Stochastic Optimization Algorithms
Report

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Home problems, set 1

Problem 1.1, Penalty method (Mandatory)

The problem is to find the minimum of the function

$$f(x_1, x_2) = (x_1 - 1)^2 + 2(x_2 - 2)^2$$

subject to the constraint

$$g(x_1, x_2) = x_1^2 + x_2^2 - 1 \leq 0$$

using the penalty method.

1.

Penalty function is given by

$$p(x_1, x_2; \mu) = \mu(max\{0, g(x_1, x_2)\})^2 = \mu(max\{0, (x_1^2 + x_2^2 - 1)\})^2 = \mu(x_1^2 + x_2^2 - 1)^2$$

The function $f_p(x_1, x_2; \mu)$ is defined as

$$f_p(x_1, x_2; \mu) = \begin{cases} (x_1 - 1)^2 + 2(x_2 - 2)^2 + \mu(x_1^2 + x_2^2 - 1)^2, & \text{for } x_1^2 + x_2^2 - 1 > 0 \\ (x_1 - 1)^2 + 2(x_2 - 2)^2, & \text{otherwise} \end{cases}$$

2.

Case 1: The gradient where the constraints are not fulfilled is given by

$$\nabla f_p(x_1, x_2; \mu) = \begin{bmatrix} \frac{\partial f_p}{\partial x_1} \\ \frac{\partial f_p}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2(x_1 - 1) + 4\mu x_1(x_1^2 + x_2^2 - 1) \\ 4(x_2 - 2) + 4\mu x_2(x_1^2 + x_2^2 - 1) \end{bmatrix} \tag{1}$$

Case 2: The gradient where the constraints are fulfilled is given by

$$\nabla f_p(x_1, x_2; \mu) = \begin{bmatrix} \frac{\partial f_p}{\partial x_1} \\ \frac{\partial f_p}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2(x_1 - 1) \\ 4(x_2 - 2) \end{bmatrix} \tag{2}$$

3.

The unconstrained minimum (i.e. for $\mu = 0$) of the function is obtained by setting the partial derivatives to zero.

$$\frac{\partial f}{\partial x_1} = 0 \Rightarrow 2(x_1^* - 1) = 0 \Rightarrow x_1^* = 1$$

$$\frac{\partial f}{\partial x_2} = 0 \Rightarrow 4(x_2^* - 2) = 0 \Rightarrow x_2^* = 2$$

Therefore, the starting point for gradient descent is $(x_1^*, x_2^*) = (1, 2)$.

4.

Matlab files.

5.

After running the program, we get the output for a sequence of μ values which is represented in table 1.

Table 1: The table presents the values of x_1^* and x_2^* for different values of μ.

μ	x_1^*	x_2^*
1	0.434	1.210
10	0.331	0.996
100	0.314	0.955
1000	0.312	0.951

Problem 1.2, Constrained optimization (Voluntary)

a)

The problem is to find the global minimum of the function

$$f(x_1, x_2) = 4x_1^2 - x_1x_2 + 4x_2^2 - 6x_2$$

on the closed set S using the analytical method.
Setting the partial derivatives of f to zero, we get

$$\frac{\partial f}{\partial x_1} = 8x_1 - x_2 = 0$$

$$\frac{\partial f}{\partial x_2} = -x_1 + 8x_2 - 6 = 0$$

There is only one stationary point $P_1(\frac{6}{63}, \frac{48}{63})$ inside the set S. Now, it remains to consider the boundary ∂S of S.
Along the line $0 < x_1 < 1$, $x_2 = 1$ and $f(x_1, 1) = 4x_1^2 - x_1 - 2$. Taking the derivative, we obtain $8x_1 - 1 = 0$ and the point is $P_2(\frac{1}{8}, 1)$.
Along the line $0 < x_2 < 1$, $x_1 = 0$ and $f(0, x_2) = 4x_2^2 - 6x_2$. Taking the derivative, we obtain $8x_2 - 6 = 0$ and the point is $P_3(0, \frac{3}{4})$.
Along the line $x_1 = x_2$, $f(x_1, x_1) = 7x_1^2 - 6x_1$. Taking the derivative, we obtain $14x_1 - 6 = 0$ and the point is $P_4(\frac{3}{7}, \frac{3}{7})$.
Vertices are $P_5(0, 0)$, $P_6(0, 1)$ and $P_7(1, 1)$.
All the points P_1, P_2, P_3, P_4, P_5, P_6 and P_7 are candidates of the global minimum.
Then, we obtain $f(P_1) = -2\frac{2}{7}$, $f(P_2) = -2\frac{1}{16}$, $f(P_3) = -2\frac{1}{4}$, $f(P_4) = -1\frac{2}{7}$, $f(P_5) = 0$, $f(P_6) = -2$ and $f(P_7) = 1$.
Therefore, the global minimum is $(x_1^*, x_2^*) = P_1(\frac{6}{63}, \frac{48}{63})$.

b)

The problem is to find the minimum of the function

$$f(x_1, x_2) = 15 + 2x_1 + 3x_2$$

subject to the constraint

$$h(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2 - 21 = 0$$

using the Lagrange multiplier method.
We define $L(x_1, x_2, \lambda)$ as

$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda h(x_1, x_2) = 15 + 2x_1 + 3x_2 + \lambda(x_1^2 + x_1x_2 + x_2^2 - 21)$$

Setting the gradient of L to zero, we get

$$\frac{\partial L}{\partial x_1} = 2 + \lambda(2x_1 + x_2) = 0$$

$$\frac{\partial L}{\partial x_2} = 3 + \lambda(x_1 + 2x_2) = 0$$

$$\frac{\partial L}{\partial \lambda} = x_1^2 + x_1x_2 + x_2^2 - 21 = 0$$

We get $Q_1(1, 4)$, $Q_2(1, -4)$, $Q_3(-1, 4)$ and $Q_4(-1, -4)$.
Then, we obtain $f(Q_1) = 29$, $f(Q_2) = 5$, $f(Q_3) = 25$ and $f(Q_4) = 1$.
Therefore, the minimum is $(x_1^*, x_2^*) = Q_4(-1, -4)$.

Problem 1.3, Basic GA program (Mandatory)

a)

Matlab files.
The minimum value of the function $g(x_1, x_2)$ is 0.333 at the point $(-0.006, -1.002)$.

b)

Table 2: The table presents the median fitness values obtained for different values of the mutation rate.

$\frac{1}{m}$	Median fitness
0.00	0.258
0.02	0.333
0.05	0.333
0.10	0.333

Table 2 represents the median fitness value obtained for each value of the mutation rate $(1/m)$ after 100 runs, each lasting 100 generations.
The conclusion drawn from the above analysis is that different mutation probabilities give good results.

c)

The function g can be written as

$$g(x_1, x_2) = g_1(x_1, x_2) * g_2(x_1, x_2) = (1 + f_1(x_1, x_2) * f_2(x_1, x_2)) * (30 + h_1(x_1, x_2) * h_2(x_1, x_2))$$

where

$$f_1(x_1, x_2) = (x_1 + x_2 + 1)^2,$$

$$f_2(x_1, x_2) = 19 - 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2^2,$$

$$h_1(x_1, x_2) = (2x_1 - 3x_2)^2,$$

$$h_2(x_1, x_2) = 18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2.$$

The gradient is defined by

$$\nabla g(x_1, x_2) = \begin{bmatrix} \frac{\partial g}{\partial x_1} \\ \frac{\partial g}{\partial x_2} \end{bmatrix} \tag{3}$$

Using the product rule for derivatives, we get

$$\begin{aligned}
\frac{\partial g}{\partial x_1} &= \frac{\partial g_1 g_2}{\partial x_1} \\
&= g_1 \frac{\partial g_2}{\partial x_1} + g_2 \frac{\partial g_1}{\partial x_1} \\
&= (1 + f_1 f_2) \frac{\partial (30 + h_1 h_2)}{\partial x_1} + (30 + h_1 h_2) \frac{\partial (1 + f_1 f_2)}{\partial x_1} \\
&= (1 + f_1 f_2)(h_1 \frac{\partial h_2}{\partial x_1} + h_2 \frac{\partial h_1}{\partial x_1}) + (30 + h_1 h_2)(f_1 \frac{\partial f_2}{\partial x_1} + f_2 \frac{\partial f_1}{\partial x_1})
\end{aligned}$$

and similarly,

$$\begin{aligned}
\frac{\partial g}{\partial x_2} &= \frac{\partial g_1 g_2}{\partial x_2} \\
&= g_1 \frac{\partial g_2}{\partial x_2} + g_2 \frac{\partial g_1}{\partial x_2} \\
&= (1 + f_1 f_2) \frac{\partial (30 + h_1 h_2)}{\partial x_2} + (30 + h_1 h_2) \frac{\partial (1 + f_1 f_2)}{\partial x_2} \\
&= (1 + f_1 f_2)(h_1 \frac{\partial h_2}{\partial x_2} + h_2 \frac{\partial h_1}{\partial x_2}) + (30 + h_1 h_2)(f_1 \frac{\partial f_2}{\partial x_2} + f_2 \frac{\partial f_1}{\partial x_2}).
\end{aligned}$$

Again,

$$\frac{\partial f_1}{\partial x_1} = 2(x_1 + x_2 + 1),$$

$$\frac{\partial f_1}{\partial x_2} = 2(x_1 + x_2 + 1),$$

$$\frac{\partial f_2}{\partial x_1} = -14 + 6x_1 + 6x_2,$$

$$\frac{\partial f_2}{\partial x_2} = -14 + 6x_1 + 6x_2,$$

$$\frac{\partial h_1}{\partial x_1} = 4(2x_1 - 3x_2),$$

$$\frac{\partial h_1}{\partial x_2} = -6(2x_1 - 3x_2),$$

$$\frac{\partial h_2}{\partial x_1} = -32 + 24x_1 - 36x_2,$$

$$\frac{\partial h_2}{\partial x_2} = 48 - 36x_1 + 54x_2.$$

Now, $f_1(0,-1) = 0$, $f_2(0,-1) = 36$, $h_1(0,-1) = 9$ and $h_2(0,-1) = -3$.
Also, $\frac{\partial f_1(0,-1)}{\partial x_1} = 0$, $\frac{\partial f_1(0,-1)}{\partial x_2} = 0$, $\frac{\partial f_2(0,-1)}{\partial x_1} = -20$, $\frac{\partial f_2(0,-1)}{\partial x_2} = -20$,
$\frac{\partial h_1(0,-1)}{\partial x_1} = 12$, $\frac{\partial h_1(0,-1)}{\partial x_2} = -18$, $\frac{\partial h_2(0,-1)}{\partial x_1} = 4$ and $\frac{\partial h_2(0,-1)}{\partial x_2} = -6$.
Substituting the above values, we obtain

$$\frac{\partial g(0,-1)}{\partial x_1} = 0, \frac{\partial g(0,-1)}{\partial x_2} = 0.$$

Therefore, $(0,-1)$ is a stationary point of the function g proved analytically.