ## At some height h above the surface:

The magnitude of force with which Earth attracts the body towards its centre is:

$$F = \frac{GM_E m}{(h+R_E)^2} \tag{21.63}$$



According to Newton's second law:

$$F = mg_h = \frac{GM_E m}{(h+R_E)^2} \tag{21.64}$$

$$g_h = \frac{GM_E}{(h+R_E)^2} \tag{21.65}$$

From here, we can find the effective value of g:

$$g_h = \frac{GM_E}{(h+R_E)^2} \tag{21.66}$$

$$= \frac{GM_E}{R_E^2\left(1+\frac{h}{R_E}\right)^2} \tag{21.67}$$

$$= \frac{\left(\frac{GM_E}{R_E^2}\right)}{\left(1+\frac{h}{R_E}\right)^2} \tag{21.68}$$

$$= \frac{g}{\left(1+\frac{h}{R_E}\right)^2} \tag{21.69}$$

For $h << R_E$, or $\dfrac{h}{R_E} << 1$

$$g_h = g\left(1+\frac{h}{R_E}\right)^{-2} \tag{21.70}$$

$$= g\left(1-\frac{2h}{R_E}\right)+\ldots\ldots \tag{21.71}$$

For $h = 0$, $_h =$

As discuused in the section of the sling shot effect, when a body of mass m is on the surface or above the surface of the Earth, the Earth works like a point particle, as if its total mass is concentrated at its centre.

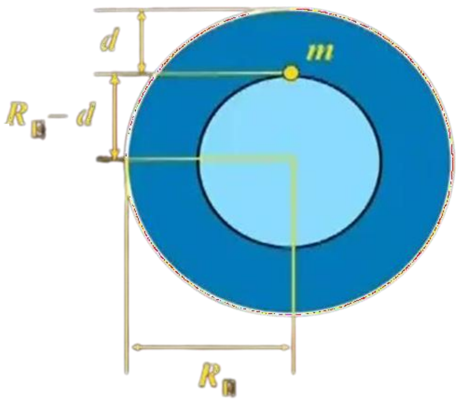
- ## Below the surface of the Earth:

As the particle lies inside the shell of radius , there is no force on the particledue to the shell. The only force exerted comes from the sphere of radius −
. The mass of this smaller sphere is:

$$M_E' = \rho_E\left(\frac{4}{3}\pi(R_E-d)^3\right) \tag{21.72}$$

But the density of this sphere is:

$$\rho_E = \frac{M_E}{\left(\frac{4}{3}\pi R_E^3\right)} \tag{21.73}$$



Now we can find out the effective acceleration due to gravity. The force acting on the mass m will be:

$$F = mg_d = \frac{GM'_E m}{(R_E - d)^2} \tag{21.74}$$

$$g_d = \frac{G\left[\rho_E\left(\frac{4}{3}\pi(R_E - d)^3\right)\right]}{(R_E - d)^2} \qquad \because \text{(Substitute Eq. (21.72}$$

$$= G\rho_E\left(\frac{4}{3}\pi(R_E - d)\right) \tag{21.76}$$

$$= G\rho_E\frac{4}{3}\pi R_E\left(1 - \frac{d}{R_E}\right) \qquad \because \text{(Take } R_E \text{ common)}$$

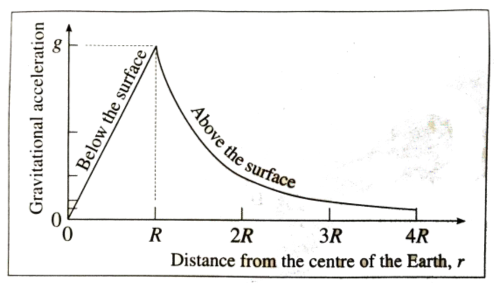
Now multiply and divide by R_E^2

$$g_d = \frac{G\rho_E\frac{4}{3}\pi R_E^3}{R_E^2}\left(1 - \frac{d}{R_E}\right) \qquad \because \text{(Multiply and Divide by } R_E^2\text{)}$$

$$= \frac{GM_E}{R_E^2}\left(1 - \frac{d}{R_E}\right) \qquad \because \text{(Substitute } M_E \text{ from Eq. (21.73))}$$

$$g_d = g\left(1 - \frac{d}{R_E}\right) \qquad \because \text{(Substitute } g = \frac{GM_E}{R_E^2}\text{)}$$

From the above equation, we realize that $g_d = g$ when $= 0$. If we increase d then decreases which implies that effective gravitational acceleration decreases as we go below the surface of the Earth. At the center of the Earth where $=$ the valueof $= 0$. Hence there is no gravitational force at the center of the Earth as shown in the figure.



Variation of the gravitational acceleration due to the Earth with distance from the centre of the Earth

Physics - PHY101 - Lecture # 22-

ELECTROSTATICS - I

During the last several lectures, we have talked to you about force. Force produces acceleration in a body. The greater the force, the greater will be the acceleration of that body. Similarly, Gravity is the force that pulls every object towards the center of the earth. Gravity is a fundamental force but we also discussed other forces like Force of friction which is not a fundamental force. It arises when two surfaces come in contact with each other and at microscopic level their charges interact/attract and this is an electric force.

Today's lecture will be related to electric force and specifically electrostatics. By electrostatics, it means that when the charges are at rest or stationery. We have knowledge about these electric force since ancient times and this is also in our daily life observation. For example, when it is winter and we comb our hair. By rubbing the comb, your hair straightens up and then you must have also observed that if you bring the same comb near the pieces of paper, then the pieces of paper rise are pulled towards it. We will observe this here as well (as demonstrated in video lecture). Surely there must be some reason why the comb attracts the pieces of paper towards itself, but before that, let's observe the same effect with another material. So, there is a glass rod and the comb is rubbed with the coat. Now as seen in the lecture (and also depicted by figure 22(a)), comb also attracts this glass rod towards itself. Is the electric force always attractive as it pulls things towards it? Now let's prove by another experiment that this is not always the case. We rub both the combs with coat and suspend one of then with the string and bring both the combs closer. They repel each other as shown in video lecture (and also depicted by figure 22(b)). This is repulsive electrostatic force as is pushes away the objects. So electrostatic forces can be both attractive as well as repulsive.

We know that because there are two different charges negative and positive. For now we are not concerned which one is positive and which one is negative, more importantly negative repel negative, positive repels positive away but negative and positive attract each other. Where these originate from? There are small units of charge and that charge is present on the electron. An electron has a charge on it and conventionally we call electrons negatively charged. The proton inside the atom is assumed to be positively charged. So, it doesn't matter who we call minus and who we call plus, but once we say that, it is determined that the electron is negative, then it becomes necessary to call the proton positive. When we rub two materials together, one becomes positively charged and the other negatively charged, so can we tell which one will be negative and which one will not be positive? It's not that easy because we need to understand the process by which charge separation occurs. Notice that every material has molecules and atoms. When rubbed together and due to friction, electrons move from one material to another material. It goes into the material where it feels more attraction and where it can reduce its energy. Now because the materials are complicated, we cannot be sure about the electrons movement if we rub silk over cotton but it can be calculated in principle.

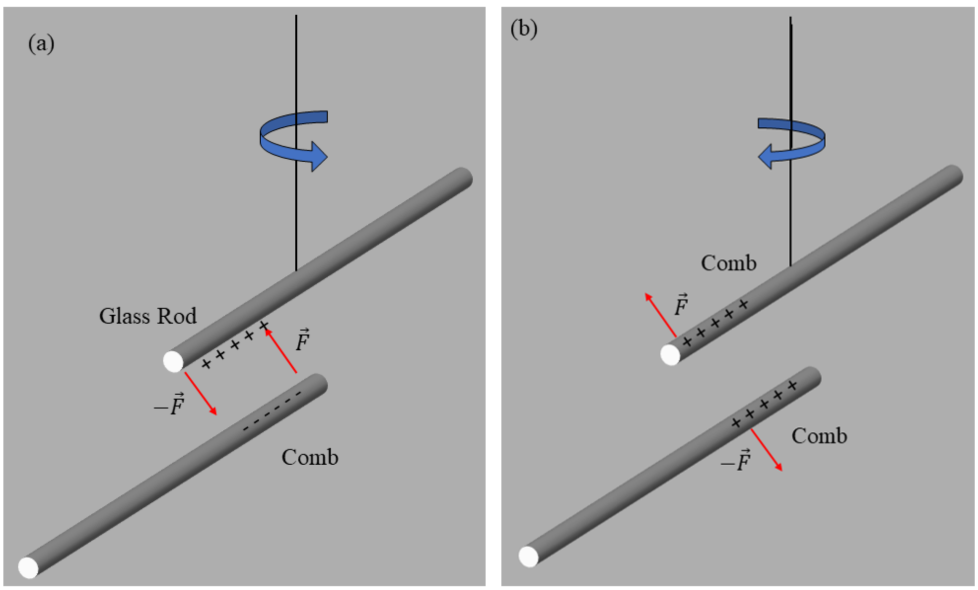


Figure 22.1: (a) Glass rod and comb attractive interaction as comb was rubbed against the coat, (b) Both combs repel each other after being rubbed against coat.

Now we would like to talk about different materials that how electrons interact in different materials. First of all, we will discuss conductors. Inside a conductor, there are a lot of electrons. One cubic centimeter (cm^3) has 10^{23} to 10^{26} number of electrons available for conduction, meaning that under the right condition electrons can be moved from one place to another without much resistance. So, conductors conduct electricity easily meaning electrons can pass through conductors very easily.

Metals are generally good conductors. But if we dissolve table salt (sodium chloride NaCl) in water, that water also becomes a conductor. Sodium chloride has two different types of atoms, sodium and chlorine. Sodium is positively charged, and Chlorine is negatively charge. When these atoms are present in water in from of ions, they move towards respective applied polarity (opposite to each other) and in this way charges flow. On the contrary, there are many such materials such as plastic, wood pure water, through which the passage of electricity or the charge/electrons flow is not possible. This is due to the fact that the electrons are bond to the atoms. It is not possible for these electrons to hop from one atom to another atom as this requires a lot of energy, so such materials are called insulators.

There is another material in between conductors and insulators which is called semiconductors. In semiconductors, the number of such electrons is very small, which can become a part of conduction, which means that there are electrons inside the semiconductor but their transportation from one atom to another is very difficult. If any impurity is included in it or light is shone on semiconductor then the number of free electrons increases in semiconductors. Semiconductors material are of great importance as the transistor (which is basic building block) of integrated circuit (IC) technology driving computers is made up from semiconductor material. Two important semiconductors are silicon and Germanium. Silicon is especially important because it proves to be more suitable.

Coulomb’s Law

When we talked about gravity, at that time it was observed that if there are two-point masses then there is a force of attraction between them which is directly proportional to the product of their masses divided by the square of distance between them. Similar to it is another law of force for the charges that was first discovered by Coulomb about 200 years ago.

According to Coulomb's law, the force between two charges will be positive if they are of the similar and will be negative if they are of different types. The magnitude of force will be proportional to the product of charges divided by the square of the distance between them. Mathematically Coulomb’s law is given as,

$$F \propto \frac{q_1 q_2}{r^2} \tag{22.1}$$

Where q_1 and q_2 are charges and r is distance between these charges. It is necessary to put a proportional constant K, so eq. 22.1 equation can be rewritten as

$$F = K\frac{q_1 q_2}{r^2} \qquad 22.2$$

In MKS system the unit of charge is coulomb (C). When we write a formula, the dimensions should be equal on both the left side and the right side. Referring to eq. 22.2, if the force is on the left side having unit Newton (N), then the units of K on the right side be such that overall units on both sides to be equal. Since the units of charge in coulombs, the value of K then becomes 8.99 x 10^9 N-m^2/C^2.

The final form of Coulomb's Law

$$F = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q_1 q_2}{r^2} \qquad 22.3$$

Where

$$\varepsilon_0 = 8.85\times10^{-12} C^2 / Nm^2 \qquad 22.5$$

and

$$K = \frac{1}{4\pi\varepsilon_0} = 8.99\times10^9 Nm^2 / C^2 \qquad 22.4$$

If we write this in vector form (as shown in fig. 22.2), then we see that the force between two charges is that along the unit vector. As we know, unit vector whose length is unit length. we can write Coulomb's Law using unit vectors as follows

$$\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q_1 q_2}{r_{12}{}^2}\hat{r}_{12} \qquad 22.5$$

This is the force exerted on 1 due to 2 (F_{12}), where

$$\hat{r}_{12} = \frac{\vec{r}_{12}}{r_{12}}$$

Contrary to this force exerted on 2 due to 1 (F_{21}) will be

$$\vec{F}_{21} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{{r_{21}}^2} \hat{r}_{21} \qquad 22.6$$

Magnitude of distance squared between both charges will be same ($r_{21}^2 = r_{12}^2$). The only difference is of the unit vector. One unit vector goes from 1 to 2 and the other unit vector goes from 2 to 1 which can be mathematically written as

$$\hat{r}_{12} = -\hat{r}_{21}$$

$$\vec{F}_{12} = -\vec{F}_{21}$$

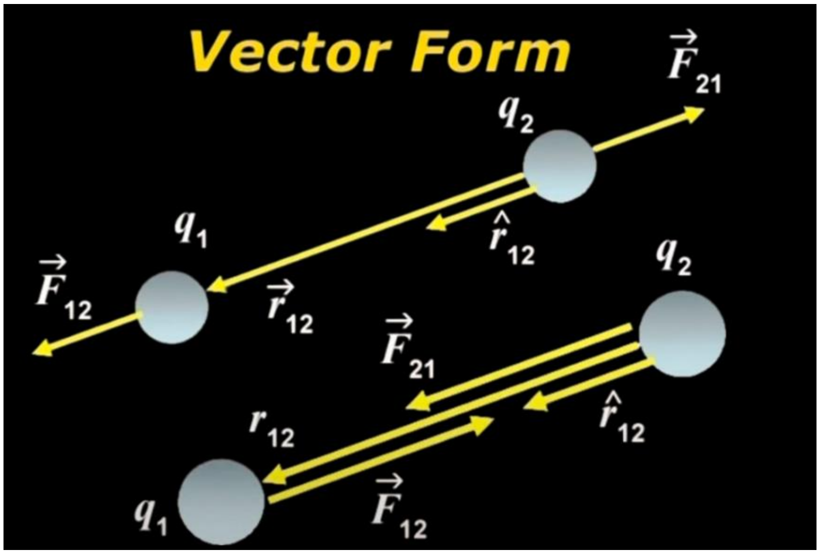


Figure 22.2: Coulomb force in vector form. Upper scenario is for similar charges and lower scenario is for dissimilar charges.

If there are many charges, then effect of each charge in from force will be added up so the force on charge number 1 (F_1) due other charges can be written as.

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \ldots\ldots..$$

Charge is Quantized

A very interesting thing about charges is that it has a minimum amount. Charge of the electron (e) is the minimum charge and charge less than has never been found. The proton also has the same charge, but it is positive, but the quantity is never less than that of charge of single electron. It can definitely be more than this. For example, if we consider any metal, its total charge can be 10^{20} times the charge of the electron (e). It means there is no upper limit to this but it cannot be less than this. This is called quantization of electric charge. Mathematically, total charge q is given by

$$q = ne$$

where

$$e = 1.602 \times 10^{-9} C$$

and

$$n = 0, \pm 1, \pm 2, \pm 3, ...$$

Neutron and proton are made of smaller particles which are called quarks and which are fractionally charged. For example, the proton is made up of three quarks, two u quarks and one d quark. u quark charge has two thirds (2/3) of the charge of the proton and d quark has a charge negative one third (-1/3) of the charge of the proton so we add up these the charge of the proton becomes equal to 1 as follows.

$$\overset{u}{\frac{2}{3} + \frac{2}{3}} - \overset{d}{\frac{1}{3}} = \frac{2+2-1}{3} = \frac{3}{3} = +1$$

Neutron has two d quark and one u quark, so net charge becomes zero and we know that neutron is neutral.

$$\overbrace{-\frac{1}{3} - \frac{1}{3}}^{d} + \overset{u}{\frac{2}{3}} = \frac{-1-1+2}{3} = \frac{0}{3} = 0$$

Conservation of Charge

Charges can neither be created nor destroyed, which means that charge is conserved. Chemical reactions and particle interactions often demonstrate fundamental principles of physics, including the conservation of charge. In chemical reactions, when a positively charged ion combines with a negatively charged ion, the result is a neutral molecule with a net charge of zero. This process is a direct demonstration of charge conservation, where the total charge before and after the reaction remains the same. On a more fundamental level, elementary particles also exhibit this principle. For instance, when a positron (a positively charged electron, e^+) collides with an electron (negatively charged, e^-), they annihilate each other, producing photons $\gamma + \gamma$. Here, gamma (γ) represents a photon, a quantum of light that is electrically neutral. This interaction conserves the total charge since the initial charge $(+1+(-1) = 0)$ is equal to the total charge of the resulting photons (neutral). Another example is the decay of a pion (π^o), a neutral particle, into two photons. In this process, the net charge remains zero on both sides of the equation. Initially, the pion has no net charge, and the two resulting photons are also neutral, preserving the charge balance. These examples underscore a critical concept in physics: charge conservation. Whether in chemical reactions, elementary particle interactions, or other phenomena involving charged particles, the total charge before and after a reaction or interaction remains constant. This principle holds universally across all systems and is a cornerstone of both classical and modern physics.

What is FIELD?

The concept of a field is fundamental in physics, especially when analyzing forces such as electric and gravitational forces. A field represents a quantity that is defined at every point in space and may also vary with time. For example, consider the gravitational force acting on a mass. Near the surface of the Earth, this force causes objects to fall downward, and its strength is greater when closer to the surface. As one moves further away into space, the gravitational force decreases with increasing distance from the Earth's center. Despite this variation, the gravitational force always acts on the mass, regardless of its position. This leads to the concept of a gravitational field, which describes how gravitational force varies with position.

In general, a field can be understood as a function of position and, sometimes, time. For instance, imagine measuring the temperature at different points using a thermometer. At one location, the temperature might be 20 degrees Celsius, while at another location, it might be 30 degrees Celsius. Additionally, the temperature at any given point can change over time due to weather fluctuations. This means temperature depends on spatial coordinates and time, making it an example of a scalar field. A scalar field assigns a single numerical value to each point in space and time without any associated direction. Temperature, density, and similar quantities are common examples of scalar fields. Some examples of scalar field are as follows.

Temperature	$T(x, y, z, t)$
Density	$\rho(x, y, z, t)$

In contrast, a vector field assigns a quantity with both magnitude and direction to every point in space and time. An example of a vector field is wind velocity. At any given location, the wind has a specific speed and direction. To describe it fully, one must specify how fast it is blowing and in which direction it moves. Thus, wind velocity is a vector field that varies across space and time. A simple analogy to visualize a vector field is a sugarcane field, where each sugarcane represents a vector. Some sugarcanes are tall, while others are short, and some may be straight while others are bent. The variation in height and direction across the field forms a vector field. Vector field in 3D is represented by three function each representing component along each axis.

$$\vec{V}(x, v, z, t) = \{V_1(x, y, z, t), V_2(x, y, z, t), V_3(x, y, z, t)\}$$

Where subscripts (1, 2 and 3) are interchangeable with x, y and z, respectively. Scalar and vector fields are powerful tools for describing physical phenomena.

Electric Field

Let us now consider a specific type of field: the electric field. If a small charge is placed in an electric field, a force acts on it. The magnitude of this force is proportional to the product of the charge and the strength of the electric field at that point. The electric field $\vec{E}$ can be defined as the force $\vec{F}$ experienced by a charge divided by the magnitude of the charge q_0.

$$\vec{E} = \frac{\vec{F}}{q_0}$$

For a unit charge, the magnitude of the force and the electric field become numerically equal. Thus, the electric field represents a region where a charge experiences a force when placed within it. Importantly, the introduction of the charge should not disturb the existing electric field. To ensure this, the charge used to measure the field is assumed to be infinitesimally small, often referred to as a "test charge." This test charge allows us to probe the electric field without altering it.

The electric field is often represented visually using electric field lines. These lines have specific properties: they originate from positive charges and terminate on negative charges. The shape and direction of the electric field lines can be determined by placing a test charge in the field. The force acting on the test charge at various points indicates the direction and shape of the electric field lines in that region. The number and density of electric field lines provide information about the strength of the electric field. A small charge generates fewer field lines, while a larger charge produces more lines. The density of these lines in a given region corresponds to the intensity of the electric field; a higher density indicates a stronger field. Additionally, the direction of the electric field at any point can be determined by the tangent to the electric field line at that point. For example, if two positive charges are placed near each other, the electric field between them forms a distinct pattern as shown in figure 22.3(a).

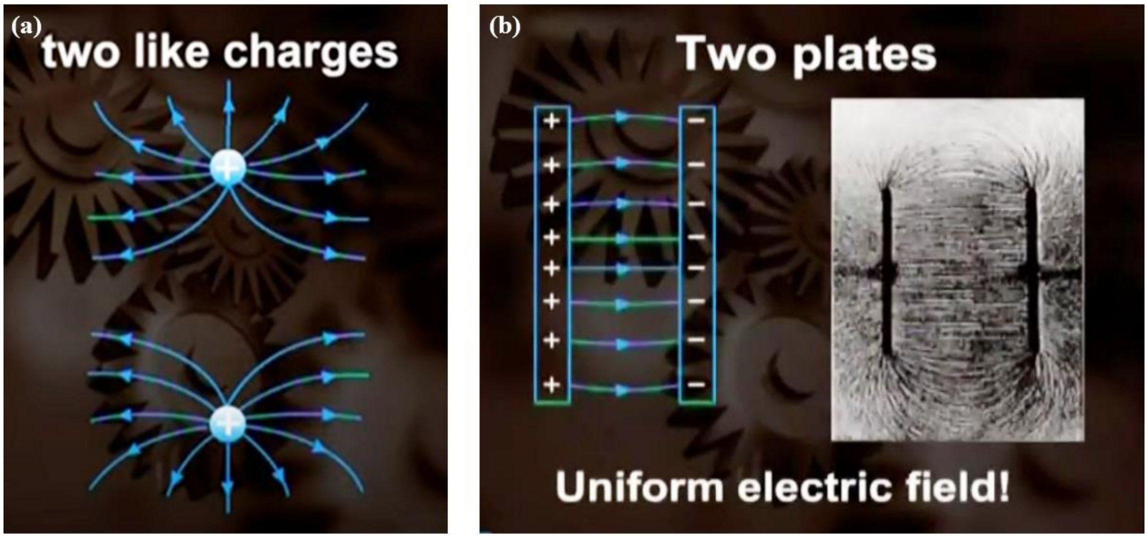


Figure 22.3: (a) Electric field lines for two positive charges. (b) Electric field line for oppositely charged parallel plates.

When two plates are charged with opposite types of charge, positive on one plate and negative on the other, electric field lines originate from the positively charged plate and terminate at the negatively charged plate as depicted in figure 22.3(b). These lines are parallel between the plates, indicating a uniform electric field in this configuration. However, if a charged plate and a charged ring are considered, a non-uniform electric field is generated. Inside the ring, the electric field is effectively zero, while outside the ring, the field strength varies from one region to another as shown in figure 22.4.

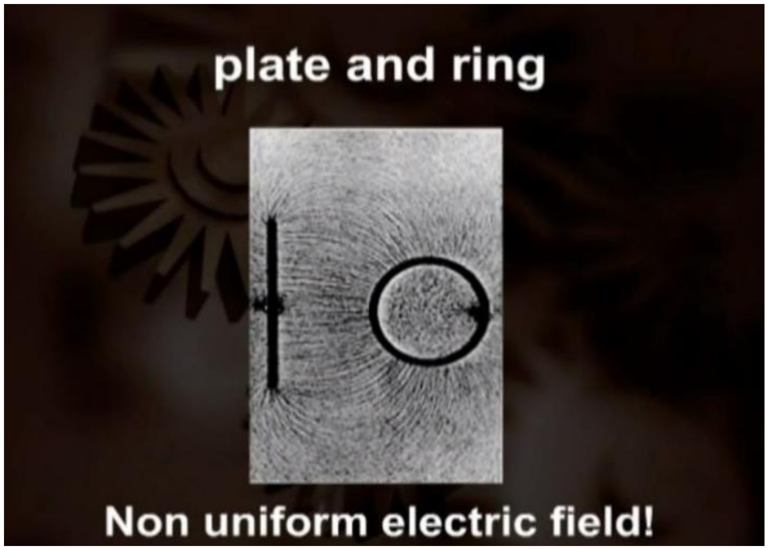


Figure 22.4: Electric field lines plate and ring.

The units of the electric field are derived from the relationship between force and charge. Force is measured in newton (N), while charge is measured in coulombs (C). As a result, the electric field is expressed in units of newton per coulomb (N/C), representing the force experienced by a unit charge in an electric field.

Typical values of Electric fields

The magnitude of the electric field depends on the specific situation. For example, inside an atom, the electric field strength is extremely high due to the proximity of charged particles like protons and electrons. Within an atom, the electric field intensity can reach values of the order of 10^{11} newton per coulomb. Protons in the nucleus create a strong positive effect near the center, while electrons orbiting at a distance partially neutralize this field. However, close to the nucleus, the positive influence of protons dominates.

The electric field plays a crucial role in numerous practical applications. For instance, in a television tube, electrons are directed toward the screen, which is coated with a phosphor material. These electrons are accelerated by an electric field of intensity approximately 10^5 newton per coulomb. Upon striking the phosphor, photons are emitted, creating the visual images we see. The electric field in this scenario is significantly weaker than the field inside an atom but remains crucial for the operation of the television.

In atmospheric conditions, the electric field near the Earth's surface typically measures around 10^2 newton per coulomb. If a measuring device were placed in this field, its free electrons would move under the influence of the field, allowing the measurement of its intensity. Electric fields are also present in conductors, such as copper wires. In such materials, the field drives the flow of electrons, creating an electric current. However, the field inside a perfect conductor would be zero because even a minimal electric field would induce an infinite current due to the conductor's ideal properties.

In air, the behavior of the electric field is distinct. Air molecules, like those in oxygen and nitrogen, tightly bind their electrons. A high electric field is required to separate these electrons from their parent atoms, and when this happens, ionization occurs. Ionization can lead to phenomena such as sparking. For instance, in a spark plug of a motorcycle, the electric field across a small gap between two electrodes becomes intense enough to ionize the air, creating a spark. This spark ignites the fuel mixture, causing combustion, which produces gases that push the piston and generate motion.

Lightning

The principle behind lightning is rooted in atmospheric electric fields. As the sun heats the Earth's surface, warm air masses rise and collide with cooler air masses, leading to ionization. During this process, atoms lose electrons, forming positively charged ions and free electrons. These charges accumulate in clouds, creating a significant electric field between the clouds and the ground. When the intensity of this electric field exceeds a critical threshold, it results in a discharge, producing lightning.

Lightning can manifest in various forms, from illuminating clouds to striking objects like trees or buildings. The intense current associated with a lightning strike can cause severe damage to anything in its path. To mitigate this, lightning rods were invented by Benjamin Franklin over 250 years ago. By providing a controlled pathway for the electric charge to dissipate harmlessly into the ground, lightning rods protect structures from the destructive effects of lightning. Franklin's discovery came during an experiment where he flew a kite during a thunderstorm. He observed that a key tied to the kite became charged, confirming the relationship between lightning and electric charge. This insight laid the foundation for modern lightning protection systems. Today, lightning rods are installed on buildings and other structures to ensure safety during storms.

Calculating Electric Field

Every electric charge generates an electric field around it. The strength and direction of this field can be represented using electric field lines, which illustrate the influence of the charge on its surroundings. The intensity of the electric field at a point can be determined using Coulomb's law. According to Coulomb's law, the force $\vec{F}$ between two charges is directly proportional to the product of their magnitudes and inversely proportional to the square of the distance between them. This relationship is governed by a proportionality constant. To quantify the electric field, we introduce a test charge q_0 at a point in the field. The electric field strength (denoted as $\vec{E}$) at that point is defined as the force experienced by the test charge per unit charge.

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r^2}$$

$$\vec{E} = \frac{\vec{F}}{q_0} = \frac{\frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r^2}}{q_0} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

Using this concept, the electric field generated by a point charge is calculated, showing that the field strength is directly proportional to charge q and decreases with the square of the distance from the charge. Electric field lines help visualize this field, emerging radially outward from a positive charge and inward toward a negative charge. These lines provide a graphical representation of the field's magnitude and direction at various points in space (as discussed earlier).

Millikan’s Experiment

The smallest known discrete charge in nature is the charge of the electron. While it is not infinitesimally small, it represents the fundamental unit of electric charge. The first precise measurement of this charge was conducted by Robert Millikan through his famous oil-drop experiment. Millikan devised an ingenious method to determine the charge of the electron by observing the behavior of tiny oil droplets in an electric field. In this experiment (as depicted by figure 22.5), oil droplets are introduced into a chamber using an atomizer, which functions like a spray gun. As the droplets enter the chamber, some of them become ionized. Ionization can occur naturally, such as through friction or rubbing.

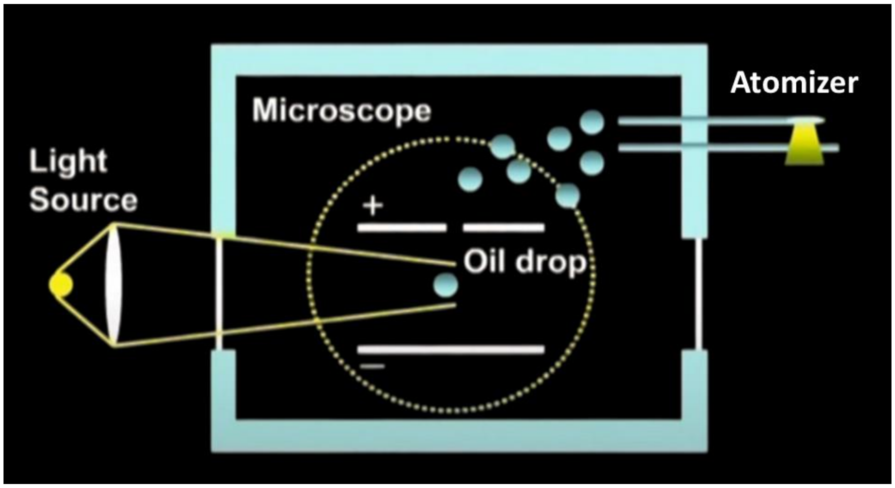


Figure 22.5: Experimental setup for Millikan oil drop experiment

Under normal circumstances, the ionized oil droplets fall downward due to gravity. However, when an electric field € is applied in the chamber, the force (eE) exerted by the electric field can counteract the gravitational pull (mg) as shown in figure 22.6. By carefully adjusting the electric field strength, a balance can be achieved where the downward gravitational force on a droplet equals the upward electric force. This balance allows for the determination of the droplet's charge (e).

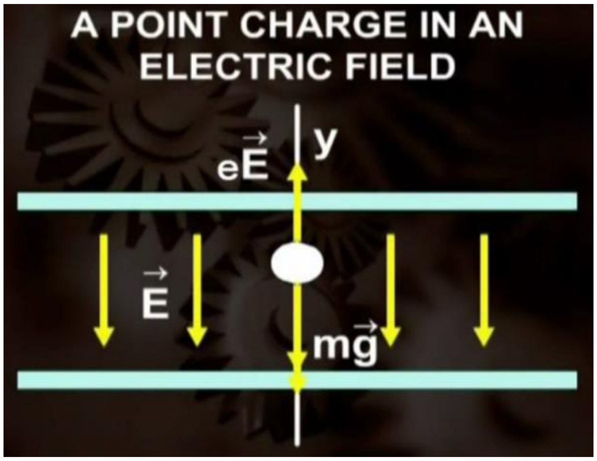


Figure 22.6: Force/Free body diagram of oil droplet in Millikan experiment.

Specifically, the relationship between the droplet's weight (mg), the electric field (E), provides a means to calculate the total charge (e) on the droplet. Mathematically for equilibrium condition,

$$\sum F = eE - mg = 0$$

$$eE = mg$$

$$e = \frac{mg}{E}$$

However, this calculation is complicated by the fact that a droplet may carry more than one electron. For instance, some droplets might have two electrons, others might have twenty, and so on. Despite this, Millikan's experiment revealed a significant result: the charge on the droplets always occurred in discrete multiples of a fundamental value. This observation led to the conclusion that electric charge is quantized, meaning it exists in fixed, indivisible units corresponding to the charge of a single electron.

Millikan’s experiment was pivotal in confirming the quantization of electric charge and provided one of the most precise measurements of the electron's charge at the time. This groundbreaking work laid the foundation for our modern understanding of atomic structure and electromagnetism.

Applications of Electrostatics

When an electric field is present, small objects like pieces of paper are attracted toward the source of the charge due to the electric field's influence. This basic principle has practical applications, including the technology used in photocopiers. It is fascinating that such a simple observation has been adapted into a process that allows us to replicate documents.

In a photocopier, the key component is a photosensitive drum as shown in figure 22.7. This drum reacts to light by creating regions with varying charges. Some areas become positively charged, while others become negatively charged. The amount of charge generated is proportional to the intensity of the light falling on the drum. This variation in charge distribution results in corresponding differences in the strength of the electric field at various points on the drum. The photocopier uses these electric fields to create an image. Inside the device, there is a container filled with extremely fine carbon powder, often referred to as toner. The toner particles are attracted to the charged regions on the drum, forming a pattern that replicates the original document. When the drum comes into contact with a sheet of paper, the toner is transferred to the paper and then fixed in place, typically through heat or pressure. This process demonstrates a direct application of electrostatics in technology.

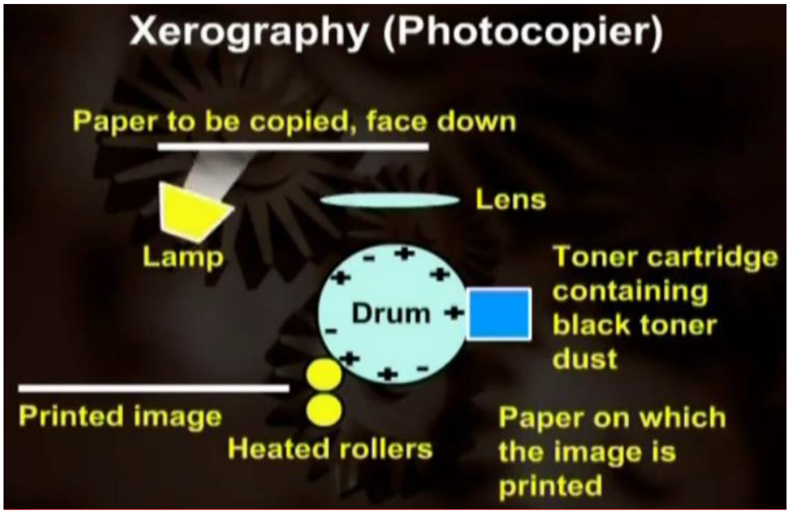


Figure 22.7: Photocopier schematic.

Another example of electrostatic applications can be found in inkjet printers. In these printers, ink is sprayed onto the paper in form of tiny droplets. Before being sprayed, the ink is charged, and the resulting charged droplets are directed by an electric field. By controlling the strength and direction of the electric field (see animation in video lecture), the movement of the droplets can be precisely managed, allowing the printer to control the density and placement of the ink on the paper. This mechanism ensures high-quality printing by accurately reproducing the desired image or text. Both photocopiers and inkjet printers highlight how the principles of electrostatics are harnessed to create reliable and efficient technologies for everyday use.

Electric Dipole

If a single charge is present in a region, its electric field can be calculated using Coulomb's law. However, when a second charge of equal magnitude but opposite in sign is introduced, the system behaves differently. If the positive and negative charges are located at the same point, their electric fields cancel each other due to their opposite directions. On the other hand, when the charges are separated by a finite distance, they form a system known as an electric dipole. The separation between the charges introduces unique properties to the resulting electric field, which becomes a fundamental aspect of studying dipoles in physics.

An electric dipole consists of two charges of equal magnitude but opposite sign, separated by a finite distance. When considering a single charge, the electric field it generates can be calculated using Coulomb's law. However, when two such charges (one positive and one negative) and are brought together but separated by a small distance, their combined system is termed an electric dipole. The separation between the charges introduces unique characteristics in the resulting electric field. To analyze the electric field due to a dipole, consider two charges +q and −q separated by a distance d as shown in figure 22.8.

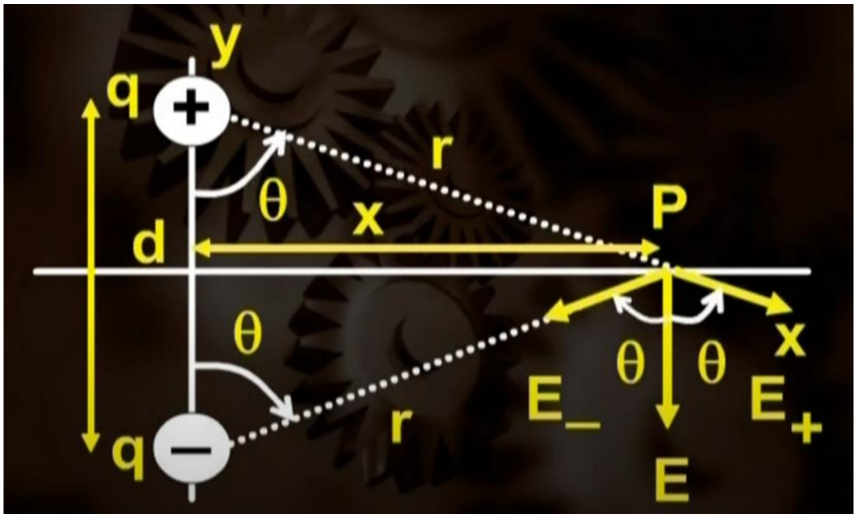


Figure 22.8: Electric dipole on xy-plane

At a point P, located at a distance x from the midpoint of the dipole along its axis, the electric field contributions from the positive and negative charges, denoted as E_+ and E_-, must be considered. Since these electric fields are vector quantities, both their magnitudes and directions influence the resultant field. The electric field generated by each charge individually can be determined using Coulomb's law.

$$\vec{E} = \vec{E}_+ + \vec{E}_-$$
$$\left|\vec{E}_+\right| = \left|\vec{E}_-\right| = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{x^2 + \left(d/2\right)^2}$$

The total electric field at point P is the vector sum of the contributions from +q and -q. Due to the geometry, the resultant field points along the axis of the dipole. Using trigonometric relationships from the diagram, the angle θ and its cosine can be expressed in terms of the dipole's geometry. Substituting these relationships into the expressions for the electric field allows simplification.

$$E = E_+ \cos\theta + E_- \cos\theta = 2E_+ \cos\theta$$
$$\cos\theta = \frac{d/2}{\sqrt{x^2 + (d/2)^2}}$$
$$E = 2\frac{1}{4\pi\varepsilon_0}.\frac{q}{x^2 + (d/2)^2}\frac{d/2}{\sqrt{x^2 + (d/2)^2}}$$
$$E = \frac{1}{4\pi\varepsilon_0}.\frac{qd}{\left[x^2 + \left(\frac{d}{2}\right)^2\right]^{\frac{3}{2}}}$$
$$as\ P = qd$$
$$E = \frac{1}{4\pi\varepsilon_0}.\frac{P}{x^3}\frac{1}{\left[x^2 + \left(\frac{d}{2}\right)^2\right]^{\frac{3}{2}}}$$
$$E = \frac{1}{4\pi\varepsilon_0}\frac{P}{x^3}\left[1 + \left(\frac{d}{2x}\right)^2\right]^{-\frac{3}{2}}$$

By binomial expansion such as $(1+y)^n = 1 + ny + \frac{n(n-1)}{2}y^2 + ...$, E becomes

$$E = \frac{1}{4\pi\varepsilon_0}.\frac{P}{x^3}\left[1 + \left(-\frac{3}{2}\right)\left(\frac{d}{2x}\right)^2 + ...\right]$$

The resulting expression for the electric field includes a factor related to the dipole moment (P = qd), where q is the magnitude of the charge, and d is the separation between the charges. For large distances (x ≫ d), the electric field simplifies further as follows.

$$E = \frac{1}{4\pi\varepsilon_0}\frac{P}{x^3}$$

Ultimately, the electric field due to a dipole at a distance x is proportional to P/x^3, where pp is the dipole moment and x is the distance from the dipole's center to the observation point. This expression also includes the factor $1/(4\pi\epsilon_o)$, which accounts for the permittivity of free space.

The concept of the dipole moment and its corresponding electric field is crucial in physics, as dipoles frequently occur in nature. For instance, in an atom, the center of the positive charge (nucleus) and the center of the negative charge (electron cloud) are often slightly displaced, creating a natural dipole. The separation d between these centers, combined with the charge magnitude q, defines the atom's dipole moment. This phenomenon is fundamental in understanding the behavior of molecules, dielectric materials, and many other physical systems.

Dipole in an Electric Field

When a dipole is placed in a uniform electric field E at an angle (as shown in figure 22.9), the positive and negative charges in the dipole experience forces that are equal in magnitude but opposite in direction. As a result, the net force on the dipole is zero.

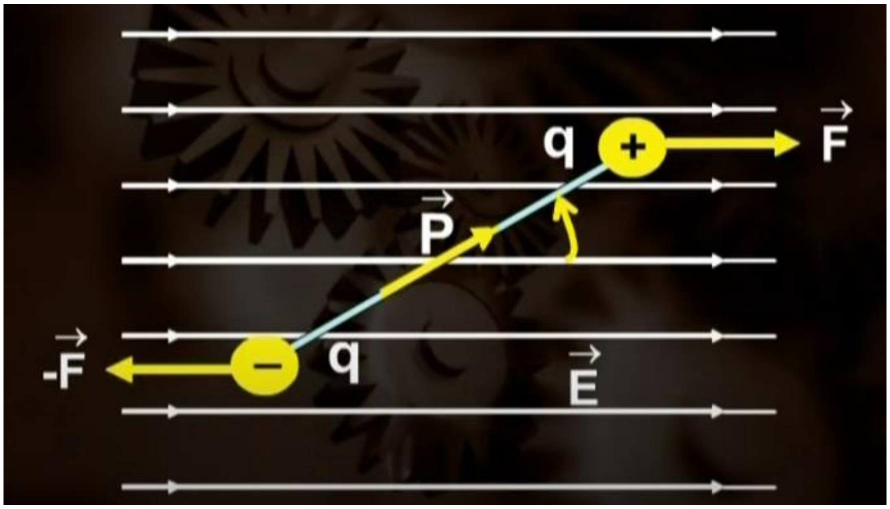


Figure 22.9: A dipole in an Electric Field

However, this does not mean the dipole remains unaffected. If the dipole is aligned perpendicularly to the electric field, it experiences a torque τ. This torque arises because the positive charge tends to move in one direction while the negative charge moves in the opposite direction, creating a pair of forces with a separation between them. This separation causes a rotational effect on the dipole.

The torque acting on the dipole is determined by the combined effects of the forces acting on the positive and negative charges and the perpendicular distance d between them. The total torque depends on the electric field strength E, the magnitude of the charges q, the distance d between the charges in the dipole, and the angle θ between the dipole axis and the electric field. The torque can also be expressed in terms of the dipole moment, which is the product of the charge and the separation distance. This torque causes the dipole to rotate and attempt to align with the electric field.

$$\tau = F\frac{d}{2}\sin\theta + F\frac{d}{2}\sin\theta$$
$$\tau = Fd\sin\theta$$
$$\tau = (qE)d\sin\theta = (qd)E\sin\theta$$
$$\tau = PE\sin\theta$$
$$\vec{\tau} = \vec{P}\times\vec{E}$$

When the torque rotates the dipole within the electric field, it performs work. The amount of work done is determined by the torque and the angle through which the dipole rotates. This work represents the energy required to change the dipole's orientation from an initial angle to a final angle. If the dipole begins at one orientation and ends in another, the work done depends on the electric field strength, the dipole moment, and the change in orientation. If the dipole starts perpendicular to the electric field and rotates to align with it, the work done is at its maximum. Conversely, if the dipole does not change its orientation, no work is performed.

$$W = \int dW = \int_{\theta_0}^{\theta} \vec{\tau}.d\vec{\theta}$$
$$W = \int_{\theta_0}^{\theta} -\tau d\theta$$
$$W = \int_{\theta_0}^{\theta} -PE\sin\theta d\theta$$
$$W = -PE\int_{\theta_0}^{\theta} \sin\theta d\theta$$

$$
\begin{aligned}
&W = PE(\cos\theta - \cos\theta_0) \\
&\Delta U = U(\theta) - U\left(\theta_0\right) = -W \\
&\Delta U = -PE(\cos\theta - \cos\theta_0) \\
&\text{take } \theta_0 = 90^o \text{ and } U\left(\theta_0\right) = 0 \\
&U = -PE\cos\theta \\
&U = -\vec{P}.\vec{E}
\end{aligned}
$$

Summary

This discussion marked the beginning of our exploration into Coulomb forces. We started by understanding the fundamental nature of electric charges. There are two types of charges: positive and negative. Like charges repel each other, while opposite charges attract. This interaction is a defining property of electric charges and forms the basis of electrostatic forces.

Charges are also the source of electric fields. An electric field is a region around a charge where other charges experience a force. The strength of this field is directly related to the magnitude of the force it exerts. The greater the force on a charge within the field, the stronger the electric field at that location. We also explored the concept of charge quantization. The charge of an electron is the smallest discrete unit of charge found in nature, and all other charges are multiples of this fundamental value. This principle highlights that electric charge exists in indivisible units, which is a cornerstone of modern physics.

Additionally, we briefly discussed practical applications of electrostatics, such as their role in devices like photocopiers and inkjet printers. These technologies rely on the precise control of electric charges to function effectively. In today’s world, where electronics play a pivotal role, managing and understanding charges is critically important. Interestingly, the term "electronics" itself is derived from the word "electron," emphasizing the central role of this particle in modern technology. Understanding these foundational concepts is essential for making sense of the physical phenomena around us and for appreciating the significance of physics in our daily lives. As we continue, we will delve deeper into these ideas and their applications.