Partial Differential Equation

June – 2014

(1) Let $x = x(s), y = y(s), u = u(s), s\epsilon\mathbb{R}$, be the characteristic curve of the PDE

$$\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) - u = 0$$

Passing through a given curve

$x = 0, y = \tau, u = \tau^2, \tau \cdot \epsilon\mathbb{K}$. Then the characteristics are given by

(a) $x = 3\tau(e^s - 1), y = \frac{\tau}{2}(e^{-s} + 1), u = \tau^2 e^{-2s}$

(b)$x = 2\tau(e^{-s} - 1), y = \tau(2e^{2s} - 1), u = \frac{\tau^2}{2}(1 + e^{-2s})$

(c) $x = 2\tau(e^s - 1), y = \frac{\tau}{2}(e^s + 1), u = \tau^2 e^{2S}$

(d) $x = \tau(e^{-s} - 1), y = -2\tau\left(e^{-s} - \frac{3}{2}\right), u = \tau^2(2e^{-2s} - 1)$

Answer: (c)

Solution: for first option (a)

$\frac{\partial u}{\partial x} = \tau^2(-2)e^{-2s} \cdot \frac{\partial s}{\partial x}$

$= \frac{-2\tau^2 e^{-s}}{3\tau\, e^s} = -\frac{2}{3}\tau e^{-2s}$

$\frac{\partial u}{\partial y} = \frac{-2\tau^2 e^{-2s}}{-\frac{\tau}{2}e^{-s}} = 4\tau\, e^{-s}$

So, $\left(\frac{\partial u}{\partial x}\right) \cdot \left(\frac{\partial u}{\partial x}\right) - u = 0$ does not satisfied.

Similarly, option (b) does not satisfy the given equation.

For the option (c)

$\frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} = \frac{2\tau^2 e^{2s}}{2\tau\, e^s} \cdot \frac{2\tau^2 e^{2s}}{\frac{\tau}{2}e^s} = x + 4y = 2\tau \cdot 2e^s = 4\tau e^s$

$\therefore (x + 4y)^2 = 16\,\tau^2 e^{2s} = 16u$

$\therefore u = \frac{1}{16}(x + 4y)^2$

$\frac{\partial u}{\partial x} = \frac{1}{8}(x + 4y), \frac{\partial u}{\partial y} = \frac{1}{2}(x + 4y)$

$\therefore \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} = \frac{1}{16}(x + 4y)^2 = u$ satisfied

So, the option (c) is correct.

(2) The initial value problem
$\frac{\partial u}{\partial t} + x\,\frac{\partial u}{\partial x} = x, 0 \le x \le 1, t > 0$ and $u(x,0) = 2x$ has
(a) A unique solution $u(x,t)$ which $\to \infty\ as\ t \to \infty$.
(b) More than one solution.
(c) A solution which remains bounded as $t \to \infty$.
(d) No solution.
Answer: (c)
Solution: $q + xp = x$ $\quad [Here\ P = x, Q = 1, R = x]$
i.e. $p\,P + q\,Q = R$
So, Lagrange's auxiliary equations are $\frac{dx}{x} = \frac{dt}{1} = \frac{du}{x}$
$\therefore \frac{du}{x} = \frac{dx}{x} \Rightarrow du = dx \Rightarrow x = u + c_1$
$u = x - c_1 = x + c$
$\frac{dx}{x} = \frac{dt}{1}$
$\log x = t + \log c_2$
$x = c_2 e^t$
$xe^{-t} = c_2$
General solution is $\phi(xe^{-t}) = u - x$
$u(x,0) = 2x \quad \Rightarrow \phi(x) = 2x - x = x$
$\phi(x) = x$
$\therefore u(x,t) = x + xe^{-t}$
$\therefore \phi(xe^{-t}) + \phi(x) = u$
$\phi(xe^{-t} + x) = x + xe^{-t} \Rightarrow u(x,t) = x + xe^{-t}$
$\lim_{t\to\infty} u\,(x,t) = x + 0 = x \quad \forall\, x\, \epsilon [0,1]$
$\therefore u(x,t)$ is bounded on $[0,1]$ as $t \to \infty$.
Hence, the option (c) is correct.

(3) Let $xyu = c_1\ and\ x^2 + y^2 - 2u = c_2$, where c_1 and c_2 are arbitrary constants, be the first integrals of the PDE.
$x(u + y^2)\frac{\partial u}{\partial x} - y(u + x^2)\frac{\partial u}{\partial y} = (x^2 - y^2)u$. Then the solution of the PDE with $x + y = 0, u = 1$ is given by
(a) $x^3 + y^3 + 2xyu^2 + 2x^2u = 0$
(b) $x^3 + yx^2 + (x^2 + xy)u = 0$
(c) $x^2 + y^2 + 2(xy - 1)u + 2 = 0$
(d) $x^2 - y^2 - u(x + y - 2) - 2 = 0$
Answer: (c)
Solution: $uxy = c_1,\ x^2 + y^2 - 2u = c_2$
$x + y = 0, u = 1$
Let $x = t, y = -t, u = 1$
$\therefore -t^2 = c_1$
$t^2 + t^2 - 2 = c_2 or, 2t^2 - 2 = c_2$
$or, 2(-c_1) - 2 = c_2$
$or, -2c_1 - 2 = c_2$
$or, 2c_1 + c_2 + 2 = 0$

$or, 2(xyu) + (x^2 + y^2 - 2u) + 2 = 0$
$or, x^2 + y^2 + 2u(xy - 1) + 2 = 0$
Hence the option (c) is correct.

(4) Let $u(x,t)$ be the solution of the equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, which tends to zero as $t \to \infty$ and has the value $\cos(x)$ when $t = 0$ then
(a) $u = \sum_{n=1}^{\infty} a_n \sin(nx + b_n)\, e^{-nt}$ where a_n , b_n are arbitrary constants.
(b) $u = \sum_{n=1}^{\infty} a_n \sin(nx + b_n)\, e^{-n^2 t}$ where a_n, are non-zero constants.
(c)$u = \sum_{n=1}^{\infty} a_n \cos(nx + b_n) e^{-nt}$ where a_n are not all zero and $b_n = 0$ for $n \geq 1$.
(d) $u = \sum_{n=1}^{\infty} a_n \cos(nx + b_n)\, e^{-n^2 t}$ where $a_1 \neq 0, a_n = 0\ for\ n > 1, and\ b_n = 0\ for\ n \geq 1.$

Answer: (d)

Solution: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$
$u(x,t) = (c_1 \cos nx + c_2 \sin nx)e^{-n^2 t}$
$u_n(x,t) = (a_n \cos nx + b_n \sin nx)e^{-n^2 t}$
$u(x,t) = \sum_{n=1}^{\infty} u_n\,(x,t)$
$\therefore u(x,t) = \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\, e^{-n^2 t}$
$u \to \infty\ as\ t \to \infty$
$u(x,0) = \cos x$
$u(x,0) = \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$
$\Rightarrow \cos x = a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots\cdots\cdots\cdots$
$\Rightarrow a_1 = 1, a_2 = a_3 = \cdots\cdots\cdots = 0, n \geq 1$
$b_n = 0$ for all $n \geq 1$
Hence, the option (d) is correct.

(5) The $PDE\ \frac{\partial^2 u}{\partial x^2} + 2\ \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$ is
(a) Parabolic and has characteristics.

$$\xi(x,y) = x + 2y, \eta(x,y) = x - 2y$$

(b) Reducible to the canonical form $\frac{\partial^2 u}{\partial \xi^2} = 0$, where $\xi(x,y) = x + 2y$.
(c) Reducible to the canonical form $\frac{\partial^2 u}{\partial \eta^2} = 0$, where $\eta(x,y) = x + y$
(d) Parabolic and has the general solution $u = (x - y)f_1(x + y) + f_2(x - y)$ where f_1, f_2 are arbitrary functions.
Answer: (c)
Solution: Here $A = 1, B = 2, C = 1$
$\therefore$ Discriminant $B^2 - 4AC = 4 - 4.1.1 = 0$
So, the given PDE is parabolic characteristics are $\frac{dy}{dx} = \frac{B}{2A} = \frac{2}{2} = 1$
Integrating, $y = x + c$
$or, y - x = c$
$\xi = y - x, \eta = y + x$
$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = \frac{-\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$
$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}$
$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}$
$2\frac{\partial^2 u}{\partial x \partial y} = 2\frac{\partial^2 u}{\partial \eta^2} - 2\frac{\partial^2 u}{\partial \xi^2}$
$\therefore 2\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}\right) + 2\left(\frac{\partial^2 u}{\partial \eta^2} - \frac{\partial^2 u}{\partial \xi^2}\right) = 0$
$or, \frac{\partial^2 u}{\partial \eta^2} = 0$
So, the option (c) is correct.

December – 2014

(1) Let $u(x,t) = e^{iwx}\, v(t)$ with $v(0) = 1$ a solution to $\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}$. Then
(a) $u(x,t) = e^{iw(x-w^2t)}$
(b) $u(x,t) = e^{iw\,x-w^2t}$
(c) $u(x,t) = e^{iw(x+w^2t)}$
(d) $u(x,t) = e^{iw^3(x-t)}$

Answer: (a)
Solution: Option (a) $\rightarrow u(x,t) = e^{iw\,(x-w^2t)}$
$\therefore \frac{\partial u}{\partial t} = e^{iw(x-w^2t)} \cdot iw \cdot (-w^2)$
$\frac{\partial u}{\partial x} = e^{iw(x-w^2t)} \cdot iw$
$\frac{\partial^2 u}{\partial x^2} = e^{iw(x-w^2t)} \cdot (iw)^2$
$\frac{\partial^3 u}{\partial x^3} = e^{iw(x-w^2t)\cdot(iw)^3}$
$= e^{iw(x-w^2t)\cdot(-iw^3)}$
So, $\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}$
So, the option (a) is correct.

(2) The Charpit’s equations for the PDE
$up^2 + q^2 + x + y = 0, p = \frac{\partial u}{\partial x}, q = \frac{\partial u}{\partial y}$ are given by
(a) $\frac{dx}{-1-p^3} = \frac{dy}{-1-qp^2} = \frac{du}{2p^2u+2q^2} = \frac{dp}{2pu} = \frac{dq}{2q}$
(b) $\frac{dx}{2pu} = \frac{dy}{2q} = \frac{du}{2p^2u+2q^2} = \frac{dp}{-1-p^3} = \frac{dq}{-1-qp^2}$
(c) $\frac{dx}{up^2} = \frac{dy}{q^2} = \frac{du}{0} = \frac{dp}{x} = \frac{dq}{y}$
(d) $\frac{dx}{2q} = \frac{dy}{2pu} = \frac{du}{x+y} = \frac{dp}{p^2} = \frac{dq}{qp^2}$

Answer: (b)

Solution: Here $f(x, y, u, p, q) = up^2 + q^2 + x + y$ Charapit's equations are
$\frac{dp}{f_x+pf_u} = \frac{dq}{f_y+q\,f_u} = \frac{du}{-pf_p-q\,f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$
$\therefore f_x = 1, f_y = 1, f_u = p^2, f_p = 2pu, f_q = 2q$
$\Rightarrow \frac{dp}{1+p^3} = \frac{dq}{1+qp^2} = \frac{du}{-2p^2u-2q^2} = \frac{dx}{-2pu} = \frac{dy}{-2q}$
So, the option (b) is correct.

(3) Consider the Cauchy problem of finding $u = u(x, t)$ such that
$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$ for $x \epsilon \mathbb{R}, t > 0 u(x, 0) = u_0(x), x \epsilon \mathbb{R}$
Which choices of the following functions for u_0 yield aC' solution $u(x, t)$ for all $x\epsilon\mathbb{R}$ and $t > 0$.
(a) $u_0(x) = \frac{1}{1+x^2}$
(b) $u_0(x) = x$
(c) $u_0(x) = 1 + x^2$
(d) $u_0(x) = 1 + 2x$

Answer: (b), (d)
Solution: $\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = 0$
Lagrange's equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$
$\therefore c\frac{dt}{1} = \frac{dy}{u} = \frac{du}{0} \Rightarrow du = 0 \Rightarrow u = c$
Also, $dt = \frac{dx}{c} \Rightarrow ct = x + c_1$
$x - ct = c_1 \rightarrow$ straight lines.
$u(x, t) = \phi(x - ct)$
$u(x, 0) = \phi(x)$
$\therefore \phi(x) = u_0(x)$
So, the option (b) and (d) are correct.

(4) Let $u = u(x, t)$ be the solution of the Cauchy problem $\frac{\partial u}{\partial t} + \left(\frac{\partial u}{\partial x}\right)^2 = 1, x\epsilon\mathbb{R}, t > 0 u(x, 0) = -x^2, x\epsilon\mathbb{R}$. Then
(a) $u(x, t)$ exists for all $x \epsilon \mathbb{R}$ and $t > 0$.
(b) $|u(x, t)| \rightarrow \alpha$ as $t \rightarrow t^*$ for some $t^* > 0$ and $x \neq 0$.
(c) $u(x, t) \leq 0$ for all $x\epsilon\mathbb{R}$ and for all $t < \frac{1}{4}$.
(d) $u(x, t) > 0$ for some $x\epsilon\mathbb{R}$ and $0 < t < \frac{1}{4}$.

Answer: (b), (d)
Solution: $\frac{\partial u}{\partial t} + \left(\frac{\partial u}{\partial x}\right)^2 = 1$
$\therefore q + p^2 = 1$, where $q = \frac{\partial u}{\partial t}, p = \frac{\partial u}{\partial x}$
Solution is $u(x, t) = ax + bt + c$
$p = a, q = b$
$\therefore b + a^2 = 1 \Rightarrow b = 1 - a^2$
$u(x, t) = ax + (1 - a^2)t + c$

$u(x, o) = -x^2$
$\therefore -x^2 = ax + c$
$c = -x^2 - ax$
$u(x, t) = ax + (1 - a^2)t - x^2 - ax = -x^2 + (1 - a^2)t$
$\frac{\partial u}{\partial a} = 0 \Rightarrow 0 = 0 + (-2a)t \Rightarrow a = 0$
$\therefore u(x, t) = -x^2 + t$
$\lim_{t\to\infty} |u(x, t)| \to \infty$
$\lim_{x\to\infty} u\,(x, t) \to \infty, u(x, t)$ does not exist.
If $t = \frac{1}{6}, x = 0, u(x, t) = \frac{1}{6} > 0$
So, the options (b) and (d) are correct.

(5) Let $u(x, t)$ satisfy for $x\, \epsilon\, \mathbb{R}$, $t > 0\, \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + 2\frac{\partial^2 u}{\partial x^2} = 0$. A solution of the form $u = e^{ix}\, v(t)$ with $v(0) = 0\ and\ u'(0) = 1$
(a) Is necessarily bounded
(b) Satisfies $|u(x, t)| < e^t$
(c) Is necessarily unbounded
(d) Is oscillatory in x.

Answer: (b), (c) and (d)
Solution: $u = e^{ix}\, v(t)$
$\frac{\partial u}{\partial t} = e^{ix} v'(t), \frac{\partial^2 u}{\partial t^2} = e^{ix} v''(t)$
$\frac{\partial u}{\partial x} = i\, e^{ix}\, v(t), \frac{\partial^2 u}{\partial x^2} = -e^{ix}\, v(t)$
So, $e^{ix}[v'' + v' - 2v] = 0 \Rightarrow v'' + v' - 2v = 0 \Rightarrow v(t) = c_1 e^{-2t} + c_2 e^t$
$v'(t) = -2c_1 e^{-2t} + c_2 e^t$
$v(0) = 0 \Rightarrow c_1 + c_2 = 0$
$v'(0) = 1 \Rightarrow -2c_1 + c_2 = 1$
$\therefore c_1 = -\frac{1}{3}, c_2 = \frac{1}{3}$
$\therefore u(x, t) = e^{ix} \cdot \frac{1}{3}[e^t - e^{-2t}]$
$= \frac{1}{3}\left[e^{ix+t} - e^{ix-2t}\right]$
So, the options (b), (c) and (d) are correct.

June–2015 (Part-B)

1. Let $a, b \epsilon \mathbb{R}$ be such that $a^2 + b^2 \neq 0$. Then the Cauchy problem $a\frac{ru}{rx} + b\frac{ru}{ry} = 1; x, y \epsilon \mathbb{R}, u(x, y) = x$ on $ax + by = 1$
(a) has more than one solution if either a or b is zero.
(b) has no solution.
(c) has a unique solution.
(d) has infinitely many solutions.

Answer: (c)

Solution:
$Pp + Qq = R$
Here $P = a,\ Q = b.$
$b \neq 0,\ (\xi,\ \frac{1-a\xi}{b},\ \xi)$
$\frac{P}{\frac{\partial x_0}{\partial \xi}}, \frac{Q}{\frac{\partial y_0}{\partial \xi}}, \frac{R}{\frac{\partial z_0}{\partial \xi}}$
i.e., $\frac{a}{1} \neq \frac{b}{-a/b} \neq \frac{1}{1}$
Jacobian (J) $= \begin{vmatrix} P & \frac{\partial x_0}{\partial \xi} \\ Q & \frac{\partial y_0}{\partial \xi} \end{vmatrix} \neq 0$
So, it has a unique solution. Option (c) is correct.

2. Consider the initial value problem $\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0,\ u(0, y) = 4e^{-2y}$. Then the value of $u(1, 1)$is
(a) $4e^{-2}$
(b) $4e^{2}$
(c) $2e^{-4}$
(d) $4e^{4}$

Answer: (b)
Solution:
$(D + 2D^1)u = 0$
Lagrange's equations are $\frac{dx}{1} = \frac{dy}{2} = \frac{dz}{0}$, $z = c_1$
$2x - y = c_2,\quad z = c_1$
Let $\left(0, \xi, 4e^{-2\xi}\right) \Rightarrow$ transformation
$c_1 = 4e^{-2\xi},\ c_2 = -\xi$
$\therefore c_1 = 4e^{2c_2}$
$z = 4e^{2(2x-y)}$
$\therefore u(1,1) = 4e^{2}$
So, option (b) is correct.

Part-C

3. For an arbitrary continuously differentiable function f, which of the following is a general solution of $z(px - qy) = y^2 - x^2$.

(a) $x^2 + y^2 + z^2 = f(xy)$
(b) $(x + y)^2 + z^2 = f(xy)$
(c) $x^2 + y^2 + z^2 = f(y - x)$
(d) $x^2 + y^2 + z^2 = f((x + y)^2 + z^2)$

Answer: (a), (b) and (d)
Solution: $zxp + (-zy)q = y^2 - x^2$
Lagrange's equations are $\frac{dx}{zx} = \frac{dy}{-zy} = \frac{dz}{y^2-x^2} \Rightarrow \frac{dx}{x} = \frac{-dy}{y}$, Integrating $xy = c_1$
Also, $\frac{dx+dy}{zx-zy} = \frac{dz}{y^2-x^2}$
$(x + y)(dx + dy) + zdz = 0$
Integrating, $\frac{(x+y)^2}{2} + \frac{z^2}{2} = \frac{c_2}{2}$
$or, z^2 + (x + y)^2 = c_2$
General solution is
$f(c_1, c_2) = 0$
$or, (xy, (x + y)^2 + z^2) = 0$
$\Rightarrow (x + y)^2 + z^2 = f(xy)$
$\Rightarrow$ Options (a), (b) and (d) are correct.

4. The second order partial differential equation $u_{xx} + xu_{yy} = 0$ is

(a) elliptic for $x > 0$
(b) elliptic for $x < 0$
(c) hyperbolic for $x > 0$
(d) hyperbolic for $x < 0$

Answer: (a), (d)
Solution: Discriminant $B^2 - 4AC = 0 - 4.1.x = -4x$
$< 0\ for\ x > 0$ → Elliptic
$> 0\ for\ x < 0$ → Hyperbolic
So, option (a) and (d) are correct.

5. Which of the following are complete integrals of the partial differential equation $pqx + yq^2 = 1$?

(a) $z = \frac{x}{a} + \frac{ay}{x} + b$

(b) $z = \frac{x}{b} + \frac{ay}{x} + b$

(c) $z^2 = 4(ax + y) + b$

(d) $(z - b)^2 = 4(ax + y)$

Answer: (a), (d)

Solution: $z = \frac{x}{a} + \frac{ay}{x} + b$

$\therefore p = \frac{1}{a} - \frac{ay}{x^2}, q = \frac{a}{x}$

So, $pqx + yq^2 = \left(\frac{1}{a} - \frac{ay}{x^2}\right) \cdot \frac{a}{x} \cdot x + y \cdot \frac{a^2}{x^2}$

$= 1 - \frac{a^2y}{x^2} + \frac{a^2y}{x^2} = 1$

$z = \frac{x}{b} + \frac{ay}{x} + b$

$\therefore p = \left(\frac{1}{b} - \frac{ay}{x^2}\right), q = \frac{a}{x}$

So, $pqx + yq^2 = \left(\frac{1}{b} - \frac{ay}{x^2}\right) \cdot \frac{a}{x} \cdot x + y \cdot \frac{a^2}{x^2} = \frac{a}{b} - \frac{a^2y}{x^2} + \frac{a^2y}{x^2} = \frac{a}{b} \neq 1$

Now, $z^2 = 4(ax + y) + b$

$or, 2zp = 4a \Rightarrow p = \frac{2a}{z}, 2zq = 4$

$q = \frac{2}{z}$

Now, $pqx + yq^2 = \frac{2a}{z} \cdot \frac{2}{z} \cdot x + y\frac{4}{z^2} = \frac{4ax+4y}{z} \neq 1$

$(z - b)^2 = 4(ax + y)$

$\therefore 2(z - b) \cdot p = 4a or, p = \frac{2a}{z-b}$

and $2(z - b)q = 4 or, q = \frac{2}{z-b}$

Now, $pqx + yq^2 = \frac{2a}{z-b} \cdot \frac{2}{z-b} \cdot x + y\frac{y}{(z-b)^2} = \frac{4ax+4y}{(z-b)^2} = \frac{4(ax+y)}{4(ax+y)} = 1$

So, the options (a) and (d) are correct.

December – 2015

1. The $PDE\, \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = X$, has
(a) Only one particular integral.
(b) A particular integral which is linear in x and y.
(c) A particular integral which is a quadratic polynomial in $x\ and\ y$.
(d) More than one particular integral.
Answer: (d)
Solution: $\left(\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) u = x$
Let $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) u = z$
$\therefore \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = x$
$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{x} \Rightarrow x - y = c_1$
and $\frac{x^2}{2} - z = c_2$
Solution is $f(c_1, c_2) = 0$
$z = \frac{x^2}{2} = f(x - y)$
Also, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{x^2}{2} + f(x - y)$
$\frac{dx}{1} = \frac{dy}{1} = \frac{dx}{\frac{x^2}{2} + f(x-y)}$
$\Rightarrow x - y = c_3$ and $\left(\frac{x^2}{2} + f(c_3)\right) dx = du$
$\Rightarrow \frac{x^3}{3} + x\, f(x - y) = u$
So, there exist more than one particular integral.
Option (d) is correct.

2. the solution of the initial value problem $(x - y)\frac{\partial u}{\partial x} + (y - x - u)\frac{\partial u}{\partial y} = u, u(x, 0) = 1$Satisfies
(a) $u^2(x - y + u) + (y - x - u) = 0$
(b) $u^2(x + y + u) + (y - x - u) = 0$
(c) $u^2(x - y + u) - (x + y + u) = 0$
(d) $u^2(y - x + u) + (x + y - u) = 0$
Answer: (b)
Solution: Lagrange's equations are
$\frac{dx}{x-y} = \frac{dy}{y-x-u} = \frac{du}{u} \Rightarrow dx + dy + du = 0 or, x + y + u = c_1$
$or, x + u = c_1 - y$
$u(x, 0) = 1 \Rightarrow x + 1 = c_1$
Also, $\frac{dy}{y-c_1+y} = \frac{du}{u} \Rightarrow \frac{dy}{2y-c_1} = \frac{du}{u} \Rightarrow \frac{1}{2}\log(2y - c_1) = \log u + \log c_2 \Rightarrow \frac{(2y-c_1)^{\frac{1}{2}}}{u} = c_2$
$\Rightarrow \frac{(2y-x-u-y)^{\frac{1}{2}}}{u} = c_2 \Rightarrow \frac{(y-x-u)^{\frac{1}{2}}}{u} = c_2$

Also, $u(x,0) = 1 \Rightarrow c_2 = (-x-1)^{\frac{1}{2}}$
$c_2^2 = -x-1$
$\therefore \frac{(y-x-u)}{u^2} = c_2^2 = -(x+1) = -c_1 = -(x+y+u)$
$or, u^2(x+y+u) + y - x - u = 0$
So, the option (b) is correct.

3. Let $u(x,t)$ satisfy the wave equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$; $x \epsilon (0, 2\pi), t > 0$
$u(x,0) = e^{iwx}$ $for\ some\ w\ \epsilon \mathbb{R}$. Then
(a) $u(x,t) = e^{iwx}e^{iwt}$
(b) $u(x,t) = e^{iwx}e^{-iwt}$
(c) $u(x,t) = e^{iwx}\left(\frac{e^{iwz}+e^{-iwz}}{2}\right)$
(d) $u(x,t) = t + \frac{x^2}{2}$

Answer: (a), (b) and (c)
Solution: Clearly, in the options (a), (b) and (c) $u(x,t)$ satisfies the equation and the initial conditions.
So, the options (a), (b) and (c) are correct.

4. Let $u(x,y)$ be the solution of the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, which tends to zero as $y \to \infty$ and has the value $\sin x$ when $y = 0$. Then
(a) $u = \sum_{n=1}^{\infty} a_n \sin(nx + b_n)\, e^{-ny}$, where a_n are arbitrary and b_n are non-zero constants.
(b) $u = \sum_{n=1}^{\infty} a_n \sin(nx + b_n)\, e^{-n^2 y}$,where $a_1 + 1$ and $a_n(n > 1), b_n$ are non-zero constants.
(c) $u = \sum_{n=1}^{\infty} a_n \sin(nx + b_n)\, e^{-ny}$,where $a_1 = 1, a_n = 0$ for $n > 1$ and $b_n = 0$ for $n \geq 1$
(d) $u = \sum_{n=1}^{\infty} a_n \sin(nx + b_n)\, e^{-n^2 y}$, where $b_n = 0$ for $n \geq 0$ and a_n are all non-zero.
Answer: (c)
Solution: $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} = 0, u \to 0\ as\ y \to \alpha$
$u(x,0) = \sin x$

<u>Separation of variables</u>
Let $u(x,y) = X(x), Y(y) \Rightarrow X''y + Y''x = 0 \Rightarrow \frac{X''}{X} = \frac{-Y''}{Y} = \lambda$
$\therefore X'' - \lambda X = 0\ and\ Y' + \lambda Y = 0$
for $\lambda < 0$ let $\lambda = -k^2, k > 0$
$X = c_1 \cos kx + c_2 \sin kx$
And $Y = c_3 e^{ky} + c_4 e^{-ky}$
$\therefore u(x,y) = (c_1 \cos kx + c_2 \sin kx) \cdot (c_3 e^{ky} + c_4 e^{-ky})$
Since $u \to 0\ as\ y \to \infty$
Then $u(x,y) = (c_1 \cos kx + c_2 \sin kx) c_4 e^{-ky} (c_3 = 0)$
$u(x.0) = (c_1 \cos kx + c_2 \sin kx) = \sin x$
Comparing, $k = 1, c_2 = 1, c_1 = 0$
$\therefore u(x,y) = \sin x\, e^{-y}$
So, the option (c) is correct.

5. A solution of the $PDE x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2-u=0$ represents

(a) An ellipse in the xyplane.
(b) An ellipsoid in xyu space.
(c) A parabola in the $u-x$ plane.
(d) A hyperbolic in the $u-y$ plane.

Answer: (c)

Solution: $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=u$

Complete integral $u=xp+yq+p^2+q^2$
$=xp+yq+f(p,q)\rightarrow$ Clairant equation
$u=ax+by+a^2+b^2$
Singular integral $\frac{\partial u}{\partial a}=0\Rightarrow x+2a=0$
$\frac{\partial u}{\partial b}=0\Rightarrow y+2b=0$
$or, b=-\frac{y}{2}$
$\therefore u=-\frac{x^2}{2}-\frac{y^2}{2}+\frac{x^2}{4}+\frac{y^2}{4}=-\frac{x^2}{4}-\frac{y^2}{4}\Rightarrow\ x^2+y^2=4u$
This is a parabola in $u-x$ plane and also.
So, the option (c) is correct.

June – 2016

(1) Let a, b, c, d be four differentiable functions defined on $\mathbb{R}^2$. Then the partial differentiable equation.
$\left(a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}\right)\left(c(x,y)\frac{\partial}{\partial x} + d(x,y)\frac{\partial}{\partial y}\right)u = 0$ is
(a) Always hyperbolic
(b) Always parabolic
(c) Never parabolic
(d) Never elliptic

Answer: (d)
Solution: Here $A = ac$
$B = ad + bc$
$C = bd$
Now, $B^2 - 4AC = (ad + bc)^2 - 4\, ac \cdot bd = (ad - bc)^2 \nless 0$
So, never elliptic
Option (d) is correct.

(2) For the Cauchy problem
$u_t - u\, u_x = 0, x\epsilon\mathbb{R}, t > 0$
$u(x, 0) = x, x\epsilon\mathbb{R}$
Which of the following statements is true?
(a) The solution u exists for all $t > 0$.
(b) The solution u exists for $t < \frac{1}{2}$ and breaks down at $t = \frac{1}{2}$.
(c) The solution u exists for $t < 1$ and breaks down at $t = 1$.
(d) The solution u exists for $t < 2$ and breaks down at $t = 2$.

Answer: (c)
Solution: $u_t - u\, u_x = 0\,, u(x, 0) = x$
Lagrange's equations are $\frac{dt}{1} = \frac{dx}{-u} = \frac{du}{0} \Rightarrow u = c_1$ and $dt = \frac{dx}{-c_1}$
$tc_1 + x = c_2$
Let, $\mathbb{C}_1 = \xi\,, c_2 = t\xi + \xi$
$c_2 = tc_1 + c_1$
$tu + x = u$
$u = \frac{x}{1-t}$
So, the option (c) is correct.

(3) Let $u: \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be a solution of the initial value problem.
$u_{tt} - u_{xx} = 0$, for $(x, t) \epsilon \mathbb{R} \times (0, \infty)$
$u(x, 0) = f(x), x \epsilon \mathbb{R}$
$u_t(x, 0) = g(x), x \epsilon \mathbb{R}$
Suppose, $f(x) = g(x) = 0$ for $x \notin [0,1]$, then we always have
(a) $u(x, t) = 0$ for all $(x, t) \epsilon (-\alpha, 0) \times (0, \alpha)$
(b) $u(x, t) = 0$ for all $(x, t) \epsilon (1, \alpha) \times (0, \alpha)$
(c) $u(x, t) = 0$ for all (x, t) satisfying $x + t < 0$
(d) $u(x, t) = 0$ for all (x, t) satisfying $x - t > 1$

Answer: (c), (d)
Solution: $u_{tt} = u_{xx}, u(x, 0) = f(x), u_t(x, 0) = g(x)$
Options (c) and (d) are correct.

(4) Consider the Cauchy problem for the Eikonal equation $p^2 + q^2 = 1; p \equiv \frac{\partial u}{\partial x}, q \equiv \frac{\partial u}{\partial y}$ $u(x, y) = 0$ on $x + y = 1, (x, y) \epsilon \mathbb{R}^2$. Then
(a) The Charpit's equations for the differential equation are
$\frac{dx}{dt} = 2p; \frac{dy}{dt} = 2q; \frac{du}{dt} = 2; \frac{dp}{dt} = -p; \frac{dq}{dt} = q$.
(b) The Charpit's equations for the differential equation are
$\frac{du}{dt} = 2p; \frac{dy}{dt} = 2q; \frac{du}{dt} = 2; \frac{dp}{dt} = 0, \frac{dq}{dt} = 0.$
(c) $u\left(1, \sqrt{2}\right) = \sqrt{2}$.
(d) $u\left(1, \sqrt{2}\right) = 1$

Answer: (b), (d)
Solution: $p^2 + q^2 = 1, u(x, y) = 0 \; on \; x + y = 1$
$F(x, y, u, p, q) = p^2 + q^2$
$\frac{dx}{dt} = F_p = 2p$
$\frac{dy}{dt} = F_q = 2q$
$\frac{du}{dt} = p\,F_p + q\,F_q = 2p^2 + 2q^2 = 2 \Rightarrow u = 2t + c_1$
$u_0 = 2.0 + c_1 \Rightarrow c_1 = 0$
$\therefore u = 2t$
$\frac{dp}{dt} = -f_x - p\,F_u = 0 - 0 = 0$
$\frac{dq}{dt} = -f_y - q\,F_u = 0 - 0 = 0$
$x = \sqrt{2}\,t + \zeta, y = \sqrt{2}\,t + 1 - \zeta$
$p = \frac{1}{\sqrt{2}}, q = \frac{1}{\sqrt{2}}$
$x = \sqrt{2}\,t + \sqrt{2}\,t + 1 - y$
$x + y - 1 = 2\sqrt{2}\,t$
$2t = \frac{x+y-1}{\sqrt{2}}$
$\therefore u = \frac{x+y-1}{\sqrt{2}}$

$\therefore u(1, \sqrt{2}) = \frac{1+\sqrt{2}-1}{\sqrt{2}} = 1$
So, the options (b) and (d) are correct.

(5) Let u be the solution of the boundary value problem
$u_{xx} + u_{yy} = 0$ for $0 < x, y < \pi$
$u(x, 0) = 0 = u(x, \pi)\ for\ 0 \leq x \leq \pi$
$u(0, y) = 0, u(\pi, y) = \sin y + sin\ 2y\ for\ 0 \leq y \leq \pi$
Then,
(a) $u\left(1, \frac{\pi}{2}\right) = \left(\sin h\,(\pi)\right)^{-1} \sin h\,(1)$
(b) $u\left(1, \frac{\pi}{2}\right) = \left(\sin h\,(1)\right)^{-1} \sin h\,\pi$
(c) $u\left(1, \frac{\pi}{4}\right) = \left(\sin h\,(\pi)\right)^{-1}\left(\sin h\,(1)\right)\frac{1}{\sqrt{2}} + \left(\sin h\,(2\pi)\right)^{-1} \cdot \sin h\,(2)$
(d) $u\left(1, \frac{\pi}{4}\right) = \left(\sin h\,(1)\right)^{-1}\left(\sin h\,(\pi)\right)\frac{1}{\sqrt{2}} + \left(\sin h\,(2)\right)^{-1} \sin h\,(2\pi)$

Answer: (a), (c)
Solution: $u(x, y) = X(x) \cdot Y(y)$
$\therefore X'' \cdot Y + Y'' \cdot X = 0 \Rightarrow \frac{X''}{X} = \frac{-Y''}{Y} = \lambda$
$X'' - \lambda X = 0\ and\ Y'' + \lambda Y = 0$
$\lambda > 0, \lambda = k^2 (say)$
$X'' - K^2 X = 0, Y'' + K^2 Y = 0$
$u(x, y) = (c_1 e^{kx} + c_2 e^{-kx})(c_3 \cos ky + c_4 \sin ky)$
$u(x, 0) = 0 \Rightarrow X(x)\, Y(0) = 0 \Rightarrow Y(0) = 0 \Rightarrow c_3 = 0$
$u(x, \pi) = 0 \Rightarrow X(x)\, Y(\pi) = 0 \Rightarrow Y(\pi) = 0 \Rightarrow c_4 \sin k\pi = 0 \Rightarrow k = n$
$u(x, y) = \left(A_n e^{nx} + B_n e^{-nx}\right) \cdot \sin ny$
$u(0, y) = 0 \Rightarrow A_n + B_n = 0 \Rightarrow A_n = -B_n$
$\therefore u(x, y) = A_n (e^{nx} - e^{-nx}) \sin ny$
$u(\pi, y) = \sin y + \sin 2y \Rightarrow A_n (e^{n\pi} - e^{-n\pi}) \sin ny = \sin y + \sin 2y$
Comparing, $A_1 = \frac{1}{e^{\pi} - e^{-\pi}}$, $A_2 = \frac{1}{e^{2\pi} - e^{-2\pi}}$
$A_n = 0\ \forall\ n \geq 3.$
$u(x, y) = \sum_{n=1}^{\infty} A_n\, (e^{nx} - e^{-nx}) \sin ny$
$= \frac{1}{e^{\pi} - e^{-\pi}} (e^{x} - e^{-x}) \sin y + \frac{1}{e^{2\pi} - e^{-2\pi}} (e^{2x} - e^{-2x}) \sin 2y$
$u\left(1, \frac{\pi}{2}\right) = \frac{e - e^{-1}}{e^{\pi} - e^{-\pi}} = \sin h\,(1)\left(\sin h\,(\pi)\right)^{-1}$
Option (a) is correct.
$u\left(1, \frac{\pi}{4}\right) = \frac{e - e^{-1}}{e^{\pi} - e^{-\pi}} \times \frac{1}{\sqrt{2}} + \frac{e^2 - e^{-2}}{e^{2\pi} - e^{-2\pi}}$
$= \sin h\,(1)\left(\sin h\,(\pi)\right)^{-1} \cdot \frac{1}{\sqrt{2}} + \sin h\,(2) \cdot \left(\sin h\,(2\pi)\right)^{-1}$
Option (c) is correct.
Hence, the option (a) and (c) are correct.

December – 2016

(1) The PDE $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} = 0$ is
(a) Hyperbolic for $x > 0, y < 0$
(b) Elliptic for $x > 0, y < 0$
(c) Hyperbolic for $x > 0, y > 0$
(d) Elliptic for $x < 0, y > 0$
Answer: (a)
Solution: Discriminant = $B^2 - 4AC = 0^2 - 4 \cdot xy = -4xy$
If $x > 0, y < 0$, then $B^2 - 4AC > 0$. This is a hyperbola.
So, the option (a) is correct.

(2) Let $u(x,t)$ satisfy the initial boundary value problem
$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}; x\epsilon(0,1), t > 0$
$u(x,0) = \sin(\pi x)\,; x\epsilon[0,1]$
$u(0,t) = u(1,t) = 0, t > 0$
Then for $x\epsilon(0,1), u\left(x, \frac{1}{\pi^2}\right)$ is equal to
(a) $e\sin(\pi x)$
(b) $e^{-1}\sin(\pi x)$
(c) $\sin(\pi x)$
(d) $\sin(\pi^{-1}x)$
Answer: (b)
Solution: $u(x,t) = \sum A_n \sin\left(\frac{n\pi x}{a}\right) e^{\frac{-n^2\pi^2}{a^2}t}$
$\sin \pi x = \sum A_n \sin(n\pi x)$
$\Rightarrow A_1 = 1$ and $A_n = 0\ \forall\ n > 1$
$\therefore u(x,t) = \sin \pi x \cdot\ e^{-\pi^2 t}$
$\therefore u\left(x, \frac{1}{\pi^2}\right) = \sin \pi x\, e^{-1} = e^{-1} \sin \pi x$
So, option (b) is correct.

(3) Consider the wave equation for $u(x,t)\,\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, (x,t)\epsilon\mathbb{R} \times (0,\alpha)$
$u(x.0) = f(x), x\epsilon\mathbb{R}$
$\frac{\partial u}{\partial t}(x,0) = g(x), x\epsilon\mathbb{R}$
Let u_i be the solution of the above problem with $f = f_i\ and\ g = g_i\ for\ i = 1,2$, where $f_i: \mathbb{R} \to \mathbb{R}$ and $g_i: \mathbb{R} \to \mathbb{R}$ are given C^2 functions satisfying $f_1(x) = f_2(x)$ and $g_1(x) = g_2(x)$, for every $x\epsilon[-1,1]$, which of the following statements are necessarily true?
(a) $u_1(0,1) = u_2(0,1)$
(b) $u_1(1,1) = u_2(1,1)$
(c) $u_1\left(\frac{1}{2}, \frac{1}{2}\right) = u_2\left(\frac{1}{2}, \frac{1}{2}\right)$
(d) $u_1(0,2) = u_2(0,2)$

Answer: (a), (c)
Solution: Options (a) and (c) are correct.

(4) $y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0$
$u = g\ on\Gamma$
Has a unique solution in a neighborhood of Γ for every differentiable function $g: \Gamma \to \mathbb{R}$ if
(a) $\Gamma = \{(x, 0): x > 0\}$
(b) $\Gamma = \{(x, y): x^2 + y^2 = 1\}$
(c) $\Gamma = \{(x, y): x + y = 1, x > 1\}$
(d) $\Gamma = \{(x, y): y = x^2, x > 0\}$

Answer: (a), (c) & (d)
Solution: Here $P = y, Q = -x, R = 0$
If $\begin{vmatrix} P & Q \\ x_0^1(x) & y_0^1(x) \end{vmatrix} \neq 0$, unique solution
(a) $\{(x, 0): x > 0\}$
$x_0 = S, y_0 = 0$
$\begin{vmatrix} P & Q \\ x_0^1 & y_0^1 \end{vmatrix} = \begin{vmatrix} 0 & -s \\ 1 & 0 \end{vmatrix} = s \neq 0$, unique solution
(b) $\{(x, y): x^2 + y^2 = 1\}$
Let $x_0 = s, y_0 = \sqrt{1 - s^2}$
$\therefore \begin{vmatrix} P & Q \\ x_0^1 & y_0^1 \end{vmatrix} = \begin{vmatrix} \sqrt{1 - s^2} & -s \\ 1 & -\frac{s}{\sqrt{1-s^2}} \end{vmatrix} = 0$
Not unique solution.
(c) $\{(x, y): x + y = 1, x > 1\}$
Let no $= s > 1, y_0 = 1 - s$
$\begin{vmatrix} P & Q \\ x_0^1 & y_0^1 \end{vmatrix} = \begin{vmatrix} 1 - s & -s \\ 1 & 1 \end{vmatrix} = 2s - 1 > 0$
Unique solution
(d) $\{(x, y): y = x^2, x > 0\}$
Let $x_0 = s \Rightarrow y_0 = s^2$
$\begin{vmatrix} P & Q \\ x_0^1 & y_0^1 \end{vmatrix} = \begin{vmatrix} s^2 & -s \\ 1 & 2s \end{vmatrix} = 2s^3 + s > 0$
Unique solution
So, the option (a), (c) and (d) are correct.

(5) Let $u: \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}^2$ be a C^2 function satisfying $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, for\ all (x,y) \neq (0,0)$.

Suppose u is of the form $u(x,y) = f\left(\sqrt{x^2+y^2}\right)$, where $f: (0,\alpha) \to \mathbb{R}$ is a non-constant function, then

(a) $\lim\limits_{x^2+y^2 \to 0} |u(x,y)| = \infty$

(b) $\lim\limits_{x^2+y^2 \to 0} |u(x,y)| = 0$

(c) $\lim\limits_{x^2+y^2 \to \infty} |u(x,y)| = \infty$

(d) $\lim\limits_{x^2+y^2 \to \infty} |u(x,y)| = 0$

Answer: (a), (c)

Solution: $u_{xx} + u_{yy} = 0 \Rightarrow u(x,y) = \phi_1(y - ix) + \phi_2(y - ix)$

Let, $\phi_1 = \frac{1}{\sqrt{x^2+y^2}}, \phi_2 = \sqrt{x^2+y^2}$

$\lim\limits_{x^2+y^2 \to 0} u(x,y) = \infty + 0 = \infty$

$\lim\limits_{x^2+y^2 \to \infty} u(x,y) = 0 + \infty = \infty$

So, the option (a) and (c) are correct.