COUNCIL OF SCIENTIFIC & INDUSTRIAL RESEARCH

Mathematical Science Code: 04

Unit – 5:

Syllabus

Sub unit – 3: Topology

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Topology

3.1. Topology.

Definition:

- Let $X = \phi$, the power set $P(X)$ is the family of all subset of X. conventionally, $\phi \in P(X), X \in P(X)$.

A subfamily $\lambda\, of\, P(X)$ having the properties listed below

(i) $\phi \in \lambda,\ X \in \lambda$

(ii) $A \in \lambda,\ B \in \lambda$

(iii) $A_\alpha \in \lambda,\ \forall\, \alpha \in \Delta \Rightarrow \bigcup_{\alpha \in \Delta} A_\alpha \in \lambda,\ \ \Delta$ being an indexed set is called a Topology on X.

And a set X , together with a topology λ defined on it is called topological space and denoted by (X, λ).

Elements of X are referred to as points, whereas elements of λ are referred to as $\lambda - open\ set$ (or simply open sets). The same set X may have different topologies. Let λ_1 and λ_2 be any two topologies on the same set X. If $\lambda_1 \subset \lambda_2$, then λ_1 is called smaller or weaker or coarser topology than λ_2. If $\lambda_1 \subset \lambda_2$, then we also say that λ_2 is longer or finer or stronger or larger than λ_1.

- Whatever the set X may be, the family $\{\phi, X\}$ is always a topology on X and is called indiscrete or trivial topology. This topology is smaller than any other topology defined for X. The set X together with indiscrete topology is called indiscrete topological space or indiscrete space.

The family $P(X)$ is always a topology on X and is called discrete topology on X. This topology is longer than any other topology defined for X , so that $P(X)$ is the strongest topology on X. The set X together with discrete topology is called a discrete topological space or simply a discrete space. In a discrete space (X, λ) , every subset of X is an open set.

e.g.,

(i) If $X = \phi\ , then\ P(X) = \{X\} = \lambda,$ here all the postulates of a topology are trivially satisfied.

(ii) Let $X = \{a, b\}$,

Then, $P(X) = \big\{\phi, X, \{a\}, \{b\}\big\}$.

If we take

(1) $\lambda = \{\phi, X\}$ (2) $\lambda = P(X)$ (3) $\lambda = \big\{\phi, X, \{a\}\big\}$ (4) $\lambda = \big\{\phi, X, \{b\}\big\}$

The result is always a topology, if we take

(1) $\lambda = \{\phi, \{a\}, \{b\}\}$

(2) $\lambda = \{X, \{a\}, \{b\}\}$

(3) $\lambda = \{\phi, \{a\}\}$

The result is not a topology.

(iii) Let X be an infinite set. Let λ be the family consisting of ϕ , X and complements of finite subsets of X. This topology is defined as cofinite topology or $\lambda_1 - topology$ or finite complement topology.

(iv) The topology generated by the family of intervals of the form $(a, \infty) = \{x \in R : x > a\}$ is called right-hand topology.

(v) The topology generated by the family or interval of the form $(-\infty, a) = \{x \in R : x < a\}$ is called left-hand topology.

(vi) If $X = (0, 1)$, then $\lambda = \{\phi, X, \{0\}\}$ is a topology on X. This topological space is called Sier Pinski space.

3.2. Open Sets and Closed Sets:

Let (X, λ) be a topological space. Any set $A \in \lambda$ is called an open subset of X or simply a open set and $X - A$ is called a closed subset of X.

A topological space (X, λ) is said to be a door space if every subset of X is either $\lambda - open$ or $\lambda - closed$.

e.g., Let $X = \{1, 2, 3\}$ and $\lambda = \{\phi, \{1, 2\}, \{2, 3\}, \{2\}, X\}$

Then, $\lambda - closed$ sets are $X, \{3\}, \{1\}, \{1, 3\}, \phi$. From this, it follows that every subset of X is either $\lambda - open$ or $\lambda - closed$. Hence (X, T) is door space.

3.3. Neighbourhood:

Let (X, λ) be a topological space. $A \subset X$ is called a neighbourhood of a point $x \in X\ if\ \exists\ G \in \lambda\ with\ x \in G\ s.t.\ G \subset A$. The word neighbourhood is, in short, written as 'nbd'. Any open set $G \subset X$ with $x \in G$ is also a neighbourhood of a point $x \in X$. If A is a neighbourhood of a point $x \in X$, then $A - \{x\}$ is called deleted neighbourhood of x.

Example (3.1): Let $\lambda = \{\phi, X, \{b\}, \{a,b\}, \{a,b,d\}\}$ be a topology on $X = \{a,b,c,d\}$.

Find $\lambda - neighbourhood$ of

(i) a (ii) b and (iii) c

Solution:

(i) $\lambda - open$ set containing a are $X, \{a,b\}, \{a,b,c\}$.

Supersets of $X\ is\ X$.

Supersets of $\{a,b\}$ are $\{a,b\}, \{a,b,c\}, \{a,b,d\}, X$.

Supersets of $\{a,b,d\}$ are $\{a,b,d\}, X$.

$\lambda - neighbourhoods$ of a are $\{a,b\}, \{a,b,c\}, \{a,b,d\}, X$.

(ii) $\lambda - open$ set containing b are $\{b\}, \{a,b\}, \{a,b,d\}, X$.

Superset of $\{a,b\}$ are $\{a,b\}, \{a,b,c\}, \{a,b,d\}, X$.

Supersets of $\{a,b,d\}$ are $\{a,b,d\}, X$.

Supersets of $\{b\}$ are $\{b\}, \{a,b\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{b,c,d\}, X$.

$\lambda - neighbourhood\ b$ are $\{b\}, \{a,b\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{b,c,d\}, X$.

(iii) $\lambda - open$ set containing c is X. Hence, $\lambda - neighbourhood$ of c is X.

Example (3.2):

Give an example of a non – trivial topological space in which open sets are exactly the same as closed sets.

Solution: In every discrete topological space (X, D) consisting of more than one element, every proper subset of X is both open and closed. Consider the following topological spaces in which every proper subset of X is both open and closed.

e.g., (i) $X = \{a,b,c\}, \lambda = \{\phi, X, \{a\}, \{b,c\}\}$. Then, (X, λ) is a topological space. Evidently, $\{a\}, \{b,c\}$ are $\lambda - open$ sets, so that $X - \{a\}$, $X - \{b,c\}\ are\ \lambda$ closed set, i.e., $\{b,c\}, \{a\}$ are closed sets.

$\therefore$ $\{a\}, \{b,c\}$ are both open sets as well as closed sets.

(ii) Suppose, $X = \{a,b,c\}$ and $\lambda = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}\}$.

Then, λ is a topology on X. Also, each proper subset of X is open as well as closed.

Example (3.3): Prove that in a topological space, an arbitrary union of open sets is open and a finite intersection of open sets in the open.

Example (3.4):

In a topological space (X, λ), prove that an arbitrary intersection of closed sets is closed and finite union of closed sets is closed.

Closure:

Let (X, λ) be a topological space and $A \subset X$. The closure of A is defined as the intersection of all closed sets which contains A and is denoted by the symbol $\bar{A}$.

Symbolically, $\bar{A} = \cap \{ F \subset X : F$ is closed, $F \supset A\}$

Evidently,

(i) $\bar{A}$ is closed for an arbitrary intersection of closed sets is closed.

(ii) $A \subset \bar{A}$. We also define $\bar{A} = \cap\ F_i\ , where\ F_i$'s are closed supersets of A.

(iii) $\bar{A}$ is the smallest closed set which contains A.

3.4. Interior:

Let (X, λ) be a topological space and $A \subset X$. A point $x \in A$ is called an interior point of A, if only if $\exists$ open set $G \in \lambda\ wiht\ x \in G\ s.t.\ G \subset A$.

The interior of A is defined on the set of all interior points of A and is denoted by A^0 or by $int\,(A)$.

Alternatively, we defined

$$A^0 = int\,(A) = \cup \{ G \in \lambda : G \subset A \}$$

Evidently,

(i) A^0 is open. For an arbitrary union of open sets is open.

(ii) $A^0 \subset A$.

(iii) A^0 is the largest open subset of A.

3.5. Limit Point:

Let (X, λ) be a topological space and $A \subset X$. A point $x \in X$ is called a limit point of A if every open set G containing x contains a point of A other than x.

symbolically, $x \in X$ is called a limit point of A if for every $G \in \lambda\ with\ x \in G$.

$$(G - \{x\}) \cap A \neq \phi$$

The following have the same meaning. The limit point, accumulation point, clusters point. The set of all accumulation points of A is called the derived set of A and is denoted by $D(A)$.

Adherent Point:

A point $x \in X$ is called an adherent point (or contact point) of $A \subset X$ if and only if every neighbourhood of x contains at least one point of A. That is to say, $x \in X$ is called an adherent point of A if only if $\exists$ $neighbourhood\ N\ of\ x\ s.t.\ N \cap A \neq \phi$. The set of all adherent points of A is denoted by $adh\ (A)$.

Isolated Point:

Let (X, λ) be a topological space and $A \subset X$. An adherent point $x \in A$ is called an isolated point of $A\ if\ x$ is not a limit point of A. In other words, $\exists\ neighbourhood\ G\ of\ x\ s.t.\ (G - \{x\}) \cap A = \phi\ or\ G \cap A = \{x\}$. If every point of a set A is an isolated point of A, then the set A is called isolated set.

3.6. Base:

Let (X, λ) be a topological space. Let $B \subset \lambda, s.t.\ B \neq \phi. B$ is said to be a base or open base or basis for the topology $\lambda\ on\ X$ if given any non – empty set $G \in \lambda \Rightarrow \exists\ B_1 \subset B\ s.t.,\ G = \cup \{B : B \in B_1\}$.

Another Definition:

B is said to be a base for the topology $\lambda\ on\ X$ if $x \in G \in \lambda \Rightarrow \exists\ B \in B\ s.t.\ x \in B \in G$. The elements of B are referred to as basis open sets. Later on, we shall show that both definitions of bases are equivalent. e.g.,

(i) The set of all open intervals in R form a base for the usual topology on R.

(ii) The set of all open intervals (r, s) with $r\ and\ s$ as rationals forms a base for the usual topology on R.

(iii) The set of all circular discs (not containing points on the circumference) with centres and radius rationals forms a base for the usual topology on R.

Sub-base:

Let (X, λ) be a topological space. Let $S \subset \lambda\ s.t.\ S \neq \phi$. S is said to be a sub-base for the topology $\lambda\ on\ X$ if finite intersections of the members of S form a base for the topology $\lambda\ on\ X$. i.e., the unions of the members of S give all the members of λ. The elements of S are referred to as sub – basic open sets.

e.g., Let $a, b \in R$ be arbitrary s.t. $a < b$. Evidently $(-\infty, b) \cap (a, \infty) = (a, b)$

The open intervals (a, b) form a base for the usual topology on R. Hence, by definition, the family of infinite open intervals creates a sub-base for the usual topology on R.

Local – base:

Let (X, λ) be a topological space. A family B_x of open subsets of X is said to be a loca – base at $x \in X$ for the topology λ *on* X if

(i) any $B \in B_x \Rightarrow x \in B$

(ii) any $G \in \lambda$ *with* $y \in G \Rightarrow \exists\, B \in B_x$ *s.t.* $y \in B \subset G$.

e.g., Let $x \in R$ be arbitrary.

Write $A_n = \left(x - \frac{1}{x}, x + \frac{1}{x}\right)$, $\forall\ n \in N$.

Take $B_x = \{A_n : x \in N\}$

Evidently, B_x is a local – base at $x \in X$ for the usual topology on R.

3.7. First countable Space:

Let (X, λ) be a topological space. The space X is said to satisfy the first axiom of countability if X has a countable local base at each $x \in X$. The space X, in the case, is called the first countable or first axiom space.

Second Countable Space:

Let (X, λ) be a topological space. The space X is said to satisfy the second axiom of countability if $\exists$ a countable base for λ *on* X.

In this case, the space X is called second countable or second axiom space.

A second countable space is also called ‘completely separable space’.

e.g., The set of all open intervals (r, s) with r *and* s as rational numbers forms a base, say, B bor the usual topology U *of* R.

Since $Q, Q \times Q$ are countable sets, and so B is a countable base for U *on* R.

$\therefore (R, U)$ is second countable.

Theorems

- A metric space is first countable.
- Every metric space is not second countable.
- Let (X, d) be a metric space and B be the collection of all open spheres $S_r(X)$, where r is any positive number and $x \in X$, then B is a bse for some topology λ *on* X. If d is a usual metric on R^n, then (R^n, d) is called Euclidean space and d is called Euclidean metric. If λ is a topology on R^n induced by d, then λ is called Euclidean topology on R^n.

Example (3.5): Prove that (R, U) is a second axiom space (second countable).

Solution: We know that Q is a countable subset of R. If we write $B = \{(a, b)\,; a < b \text{ and } a, b \in Q\}$. Then, B forms a countable base for the usual topology U *on* R, so that R is second countable.

Example (3.6) : Prove that (R^2, U) is second countable.

Solution: If we write $B = \{S, (x) : x,\ r \in Q\}$ then B forms a countable base for the usual topology $U\ on\ R^2$. Hence, (R^2, U) is a second countable space.

Example (3.7): Let $X = \{1, 2, 3, 4\}$. Let $A = \{\{1, 2\}, \{2, 4\}, \{3\}\}$. Determine the topology on X generated by the elements of A and hence determine the base for this topology.

Solution:

Let $X = \{1, 2, 3, 4\}$ and $A = \{\{1, 2\},\ \{2, 4\},\ \{3\}\}$. Finite intersections of the members of A form the class B given by $B = \{\{1, 2\},\ \{3\},\ \{2, 4\},\ \phi,\ \{2\},\ X\}$. The unions of the members of B form the class λ given by $= \{\{1, 2\}, \{3\}, \{2, 4\}, \phi, \{2\}, X, \{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 4\}, \{3, 2, 4\}, \{3, 2\}\}$. It can be easily verified that B is base for the topology $\lambda\ on\ X$.

Example (3.8): Let (X, λ) be a discrete space. Let $B = \{\{x\} : x \in X\}$ show that any family B^* of subsets of X is a base for $\lambda\ on\ X$ if and only if it is a superset of B.

Solution: Let (X, λ) be a discrete space. Let $B = \{\{x\} : x \in X\}$. Let B^* be a family of all subsets of $X\ s.t. B \subset B^*$. To prove that B^* is a base for the topology $\lambda\ on\ X$. Since, X is a discrete space, and therefore A is an open subset of $X\ \forall\ A \subset X$.

In particular, $\{x\}$ is open in $X, \forall\ x \in X$. We can write $A = U\{\{x\} : x \in A\}$. $Also\ B = \{\{x\} : x \in X\}$ This means that every non – empty open set is expressible as a union of some members of B. Hence, by definition B is a base for the topology $\lambda\ on\ X$.

Furthermore, $\phi \neq G \in \lambda \Rightarrow \exists\ B_1 \subset B^*\ s.t.\ G = U\{B : B \in B_1\}$

$$\Rightarrow \exists\ B_1 \subset B^*\ s.t. G = U\{B : B \in B_1\}$$

For $B \subset B^*$

Finally, $\phi = G \in \lambda \Rightarrow \exists\ B_1 \subset B^*\ s.t.\ G = U\{B : b \in B_1\}$

This $\Rightarrow$ $B^*$is a base for the topology $\lambda\ on\ X$.

Conversely, suppose that B^* is a base for the discrete topology $\lambda\ on\ X$.

Also, suppose $B = \{\{x\} : x \in X\}$

To prove that $B \subset B^*$

Let $\{x\} \in B$ be arbitrary, so that $\{x\}$ is a non – empty open subset of X. Since, B^* is a base for the topology $\lambda on\ X$ and hence $\{x\}$ must be expressed as a union of some members of B^*. Since, any singleton set is a union of itself or itself with the set ϕ, showing thereby $\{x\} \in B^*$

$\therefore\ any\ \{x\} \in B \Rightarrow \{x\} \in B^*$ This, $\Rightarrow B \subset B^*$

3.8. Continuous Functions:

1. Let (X,λ) and (Y,U) be any two topological spaces. Let $f : (X,\lambda) \to (Y,U)$ be a map. The map f is said to be continuous at $x_0 \in X$ if given any $U - open$ set H containing $f(x_0)$, $\exists$ a $\lambda - open$ set G containing x_0 $s.t.$ $f(G) \subset H$.

If the map is continuous at each $x \in X$, then the map is called a continuous map.

2. Let (X,d) and (X,P) be metric spaces. A map $f : (X,d) \to (Y,P)$ is said to be continuous at $x_0 \in X$ if only if given $\varepsilon > 0$, $\exists$ $\delta > 0$ $s.t.$ $d(x,x_0) < \delta \Rightarrow P\,[\,f(x), f(x_0)] < \varepsilon$ or equivalently, f is continuous at x_0 if only if to each open sphere $S_\varepsilon\big(f(x_0)\big)$, $\exists$ open sphere $S(x_0,\delta)$ s.t., $f[S(x_0,\delta)] \subset S_\varepsilon\big(f(x_0)\big)$

f is continuous X if only if it is continuous at every point of X.

3. Let $f : (X,\lambda) \to (Y,U)$ be a map. f is sequentially continuous at $x_0 \in X$ if for every sequence $< x_n \in X : n \in N >$ converging to x_0, the sequence $< f(x_n) \in Y : n \in N >$ converges to $f(x_0)$, i.e., if $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$

4. A map $f : (X,\lambda) \to (Y,U)$ is said to be homeomorphism or topological mapping if

(i) f is one – one onto.

(ii) f and f^{-1} are continuous.

In this case, the spaces X and Y are said to be homeomorphism or topological equivalent to one another and Y is called the homeomorphic image of X.

e.g., Let λ denoted the usual topology on R and a any non – zero real number. Then each of the following maps is a homeomorphism.

(i) $f : (R,\lambda) \to (R,\lambda)$ $s.t.$ $f(x) = a + x$

(ii) $f : (R,\lambda) \to (R,\lambda)$ $s.t. f(x) = ax$

(iii) $f : (R,\lambda) \to (R,\lambda)$ $s.t. f(x) = x^3, where\ x \in R$

5. A map f is called a bicontinuous map if f and f^{-1} both are continuous maps.

6. A map $f : (X,\lambda) \to (Y,U)$ is called an open or interior map if it maps open sets onto open sets, i.e., if any $G \in \lambda \Rightarrow f(G) \in U$

7. A map $f : (X,\lambda) \to (Y,U)$ is called a closed map if any $\lambda - clsoed$ set $F \Rightarrow f(F)$ is $U -$ $closed$ set. e.g.,

(i) Let λ denoted the usual topology on R, let a be any non – zero real number. Then, each of the following map is open as well as closed.

(a) $f : (R,\lambda) \to (R,\lambda)$ $s.t.$ $f(x) = a + x$

(b) $f : (R,\lambda) \to (R,\lambda)$ $s.t. f(x) = ax$

In this case, if $a = 0$, then this map is closed but not open.

(ii) The identity map $f\ :(X,\lambda) \rightarrow (X,\lambda)$ is open and as well as closed.

(iii) A map from an indiscrete space into a topological space is open as well as closed.

(iv) A map from a topological space into a discrete space is open as well as closed.

➢ The function $f\ :(X,\lambda) \rightarrow (Y,U)$ is continuous if only if $f^{-1}(V)$ is open in X for every open set $V\ in\ Y$.

Example (3.9): If $X = \{a,b\}\ and\ \lambda = \{\phi,\{a\},\ X\}$. Is space $(X,\lambda)\ a\ T_0 -$ space or a $T_1 -$ space.

Solution:

(i) (X,λ) is $T_0 -$ space.

For given $a,b \in X,\ \exists\ \{a\} \in \lambda\ \ s.t.\ \ a \in \{a\}, b \in \{a\}$

$a,b \in X, \exists\ \{a\}\ and\ X \in \lambda\ \ s.t.$

$a \in \{a\}\ ,\ b \notin \{a\}\ and\ b \in X,\ \ a \in X.$

Example (3.10): Prove that (R,λ) is a Hausdorff Space.

Solution: Let $a,b \in R$ be arbitrary s.t.$|a - b| = \varepsilon > 0$.

Write $G = \left(a - \frac{\varepsilon}{3}, a + \frac{\varepsilon}{3}\right)$, $H = \left(b - \frac{\varepsilon}{3}, b + \frac{\varepsilon}{3}\right)$ let λ be the usual topology on R. Then –

$G,H \in \lambda\ \ s.t.\ \ \ a \in G,\ \ b \in H,\ \ G \cap H = \phi$

By definition, this implies that (R,λ) is a Hausdorff Space.

Example (3.11): Show that every discrete space is a Hausdorff Space.

Solution: Let (X,λ) be a discrete space and $x,y \in X$ be arbitrary s.t. $x \neq y$.

By definition of discrete space, $\{x\}\ and\ \{y\}$ are $\lambda -$ open sets.

Obviously, $\{x\} \cap \{y\} = \phi, x \in \{x\}\ , y \in \{y\}$.

Thus, $\exists$ disjoint open sets $\{x\}\ and\ \{y\}$ containing $x\ and\ y$, respectively. Consequently, (X,λ) is a Hausdorff Space.

3.9. Separation axioms:

We saw earlier how the ideas of convergence could be interpreted in a topological rather than a metric space: A sequence (a_i) converges to α if every open set containing α contains all but a finite number of the $\{a_i\}$. Unfortunately, this definition does not give some of the "nice" properties we get in a metric space.

For example, if a sequence in a metric space converges, it has a unique limit, but in a topological space this need not happen. For example, in R with the trivial topology, every sequence converges to every point.

To recovery the nice properties of convergence, we need to have "enough" open sets in the topology. Topologists have devised various separation axioms to classify how this happens.

Definition

A topological space X is called Hausdorff if every pair of points can be separated by open sets. That is, if $x_1 \neq x_2 \in X$ then there are disjoint open sets U_1 and U_2 with $x_1 \in U_1$ and $x_2 \in U_2$.

Remarks

1. Felix Hausdorff (1869 to 1942) introduced this idea. He was also responsible for the first formulation of the idea of fractional dimension encountered in fractal Geometry.
2. The Hausdorff condition is sometimes called T_2 .This axiom is one of a number of separation axioms: T_0, T_2, T_2, T_3, $T_{3\frac{1}{2}}$, T_4 .

   These were named by Heinreich Tietze (1880 to 1964) in 1923. The T stands or trennung (= separation in German). Some references call them the Tychonoff axioms after Andrei Tychonoff (1906 to 1993).

   Just for the record, T_0 spaces are sometimes called Kolmogorov spaces, T_1 spaces are called Fréchet spaces, T_2 spaces are Hausdorff, T_3 spaces are regular, $T_{3\frac{1}{2}}$ spaces are completely regular. We will see about T_4 spaces shortly.

   See this link for more details.
3. In a Hausdorff space, distinct points are "housed off" from one another by open sets.

Theorem

Every metric space is Hausdorff.

Theorem

In any Hausdorff space sequences have at most one limit.

Theorem

In a Hausdorff space every point is a closed set.

Remark

It follows that every finite set is closed in a Hausdorff space and the topology is, therefore, stronger than the cofinite topology.

The other separation axiom we will consider is:

Definition

A topological space X is called normal if every disjoint pair of closed sets can be separated by open sets. That is, if A_1 *and* A_2 are disjoint closed subsets of X then there are disjoint open sets U_1 *and* U_2 with $A_1 \subset U_1$ and $A_2 \subset U_2$.

Remark:- If every point is a closed set (that is T_1) then such a normal space is Hausdorff. $[normal + T_1 = T_4]$

Theorem:- Every metric space is normal.

Remark: Note that the distance between disjoint closed sets maybe 0 (but they can still be separated by open sets).

Examples (3.12)

1. As above, all metric spaces are both Hausdorff and normal.
2. The space $X = \{a, b\}$ with $\mathcal{T} = \{\phi, X, \{a\}\}$ is not Hausdorff since a, b cannot be separated by open sets.
   It is, however, normal since there are no non-empty disjoint closed sets.

Remarks

1. Finding a Hausdorff space which is not normal is possible, but tricky!
2. By demanding more separation axioms one gets closer to a metric space. Paul Urysohn (1898 to 1924) proved in 1923 that any T_4 space with a countable basis is metrisable (that is, the topology may be obtained from a metric).
   In fact this is not a necessary condition for metrisability. For example, R with the discrete topology is metrisable but does not have a countable basis. Marshall Stone (1903 to 1989) and R. H. Bing (1914 to 1986) found a necessary and sufficient condition for metrisability in 1950.

3.10. Connectedness:

A topological space is said to be connected if it is not the union of two disjoint nonempty open sets. A set is open if it contains no point lying on its boundary; thus, in an informal, intuitive sense, the fact that space can be partitioned into disjoint open sets suggests that the boundary between the two sets is not part of the space, and thus splits it into two separate pieces.

One way of distinguishing between different topological spaces is to look at the way thay "split up into pieces". To make this idea rigorous, we need the idea of connectedness.

Definition

A space which is a union of two disjoint non-empty open sets is called disconnected.

Equivalently

A space X is connected if the only subsets of X which are both open and closed (= clopen) are ϕ and X.

Proof of equivalence

If $X = A \cup B$ with A and B open and disjoint, then $X - A = B$ and so B is the complement of an open set and hence is closed. Similarly, B is clopen.

Conversely, if A is a non-empty, proper open subset then A and $X - A$ disconnect X.

A subset of a topological space is called connected if it is connected in the subspace topology.

Example (3.13): The set $[0,1] \cup [2,3] \subset \mathbb{R}$ with its usual topology is not connected since the sets $[0,1]$ and $[2,3]$ are both open in the subspace topology.

Theorem:- The interval $(0,1) \subset \mathbb{R}$ with its usual topology is connected.

A similar proof shows that any interval is a connected subset of $\mathbb{R}$. In fact, we have:

Theorem:- Intervals are the only connected subsets of $\mathbb{R}$ with the usual topology.

The most important property of connectedness is how it affected by continuous functions.

Theorem:- The continuous image of a connected space is connected.

Corollary:- Connectedness is preserved by a homeomorphism.

We may use this fact to distinguish between some non-homeomorphic spaces.

Example (3.14)

The spaces $[0,1]$ and $(0,1)$ (both with the subspace topology as subsets of $\mathbb{R}$) are not homeomorphic.

Proof

Removing any point from $(0,1)$ gives a non-connected space, whereas removing an end-point from $[0,1]$ still leaves an interval which is connected.

A similar method may be used to distinguish between the non-homeomorphic spaces obtained by thinking of the letters of the alphabet as in Exercises 1 question 1.

Corollary (The Intermediate Value Theorem)

If $f: \mathbb{R} \to \mathbb{R}$ is continuous then for any $a, b \ in \ \mathbb{R}$, f attains any value between $f(a)$ and $f(b)$ at some point between $a \ and \ b$.

Proof: -Since the interval $[a, b]$ is connected, so is its image $f([a, b])$ and so this too is an interval.

Remark:- A similar result holds for a continuous real-valued function on any connected space.

Fastening together connected space "with an overlap" gives a conected space:

Theorem

If A *and* B are connected and $A \cap B \neq \phi$ then $A \cup B$ is connected.

Definition

The maximal connected subsets of a space are called its components.

Note that every point of a space lies in a unique component and that this is the union of all the connected sets containing the point (This is connected by the last theorem.)

Examples (3.15)

1. The components of the space $[0,1] \cup [2,3]$ with the subspace topology inherited from $\mathbb{R}$, are the subspaces $[0,1]$ *and* $[2,3]$.
2. Components of $\mathbb{Z}$ (with the subspace topology from $\mathbb{R}$) are the single points.
3. (Harder!) Components of $\mathbb{Q}$ (with the subspace topology from $\mathbb{R}$) are also single points.

   Proof

   If $r < s$ are rationals, choose an irrational x between them. Then $\mathbb{Q} \cap (-\infty, x)$ and $\mathbb{Q} \cap (x, \infty)$ disconnect $\mathbb{Q}$ and so r, s are in different components.

3.11. Compactness (Important Results):

1. A closed subset of a compact space is compact. i.e, if (X, λ) be a compact space and $A \subset X$ be closed, then A is compact.

2. Let (A, U) be a subspace of $\{X, \lambda\}$, then A is compact with respect to the topology U if only if A is compact with respect to the topology λ *on* X.

3. Every cofinite topological space (X, λ) is compact.

4. A countably compact topological space has a BWP.

5. A closed subset of a countably compact space is countably compact.

6. If (X, λ) is a compact topological space, then every basic open cover of it is reducible to afinite sub-cover.

7. A topological space is compact if every basic open cover has a finite sub-cover.

8. A continuous image of a sequentially compact set is sequentially compact.

9. (X, λ) is compact $\Leftrightarrow$ each family of closed sets with finite intersection property has a non – empty intersection.

10. A topological space is compact $\Leftrightarrow$ every class of closed sets which has empty intersection, has a finite sub – class with empty intersection.

11. Suppose, A is compact set in a regular space (X, λ) and $G \subset X$ is a neighbourhood of A. Then, $\exists$ a neighbourhood $V \subset X$ *of* A *s.t.* $\bar{V} \subset G$.

12. Let A be a compact subsets of a regular (X, λ). If G is an open subset containing A, then $\exists$ a closed set H containing A $s.t.$ $A \subset H \subset G$.

13. Every compact regular space is normal.

14. A compact Hausdorff Space is normal.

or

If A and B be disjoint compact subset of a Hausdorff Space X, then there exist open sets G and H $s.t.$ $A \subset G$, $B \subset H$.

15. A compact Hausdorff Space is regular.

16. A compact subset of a Hausdorff Space is closed.

17. Let A and B be disjoint compact subsets of a Hausdorff Space (X, λ), then there exist disjoint open sets G and H such that $A \subset G$, $B \subset H$.

or

Topological space (X, λ) is a Hausdorff Space if only if disjoint compact subsets of X can be separated by disjoint open sets.

18. A continuous image of a compact space is compact.

19. Compactness is a topological property. i.e., if $f : (X, \lambda) \to (Y, U)$ is a continuous map, then any compact set $A \subset X \Rightarrow f(A) \subset Y$ is compact.

20. **(i)** A closed and bounded subset (subspace) of R is compact.

(ii) Every closed interval [a, b] is compact.

(iii) Cantor's set is compact. Cantor's set is given by

$\Gamma = \bigcap_{n=1}^{\infty} F_n$ where

$F_0 = [0\ 1]$, $F_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$

$F_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$ and so on.

i.e., F_n is the union of 2^{nd} disjoint closed intervals each of length $\frac{1}{3^n}$. Being finite union of closed sets, F_n is closed also Γ is bounded as $\Gamma \subset [0, 1]$

Given X, also known as the product space, such that

3.12. The product topology:

Given topological spaces X and Y we want to get an appropriate topology on the Cartesian product $X \times Y$.

Obvious method

Call a subset of $X \times Y$ open if it is of the form $A \times B$ with A open in X and B open in Y.

Difficulty

Taking $X = Y = \mathbb{R}$ would give the "open rectangles" in $\mathbb{R}^2$ as the open sets. These subsets are open, but unfortunately, there are lots of other sets which are open too. We are, therefore forced to work a bit differently.

Definition

A set of subsets $\mathbb{B}$ is a basis of a topology $\mathcal{T}$ if every open set in $\mathcal{T}$ is a union of sets of $\mathbb{B}$.

Example (5.16): In any metric space the set $\mathbb{B}$ of all ε-neighbourhoods (for all different values of ε) is a basis for the topology.

Remark

This is a very helpful concept. For example, to check that a function is continuous you need only verify that $f^{1}(B)$ is open for all sets B in a basis -- usually much smaller than the whole collection of open sets.

We can now define the topology on the product.

Definition

If X *and* Y are topological spaces, the product topology on $X \times Y$ is the topology whose basis is $\{A \times B \mid A \in \mathcal{T}X, B \in \mathcal{T}Y\}$.

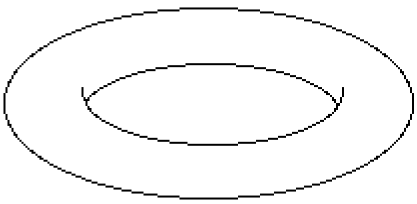
Examples (3.17)

1. The topology on $\mathbb{R}^2$ as a product of the usual topologies on the copies of $\mathbb{R}$ is the usual topology (obtained from, say, the metric d_2).

   Proof

   The sets of the basis are open rectangles, and an ε-neighbouhood U in the metric d_2 is a disc. It is easy to see that every point of U can be contained in a small open rectangle lying inside the disc. Hence U is a union of (infinitely many!) of these rectangles and hence is in the product topology. Since every open set in the d_2 metric is a union of ε-neighbourhoods, every open set can be written as a union of the open rectangles.

2. A torus is the surface in $\mathbb{R}^3$:

   It can also be regarded as the product $S^1 \times S^1$ where S^1 is a circle (the curve, not the interior) in $\mathbb{R}^2$. In this way it can be thought of as a subset of $\mathbb{R}^4$.

   The topology on S^1 is the subspace topology as a subset of $\mathbb{R}^2$ and so we get the product topology on $S^1 \times S^1$.

   Fortunately this is the same as the topology on the torus thought of as a subset of $\mathbb{R}^3$.

Proof

A basis for the subspace topology on S^1 is the set of "arcs"

Hence a basis for the product topology on $S^1 \times S^1$ is sets of the form:

A basis for the subspace topology on the torus as a subset of $\mathbb{R}^3$ is the intersection of the torus with ε-neighbourhoods of $\mathbb{R}^3$ (which are "small balls") and hence are sets of the form: As before, one can get these "ovals" as unions of the small "bent rectangles".

3. Take the topology $\mathcal{T}\{\phi, \{a, b\}, \{a\}\} on\ X = \{a, b\}$.

Then the product topology on $X \times X\ is\ \{\phi,\ X \times X, \{(a, a)\}, \{(a, a), (a,\ b)\}, \{(a, a), (b, a)\},\ \{(a, a), (a, b), (b, a)\}\ \}$ where the last open set in the list is not in the basis.

Remark

Given any product of sets $X \times Y$, there are projection maps $\pi_X\ and\ \pi_Y$ from $X \times Y$ to X and to Y given by $(x,\ y) \ \rightarrow\ x\ and\ (x,\ y)\ \rightarrow\ y$.

The product topology on $X \times Y$ is the weakest topology (fewest open sets) for which both these maps are continuous.

Example (3.18)

Let $X = \{1, 2, 3, 4, 5\}$. Then , which of the following is topology on X?

(a) $T_1 = \{X,\ \phi,\ \{1,\ 2,\ 3\},\ \{2,\ 3,\ 4\},\ \{1,\ 5\}\}$ is topology on X.

(b) $T_2 = \{X,\ \phi,\ \{1\},\ \{2\},\ \{1,\ 2\},\ \{4,\ 5\},\ \{1,\ 2,\ 4,\ 5\},\ \{1,\ 4,\ 5\}\}$

(c) $T_3 = \{X,\ \phi,\ \ \{1,\ 2,\ 3\},\ \{1,\ 2,\ 4\},\ \{1,\ 3,\ 5\}\}$

(d) None of the above.

Solution : (b)

Hence, $\phi,\ X \in\ T_1 \cap T_2 \cap\ T_3$

Now,

For each $G_1, G_2\ \in\ T_2$, we find $G_1 \cup\ G_2 \in\ T_2\ and\ G_1 \cap\ G_2 \in\ T_2$. Hence T_2 is topology on X but T_1 and T_3 are not topology on X as $\{1, 2, 3\}, \{2, 3, 5\}\ \in\ T_1$ but $\{1, 2, 3\} \cap \{2, 3, 5\} = \{2, 3\} \notin\ T_1$ and $\{1, 2, 3\},\ \{1, 2, 4\} \in\ T_3$ but $\{1, 2, 3\} \cap \{1, 2, 4\} = \{1, 2\} \notin\ T_3$.

Example (3.19)

Let (X, T) be a topological space where $X = \{1, 2, 3, 4, 5\}$ and $T_2 = \{X, \phi, \{1\}, \{2\}, \{1, 2\}, \{4, 5\}, \{1, 2, 4, 5\}, \{1, 4, 5\}, \{2, 4, 5\}\}$. Then , which of the following is neighbourhood of 2 ?

(a) $\{1, 3\}$ (b) $\{3, 4, 5\}$ (c) $\{1, 2, 4, 5\}$ (d) None of these.

Solution – (c)

Since, $\{1, 2, 3, 5\} \in T$.

Hence $\{1, 2, 3, 5\}$ is open set.

Hence, $\{1, 2, 3, 5\}$ is neighbourhood of each of element belonging $\{1, 2, 3, 5\}$. Hence, $\{1, 2, 3, 5\}$ is neighbourhood of 2.

Example (3.20)

Which of the following (s) is/are true?

(a) The collection $S = \{t_1^{-1}(U) | U \ open \ in \ X\} \cup \{t_2^{-1}(V) | V \ open \ in \ V\}$, is a sub – basis for the product topology on $X \times Y$.

(b) If $\mathbb{B}$ is a basis for the topology to X, then $B_y = \{B \cap Y | B \in \mathbb{B}\}$

(c) If Y is a subspace of X. If U is open in Y and V is open in X, then U is open in X.

(d) None of the above.

Solution: (a), (b), (c)

(a) Let T denote the product topology on $X \times Y$. Let T be the topology generated by the S. Since, every element of S belongs to T, so do arbitrary unions of finite intersections of elements of S. Thus, $T' \subset T$ on the other hand, every basis element U_{XV}. For the topology T is a finite intersection of elements of S, since $U_{XV} = t_1^{-1}(U) \cap t_2^{-1}(V)$.

Hence, $U_{XV} \in T'$, hence $T \subset T'$. Hence, S is a sub – basis for product topology on $X \times Y$.

(b) Given, U open X and given $Y \in U \cap Y$, we can choose an element B of such that $Y \in B \subset U$. Then $Y \in B \cap V \subset U \cap Y$

Hence, $\mathbb{B}_y$ is a basis for subspace topology on Y.

(c) Since, U is open in $Y, U = Y \cap V$ for some set V open in X. Since, $Y \ and \ V$ are both in $X, so \ is \ Y \cap V$.

Example (3.21)

Which of the following $g(s)$ is/are true?

(a) The subset $[a, b]$ of R is closed.

(b) In the plane $\mathbb{R}^2$, the set $\{X \times Y \,|X \geq 0 \;\; and \;\; Y \geq 0\}$ is closed.

(c) In the discrete topology on the set X, every set is open.

(d) None of the above.

Solution : (a), (b), (c)

(a) Since, I $\mathbb{R} - [a, b] = (-\infty, a) \cup (b, \infty)$ is open. Hence, $[a, b]$ is closed in $\mathbb{R}$.

(b) Since the complement of $\{X \times Y \,|\, X \geq 0 \; and \;\; Y \geq 0\}$ in $\mathbb{R}^2$ is the union of the two sets $(-\infty, 0) \times \mathbb{R} \;\; and \;\; \mathbb{R} \times (-\infty, 0)$ each of which is a product of open set of $\mathbb{R}$ and is, therefore, open in $\mathbb{R}$.

(c) Obviously, in the discrete topology on the set X, every set is open.

Example (3.22)

Let X be the real line $\mathbb{R}$. If $A = (0, 1)$. Then

(a) $\bar{A} = (0, 1]$ (b) $\bar{A} = (0, 1)$ (c) $\bar{\bar{A}} = [0, 1]$ (d) $\bar{A} = [0, 1)$

Solution : (c)

Since, every neighbourhood of O intersects A, while every point outside [0, 1] has a neighbourhood disjoint form A. Hence, $\bar{A} = [0, 1]$.

Example (3.23)

Let $F : A \to X \times Y$ be given by the equation $F(a) = \big(F_1(a), F_2(a)\big) \;\; and \;\; F|S$ continuous, then –

(a) $F_1 : A \to X$ is continuous.

(b) $F_1 : A \to X$ need not be continuous.

(c) $F_2 : A \to X$ is continuous.

(d) $F_2 : A \to X$ need not be continuous.

Solution: (a), (c)

Let $t_1 : X \times Y \to X$ and $t_2 : X \times Y \to Y$ projection onto the first and second that for each $a \in A$.

$F_1(a) = \pi_1\big(F(a)\big) \; and \; F_2(a) = \pi_2\big(f(a)\big)$. If F is continuous functions and therefore continuous.

Example (3.24)

Which of the following (s) is/are correct?

(a) If A be a connected subspace of X and $A \subset B \subset \bar{A}$, then B is also connected.

(b) The union of a collection of connected subspaces of X that have a point in common is connected.

(c) The image of a connected space under a continuous map is connected.

(d) None of the above.

Solution: (a), (b), (c)

(a) Let A be connected and $A \subset B$ suppose that $B = C \cup D$ is a separation of B. Hence, the set A must lie entirely in C or in D. Suppose the ,$A \subset C$. Then, $A \subset C$; since C and D are disjoint, B cannot intersect D. This contradict the fact that D is a non-empty subset of B. Hence, B is also connected.

(b) Let $\{A_a\}$ be a collection of connected subspace of a space X. Let $P \in \cap A_a$. We prove that space $Y = U A_a$ is connected. Suppose that $Y = C \cup D$ is a separation of Y. The point P is in one of the sets C or D; suppose $P \in C$. Since, A_a is connected, it must lie entirely in either C or D and it cannot lie in D as it contains the point P to C.

Therefore, $A_a \subset C$, $\forall$ a. Hence, $U A_a C$, which contradict the fact that D is non-empty. Hence, Y is connected. Hence the arbitrary union of connected set is connected.

(c) Let $f : X \to Y$ be a continuous map and X be connected. Since, the map obtained from f by restricting its range to the space Z is also continuous, it suffices to consider the case of a continuous surjective map $g : X \to Z$.

Suppose that, $Z = A \cup B$ is a separation of Z into two disjoint non-empty sets, open in Z. Then, $g^{-1}(A)$ and $g^{-1}(B)$ are disjoint sets, whose union is X; they are open in X because g is continuous and non-empty because g is surjective. Hence, they form a separation of X, which contradict the assumption of X is connected. Hence, image of a connected space under a continuous map is connected.

Exercise – 2

1. Let a, b, c be arbitrary distinct points of the set R. Let $X = \{a, b, c\}$ and $\mathcal{T} = \{\phi, X, \{a\}, \{b, c\}\}$, then which of the followings is/are true?

(a) $(X, \mathcal{T})$ is a topological space.

(b) $(X, \mathcal{T})$ is a T_1 – space.

(c) $(X, \mathcal{T})$ is a regular space.

(d) $(X, \mathcal{T})$ is a normal space.

Ans:a, b, c be arbitrary distinct points of the set R and $X = (a, b, c)$ $and\ \mathcal{T} = \{\phi, X, \{a\}, \{b, c\}\}$. Then, $(X,\ \mathcal{T})$ is a topological space. Evidently, $\{a\}, \{b, c\}$ $and\ \mathcal{T} - open$ sets.
$\therefore X - \{a\},\ X - \{b, c\}$ are $\mathcal{T} - closed$ sets. i.e., $\{b, c\}, \{a\}$ are $\mathcal{T} - closed$ sets. Thus, $\{a\}$ and $\{b, c\}$ are $\mathcal{T} - closed\ sets$ as well as $\mathcal{T} - closed\ sets$. Consider a pair of distinct elements $b, c \in X$. Then, the only open sets containing either of the elements $b, care\ X, \{b, c\} S.t. \in X, b \in \{b, c\};\ \ e \in X, C \in \{b, c\}$.

This proves that $(X, \mathcal{T})$ is not a $F_1 - space$. Given, a pair of disjoint closed sets $\{a\}, \{b, c\} \subset X$, we can find a pair of disjoint open sets $\{a\},\ \{b, c\} \subset X\ \ S.t.$

Closed set $\{a\} \subset open\ set\ \{a\}$.

Closed set $\{b, c\} \subset open\ set\ \{b, c\}$.

This proves that $(X, \mathcal{T})$ is a normal space. Repeating the same arguments we can show this example, it is clear from a regular space need not be a $T_1 - space$ as well as normal space need not be a $T_1 - space$.

2. Define topology $\mathcal{T}$ on $N\ \ S.t.$

(i) $\phi\ N \in \mathcal{T}$

(ii) $A_n \in \mathcal{T},\ \forall\, n \in N, where\ \ A_n = \{1, 2, 3, \ldots\ldots, n\}$, then $(N, \mathcal{T})$ is

(a) $T_0 - space$

(b) $T_1 - space$

(c) $T_0\ and\ T_1 - space$

(d) Neither to nor $T_1 - space$.

Ans: - Consider $2, 4 \in N,\ \ A_2 \in \mathcal{T},\ 2 \in A_2\ and\ 4 \notin A_2$.

Given, two distinct numbers $2, 4, \exists$ open set A_2S. t. $2 \in A_2\ but\ \ 4 \notin A_2$.

$$\therefore\ (N, \mathcal{T})\ is\ T_0 - space.\ A_4 \in \mathcal{T},\qquad 2 \in A_4, 4 \in A_4$$

Hence, $\exists$ no open set which contains 4 but not 2.

Thus, $(N, \mathcal{T})\ is\ not\ T_1 - space$.

3. Let $X = \{a, b, c\},\ \mathcal{T} = \{\phi, \{a\}, \{a, b\}, X\},\ then\ (X, \mathcal{T})$ is a

(a) compact space

(b) nausdorft space

(c) connected space

(d) disconnected space

Ans: Since, every topological space $(X, \mathcal{T})$ is compact if X is finite also,

$$\{a\} \cap \{a, b\} = \{a\} \neq \phi$$

$$\{a\} \cup \{a, b\} = \{a, b\} \neq X$$

4. Let $X\ and\ Y$ be topological spaces and $f : X \to Y$ is continuous. Then ,

(a) for every subset $A\ of\ X$, one has $f(\bar{A}) \subseteq \overline{f(A)}$

(b) for every closed set $B\ of\ Y$, the set $f^{-1}(B)$ is closed in X.

(c) for each $X \in X$ and each neighbourhood $V\ of\ f(x)$, there is a neighbourhood $U\ of\ X$ such that $f(U) \subset V$.

(b) None of the above.

Ans: Assume that f is continuous. Let A be a subset of X. We show that if $X \in A,\ then\ f(X) \in (A)$. Let V be a neighbourhood of $f(X)$. Then, $f^{-1}(V)$ is an open set of X continuous X; it must intersect A in some point Y. Then a intersect $f(A)$ in the point $f(Y)$. Hence, $f(X) \in f(A)$.

Hence, $f(\bar{A}) \subseteq \overline{f(A)}$

Now, Let B be closed set in Y and let $A = f^{-1}(B)$. We wish to prove that A is closed in X. We should that $\bar{A} = A$, by elementary set ------,

We have ,$f(A) = f\left(f^{-1}(B)\right) \subset B$

Therefore if $X \in \bar{A}$

$f(X) \in f(\bar{A}) \subset \overline{f(A)} \subset \bar{B} = B \Rightarrow X \in f^{-1}(B) = A$

Hence, $\bar{A} \subset A \Rightarrow \bar{A} = A$

Hence, $f^{-1}(B)$ is closed in X.

Now, let $X \in X\ and\ V$ be a neighbourhood of $f(X)$. Then , the set $U = f^{-1}(V)$ is a neighbourhood of X such that $f(U) \subset V$.

5. Let $X, Y\ and\ Z$ be topological spaces, then

(a) If $f : X \to Y$ maps all of X in to the single point Y_0 of Y, then its continuous.

(b) If A is subspace of X, the inclusion function $j : A \to X$ is continuous.

(c) If $f : X \to X\ and\ g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous

(d) None of the above.

Ans: (a) Let $f(x) = Y_0, \forall\ X \in X$. Let V be open in Y. The set $f^{-1}(V)$ equal to $X\ or\ \phi$, depending g on where then V contains $Y_0\ or\ not$, in either case, it is open, Hence, f is continuous.

(b) Let U be open set in X, then $j^{-1}(U) = U \cap A$, which is open in A by definition of subspace topology.

(c) Let U be open set in Z, then $g^{-1}(U)$ is open in Y and $f^{-1}(g^{-1}(U))$ is open in X. But $f^{-1}\left(g^{-1}(U)\right) = (g \circ f)^{-1}(U)$.

Hence, $g \circ f$ is continuous.

6. Let $X, Y\ and\ Z$ be topological spaces. Then,

(a) if $f : X \to Y$ is continuous and if A is a subspace of X, then the restricted function $f|A : A \to Y$ is continuous.

(b) The maps $f : X \to Y$ is continuous, if X can be written as the union of open sets U_a such that $f\,|U_a$ is continuous for each a.

(c) Both (a) and (b) are true.

(d) Neither (a) nor (b) is true.

Ans: The function $f \mid A$ equals the composite of the inclusion map $j : A \to X$ and the maps $f : X \to Y$ both of which are continuous.

Next, by Hypothesis; we can write X as a union of open sets U_a, such that $f \mid U_a$ is continuous for each a, let V be an open set in Y. Then, $f^{-1}(V) \cap U_a = (f\,|U_a)^{-1}(V)$ because both expressions represent set of those points X lying in U_a for which $f(X) \in V, since\ f \mid U_a$ is continuous, this set is open in U_a and hence open in X. But

$$f^{-1}(V) = U_a(f^{-1}(V) \cap U_a)$$

Hence, $f^{-1}(V)$ is also open in X.

7. If X is a topological space and if $f, g : X \to R$ are continuous functions. Then,

(a) $f + g$ is continuous.

(b) $f - g$ is continuous.

(c) $f \cdot g$ is continuous.

(d) $\frac{f}{g}$ is continuous provided $g(X) \neq 0, \forall\ X$

Ans: The map $h : \mathbb{R} \times \mathbb{R}$ given by $h(X) = f(X) \times g(X)$ is continuous. The function $f + g$ equals the composite of h and the addition operation $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; therefore $f + g$ is continuous. Similarly, $f - g,\ f \cdot g\ and\ \frac{f}{g}$ are continuous maps.

8. Which of the following(s) is/are connected?

(a) The subspace $[-1, 1]$ of the real $|inc\ \mathbb{R}$.

(b) The subspace of rational of real $|inc\ \mathbb{R}$.

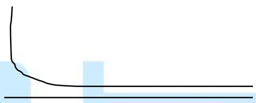
(c) The subspace $X = \{X \times Y\ |Y = 0\} \cup \{X \times Y\ |X > 0\ and\ Y = \frac{1}{x}\}$ of plane $\mathbb{R}^2$.

(d) None of the above.

Ans: (a) The sets $[-1, 0]$ and $[0, 1]$ are disjoint and non-empty but the do not form a separation of X because the first set is not open in $[-1, 1]$. Hence, the subspace $[-1, 1]$ of real $|inc\ \mathbb{R}$ is connected.

(b) Let Y be a subspace of Q containing two points $p\ and\ q$, one can choose an irrational number a lying between $p\ and\ q$, and write Y as the union of the open sets $Y \cap (-\infty, a)\ and\ Y \cap (a, \infty)$. Hence, Q is disconnected.

(c) X is not connected as the two indicated sets form separation of X because neither contains a limit point of other shown in the following figure.



9. Let X be a topological space. Then,

(a) X is locally connected implies for every open set $U\ of\ X$, each component of $U$is open in X.

(b) If for every open set $U\ of\ X$, each component of U is open in X, then is locally connected.

(c) Exactly only one of above.

(d) Neither (a) nor (b) is true.

Ans: Suppose that, X is locally connected. Let U be an open set in X, let C be a component of U. If X is a point of C, we can choose a connected neighbourhood $V\ of\ X$ such that $V \subset U$. Since, V is connected, it must lie entirely in the component $C\ of\ U$. Hence, C is open in X.

Next, suppose that, components of open sets in X are open. Given, a point $X \circ F \circ X$ and a neighbourhood $U\ of\ X$, let C be the component of U continuing X. Now, C is connected. Since, it is open in X by hypothesis. Hence, X is locally connected at X.

10. Which of the following(s) is/are compact?

(a) The real $|inc$ $\mathbb{R}$.

(b) The subspace $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$ of $\mathbb{R}$.

(c) The interval $(0, 1]$.

(d) None of the above.

Ans: (a) The real $|inc$ $\mathbb{R}$ is not compact, for the covering of $\mathbb{R}$ of open intervals $A = \{(n, n+2) | \, n \in \mathbb{Z}\}$ contains no finite sub-collection that covers $\mathbb{R}$.

(b) Given, an open covering A of X, there is an element U of A containing U. The set U contains all but finitely many of the points $\frac{1}{n}$; choose for each point of X not in U, an element of A containing it. The collection consisting of these elements of A, along with the element U, is a finite sub-collection of A that covers X.

(c) The open covering $A = \left\{\left[\frac{1}{n}, 1\right] n \in \mathbb{Z}_+\right\}$ contains no finite sub-collection covering $(0, 1]$.