COUNCILE OF SCIENTIFIC & INDUSTRIAL RESEARCH

Mathematical Science **Code : 04**

Unit – 3

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Sub Unit – 4: Calculus of Variations

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Calculus of variations

4.1. Calculus of variations:

One of the main problem of the calculus of variation is determine that curve connection two given points which either maximizes or minimizes some given integral.

Consider the curve $y = y(x), where\ \ y(x_1) = y_1\ \ and\ y(x_2) = y_2$ such that for some given known function $F(x, y, y')$ of variables x, y, y' the integral

$$\int [y(x)] = \int_{x_1}^{x_2} F\ (x, y, y')\ dx$$

is either maximum or minimum, also called an extremum or stationary values. A curve $y = y(x)$ which satisfies this property is called an extremal.

An integral such as Eq. (i) which assumes a numerical value for some class of functions $y(x)$ is said to be functional. This assumed the function $y(x)$ is taken from a class of functions on which the functional L$[y(x)]$ is defined.

4.2. Linear Functional:

A linear functional is a functional $L[y(x)]$ if it satisfies the following two conditions.

(i) $L\ [Cy(x)] = C\ L[y(x),\ \ C$ is being an arbitrary constant.

(ii) L$[y_1(x) + y_2(x)] =$ L$[y_1(x)] +$ L$[y_2(x)]$ e.g., The functional

$|[y(x)] = \int_a^b \{y'(x) +\ y(x)\}\, dx$ is linear in the class $C'[a, b]$ but the functional

$|[y(x)] =\ \int_a^b [p(x)\{\ y'\,(x)\{y'(x)\}^2 +\ q(x)\{y(x)\}^2]dx$ is non-linear. Here, $C'[a, b]$ denotes the set of all functions varying continuous first order derivatives in $[a, b]$.

4.2.1. Theorem:

If a functional $|[y\ (x)]$ having a variation attains a maximum or minimum on $y = y_0(x)$, where, $y\ (x)$ is an interior point of the domain of definition of the functional, at

$y = y(x), \delta / = 0.$

4.2.2. Theorem:

If for every continuous function $\eta(x), \int_{x_1}^{x_2} \phi\ (x)\ \eta\ (x) dx = 0$, where the $\phi(x)$ is continuous on the interval $[x_1, x_2]$ then $\phi(x) = 0$ on that interval.

4.3. Euler – Lagrange Equation:

A necessary condition for $|[y(x)] = \int_{x_1}^{x_2} F\left(x; y(x), y'(x)\right) dx$ to be an extremum is that

$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y}\right) = 0$ which is called Euler's equation or Euler – Lagrange equation.

Example (4.1):

Find the extremal of the functional $\int_0^{\frac{\pi}{2}} (y'^2 - y^2 + 2xy)\, dx$ the satisfy the boundary condition $y(0) = 0,\ y\left(\frac{\pi}{2}\right) = 0$

Solution:

Here, $F\,(x, y, y') = y'^2 - y^2 + 2xy$

$\Rightarrow\ \frac{\partial F}{\partial y} = 0 - 2y + 2x = 2(x - y)$

$\frac{\partial F}{\partial y\prime} = 2y' - 0 + 0 = 2y'$

Since, Euler's - Lagrange equation is

$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y}\right) = 0$

$\Rightarrow 2(x - y) - \frac{d}{dx}(2y') = 0$

$\Rightarrow x - y - y'' = \ 0$

$\Rightarrow m^2 + 1 = 0 \ \ \Rightarrow m = \ \pm i$

$\therefore\ CF = C_1 \cos x + C_2 \sin x$

$PI = \frac{1}{D^2+1}(x) = (1 + D^2)^{-1}x = (1 -\ D^2 + \ldots\ldots.)x\ = x$

Thus, $y = C_1 \cos x = \ C_2 \sin x + x$

From boundary conditions

$y(0) = 0\,,\ y\left(\frac{\pi}{2}\right) = 0\ \Rightarrow 0 = C_1 + 0 + 0,\ 0 = 0 + C_2 + \frac{\pi}{2}\ \Rightarrow\ C_1 = 0, C_2 = \ -\frac{\pi}{2}$

Hence, required extremal is

$y = -\frac{\pi}{2} \sin x + x \quad or \quad y = x - \frac{\pi}{2} \sin x$

Example (4.2):

Find the extremal of the functional $\int_0^1 (y^2 + y'^2)\, dx$, where $y(0) = 0,\ \ y(1) = 1$

Solution:

Here, $F\ (x, y, y') = y^2 + y'^2$

$\Rightarrow \frac{\partial F}{\partial y} = 2y + 0 = 2y$

$\frac{\partial F}{\partial y\prime} = 0 + 2y' = 2y'$

From Euler's equation, we get

$$2y + 2y'' = 0 \Rightarrow y'' - y = 0$$

Alternatively, here x is absent in F

$\Rightarrow \frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0$

$\Rightarrow \frac{d}{dx}(y^2 + y'^2 - 2y'^2) = 0$

$\Rightarrow \frac{d}{dx}(y^2 + y'^2) = 0$

$\Rightarrow\ 2yy' - 2y'y'' = 0 \Rightarrow\ y'' - y = 0 \qquad (\because y' \neq 0)$

AE is $m^2 - 1 = 0\ \ \Rightarrow m = \pm 1$

$\therefore$ Solution of equation (i) is $y = C_1 \cosh x + C_2 \sinh x$. where C_1 and C_2 are arbitrary constants.

From given boundary conditions

$y\ (0) = 0,\ y\ (1) = 1 \Rightarrow\ C_1 = 0, 1 = C_1 \cosh 1 + C_2 \sinh 1 \Rightarrow\ C_1 = 0,\ C_2 = \frac{1}{\sinh 1}$

$\therefore$ *Required extremal is* $y = \frac{\sinh x}{\sinh 1}$

Example (4.3):

Find the extremals of the functional $\int_{x_0}^{x_1} \frac{y'^2}{x^3}\,dx$.

Solution: $F(x, y, y') = \frac{y'^2}{x^3} \Rightarrow \frac{\partial F}{\partial y} = 0,\ \frac{\partial F}{\partial y'} = \frac{2y'}{x^3}$

Here, y is absent.

$\therefore\ \frac{\partial F}{\partial y} = constant$

$\Rightarrow \frac{2y'}{x^3} = constant$

$\Rightarrow y' = c\,x^3$

$\Rightarrow dy = \ c\,x^3\,dx$

On integrating both sides, we get $y = c\frac{x^4}{4} + c'$

Hence, $y = k_1x\ + k_2$ is the required extremal, where k_1 and k_2 are constants.

Example (4.4):

Find the extremal of the functional $\int_1^3 y\,(3x - y)dy$ that satisfy the boundary conditions $y\,(1) = 1,\ y(3) = \frac{9}{2}$.

Solution: $F\,(x, y, y') = 3xy - y^2$

$\frac{\partial F}{\partial y} = 3x - 2y,\ \frac{\partial F}{\partial y'} = 0$

$\therefore\ From\ Euler's\ equation\ \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$\Rightarrow (3x - 2y) - \frac{d}{dx}(0) = 0\ \Rightarrow 3x - 2y = 0$

Which is the only extremal if the given functional.

Given, $y(1) = 1\ \Rightarrow 3x[-2x] = 0\ \ \Rightarrow 1 = 0$

which is the absurd. Hence, the given functional has on extremal.

Example (4.5):

Show that the shortest distance between two fixed points in a plane is a straight line.

Solution:

Let $PA = S \quad and \quad AB = ds$

Then, arc length

$$PQ = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{(dx)^2 + (dy)^2} = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} . dx = \int_{x_1}^{x_2} (1 + y^2)^{\frac{1}{2}} \, dx$$

$F\,(x, y, y') = (1 + y'^2)^{\frac{1}{2}}$

Here, F is independent of $\,x \; and \; y$

$\Rightarrow y'' \frac{\partial^2 F}{\partial y'^2} = 0 \quad but \;\; \frac{\partial^2 F}{\partial y'^2} \neq 0 \quad \Rightarrow y'' = 0$

On integration giving

$y = c_1\, x - c_2$ where is a straight line joining $P \; and \; Q$.

$\Rightarrow Shortest\ distance\ between\ two\ fixed\ points\ in\ a\ plane\ is\ a\ straight\ line.$

Example (4.6):

Find the path on which a part will take minimum time in sliding from a point (x_1, y_1) to another point (x_2, y_2), if its rate of motions $V\left(=\frac{dx}{dt}\right)$ is equal to x.

It is known as the problem of Brachistochroen (shortest line) in horizontal plane.

Solution:

Since, $ds = (1+y^2)^{\frac{1}{2}}\,dx$

Given, $V = \frac{ds}{dt} = x \;\Rightarrow dt = \frac{ds}{x} \;\Rightarrow f[y(x)] = \int_{x_1}^{y_2} \frac{(1+y'^2)^{\frac{1}{2}}}{x}\,dx \ldots\ldots\ldots\ldots\ldots (i)$

Where $f[y(x)]$ is the line taken in sliding from (x_1, y_1) to (x_2, y_2).

Hence,

$F = \frac{(1+y'^2)^{\frac{1}{2}}}{x}$

$\frac{\partial F}{\partial y'} = \frac{y'}{(1+y'^2)^{\frac{1}{2}}}$

Here, y is absent.

$\Rightarrow \frac{\partial F}{\partial y'} = c$

$\Rightarrow \frac{m}{x(1+m^2)^{\frac{1}{2}}} = c \quad (put\;\; y' = m)$

$\Rightarrow \;\; m^2 = x^2(1+m^2)c^2 = x^2c^2 + x^2m^2c^2$

$\Rightarrow m^2(1 - c^2x^2) = x^2c^2$

$\Rightarrow m = \frac{cx}{\sqrt{1-c^2x^2}} \qquad\qquad (taking\; m > 0)$

$\Rightarrow y' = \int \frac{x}{\sqrt{\frac{1}{c^2}-x^2}}\,dx = \int \frac{x\,dx}{\sqrt{k^2-x^2}} \left(\because k = \frac{1}{c}\right)$

Put $x = k\sin\theta$

$\Rightarrow \;\; dx = k\cos\theta\;d\theta \ldots\ldots\ldots\ldots\ldots\ldots (ii)$

$\therefore \;\; y = \int \frac{k^2 \sin\theta\cos\theta\,d\theta}{k\cos\theta} = k\int \sin\theta\,d\theta = -k\cos\theta + k'$

$\Rightarrow y - k' = -k\cos\theta$

$\Rightarrow (y-k')^2 = k^2 - k^2\sin\theta = k^2 - x^2 \quad [from\; equation\; (ii)]$

$\Rightarrow x^2 + (y-k')^2 = k^2$

Which is a family of circle so that particle takes minimum time in sliding from a point (x_1, y_1) to (x_2, y_2).

Example (4.7):

Find the curve passing through the points (x_1, y_1) *and* (x_2, y_2) which when rotated about the $x - axis$ gives a minimum surface area.

Solution:

Surface area = $2\pi \int_{x_1}^{x_2} y\, ds \Rightarrow S[y(x)] = 2\pi \int_{x_1}^{x_2} y\sqrt{1+y'^2}\, dx$

Here, $F = y\sqrt{1+y'^2}$

$$\Rightarrow \frac{\partial F}{\partial y} = \sqrt{1+y'^2}, \Rightarrow \frac{\partial F}{\partial y} = \frac{yy'}{\sqrt{1+y'^2}}$$

Since, x is absent.

$$\therefore F - y', \frac{\partial F}{\partial y} = constant$$

$$\Rightarrow y\sqrt{1+y'^2} - \frac{yy'}{\sqrt{1+y'^2}} = c$$

$$\Rightarrow \frac{y}{\sqrt{1+y'^2}} = c \Rightarrow y^2 = c^2(1+y'^2) \Rightarrow \frac{y^2-c^2}{c^2} = y'^2$$

$$\Rightarrow y' = \frac{\sqrt{y^2-c^2}}{c} \Rightarrow dy = \frac{\sqrt{y^2-c^2}}{c}\, dx \Rightarrow \frac{dy}{\sqrt{y^2-c^2}} = \frac{dx}{c}$$

On integrating both sides, we get

$$\operatorname{csch}^{-1}\left(\frac{y}{c}\right) = \frac{1}{c}\, x + D = \frac{x+B}{c}$$

$$\Rightarrow \frac{y}{c} = \operatorname{csch}^{-1}\left(\frac{x+B}{c}\right)$$

$$\Rightarrow y = c\, \operatorname{csch}^{-1}\left(\frac{x+B}{c}\right) \ldots\ldots\ldots (i)$$

Which is a catenary, where B *and* c are two arbitrary constants equation (i) is the family of catenaries of two parameter having one, two or no solution.

4.4. Geodesics:

A geodesics on a surface is a curve along which the distance between two points on the surface is a minimum. To find the geodesics on a surface is a variational problem involving conditional extremum.

(i) Geodesics on a plane are straight lines.

(ii) Geodesics on a sphere of fixed radius are its great circles.

(iii) Geodesics on a circular cylinder of fixed radius are circular helix.

4.5. Isoperimetric Problems:

Isoperimetric problem includes finding extremals of $/[y(x)] = \int_{x_1}^{x_2} F(x, y, y')dx$

when another functional $|[y(x)] = \int_{x_1}^{x_2} G(x, y, y')dx = constant$

Such problems are generally solved by method of Lagrange multipliers.

Example (4.8)

Find the curve of Length L which passes through the points (0,0) and (1,0) for which the area between the curve and the $x - axis$ is a maximum.

Solution:

Here, l is the fixed perimeter of a plane curve passing through two given points $O(0,0)$ and $A(1,0)$. Let the area enclosed by that plane curve and $x - axis$. Then, we are to maximize.

$$S[y(x)] = \int_0^1 y\, dx$$

Subject to the constraint.

$$L[y(x)] = \int_0^1 (1 + y'^2)^{\frac{1}{2}}\, dx = 1$$

and the boundary conditions

$y(0) = 0$ and $y(1) = 0$

Here, $F = y$ and $G = \sqrt{1 + y'^2}$

Let $H = F + xG = y + x\sqrt{1 + y'^2}$

Now, H satisfies Euler's equation i.e.,

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0 \Rightarrow 1 - \frac{d}{dx}\left(\frac{xy'}{\sqrt{1+y'^2}}\right) = 0$$

Integrating with respect to x, we get

$$x - \frac{xy'}{\sqrt{1+y'^2}} = c \Rightarrow (x - c)^2 = \frac{x^2y'^2}{1+y'^2} \Rightarrow \frac{1+y'^2}{y'^2} = \frac{x^2}{(x-c)^2}$$

$$\Rightarrow y'^2 = \frac{(x-c)^2}{x^2-(x-c)^2} \Rightarrow \frac{dy}{dx} = \frac{(x-c)}{\sqrt{x^2-(x-c)^2}} \qquad (with\ positive\ sign)$$

$$\Rightarrow dy = \frac{(x-c)}{\sqrt{x^2-(x-c)^2}}\, dx$$

Integrating, we get

$$y = -\sqrt{x^2 - (x - c)^2} + c'$$

$$\Rightarrow (y - c')^2 = x^2 - (x - c)^2 \Rightarrow (x + c)^2 + (y - c')^2 = x^2$$

Which is a circle.

4.6. System of Euler's Equations:

We now extend the variational problem to a problem with several variables as function of a single independent variable, consider the functional

$$I[y_1, y_2, \ldots\ldots ..\ , y_n] = \int_{x_1}^{x_2} F\left(x_1, y_1, y_2, \ldots\ldots y_n, y'_1, y'_2, ..\ \ldots y'_n\right)\, dx \ldots\ldots\ldots\ldots\ldots (i)$$

Where boundary conditions of all functions $y_1, y_2, \ldots\ldots ..\ , y_n$ are given.

A necessary condition for equation (i) to be an extremum is that

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'_i}\right) = 0,\ i = 1,2, \ldots\ldots, n$$

Which is the system of n Euler's equations corresponding to n a variables $y_1, y_2, \ldots\ldots ..\ , y_n$.

4.7. Euler's Equation for Higher Order Derivatives:

A necessary condition for the functional $I[y(x)] = \int_{x_1}^{x_2} F(x, y, y', y'', \ldots\ldots y^n)\, dx$ to be extremum is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) - \ldots\ldots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) = 0$$

In particular for $n = 2$, we get

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$$

Example (4.9): Find the external of the functional $\int_0^1 (1 + y''^2)^{\frac{1}{2}}\, dx$ which satisfy $y(0) = 0, y'(0) = 1, y(1) = 1, y'(1) = 1$.

Solution: Hence, $F = 1 + y''^2$

From Euler's equation for higher derivative

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0 \Rightarrow 0 - 0 + \frac{d^2}{dx^2}(2y'') = 0 \ \Rightarrow\ 2y^{iv} = 0 \ \Rightarrow\ y^{iv} = 0$$

$$\Rightarrow \frac{d^4 y}{dx^4} = 0$$

By integrating successively four times, we get

$$y = c_1 x^3 + c_2 x^2 + c_3 x + c_4 \ \Rightarrow y' = 3c_1 x^2 + 2c_2 x + c_3$$

From boundary conditions $y(0) = 0,\ y'(0) = 1,\ y(1) = 1,\ y'(1) = 1$

We get

$$c_4 = 0,\ c_3 = 1,\ c_1 + c_2 + c_3 + c_4 = 1,\ 3c_3 + 2c_3 + c_3 = 1$$

$$\Rightarrow c_1 = c_2 = c_4 = 0, c_3 = 1$$

Hence, required extremal is $y = x$

4.8. Sufficient conditions for Extrema proper field:

A family of curves $y = y(x, c)$ is said to form a proper field in a given region D of the $xy - plane$ if one and only one curve of the family passes through every point of the region D. e.g., inside the circle $x^2 + y^2 = 1$, the family of parallel lines $y = x + c$ $(c\ being\ an\ arbitrary\ constant)$ rms a proper field, since through any point of the above circle there passes one and only one straight line of the family as shown in Fig. (i). On the other hand, the family of parabolas $(x - c)^2 = y + 1$ inside the same circle $x^2 + y^2 = 1$ does not form a proper field, since the parabolas of this family intersect inside the circle as shown in Fig.(ii).

Figure:

4.9. Central Field:

If all the curves of the family $y = y(x, c)$ pass through a certain point (x_0, y_0), i.e., if the form a pencil of curves, then they do not form a proper field in the region D, if the centre of the pencil (x_0, y_0) belongs to D. However, if the curves of the pencil cover the entire region D and do not intersect anywhere in this region, with the exception of the centre of the pencil (x_0, y_0), then the family $y = y(x, c)$ is said to form a central field as show in Fig. (iii).

For example, the pencil of curve $y = e\sin x \;\; for \;\; 0 \le x \le a, \; a < \pi$ forms a central field (Fig. (iv)). But the pencil of curves forms a proper field in a sufficiently small neighbourhood of the segment of $x - axis$ for $\delta \le x \le a, \; \delta > 0, \; a < \pi$.

Pencil of curves does not form a proper field in a neighbourhood of the segment of the $x - axis$, for $0 \le x \le a_1, a_1 > \pi$.

4.10. Extremal Field:

If a proper field or central field is formed by a family of extremals of a given variational problem, then it is called an extremal field.

Example (4.10):

Discuss the extremal field for the functional

$|[y(x)] = \int_0^a (y'^2 - y^2)\, dx. y(0) = 0, y(a) = 0.$

Solution: $F = y'^2 - y^2$

Euler's equation is $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \Rightarrow -2y - \frac{d}{dx}(2y') = 0 \;\; \Rightarrow y'' + y = 0$

$\Rightarrow (D^2 + 1)y = \; 0 \ldots\ldots\ldots\ldots (i)$

The general solution of equation (i) is $y = c_1 \cos x + c_2 \sin x \;\; \ldots\ldots\ldots\ldots\ldots\ldots (ii)$

Since, the required extremal passes through (0,0)

$\therefore$ From equation (ii), we get $0 = c_1 + 0 \;\Rightarrow\; c_1 = 0$

$\therefore$ Equation (ii) reduces to $y = c_2 \sin x \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (iii)$

Which pencil curves form as central field on $0 \le a \le x, \; a < \pi$ including for $c_2 = 0$, the extremalis $y = 0$.

For $a \ge \pi$, the family of extremals equation (iii) does not form an extremal field.

Example (4.11):

Show that the extremal to the variational problem $\int_0^2 (y'^2 - y^2)\, dx,\ y(0) = 0, y(2) = 3$ is included in proper field of extremals of the given functional.

Solution: Here, $F = y'^2 - x^2$

$\therefore$ Euler's equation is $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \ \Rightarrow 0 - \frac{d}{dx}(2y') = 0 \ \Rightarrow \frac{dy'}{dx} = 0 \ \Rightarrow \ y^2 = c_1$

$\Rightarrow y = c_1 x + c_2 \ \dots\dots\dots\dots\dots\dots.(i)$

Equation (i) is the equation of the extremals. Using the given boundary condition $y(0) = 1$ and $y(2) = 3$, equation (i) yields $c_2 = 1$ and $2c_1 + c_2 = 3$

So that $c_1 = c_2 = 1$.

Hence, equation (i) $\Rightarrow y = x + 1$, which is the extremal of the given variational problem.

For $c_1 = 1$, equation (i) reduces to $y = x + c_2$, which is a proper field of the extremals in the domain $0 \le x \le 2$. For $c_2 = 1, y = x + c_2,\ yields\ y = x + 1$, showing that the extremal $y = x + 1$ is included in the proper field of extremals$y = x + c_2$.

<u>Working Rule for Testing for a weak minimum/weak. Maximum and strong minimum/strong maximum of an Extremal of a given Functional with help of Legendre condition.</u>

Legendre condition Let $|[y(x)] = \int_{x_1}^{x_2} F(x, y, y')\, dx \dots\dots\dots\dots\dots\dots..(i)$ be a functional with fixed boundaries $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. Let $F(x, y, y')$ possess a continuous partial derivative $F_{y'y'}$, i.e.,$\frac{\partial^2 F}{\partial y' \partial y'}$. Let C can be the curve of the extremal of the given functional which passes through $P_1\ and\ P_2$. Also, assume that the extremal of the curve C is included in the field of extremals. Then, the following table provides the sufficient conditions for the nature of the extremal of the given functional.

Sl. No.	Nature of the extremal of the given functional.	Sufficient Conditions.
1.	For a weak minimum to be attained on the.	$F_{y'y'} > 0$ on C.
2.	For a weak maximum to be attained on the extremal on the curve C.	$F_{y'y'} < 0$ on C.
3.	For a strong minimum to be C attain on the extremal on the curve C.	$F_{y'y'} \ \geq 0$ at points close to C and also for arbitrary values of y' .
4.	For a strong maximum to be attain on the extremal on the curve C	$F_{y'y'} \ \leq 0$ at points close to C and also for arbitrary values of y'.

Example (4.12)

Test for an extremum the functional

$|[y(x)] = \int_0^a \left(y'^2 - y^2\right) dx,\ a > 0,\ y(0) = 0,\ y(a) = 0.$

Solution:

Comparing the given functional with $\int_0^a F(x, y, y')\ dx, F(x, y, y') = y'^2 - y^2 \ldots\ldots\ldots\ldots..(i)$

Euler's equation is $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0\ \ldots\ldots\ldots..\ ..\ \ldots\ldots\ldots..(ii)$

From equation (i) $\frac{\partial F}{\partial y} = -2y,\ \frac{\partial F}{\partial y'} = 2y'\,and\ \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{d(2y')}{dx} = 2y'$

Using these values, equation (ii) reduces to

$-2y - 2y^2 = 0\ \ or\ \ (D^2 + 1)y = 0 \ldots\ldots\ldots\ldots\ldots\ldots.(iii)$

The general solution of equation (iii) is

$y = c_1 \cos x + c_2 \sin x\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(iv)$

Where c_1 and c_2 are arbitrary constants.

Since, $y(0) = 0\ and\ y(a) = 0$ equation (iv) gives $0 = c_1$ and $0 = c_2 \sin a$, if $c_2 = 0$.Thus, for $a \neq n\pi$ an extremum may be attained only on the straight line $y = 0$. If $a < \pi$, then the pencil of extremals $y = c_2 \sin x$with centre at (0,0) forms accrual field.

For $a > \pi$ extremals $y = c_2 \sin x$ neither forms proper field nor central field.

Now,

$Fy'y = 2 > 0,\ \forall\ y'$

$\Rightarrow A\ storng\ minimum\ is\ attained\ or\ y = 0\ for\ a < \pi.$

But for $a > \pi$, minimum is not attained the $y = 0$