Stat 261 Assign # 3, 2018.

1. 90th percentile γ such that $P(X \le \gamma) = .90$

$X \sim \exp(\theta \text{ mean})$. lifetime of transistor

Solve for γ ; $P(X \le \gamma) = \int_0^\gamma \frac{1}{\theta} e^{-x/\theta} dx = 1 - e^{-\gamma/\theta} = .90$

$\Rightarrow .1 = e^{-\gamma/\theta} \Rightarrow \frac{\gamma}{\theta} = -\ln .1 \Rightarrow \gamma = -\theta \ln .1$

$\Rightarrow \gamma = \theta \ln\left(\frac{1}{10}\right)^{-1} \Rightarrow \gamma = \theta \ln 10.$

To compute the MLE of θ :

$L(\theta) = \prod_{i=1}^n \frac{1}{\theta} e^{-x_i/\theta} = \theta^{-n} e^{-\frac{1}{\theta}\sum_1^n x_i}$

$\ell(\theta) = -n \ln \theta - \frac{1}{\theta}\left(\sum_1^n x_i\right)$

$\ell'(\theta) = -\frac{n}{\theta} + \frac{1}{\theta^2}\left(\sum_{i=1}^n x_i\right) = 0 \Rightarrow \hat{\theta} = \frac{\sum^n x_i}{n}$

here $\hat{\theta} = \frac{180}{10} = 18.$ By the invariance property, $\hat{\gamma} = \hat{\theta} \ln 10 = 41.44$ is the MLE of the 90th percentile.

2.

Table 1

G Y \ R A	R	A
G	55 (56.24)	19 (19.76)
Y	21 (17.76)	5 (6.24)

(expected frequencies in brackets)

multinomial ($n = 100$, $p_1 = \alpha\beta$, $p_2 = \alpha(1-\beta)$, $p_3 = (1-\alpha)\beta$, $p_4 = (1-\alpha)(1-\beta)$)

$L(\alpha, \beta) = (\alpha\beta)^{55} [\alpha(1-\beta)]^{21} [(1-\alpha)\beta]^{19} [(1-\alpha)(1-\beta)]^5$

$= \alpha^{55+21} (1-\alpha)^{19+5} \beta^{55+19} (1-\beta)^{21+5}$

$= \alpha^{76} (1-\alpha)^{24} \beta^{74} (1-\beta)^{26}.$

$$l(\alpha,\beta) = 76\ln\alpha + 24\ln(1-\alpha) + 74\ln\beta + 26\ln(1-\beta)$$

① $\dfrac{\partial l}{\partial \alpha} = \dfrac{76}{\alpha} - \dfrac{24}{1-\alpha}$ $\Rightarrow$ $\hat{\alpha} = \dfrac{76}{100} = .76$.

② $\dfrac{\partial l}{\partial \beta} = \dfrac{74}{\beta} - \dfrac{26}{1-\beta}$ $\Rightarrow$ $\hat{\beta} = \dfrac{74}{100} = .74$.

Estimated expected frequencies:

$e_1 = n\hat{p}_1 = n\hat{\alpha}\hat{\beta} = 100 \times .76 \times .74 = 56.24$

$e_2 = n\hat{p}_2 = n\hat{\alpha}(1-\hat{\beta}) = 100 \times .76 \times .26 = 19.76$.

$e_3 = n\hat{p}_3 = n(1-\hat{\alpha})\hat{\beta} = 100 \times .24 \times .74 = 17.76$.

$e_4 = n\hat{p}_4 = n(1-\hat{\alpha})(1-\hat{\beta}) = 100 \times .24 \times .26 = 6.24$

From Table 1, the estimated frequencies are close to the observed frequencies

3. $X_i \sim N(\mu b_i, \sigma^2)$

$$L(\mu,\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sigma} \exp\left\{-\frac{1}{2\sigma^2}(x_i - \mu b_i)^2\right\}$$

$$= \frac{1}{\sigma^n} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu b_i)^2\right\}.$$

$$l(\mu,\sigma^2) = -n\ln\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu b_i)^2$$

① $\dfrac{\partial l}{\partial \mu} = +\dfrac{1}{2\sigma^2}\sum(x_i - \mu b_i)\,b_i$

② $\dfrac{\partial l}{\partial \sigma} = \dfrac{-n}{\sigma} + \dfrac{1}{\sigma^3}\sum_{i=1}^{n}(x_i - \mu b_i)^2$

from equation 1 $\quad \sum (x_i - \hat{\mu} b_i) b_i = 0$

$$\Rightarrow \quad \sum_1^n (x_i b_i) = \hat{\mu} \sum_1^n b_i^2$$

$$\Rightarrow \quad \hat{\mu} = \frac{\sum_1^n (x_i b_i)}{\sum_1^n b_i^2}$$

Equation 2 $\times \sigma^3$

$$\sum (x_i - \hat{\mu} b_i)^2 = n \hat{\sigma}^2$$

$$\Rightarrow \quad \hat{\sigma}^2 = \frac{\sum_1^n (x_i - \hat{\mu} b_i)^2}{n}$$

$$= \frac{\sum_{i=1}^n \left[x_i - \hat{\mu} \left(\frac{\sum x_i b_i}{\sum b_i^2} \right) \right]^2}{n}.$$