Math 212, Assignment 6

Solutions

Question 3 is worth 20 marks; all other questions are worth 10 marks each. They will be marked for correctness and clarity of explanation.

1. Use de Moivre's Theorem to compute the following (with answers in the form $x + iy$):

   (a) $(-\sqrt{3} + i)^8$

   Solution: First, write this in polar form:

$$r = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2,$$

   while $\theta = \frac{5\pi}{6}$. Then

$$\begin{aligned}
(-\sqrt{3} + i)^8 &= \left(2(\cos(\frac{5\pi}{6}) + i\sin(\frac{5\pi}{6}))\right)^8 \\
&= 2^8\left(\cos(\frac{8\cdot 5\pi}{6}) + i\sin(\frac{8\cdot 5\pi}{6})\right) \\
&= 256\left(\cos(\frac{40\pi}{6}) + i\sin(\frac{40\pi}{6})\right) \\
&= 256\left(\cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3})\right) \\
&= 256\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\
&= -128 + i128\sqrt{3}
\end{aligned}$$

   (b) $(-1 - i)^9$

   Solution: First, write this in polar form:

$$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2},$$

while $\theta = \frac{5\pi}{4}$. Then

$$\begin{aligned}
\left(\sqrt{2}(\cos(\frac{5\pi}{4} + i\sin(\frac{5\pi}{4})\right)^9 &= 2^{\frac{9}{2}}\left(\cos(\frac{45\pi}{4} + i\sin(\frac{45\pi}{4}\right) \\
&= 2^{\frac{9}{2}}\left(\cos(\frac{5\pi}{4} + i\sin(\frac{5\pi}{4}\right) \\
&= -2^4 - i2^4.
\end{aligned}$$

2. Find all solutions to $w^6 = -4$, and plot them in the complex plane. Include both the polar form and the rectangular form for the first two solutions. You may leave the others in polar form.

   Solution: First, we write the complex number -4 in polar form:
   $-4 = 4(\cos\pi + i\sin\pi)$.

   The the six solutions to the equation are w_0, $w_1, \cdots, w_5$, where

   $$w_k = 4^{\frac{1}{6}}(\cos\theta_k + i\sin\theta_k),$$

   and $\theta_k = \frac{\pi}{6} + \frac{2k\pi}{6}$. Then we have

   $$\begin{aligned}
   \theta_0 &= \frac{\pi}{6} \\
   \theta_1 &= \frac{\pi}{6} + \frac{2\pi}{6} = \frac{\pi}{2} \\
   \theta_2 &= \frac{\pi}{6} + \frac{4\pi}{6} = \frac{5\pi}{6} \\
   \theta_3 &= \frac{\pi}{6} + \frac{6\pi}{6} = \frac{7\pi}{6} \\
   \theta_4 &= \frac{\pi}{6} + \frac{8\pi}{6} = \frac{3\pi}{2} \\
   \theta_5 &= \frac{\pi}{6} + \frac{10\pi}{6} = \frac{11\pi}{6}.
   \end{aligned}$$

Note that $4^{\frac{1}{6}} = 2^{\frac{1}{3}}$, so

$$\begin{aligned}
w_0 &= 2^{\frac{1}{3}}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\\
w_1 &= 2^{\frac{1}{3}}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\\
w_2 &= 2^{\frac{1}{3}}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)\\
w_3 &= 2^{\frac{1}{3}}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)\\
w_4 &= 2^{\frac{1}{3}}\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)\\
w_1 &= 2^{\frac{1}{3}}\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right).
\end{aligned}$$

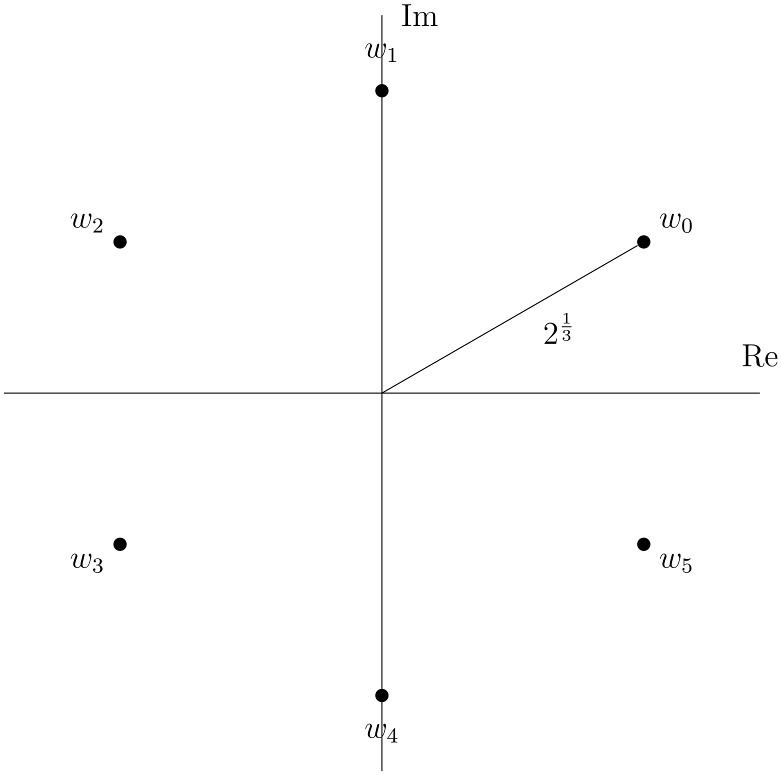
In rectangular form,

$$w_0 = 2^{\frac{1}{3}}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \sqrt{3}\cdot 2^{-2/3} + 2^{-2/3}i$$

and

$$w_1 = 2^{\frac{1}{3}}(0+i) = 2^{\frac{1}{3}}i.$$

Plotting $w_0, \dots, w_5$ in the complex plane, we get



3. Let n be a positive integer and let

$$H = \{z \in \mathbb{C} \mid z^n = 1\}.$$

(a) Write out all elements of H in polar form.

Solution: Start by writing

$$1 = 1\left(\cos(0) + i\sin(0)\right).$$

Use $1^{1/n} = 1$ and $0/n = 0$, to get all roots:

$$z_k = \cos(2k\pi/n) + i\sin(2k\pi/n), k = 0, 1, 2, \ldots, n-1.$$

or, if you prefer,

$$\begin{array}{l} 1 \\ \cos(2\pi/n) + i\sin(2\pi/n), \\ \cos(4\pi/n) + i\sin(4\pi/n), \\ \cdots \\ \cos(2(n-1)\pi/n) + i\sin(2(n-1)\pi/n). \end{array}$$

(b) Prove that with multiplication of complex numbers, H is a group.

Solution: We know that $\mathbb{C}^{\times}$, the non-zero complex numbers, with multiplication is a group. We show that H is a subgroup. It is obvious from above that H is not empty. Let a, b be in H. Let us check that $a^{-1}b$ is in H:

$$(a^{-1}b)^n = a^{-n}b^n = (a^n)^{-1}b^n = 1 \cdot 1 = 1.$$

Hence $a^{-1}b$ is in H.

(c) Prove that this group is cyclic.

Solution: Notice that (using the notation above), for any $k = 1, 2, \ldots$, we have (using de Moivre's Theorem), that

$$\begin{aligned} z_1^k &= \left(\cos(2\pi/n) + i\sin(2\pi/n)\right)^k \\ &= \cos(2k\pi/n) + i\sin(2k\pi/n) \\ &= z_k. \end{aligned}$$

Hence, every element of H is in the cyclic subgroup generated by z_1 and H is cyclic.

(d) Find a more familiar group which is isomorphic to H and give an explicit isomorphism. (Hint: Theorem 3.12.4.)

Solution: By Theorem 3.12.4, any cyclic group with n elements is isomorphic to $\mathbb{Z}_n$. Hence, $H \cong \mathbb{Z}_n$. In fact, once you know a generator, in this case z_1, 3.12.4 actually gives you a formula for the isomorphism, $f : \mathbb{Z}_n \to H$:

$$f([k]_n) = z_1^k = z_k,$$

for $0 \leq k < n$.

(e) Let s be the sum of all elements of H. Prove that, for any z in H, $zs = s$.

Solution: We can assume that $z = z_1^k$, for some $0 \leq k < n$. The point here is that if we multiply all the elements of H by z, we just end up with the same set of elements:

$$\begin{aligned} zs &= z_1^k \left(1 + z_1 + z_1^2 + \cdots + z_1^{n-1}\right) \\ &= \left(z_1^k + z_1^{k+1} + \cdots + z_1^{n-1} + 1 + z_1 + \cdots z_1^{k-1}\right) \\ &= s \end{aligned}$$

(f) Prove that the sum of all the elements of H is 0.

Solution: We have

$$zs = s \Rightarrow (z-1)s = 0 \Rightarrow s = 0,$$

since we can choose $z - 1 \neq 0$ (provided $n > 1$).

(g) Prove that the product of all the elements of H is 1 if n is odd and -1 if n is even.

Solution: If z_k is in the group and is not a real number, then z_{n-k} is also in the group and $z_{n-k} = \overline{z_k} \neq z_k$. Moreover, their product $z_{n-k}z_k = |z_k|^2 = 1$. So all of the elements of H which are not real numbers pair off and their product is just 1. It remains only to see that when n is even, both 1 and -1 are in H and their product is -1. If n is odd, then 1 is in H, but -1 is not and so the product is 1.

4. The subset of the complex numbers $\{m + ni \mid m, n \in \mathbb{Z}\}$ is usually called the Gaussian integers and is written $\mathbb{Z}[i]$.

   (a) Prove that this is a subring of the complex numbers.

   Solution: Obviously this set, which we call S, contains 0 so it is not empty. Let m, n, k, l be integers so $a = m + in, b = k + il$ are in S. First, we check $a - b$ is in S:

$$a - b = m + in - (k + il) = (m - k) + i(n - l),$$

   which is clearly in S since $m - k$ and $n - l$ are integers. Next, we check that ab is in S:

$$a \cdot b = (m + in)(k + il) = (mk - nl) + i(ml + nk),$$

   and as $mk - nl$ and $ml + nk$ are both integers ab is in S.

   (b) Find all units in this ring.

   Solution: Each element of S has an inverse in the complex numbers (except 0). The question is whether or not the inverse remains in S:

$$\begin{aligned}
(m + in)^{-1} &= \frac{1}{m + in} \\
&= \frac{1}{m + in} \frac{m - in}{m - in} \\
&= \frac{m - in}{m^2 + n^2} \\
&= \frac{m}{m^2 + n^2} - i \frac{n}{m^2 + n^2}
\end{aligned}$$

   If m and n are both non-zero, then $m^2 + n^2$ is greater than both of them, in absolute value and so the fractions above cannot be integers and so $m + in$ does not have an inverse in S. Similarly, if either m or n is at least 2 in absolute value, then one of the denominators is greater than the numerator and again the inverse is not in S.

   On the other hand, if one of m, n is 0 and the other is ± 1, then $m + in$ has an inverse:

$$1^{-1} = 1,\ (-1)^{-1} = -1,\ i^{-1} = -i,\ (-i)^{-1} = i.$$

   These are exactly the units of S.

5. Apply the division algorithm in each of the following.

   (a) In the ring $\mathbb{Q}[x]$, divide $2x^2 - 1$ into $x^4 + 2x^3 - 5$.

(b) In the ring $\mathbb{Z}_2[x]$, divide x^2+x+1 into x^3+1.

(c) In the ring $\mathbb{Z}_5[x]$, divide $4x+1$ into x^3+3x^2+x+2.

Solution:

(a) $x^4+2x^3-5 = \left(\frac{1}{2}x^2+x+\frac{1}{4}\right)(2x^2-1)+\left(x-\frac{19}{4}\right)$

(b) $x^3+1=(x+1)(x^2+x+1)$

(c) $x^3+3x^2+x+2=(4x^2+x)(4x+1)+2$

6. In the polynomial ring $\mathbb{Z}_5[x]$ decide which of $f(x)=x^2+1$ and $g(x)=x^2+2$ is irreducible. Explain your answer and, if either is reducible, factor it (in a non-trivial way).

Solution:

Since these polynomials are degree two, irreducible is the same as having not roots. So we look for roots:

$$\begin{aligned} f(0) &= 1, \\ f(1) &= 1+1=2, \\ f(2) &= 4+1=0, \\ f(3) &= 9+1=0, \\ f(4) &= 16+1=2. \end{aligned}$$

So both $x-2=x+3$ and $x-3=x+2$ are factors and

$$f(x)=(x+3)(x+2).$$

$$\begin{aligned} g(0) &= 2, \\ g(1) &= 1+2=3, \\ g(2) &= 4+2=1, \\ g(3) &= 9+2=1, \\ g(4) &= 16+2=3. \end{aligned}$$

So $g(x)$ is irreducible.

7. In the ring $\mathbb{Q}[x]$, factor $x^3 - 2x + 1$ into irreducible polynomials. Explain your answer.

Solution: By Theorem 5.3.11, the only possible roots are $\frac{p}{q}$, where p divides 1 and q divides 1. So we need only check $-1, 1$ as possible roots.

$$\begin{aligned} f(-1) &= -1 + 2 + 1 = 2, \\ f(1) &= 1 - 2 + 1 = 0 \end{aligned}$$

So $x - 1$ is a factor.

We next need to divide $x^3 - 2x + 1$ by $x - 1$:

$$\begin{array}{rr|rrrr} & & x^2 & +x & -1 & \\ \cline{3-6} x & -1 & x^3 & & -2x & +1 \\ & & x^3 & -x^2 & & \\ \cline{3-6} & & & & & \\ & & & x^2 & -2x & +1 \\ & & & x^2 & -x & \\ \cline{4-6} & & & & -x & +1 \\ & & & & -x & +1 \\ \cline{5-6} & & & & & 0 \end{array}$$

We have factored $x^3 - 2x + 1 = (x - 1)(x^2 + x - 1)$. We now have to see if $x^2 + x - 1$ has any roots. As the only root of the original polynomial is 1, the only possible root here is also 1:

$$1^2 + 1 - 1 = 1 \neq 0.$$

so $x^2 + x - 1$ is irreducible.