Homework Report

Shawn Seymour
University of Minnesota Morris

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Heuristics

In this report, I will discuss multiple vertex coloring heuristics. The input for all heuristics defined here is a simple graph $G = (V, E)$. I will define them as follows:

Heuristic A

Heuristic A is the greedy algorithm.

Algorithm 1 Greedy algorithm

```
Label each vertex in V, i.e. v_1, v_2, ..., v_n
for each v ∈ V do
    Assign a color p_i to v_i using the smallest available p_i
```

Heuristic B

Heuristic B is the greedy algorithm with degree sequencing. It orders the vertices according to the decreasing value of their degree. This is also known as the Welsh-Powell algorithm, which is defined in [5].

Algorithm 2 Welsh-Powell algorithm

```
Label each vertex in V, i.e. v_1, v_2, ..., v_n, such that d_G(v_1) ≥ d_G(v_2) ≥ ··· ≥ d_G(v_n)
for all v ∈ V do
    Assign a color p_i to v_i using the smallest available p_i
```

Heuristic C

Heuristic C colors a graph by finding maximal independent sets of G.

Algorithm 3 Coloring via maximal independent set algorithm

```
while G is non-empty do
    Find a maximal independent set of G, i.e. S_i
    Color all vertices in S_i with color p, where p is the smallest color available
    Let G ← G[V \ S_i]
```

Findings

Proposition 1. *Heuristic A and Heuristic B do not always produce an optimal solution, i.e. they do not always produce a minimum coloring of a graph.*

I will show this for *heuristic B*. It is trivial to show the same for *heuristic A* as *heuristic B* includes the same steps as *heuristic A*. I will construct a simple graph, $G = (V, E)$, such that:

$$|V| \geq 8 \tag{1}$$
$$\chi(G) > \chi^*(G) \tag{2}$$

Let χ refer to the coloring of G generated by *heuristic B*. Let χ^* be the optimal coloring of G.

Example 1

I constructed a simple graph, $G = (V, E)$, such that $\Delta(G) = 3$ and $\delta(G) = 1$. I've let $|V| = 8$ for this example.

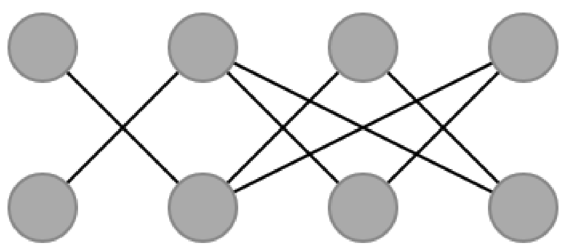

Figure 1: Uncolored original graph G

By applying *heuristic B*, we get the following coloring. The numbers indicate the ordering of vertices before applying the heuristic. Any vertex of the same degree got assigned arbitrarily. This results in $\chi(G) = 3$.

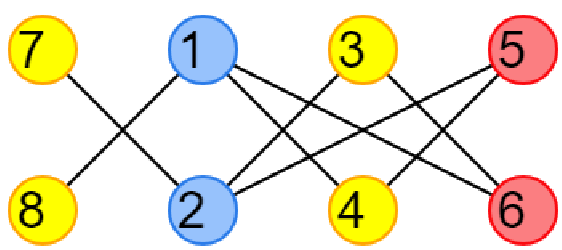

Figure 2: Coloring from applying heuristic B

Definition (Bipartite graph)**.** *A bipartite graph is one whose vertex set can be partitioned into two subsets X and Y such that each edge has one end in X and one end in Y*

We can see that this is a bipartite graph, defined above by [1]. Thus, G is *2-colorable*. This means $\chi^*(G) = 2$. G is an example graph that satisfies conditions (1) and (2). The optimal coloring is shown below.

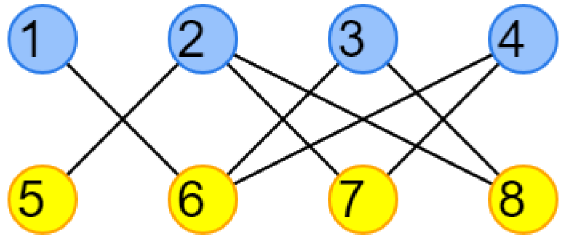

Figure 3: Optimal coloring of graph G

Proposition 2. *Heuristic C also does not always produce an optimal solution, i.e. it does not always produce a minimum coloring of a graph.*

Example 2

To show this, I will create another example that satisfies conditions (1) and (2) from above. I've constructed a graph $K = (V, E)$ for this example.

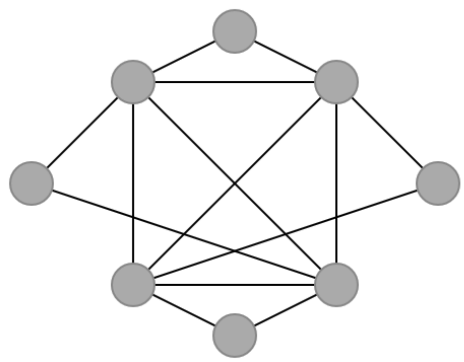

Figure 4: Uncolored original graph K

By applying heuristic C and finding maximal independent sets, we see that K is *5-colorable*.

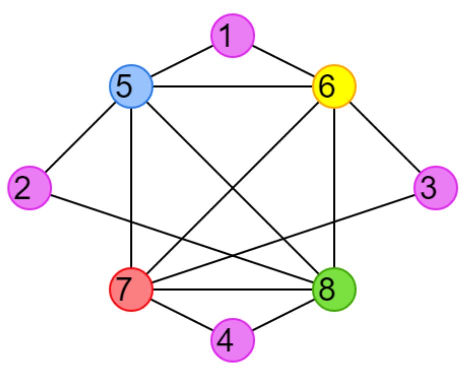

Figure 5: Coloring of K after applying heuristic C

This is a non-optimal solution. We can see that the subgraph created by vertices $\{5, 6, 7, 8\}$ is complete and thus we need at least 4 colors. This graph is *4-colorable* however, meaning $\chi^*(K) = 4$. Thus, we can see that heuristic C does not always give us an optimal solution.

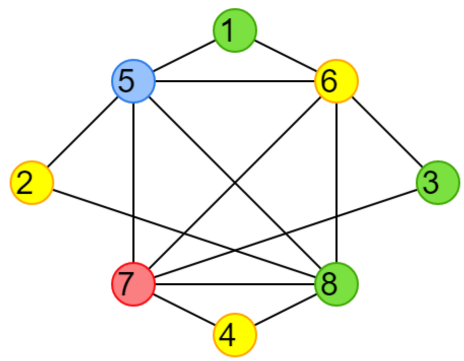

Figure 6: Optimal coloring of K

Proposition 3. *Heuristic A and heuristic B have an upper bound of* $\Delta + 1$.

To show this, I will create another example that satisfies conditions (1) and (2) from above. After doing some research into bipartite graphs, I learned that *crown graphs* are excellent at showing how bad greedy heuristics can be, as shown and defined in [3].

Definition (Crown Graph)**.** *A crown graph* $CR_n = (V, E)$ *is an undirected graph with two sets of vertices where* $V = V_1 \cup V_2$ *with an edge from* $v_i \in V_1$ *to* $v_j \in V_2$ *whenever* $i \neq j$. *A crown graph can also be viewed as a complete bipartite graph where the edges of a perfect matching have been removed.*

Example 3

I constructed a simple crown graph, $H = (V, E)$. I've let $|V| = 10$ for this example.

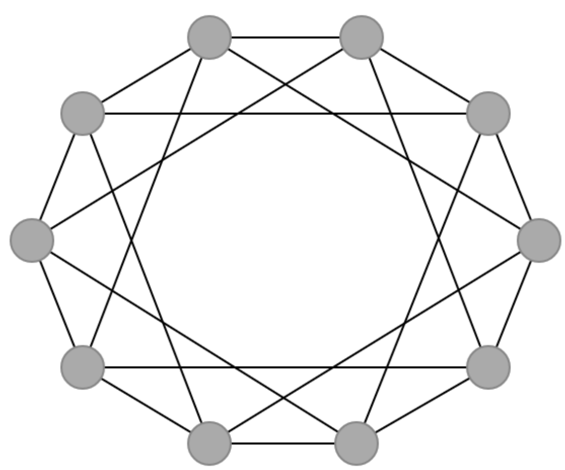

Figure 7: Uncolored original graph H

We can see that $\Delta(G) = 4$. We can also see $d_G(v) = 4$ for all $v \in V$. Thus, in both heuristics A and B, the greedy algorithm would pick an order arbitrarily. We can show using this crown graph the worst-case scenario of these heuristics.

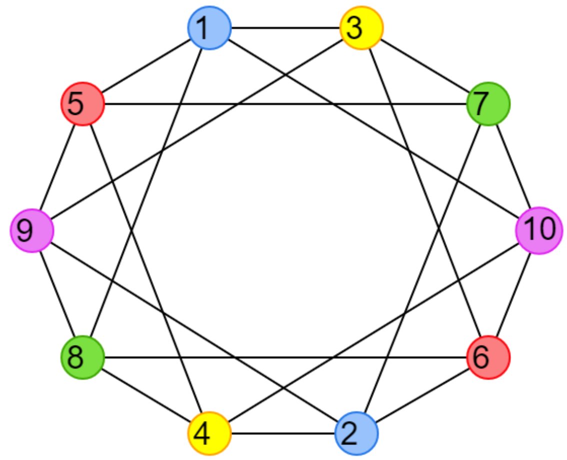

Figure 8: Worst-case coloring of H using either heuristic A or B

In Figure 5, using either heuristic with this ordering, we get $\chi(H) = 5$. This gives us $\frac{|V|}{2}$ colors. This is the worst-case for a crown graph as shown in [2], but this graph also demonstrates the upper bounds for these heuristics.

In General: Let's take a look at how this looks in general. Let $P = (V, E)$ be a simple, complete graph.

We can see that $\chi(H) = \Delta(H) + 1$. This is very easy to see in a *complete* graph, where all vertices are connected to every other vertex. This means all vertices have degree $|V| - 1$. Thus, every time we color a node, a new color is needed. And since we have $\Delta(P) = |V| - 1$ and $|V|$ vertices, we will need $\Delta(P) + 1$ colors. This is stated in *Brooks' Theorem*.

Theorem (Brooks' Theorem)**.** *For any connected undirected graph G with maximum degree Δ, the chromatic number of G is at most Δ unless G is a complete graph or an odd cycle, in which case the chromatic number is $\Delta + 1$.*

The proof of *Brook's Theorem* can be found in [4]. Overall, heuristic A and heuristic B can produce some very undesirable results. In graph H, at the worst case, these heuristics produce $\chi(H) = 5$ when $\chi^*(H) = 2$ as it is bipartite. This is shown below.

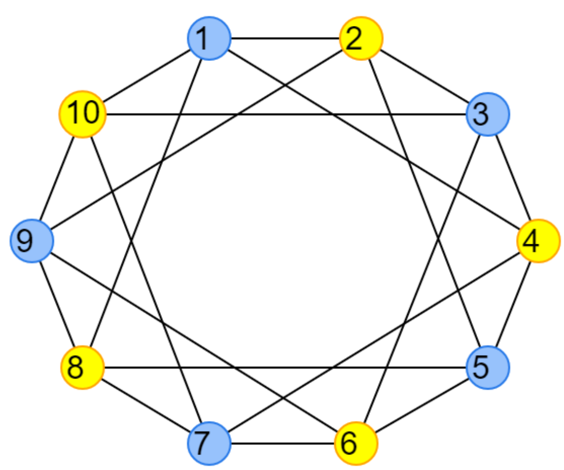

Figure 9: Optimal coloring of H

References

[1] Bondy, J.A. and U.S.R. Murty [1976], *Graph Theory with Applications*, American Elsevier Publishing, New York, NY.

[2] Johnson, D. S. [1974], *Worst-case behavior of graph coloring algorithms*, Proc. 5th Southeastern Conf. on Combinatorics, Graph Theory, and Computing, Utilitas Mathematicae, Winnipeg, pp. 513–527.

[3] Kordecki, W. and A. Łyczkowska-Hanćkowiak [2016], *Greedy online colouring with buffering*, arXiv preprint arXiv:1601.00252.

[4] Lovász, L. [1975], *Three short proofs in graph theory*, Journal of Combinatorial Theory, Series B, 19(3), 269-271.

[5] Welsh, D. J. and Powell, M. B. [1967], *An upper bound for the chromatic number of a graph and its application to timetabling problems*, The Computer Journal, 10(1), 85-86.