Solving the Maximum Nash Welfare Problem: A Cutting Plane Approach

Devpriyo Ray

Supervised by: Prof. Sachin Jayaswal

1 Introduction

Let $I = \{1, ..., m\}$ be the set of m agents, $J = \{1, ..., n\}$ be a set of n indivisible items, and each agent has a utility/valuation function $v_i(S)$, defined for all $S \subseteq J$. We assume the preferences to be additive, i.e., $v_i(S) = \sum_{j \in J} v_i(j)$, for all $S \subseteq J$. Further, we assume monotonic preferences, i.e., $v_i(S) \geq v_i(T)$, for all $T \subseteq S \subseteq J$. We consider the items to be goods, i.e., $v_i(j) \geq 0$, $\forall j \in J$. Let $\mathbf{A} = \{A_1, ..., A_m\}$ be the allocation of the n goods to m agents, where each agent i receives a bundle A_i.

Given the above setup, we want to allocate the n indivisible goods amongst the m agents, with respect to a fairness criteria. We say that the allocation is *Envy-free upto one good*, or $EF1$, if for all $i, k \in I$, the following inequality holds-

$$v_i(A_i) \geq v_i(A_k \setminus j), \quad \text{for some } j \in A_k$$

To ensure that $EF1$ fairness is met, we obtain the allocation by maximizing the following product, also known as the Maximum Nash Welfare(MNW)(Caragiannis et al., 2019)-

$$\Big(\prod_{i=1}^{m} v_i(A_i)\Big)^{\frac{1}{n}} \tag{1}$$

2 Maximum Nash-Welfare As An Optimization Problem

To find out an MNW allocation, we convert (1) into a non-linear integer program(NLIP). For ease of notation, we write $v_i(j)$ as v_{ij}, for all $i \in I$, $j \in J$. Let x_{ij} be the binary variable, which is equal to 1 if agent $i \in I$ gets item $j \in J$, 0 otherwise. Our NLIP formulation is now as follows -

$$\max_{\mathbf{x}} \quad \Big(\prod_{i \in I} \Big(\sum_{j \in J} v_{ij} x_{ij}\Big)\Big)^{\frac{1}{n}} \tag{2}$$

$$\text{subject to:} \quad \sum_{j} x_{ij} = 1, \quad \forall i \in I \tag{3}$$

$$x_{ij} \in \{0, 1\}, \quad \forall (i, j) \in I \times J \tag{4}$$

To solve this problem, we can proceed via branch and bound, which involves relaxing the integer variable x_{ij}. The relaxed problem would now be -

$$\max_{\mathbf{x}} \quad \Big(\prod_{i \in I} \Big(\sum_{j \in J} v_{ij} x_{ij}\Big)\Big)^{\frac{1}{n}} \tag{5}$$

$$\text{subject to:} \quad \sum_{j} x_{ij} = 1, \quad \forall i \in I \tag{6}$$

$$x_{ij} \in [0, 1], \quad \forall (i, j) \in I \times J \tag{7}$$

Firstly, we note that the above NLIP cannot be handled by modern solvers like CPLEX, Gurobi, etc. **directly**, as these solver can at most handle Quadratically Constrained Quadratic Programs(QCQP). We can but convert our problem into a quadratic optimization problem using standard reformulation and linearisation techniques(**RLT**), but that will introduce a factorial number of new integer variables in the model, and increase the complexity of the model exponentially.

CPLEX and Gurobi, however, do allow for piecewise linearisation of non-linear functions of one variable. This, however will again require us to convert the multivariable problem into a single variable problem using RLT, and then use finite constraint linearisation upto the desired accuracy. This will increase both the number of variables and constraints in the model exponentially, and hence increase the complexity by a great deal.

Another reason why MNW is an interesting problem is because finding an MNW allocation is a Strongly NP-Hard problem. Hence, MNW becomes an interesting optimization problem.

3 Solution Procedure

We start by making the problem linear. Firstly, we note that $\sum_{j \in J} v_{ij} x_{ij}$ is an increasing function in x_{ij}, for all i. Moreover, $\sum_{j \in J} v_{ij} x_{ij} \geq 0$ by the definition of goods. Hence, we can always use a monotonic transformation $f(z)$ to transform the objective of our NLIP. As an immediate consequence of this fact, we note that maximizing the geometric mean $\left(\prod_{i=1}^{m} v_i(A_i)\right)^{\frac{1}{n}}$ is equivalent to just maximizing the product $\left(\prod_{i=1}^{m} v_i(A_i)\right)$.

For our problem, let $z = \prod_{i\in I}\sum_{j\in J} v_{ij}x_{ij}$. We use the monotonic transformation $f(z) = \log(z+1)$. Adding 1 is not a coincidence, we do so to ensure that whenever $z \to 0$, $\log(z) \nrightarrow -\infty$ (as $\log(1) = 0$). We can now rewrite our objective as

$$\sum_{i\in I} \log\left(1 + \sum_{j\in J} v_{ij}x_{ij}\right)$$

Next, we state and prove the following proposition-

Proposition 1. *The function* $\log(\mathbf{a}^\intercal\mathbf{x} + b)$ *is concave in* $\mathbf{x}$*, where* $\mathbf{x} \in \mathbb{R}^n$*,* $\mathbf{a} \in \mathbb{R}^n$*, and* $b \in \mathbb{R}$*.*

Proof. We know that $f(\mathbf{x}) = \mathbf{a}^\intercal\mathbf{x} + b$ is an affine function, and is hence trivially concave(and convex). It is also trivial to show that $g : D(\subseteq \mathbb{R}_{\geq 0}) \to \mathbb{R}$, $g(x) = \log(x)$ is a concave function. To show that $\log(f(\mathbf{x}))$ is concave, note that -

$$\log(f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y})) = \log(\lambda f(\mathbf{x}) + (1-\lambda)f(\mathbf{y})) \geq \lambda\log(f(\mathbf{x})) + (1-\lambda)\log(f(\mathbf{y}))$$

which proves the concavity of $\log(\mathbf{a}^\intercal\mathbf{x} + b)$. ■

Proposition 1 tells us that $\forall i \in I$, $\log\left(1 + \sum_{j\in J} v_{ij}x_{ij}\right)$ is a concave function(where x_{ij} is relaxed). Hence, we can approximate it arbitrarily well using tangent hyperplanes to it. Now, let $W_i = \log\left(1 + \sum_{j\in J} v_{ij}x_{ij}\right) \geq 0$, $\forall i \in I$. With this notation, we now write our optimization problem as -

$$\begin{aligned}
\max \quad & \sum_{i\in I} W_i && (8)\\
\text{subject to:} \quad & W_i = \log\left(1 + \sum_{j\in J} v_{ij}x_{ij}\right), \quad i \in \{1, ...m\} && (9)\\
& \sum_{j} x_{ij} = 1, \quad \forall i \in I && (10)\\
& x_{ij} \in \{0,1\}, \quad \forall (i,j) \in I \times J && (11)\\
& W_i \geq 0, \quad \forall i \in I && (12)
\end{aligned}$$

For any i, let $\mathbf{x}_i = (x_{i1}, ..., x_{in}) \in [0,1]^n$, $\mathbf{v}_i = (v_{i1}, ..., v_{in})$, and $f(\mathbf{x}_i) = \log(1 + \mathbf{x}_i^\intercal\mathbf{v}_i)$. Let $\mathbf{x}_i^0 \in [0,1]^n$ be a point in the domain of $f(.)$, on which we wish to calculate the tangent.

From the differentiability criteria of concavity, we know that if $f : \Omega(\subseteq \mathbb{R}^n) \to \mathbb{R}$ is a differentiable concave function, then for any $x, y \in \mathbb{R}$, we have(Hiriart-Urruty and Lemaréchal, 2004)-

$$f(y) \leq f(x) + \nabla f(x)^\intercal (y - x)$$

Hence, in our case, for any $\mathbf{x}_i^0$, we have

$$\log(1 + \mathbf{x}_i^\intercal \mathbf{v}_i) \leq \log(1 + (\mathbf{x}_i^0)^\intercal \mathbf{v}_i) + \nabla(\log(1 + (\mathbf{x}_i^0)^\intercal \mathbf{v}_i))^\intercal (\mathbf{x}_i - \mathbf{x}_i^0) \tag{13}$$

Now, we can use a sufficient number of points H_i from the domain of $\log(1+\mathbf{x}_i^\intercal \mathbf{v}_i)$ to approximate it within our required precision. We may also note that $W_i = \log(1+\mathbf{x}_i^\intercal \mathbf{v}_i) \leq \log(1+\max_{i,j}\{v_{ij}\} * n)$. Let $\log(1 + \max_{i,j}\{v_{ij}\} * n) = M$.

Using the above ideas, we now finally linearize our constraint, and obtain the following formulation-

$$\textbf{MNW-L:} \quad \max \quad \sum_{i \in I} W_i \tag{14}$$

$$\text{subject to:} \quad W_i \leq \log(1 + (\mathbf{x}_i^h)^\intercal \mathbf{v}_i) + \nabla(\log(1 + (\mathbf{x}_i^h)^\intercal \mathbf{v}_i))^\intercal (\mathbf{x}_i - \mathbf{x}_i^h), \quad \forall h \in H_i, \ \forall i \in I \tag{15}$$

$$\sum_j x_{ij} = 1, \quad \forall i \in I \tag{16}$$

$$x_{ij} \in \{0, 1\}, \quad \forall (i, j) \in I \times J \tag{17}$$

$$0 \leq W_i \leq M, \quad \forall i \in I \tag{18}$$

4 Algorithm

Although we have our linear formulation **MNW-L**, we would, however, not define the set H_i apriori. Instead, we would generate tangent hyperplanes as per every run of **MNW-L**.

Algorithm 1 Cutting Plane Procedure

Initialize ϵ ▷ Precision level of optimal objective
Define $Curr$, $Prev$ ▷ For current and previous objective values
Set $Prev \leftarrow 0$
while $|Curr - Prev| > \epsilon$, **do**
    Solve **MNW-L**
    Let $\mathbf{x} \in \arg\max\{$**MNW-L**$\}$. Generate Tangent Hyperplanes using $\mathbf{x}$ and (13).
    $Prev \leftarrow Curr$
    $Curr \leftarrow$ Optimal value of **MNW-L**
end while

5 Computational Results

We solved the following instances-

- $J \leftarrow 20$ to 270, with an increment of 10. $I \leftarrow 10$ to J, with an increment of 10, for all J(refer to Results1.txt).
- **(Significantly)Larger Instances:** $J \leftarrow 500$. $I \leftarrow 300$ to J, with an increment of 50(refer to Results2.txt).
- **(Large J, Smaller I):** $J \leftarrow 500$. $I \leftarrow 10$ to 100, with an increment of 10(refer to Results3.txt).

All the instances were solved upto optimality.

References

Ioannis Caragiannis, David Kurokawa, Hervé Moulin, Ariel D Procaccia, Nisarg Shah, and Junxing Wang. The unreasonable fairness of maximum nash welfare. *ACM Transactions on Economics and Computation (TEAC)*, 7(3):1–32, 2019. (Cited on page 1.)

Jean-Baptiste Hiriart-Urruty and Claude Lemaréchal. *Fundamentals of convex analysis*. Springer Science & Business Media, 2004. (Cited on page 4.)