

**MTH 166**

**Lecture-10**

**Solution of 2<sup>nd</sup> Order Homogeneous LDE  
with Constant Coefficients-II**

**Topic:**

Solution of 2<sup>nd</sup> order Homogeneous LDE with Constant coefficients

**Learning Outcomes:**

1. Formulation of 2<sup>nd</sup> order homogeneous LDE when roots are given.
2. Solution of Initial Value Problems (IVP) and Boundary Value Problems (BVP).

**Formulation of LDE of the form:  $ay'' + by' + cy = 0$  when Roots are given:**

Let the two given roots be:  $m_1$  and  $m_2$ .

Then required 2<sup>nd</sup> order homogeneous LDE is:

$$y'' - (\text{sum of roots}) y' + (\text{Product of roots}) y = 0$$

$$\text{i.e. } y'' - (m_1 + m_2)y' + (m_1 m_2)y = 0$$

or

Then required 2<sup>nd</sup> order homogeneous LDE is:

$$(D - m_1)(D - m_2)y = 0 \text{ where } D \equiv \frac{d}{dx}$$

**Find a LDE of the form:  $ay'' + by' + cy = 0$  for which the following functions are solutions:**

**Problem 1.**  $(e^{3x}, e^{-2x})$

**Solution:** Comparing with:  $(e^{m_1x}, e^{m_2x})$

We have:  $m_1 = 3, m_2 = -2$

Then required 2<sup>nd</sup> order homogeneous LDE is:

$$y'' - (\text{sum of roots})y' + (\text{Product of roots})y = 0$$

$$\text{i.e. } y'' - (m_1 + m_2)y' + (m_1m_2)y = 0$$

$$\Rightarrow y'' - (3 - 2)y' + (3)(-2)y = 0$$

$$\Rightarrow y'' - y' - 6y = 0 \quad \textbf{Answer.}$$

**Problem 2.**  $(1, e^{-2x})$

**Solution:** Here  $(1, e^{-2x}) = (e^{0x}, e^{-2x})$

Comparing with:  $(e^{m_1x}, e^{m_2x})$

We have:  $m_1 = 0, m_2 = -2$

Then required 2<sup>nd</sup> order homogeneous LDE is:

$$y'' - (\text{sum of roots}) y' + (\text{Product of roots})y = 0$$

$$\text{i.e. } y'' - (m_1 + m_2)y' + (m_1m_2)y = 0$$

$$\Rightarrow y'' - (0 - 2)y' + (0)(-2)y = 0$$

$$\Rightarrow y'' + 2y' = 0 \quad \textbf{Answer.}$$

**Problem 3.**  $(e^{2x}, xe^{2x})$

**Solution:** Comparing with:  $(e^{m_1x}, xe^{m_2x})$

We have:  $m_1 = 2, m_2 = 2$

Then required 2<sup>nd</sup> order homogeneous LDE is:

$$y'' - (\text{sum of roots}) y' + (\text{Product of roots})y = 0$$

$$\text{i.e. } y'' - (m_1 + m_2)y' + (m_1m_2)y = 0$$

$$\Rightarrow y'' - (2 + 2)y' + (2)(2)y = 0$$

$$\Rightarrow y'' - 4y' + 4y = 0 \quad \textbf{Answer.}$$

**Problem 4.**  $(e^{-3ix}, e^{3ix})$

**Solution:** Comparing with:  $(e^{m_1x}, e^{m_2x})$

We have:  $m_1 = -3i, m_2 = 3i$

Then required 2<sup>nd</sup> order homogeneous LDE is:

$$y'' - (\text{sum of roots}) y' + (\text{Product of roots})y = 0$$

$$\text{i.e. } y'' - (m_1 + m_2)y' + (m_1m_2)y = 0$$

$$\Rightarrow y'' - (-3i + 3i)y' + (-3i)(3i)y = 0$$

$$\Rightarrow y'' + 9y = 0 \quad \textbf{Answer.} \quad (i^2 = -1)$$

**Problem 5.**  $(e^{(5+3i)x}, e^{(5-3i)x})$

**Solution:** Comparing with:  $(e^{m_1x}, e^{m_2x})$

We have:  $m_1 = 5 + 3i, m_2 = 5 - 3i$

Then required 2<sup>nd</sup> order homogeneous LDE is:

$$y'' - (\text{sum of roots}) y' + (\text{Product of roots})y = 0$$

$$\text{i.e. } y'' - (m_1 + m_2)y' + (m_1m_2)y = 0$$

$$\Rightarrow y'' - [(5 + 3i) + (5 - 3i)]y' + (5 + 3i)(5 - 3i)y = 0$$

$$\Rightarrow y'' + 10y' + 34y = 0 \quad \textbf{Answer.} \quad (i^2 = -1)$$



**Problem:** Solve the Initial value problem:  $y'' - y = 0$ ,  $y(0) = 0$ ,  $y'(0) = 2$ .

**Solution:** The given equation is:

$$y'' - y = 0 \quad (1)$$

Such that:  $y(0) = 0$ ,  $y'(0) = 2$

**S.F. :**  $(D^2 - 1)y = 0$  where  $D \equiv \frac{d}{dx}$

**A.E. :**  $(D^2 - 1) = 0 \Rightarrow D^2 = 1 \Rightarrow D = \pm 1$  (real and unequal roots)

Let  $m_1 = 1$  and  $m_2 = -1$

$\therefore$  General Solution of equation (1) is given by:

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$\Rightarrow y = c_1 e^{1x} + c_2 e^{-1x}$$

$$\Rightarrow y(x) = c_1 e^x + c_2 e^{-x} \quad (2)$$

$$\Rightarrow y'(x) = c_1 e^x - c_2 e^{-x} \quad (3)$$

Using  $y(0) = 0$  in equation (2), we get:

$$y(0) = c_1 e^0 + c_2 e^{-0} \quad \Rightarrow 0 = c_1 + c_2 \quad (4)$$

Using  $y'(0) = 2$  in equation (3), we get:

$$y'(0) = c_1 e^0 - c_2 e^{-0} \quad \Rightarrow 2 = c_1 - c_2 \quad (5)$$

Solving equations (4) and (5), we get:  $c_1 = 1, c_2 = -1$

Putting these values of  $c_1$  and  $c_2$  in equation (2), we get:

$$y(x) = e^x - e^{-x} \quad \textbf{Answer.}$$

**Problem:** Solve the Boundary value problem:  $y'' - 4y' + 3y = 0$  such that  $y(0) = 1, y(1) = 0$ .

**Solution:** The given equation is:

$$y'' - 4y' + 3y = 0 \quad (1)$$

Such that:  $y(0) = 1, y(1) = 0$

**S.F. :**  $(D^2 - 4D + 3)y = 0$  where  $D \equiv \frac{d}{dx}$

**A.E. :**  $(D^2 - 4D + 3) = 0 \Rightarrow (D - 1)(D - 3) = 0$

$\Rightarrow D = 1, 3$  (real and unequal roots)

Let  $m_1 = 1$  and  $m_2 = 3$

$\therefore$  General Solution of equation (1) is given by:

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$\Rightarrow y = c_1 e^{1x} + c_2 e^{3x}$$

$$y(x) = c_1 e^{1x} + c_2 e^{3x} \quad (2)$$

Using  $y(0) = 1$ , we get:

$$y(0) = c_1 e^0 + c_2 e^0 \quad \Rightarrow 1 = c_1 + c_2 \quad (3)$$

Using  $y(1) = 0$ , we get:

$$y(1) = c_1 e^1 + c_2 e^3 \quad \Rightarrow 0 = c_1 e^1 + c_2 e^3 \quad (4)$$

Solving equations (3) and (4), we get:  $c_1 = \frac{e^2}{e^2-1}$  and  $c_2 = \frac{1}{e^2-1}$

Putting these values of  $c_1$  and  $c_2$  in equation (2), we get:

$$y(x) = \frac{e^2}{e^2-1} e^x + \frac{1}{e^2-1} e^{3x} \quad \textbf{Answer.}$$



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