Math 212, Assignment 5

Due Tuesday, March 27

All questions are equally weighted. They will be marked for correctness and clarity of explanation.

1. Consider the set $\mathbb{R}^3$ with componentwise addition, and with multiplication defined by
$$(x, y, z) \cdot (x', y', z') = (0, yy', 0).$$
Is $\mathbb{R}^3$ a ring with these operations? If so, is it commutative? Does it have an identity? If it has an identity, determine which elements are units.

2. Prove that the set
$$S = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$
is a subring of $M_2(\mathbb{R})$ with its usual addition and multiplication. Is S a commutative ring? Does it have an identity? If so, find the identity and determine which elements are units of S.

3. For each of the following, decide if the set S is a subring of the real numbers, with its usual operations of addition and multiplication. If it is, give a proof, and if not, explain why.
   (a) $S = \{\frac{n}{4} : n \in \mathbb{Z}\}$
   (b) $S = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} : a, b, c \in \mathbb{Z}\}$.

4. TRUE or FALSE. (If true, give a proof and if false, give an explicit counterexample.)
If R is a field and S is a subring of R, then S is also a field.

5. Let R be a commutative ring with identity $1 \neq 0$, and suppose the cancellation rule holds in R. That is, for all $a, b, c \in R$, with $a \neq 0$, if $ab = ac$ then $b = c$. Prove that if R is finite, then R is a field.

6. Prove that $F = \{a + b\sqrt{5} : a, b \in \mathbb{Q}\}$, with ordinary addition and multiplication, is a field.

7. Let $R = \{i + j\sqrt{5} : i, j \in \mathbb{Z}\}$, and let Q be the field of quotients of R.

   (a) Prove that if $[a, b]$ is in Q, then there exists c in R and $k \in \mathbb{Z}$, $k \neq 0$, such that $[a, b] = [c, k]$.

   (b) Define an isomorphism from the field F of problem 6 to Q. You do not need to prove it is an isomorphism, but use part (a) to prove it is surjective.

8. For each of the following subsets of the rational numbers, determine if the set has (i) a maximum, (ii) an upper bound, (iii) a least upper bound. (This means within the set of *rational* numbers.) In parts (i) and (iii) if the answer is yes, name one. In part (ii) if the answer is yes, name three.

   (a) $\{\frac{1}{2n} : n = 1, 2, 3, \ldots\}$.

   (b) $\{\frac{a}{b} \mid a, b \in \mathbb{Z}, 0 < a < b\}$.

   (c) $\{\frac{n^3+5n}{n^2} : n = 1, 2, 3, \ldots\}$.

   (d) $\{\frac{a}{b} : a, b \in \mathbb{Z}^+, a^2 + 3ab - b^2 < 0\}$. (You may assume there is no rational number r with $r^2 = 13$.)

Rules for group assignments. Make sure you follow the universal rules for group assignments (below) and any additional rules/procedures laid out in your Group Contract.

1. Each group member is expected to contribute to the best of their ability, and assignment submissions should only include the names of group members who meet this expectation.

2. Each group member should be able to explain the group's solution to me and answer any questions I may have about it. It is the whole group's responsibility to ensure that this standard is met.

3. The task of composing final solutions and writing them up in good copy must be shared equally among all group members (after a collaborative problem-solving process).

4. After good copy solutions are complete, they should be shared among all group members to be double-checked and proofread. This should be done in advance of the due date, to allow time for any necessary corrections. Corrections should be completed by the person who wrote the original solution.