Combinatorial Game Theory

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1 Introduction

In this section, we will go over basic definitions and the necessary mathematical foundations needed to fully comprehend the impartial game Nim and the results that can be derived from Nim. To start off, in the heart of this paper, we will be talking about impartial combinatorial games, which are games with the following properties:

- There are two players who alternate turns until the game ends
- There is a finite set of positions and operations, and the game must end, so no loops are allowed
- An allowable move at a given turn is only dependent on the position in the game, not on which of the player's turn it is, meaning neither player has a unique state independent from their position (i.e. unique pieces, cards, resources, or anything of the like)
- The game ends when one of the players cannot make a move and the game must result in a winner
- The validity of a move cannot be dependent on chance

In impartial games, there are two ways to play: Normal or Misere. Since the players must alternate turns until the game ends, in Normal Play, we say the last player to move in a game wins the game. Alternatively, in Misere Play, we say that the last player to move in the game loses the game. For the remainder of the paper, we will be focusing solely on Normal Play, although some results shown below have equivalent results for Misere Play.

1.1 Impartial Game Positions

In an impartial game, we define the following three positions that a game can be in:

1. P-Position: If the game is in a P-position, then the player who just moved is guaranteed to win, should they play optimally for the remainder of the game
2. N-Position: If the game is in a N-position, then the next player to move is guaranteed to win, should they play optimally for the remainder of the game
3. Terminal Position: The game is said to be in a terminal position if there are no possible moves left

We will now prove that these are in the fact the only positions a game can be in.

Theorem: Every position in an impartial game is either a N-Position or a P-Position.

Proof: From our definition of an impartial game, every game must end. Therefore, for any position x_1 in an impartial game, every valid sequence of moves $x_1, x_2, \ldots, x_n, t$ starting at x_1 should result in a terminal state t. We will now prove by induction that every position in our sequence is an N-Position or a P-Position.

For our base case, we will consider position t. If our game is Normal Play, then this is a P-Position, as by definition of a Terminal position there are no possible moves left, meaning the next player will lose (as they cannot make a move) and thus the player who just moved will win. Otherwise, if the game is Misere Play, then this is an N-Position by the same reasoning.

Now, assume our inductive hypothesis holds for position x_{i+1}, and consider position x_i. From our assumption, we know that position x_{i+1} is either a N-Position or a P-Position. We also know that the next person to move at position x_i will become the person who moved previously at position x_{i+1}, and the person who moved previously at position x_i will become the next person to move at position x_{i+1}.

We will now break into cases on whether x_{i+1} is a N-Position or a P-Position.

- Case 1 - x_{i+1} is a P-Position. Then, by definition of a P-Position, the previous person who moved at position x_{i+1} is guaranteed a winning strategy. Thus, since the person to next move at position x_i will become the person who previously moved at position x_{i+1}, the next person to move at position x_i will be guaranteed a winning strategy. Therefore, x_i is a N-Position.

- Case 2 - x_{i+1} is a N-Position. Then, by definition of a N-Position, the next person to move at position x_{i+1} is guaranteed a winning strategy. Thus, since the person who previously moved at position x_i will become the next person to move at position x_{i+1}, the person who previously moved at position x_i will be guaranteed a winning strategy. Therefore, x_i is a P-Position.

To see a position cannot be both a N-Position and a P-Position, observe that if a position is a N-Position, then the player who next moves can guarantee the next position will be one from which they are guaranteed to win. Therefore, this will be a position for which the other player is guaranteed to lose, meaning this position cannot simultaneously be a P-Position, as the person who moved previously is not guaranteed to win.

From this discussion, we have a theoretical guarantee that every position in a game is either a N-Position or a P-Position. However, while it is clear when the game is in a terminal state, for some impartial games it can be difficult and time-consuming to practically calculate whether the state of the game is a P-Position or an N-Position. For example, the winning states for the impartial game Sprouts has only been calculated for up to 53 spots, despite the fact that Sprouts can be played with any number of spots. We will now describe an algorithm for determining what state a game is in given the position.

Start at the terminal positions, and label them P. Then, label all states that can reach the terminal state N. Now, label all states reachable by these N states P, and then proceed to label all states reachable by these P states N. Repeat this process until every state has been labeled. This sort of graphical representation of the impartial games is described more in depth in Section 3.

1.2 XOR Arithmetic

In impartial games, many of the results rely heavily on the XOR function: for two integers written in their binary expansions, x and y, $XOR(x, y) = z$, where the i_{th} bit in z is defined to be

$$z[i] = \begin{cases} 0\ if\ x[i] = y[i] \\ 1\ if\ x[i] \neq y[i] \end{cases}$$

Notationally, we will write $XOR(x, y) = x \oplus y$. From this definition, it is clear to see that $x \oplus y = 0$ if and only if $x = y$. Another way of thinking about XOR is if there are an odd number of 1's in i_{th} digit, then the output will have a 1 in the i_{th} digit and if there are an even number of 1's in the i_{th} digit, the output will have a 0 in the i_{th} digit.

Theorem 1: The XOR function is commutative.

Proof: Let x and y be arbitrary integers written in their binary notation. Let $b = x \oplus y$ and $a = y \oplus x$. Then, the i_{th} digit in b will be 0 if $x[i] = y[i]$ and 1 if $x[i] \neq y[i]$ and the i_{th} digit in a will be 0 if $y[i] = x[i]$ and 1 if $y[i] \neq x[i]$. So, $b[i] = a[i]$ for all i.

Theorem 2: The XOR function is associative.

Proof: Let x, y, and z be arbitrary integers written in their binary notation. Let $b = (x \oplus y) \oplus z$ and $a = z \oplus (y \oplus z)$. Then, the i_{th} digit in b will be 0 if there is an even number of 1's in the i_{th} digit across x, y, and z and it will be 1 otherwise. The i_{th} digit in a will also be 0 if there is an even number of 1's in the i_{th} digit across x, y, and z and it will be 1 otherwise. So, $a = b$.

Theorem 3: $XOR(x, 0) = XOR(0, x) = 0$ for all integers x.

Proof: We only need to prove that $x \oplus 0 = x$, as we have already proved that XOR is commutative. Since the binary expansion of 0 is 0, by definition, $x \oplus 0$ will have a 1 in the i_{th} digit if $x[i] \neq 0$ and it will have a 0 if $x[i] = 0$. So, each digit will match x, meaning $x \oplus 0 = x$.

1.3 Minimum Excluded Value Function ($mex()$)

The mex function takes in a subset s of a well-ordered set S and returns the smallest value $x \in S$ such that $x \notin s$. In other words, mex returns the minimum value in the complement of s, meaning $mex(s) = \min S \setminus s$. As an example, let S be the set of non-negative integers, and let $s = [0, 1, 3, 4]$. Thus, $mex(s) = 2$, as the minimum non-negative integer not in s is 2.

This function will be relied upon heavily in our discussion of the Sprague-Grundy Function and the Sprague-Grundy Theorem. In our discussions, we will focus on S being the set of non-negative integers.

2 Our First Game: Nim

Now that we have reviewed some necessary prerequisites, we can look at our first impartial game: Nim! Although it is quite simple, this game has formed the basis for some of the most important developments in combinatorial game theory.

2.1 How To Play

To start, there are n piles of stones, where the number of stones in each pile is a positive integer and need not be the same number of stones as the other piles. Typically, $n = 3$. As this is a combinatorial impartial game, there are two players, Player A and Player B, and the players alternate turns, removing any number of stones from a single pile on a single turn. The game ends when there are no stones left in any pile. Since we are playing under normal rules and not misere, the player who takes the last stone wins the game. The following is a short example of a game of Nim:

Pile 1	Pile 2	Pile 3	Moves
1	6	4	Player A takes 3 stones from Pile 2
1	3	4	Player B takes 1 stone from Pile 1
0	3	4	Player A takes 3 stones from Pile 3
0	3	1	Player B takes 2 stones from Pile 2
0	1	1	Player A takes 1 stone from Pile 2
0	0	1	Player B takes 1 stone from Pile 3 and wins the game

Table 1: An Example Game of Nim

2.2 The Winning Strategy

First, let's define the **Nim-sum** of a position in a game of Nim to be the XOR of the number of stones in each pile at that state of the game. So, if there are 3 stones in the first pile, 6 stones in the second, and 5 in the third, the Nim-sum of this position is $3 \oplus 6 \oplus 5 = (011) \oplus (110) \oplus (101) = 0$. It is clear to see that when all piles are empty (i.e. the game has finished), the Nim-sum is 0; however, as our example showed, this is not the only state of the game in which the Nim-sum is 0. Our winning strategy will rely on these facts. In the formulation of our winning strategy, we first prove a pair of lemmas:

Lemma 1: If the Nim-sum is 0 and the game is not yet over, then the next move must result in a non-zero Nim-sum.

Proof: Assume that the game is not yet over and the Nim-sum of the current state of the game, denoted as s, is 0. Let there be n piles of stones, and in pile i, there are x_i stones remaining. So, by definition of Nim-sum, $s = x_1 \oplus x_2 \oplus \cdots \oplus x_n = 0$. Now, the next player to go picks k stones from pile j, resulting in the new Nim-sum $t = x_1 \oplus x_2 \oplus \cdots \oplus y_j \oplus x_{j+1} \oplus \cdots \oplus x_n$, where y_j is the number of stones remaining in pile j after removing k stones. Then, using XOR arithmetic and the fact that $s = 0$, we derive the following:

$$
\begin{aligned}
t &= 0 \oplus t && (1)\\
&= s \oplus s \oplus t && (2)\\
&= s \oplus (x_1 \oplus x_2 \oplus \cdots \oplus x_n) \oplus (x_1 \oplus x_2 \oplus \cdots \oplus y_j \oplus x_{j+1} \oplus \cdots \oplus x_n) && (3)\\
&= s \oplus (x_1 \oplus x_1) \oplus (x_2 \oplus x_2) \oplus \cdots \oplus (x_j \oplus y_j) \oplus \cdots \oplus (x_n \oplus x_n) && (4)\\
&= s \oplus (x_j \oplus y_j) && (5)\\
&= x_j \oplus y_j && (6)
\end{aligned}
$$

Here, lines (1) and (2) comes directly from $x \oplus x = 0$ and $x \oplus 0 = x$, line (3) follows by definition of s and t, line (4) comes from XOR being associative, line (5) follows by $x \oplus x = 0$, and line (6) comes from $s = 0$. Since $y_j \neq x_j$ as we removed $k \geq 1$ stones, $x_j \oplus y_j$ is non-zero. Therefore, t is non-zero. Since t was an arbitrary next move in the game, Lemma 1 holds true.

Lemma 2: If the Nim-sum of the game is non-zero at the start of a turn, there exists a move such that at the end of the turn, the Nim-sum is zero.

Proof: Let there be n piles of stones, and in pile i, there are x_i stones remaining. Let s be the non-zero Nim-sum at the start of Player A's turn: $s = x_1 \oplus x_2 \oplus \cdots \oplus x_n \neq 0$. Let k be the most significant bit in the binary expansion of s. By the definition of XOR, there must be a pile j such that the binary expansion of x_j has a nonzero k_th bit, as otherwise, k would not be the most significant bit. Then, let Player A remove $x_j - (s \oplus x_j)$ stones from pile x_j. Note that this is a valid move as all bits more significant than k in x_j are the same in $(s \oplus x_j)$ and the k_th bit will be 0 in $(s \oplus x_j)$, so $x_j > (s \oplus x_j)$. Using the notation defined in the proof of Lemma 1 and jumping to line (5) in that proof, $y_j = s \oplus x_j$ and $t = s \oplus x_j \oplus y_j$. Then, by commutativity of XOR, $t = s \oplus s \oplus x_j \oplus x_j = 0$. So, after Player A's turn, the Nim-sum of the game is 0.

With those Lemma's in our back pockets, we can now prove the winning strategy for Nim:

Theorem: Assuming that the Nim-sum is non-zero at the start of one of Player A's turns, if Player A makes the Nim-sum equal to 0 at the end of each of their turns, then Player A will be guaranteed to win.

Proof: Assume that Player A has a turn where the Nim-sum is non-zero at the start of the turn. Then, by Lemma 2, Player A has a move to make the Nim-sum equal 0 after they move. By Lemma 1, any move Player B makes must result in a non-zero Nim-sum. This means that Player A will always be able to make a move to ensure the Nim-sum is 0 at the end of their turn, as Player B will be forced to make the Nim-sum non-zero. Since there is a finite number of stones in each pile and the

Nim-sum when the game is over is 0, after a finite number of turns removing stones, the game must reach the state where Player A takes the last stone, meaning Player A will be guaranteed to win using this strategy.

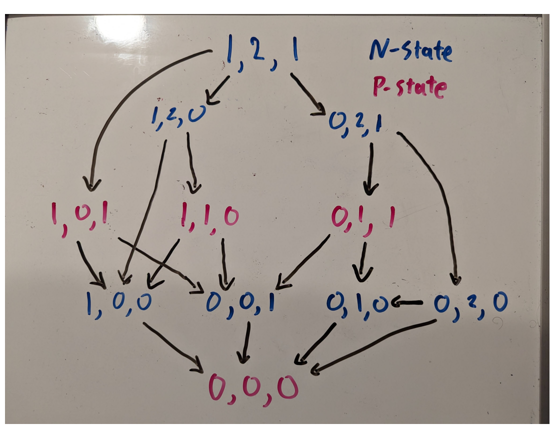
Observation: There are variants of Nim that allow the addition of stones to a pile, the creation of a new pile, or both. As in all impartial games, the number of turns in a game must be finite, and thus, we include the some constraint in the total number of stones that may be added over the course of the game. Given this constraint, we note that our game winning strategy for Nim-sum remains true. Following the logic in our theorems, if the Nim-sum was previously 0, then any added stones would have a non-zero Nim-sum. Given that a player may not both add and take stones in the same turn, that means that if a turn starts with a Nim-sum of 0, it must end with a non-zero Nim-sum. Similarly, we already proved that if the Nim-sum was previously non-zero, there is always the option to remove stones from a pile to give a zero Nim-sum. Thus, the winning strategy is still to maintain a Nim-sum of 0.

3 The Sprague-Grundy Theorem

In this section we will show a relationship between Nim and any other impartial game g. In order to establish this relationship, we need to formulate the games in a common structure, a graph. We will transform game g into $G = (X, E)$ defined as follows:

1. X is the set of all possible positions in game g
2. E is the set of edges, where each $x \in X$ has a directed edge to $y \in X$ if it is possible to move from position x to position y in game g. We will call such y *followers* of x, and denote the set of followers of x to be $F(x)$. If x has no outgoing edges, then x is a terminal position.
3. The starting position of g is represented as $x_0 \in X$. To begin the game, Player A picks where in the graph G to move to from vertex x_0
4. Players A and B alternating moves corresponds to Players A and B alternating choosing which neighboring vertex $y \in F(x)$ to move to from x.
5. Under Normal Play, the player who must make a move from a vertex $x \in X$ with no outgoing edges loses. So, $F(x) = \emptyset$.

From our requirements for being an impartial game, the game must end, so the graph G is finite and has no cycles. As an example graph, below is the graph of Nim for a starting state of Nim=$[1, 2, 1]$, with P-State positions marked in pink and N-State positions marked in Blue.



3.1 The Sprague-Grundy Function

The Sprague-Grundy function, $SG : G \to \mathbb{Z}_{\geq 0}$, is a recursive function that takes in a vertex $x \in X$ and returns the *mex* of the SG values of $F(x)$. In mathematical notation,

$$SG(x) = mex\{SG(y) : y \in F(x)\}$$

For a base case, we will set $SG(x) = 0$ for all x where $F(x) = \emptyset$. On our game graph G, we can start at the vertices with no outgoing edges and give them an SG value of 0, as defined in the base case. Then, we can traverse the graph in reverse, finding all vertices than can reach the end vertices, and give them an SG value of 1. We can continue doing this process of traversing to the unmarked neighbors and assigning it the correct SG value until all of the vertices in the graph have been assigned a value.

3.2 Adding Graphs Together

By using the graph representation and the Sprague-Grundy Function, we can add graphs representing games together to make one big game-representing graph. Think of a graph G_1 representing a game of Nim with m piles and G_2 representing a game of Nim with n piles. By adding these graphs together in a specific way, we can represent a game of Nim with $n + m$ piles. This addition of two games is called the *disjunctive sum* of two games. In order for the addition to make sense, we say that in the disjunctive sum of games G and H, during a player's turn, they can either perform their turn in game G or in game H but not both. The full game $G + H$ ends when both G and H have ended. More concretely, we say that the disjunctive sum of games $G_1(X_1, E_1), G_2(X_2, E_2), \ldots, G_n(X_n, E_n)$ is defined to be $G(X, E) = G_1 + G_2 + \cdots + G_n$, where

- $X = X_1 \times X_2 \times \cdots \times X_n$, i.e the cartesian product of all X_i.
- If there is an edge between states y_i and y_j in game G_k, then there is an edge between $(x_1, \ldots, x_{k-1}, y_i, x_{k+1}, \ldots, x_n)$ and $(x_1, \ldots, x_{k-1}, y_j, x_{k+1}, \ldots, x_n)$ in G.

3.3 The Sprague-Grundy Theorem

Sprague-Grundy Theorem: If SG_i is the Sprague-Grundy function of G_i for $i \in [n]$, then $G = G_1+G_2+\cdots+G_n$ has Sprague-Grundy function $SG(x_1, \ldots, x_n) = SG_1(x_1) \oplus SG_2(x_2) \oplus \cdots \oplus SG_n(x_n)$.

In proving the Sprague-Grundy Theorem, we are also proving that any position x of an impartial game G is equivalent to a Nim pile of size $SG(x)$. This is because the Sprague-Grundy function for G is just the Nim-sum of the Sprague-Grundy functions of its components, so the possible moves that can be made in graph G is equivalent to the number of stones that can be picked in a pile of size $SG(x)$.

Proof: Let $x = (x_1, \ldots, x_n)$ be an arbitrary position/vertex in graph G and let $b = SG_1(x_1) \oplus SG_2(x_2) \oplus \cdots \oplus SG_n(x_n)$. In order to prove $SG(x_1, \ldots, x_n) = b$, we must prove that $mex\{SG(y) : y \in F(x_1, \ldots, x_n)\} = b$, by definition of SG. This means we must prove the following two properties:

1. For all $a \in \mathbb{Z}_{\geq 0}$ where $a < b$, there exists a follower $y \in F(x_1, \ldots, x_n)$ such that $SG(y) = a$. This shows that $SG(x_1, \ldots, x_n) \geq b$.
2. For all $y \in F(x_1, \ldots, x_n)$, $SG(y) \neq b$. This shows that $SG(x_1, \ldots, x_n) \leq b$.

Property 1: Let $a \in \mathbb{Z}_{\geq 0}$ where $a < b$ and let $d = a \oplus b$. In the binary expansion of d, let k be the position of the most significant bit. So, $d[k] = 1$, as otherwise, k would not be the most significant bit. Then, we claim that b must have a 1 in the k_{th} position but a must be 0 in the k_{th} position. This is true because $a < b$, so b must must have a 1 in the k_{th} position, and if a also had a 1 in the k_{th} position, then in the XOR of a and b, there would be a 0 in the k_{th} position, contradicting k being the most significant bit.

Since b is defined to be $b = SG_1(x_1) \oplus SG_2(x_2) \oplus \cdots \oplus SG_n(x_n)$, there must be at least one x_i such that the k_{th} position of $SG_i(x_i)$ is 1 in its binary expansion. Now, we claim that $d \oplus SG_i(x_i) < SG_i(x_i)$. This is true because all of the bits more significant than k in $SG_i(x_i)$ stay the same, as they are 0's in d, and the $k_t h$ bit goes from a 1 to a 0 since both d and $SG_i(x_i)$ have a 1 in that position. Now, since $d \oplus SG_i(x_i) < SG_i(x_i)$, there must be a move from x_i to y_i in graph G_i such that $SG(y_i) = d \oplus SG_i(x_i)$, by how SG is defined. Then, by how we defined the disjunctive sum of games, the move $(x_1, \ldots, x_i, \ldots, x_n) \to (x_1, \ldots, y_i, \ldots, x_n)$ is a valid move in the game G, and

$$\begin{aligned} SG_1(x_1) \oplus \cdots \oplus SG_i(y_i) \oplus \cdots \oplus SG_n(x_n) &= SG_1(x_1) \oplus \cdots \oplus (d \oplus SG_i(x_i)) \oplus \cdots \oplus SG_n(x_n) \\ &= d \oplus SG_1(x_1) \oplus \cdots \oplus SG_i(x_i) \oplus \cdots \oplus SG_n(x_n) \\ &= d \oplus b \\ &= a \oplus b \oplus b \\ &= a \end{aligned}$$

Property 2: We will show that for all $y \in F(x_1, \ldots, x_n)$, $SG(y) \neq b$. Assume for contradiction that there exists $(x_1, x_2, \ldots, y_i, x_{i+1} \ldots, x_n) \in F(x_1, \ldots, x_n)$ such that $SG(x_1, x_2, \ldots, y_i, x_{i+1} \ldots, x_n) = b$. By how we defined the disjunctive sum, this means that there exist a move from x_i to y_i in game G_i. Since $b = SG_1(x_1) \oplus SG_2(x_2) \oplus \cdots \oplus SG_n(x_n)$ and $SG(x_1, x_2, \ldots, y_i, x_{i+1} \ldots, x_n) = b$, then

$$SG_1(x_1) \oplus \cdots \oplus SG_i(x_i) \oplus \cdots \oplus SG_n(x_n) = SG_1(x_1) \oplus \cdots \oplus SG_i(y_i) \oplus \cdots \oplus SG_n(x_n)$$

XOR-ing $SG_1(x_1) \oplus \cdots \oplus SG_{i-1}(x_{i-1}) \oplus SG_{i+1}(x_{i+1}) \oplus \cdots \oplus SG_n(x_n)$ to both sides results in $SG_i(x_i) = SG_i(y_i)$. But then the move x_i to y_i is not a valid move in the game G_i, as it is impossible to have a follower with the same SG value, by definition of SG. Contradiction! So, it must be the case that for all $y \in F(x_1, \ldots, x_n)$, $SG(y) \neq b$.

Corollary 1: Any position of an impartial game is equivalent to a position in a game of Nim with a single heap.

Proof: It is clear that the SG value of a game of Nim with a single stone pile of size n is n. Therefore, the SG value of a single heap is just it's Nim-sum. So, say in an impartial game G, we are at a position with an SG value of x and we move to a position with an SG value of y. If $y < x$, this is equivalent to picking $x - y$ stones from the heap. If $y > x$, then there is a move in game G to get back to a position with SG value x, so it is as if the turn never happened. By definition of SG value, $x \neq y$, so this covers all of the cases. Therefore, any position of an impartial game is equivalent to a position in a game of Nim with a single heap.

In the following sections, we will see the use of the Sprague-Grundy Theorem through two fun examples.

4 A Worked Example: P Little Pigs

In Pigland, there is one street where P Little Pigs live, each in their own house made of straw. Unfortunately, right next to Pigland is Wolfland, and the wolves in Wolfland like to play the following game to annoy their pig neighbors. For some strange reason, wolves in Wolfland always travel in packs of two, so this game is always played with only two wolves. Let the two wolves be Wolf A and Wolf B. The game consists of the following rules:

- The wolves must alternate taking turns
- A turn consists of either building a new pig house on the far right side of the street OR (blowing over one of the houses h and, if desired, either blowing down one of the houses to the left of h or rebuilding one of the houses to the left of h)

- They can only choose to build a new house on the far right at most n times
- The wolf that blows over the last house wins

Because this is a popular game amongst the wolves, it is often the case that some of the houses are still blown down from other wolves who passed by earlier and played.

Knowing the rules of the game, and realizing that this is an impartial game, we will prove a winning strategy by reducing this game to Nim.

4.1 The Winning Strategy

First, we will number the houses $1, \ldots, n$ from the left side of the street to the right. For each house i that is not blown over, we will create a Nim heap of size i. For each house that is blown over, we will ignore it. So, if B represents a house that is blown over and S represents a house that is still standing, the street $SBBSSBSB$ will correspond to a Nim position with heaps of size $1, 4, 5$, and 7. Just like in Nim, if the Nim-sum is non-zero at the start of Wolf A's turn, there is a strategy that guarantees Wolf A wins the game, and that strategy is always making the Nim-sum equal to 0 at the end of each turn.

We will prove this by showing two facts: first, there is always a move that Wolf A can do to make the Nim-sum equal to zero at the end of the turn, assuming that the Nim-sum is non-zero at the start of the turn, and second, if the Nim-sum is zero at the start of Wolf B's turn, then any move Wolf B performs results in a nonzero Nim-sum. As the game is over when all houses are blown down, in our reduction, this corresponds to all piles being empty. Therefore, just like in Nim, if the game is over then the Nim-sum must be equal to 0. So, if we can show these two facts, we are showing that Wolf A is guaranteed to win the game with this strategy.

Theorem 1: Assuming that the Nim-sum is initially non-zero, there is always a move that Wolf A can do to make the Nim-sum equal to zero at the end of the turn.

Proof: Assume the Nim-sum is non-zero. Then, in our reduction to Nim, by Lemma 2 of Section 2.2, there exists a move to make the resulting Nim-sum zero. We will show that this move by Player A in Nim corresponds to a valid move by Wolf A in P Little Pigs.

- Player A removes a whole pile of size n. Since each pile represents a house, this move is equivalent to Wolf A only blowing down the n_{th} house. As we know from the rules of P Little Pigs, doing just this is a valid move.
- Player A removes $1 \leq x < n$ stones from a pile, resulting in a pile of size $n - x$, and there is no other pile with $n - x$ stones in it. This is equivalent to Wolf A blowing down the n_th house and then rebuilding the $n - x$ house, which is again a valid move.
- Player A removes $1 \leq x < n$ stones from a pile, resulting in a pile of size $n - x$, and there exists another pile with $n - x$ stones in it. Since the XOR of $n - x$ with $n - x$ is 0, in terms of the Nim-sum, this is equal to not having either pile left in the game. So, in P Little Pigs, this is equivalent to blowing down house n and also blowing down house $n - x$, which is a valid move.

Since these are all of the possible moves that can be made in Nim, and there exists a move in Nim to make the Nim-sum equal to zero, there is then a valid corresponding move in P Little Pigs to make the Nim-sum equal to zero.

Theorem 2: If the Nim-sum is equal to zero at the start of Wolf B's turn, then any move Wolf B performs results in a nonzero Nim-sum.

Proof: Assume the Nim-sum is equal to zero at the start of Wolf B's turn and the game is not yet over. We will show that for any move that Wolf B can make, the resulting Nim-sum is non-zero.

- Wolf B builds a new house to the right of the right-most house. By our reduction to Nim, this is adding a pile of size $n+1$ to the game, if there were n houses on the street before building this one. Since the Nim-sum was originally 0, by adding this new pile, the new Nim-sum is equal to $0 \oplus (n+1) = n+1 \neq 0$. So, after making this move, the Nim-sum is non-zero.

- Wolf B only blows down house i. In our reduction, this is equivalent to Wolf B picking the entire pile of size i. As the Nim-sum was 0 before Wolf A's turn, by taking this whole pile, as we know from Lemma 1 of Section 2.2, this must result in the Nim-sum being non-zero after Wolf A's turn.

- Wolf B blows down house i and rebuilds house j to the left of house i. In our reduction to Nim, this is equivalent to taking $i-j$ stones from the pile with i stones in it. This is because we are taking all i stones from the pile and then creating a new pile with j stones by our reduction, and these two formulations are equivalent. By Lemma 1 of Section 2.2, this move results in a non-zero Nim-sum.

- Wolf B blows down house i and also blows down house j to the left of house i. Since in Nim, it is not legal to take from two different piles on the same turn, this turn by Wolf B is equivalent to taking $i-j$ stones from pile i to result in two piles of size j. This is equivalent because $j \oplus j = 0$ for all j, so it is as if these two piles were removed. As with the other cases, by Lemma 1, this results in a non-zero Nim-sum.

Therefore, for any move that Wolf B performs, the state of the game after the move will have a non-zero Nim-sum.

Therefore, if the Nim-sum is non-zero at the start of their turn, Wolf A is guaranteed to win the game by making the Nim-sum equal to 0 at the end of each turn.

5 Additional Example: The Great Escape

Once again, all is not well in Pigland. After the Wolves blew down all the houses of the P Little Pigs, they were pursued by an angry mob of hogs wanting revenge for their high home insurance premiums. As such, the Wolves need to figure out how to evade the pigs through the streets of Pigland, modeled as a graph $G = (V, E)$. Using their superior athletics, the Wolves can scale the walls of a dead end, allowing them to end the chase. However, if the pigs corner the wolves, then they'll certainly wish they hadn't been so reckless with their leisure time. Thus, both sides are trying to lead the other into a dead end. Therefore, with these considerations, we can model the escape of the Wolves as a game with the following rules:

- The Wolves choose where to begin their escape, meaning they choose a vertex v as their initial location
- The Pigs and Wolves alternate taking turns
- On each turn, either the Pigs or Wolves select a vertex adjacent to their current location that they haven't already traversed to move to
- The game ends when either the Pigs or Wolves have no vertices to move to which they haven't already been traversed, in which case they lose

Graph theoretically, starting from a vertex v, the players take turns adding vertices to a path starting at v, which ends when no more vertices can be added to that path. We will now analyze the winning strategies of this game, beginning with the Wolves.

5.1 The Winning Strategy

Theorem: If G does not have a perfect matching, then the Wolves have a winning strategy.

Proof: Take a maximal matching M' of G, meaning there does not exist any $e \in E$ such that $M' \cup \{e\}$ is a valid matching of G. Since G does not have a perfect matching, there must exist some $u \in V$ such that u is not matched in M'. The Wolves should choose this vertex as their initial location. Next, on each of their turns, they should choose the vertex v^* matched to their current location v in M' (as long as it has not already been used).

To show that this is a valid strategy, we need to show (1) that such a vertex v^* always exists, and (2) that it will not have already been traversed.

For (1), to see that such a vertex v^* will always exist, assume for sake of contradiction that it does not, meaning the Pigs chose a vertex v' not matched in M'. Let $P = vv_1 \cdots v_{2k}v'$ be the path up to this point, meaning v_i and v_{i+1} are matched in M' for $i = 1, 3, \ldots, 2k-1$ (as this is the first time the Pigs chose a vertex not matched in M'). However, this is a path in G where the first and last vertices are not matched in M', meaning in the language of Bona it is an M'-augmenting path. However, by Theorem 11.16 of Bona, M' is maximal if and only if G has no M' augmenting paths, contradicting our assumption that M' was maximal. Therefore, vertex v^* will always exists.

Now, for (2), assume for sake of contradiction that the Pigs choose a vertex v' such that the vertex v^* it is matched to in M' has already been traversed. We will now break into cases on whether the Wolves or Pigs traversed v^*:

- Case 1 - The Wolves traversed v^*: By definition of our strategy for the Wolves, they only ever pick the vertex matched to their current location in M'. Thus, since they move to v^* on their turn, they must have started their turn in v', meaning we have already traversed v' and thus the Pigs could not have moved there.
- Case 2 - The Pigs traversed v^*: If the Pigs traversed v^*, then again by definition of our strategy for the Wolves, on their turn they would have moved to v'. However, then we have already traversed v', meaning the Pigs could not have moved there.

Thus, in either case, we have a contradiction, meaning such a vertex will never have already been traversed.

Combining (1) and (2), our described strategy is always valid. Therefore, the Wolves can always move. Thus, since we have a finite number of vertices, we must eventually reach a maximal path P, which must end on the Pigs turn or else we could expand it by following the strategy described.

We will now describe a winning strategy for the Pigs.

Theorem: If G has a perfect matching M, then the Pigs have a winning strategy.

Proof: If G has a perfect matching, then the Pigs have a winning strategy by always choosing to move to the vertex matched to their current location in M.

We will prove this by induction on $m = |M|$, the size of our perfect matching.

For our base case, we will take $m = 0$. Then, since M must match every vertex in G, M must have no vertices. However, then the Wolves lose immediately, as they cannot start the game by picking an initial location.

Now, assume our inductive hypothesis holds for $m - 1$, and consider a graph G with a perfect matching M of size m. Then, the Wolves will pick a starting location v, and the Pigs will pick the vertex v' matched to v in M. We can observe that the graph $G' = G - v - v'$ will have a perfect matching $M' = M \setminus \{v, v'\}$ of size $m - 1$. Thus, since the Pigs and Wolves cannot use v or v' in the remainder of their path, G' represents the remainder of our playable graph, meaning the rest of the game is equivalent to our original game played on G' where the Wolves choose v' as this initial location. Thus, by our inductive hypothesis, this strategy will allow the Pigs to win the game.

5.2 Reduction to Nim

We can observe that our game is an impartial one, as the players alternate turns, there are a finite set of positions and operations, the game must end, and the allowable moves only depend on the position in the game. Therefore, by the Sprague Grundy Theorem, our game of The Great Escape is equivalent to a game of Nim.

To actually construct this equivalence, we can first represent our game in the graph form introduced in Section 3. To do this, we must determine X, the set of all positions in our game. We can observe that a position is entirely determined by the vertex the Pigs and Wolves currently occupy, as well as the vertices they have already traversed.

Once the set X is defined and the graph G is created, starting at the terminal positions of the graph, we can assign each vertex its SG value, eventually valuating each node. Once completed, each position on the graph can be represented by a game of Nim with a single heap.

As we mentioned in Section 1.1, some impartial games are more difficult to make equivalent to Nim than others. With P Little Pigs, with a little thought, we were able to fully make the reduction. This game is much more complex (i.e. dissimilar to Nim), meaning coming up with all of the different positions and evaluating the SG values for those positions is, in turn, also significantly more difficult. Thankfully, with the Sprague-Grundy Theorem, we know it is possible.

6 References

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