Computing Equilibria in Single Leader Multiple Follower Games

1 Introduction

We look at situations in which we have a single leader in the upper level, and multiple followers at the lower level. The players at the lower level are playing a non-cooperative game. We aim to find out a *subgame-perfect equilibrium*, either in pure strategies, or mixed strategies.

We now start defining our problem. Let $N = \{1, ..., n\}$ be the number of players in the Stackelberg game. For $p \in N$, A_p is the action space, with $m_p = |A_p|$. Let Δ_p be the set of all probability distributions for the action space of player p, and let $x_p = (x_p^1, \cdots, x_p^{m_p}) \in \Delta_p$ one such probability distribution. If $x_p \in \{0, 1\}^{m_p}$, then such a strategy will define a *pure strategy* equilibrium, otherwise we have a mixed strategy equilibrium. Let $U_p \in \mathbb{Q}^{m_1 \times \cdots \times m_n}$ be the n dimensional utility matrix associated with player $p \in N$. Any element in the matrix U_p can be written as $U_p^{a_1 \cdots a_n}$, which is precisely the utility obtained by player p, when player $i \in N$ is playing strategy a_i. Given this, the expected utility can be written as -

$$U_p(x_1, ..., x_n) = \sum_{a_1 \in A_1} \cdots \sum_{a_n \in A_n} U_p^{a_1 \cdots a_n} x_1^{a_1} \cdots x_n^{a_n}$$

2 Solution Strategy

For ease of notation, we consider $n = 3$. Note that the procedure described below can be easily adapted for any n. WLOG, assume $m_1 = m_2 = m_3 = m$. We now have the following problem:
LEADER:

$$\max_{x_1 \times x_2 \times x_3 \in \Delta_1 \times \Delta_2 \times \Delta_3} \sum_{i \in A_1} \sum_{j \in A_2} \sum_{k \in A_3} U_3^{ijk} x_1^i x_2^j x_3^k \tag{1}$$

subject to followers problems-

FOLLOWER:

$$\max_{x_1 \in \Delta_1} \sum_{i \in A_1} \sum_{j \in A_2} \sum_{k \in A_3} U_1^{ijk} x_1^i x_2^j x_3^k \tag{2}$$

$$\max_{x_2 \in \Delta_2} \sum_{i \in A_1} \sum_{j \in A_2} \sum_{k \in A_3} U_2^{ijk} x_1^i x_2^j x_3^k \tag{3}$$

For the lower level problem, consider $\widetilde{U}_1^{ij} = \sum_{k \in A_3} U_1^{ijk} x_3^k$, $\widetilde{U}_2^{ij} = \sum_{k \in A_3} U_2^{ijk} x_3^k$. Our lower level problems can now be written as-

$$\max_{x_1 \in \Delta_1} \sum_{i \in A_1} \sum_{j \in A_2} \widetilde{U}_1^{ij} x_1^i x_2^j \tag{4}$$

$$\max_{x_2 \in \Delta_2} \sum_{i \in A_1} \sum_{j \in A_2} \widetilde{U}_2^{ij} x_1^i x_2^j \tag{5}$$

We convert (4) and (5) into their respective duals, and write down the complementarity problem as follows-

$$(v_1 - \sum_{j \in A_2} \widetilde{U}_2^{ij} x_2^j) x_1^i = 0, \quad \forall i \tag{6}$$

$$v_1 \geq \sum_{j \in A_2} \widetilde{U}_2^{ij} x_2^j \tag{7}$$

$$(v_2 - \sum_{i \in A_1} \widetilde{U}_1^{ij} x_1^i) x_2^j = 0, \quad \forall j \tag{8}$$

$$v_2 \geq \sum_{i \in A_1} \widetilde{U}_1^{ij} x_1^i \tag{9}$$

(1), (6), (7), (8), (9) together now describe a single level problem, with non-linear constraints.

We now convert the complementary slackness constraints (6), (8) into linear constraints by introducing binary variables $s_1,\ s_2 \in \{0,1\}^m$, and a large constant M, as follows-

$$x_1^i \leq 1 - s_1^i,\ \forall i \tag{10}$$

$$v_1 - \sum_{j \in A_2} \widetilde{U}_2^{ij} x_2^j \leq M s_1^i,\ \forall i \tag{11}$$

$$x_2^j \leq 1 - s_2^j,\ \forall j \tag{12}$$

$$v_2 - \sum_{i \in A_1} \widetilde{U}_1^{ij} x_1^i \leq M s_2^j,\ \forall j \tag{13}$$

Finally, we use RLT as follows-

$$y_{23}^{jk} = x_2^j x_3^k, \quad \forall j, k \tag{14}$$

$$y_{13}^{ik} = x_1^i x_3^k, \quad \forall i, k \tag{15}$$

$$z^{ijk} = x_1^i y_{23}^{jk}, \quad \forall i, j, k \tag{16}$$

We also note that-

$$\sum_{i,k} y_{13}^{ik} = 1 \tag{17}$$

$$\sum_{j,k} y_{23}^{jk} = 1 \tag{18}$$

$$\sum_{i,j,k} z^{ijk} = 1 \tag{19}$$

Our final model is now-

$$\begin{aligned}
\max \quad & \sum_{i \in A_1} \sum_{j \in A_2} \sum_{k \in A_3} U_3^{ijk} z_{ijk} && (20)\\
\text{subject to} \quad & (10), (11), (12), (13), (7), (9), (15), (14), (16), (17), (18), (19) && (21)\\
& y_{13}, y_{23} \in [0,1]^{m \times m} && (22)\\
& x_1 \in \Delta_1, \ , x_2 \in \Delta_2, \ , x_3 \in \Delta_3 && (23)\\
& s_1, s_2 \in \{0,1\}^m && (24)\\
& z \in [0,1]^{m \times m \times m} && (25)\\
& v_1, v_2 \text{ free} && (26)
\end{aligned}$$