Math 212, Assignment 2

Due Friday, February 9, 2018

All questions are equally weighted. They will be marked for correctness and clarity of explanation.

1. (a) Let G and H be groups, with $g \in G$ and $h \in H$. Prove that the order of (g, h) in $G \oplus H$ is the least common multiple of $o(g)$ and $o(h)$.

   (b) For which pairs of natural numbers m and n is $\mathbb{Z}_m \oplus \mathbb{Z}_n$ cyclic? Explain.

2. In S_{12}, let $\sigma = (5\ 11)$ and $\pi = (3\,4\,5\,6)$.

   (a) Compute
   $$\sigma^{-1}\pi\sigma,$$
   writing the answer as a product of disjoint cycles.

   (b) Based on part (a), find $\tau \in S_{12}$ such that
   $$\tau^{-1}\pi\tau = (9\ 10\ 11\ 12).$$

   (c) Let $(a_1\, a_2\, \ldots a_k)$ and $(b_1\, b_2\, \ldots b_k)$ be cycles in S_n. Give a permutation $\tau \in S_n$ that satisfies
   $$\tau^{-1}(a_1\, a_2 \ldots a_k)\tau = (b_1\, b_2\, \ldots b_k),$$
   and explain why your choice of τ works.

3. For $n \geq 2$, let A_n be the subset of S_n consisting of all even permutations in S_n. Prove that A_n is a group.

4. Let $n \geq 2$. Show that exactly half of the permutations in S_n are even by finding a bijection from the set of all even permutations in S_n to the set of all odd permutations in S_n.

5. In the group D_8, give an algebraic proof that $r^3 j$ has order 2, and also a geometric proof of the same fact.

6. For $n \geq 4$, is D_n cyclic? Explain your answer.

7. (a) How many generators does the group $\mathbb{Z}_{15}$ have?

   (b) Let p and q be distinct primes. How many generators does the group $\mathbb{Z}_{pq}$ have?

8. Is U_{10} cyclic? Is U_{12}? For each, find a generator or prove that it is not cyclic.

Rules for group assignments. Make sure you follow the universal rules for group assignments (below) and any additional rules/procedures laid out in your Group Contract.

1. Each group member is expected to contribute to the best of their ability, and assignment submissions should only include the names of group members who meet this expectation.

2. Each group member should be able to explain the group's solution to me and answer any questions I may have about it. It is the whole group's responsibility to ensure that this standard is met.

3. The task of composing final solutions and writing them up in good copy must be shared equally among all group members (after a collaborative problem-solving process).

4. After good copy solutions are complete, they should be shared among all group members to be double-checked and proofread. This should be done in advance of the due date, to allow time for any necessary corrections. Corrections should be completed by the person who wrote the original solution.