CSC 423 – A4

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1. **Exercise 6.2**

   a) Construct a uniform probability space over all possible 2-colorings by independently assigning each edge to color 1 with probability 1/2 and color 2 with probability 1/2. Let X be the number of monochromatic K_4s for this coloring. A particular K_4 gets the same color with probability $2*2^{-4\mathbf{C}2} = 2^{-5}$, and linearity of expectation gives $E[X] = n\mathbf{C}4*2^{-5}$. By the expectation argument, there exists a 2-coloring with at most $n\mathbf{C}4*2^{-5}$ monochromatic K_4s.

   b) Each trial of the randomized algorithm independently picks a random 2-coloring as in part (a) and checks if the coloring satisfies the desired property. The trials are repeated until one succeeds. (Note, that counting the number of monochromatic K_4s can be done in $O(n^4)$ time.) Each trial fails with probability $Pr[X \geq n\mathbf{C}4*2^{-5} + 1] \leq ((n\mathbf{C}4*2^{-5})/(n\mathbf{C}4*2^{-5} + 1))$ using Markov' inequality. The number of trials thus follows a geometric distribution with parameter $1/(n\mathbf{C}4*2^{-5} + 1)$, and has expectation $n\mathbf{C}4*2^{-5} + 1 = O(n^4)$.

2. **Exercise 6.10**

   1. Choose every subset of size $\lfloor n/2 \rfloor$.

   2. Following the hint, choose a random permutation of (1, . . . , n). Let $X_k = 1$ if the first k numbers yield a set in F, and let $X = \sum^{n}_{k=0} X_k$. Note that $Pr(X_k = 1) = f_k /(n\mathbf{C}k)$. Furthermore, for only one value of k can $X_k = 1$, which means that $E[X] \leq 1$. Therefore,
      $E[X] = \sum^{n}_{k=0} E[X_k]$
      $= f_k /(n\mathbf{C}k)$
      ≤ 1

   3. For a fixed n, the binomial coefficient is maximized at $n\mathbf{C}\lfloor n/2 \rfloor$. Therefore,
      $(1/ n\mathbf{C}\lfloor n/2 \rfloor)\sum^{n}_{k=0} f_k \quad \leq \quad \sum^{n}_{k=0} (f_k/(n\mathbf{C}k)) \quad \leq \quad 1$
      implying
      $\sum^{n}_{k=0} f_k \quad \leq \quad n\mathbf{C}\lfloor n/2$
      The result follows since $|F| = \sum^{n}_{k=0} f_k$

3. **Exercise 6.17**

   There are $n\mathbf{C}k$ K_k cliques in K_n. We let A_i be a bad event such that the i^{th} clique K_k is monochromatic. Since each clique K_k has $k\mathbf{C}2$ edges, $Pr[A_i] = 2/2^{(k\mathbf{C}2)} = 2^{1-(k\mathbf{C}2)}$. We can construct a dependency graph G = (V, E), where each vertex $v_i \in V$ corresponds to the event A_i. Furthermore, $(v_i , v_j) \notin E$ iff A_i and A_j are independent. Note that for a fixed clique, the number of other cliques sharing at least two edges with it is at most $(k\mathbf{C}2)*((n-2)\mathbf{C}(k-2)) < (k\mathbf{C}2)*(n\mathbf{C}(k-2))$, so we know that each vertex in the dependency graph has degree at most $(k\mathbf{C}2)*(n\mathbf{C}(k-2))$, i.e., $d \leq (k\mathbf{C}2)*(n\mathbf{C}(k-2))$. Let p denote $Pr[A_i]$. Since $4*(k\mathbf{C}2)*(n\mathbf{C}(k-2))* 2^{1-(k\mathbf{C}2)} \leq 1$, we have $4dp \leq 1$. Hence by Lovasz local lemma, it is possible that none of the bad events (i.e., A_i's) happens, that is, there exists a monochromatic K_k subgraph in K_n.

4. **Exercise 6.19**

As the hint given in the problem description, we let $A_{u,v,c}$ be the bad event that u and v, where u and v are adjacent, are both colored with color c. It is clear that the event happens only when the color c lies in both S(u) and S(v). If $c \notin S(u)$ or $c \notin S(v)$, then $Pr[A_{u,v,c}] = 0$, so we consider the case that $c \in S(u)$ and $c \in S(v)$. We derive that

$Pr[A_{u,v,c}] \leq 1/(8r)^2 = 1/64r^2$

We can construct a dependency graph G = (V, E), where V consists of the events $\{A_{u,v,c} \mid (u, v) \in E\}$. Since for each $v \in V$ and $c \in S(v)$ there are at most r neighbors u of v such that c lies in S(u), we have that $A_{u,v,c}$ has dependency on at most $8r*r + 8r*r = 16r^2$ other events, the degree of G, i.e., d, is at most $16r^2$. Since $4*Pr[A_{u,v,c}]*d \leq 4*(1/64r^2)*16r^2 \leq 1$, by Lovasz local lemma, we have the desired result.

5. **Exercise 7.2**

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6. **Exercise 7.12**

Let $Y_t = X_t \pmod k$, meaning that Y_t is the remainder of X_t divided by k. Then $Y_0, Y_1, \ldots$ is a Markov chain with k states, and $Y_0 = 0$. The transition probabilities are

$P_{i,j} = 1/6\sum^{6}_{a=1} I[i + a = j \pmod k]$

X_t is divisible by k if and only if $Y_t = 0$. And we know that $Pr[Y_t = 0] = P^t_{0,0}$. This is a finite Markov chain. This is clearly irreducible, since you can reach from state i to state j by rolling $j - i \pmod k$ 1s on the die. It is also aperiodic because from any state i, there is a path to i of length k (k rolls of 1) and $k - 1$ ($k - 2$ rolls of 1, and 1 roll of 2). Thus, the chain is ergodic and hence has a unique stationary distribution. It is easy to check that the uniform distribution is indeed a stationary distribution.

7. **Exercise 7.22**

Following the hint, we formulate a new Markov chain with n^2 states of the form $(i, j) \in [1, n]^2$. Each node (i, j) in the new chain is connected to N(i)N(j) neighbors, where N(i) denotes the number of neighbors of state i in the old Markov chain. Hence the number of edges in the new chain comes to

$2|E| = \sum_i\sum_j N((i, j)) = \sum_i\sum_j N(i)N(j) = (\sum_i N(i)) (\sum_j N(j)) = 4m^2$

By Lemma 16, if an edge exists between nodes $u = (i_1, j_1)$ and $v = (i_2, j_2)$, then $h_{u,v} \leq 2|E| = 4m^2$. In order to obtain the $O(m^2n)$ upper bound, we need to show that for any node (i, j), there exists a path of length O(n) connecting it to some node of the form (v, v). In fact, we show that there exists a length O(n) path between (i, j) and (i, i). Since the graph is undirected, the cat can always go back to node i in two steps. At the same time, because the graph is connected, there's a path of length $k < n$ from j to i. If k is even, then the mouse will run into the cat. If k is odd, then the mouse will get to node i when the cat is away. But since the chain is non-bipartite, there must be a path of odd length from i back to itself; let the mouse follow this path, and it will run into the cat on the next return to i. Thus, the total length of this path from (i, j) to (i, i) is at most 3n. Each edge on this path requires at most $4m^2$ steps, thus the desired upper bound on the time to collision is $O(m^2n)$ steps