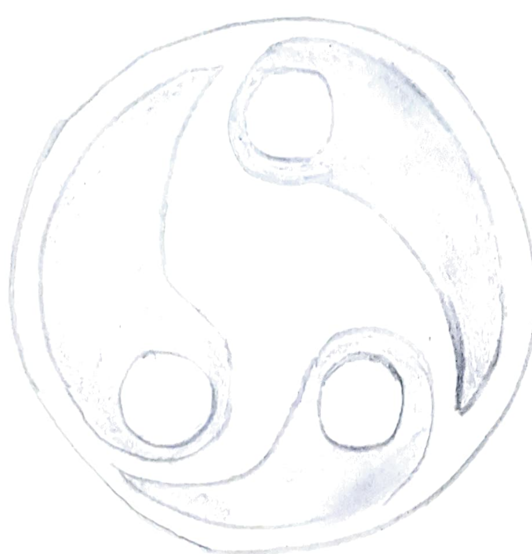
INDIAN INSTITUTE OF TECHNOLOGY, GUWAHATI

Department of Mechanical Engineering – ME 670

Advanced Computational Fluid Dynamics

Assignment – 2

(i. V-cycle program for 1-D model problem
ii. Full – Multigrid Subroutine)



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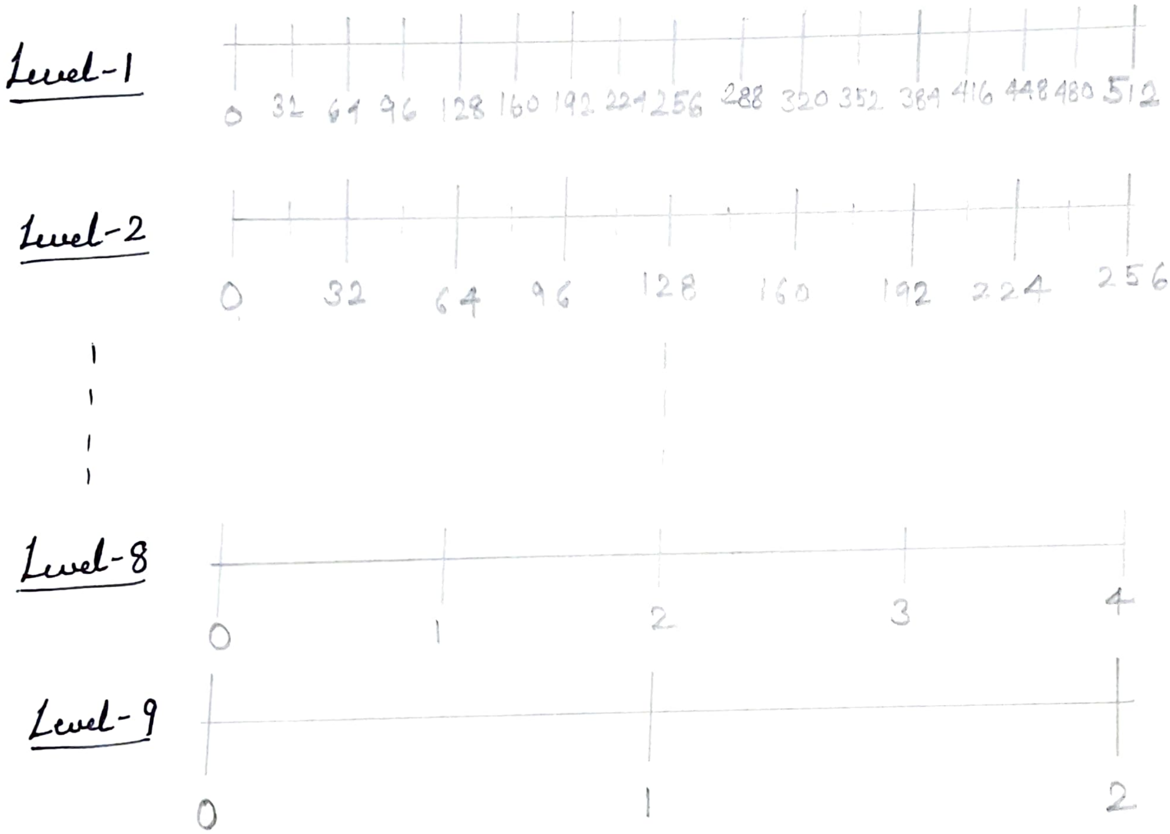
Grid Detail:

The given problem is a one-dimensional problem and is required to be solved using the Finite-difference Method.



In order to discretize a domain into grid size 'M' we need (M+1) nodes. These are labelled from 0 to M.

For level 1 $\rightarrow$ $M = 512$

level 2 $\rightarrow$ $M = 256$

⋮

level 9 $\rightarrow$ $M = 2$

Grid Size $(h) = \dfrac{1}{M}$

$x_i = ih \quad ; \; i = 0, 1, 2 \cdots M$

Discretized Equations Detail:

The model problem required to be solved is –

$$-u''(x) + \sigma u(x) = f(x) \quad ; \; 0 < x < 1$$

In order to obtain the discretized equation, we replace the derivative term in the above equation with its finite difference approximation –

$$u''(x) = \frac{d^2u}{dx^2} = \frac{U_{i+1} - 2U_i + U_{i-1}}{(\delta x)^2 \; [= h^2]} \quad \text{[Central Difference]}$$

$$(U(x_i) = U_i)$$

Substituting this approximation in the given equation we get,

$$-\left(\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}\right) + \sigma U_i = f_i \qquad (f(x_i) = f_i)$$

$$\Rightarrow \frac{1}{h^2}\left(-U_{i+1} - U_{i-1} + (2+\sigma h^2)U_i\right) = f_i \quad ; \; 0 < i < M$$

In the given problem, $\sigma = 1$, therefore we can write

$$\frac{1}{h^2}\left(-U_{i-1} + (2+h^2)U_i - U_{i+1}\right) = f_i = C\sin(k\pi x_i) = C\sin(k\pi h i)$$

$$; \text{where } C = \pi^2k^2 + \sigma$$

This can be rewritten as:

$$[A]\{U\} = \{f\}$$

where $[A] = \frac{1}{h^2}\begin{bmatrix} (2+h^2) & -1 & 0 & \cdots & 0 \\ -1 & (2+h^2) & -1 & 0 \cdots & 0 \\ 0 & -1 & (2+h^2) & -1 \cdots & 0 \\ & & \vdots & & \\ 0 & \cdots & 0 & -1 \; (2+h^2) & -1 \\ 0 & \cdots & 0 & 0 \; -1 & (2+h^2) \end{bmatrix}_{(M-1)\times(M-1)}$

$$\{v\} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{M-1} \end{bmatrix} \qquad \{f\} = C\begin{bmatrix} \sin(k\pi h) \\ \sin(k\pi 2h) \\ \vdots \\ \sin(k\pi(M-1)h) \end{bmatrix}$$

For solving the above system of equations we can use both Gauss-Seidel or Weighted Jacobi for relaxation. The discretized equations for each case is given below.

Gauss Seidel :

By rearranging the terms in the previous discretized equation (shifting v_i on the LHS & remaining terms on RHS) we can write -

$$v_i^{k+1} = \left[Ch^2 \sin(k\pi h i) + v_{i-1}^{k+1} + v_{i+1}^{k}\right] / (2+h^2)$$

Jacobi Method -

$$v_i^{k+1} = \left(Ch^2 \sin(k\pi hi) + v_{i-1}^k + v_{i+1}^k\right) / (2 + h^2)$$

Weighted jacobi method -

The general expression for weighted jacobi method is given as -

$$v_i^{k+1} = (1-\omega)\, v_i^k + \omega\, v_i^*$$

where v_i^* is v_i approximated using jacobi method.

So, we can now write -

$$v_i^{k+1} = (1-\omega)\, v_i^k + \frac{\omega}{2+h^2}\left[Ch^2 \sin(k\pi hi) + v_{i-1}^k + v_{i+1}^k\right]$$

For higher levels ($l > 1$) / coarser grids, we make a small modification to our equation as follows -

Considering initial guess (v_0) $\longrightarrow$ $A v_0 = f$

residue $\longrightarrow$ $r = f - A v_0$

Now, initial error $e_0 = U - V_0$

Solving the equation $A e_0 = r$ for initial guess zero, is same as writing : $r = 0$ or $f = A v_0$.

Therefore, at the coarser grids, instead of solving

$Av = f$ we go for solving $Ae = r$

where $r = f - Av$.

In discretized form,

$$r_i = C\sin(k\pi h i) + \left[v_{i+1} + v_{i-1} - 2h^2 v_i\right]/h^2$$

Boundary Condition Implementation Detail

In the model problem presented, it is given that scenario involves Homogeneous Boundary Conditions.
Now, since the domain of x is given as $0 \leq x \leq 1$
we can write

$$u\Big|_{x=0} = 0 \quad \& \quad u\Big|_{x=1} = 0$$

Further, since we are using the approximate solution 'v' as a 1D array, we can write

$$v[0] = 0 \quad \& \quad v[M] = 0$$; where M is size of grid.

Also, as we ~~av~~ have f in the form resembling Fourier modes $[f = C \sin(k\pi x)]$, it is quite evident that $f[0] = f[M] = 0$.

Additionally, when we proceed to a coarse grid, we calculate the residue as : $r = \{f\} - [A][v]$

At the boundaries,

$$r[0] = f[0] - A\,v[0] = 0$$
$$r[M] = f[M] - A\,v[M] = 0$$

Now, the error equation is $[A]\{E\} = \{r\}$

subject to $r[0] = 0$ & $r[M] = 0$