PROJECT - 2 : Two-Dimensional Finite Element Methods

Report submitted by

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Chapter 1

Problem Description

The present report consists of analysis of two engineering problems aimed towards understanding and demonstration of the application of 2D finite element methods. The problems considered for the study are described below :

1.1 Rectangular plate with circular hole

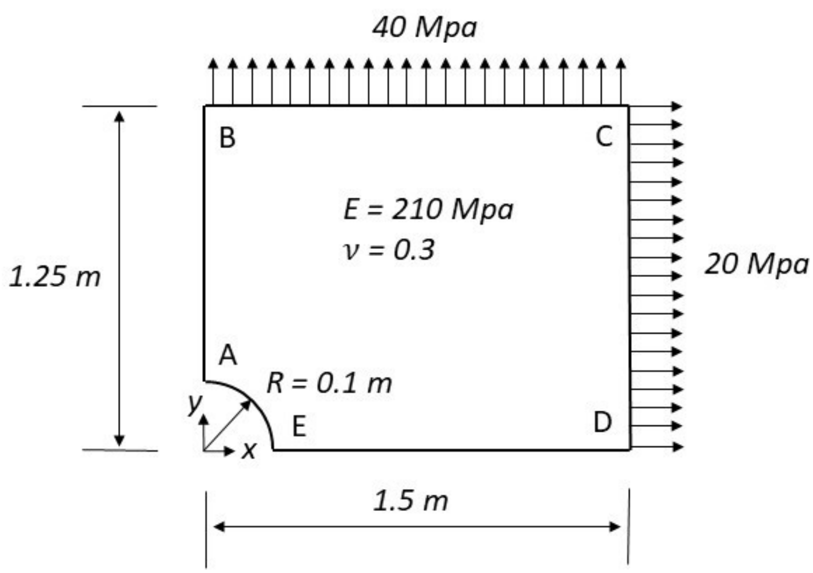


Figure 1.1: A plate subjected to biaxial loading. The material property is shown in the figure.

This problem involves static analysis of a quarter section of a rectangular plate with circular hole(symmetrical about the two coordinate axes). The plate is subjected to biaxial tension applied in the form of uniformly distributed load over the outer edges as shown in the fig. Gravitational force is assumed to be absent in this problem.

The parameter values for the given problem are specified as follows:

$$\begin{aligned} l &= 1.5 \text{ m} \\ h &= 1.25 \text{ m} \\ R &= 0.1 \text{ m} \\ E &= 210 \text{ GPa} \\ \nu &= 0.3 \end{aligned}$$

The bar, in the given problem, is subjected to the following boundary conditions:

$$\begin{aligned} &\text{at } x = 0, \quad && u_x = 0 \text{ m} \\ &\text{at } y = 0, \quad && u_y = 0 \text{ m} \\ &\text{at } x = l, \quad && \sigma_{xx} = 20 \text{ MPa} \\ &\text{at } y = h, \quad && \sigma_{yy} = 40 \text{ MPa} \end{aligned}$$

1.2 Static analysis of a Dam

The second problem presented herein, is that of a dam structure whose wall is subjected to pressure of water contained in the reservoir. This pressure is modelled as a uniformly varying load and its magnitude at a given height can be evaluated using Pascal's law. Also, the dam is under the action of body forces due to self weight.

The right edge of the dam structure is assumed to be in the form of a circular arc whose dimensions are shown in the figure below.

The material for the dam section is assumed to be concrete. Parameter values for this problem are given below:

$$
\begin{aligned}
L_x &= 150 \text{ m} \\
L_y &= 1.25 \text{ m} \\
H_{water} &= 75 \text{ m} \\
r &= 105 \text{ m} \\
R &= 10 \text{ m} \\
E &= 25 \text{ GPa} \\
\nu &= 0.15 \\
\rho_{water} &= 1.0\text{x}10^3 \,\text{kg/m}^3 \\
\rho_{concrete} &= 2.5\text{x}10^3 \,\text{kg/m}^3
\end{aligned}
$$

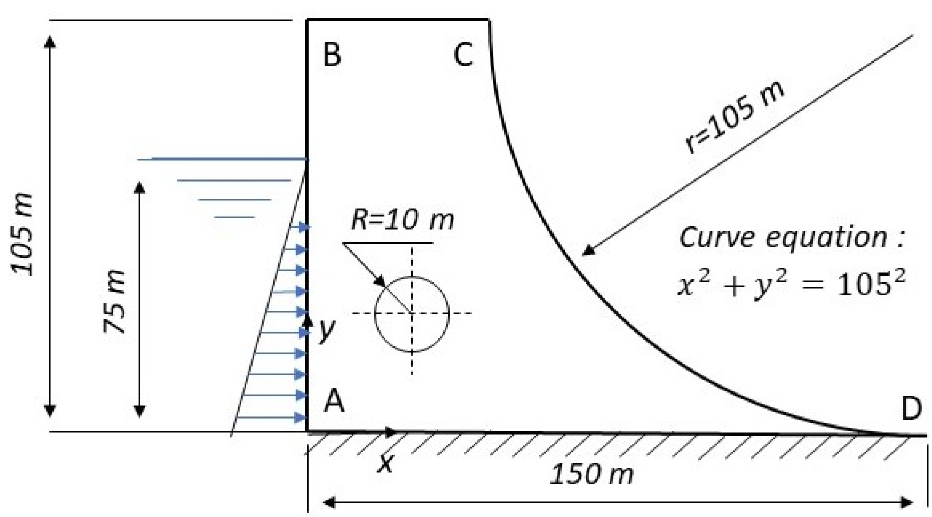


Figure 1.2: A dam subjected to water pressure and self-weight.

The dam section is subjected to the following boundary conditions:

$$\begin{aligned}
\text{at } x = 0, \qquad \sigma_{xx} &= \rho g(H_{water} - y) \ \text{MPa} \\
\text{at } y = 0, \qquad u_x &= u_y = 0 \ \text{m}
\end{aligned}$$

Chapter 2

Problem Formulation

The governing equations for two dimensional deformation of a body under biaxial loading are given by:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x = 0 \tag{2.1}$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + b_y = 0 \qquad \text{*for static cases} \tag{2.2}$$

which can be written in a condensed form as:

$$[S]^T [\sigma] + [b] = 0 \tag{2.3}$$

For the problems included in this project, we are required to model them using plane strain assumption. So, the governing equations can further be modified. Recall that,

$$[\sigma] = [D] [\epsilon] \qquad \text{*neglecting pre strain.} \tag{2.4}$$

where,

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

Therefore, the governing equation can be rewritten as:

$$G(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + \frac{G}{1-2\nu}\frac{\partial}{\partial x}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + b_x = 0 \tag{2.5}$$

$$G(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) + \frac{G}{1-2\nu}\frac{\partial}{\partial y}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + b_y = 0 \tag{2.6}$$

where,

$$G = \frac{E}{2(1+\nu)}$$

Next in order to get the weak form of the governing equation, we use the Method of Weighted Residuals.

$$I(u, w) = \int_{\Omega} [w] \Big[-[S]^T [\sigma] - [b] \Big] \, d\Omega \tag{2.7}$$

Here, we make use of the Principle of Virtual Work, where the arbitrary weight function is the virtual displacement.

$$\delta\Pi_{eq} = \int_{\Omega} [\delta u] \Big[-[S]^T [\sigma] - [b] \Big] \, d\Omega \tag{2.8}$$

The above equation can be simplified to get the final form resulting as :

$$\delta\Pi_{eq} = \int_{\Omega} \delta \left[[S] \, [u]\right]^T [\sigma] \, d\Omega - \int_{\Gamma t} [\delta u]^T [\bar{t}] \, d\Gamma - \int_{\Omega} [\delta u]^T [b] \, d\Omega \tag{2.9}$$

We can now proceed with using Galerkin FEM. For this, the geometry is discretized into 'n' number of elements. Since we are considering a two dimensional problem, we use a 4-noded quadrilateral element for representation of geometry. A linear approximation of displacement(u) over an element is given by:

$$[u] \approx [\hat{u}]^e = [N]^{eT} [\tilde{u}]^e \tag{2.10}$$

Similarly the variation in displacement can be represented as:,

$$[\delta\hat{u}]^e = [N]^{eT} [\delta\tilde{u}]^e \tag{2.11}$$

The weak form now expressed over an element of the discretized geometry is given by

$$\delta\hat{\Pi}_{eq}^{\ e} = \int_{\Omega^e} \left[[B]^{eT}[\delta\tilde{u}]^e\right]^T [\sigma]\, d\Omega - \int_{\Gamma_t^e} [\delta\tilde{u}]^{eT}[N]^e[\bar{t}]\, d\Gamma - \int_{\Omega^e} [\delta\hat{u}]^{eT}[N]^e[b]\, d\Omega \quad (2.12)$$

where,

$$[B]^{eT} = [S]\,[N]^{eT}$$

The above equation can further besimplified as:

$$\delta\hat{\Pi}_{eq}^{\ e} = [\delta\tilde{u}]^{eT}[f_{int}]^e - [\delta\tilde{u}]^{eT}[f_{ext}]^e - [\delta\tilde{u}]^{eT}[f_{body}]^e; \quad (2.13)$$

$$\begin{aligned}
[f_{int}]^e &= \int_{\Omega^e} [B]^e[\sigma]\, d\Omega = \int_{\Omega^e} [B]^e[D][B]^{eT} d\Omega\,[\tilde{u}]^e = [K]^e[\tilde{u}]^e \\
[f_{ext}]^e &= \int_{\Gamma_t^e} [N]^e[\bar{t}]\, d\Gamma \\
[f_{body}]^e &= \int_{\Omega^e} [N]^e[b]\, d\Omega
\end{aligned}$$

Mapping the coordinates from physical space to master space,

$$\begin{aligned}
[K]^e &= \int_{-1}^{1}\int_{-1}^{1} [B']^e[D][B']^{eT}[J]\, d\xi d\eta \\
[f_{ext}]^e &= \int_{-1}^{1} [N']^e[\bar{t}][J]\, d\xi \\
[f_{body}]^e &= \int_{-1}^{1}\int_{-1}^{1} [N']^e[b][J]\, d\xi d\eta
\end{aligned} \quad (2.14)$$

Using Gauss Quardrature formula to numerically evaluate the integrals,

$$\begin{aligned}
[K]^e &= \sum_{\xi_1}^{n\xi}\sum_{\eta_1}^{n\eta} [B']^e[D][B']^{eT}[J]\, w_{ij} \\
[f_{ext}]^e &= \sum_{\xi_1}^{n\xi} [N']^e[\bar{t}][J]\, w_i \\
[f_{body}]^e &= \sum_{\xi_1}^{n\xi}\sum_{\eta_1}^{n\eta} [N']^e[b][J]\, w_(ij)
\end{aligned}$$

Next we carry out assembly over all the elements,

$$\begin{aligned} [K]^g &= A_{e=1}^{ne}\,[K]^e \\ [f_{ext}]^g &= A_{e=1}^{surfe}\,[f_{ext}]^e \\ [f_{body}]^g &= A_{e=1}^{ne}\,[f_{body}]^e \end{aligned} \tag{2.15}$$

Finally, setting the variation in total energy equal to zero gives us,

$$\begin{aligned} &[\delta\tilde{u}]^T\,[[f_{int}]^g - [f_{body}]^g - [f_{ext}]^g] = 0 \\ &\text{*since } \delta\tilde{u} \text{ is arbitrary,} \qquad [f_{int}]^g = [f_{body}]^g + [f_{ext}]^g \\ &[K]^g\left[\tilde{U}\right] = [F]^g \end{aligned} \tag{2.16}$$

Enforcing essential boundary condition,

$$[\hat{K}]^g[\hat{U}] = [\hat{F}]^g \tag{2.17}$$

Chapter 3

Results and Discussion

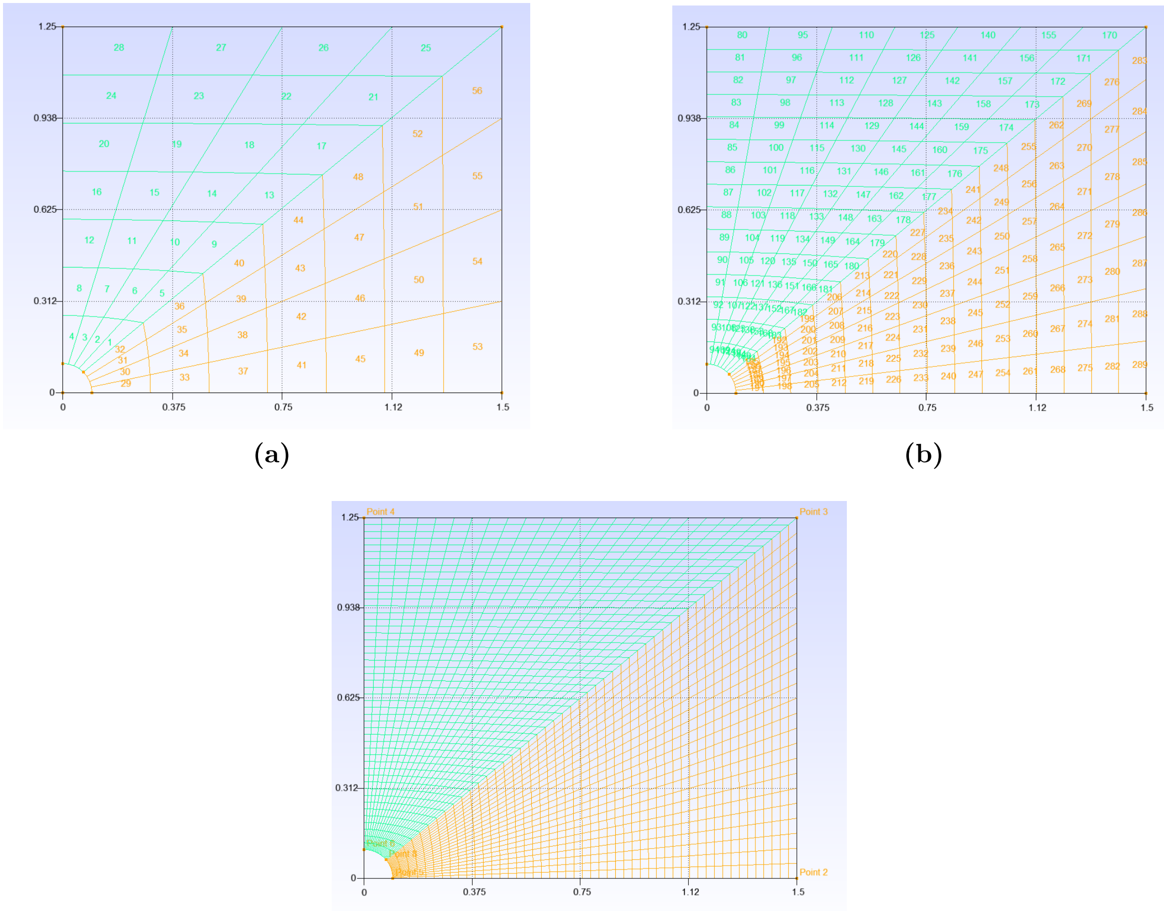
The problems were solved using a MATLAB program. The geometry was sketched and a mesh was generated(with different refinement for multiple cases) using GMSH (open source software). For mesh generation, care was taken to create a structured mesh(for ease of node assignment to boundary) by subdividing the sketch into 4-sided polygons. Then using transfinite operation on opposite edges, equal number of subdivisions were created.

The number of nodes per element was taken as 4 and the degrees of freedom per node was kept equal to 2. The number of gauss points was chosen as 2(since the highest degree of polynomial appearing in the integral formulation is 2) for the integrals over the bulk domain and number of gauss points over boundary integrals was taken as 5.

Below, we present the results obtained for each of the two problems mentioned in this project. For the 1^{st} case(plate with hole problem), the numerical results have also been compared with the analytical solution.

3.1 Rectangular Plate with Hole

The different meshes generated for analysing this problem are shown below.



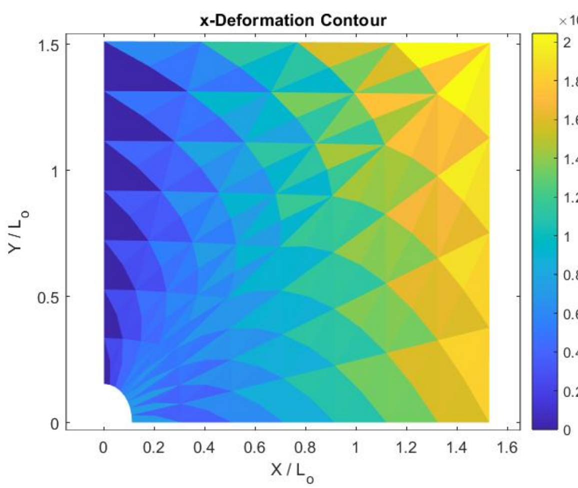
(a) **(b)** **(c)**

Figure 3.1: Meshes for Rectangular Plate(a) M1 (b) M2 (c) M3

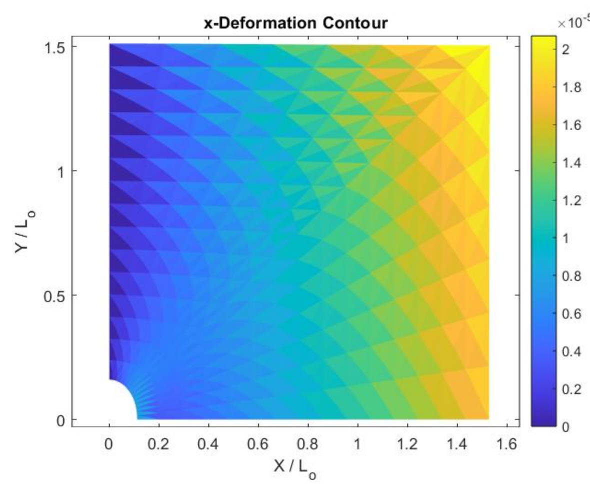
Mesh no.	No. of nodes in radial direction (n_r)	No. of nodes in tangential direction (n_t)	Total no. of nodes (n_{total})	Total no. of elements (e)
M1	8	9	72	56
M2	16	15	240	210
M3	50	49	2450	2352

3.1.1 Deformation Contours

The deformation contours were plotted for directional deformation along x and y directions respectively. The results obtained for each of the cases are shown below. For the sake of visualization, the deformation has been scaled by a factor of 1.5×10^3 in these plots.



(a)



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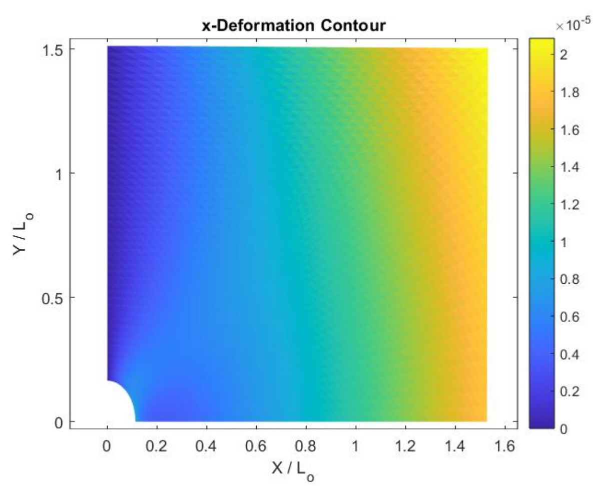
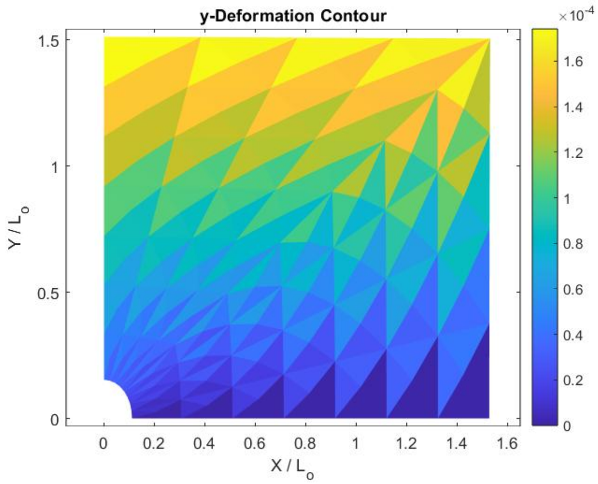


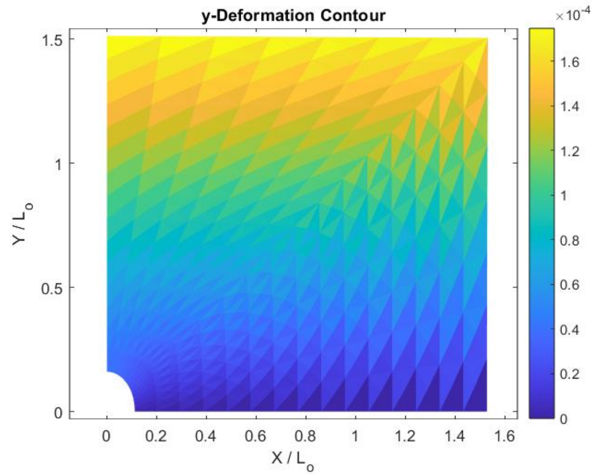
(c)

Figure 3.2: X-Deformation Contours for plate

Looking at the above contours, it can be seen that the x displacement is maximum near the right edge of the plate where the tractions are applied. The peak value of displacement is found to be 2.044×10^{-5} m.



(a)



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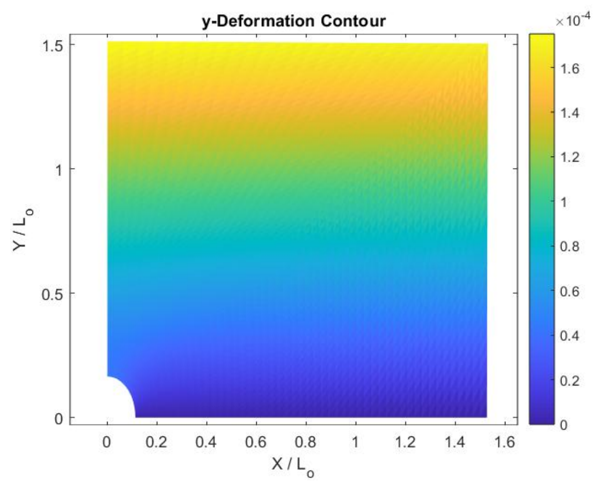

(c)

Figure 3.3: Y-Deformation Contours for plate

Observing the Y-deformation contours, we find that the magnitude of deformation is maximum at the top edge of the plate and decreases to zero at the bottom edge. The peak value of deformation is found to be 1.741×10^{-4} m.

Both the above deformation contours demonstrate the expected variation of displacements along a plate subjected to biaxial tension.

3.1.2 Stress Contours

Stress contours are plotted for the normal stresses($\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$), shear stress($\sigma_{xy}$) and the equivalent stress(following Von-Mises criterion) for each of the cases.

The obtained results have also been compared with the analytical solution for stresses over an infinite plate with hole. The following expressions were used to evaluate the corresponding component of stresses.

$$\begin{aligned}
\sigma_{xx} &= t_x - (3t_x - t_y)\frac{1}{2}\frac{R^2}{r^2}cos2\theta + (t_x - t_y)(\frac{3}{2}\frac{R^4}{r^4} - \frac{R^2}{r^2})cos4\theta \\
\sigma_{yy} &= t_y - (t_x - 3t_y)\frac{1}{2}\frac{R^2}{r^2}cos2\theta - (t_x - t_y)(\frac{3}{2}\frac{R^4}{r^4} - \frac{R^2}{r^2})cos4\theta \\
\sigma_{xy} &= (t_x + t_y)(\frac{3}{2}\frac{R^4}{r^4} - \frac{R^2}{r^2})sin4\theta - (t_x - t_y)\frac{1}{2}\frac{R^2}{r^2}sin2\theta
\end{aligned} \tag{3.1}$$

The results obtained for each of the cases are shown below.

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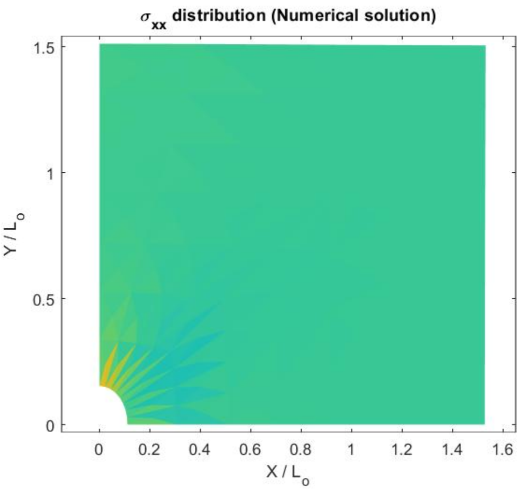
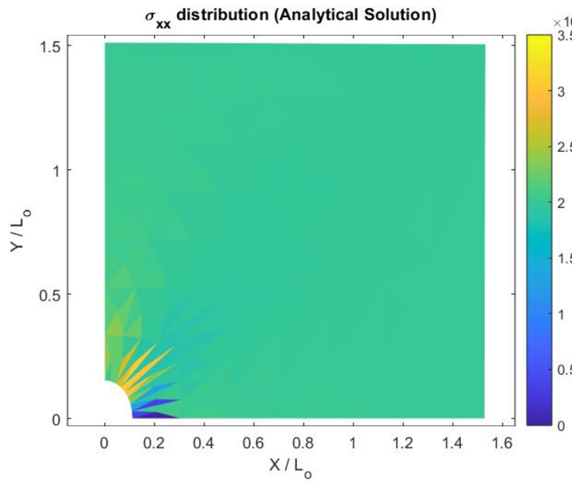
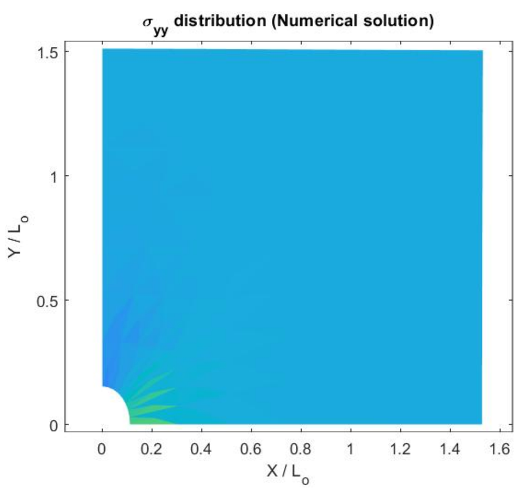


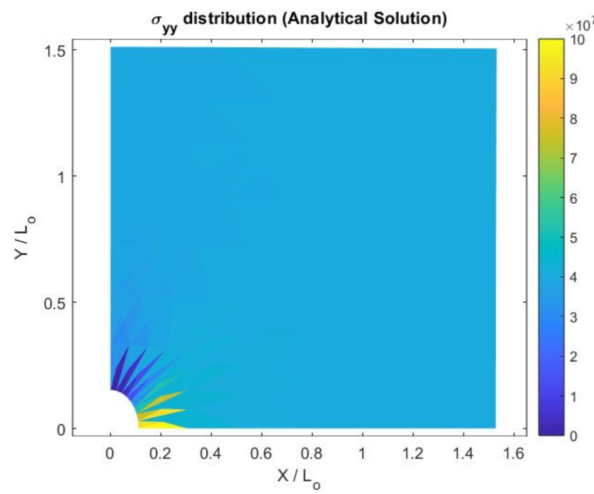
(a)



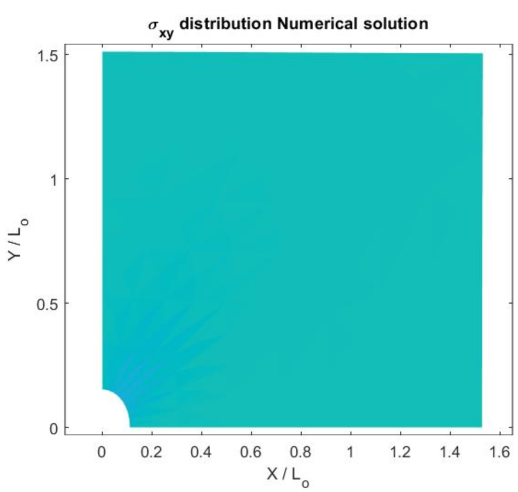
(b)



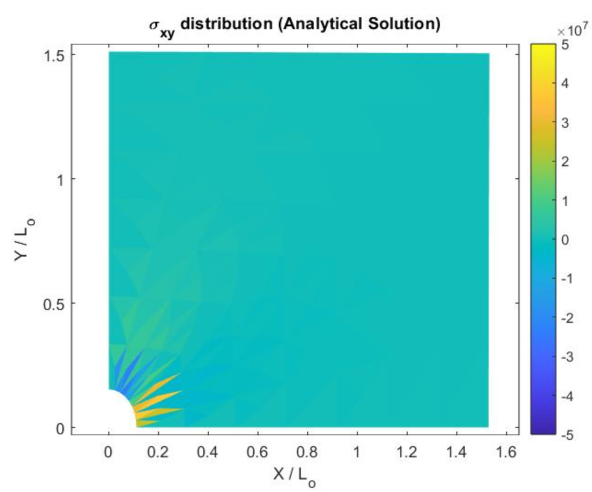
(c)



(d)



(e)



(f)

Figure 3.4: Stress Contours for mesh M1 (Plate)

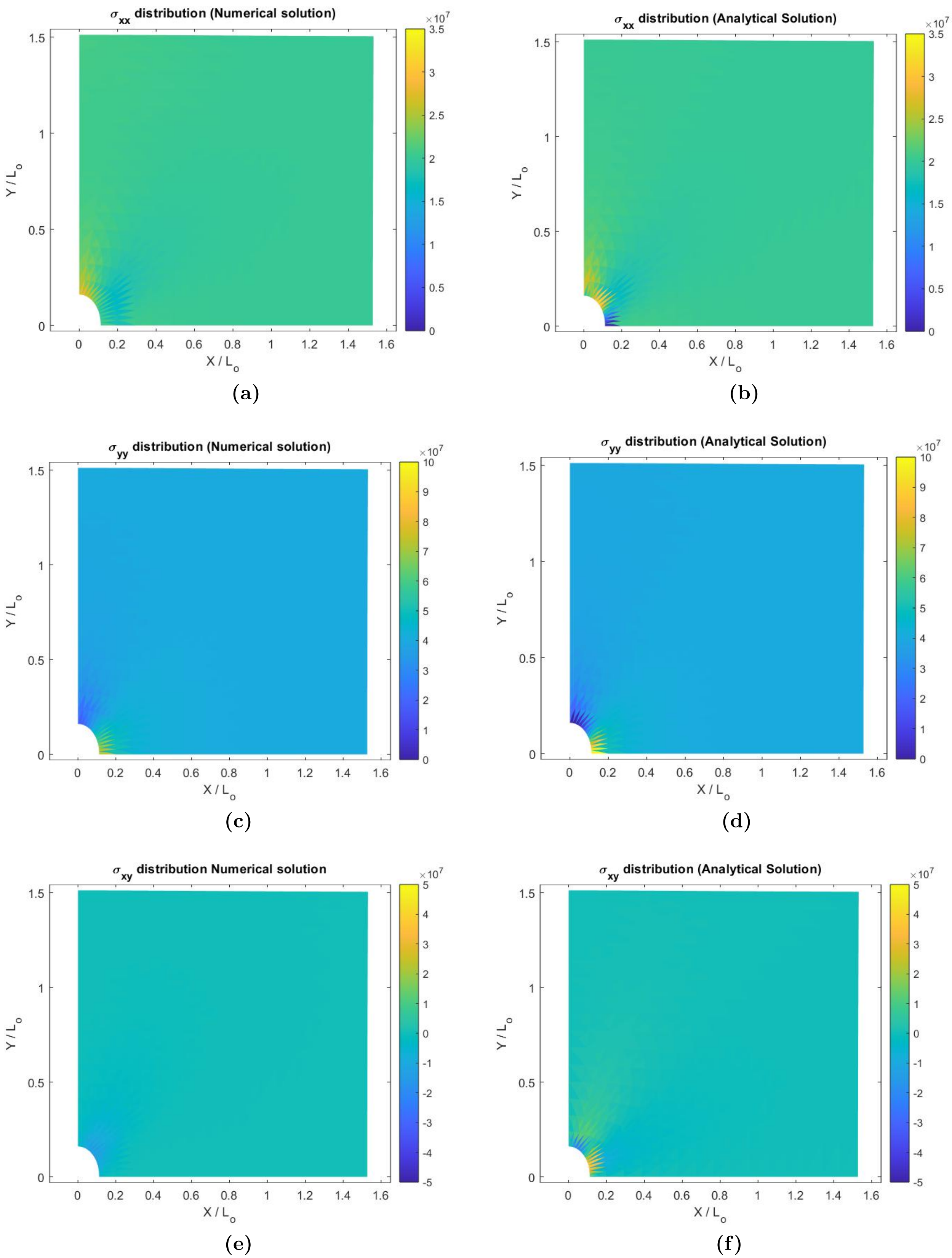


Figure 3.5: Stress Contours for mesh M2 (Plate)

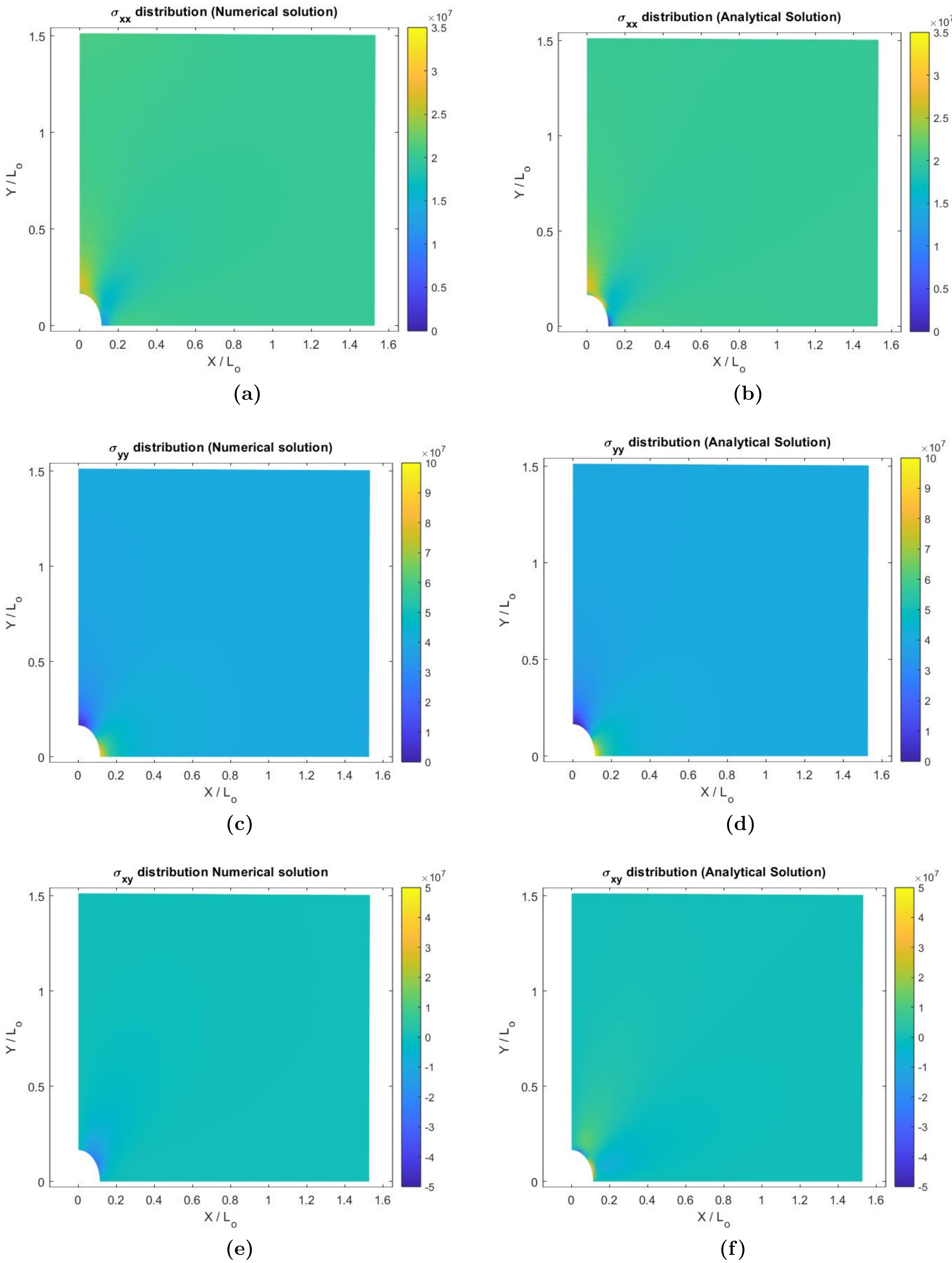


Figure 3.6: Stress Contours for mesh M3 (Plate)

Looking at these plots we observe that the variation between the numerical solution and the analytical solution goes on diminishing as the mesh gets refined.

One clear observation is that, at distances far from the centre of the hole the stresses are constant over the plate. For the plot of σ_{xx}, it is found that this constant value of stress is equal to 20 MPa(the traction applied on the right edge). Similarly, For the plot of σ_{yy}, it is found that the constant value of stress is equal to 40 MPa(the traction applied at the top edge). σ_{xy} can be seen to have a constant value(less than 1 MPa) over most of the plate .

It is also observed that the in all cases, the stress have their peak values at the edge of the hole. The peak magnitudes of the stresses are found to be 27.224 MPa for σ_{xx}, 90.492 MPa for σ_{yy} and 23.502 MPa for σ_{xy}. The reason for the significant increase in the values can be attributed to stress concentration.

Since, we have assumed plane strain condition we have σ_{zz} present as the third component of normal stress . The plot for σ_{zz} for each of the three cases is shown below.

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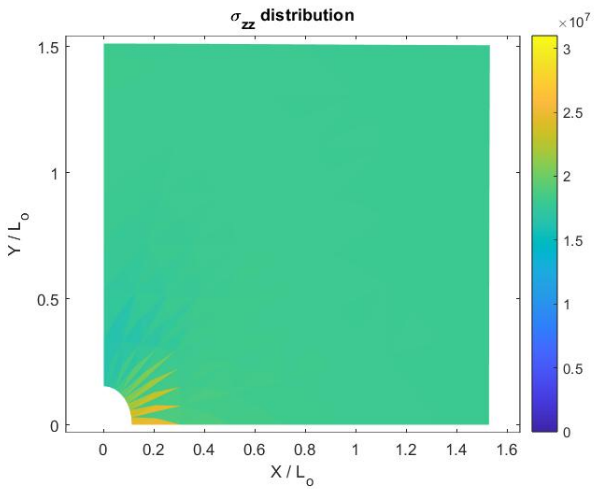
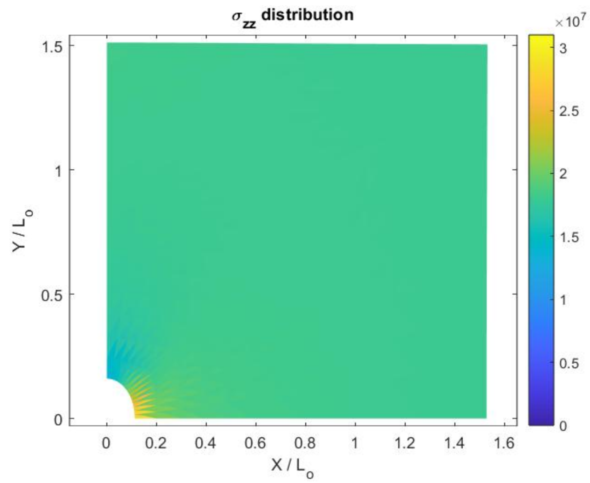
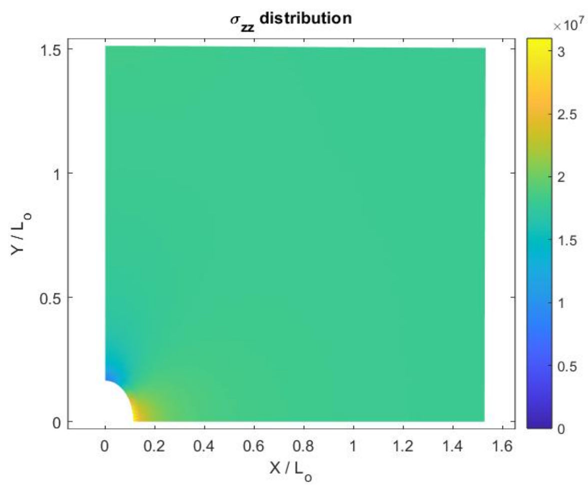


(a)



(b)

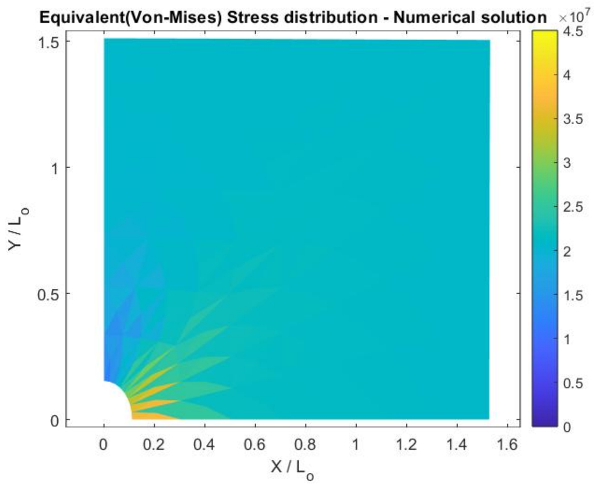


(c)

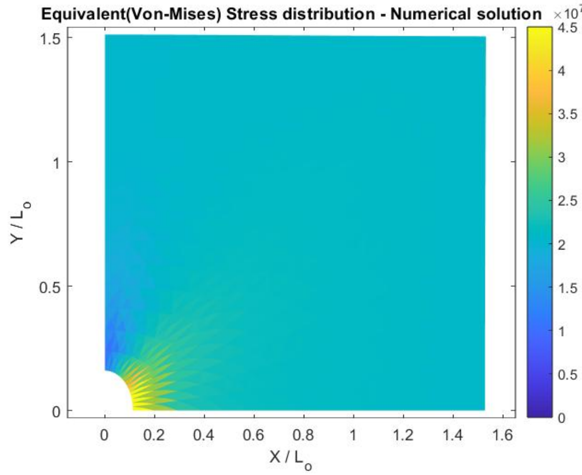
Figure 3.7: σ_{zz} Stress Contours(Plate) for (a) M1 (b) M2 (c) M3

From the plots, we note that at regions away from the hole the stress has a constant value(nearly equal to 20 MPa). The peak value of σ_{zz} occurs at the edge of the hole and is equal to 31.108 MPa.

We have also plotted the contours for the Von-Mises stress for each of the three cases. The plots are shown below.



(a)



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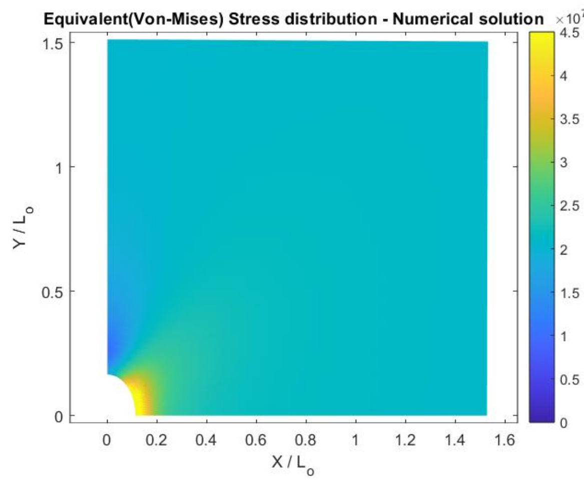


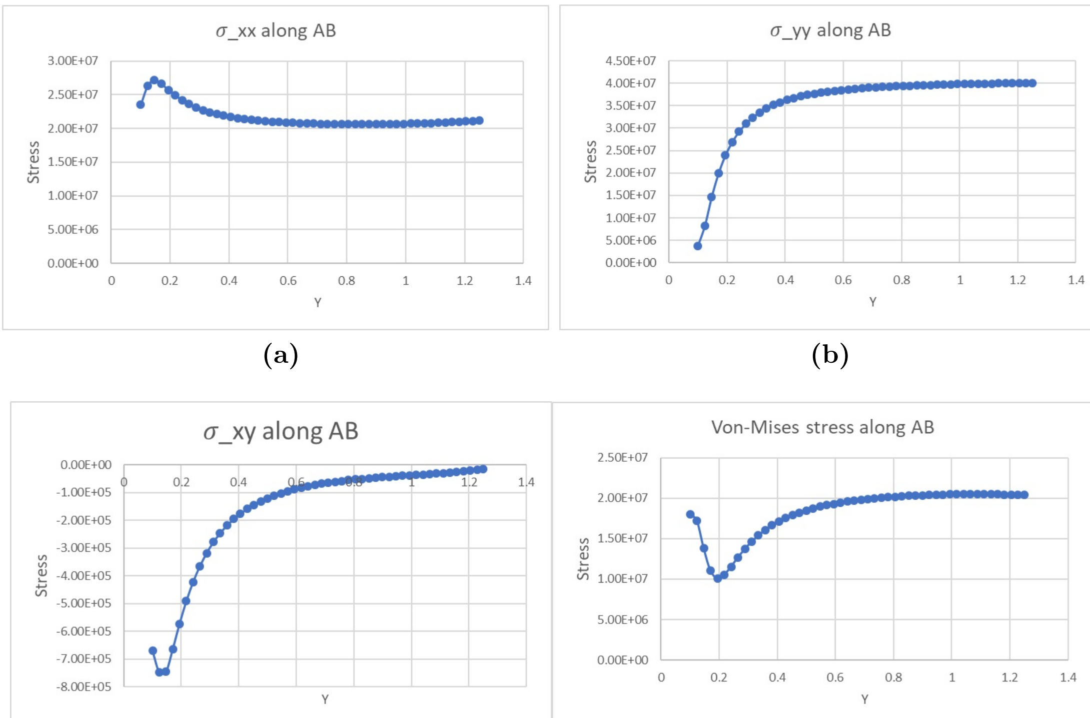
(c)

Figure 3.8: Von-Mises Stress Contours(Plate) for (a) M1 (b) M2 (c) M3

Like all the previous plots the stress has a constant value for greater region of the plate which is nearly equal to 25 MPa. The peak value of the stress observed is equal to 70.083 MPa.

3.1.3 Graphs

In order to visualize the stress distribution along edges AB and ED of the plate, following graphs have been plotted:



(a) (b) (c) (d)

Figure 3.9: Stress distribution along edge AB of plate

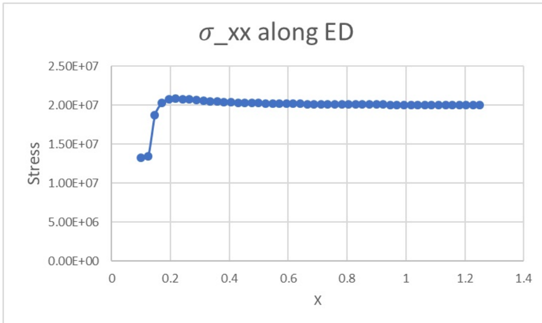
The following observations can be made:

1. σ_{xx} has the highest value near the hole and then decreases to a constant value over the rest of the edge.

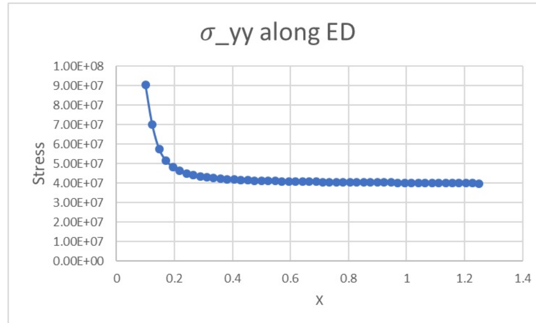
2. σ_{yy} is zero near the hole and has the highest value at the top end of the plate.

3. σ_{yy} has the highest value near the hole and decreases to become zero at the top edge of the plate.

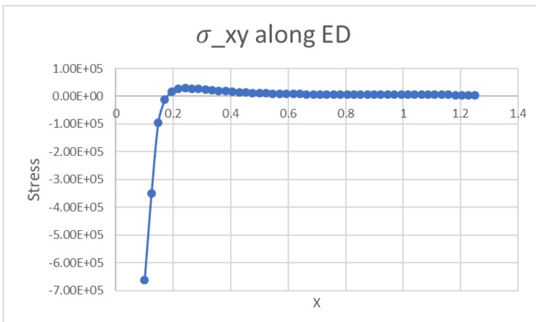
4. The Von-Mises stress is slightly higher at the top end of the plate.



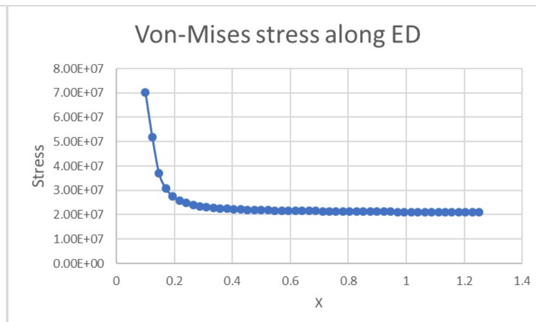
(a)



(b)



(c)



(d)

Figure 3.10: Stress distribution along edge ED of plate

The following observations can be made:

1. σ_{xx} has the lowest value near the hole and then increases to a constant value over the rest of the edge.

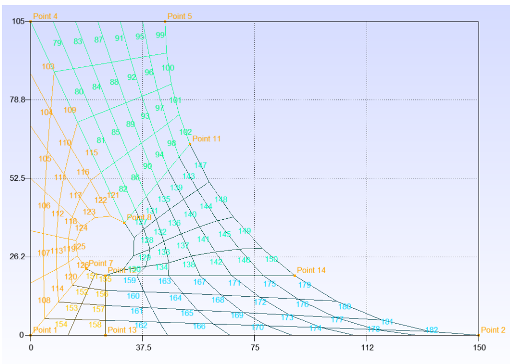
2. σ_{yy}has the highest value near the hole and then decreases to a constant value over the rest of the edge.

3. σ_{yy} has the highest value near the hole and decreases to become zero at the right edge of the plate.

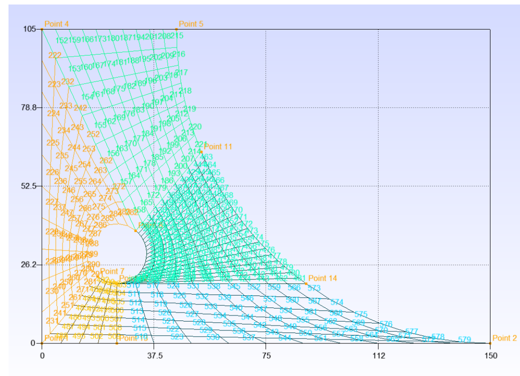
4. The Von-Mises stress is highest at the hole and then decreases to a constant value over the edge.

3.2 Dam with water in reservoir

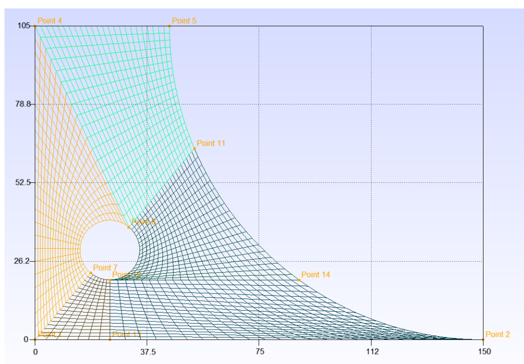
Three different meshes were generated for analysing this problem in GMSH. These are shown below.



(a)



(b)



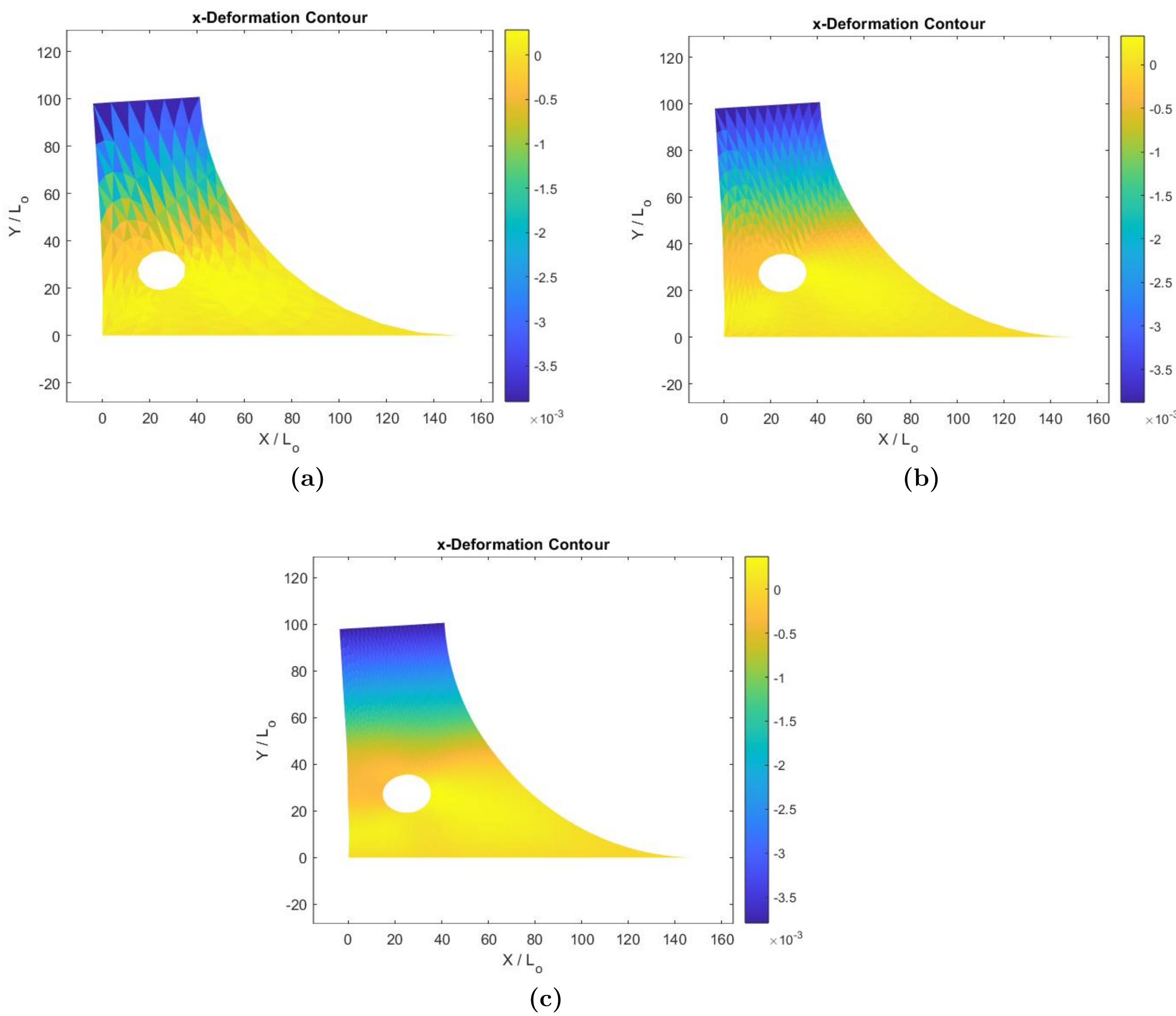
(c)

Figure 3.11: Meshes for Dam (a) M1 (b) M2 (c) M3

Mesh no.	Total no. of nodes (n_{total})	Total no. of elements (e)
M1	126	104
M2	478	428
M3	1970	1878

3.2.1 Deformation Contours

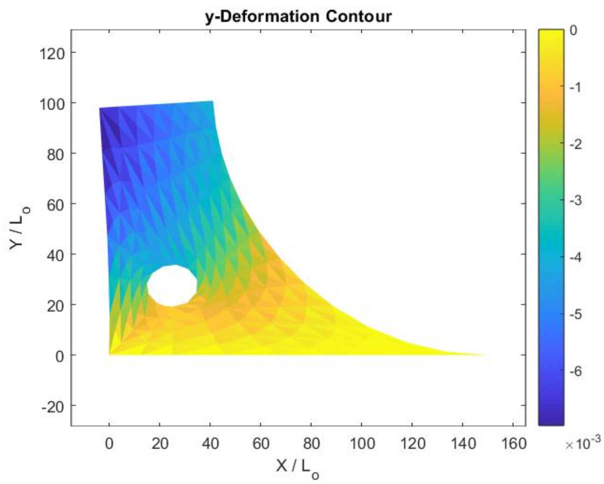
The deformation contours were plotted for directional deformation along x and y directions respectively. The results obtained for each of the cases are shown below. For the sake of visualization, the deformation has been scaled by a factor of 1.0×10^3 in these plots.



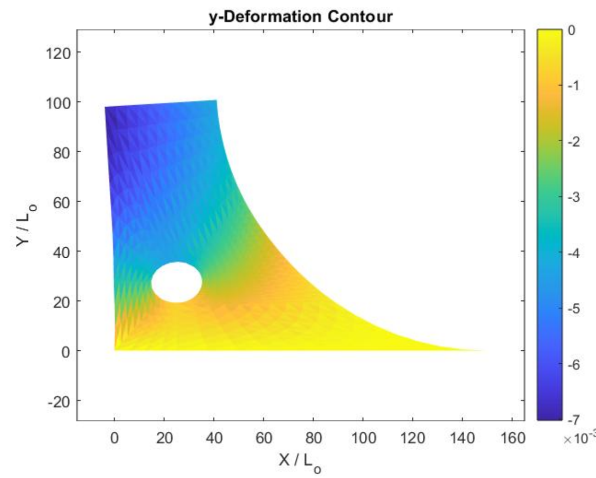
(a) (b) (c)

Figure 3.12: X-Deformation Contours for Dam

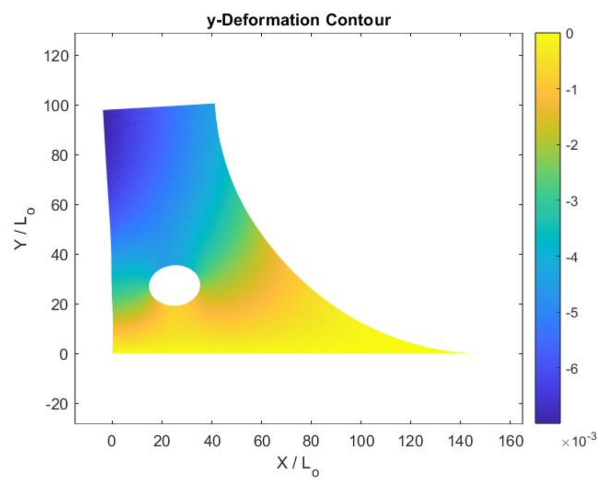
Looking at the above contours, it can be seen that the value of x displacement is maximum near the top edge of the dam. The peak value of displacement is found to be -3.8 mm. The deformed shape of the dam shows the top left zone getting displaced towards the negative x direction. This can be attributed to the fact that the weight of the dam induces compressive stresses along y direction which in turn results in a positive strain(due to non-zero poisson's ratio) along the negative x direction. We have zero displacements at the bottom as the bottom surface is fixed to the foundation.



(a)



(b)



(c)

Figure 3.13: Y-Deformation Contours for Dam

Observing the Y-deformation contours, we find that the magnitude of deformation is maximum at the top left corner of the dam and decreases to zero at the bottom edge. The peak value of deformation is found to be 7 mm.

We observe zero displacement value at the bottom surfaces in the figure. For the rest of the dam, since the dam is under the action of gravitational forces, we have all the y-displacements in negative direction.

3.2.2 Stress Contours

Stress contours are plotted for the normal stresses($\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$), shear stress($\sigma_{xy}$) and the equivalent stress(following Von-Mises criterion) for each of the cases.

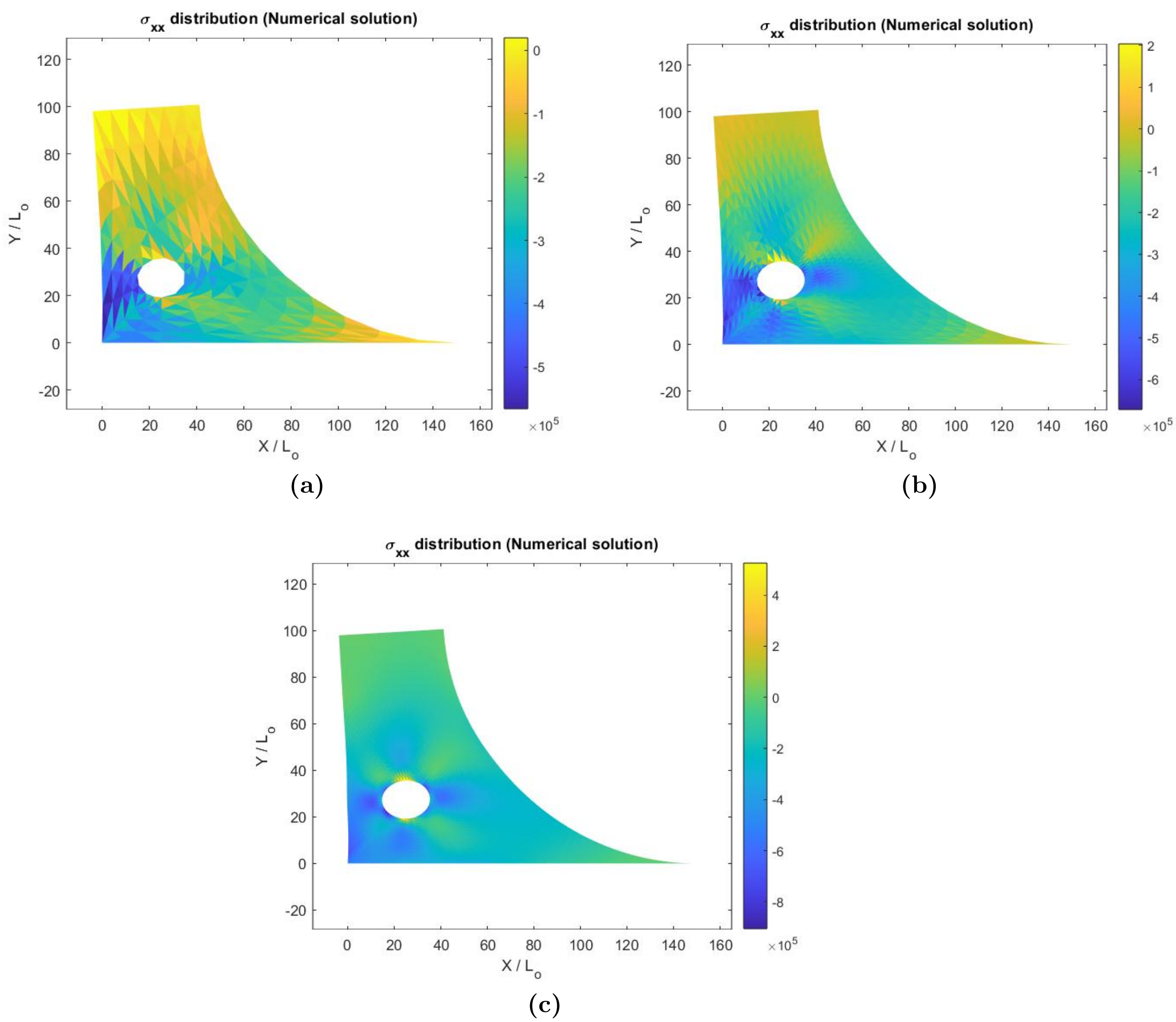


Figure 3.14: σ_{xx} Stress Contours(Dam) (a) M1 (b) M2 (c) M3

From the plots we observe that the value of the stress is minimum at the top surface(equal to zero) and is maximum near the bottom left zone of the dam.The stresses are compressive in nature(can be inferred from the negative values). The peak magnitude of the stress is found to be -0.771 MPa for σ_{xx}.

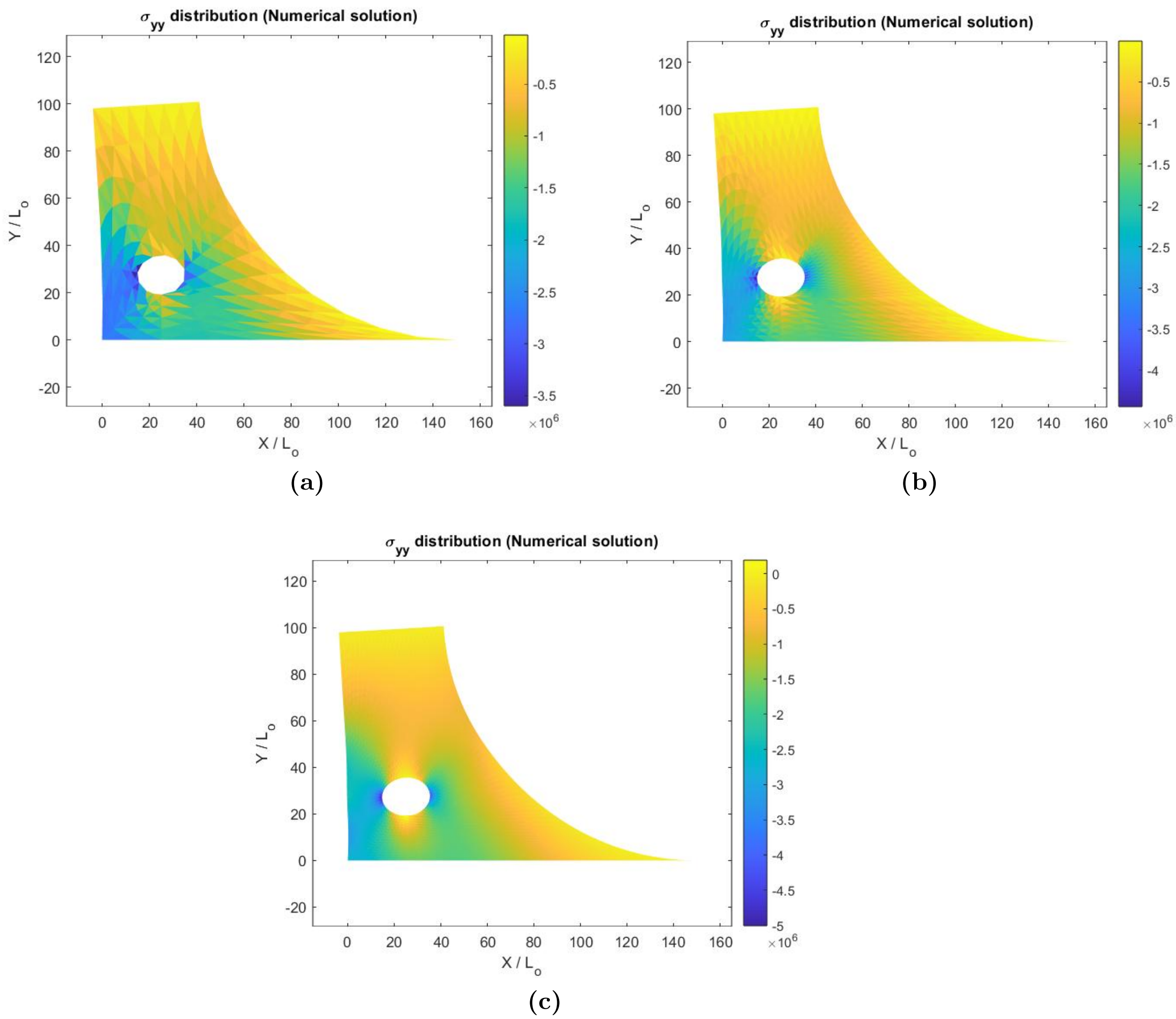
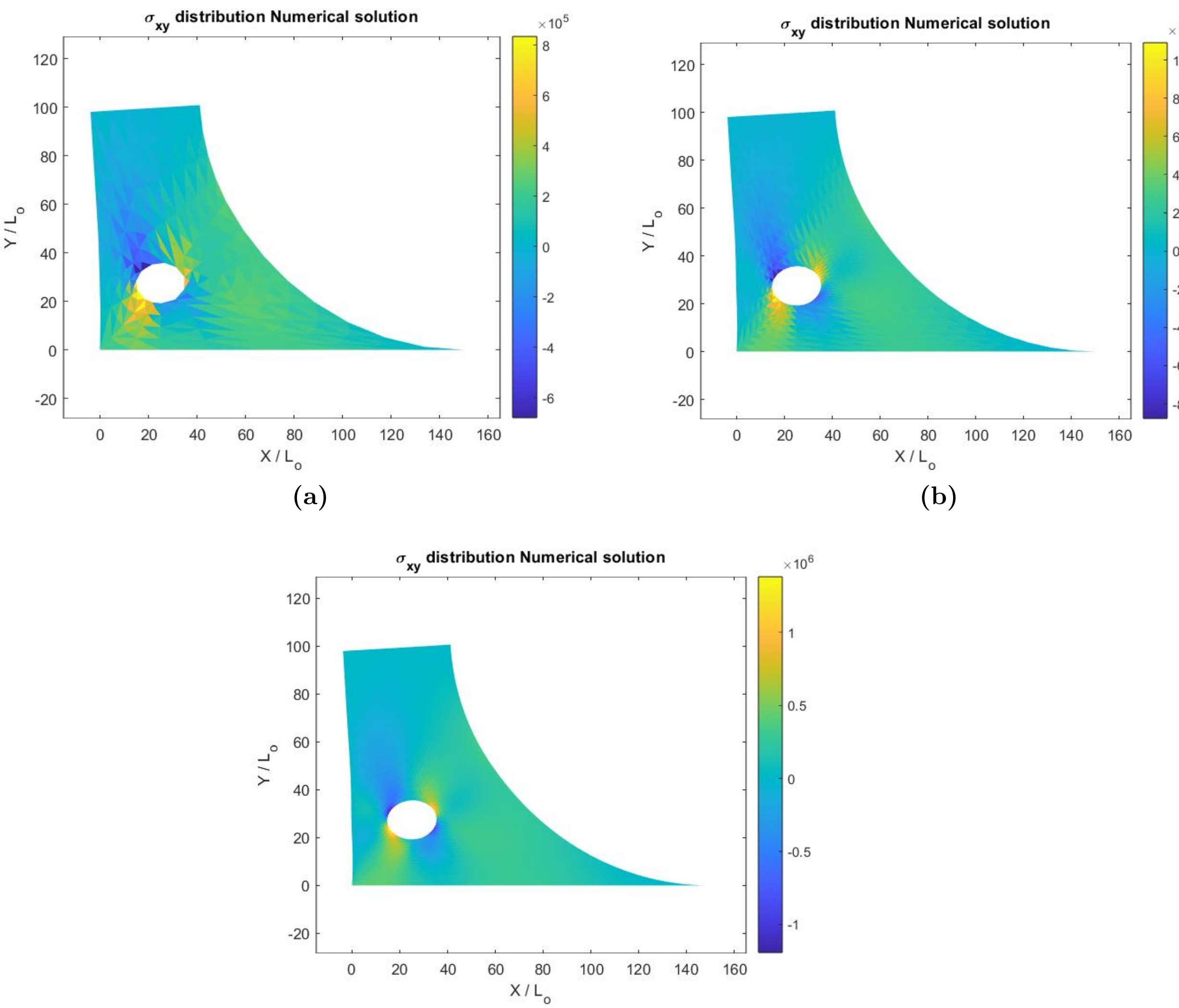


Figure 3.15: σ_{yy} Stress Contours(Dam) (a) M1 (b) M2 (c) M3

It is observed that the stresses are zero at the top surface as well as at the bottom right corner of the dam. The stresses produced in the rest of the dam are compressive in nature. The stress peaks around the edges of the tunnel inside the dam and is numerically equal to -4.44 MPa.

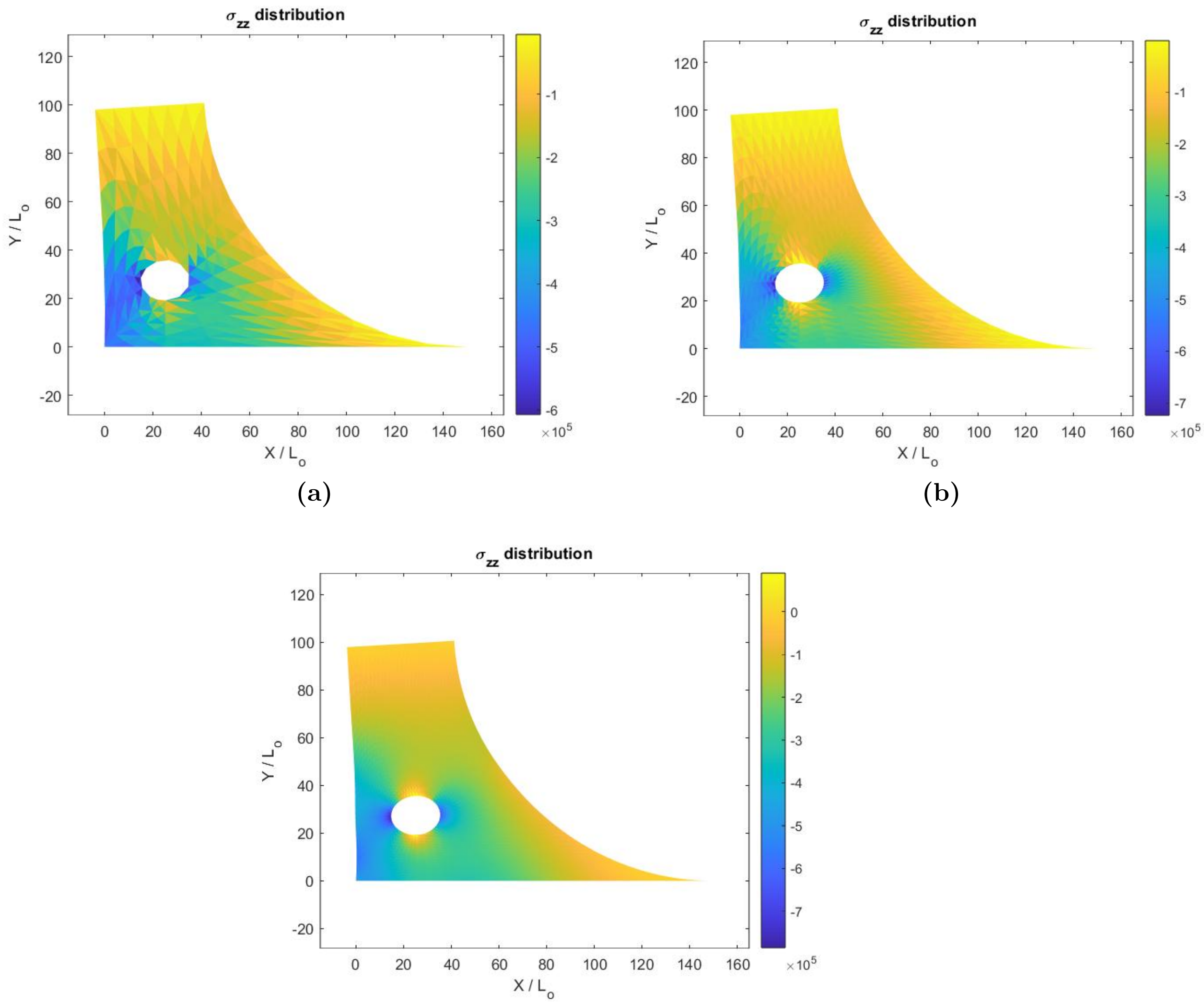


(a) **(b)** **(c)**

Figure 3.16: σ_{xy} Stress Contours(Dam) (a) M1 (b) M2 (c) M3

Shear stresses are zero at the top surface of the dam. The maximum stresses are found at the edge of the tunnel (blue shows clockwise and yellow anti-clockwise), the peak value being equal to 1.19 MPa. The reason for this can be attributed to stress concentration.

Since, we have assumed plane strain condition we have σ_{zz} present as the third component of normal stress . The plot for σ_{zz} for each of the three cases is shown below.

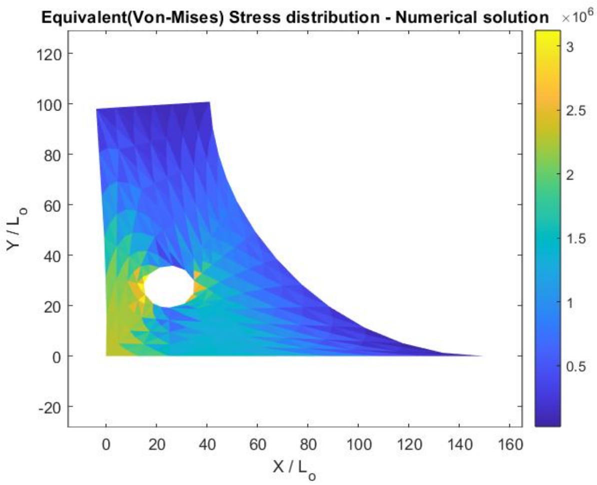


(a) (b) (c)

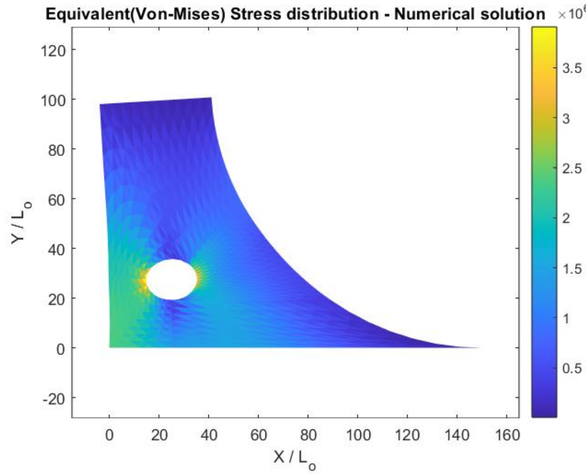
Figure 3.17: σ_{zz} Stress Contours (Dam) for (a) M1 (b) M2 (c) M3

From the plots, we note that the stress has a zero value at the top surface as well as at the btoom right corner.The stresses are compressive in nature over the rest of the dam. The peak value of σ_{zz} occurs at the edge of the tunnel and is equal to -0.723 MPa.

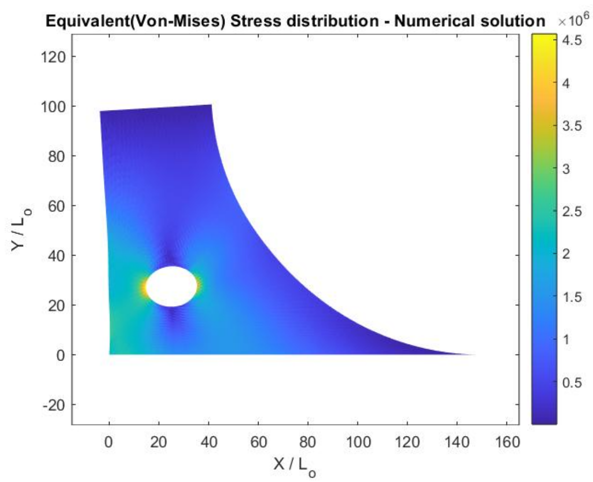
We have also plotted the contours for the Von-Mises stress for each of the three cases. The plots are shown below.



(a)



(b)



(c)

Figure 3.18: Von-Mises Stress Contours(Dam) for (a) M1 (b) M2 (c) M3

The above plot shows that the stress has zero value at four locations - at the top surface, at the bottom right corner, near the top and bottom spots at the edge of the tunnel. The peak value of the stress observed is equal to 3.909 MPa(near the left edge of the tunnel).

Chapter 4

Conclusion

The project demonstrates the application of 2D Finite Element Methods for obtaining the solutions of real life engineering problems. From the second problem included in this project, we realize that FEM serves as an excellent approach for obtaining solutions to cases where analytical solutions are very hard to obtain. From the first problem(plate with hole), we have also verified that by taking a sufficiently refined mesh, we can get numerical solutions that are closely in agreement with the analytical values.

The author is thankful to Dr. Sachin Singh Gautam for providing the basic MATLAB program which has assisted in developing a clear understanding of the implementation of various steps involved in FEM.