Indian Institute of Technology, Guwahati.

ME 670 Advanced Computational Fluid Dynamics

(Dept. of Mechanical Engg.)

Assignment – 1

(Solve Lid-driven Cavity problem using SIMPLE Algorithm)

Submitted by:

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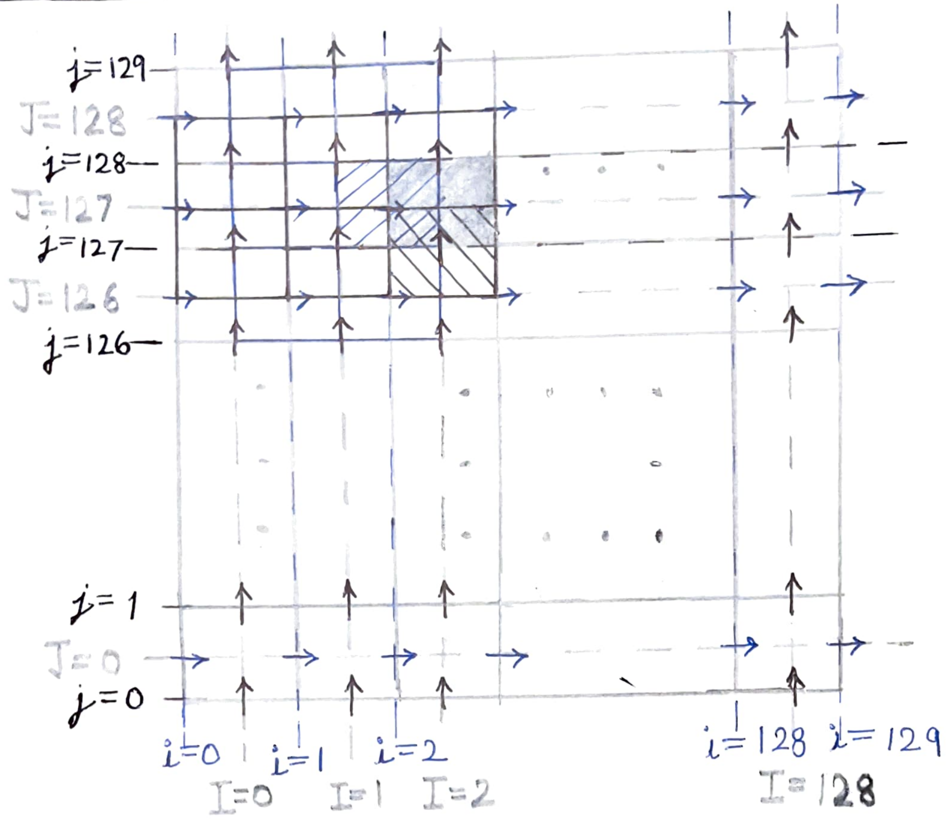
Grid Detail :



$0 \leq I, J \leq 128$ (Total 129 grid points) – for p-control volume

$0 \leq i \leq 129$ (Total 130 grid points) – for u-control volume

$0 \leq j \leq 129$ (Total 130 grid points) – for v-control volume.

$P_{I,J}$ – denotes centre of p-control volume

$P_{i,J}$ – denotes centre of u-control volume

$P_{I,j}$ – denotes centre of v-control volume

Discretized Equation detail:

The given problem requires us to solve two dimensional, incompressible, steady and isothermal flow inside a lid-driven cavity.

We start by writing the non-dimensional steady Navier stokes equations:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} = 0 \quad (\underline{\text{continuity equation}})$$

$$\Rightarrow \underline{\nabla}^* \cdot \underline{u}^* = 0 \quad (\text{compact form})$$

; where $\underline{\nabla}^* = L\underline{\nabla}$

$$\underline{u}^* = \frac{\underline{u}}{U}$$

$$(\underline{u} = u\underline{i} + v\underline{j} + w\underline{k})$$

$$(\underline{u}^* \cdot \underline{\nabla}^*)\underline{u}^* = -\underline{\nabla}^* p^* + \frac{1}{Re}\underline{\nabla}^{*2}\underline{u}^* \quad (\text{Momentum equation})$$

The term on LHS can be rewritten as:

$$(\underline{u}^* \cdot \underline{\nabla}^*)\underline{u}^* = \underline{\nabla}^* \cdot (\underline{u}^*\underline{u}^*)$$

$$\because \left[\underline{\nabla}^* \cdot (\underline{u}^*\underline{u}^*) = u^*(\underbrace{\cancel{\underline{\nabla}^* \cdot \underline{u}^*}}_{0 \text{ (continuity eq}^n)}) + (\underline{u}^* \cdot \underline{\nabla}^*)\underline{u}^* \right]$$

Therefore, we can write: $\underline{\nabla}^* \cdot (\underline{u}^*\underline{u}^*) = -\underline{\nabla}^* p^* + \frac{1}{Re}\underline{\nabla}^* \cdot (\underline{\nabla}^*\underline{u}^*)$

In order to discretize this equation, we will use the finite volume method. The first step is to divide the flow domain into a finite no. of control volumes.

The Navier Stokes equations are then integrated over the control volume as follows -

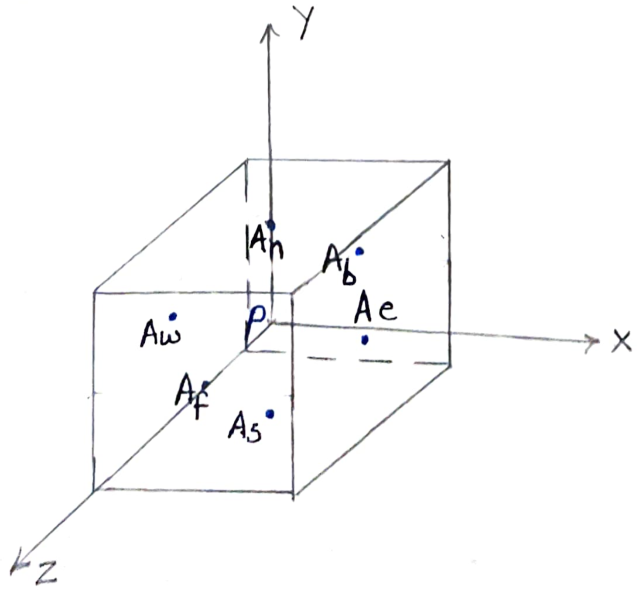
$$\int_{CV} \underline{\nabla}^* \cdot (\underline{u}^* \underline{u}^*) \, dV = \int_{CV} -\underline{\nabla}^* p^* \, dV + \int_{CV} \frac{1}{Re} \underline{\nabla}^* \cdot (\underline{\nabla}^* \underline{u}^*) \, dV$$

Using the Gauss divergence theorem, $\int_{CV} \nabla \cdot F \, dV = \oint_{CS} F \cdot \hat{n} \, ds$

we can rewrite the 1st and 3rd term in the above equation as follows:

$$\oint_{CS} (\underline{u}^* \underline{u}^*) \cdot \underline{n} \, ds = -\underline{\nabla}^* p^* \Delta V + \frac{1}{Re} \oint_{CS} (\underline{\nabla}^* \underline{u}^*) \cdot \underline{n} \, ds$$

Now, in order to proceed, we must split the integral into the constituent control surfaces which enclose the control volume and integrate over each of these surfaces individually. Consider the control volume shown below -



The integral containing the convective term on the LHS can be split as:

$$(\underline{u}^*\underline{u}^*)\cdot\underline{i}\,A_e + (\underline{u}^*\underline{u}^*)\cdot(\underline{-i})A_w + (\underline{u}^*\underline{u}^*)\cdot\underline{j}\,A_n$$
$$+ (\underline{u}^*\underline{u}^*)\cdot(\underline{-j})A_s + (\underline{u}^*\underline{u}^*)\cdot\underline{k}\,A_f + (\underline{u}^*\underline{u}^*)\cdot(\underline{-k})A_b$$

Here, we introduce a variable F to represent the convective mass flux per unit area:

$$\underline{F} = \underline{u}^*$$

$\not\Rightarrow$ Also, since we have a uniform grid,

$$A_e = A_w = A_n = A_s = A_f = A_b = A$$

Now, $\underline{u}^* = u^*\underline{i} + v^*\underline{j} + w^*\underline{k}$

$$\underline{u}^*\cdot\underline{i} = u^*$$
$$\underline{u}^*\cdot\underline{j} = v^*$$
$$\underline{u}^*\cdot\underline{k} = w^*$$

$F_e = u^*_e$
$F_w = u^*_w$
$F_n = v^*_n$
$F_s = v^*_s$
$F_f = w^*_f$
$F_b = w^*_b$

Applying these substitutions, we get

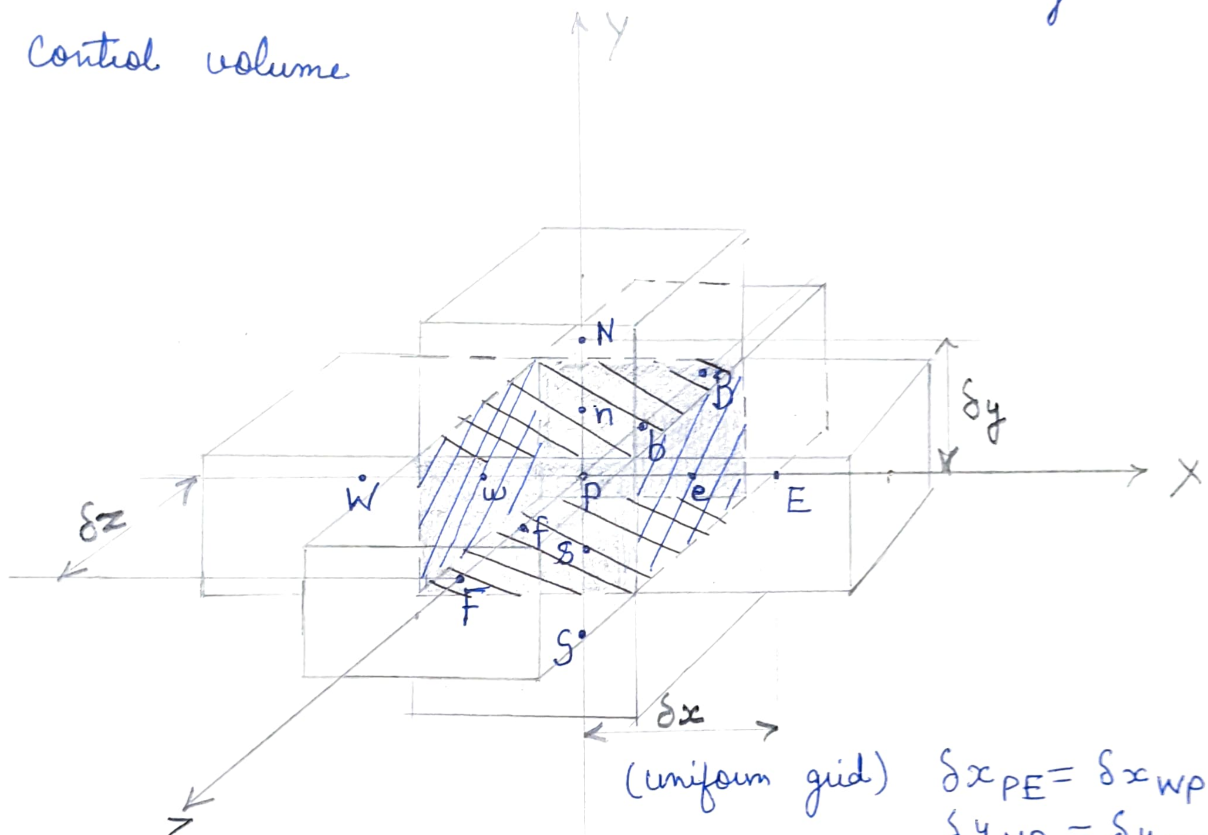
$$(F\underline{u}^*)_e A - (F\underline{u}^*)_w A + (F\underline{u}^*)_n A - (F\underline{u}^*)_s A$$
$$+ (F\underline{u}^*)_f A - (F\underline{u}^*)_b A$$

The integral containing the diffusion term on the right hand side of the eq^n can be split as:

$$\frac{1}{Re}\left(\frac{\partial u^*}{\partial x}\Big|_e A_e - \frac{\partial u^*}{\partial x}\Big|_w A_w + \frac{\partial u^*}{\partial y}\Big|_n A_n - \frac{\partial u^*}{\partial y}\Big|_s A_s \right.$$

$$\left. + \frac{\partial u^*}{\partial z}\Big|_f A_t - \frac{\partial u^*}{\partial z}\Big|_b A_b \right)$$

Here, we assume that u^* varies linearly over the control volume



(uniform grid) $\delta x_{PE} = \delta x_{WP} = \delta x$
$\delta y_{NP} = \delta y_{PS} = \delta y$

Note that the velocity is difined only at the cell centres P, E, N, S, etc. but not at the face centres e, w, n.

Assuming linear interpolation, we can now write -

$$\frac{\partial u^*}{\partial x}\Big|_e = \frac{u^*_E - u^*_P}{\delta x}, \quad \frac{\partial u^*}{\partial x}\Big|_w = \frac{u^*_P - u^*_W}{\delta x}$$

$$\frac{\partial u^*}{\partial y}\Big|_n = \frac{u^*_N - u^*_P}{\delta y}, \quad \frac{\partial u^*}{\partial y}\Big|_s = \frac{u^*_P - u^*_S}{\delta y}$$

$$\left.\frac{\partial u^*}{\partial z}\right|_f = \frac{U_F^* - U^*_P}{\delta z} \quad , \quad \left.\frac{\partial u^*}{\partial z}\right|_b = \frac{U^*_P - U^*_B}{\delta z}$$

Now, we introduce another variable D, diffusion conductance given by →

$$D_e = \frac{1}{Re\,\delta x} = D_w$$

$$D_n = \frac{1}{Re\,\delta y} = D_s$$

$$D_f = \frac{1}{Re\,\delta z} = D_b$$

Pressure term: (using Backward diff^n)

$$-\nabla^* p^* \Delta V = -\underline{i}\frac{\partial p^*}{\partial x} - \underline{j}\frac{\partial p^*}{\partial y} - \underline{k}\frac{\partial p^*}{\partial z}$$

$$= -\left[\frac{\underline{i}(P_P^* - P_W^*)\Delta V}{\delta x} + \frac{\underline{j}(P_P^* - P_S^*)\Delta V}{\delta y} + \frac{\underline{k}(P_P^* - P_B^*)\Delta V}{\delta z}\right]$$

$$= -\underline{\Delta p^*} A \quad ; \text{ where } \underline{\Delta p^*} = \begin{bmatrix} P_P^* - P_W^* \\ P_P^* - P_S^* \\ P_P^* - P_B^* \end{bmatrix}$$

Using these substitutions the RHS of the equation, can be rewritten as:

$$(\underline{U_E}^* - \underline{U_P}^*) D_e A_e - (\underline{U_P}^* - \underline{U_W}^*) D_w A_w + (\underline{U_N}^* - \underline{U_P}^*) D_n A_n - (\underline{U_P}^* - \underline{U_S}^*) D_s A_s + (\underline{U_F}^* - \underline{U_P}^*) D_f A_f - (\underline{U_P}^* - \underline{U_B}^*) D_b A_b - \underline{\Delta p}^* A$$

The momentum equation can now be rewritten as:

$$(F\underline{u}^*)_e A - (F\underline{u}^*)_w A + (F\underline{u}^*)_n A - (F\underline{u}^*)_s A + (F\underline{u}^*)_f A - (F\underline{u}^*)_b A$$

$$= (\underline{U_E}^* - \underline{U_P}^*) D_e A - (\underline{U_P}^* - \underline{U_W}^*) D_w A + (\underline{U_N}^* - \underline{U_P}^*) D_n A - (\underline{U_P}^* - \underline{U_S}^*) D_s A + (\underline{U_F}^* - \underline{U_P}^*) D_f A - (\underline{U_P}^* - \underline{U_B}^*) D_b A - \underline{\Delta p}^* A$$

[Since $A_e = A_w = A_n = A_s = A_f = A_b = A$ for uniform grid]

Dividing throughout by A,

$$\Rightarrow (F\underline{u}^*)_e - (F\underline{u}^*)_w + (F\underline{u}^*)_n - (F\underline{u}^*)_s + (F\underline{u}^*)_f - (F\underline{u}^*)_b =$$

$$(\underline{U_E}^* - \underline{U_P}^*) D_e - (\underline{U_P}^* - \underline{U_W}^*) D_w + (\underline{U_N}^* - \underline{U_P}^*) D_n - (\underline{U_P}^* - \underline{U_S}^*) D_s + (\underline{U_F}^* - \underline{U_P}^*) D_f - (\underline{U_P}^* - \underline{U_B}^*) D_b - \underline{\Delta p}^*$$

This is a vector equation in U^* and P^*. We can split this equation into its component form as follows –

X – momentum equation : ($U^* = u^*$ & $\Delta P^* = P_P^* - P_W^*$)

$$(Fu^*)_e - (Fu^*)_w + (Fu^*)_n - (Fu^*)_s + (Fu^*)_f - (Fu^*)_b =$$

$$(u_E^* - u_P^*)De + (u_P^* - u_W^*)Dw + (u_N^* - u_P^*)Dn - (u_P^* - u_S^*)Ds$$

$$+ (u_F^* - u_P^*)Df - (u_P^* - u_B^*)Db - (P_P^* - P_W^*)$$

Now, u^* is defined only at cell centres. So, we need to write U_e^*, U_w^*, etc. in terms of velocities at cell centres. For this we employ the Hybrid differencing scheme.

Consider the east face of the control volume,

We first evaluate the Peclet no. $Pe = \frac{Fe}{De}$

$$Fe\, u_e^* = Fe\, u_E^* + 0 \cdot u_P^* \quad ; \quad Pe \geq 2$$

$$Fe\, U_e^* = Fe\, u_P^* + 0 \cdot u_E^* \quad ; \quad Pe \leq -2$$

$$Fe\, U_e^* = \left(\frac{Fe}{2} + De\right) U_P^* + \left(\frac{Fe}{2} - De\right) U_E^* \quad ; \quad -2 < Pe < 2$$

The above relations can be written in a compact form as:

$$Fe\, U_e^* = -\max\left[-Fe, 0, \left(-\frac{Fe}{2} + De\right)\right] U_E^* + \max\left[Fe, 0, \left(De + \frac{Fe}{2}\right)\right] U_P^*$$

$$= -a_E U_E^* + \left(\max\left[0, -Fe, \left(De - \frac{Fe}{2}\right)\right] + Fe\right) U_P^*$$

$$= -a_E u^*_E + (a_E + F_e)\, u^*_P$$

Similarly, $(Fu^*)_w = \max\left[0, F_w, \left(\frac{F_w}{2} + D_w\right)\right] u^*_W + \max\left[0, F_w, \left(\frac{F_w}{2} - D_w\right)\right] u^*_P$

$$= a_W u^*_W - \left(\max\left[F_w, 0, \left(D_w + \frac{F_w}{2}\right)\right] - F_w\right) u^*_P$$

$$= a_W u^*_W - (a_W - F_w)\, u^*_P$$

Likewise, $(Fu^*)_n = -a_N u^*_N + (a_N + F_n)\, u^*_P$

$$(Fu^*)_s = a_S u^*_S - (a_S - F_s)\, u^*_P$$

$$(Fu^*)_f = -a_F u^*_F + (a_F + F_f)\, u^*_P$$

$$(Fu^*)_b = a_B u^*_B - (a_B - F_b)\, u^*_P$$

Substituting these terms in the equation and bringing u^*_P terms on one side we get, (setting diffusion terms equal to zero)

$$u^*_P\left[a_E + F_e + a_W - F_w + a_N + F_n + a_S - F_s + a_F + F_f + a_B - F_b\right] =$$
$$u^*_E a_E + a_W u^*_W + a_N u^*_N + a_S u^*_S + a_F u^*_F + a_B u^*_B + P^*_W - P^*_P$$

$$\Rightarrow \quad a_P u^*_P = a_E u^*_E + a_W u^*_W + a_N u^*_N + a_S u^*_S + a_F u^*_F + a_B u^*_B + P^*_W - P^*_P$$

$$;\quad a_P = a_E + a_W + a_N + a_S + a_F + a_B + F_e - F_w + F_n - F_s + F_f - F_b$$

So, our discretized x-momentum eq^n is:

$$a_P u^*_P = \Sigma a_{nb} u^*_{nb} + P^*_W - P^*_P \quad ; \quad a_P = \Sigma a_{nb} + F_e - F_w + F_n - F_s + F_f - F_b$$

Y- momentum equation: ($\underline{U}^* = v^*$ & $\underline{\Delta P}^* = P_P^* - P_S^*$)

$$(Fv^*)_e - (Fv^*)_w + (Fv^*)_n - (Fv^*)_s + (Fv^*)_f - (Fv^*)_b =$$

$$(v_E^* - v_P^*)D_e + (v_P^* - v_W^*)D_w + (v_N^* - v_P^*)D_n - (v_P^* - v_S^*)D_s$$

$$+ (v_F^* - v_P^*)D_f - (v_P^* - v_B^*)D_b - (P_P^* - P_S^*)$$

Again to compute v^*_e, v^*_w, v^*_n, etc. we use the hybrid differencing scheme, just like the case for x- momentum equation.

Substituting the hybrid differencing convective terms in the equation and rearranging to get v^*_P on LHS (setting diffusion terms equal to zero):

$$v_P^*\left[a_E + a_W + a_N + a_S + a_F + a_B + F_e - F_w + F_n - F_s + F_f - F_b\right]$$

$$= a_E v_E^* + a_W v_W^* + a_N v_N^* + a_S v_S^* + a_F v_F^* + a_B v_B^* + P_S^* - P_P^*$$

The discretized y- momentum equation is:

$$a_P v_P^* = \Sigma a_{nb} v_{nb}^* + P_S^* - P_P^* \quad ; a_P = \Sigma a_{nb} + F_e - F_w + F_n - F_s + F_f - F_b$$

Z- momentum equation: ($\underline{U}^* = w^*$ & $\underline{\Delta P}^* = P_P^* - P_B^*$)

$$(Fw^*)_e - (Fw^*)_w + (Fw^*)_n - (Fw^*)_s + (Fw^*)_f - (Fw^*)_b =$$

$$(w_E^* - w_P^*)D_e + (w_P^* - w_W^*)D_w + (w_N^* - w_P^*)D_n - (w_P^* - w_S^*)D_s$$

$$+ (w_F^* - w_P^*)D_f - (w_P^* - w_B^*)D_b - (P_P^* - P_B^*)$$

Similar to the previous cases we approximate the velocities at face centres using the hybrid differencing scheme.

Substituting the hybrid difference convective terms in the equation and rearranging to get w_P^* on LHS (setting diffusion terms equal to zero) we get,

$$w_P^* \left[a_E + a_W + a_N + a_S + a_F + a_B + F_e - F_w + F_n - F_s + F_f - F_b\right]$$

$$= a_E w_E^* + a_W w_W^* + a_N w_N^* + a_S w_S^* + a_F w_F^* + a_B w_B^* + P_B^* - P_P^*$$

The discretized z-momentum equation is :

$$a_P w_P^* = \Sigma a_{nb} w_{nb}^* + P_B^* - P_P^* \quad ; \quad a_P = \Sigma a_{nb} + F_e - F_w + F_n - F_s + F_f - F_b$$

Next, we need to find the discretized continuity eq^n. For this, we again use the finite volume method and integrate the continuity equation over the control volume as follows -

$$\int_{CV} \underline{\nabla}^* \cdot \underline{u}^* = 0 = \oint_{CS} \underline{u}^* \cdot \underline{n}\, ds$$

[Using Gauss divergence theorem]

To proceed, we split the integral into the constituent control surfaces which enclose the control volume just like done before and integrate over each of these surfaces individually

$$\underline{u}^* \cdot \underline{i}\, A_e + \underline{u}^* \cdot (\underline{-i})\, A_w + \underline{u}^* \cdot \underline{j}\, A_n + \underline{u}^* \cdot (\underline{-j})\, A_s$$

$$+ \underline{u}^* \cdot \underline{k}\, A_f + \underline{u}^* \cdot (\underline{-k})\, A_b = 0$$

$$u^*(A_e - A_w) + v^*(A_n - A_s) + w^*(A_f - A_b) = 0$$

Assuming $F_e = u^*_e$, $F_w = u^*_w$, $F_n = v^*_n$

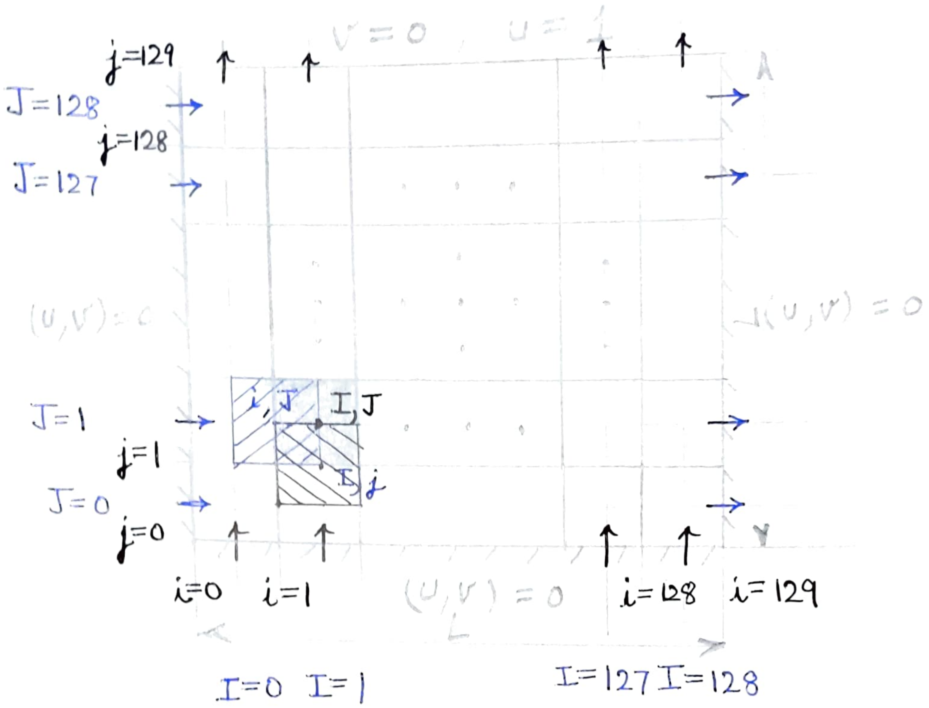
$F_s = v^*_s$, $F_f = w^*_f$, $F_b = w^*_b$

Also, $A_e = A_w = A_n = A_s = A_f = A_b = A$ (for uniform grid)

$$\Rightarrow (F_e - F_w + F_n - F_s + F_f - F_b)A = 0$$

$$\therefore \quad F_e - F_w + F_n - F_s + F_f - F_b = 0$$

Boundary Condition Implementation Detail :



For the given problem, we have discretized the control volume into a grid size of 129 X 129 such that the edges of the pressure control volume coincide with the walls of the given problem.

The boundary conditions at the four walls are :

Left Wall : Given $u = 0$, $v = 0$

since u is defined explicitly at the ~~bottom~~ Left wall, we can write

$$u_{0,J} = 0$$

Now, v is not defined explicitly at left wall, v-control volume has its edge coinciding with the wall, it experiences a shear force from the wall which can be modelled as:

$$F_L = -\tau_L A$$

$$= -\frac{1}{Re}\frac{(v_{0,j}-0)}{\frac{\delta x}{2}} A = -\frac{2}{Re\,\delta x} v_{0,j} A$$ [Here $v_p = v_{0,j}$]

$\therefore S_p = -\frac{2}{Re\,\delta x}$ [Since, we divided discretized v-momentum eq^n throughout by A]

Also, $aW = 0$; [Since v at $I = -1$ is not defined]

$$\therefore aP = aE + aN + aS + Fe + F_n - F_w - F_s - Sp$$

Right Wall: Given $u=0$, $v=0$

Since u is defined explicitly at the right wall, we can write

$$u_{129,J} = 0$$

Now, v is not defined explicitly at right wall but, the v-control volume has its right edge coinciding with the wall and experiences a shear force which can be modelled as:

$$F_R = -\tau_R A$$

$$= -\frac{1}{Re}\frac{(v_{128,j}-0)}{\frac{\delta x}{2}} A = -\frac{2}{Re\,\delta x} v_{128,j} A$$

[Here $v_p = v_{128,j}$]

$\therefore S_p = \frac{-2}{Re\,\delta x}$ [Since, we divided discretized eq^n throughout by A]

Also, $aE = 0$ [Since v at $I = 129$ is not defined
defined for $[0, 128] \rightarrow$ total 129 points]

$\therefore aP = aW + aN + aS + F_e + F_n - F_w - F_s - S_p$

Bottom Wall : given $u = 0$, $v = 0$

since, v is defined explicitly at the bottom wall we can write,

$$v_{I,0} = 0$$

Now, u is not defined explicitly at the bottom wall, the u-control volume has its bottom edge coincident with the wall, it experiences a shear force which can be modelled as -

$$F_B = -\tau_B A$$

$$= -\frac{1}{Re}\frac{(u_{i,0} - 0)}{\frac{\delta y}{2}} A = -\frac{2}{Re\,\delta y} u_{i,0} A$$

[Here $u_{i,0} = u_P$]

$\therefore S_p = -\frac{2}{Re\,\delta y}$ [since, we divided the discretized u-momentum eq^n throughout by A]

Also, $aS = 0$; [since u at $J = -1$ is not defined]

$\therefore aP = aW + aE + aN + F_e + F_n - F_w - F_s - S_p$

Top Wall : Given $u = 1$, $v = 0$

since v is defined explicitly at the top wall, we can write

$$v_{I,129} = 0$$

Now, u is not defined explicitly at the top wall, the u-control volume has its top-edge coinciding with the wall and experiences a shear force which can be modelled as:

$$F_T = -\tau_T A$$

$$= -\frac{1}{Re} \frac{(u_{i,128} - 1)}{\frac{\delta y}{2}} A$$

$$= \left(-\frac{2}{Re}\frac{u_P}{\delta y} + \frac{2}{Re\,\delta y}\right) A \qquad [\text{Here } u_P = u_{i,128}]$$

$$\therefore S_P = \frac{-2}{Re\,\delta y} \quad \& \quad S_u = \frac{2}{Re\,\delta y}$$

[Since, we divided the discretized u-momentum equation throughout by A]

Also, $a_N = 0$ [Since u at $J = 129$ is not defined
u defined for $[0, 128] \rightarrow$ total 129 points]

$$\therefore a_P = a_W + a_E + a_S + F_e + F_n - F_w - F_s - S_P$$

& Finally,
$$u_P^* = \frac{(a_W u_W^* + a_E u_E^* + a_S u_S^* + p_W^* - p_P^* + S_u)}{a_P}$$