Math 212, Assignment 1

Due Friday, January 26, 2018

All questions are equally weighted. They will be marked for correctness and clarity of explanation.

1. For each of the following sets, determine if the given operation is a binary operation or not. Explain your answers.

   (a) The set of all 2×2 matrices with real entries whose second row is twice its first row:
   $$X = \left\{ \begin{bmatrix} a & b \\ 2a & 2b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$
   with matrix multiplication.

   (b) Vectors in $\mathbb{R}^3$ with dot product: $(x, y, z) \cdot (x', y', z') = xx' + yy' + zz'$.

2. Let S be any set. Consider the binary operation *intersection* on the power set of S, $\mathcal{P}(S)$. Is this operation associative? Is it commutative? Does it have an identity? Find it, or explain why it doesn't have one.

3. Prove that matrix multiplication is a binary operation on the set
   $$X = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in R, ad - bc \neq 0 \right\}.$$
   (This set is $GL_2(\mathbb{R})$, but don't use that here.)

4. **Definition:** Let X be a set with a binary operation, and let a be an element of X. We say that an element $b \in X$ is a *left inverse* of a if $ba = e$. We say that b is a *right inverse* of a if $ab = e$. (Therefore b is an inverse of a if it both a left inverse and a right inverse of a.)

   Let A be a non-empty set and $f : A \to A$. Prove that f has a right inverse in F_A if and only if f is surjective (onto).

5. For each of the following, determine (with proof) whether the given set is a group with respect to the given operation.

   (a) The set P of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$ with the operation composition. That is, the set
   $$P = \{f : \mathbb{R} \to \mathbb{R} \mid f(x) = a_0 + a_1 x + \cdots a_n x^n, \text{ where } a_0, \ldots, a_n \in \mathbb{R}\},$$
   with composition.

(b) The set $\mathbb{R} \setminus \{-1\}$ with the operation $*$ defined by

$$a * b = a + b - ab.$$

(c) The power set $\mathcal{P}(S)$ of a set S, with the operation intersection.

6. Find, with proof, the order of $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ in $GL_2(\mathbb{R})$?

7. Let k be a positive integer and let a be a negative integer. Prove using the axioms for the integers that $a^k = (-a)^k$ if and only if k is even, using the axioms for the integers. (You may also use the results from Chapter 1 and the results from Chapter 2 that follow from the axioms.)

8. Let G be a finite group of even order. Prove that there is an element $a \neq e$ of G such that $a^2 = e$.

Rules for group assignments. Make sure you follow the universal rules for group assignments (below) and any additional rules/procedures laid out in your Group Contract.

1. Each group member is expected to contribute to the best of their ability, and assignment submissions should only include the names of group members who meet this expectation.

2. Each group member should be able to explain the group's solution to me and answer any questions I may have about it. It is the whole group's responsibility to ensure that this standard is met.

3. The task of composing final solutions and writing them up in good copy must be shared equally among all group members (after a collaborative problem-solving process).

4. After good copy solutions are complete, they should be shared among all group members to be double-checked and proofread. This should be done in advance of the due date, to allow time for any necessary corrections. Corrections should be completed by the person who wrote the original solution.