

**MTH 166**

**Lecture-15**

**Method of Variation of Parameters**

**Topic:**

Solution of Non-Homogeneous LDE with Constant coefficients.

**Learning Outcomes:**

To use Method of Variation of Parameters to solve Non-Homogeneous LDE with constant coefficients.

## Method of Variation of Parameters:

Let us consider 2<sup>nd</sup> order Non-homogeneous LDE with constant coefficients as:

$$ay'' + by' + cy = r(x) \quad (1)$$

Let two solutions of C.F.  $(aD^2 + bD + c) = 0$  be  $y_1$  and  $y_2$

i.e.  $y_c = c_1y_1 + c_2y_2$

Wronskian,  $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

P.I.  $y_p = -y_1 \int \frac{y_2 r(x)}{W} dx + y_2 \int \frac{y_1 r(x)}{W} dx$

General solution is:  $y = C.F. + P.I.$

i.e.  $y = y_c + y_p$

**Problem 1.** Find the general solution of:  $y'' + y = \sec x$

**Solution:** The given equation is:

$$y'' + y = \sec x \quad (1)$$

S.F.:  $(D^2 + 1)y = \sec x$  where  $D \equiv \frac{d}{dx}$

$$\Rightarrow f(D)y = r(x) \quad \text{where } f(D) = (D^2 + 1) \text{ and } r(x) = \sec x$$

To find Complimentary Function (C.F.):

A.E.:  $f(D) = 0 \Rightarrow (D^2 + 1) = 0 \Rightarrow D^2 = -1 \Rightarrow D = i, -i$

$\therefore$  Complimentary function is given by:

$$y_c = e^{0x}(c_1 \cos x + c_2 \sin x)$$

$$\Rightarrow y_c = (c_1 \cos x + c_2 \sin x)$$

To find Particular Integral (P.I.):

Comparing  $y_c$  with  $y_c = c_1 y_1 + c_2 y_2$

Here  $y_1 = \cos x, y_2 = \sin x$

$$\text{Wronskian, } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = (\cos^2 x + \sin^2 x) = 1$$

$$\text{P.I. is given by: } y_p = -y_1 \int \frac{y_2 r(x)}{W} dx + y_2 \int \frac{y_1 r(x)}{W} dx$$

$$\Rightarrow y_p = -\cos x \int \frac{\sin x(\sec x)}{1} dx + \sin x \int \frac{\cos x(\sec x)}{1} dx$$

$$\Rightarrow y_p = -\cos x \int \tan x dx + \sin x \int dx$$

$$\Rightarrow y_p = -\cos x(-\log|\cos x|) + \sin x(x)$$

$\therefore$  General solution is given by:  $y = \text{C.F.} + \text{P.I.}$

$$\text{i.e. } y = y_c + y_p$$

$$\Rightarrow y = (c_1 \cos x + c_2 \sin x) + \cos x(\log|\cos x|) + x \sin x \quad \text{Answer.}$$

**Problem 2.** Find the general solution of:  $y'' + y = \operatorname{cosec} x$

**Solution:** The given equation is:

$$y'' + y = \operatorname{cosec} x \quad (1)$$

S.F.:  $(D^2 + 1)y = \operatorname{cosec} x$  where  $D \equiv \frac{d}{dx}$

$$\Rightarrow f(D)y = r(x) \quad \text{where } f(D) = (D^2 + 1) \text{ and } r(x) = \operatorname{cosec} x$$

To find Complimentary Function (C.F.):

A.E.:  $f(D) = 0 \Rightarrow (D^2 + 1) = 0 \Rightarrow D^2 = -1 \Rightarrow D = i, -i$

$\therefore$  Complimentary function is given by:

$$y_c = e^{0x}(c_1 \cos x + c_2 \sin x)$$

$$\Rightarrow y_c = (c_1 \cos x + c_2 \sin x)$$

To find Particular Integral (P.I.):

Comparing  $y_c$  with  $y_c = c_1 y_1 + c_2 y_2$

Here  $y_1 = \cos x, y_2 = \sin x$

$$\text{Wronskian, } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = (\cos^2 x + \sin^2 x) = 1$$

$$\text{P.I. is given by: } y_p = -y_1 \int \frac{y_2 r(x)}{W} dx + y_2 \int \frac{y_1 r(x)}{W} dx$$

$$\Rightarrow y_p = -\cos x \int \frac{\sin x(\operatorname{cosec} x)}{1} dx + \sin x \int \frac{\cos x(\operatorname{cosec} x)}{1} dx$$

$$\Rightarrow y_p = -\cos x \int dx + \sin x \int \cot x dx$$

$$\Rightarrow y_p = -\cos x(x) + \sin x (\log|\sin x|)$$

∴ General solution is given by:  $y = \text{C.F.} + \text{P.I.}$

$$\text{i.e. } y = y_c + y_p$$

$$\Rightarrow y = (c_1 \cos x + c_2 \sin x) + \sin x(\log|\sin x|) - x \cos x \quad \text{Answer.}$$

**Problem 3.** Find the general solution of:  $y'' + y = \tan x$

**Solution:** The given equation is:

$$y'' + y = \tan x \quad (1)$$

S.F.:  $(D^2 + 1)y = \tan x$  where  $D \equiv \frac{d}{dx}$

$$\Rightarrow f(D)y = r(x) \quad \text{where } f(D) = (D^2 + 1) \text{ and } r(x) = \tan x$$

To find Complimentary Function (C.F.):

A.E.:  $f(D) = 0 \Rightarrow (D^2 + 1) = 0 \Rightarrow D^2 = -1 \Rightarrow D = i, -i$

$\therefore$  Complimentary function is given by:

$$y_c = e^{0x}(c_1 \cos x + c_2 \sin x)$$

$$\Rightarrow y_c = (c_1 \cos x + c_2 \sin x)$$

To find Particular Integral (P.I.):

Comparing  $y_c$  with  $y_c = c_1 y_1 + c_2 y_2$

Here  $y_1 = \cos x, y_2 = \sin x$

$$\text{Wronskian, } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = (\cos^2 x + \sin^2 x) = 1$$

$$\text{P.I. is given by: } y_p = -y_1 \int \frac{y_2 r(x)}{W} dx + y_2 \int \frac{y_1 r(x)}{W} dx$$

$$\Rightarrow y_p = -\cos x \int \frac{\sin x(\tan x)}{1} dx + \sin x \int \frac{\cos x(\tan x)}{1} dx$$

$$\Rightarrow y_p = -\cos x \int \frac{\sin^2 x}{\cos x} dx + \sin x \int \sin x dx$$

$$\Rightarrow y_p = -\cos x \int \frac{1-\cos^2 x}{\cos x} dx + \sin x(-\cos x)$$

$$\Rightarrow y_p = -\cos x \int (\sec x - \cos x) dx - \sin x \cos x$$

$$\Rightarrow y_p = -\cos x \log|\sec x + \tan x| + \cos x \sin x - \sin x \cos x$$

∴ General solution is given by:  $y = \text{C.F.} + \text{P.I.}$

$$\text{i.e. } y = y_c + y_p$$

$$\Rightarrow y = (c_1 \cos x + c_2 \sin x) - \cos x(\log|\sec x + \tan x|) \quad \text{Answer.}$$

**Problem 4.** Find the general solution of:  $y'' + y = \cot x$

**Solution:** The given equation is:

$$y'' + y = \cot x \quad (1)$$

S.F.:  $(D^2 + 1)y = \cot x$  where  $D \equiv \frac{d}{dx}$

$$\Rightarrow f(D)y = r(x) \quad \text{where } f(D) = (D^2 + 1) \text{ and } r(x) = \cot x$$

To find Complimentary Function (C.F.):

A.E.:  $f(D) = 0 \Rightarrow (D^2 + 1) = 0 \Rightarrow D^2 = -1 \Rightarrow D = i, -i$

$\therefore$  Complimentary function is given by:

$$y_c = e^{0x}(c_1 \cos x + c_2 \sin x)$$

$$\Rightarrow y_c = (c_1 \cos x + c_2 \sin x)$$

To find Particular Integral (P.I.):

Comparing  $y_c$  with  $y_c = c_1 y_1 + c_2 y_2$

Here  $y_1 = \cos x, y_2 = \sin x$

$$\text{Wronskian, } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = (\cos^2 x + \sin^2 x) = 1$$

$$\text{P.I. is given by: } y_p = -y_1 \int \frac{y_2 r(x)}{W} dx + y_2 \int \frac{y_1 r(x)}{W} dx$$

$$\Rightarrow y_p = -\cos x \int \frac{\sin x(\cot x)}{1} dx + \sin x \int \frac{\cos x(\cot x)}{1} dx$$

$$\Rightarrow y_p = -\cos x \int \cos x dx + \sin x \int \frac{\cos^2 x}{\sin x} dx$$

$$\Rightarrow y_p = -\cos x (\sin x) + \sin x \int \frac{1-\sin^2 x}{\sin x} dx$$

$$\Rightarrow y_p = -\cos x (\sin x) + \sin x \int (\cosec x - \sin x) dx$$

$$\Rightarrow y_p = -\cos x (\sin x) + \sin x (-\log|\cosec x + \cot x|) + \sin x (\cos x)$$

∴ General solution is given by:  $y = \text{C.F.} + \text{P.I.}$

$$\text{i.e. } y = y_c + y_p$$

$$\Rightarrow y = (c_1 \cos x + c_2 \sin x) - \sin x (\log|\cosec x + \cot x|) \quad \text{Answer.}$$

**Problem 5.** Find the general solution of:  $y'' - 2y' - 3y = e^x$

**Solution:** The given equation is:

$$y'' - 2y' - 3y = e^x \quad (1)$$

S.F.:  $(D^2 - 2D - 3)y = e^x$  where  $D \equiv \frac{d}{dx}$

$$\Rightarrow f(D)y = r(x) \text{ where } f(D) = (D^2 - 2D - 3) \text{ and } r(x) = e^x$$

To find Complimentary Function (C.F.):

A.E.:  $f(D) = 0 \Rightarrow (D^2 - 2D - 3) = 0 \Rightarrow (D - 3)(D + 1) = 0$

$$\Rightarrow D = 3, -1 \quad (\text{real and distinct roots})$$

$\therefore$  Complimentary function is given by:

$$y_c = c_1 e^{3x} + c_2 e^{-1x}$$

To find Particular Integral (P.I.):

Comparing  $y_c$  with  $y_c = c_1 y_1 + c_2 y_2$

Here  $y_1 = e^{3x}, y_2 = e^{-x}$

$$\text{Wronskian, } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & e^{-x} \\ 3e^{3x} & -e^{-x} \end{vmatrix} = (-e^{3x}e^{-x} - 3e^{3x}e^{-x}) = -4e^{2x}$$

$$\text{P.I. is given by: } y_p = -y_1 \int \frac{y_2 r(x)}{W} dx + y_2 \int \frac{y_1 r(x)}{W} dx$$

$$\Rightarrow y_p = -e^{3x} \int \frac{e^{-x}(e^x)}{-4e^{2x}} dx + e^{-x} \int \frac{e^{3x}(e^x)}{-4e^{2x}} dx$$

$$\Rightarrow y_p = \frac{e^{3x}}{4} \int e^{-2x} dx - \frac{e^{-x}}{4} \int e^{2x} dx$$

$$\Rightarrow y_p = \frac{e^{3x}}{4} \left( \frac{e^{-2x}}{-2} \right) - \frac{e^{-x}}{4} \left( \frac{e^{2x}}{2} \right) = -\frac{1}{4} e^x$$

∴ General solution is given by:  $y = \text{C.F.} + \text{P.I.}$

$$\text{i.e. } y = y_c + y_p$$

$$\Rightarrow y = (c_1 e^{3x} + c_2 e^{-1x}) - \frac{1}{4} e^x \quad \text{Answer.}$$



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