CSC 423 - A1

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1. **Exercise 1.2**
We roll two standard six-sided dice. Find the probability of the following events, assuming that the outcomes of the rolls are independent.

(a) The two dice show the same number.

Exercise 1.2:-

a) (1, 1) or (2,2) or (3,3) or (4,4) or (5,5) or (6,6)

$\Rightarrow \left(\frac{1}{6} \times \frac{1}{6}\right) + \left(\frac{1}{6} \times \frac{1}{6}\right) + \ldots\ldots + \left(\frac{1}{6} \times \frac{1}{6}\right)$

$\Rightarrow 6\left(\frac{1}{6}\right)^2$

$\Rightarrow 6 \times \frac{1}{36}$

$\therefore$ Ans $\Rightarrow$ $1/6$

(b) The number that appears on the first die is larger than the number on the second.

b) (2,1) or (3,2) or (3,1) or (4,3) or (4,2) or (4,1) or (5,4) or
(5,3) or (5,2) or (5,1) or (6,5) or (6,4) or (6,3) or (6,2) or (6,1)

$\Rightarrow 15\left(1/6\right)^2$

$\Rightarrow 15 \times \frac{1}{36}$

$\therefore$ Ans $\Rightarrow$ $5/12$

(c) The sum of the dice is even.

c) (1,3), (1,5), (2,2), (2,4), (2,6), (3,1), (3,3) (3,5), (4,2), (4,6)
(5,1), (5,3), (5,5), (1,1), (6,2), (6,4), (6,6), (4,4)

$\Rightarrow 18\left(1/6\right)^2$

$\Rightarrow 18\left(1/36\right)$

$\therefore$ Ans $\Rightarrow$ $1/2$

(d) The product of the dice is a perfect square.

d) $1 \to (1,1),(1,1) \longrightarrow 2(1/6)^2$
$4 \to (1,4),(4,1),(2,2),(2,2) \longrightarrow +4(1/6)^2$
$9 \to (3,3),(3,3) \longrightarrow +2(1/6)^2$
$16 \to (4,4),(4,4) \longrightarrow +2(1/6)^2$
$25 \to (5,5),(5,5) \longrightarrow +2(1/6)^2$
$36 \to (6,6),(6,6) \longrightarrow +2(1/6)^2$

$14(1/6)^2$

$\Rightarrow \cancel{14}^{7}(1/\cancel{36}_{18})$

$\therefore$ Ans $\Rightarrow$ $7/18$

2. **Exercise 1.8**

I choose a number uniformly at random from the range [1, 1,000,000]. Using the inclusion–exclusion principle, determine the probability that the number chosen is divisible by one or more of 4, 6, and 9.

Exercise 1.8:-

Let x be a uniform number in the range $[1, 1000000]$
Let E_i be the event where x is divisible by integer i.
Set of divisors Y:-

$$Pr[\cap_{i \in Y} E_i] = \frac{\lfloor 1000000/LCM(Y) \rfloor}{1000000}$$ (LCM = Lowest common multiple)

$$Pr(E_4 \cup E_6 \cup E_9) = \{Pr(E_4) + Pr(E_6) + Pr(E_9)\} - \{Pr(E_4 \cap E_6) + Pr(E_4 \cap E_9) + Pr(E_6 \cap E_9)\} + Pr(E_4 \cap E_6 \cap E_9)$$

$$= \frac{(250000 + 166666 + 111111) - (83333 + 27777 + 55555)}{1000000} + \frac{27777}{1000000}$$

$$= \frac{527777 - 166665 + 27777}{1000000}$$

$$= \frac{361112}{1000000}$$

$\therefore$ Ans $\Rightarrow Pr(E_4 \cup E_6 \cup E_9) = \frac{45139}{125000}$ or 0.361112

3. **Exercise 1.11**

I am trying to send you a single bit, either a 0 or a 1. When I transmit the bit, it goes through a series of n relays before it arrives to you. Each relay flips the bit independently with probability p.

(a) Argue that the probability you receive the correct bit is

$$\sum_{k=0}^{(n/2)} \binom{n}{2k} * (p^{2k}) * (1-p)^{n-2k}$$

Exercise 1.11:-

a) Probability of k flips, given n relays: $\binom{n}{k}p^k(1-p)^{n-k}$ for $0 \leq i \leq n$.

$\binom{n}{k}$ are all the possible configurations of n relays where a bit is flipped k times.

$p^k(1-p)^{n-k}$ = actual probability of k flips, n relays

Probability of an even integer $2k \leq n$ of flips occurring: $\binom{n}{2k}p^{2k}(1-p)^{n-2k}$ for $0 \leq k \leq \lfloor n/2 \rfloor$

We can only receive the correct bit if the bit has undergone an even number of flips after transmission. This can be calculated as the sum of the probabilities of 2k flips less than n. This brings us to the formula which we were initially asked to prove:

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} p^{2k}(1-p)^{n-2k}$$

(b) We consider an alternative way to calculate this probability. Let us say the relay has bias q if the probability it flips the bit is (1 − q)/2. The bias q is therefore a real number in the range [−1, 1]. Prove that sending a bit through two relays with bias q1 and q2 is equivalent to sending a bit through a single relay with bias q1q2.

b) For every relay with bias q, the probability of not flipping a bit is $(1+q)/2$. A bit will be received correctly if a flip occurs in either both or none of q_1 & q_2.

$$Pr(\text{correct}) = \frac{(1+q_1)(1+q_2)}{2} + \frac{(1-q_1)(1-q_2)}{2}$$

$$= \frac{(1+q_1q_2)}{2}$$

$$\Pr(\text{Incorrect}) = 1 - \Pr(\text{correct})$$
$$= 1 - \frac{(1+q_1q_2)}{2} = \frac{(2-1-q_1q_2)}{2}$$
$$= \frac{(1-q_1q_2)}{2}$$

The Pr(Incorrect) seems accurate based on the information provided in the question. We started our calculations with q_1 & q_2 and ended with an accurate solution in terms of q_1q_2. We would have been unable to make this observation had sending a bit through q_1 & q_2 had not been equivalent to sending a bit through q_1q_2.

(c) Prove that the probability you receive the correct bit when it passes through n relays as described before (a) is

$$\mathbf{\frac{1 + (1 - 2p)^n}{2}}$$

c) (Base Case) $n=1$:-

$$\Pr(\text{Receiving correct bit}): \frac{1+(1-2p)^1}{2} = \frac{\cancel{2}(1-p)}{\cancel{2}} = 1-p$$

(Induction) $n>1$:-

Hypothesis must hold true for $n>1$ (for $n+1$ as well).
The probability of receiving the correct bit, with $n+1$ relays can be calculated as the probability of the bit arriving correctly till the n^{th} relay & NOT getting flipped at the $n+1^{th}$ relay, plus the probability of

the bit arriving incorrectly till the n^{th} relay then getting flipped at the $n+1^{th}$ relay.

$\therefore$ The probability of receiving the bit correctly:

$$\frac{1+(1-2p)^n}{2}(1-p) + \frac{1-(1-2p)^n}{2}p$$

$$\Rightarrow \frac{1+(1-2p)^n(1-p)}{2} + \frac{1-(1-2p)^n p}{2}$$

$$\Rightarrow \frac{(1-2p)^n((1-p)-p)+1}{2}$$

$$\Rightarrow \frac{(1-2p)^n(1-2p)+1}{2}$$

$$\therefore \text{Ans} \Rightarrow \frac{(1-2p)^{n+1}+1}{2}$$

4. **Exercise 1.13**
A medical company touts its new test for a certain genetic disorder. The false negative rate is small: if you have the disorder, the probability that the test returns a positive result is 0.999. The false positive rate is also small: if you do not have the disorder, the probability that the test returns a positive result is only 0.005. Assume that 2% of the population has the disorder. If a person chosen uniformly from the population is tested and the result comes back positive, what is the probability that the person has the disorder?

Exercise 1.13:-
Let E_D be the event where a person has a disorder
Let E_T be the event where the test is positive.

$$Pr(E_D|E_T) = \frac{Pr(E_D \cap E_T)}{Pr(E_T)} = \frac{Pr(E_T|E_D) \times Pr(E_D)}{(Pr(E_T|E_D) \times Pr(E_D)) + (Pr(E_T|\neg E_D) \times Pr(\neg E_D))}$$

$$= \frac{0.999 \times 2/100}{(0.999 \times 2/100) + (0.005 \times 98/100)}$$

$$= \frac{1998/100000}{(1998+490)/100000} = \frac{1998}{2488}$$

$$\therefore \text{Ans} \Rightarrow Pr(E_D|E_T) = \frac{999}{1244} \text{ or } 0.8030547$$

5. **Exercise 1.15**
Suppose that we roll ten standard six-sided dice. What is the probability that their sum will be divisible by 6, assuming that the rolls are independent? (Hint: Use the principle of deferred decisions, and consider the situation after rolling all but one of the dice.)

Exercise 1.15:-

Let $P(k,n)$ be the probability on rolling a total with remainder k when divided by 6, (ie:- $0 \le k \le 5$), using n die.

$P(k,1)$ = Probability of rolling a k if $k \neq 0$ or 6.

$P(k,1)$ = Probability of rolling a k if $k = 6$.

$P(k,1)$ = Probability of rolling $(6-k) = 1/6$

For $n>1$: $P(k,n) = \sum_{k=0}^{5} P(k,n-1) \times P(\text{Rolling } 6-k)$

$= \sum_{k=0}^{5} P(k,n-1) \times 1/6$

$= 1/6 \sum_{k=0}^{5} P(k,n-1)$

$= 1/6 \times 1$

$\therefore$ Ans $\Rightarrow$ $= 1/6$

6. **Exercise 1.18**

We have a function F : {0, . . . , n − 1} → {0, . . . ,m − 1}. We know that, for 0 ≤ x, y ≤ n − 1, F((x + y)mod n) = (F(x) + F(y))mod m. The only way we have for evaluating F is to use a lookup table that stores the values of F. Unfortunately, an Evil Adversary has changed the value of 1/5 of the table entries when we were not looking.

Describe a simple randomized algorithm that, given an input z, outputs a value that equals F(z) with probability at least 1/2. Your algorithm should work for every value of z, regardless of what values the Adversary changed. Your algorithm should use as few lookups and as little computation as possible.

Suppose I allow you to repeat your initial algorithm three times. What should you do in this case, and what is the probability that your enhanced algorithm returns the correct answer?

Exercise 1.18:-

Choose x (uniformly) from $(0, \ldots, n-1)$

Let $y = z - x \bmod n$.

$F(z) = F(x) + F(y) \bmod m$. $\left.\begin{matrix} x \\ y = z - x \end{matrix}\right\}$ ① Uniform rv between 0 & n-1 ② Dependent

$Pr(F(x) \text{ is incorrect}) = Pr(F(y) \text{ is incorrect}) = 1/5$

By union bound:

$Pr(\text{error}) = Pr(F(x) \text{ is incorrect} \cup F(x) \text{ is incorrect}) \leq 2/5$

ie:- $Pr(\text{correct}) = 1 - Pr(\text{error})$

$= 1 - 2/5$

$\therefore$ Ans $\Rightarrow$ $Pr(\text{correct}) = \underline{\underline{3/5}}$ $(> 1/2)$

If the algorithm is repeated thrice then:

① Pick the mode, if a result is repeated, or

② Pick the first result, if no repetition.

Then $Pr(\text{error})$ needs atleast 2 runs to produce an incorrect result. The probability of the algorithm running once on some input i, is P_i. $P_i \geq 3/5$ (from our calculation above). Building on this, the probab-ility of the algorithm giving right ans atleast 2 out of the 3 times it's run:

$\binom{3}{2} P_i^2 (1 - P_i) + \binom{3}{3} P_i^3$

$\Rightarrow \binom{3}{2} P_i^2 (1 - P_i) + \binom{3}{3} P_i^3 \leq \binom{3}{2} (3/5)^2 (2/5) + \binom{3}{3} (3/5)^3$

$\Rightarrow \leq (3 \times 9/25 \times 2/5) + 27/125$

$\Rightarrow \leq 54/125 + 27/125$

$\Rightarrow \leq 81/125$ or 0.648

$\therefore$ The probability of getting the correct result upon running the algorithm thrice is $81/125$ or 0.648