Math 212, Assignment 3

Due Friday, March 2, 2018

All questions are equally weighted. They will be marked for correctness and clarity of explanation.

1. Let
$$H = \{(a, b, c) : a, b, c \in \mathbb{Z}, a + b + c = 0\}.$$
Prove that H is a subgroup of $\mathbb{Z}^3$.

2. Exercise 3.7.16, Part 1.

3. Let G be a group and let H_1 and H_2 be subgroups of G.
   (a) Prove that $H_1 \cap H_2$ is a subgroup of G.
   (b) Suppose that G is finite and H_1 and H_2 have orders p and q, respectively, where p and q are distinct primes. Prove that $H_1 \cap H_2 = \{e\}$.

4. Exercise 3.8.10

5. Let θ be a real number and consider
$$A = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$
   (a) Verify that A is in $O_2(\mathbb{R})$.
   (b) Using induction, prove that
$$A^n = \begin{bmatrix} \cos(n\theta) & \sin(n\theta) \\ -\sin(n\theta) & \cos(n\theta) \end{bmatrix}$$
   for all $n = 1, 2, 3, \ldots$.
   (c) For which values of θ does A have finite order in $O_2(\mathbb{R})$?

6. Let $G = D_6$ and $H =< r^2 >= \{e, r^2, r^4\}$ (which is a subgroup – you should be able to prove this, but you do not need to include the proof on your assignment). List all of the left cosets of H in G. Find $[G : H]$.

7. Let $G = GL_2(\mathbb{R})$ and $H = SL_2(\mathbb{R})$. Prove that, for any a, b in G, the cosets aH and bH are equal if and only if $det(a) = det(b)$.

8. (a) Let G be a cyclic group of order n. Prove that G has a subgroup of order k for every positive divisor k of n. (In other words, prove that the converse of Corollary 3.9.12 holds for cyclic groups.)

   (b) What are the possible orders of subgroups of A_4 (the alternating group on n elements). Does there actually exist a subgroup of each of these orders? (You may look up a list of the subgroups of A_4.) What conclusion can you draw about Corollary 3.9.12?

Rules for group assignments. Make sure you follow the universal rules for group assignments (below) and any additional rules/procedures laid out in your Group Contract.

1. Each group member is expected to contribute to the best of their ability, and assignment submissions should only include the names of group members who meet this expectation.

2. Each group member should be able to explain the group's solution to me and answer any questions I may have about it. It is the whole group's responsibility to ensure that this standard is met.

3. The task of composing final solutions and writing them up in good copy must be shared equally among all group members (after a collaborative problem-solving process).

4. After good copy solutions are complete, they should be shared among all group members to be double-checked and proofread. This should be done in advance of the due date, to allow time for any necessary corrections. Corrections should be completed by the person who wrote the original solution.