Math 212, Assignment 4

Due Tuesday, March 13, 2018

All questions are equally weighted. They will be marked for correctness and clarity of explanation.

1. Let A be a cube and let G be the group of rotations that are symmetries of the cube. (Do not include any reflections.) For each point x given below, describe the set $G(x)$ and the subgroup G_x. Find $\#G(x)$ and $\#G_x$ in each case, where $\#X$ denotes the number of elements in the set X.

   (a) a corner of the cube.

   (b) the mid-point of an edge of the cube.

   (c) the centre of a face of the cube.

2. Let L be the line
$$L = \left\{ \begin{bmatrix} x \\ 1 \end{bmatrix} \mid x \in \mathbb{R} \right\}$$
and let G be the $ax+b$-group which we view as a subgroup of S_L as in 3.8.13. Let
$$x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$
Find $G(x)$ and G_x.

3. Kevin is making beaded children's bracelets to sell at the community market. He has an unlimited supply of purple, blue, green and yellow beads. If each bracelet will have 10 beads, how many bracelet designs are possible?

4. Let G, H, and K be groups and let $\phi : G \to H$ and $\psi : H \to K$ be isomorphisms.

   (a) Show that ϕ^{-1} and $\psi \circ \phi$ are both isomorphisms. (The first part of this amounts to proving Theorem 3.12.3.)

   (b) Using (a), show that the isomorphism of groups determines an equivalence relation on the set of all groups. That is, show that $\cong$ is an equivalence relation on the set of all groups.

5. Let G and H be groups and let $f : G \to H$ be an isomorphism. Prove that G is abelian if and only if H is abelian. (This is part 7 of Theorem 3.12.2.)

6. Let G be the set of matrices of the form
$$\begin{bmatrix} (-1)^j & k \\ 0 & 1 \end{bmatrix},$$

where $j \in \{0,1\}$ and all entries of the matrix are elements of $\mathbb{Z}_n$ (so all operations on the entries are done modulo n); that is, the entries of the matrix are really $[-1]_n^j$, $[k]_n$, $[0]_n$ and $[1]_n$. With matrix multiplication, G is a group (you do not need to prove this). Prove that G is isomorphic to D_n by finding an explicit isomorphism (and proving that it is an isomorphism).

7. Which of the following groups are isomorphic?

   - $\mathbb{Z}$ (with addition)
   - The subgroup $\langle r^6 \rangle$ of D_8 (with composition)
   - $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (with componentwise addition)
   - $8\mathbb{Z} = \{\ldots, -16, -8, 0, 8, 16, \ldots\}$ (with addition)
   - $\mathbb{Z}_4$ (with addition)
   - $G = \{f_n | n \in \mathbb{Z}\}$, where f_n is the function on $\mathbb{Z}$ defined by $f_n(x) = n + x$ for all $x \in \mathbb{Z}$ (with composition)
   - U_5 (with multiplication)
   - U_8 (with multiplication)

   By problem 4(b), $\cong$ is an equivalence relation on the set of all groups, so this question amounts to sorting the above groups into their equivalence classes. Each time you add a new group to an equivalence class, give a specific isomorphism from that group to one of the groups already in the equivalence class. (You must be sure that your function actually is an isomorphism, but you do not need to include the proof in your answer.) Each time you add a new equivalence class to your list, give a reason why the groups in the new class do not belong in any of the equivalence classes you have already listed.

8. Let $\mathbb{R}^\times$ be the set of nonzero real numbers with multiplication. Following the proof of Cayley's Theorem, $\mathbb{R}^\times$ is isomorphic to the subgroup

   $$H = \{f_g \mid g \in \mathbb{R}^\times\}$$

   of $S_{\mathbb{R}^\times}$.

   (a) Describe the element f_{10} of H, and the inverse of this element.

   (b) Find an element of $S_{\mathbb{R}^\times}$ which is not in H.

   (c) Without using your answer from (b), give another reason why H cannot be all of $S_{\mathbb{R}^\times}$. (If it is, then $S_{\mathbb{R}^\times}$ is isomorphic to $\mathbb{R}^\times$. Using group properties, explain why this cannot be the case.)

Rules for group assignments. Make sure you follow the universal rules for group assignments (below) and any additional rules/procedures laid out in your Group Contract.

1. Each group member is expected to contribute to the best of their ability, and assignment submissions should only include the names of group members who meet this expectation.

2. Each group member should be able to explain the group's solution to me and answer any questions I may have about it. It is the whole group's responsibility to ensure that this standard is met.

3. The task of composing final solutions and writing them up in good copy must be shared equally among all group members (after a collaborative problem-solving process).

4. After good copy solutions are complete, they should be shared among all group members to be double-checked and proofread. This should be done in advance of the due date, to allow time for any necessary corrections. Corrections should be completed by the person who wrote the original solution.