Previous Year Questions & Solution

Linear Algebra

June - 2014

Part – B

1. Let A be a 5×5 matrix with real entries such that the sum of the entries of in each row of $A\ is$ 1. Then the sum of the entries of all the entries in A^3 is

1) 3

2) 15

3) 5

4) 125

Sol:

Take $A = 1 \ \Rightarrow A^3 = 1$.

Sum of all entries of $A^3 = 5$

So, option (3) is correct.

(As all other options are eliminated).

2. Let J denote a 101×101 matrix with all the entries equal to 1 and let I denote the identity matrix of order 101. Then the determinant of $J - I$ is

1) 101

2) 1

3) 0

4) 100

Sol:

$[J]_{101\times101}$ matrix with all the entries equal to 1.

The eigenvalues of $J = \left(trace\ J, \underbrace{0, 0, \ldots\ldots, 0}_{100\ times}\right) = (101, 0, 0, \ldots\ldots, 0)$

The eigenvalues of $J - I = (101 - 1, 0 - 1, 0 - 1, \ldots\ldots\ldots, 0 - 1)$

$= (100, \underbrace{-1, -1, \ldots\ldots, -1}_{100\ times}$

Now,

$$\det(J - I) = Product\ of\ the\ eigenvalues\ of\ (J - I)$$

$$= (100, \underbrace{-1, -1, \ldots\ldots, -1}_{100\ times} = 100$$

3. Let $M_{m\times n}(\mathbb{R})$ be the set of all $m \times n$ matrices with real entries. Which of the following statements is correct?

1) There exists $A \in M_{2\times 5}(\mathbb{R})$ such that the dimension of the null space of $A\ is$ 2.

2) There exists $A \in M_{2\times 5}(\mathbb{R})$ such that the dimension of the null space of $A\ is$ 0.

3) There exists $A \in M_{2\times 5}(\mathbb{R})$ and $B \in M_{5\times 2}(\mathbb{R})$ such that AB is the 2×2 identity matrix.

4) There exists $A \in M_{2\times 5}(\mathbb{R})$ whose null space is

$\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 :\ x_1 = x_2,\ x_3 = x_4 = x_5\}$

Sol:

Let A be a 2×5 real matrix $\Rightarrow R(A) \le \min\{2,5\},\ i.e., R(A) \le 2$

For option (1).

If there exists $A \in M_{2\times 5}(\mathbb{R})$ such that $\dim(null\ space\ of\ A) = 2$ i.e., $nullity\ A = 2.$

$$\Rightarrow R(A) = 3\ \ (Using\ rank\ nullity\ theorem).$$

Which is absurd as $R(A) \le 2$.

So, option (1) is incorrect.

For option (2),

If there exists $A \in M_{2\times 5}(\mathbb{R})$ such that $\dim(nulll\ space\ of\ A) = 0\,, i.e., nullity(A) = 0.$

$\Rightarrow R(A) = 5\ (Using\ rank\ nullity\ theorem)$, which is absurd as $R(A) \le 2$.

So, option (2) is incorrect.

For option (3),

There exist $A \in M_{2\times 5}(\mathbb{R})$ and $B \in M_{5\times 2}(\mathbb{R})$ such that $AB = I_{2\times 2}(The\ identity\ matrix)$.

Take, $A = \begin{bmatrix} 1 & 0 & 00 & 0 \\ 0 & 1 & 00 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 10 \\ 01 \\ 00 \\ 00 \\ 00 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So, option (3) is correct.

For option (4).

If there exists a matrix $A \in M_{2\times 5}(\mathbb{R})$ whose null space is

$\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 :\ x_1 = x_2,\ x_3 = x_4 = x_5\}$,

i.e., $\dim(null\ space\ of\ A) = 2\ \Rightarrow Nullity(A) = 2 \Rightarrow R(A) = 3.$

Which is absurd as $R(A) \le 2$.

So, option (4) is incorrect.

4. For the matrix A as given below, which of them satisfy $A^6 = I$?

1) $A = \begin{pmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} & 0 \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2)$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ 0 & -\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix}$

3)$A = \begin{pmatrix} \cos\frac{\pi}{6} & 0 & \sin\frac{\pi}{6} \\ 0 & 1 & 0 \\ -\sin\frac{\pi}{6} & 0 & \cos\frac{\pi}{6} \end{pmatrix}$

4)$A = \begin{pmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} & 0 \\ -\sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Sol:

For option (1)

$$A = \begin{pmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} & 0 \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since, A is an orthogonal matrix.

So, $A^6 = \begin{pmatrix} \cos 6\frac{\pi}{4} & \sin 6\frac{\pi}{4} & 0 \\ -\sin 6\frac{\pi}{4} & \cos 6\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

So, option (1) is incorrect.

For option (2),

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ 0 & -\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix}$$

$$A^6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\pi & \sin 2\pi \\ 0 & -\sin 2\pi & \cos 2\pi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So, option (2) is correct.

Part – C

5. Let V denote a vector space over a field F and with a basis $B = \{e_1, e_2, \dots . e_n\}$.

Let $x_1, x_2, \dots \dots, x_n \in F$.

Let $C = \{x_1 e_1,\ x_1 e_1 + x_2 e_2, \dots \dots, x_1 e_1 + x_2 e_2 + \dots \dots . + x_n e_n\}$. Then

1)C is linearly independent that $x_i \neq 0$ for every $i = 1, 2, \dots \dots ., n$.

2)$x_i \neq 0$ for every $i = 1, 2, \dots \dots ., n$ implies that C is linearly independent.

3) The linear span of $C\ is\ V$ implies that $x_i \neq 0$ for every $i \in \mathbb{N}$

4)$x_i \neq 0$ for every $i = 1, 2, \dots \dots ., n$ implies that the linear span of $C\ in\ V$.

Sol:

Let $C = \{x_1 e_1,\ x_1 e_1 + x_2 e_2, \dots \dots, x_1 e_1 + x_2 e_2 + \dots \dots . + x_n e_n\}$

We can write it as $C = \{(x_1, 0, 0, \dots ., 0), (x_1, x_2, 0, 0, \dots .., 0), \dots .. (x_1,\ x_1, \dots .., x_n)\}$

Given that $B = \{e_1, e_2, \dots . e_n\}$ is a basis of $V \Rightarrow \dim V = n$.

C is linear independent set of V if only if determinant of matrix formed by vectors of C as row vectors is non – zero, i.e., C is linearly independent if only if $\det A \neq 0$ where

$$A = \begin{pmatrix} x_1 & 0 & 0 \dots \dots & 0 \\ x_1 & x_2 & 0 \dots \dots & 0 \\ x_1 & x_2 & x_3 \cdots \cdots & 0 \\ \dots & \dots & \dots \cdots \cdots & \dots \\ x_1 & x_2 & x_3 \cdots \cdots & x_n \end{pmatrix}_{n \times n}$$

i.e., C is linearly independent if only if $x_1,\ x_1, \dots .., x_n \neq 0$.

i.e., C is linearly independent if only if $x_i \neq 0$ for every $i = 1, 2, \dots \dots ., n$

So, option (1), (2) is correct.

We know that, if V is vector space of dimension 'n' and $S \subseteq V\ such\ that\ |S| = n$ and S is linearly independent subset of V.

$\Rightarrow L(S) = V$ i.e., $S\ generates\ V \Rightarrow$ linear span of $S\ is\ V$.

Since, $\dim V = n$ and $C \subseteq V\ such\ that\ |C| = n$ and C is linearly independent set if only if $x_i \neq 0 \forall i = 1, 2, \dots \dots ., n$.

$\Rightarrow$ linear span of C is V if only if $x_i \neq 0 \forall i = 1, 2, \ldots\ldots., n$

So, option (3) and (4) are correct.

6. Let V denote the vector space of all polynomials over $\mathbb{R}$ of degree less than or equal to n. which of the following defines a norm of on V?

1)$\|P\|^2 = |P(1)|^2 + \ldots\ldots\ldots + |P(n+1)|^2, P \in V$

2)$\|P\| = \sup_{t\in[0,1]} |P(t)|, P \in V$

3)$\|P\| = \int_0^1 |P(t)|\, dt, P \in V$

4)$\|P\| = \sup_{t\in[0,1]} |P'(t)|, P \in V$

Sol:

A vector space V on which a norm is defined is called normed vector space.

Given that, V is a vector space of all polynomials over $\mathbb{R}$ of degree less than or equal to 'n'.

For option (1).

Norm on V is defined as $\|p\|^2 = |p(1)|^2 + \ldots\ldots\ldots + |p(n+1)|^2, for\ \ p \in V$

$$\Rightarrow \|p\| = \sqrt{|p(1)|^2 + \ldots\ldots\ldots + |p(n+1)|^2}$$

(i) Clearly, $\|p\| \geq 0\ \ and\ \|p\| = 0\ if\ only\ if\ p = 0, i.e.$, every non – zero vector has positive length and only zero vector has zero length.

(ii) $\|\alpha\, p(x)\| = \sqrt{|\alpha\, p(1)|^2 + |\alpha\, p(2)|^2 + \ldots\ldots\ldots + |\alpha\, p(n+1)|^2}$

$$= |\alpha|\sqrt{|p(1)|^2 + |p(2)|^2 + \ldots\ldots\ldots + |p(n+1)|^2}$$

$$= |\alpha|\|\, p(x)\| \ \forall\ \alpha \in \mathbb{R}, \forall\ p(x) \in V.$$

(iii) $\|\, p(x) + q(x)\| = \sqrt{|(p+q)(1)|^2 + \ \ldots\ldots\ldots + |(p+q)\ (n+1)|^2}$

$$= \sqrt{|p(1)|^2 + |p(2)|^2 + \ldots\ldots\ldots + |p(n+1)|^2 + |q(1)|^2 + |q(2)|^2 + \ldots\ldots\ldots + |q(n+1)|^2}$$

$$\leq \sqrt{|p(1)|^2 + |p(2)|^2 + \ldots\ldots. + |p(n+1)|^2} + \sqrt{|q(1)|^2 + |q(2)|^2 + \ldots\ldots. + |q(n+1)|^2}$$

$$= \|\, p(x)\| + \|\, q(x)\| (\because \sqrt{a+b}\ \leq \sqrt{a} + \sqrt{b}\ \forall\ a, b\ \geq 0)$$

So, all properties of norm are satisfied.

So, option (1) is correct.

By similar reason, options (2), (3) are correct.

For option (4).

Take $p(x) = 1$

So, $\|\, p(x)\| = \|\, 1\| = \sup_{x\in[0,1]} |0|$

$\Rightarrow \|\, 1\| = 0,$ which is wrong. (Norm of every non – zero vector should be positive).

So, option (4) is correct.

7. Let u, v, w be vector in an inner product space V, satisfying $\| u\| = \| v\| = \| w\| = 2$ and $\langle u, v\rangle = 0, \langle u, w\rangle = 1, \langle v, w\rangle = -1$. Then which of the following are true?

1)$\| w + v - u\| = 2\sqrt{2}$

2) $\left\{\frac{1}{2}u, \frac{1}{2}v\right\}$forms an orthogonal basis of two - dimensional subspace of V.

3)$W\ and\ 4u - w$ are orthogonal to each other.

4)u, v, w are linearly independent (necessarily)

Sol:

V is an inner product space and $u, v, w \in V$ such that $\| u\| = \| v\| = \| w\| = 2$and

$$\langle u, v\rangle = 0, \langle u, w\rangle = 1, \langle v, w\rangle = -1$$

For option (1)

Consider,

$$\langle w + v - u, w + v - u\rangle = \langle w, w + v - u\rangle + \langle v, w + v - u\rangle - \langle u, w + v - u\rangle$$

$$= \langle w, w\rangle + \langle w, v\rangle - \langle w, u\rangle + \langle v, w\rangle + \langle v, v\rangle - \langle v, u\rangle - \langle u, w\rangle - \langle u, v\rangle + \langle u, u\rangle$$

$$= \|w\|^2 + (-1) - (1) + (-1) + \|v\|^2 - 0 - 1 - 0 + \|u\|^2$$

$$= 4 - 1 - 1 - 1 + 4 - 1 + 4 = 8$$

$$\Rightarrow \ \|w + v - u\|^2 = 8 \ \Rightarrow \ \|w + v - u\| = 2\sqrt{2}$$

So, option (1) is correct.

For option (2)

Let $S = \left\{\frac{1}{2}u, \frac{1}{2}v\right\}$,

$$\langle\frac{1}{2}u, \frac{1}{2}u\rangle = \frac{1}{2}, \frac{1}{2}\langle u, u\rangle = \frac{1}{4}\|u\|^2 = 1$$

$$\langle\frac{1}{2}v, \frac{1}{2}v\rangle = \frac{1}{2}, \frac{1}{2}\langle v, v\rangle = \frac{1}{4}\|v\|^2 = 1$$

$$\langle\frac{1}{2}u, \frac{1}{2}v\rangle = \frac{1}{2}, \frac{1}{2}\langle u, v\rangle = \frac{1}{4} \cdot 0 = 0$$

$\Rightarrow S$ forms an orthogonal set in V and we know that every orthogonal set is linearly independent.

$\Rightarrow S$ forms an orthogonal basis of two - dimensional subspace of v.

For option (3)

$$\langle w, 4u - w\rangle = \langle w, 4u\rangle - \langle w, w\rangle$$

$$= 4\langle w, 4\rangle - \langle w, w\rangle = 4 \cdot 1 - \|w\|^2 = 4 - 4 = 0$$

$\Rightarrow w \; and \; 4u - w$ are orthogonal to each other.

For option (4),

Let $S = \left\{\frac{1}{2}u, \frac{1}{2}v\right\}$

Consider, $\alpha_1 \, u + \alpha \,_2 v + \alpha_3 \, w = 0$

$$\Rightarrow \langle u, \alpha_1 \, u + \alpha_2 v + \alpha_3 \, w\rangle = 0, \langle v, \alpha_1 \, u + \alpha \,_2 v + \alpha_3 \, w\rangle = 0, \langle w, \alpha_1 \, u + \alpha \,_2 v + \alpha_3 \, w\rangle = 0$$

$$\Rightarrow \alpha_1 \|u\|^2 + \alpha_2 \langle u, v\rangle + \alpha_3 \langle v, u\rangle = 0, \alpha_1 \langle u, v\rangle + \alpha_2 \|v\|^2 + \alpha_3 \langle u, v\rangle = 0,$$

$$\alpha_1 \langle u, v\rangle + \alpha_2 \langle u, v\rangle + \alpha_3 \|w\|^2 = 0$$

$$\Rightarrow 4\alpha_1 + \alpha_3 = 0, 4\alpha_2 - \alpha_3 = 0, \qquad \alpha_1 - \alpha_2 + 4\alpha_3 = 0$$

Solving we get $\alpha_1 = \alpha_2 = \alpha_3 = 0$

$\Rightarrow \{u, v, w\}$ are linearly independent.

So, option (4) is correct.

8. Let A be a 4×4 matrix over $\mathbb{C}$ such that $rank(A) = 2$ and $A^3 = A^2 \neq 0$. Suppose that A is not diagonalizable. Then

1) One of the Jordan blocks of the Jordan canonical form of A is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

2)$A^2 = A \neq 0$

3) There exists a vector v such that $Av \neq 0$ but $A^2 v = 0$.

4) The characteristic polynomial of A is $x^4 - x^3$.

Sol:

A is a 4×4 matrix over $\mathbb{C}$ such that $R(A) = 2$ and $A^3 = A^2 \neq 0$

So, A satisfy $x^3 - x^2 = x^2(x-1)$ $\Rightarrow$ the minimal polynomial of A divides $x^2(x-1)$.

Possibilities of $m_A(x) = x, x^2, x-1, x(x-1), x^2(x-1)$

But $m_A(x)$ cannot be $x, x-1, x(x-1)$ because it is given that A is not diagonalizable.

Also, $m_A(x)$ cannot be x^2, because given that $A^2 \neq 0$

So, $m_A(x) = x^2(x-1) and \; R(A) = 2$

Only possible Jordan canonical form of A is given by

$$\begin{pmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} [0] & 0 \\ 0 & [1] \end{matrix} \end{pmatrix}$$

One of the Jordan blocks of Jordan canonical form of A is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

So, option (1) is correct.

Since, $A = \begin{pmatrix} 0 & 1 & 00 \\ 0 & 0 & 00 \\ 0 & 0 & 00 \\ 0 & 0 & 01 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} 0 & 0 & 00 \\ 0 & 0 & 00 \\ 0 & 0 & 00 \\ 0 & 0 & 01 \end{pmatrix} \neq A$.

So, option (2) is incorrect.

There exists a vector $v = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ such that

$$Av \neq 0, i.e., \begin{pmatrix} 0 & 1 & 00 \\ 0 & 0 & 00 \\ 0 & 0 & 00 \\ 0 & 0 & 01 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

But $A^2 v = 0 \; i.e., \begin{pmatrix} 0 & 0 & 00 \\ 0 & 0 & 00 \\ 0 & 0 & 00 \\ 0 & 0 & 01 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

So, option (3) is correct.

Also, the characteristic polynomial of A is $x^3(x-1) = x^4 - x^3$

So, option (4) is correct.

9. Let $\phi : \mathbb{R}^2 \to \mathbb{C}$ be the map defined by $\phi(x, y) = Z, where\ Z = x + iy$.

Let $f : \mathbb{C} \to \mathbb{C}$ be the function $f(z) = z^2$ and $F = \phi^{-1} f \phi$. Which of the following is correct?

1) The linear transformation $T(x, y) = 2\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ represents the derivative of F at (x, y).

2) The linear transformation $T(x, y) = 2\begin{pmatrix} x & y \\ y & x \end{pmatrix}$ represents the derivative of F at (x, y).

3) The linear transformation $T(x, y) = 2z$ represents the derivative of f at $z \in \mathbb{C}$.

4) The linear transformation $T(x, y) = 2z$ represents the derivative of f at 0.

Sol:

Given $\phi : \mathbb{R}^2 \to \mathbb{C}$ be the map defined by $\phi(x, y) = z$ where $z = x + iy \ and\ f : \mathbb{C} \to \mathbb{C}$ be the function $f(z) = z^2$ and $F = \phi^{-1} f \phi \Rightarrow F(x, y) = (\phi^{-1} f \phi)(x, y) = \phi^{-1} f(z^2) = \phi(z)$

$$= \phi^{-1}\big((x^2 - y^2) + i2xy\big) = (x^2 - y^2, 2xy)$$

$$\Rightarrow F(x, y) = (x^2 - y^2, 2xy) = \big(f_1(x, y), f_2(x, y)\big), where\ f_1(x, y) = x^2 - y^2,$$

$$and\ \ f_2(x, y) = 2xy$$

The linear transformation which represents the derivative of F at (x, y) is given by

$$T(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix} = 2\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

So, option (1) is correct and option (2) is incorrect.

The linear transformation which represents the derivative of ‘f’ at $z \in \mathbb{C}$ is given by

$$T(z) = \frac{\partial f}{\partial z} = 2z \; \forall \; z \in \mathbb{C}$$

So, option (3) is correct and option (4) is incorrect.

10. Let V be a vector space of polynomial over $\mathbb{R}$ of degree less than or equal to n. For $p(x) = a_0 + a_1\,x + \ldots\ldots\ldots + a_n x^n$ in V, define a linear transformation $T: V \to V$ by

$T\big(p(x)\big) = a_0 - a_1\,x + a_2 x^2 - \ldots\ldots\ldots + a_n x^n$. Then which of the following is correct?

1) T is one to one.

2) T is onto.

3) T is invertible.

4)$\det T = 0$

Sol:

The matrix ‘T’ relative to basis ‘B’ is given by

$$[T]_B = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & (-1)^n \end{pmatrix}_{(n+1)\times(n+1)}$$

$\Rightarrow \det T = (-1)$for $m \in z \;\Rightarrow \det T \;\neq 0 \;\Rightarrow T$ is invertable.

So, option (3) is correct and (4) is incorrect.

Since, $\det T \neq 0 \;\Rightarrow Rank\;(T) = n+1 \Rightarrow d(I_m T) = n+1$ and

$$\dim(kerT) = 0\;(by\;Rank - nullity\;theorem)$$

$\Rightarrow T$ is onto and T is one to one.

So, option (1) and (2) are correct.

11. Consider a homogeneous system of linear equations $AX = 0$, where A is an $m \times n$ real matrix and $n > m$. Then which of the following statements are always true?

1) $AX = 0$ has a solution.

2) $AX = 0$ has no non – zero solution.

3) $AX = 0$ has a non – zero solution.

4) Dimension of the space of all solutions is at least $n - m$.

Sol:

$AX = 0$ (Homogeneous system of equations is always consistent)

($\because x = 0$ is always a solution of $AX = 0$)

So, option (1) is correct.

Since, A is an $m \times n$ matrix and $m < n$.

$$R(A) \leq \min(m, n) = m \Rightarrow R(A) \leq m < n \; and \; R(A) = R(A:0) < n.$$

So, $AX = 0$ has non – trivial solution (non – zero solution).

So, option (2) is incorrect and option (3) is correct.

Since, $R(A) \leq m$.

By Rank – nullity theorem, $R(A) + N(A) = Number \; of \; columns \; of \; A = n$

$$\Rightarrow N(A) = n - R(A) \geq n - m$$

$$\dim(Null \; space \; of \; A) \geq n - m$$

$$\Rightarrow \dim\{X: AX = 0\} \geq n - m \Rightarrow \dim of \; space \; of \; all \; solution \; is \; at \; least \; n - m.$$

So, option (4) is correct.

Answer

Sl. No	Answer	Sl. No	Answer	Sl. No	Answer
1.	3	2.	4	3.	3
4.	2	5.	1, 2, 3 & 4	6.	1, 2 & 3
7.	1, 2, 3 & 4	8.	1, 3 & 4	9.	1 & 3
10.	1, 2 & 3	11.	1, 3 & 4		

Previous Year Questions & Solution

Linear Algebra

December - 2014

Part – B

1. Let A, B be $n \times n$ matrices such that $BA + B^2 = I - BA^2$, where I is the $n \times n$ identity matrix which of the following is always true?

1)$A\ is\ non - singular$

2)$B\ is\ non - singular$

3)$A + B\ is\ non - singular$

4)$AB\ is\ non - singular.$

Sol:-

Let A, B be $n \times n$ matrices such that $BA + B^2 = I - BA^2$

Take $A = 0, B = I$

A, B satisfy the given condition.

$$\Rightarrow A\ and\ AB\ cannot\ be\ non - singular.$$

So, option (1) and (2) are incorrect.

Now, take $\mathrm{A} = -\mathrm{I}, \mathrm{B} = \mathrm{I}$

$\Rightarrow A + B$ satisfy the given condition $\Rightarrow A + B$ cannot be non – singular

So, option (3) is incorrect.

So, option (2) is correct. (as all the options are eliminated).

2. Which of the following matrices has the same row space as the matrix $\begin{pmatrix} 4 & 8 & 4 \\ 3 & 6 & 1 \\ 2 & 4 & 0 \end{pmatrix}$

1)$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2)$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3)$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

4)$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

Sol:- Given matrix is

$$\begin{pmatrix} 4 & 8 & 4 \\ 3 & 6 & 1 \\ 2 & 4 & 0 \end{pmatrix} \xrightarrow{R_1 - R_2} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 1 \\ 2 & 4 & 0 \end{pmatrix} \xrightarrow[R_3 - 2R_1]{R_2 - 3R_1} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 1 \\ 2 & 4 & 0 \end{pmatrix} \xrightarrow[-\frac{1}{6}R_3]{-\frac{1}{8}R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[R_3 - R_2]{R_1 - 3R_3} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The given matrix and $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ matrix have same non – zero rows.

So, option (1) is correct.

3. The determinant of the $n \times n$ permutation matrix

1) $(-1)^n$

2) $(-1)^{\frac{n}{2}}$

3) -1

4) 1

Sol:-

Take $n = 1 \Rightarrow A = [1]_{n\times n} \Rightarrow \det A = 1$

So, option (1) and (3) are incorrect.

Take $n = 2,\ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow detA = 1$

So, option d id incorrect.

So, only option (b) is correct.

4. The determinant $\begin{vmatrix} 1 & 1+x & 1+x+x^2 \\ 1 & 1+y & 1+y+y^2 \\ 1 & 1+z & 1+z+z^2 \end{vmatrix}$ is equal to

1) $(z-y)(z-x)(y-x)$

2) $(x-y)(x-z)(y-z)$

3) $(x-y)^2(y-z)^2(z-x)^2$

4) $(x^2-y^2)(y^2-z^2)(z^2-x^2)$

Sol:-

$$\begin{vmatrix} 1 & 1+x & 1+x+x^2 \\ 1 & 1+y & 1+y+y^2 \\ 1 & 1+z & 1+z+z^2 \end{vmatrix} \to \begin{vmatrix} 1 & 1+x & (1+x)+x^2 \\ 0 & 1+y & (1+y)+(y^2-z^2) \\ 0 & 1+z & (1+z)+(z^2-x^2) \end{vmatrix}$$

$$\to \begin{vmatrix} 1 & 1+x & 1+x+x^2 \\ 0 & y-z & (y-z)(1+y+z) \\ 0 & z-x & (z-x)(1+z+x) \end{vmatrix} = A$$

$$\therefore\ |A| = (y-z)(z-x)\{1+z+x-1-y-z\} = (x-y)(y-z)(z-x)$$

$$= (z-y)(z-x)(y-x)$$

So, option (1) is correct.

5. Which of the following matrices is not diagonalizable over $\mathbb{R}$?

1) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

3) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

4) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Sol:-
For option (1) the minimal polynomial is $(x-1)(x-2)$.
Since the minimal polynomial is linear, so (1) is diagonalizable.
For option (2) the minimal polynomial is $(x-1)(x-2)(x-3)$
So, option (2) is incorrect.
For option (3) the minimal polynomial is $(x-1)(x-2)$ i.e., the minimal polynomial is quadratic, so the matrix is not diagonalizable.
So, option (3) is correct.

6. Let P be a 2×2 complex matrix, such that P^*P is the identity matrix, where P^* is the conjugate transpose of P^*. Then the eigenvalues of P are
1) real
2) complex conjugate of each other
3) reciprocal of each other
4) of modulus 1.

Sol:- Take $P = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \Rightarrow P^* = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}$

$$P^* \ P = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

Since eigen values of P are i, i
So eigen values of P are not
(i) real
(ii) complex conjugate of each other
(iii) reciprocals of each other.
But, the eigen values of P are modulus 1.
So, option (1), (2), (3) are incorrect.
So, option (4) is correct.

Part – C

7. Let A be a real $n \times n$ orthogonal matrix, that $A^t A = AA^t = I_n$s, the $n \times n$ identity matrix, which of the following statements are necessarily true?

1) $\langle A_x, A_y \rangle = \langle x, y \rangle \; \forall \; x, y \in \mathbb{R}^n$
2)All the eigen values of A are either $+1 \; or \; -1$.
3) The rows of A form on orthogonal basis of $\mathbb{R}^n$
4) A is diagonalizable over $\mathbb{R}^n$.

Sol:-

A is a real orthogonal matrix i.e., $A^t A = AA^t = I_n$.
For option (1)

$$\langle A_x, A_y \rangle = \langle x, A^t A \; y \rangle = \langle x, Iy \rangle \langle x, y \rangle; \; \forall \; x, y \in \mathbb{R}^n$$

So , option (1) is correct.
For option (2)
Take $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
A is on orthogonal matrix but eigen values of A are i and -i.
So, option (2) is incorrect.
For option (3)
Let $A = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$, where x_i is $(1 \times n) row$ vector.
We get $\langle X_i, X_j \rangle = \begin{cases} 1, & i \neq j \\ 0, & i = j \end{cases}$ by using $A^T A = I_n$
$\Rightarrow \; X_1 , X_2, \ldots\ldots\ldots, X_n$ forms an orthogonal basis of $\mathbb{R}^n$
So, option (3) is correct.
Take $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
Hence eigen values of A are $i \; , -i$
A is diagonalizable over $\mathbb{C}$ but not $\mathbb{R}$.
So, option (4) is incorrect.

8. Which of the following matrices have Jordan canonical form equal to $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$?

1) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
2) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
3) $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
4) $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Sol:-

Let $A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

A_1, A_2, A_3 are rank 1 and nilpotent matrices.

Jordan canonical form or rank 1 nilpotent matrix is given by $E_{12} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

So, option (1), (2), (3) are correct.

option (4) is incorrect, as $A_4 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is rank 2 matrix.

9. Let A be a 3×4 and b be a 3×1 matrix with integer entries, suppose that the system $AX = b$ has a complex solution. Then

1) $AX = b$ has an integer solution.
2) $AX = b$ has a rational solution.
3) The set of real solutions to $AX = 0$ has a basis consisting of rational solutions.
4) If $b \neq 0$, then A has positive rank.

Sol:-

For option (1)

Consider $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}_{3\times4}$ $\quad and \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

So, $AX = b \quad \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

For option (2) and (3),

Since $AX = b$ has complex solution where $A_{3\times4} (3 < 4)$

$\Rightarrow AX = b$has infinity many solutions.

If $X_1 \; and \; X_2$ are two solutions of $AX = b$, then $\lambda X_1 + (1-\lambda)X_2$ also solution of $AX = b \;\; \forall \, \lambda \in \mathbb{R}$.

⇒ Solution of system of equation is convex set.

⇒ Solution set of $AX = b$ is connected set $\because$ $convex\ set \;\Rightarrow connected\ set)$

If $AX = b$ has no rational solution then solution set become disconnected set which is contradiction.

So, $AX = b$ must have rational solution.

It also clearly imply that set of real solution of $AX = 0$ has a basis of consist of rational solution.

So, option (2) and (3) are correct.

For option (4),

Given $b \neq 0$

Since $AX = b$ has solution (Infinitely many)

$$\Rightarrow \quad R(A) = R(A:v)$$

Since, $b \neq 0$

So, $R(A:b)$ can never zero $\Rightarrow \quad R(A:b) \geq 1 \Rightarrow R(A) \geq 1 \Rightarrow A$ has positive rank.

So, option (4) is correct.

10. Let f be a non – zero symmetric bilinear form on $\mathbb{R}^3$. Suppose that there exist linear transformation $T_i: \mathbb{R}^3 \to \mathbb{R}, i = 1, 2,$ such that for all $\alpha, \beta \in \mathbb{R}^3, f(\alpha, \beta) = T_1(\alpha)\, T_2(\beta)$. Then

1) $Rank\ f = 1$

2) $\dim\{\beta \in \mathbb{R}^3 : f(\alpha, \beta) = 0\ for\ all\ \alpha \in\ \mathbb{R}^3\} = 2.$

3) f is positive semi – definite or negative semi – definite.

4) $\{\alpha, f(\alpha, \alpha) = 0\}$is a linear subspace of dimension 2.

Sol:-

Let $T_1: \mathbb{R}^3 \to \mathbb{R}$ be a linear transformation defined by $T_1(x, y, z) = x$

Similarly, $T_2: \mathbb{R}^3 \to \mathbb{R}$ be a linear transformation defined by $T_2(x, y, z) = x$

Given: $f(\alpha, \beta) = T_1(\alpha) T_2(\beta)$, where $\alpha, \beta \in \mathbb{R}^3$.

Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$

$$\therefore f(e_1, e_2) = T_1(e_1) T_2(e_2) = 1.$$

$$f(e_1, e_2) = 0\ \forall\ i, j\ (i \neq 1, j \neq 1)$$

$$\begin{pmatrix} f(e_1, e_1) & f(e_1, e_2) & f(e_1, e_3) \\ f(e_2, e_1) & f(e_2, e_2) & f(e_2, e_3) \\ f(e_3, e_1) & f(e_3, e_2) & f(e_3, e_3) \end{pmatrix}$$

$$\therefore f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Clearly, f is non – zero symmetric matrix

Here, $rank\ f\ =\ 1$.

$$\therefore \quad option\ (1) is\ correct.$$

Here, $f(e_1, e_2) = 0, f(e_1, e_3) = 0$

$$\therefore dim\ \{\beta \in \mathbb{R}^3 : f(\alpha, \beta) = 0\ \forall\ \alpha \in \mathbb{R}^3\} = 2$$

Thus (2) is correct.

Further, 'f' defined above is positive semi – defined.

If, we define $T: \mathbb{R}^3 \to \mathbb{R}$ as $T\ (x, y, z) = z$ and $T(x, y, z) = -x$.

Then, 'f' becomes negative definite.

Option, (3) is correct.

Since, $f(e_2, e_2) = 0\ and\ , f(e_3, e_3) = 0$

And, $f(e_1, e_1) = 1 \neq 0$

$\therefore$ { $\alpha \in \mathbb{R} : f(\alpha, \alpha) = 0$ } is a linear subspace of dimension 2.

$\therefore$ option (4) is correct.

11. The matrix $A = \begin{pmatrix} 5 & 9 & 8 \\ 1 & 8 & 2 \\ 9 & 1 & 0 \end{pmatrix}$ satisfies

1) A is invertable and the inverse has all integer entries.
2) $det(A)$ is odd.
3) $det(A)$ is divisible by 13.
4) $det(A)$ has at least two prime divisors.

Sol:-

$$A = \begin{pmatrix} 5 & 9 & 8 \\ 1 & 8 & 2 \\ 9 & 1 & 0 \end{pmatrix}$$

$|A| = 5(-2) - 9(-18) + (1 - 72 = -10 + 162 - 568 = -416.$ ($\because$ $\det(A)$is even and $\det(A) \neq \pm 1$. So, A^{-1} cannot have integer entries)

So, options (1) and (2) are not correct and $det(A)$ is divisible by 13 and $det(A)$ has at least two prime divisors.

So, options (3), (4) are correct.

12. Let A be 5×5 matrix and let B be obtained by changing one element of A. Let $r\ and\ s$ be the ranks of $A\ and\ B$ respectively.

1) $s \leq r + 1$
2) $r - 1 \leq s$
3) $s = r - 1$
4) $s \neq r$

Sol:-

Take $A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow R(A) = 2$

If we change one element of A as following, we get,

(i) $a_{12} = 0\ to\ a'_{12} = 1.$

$$R(A) = 2, i.e., rank\ will\ be\ same.$$

(ii) $a_{22} = 1\ to\ a'_{22} = 0$

$$R(A) = 1, i.e., rank\ will\ be\ decreased\ by\ 1.$$

(iii) $a_{33} = 0\ to\ a'_{33} = 1$

$$R(A) = 3, i.e., rank\ will\ be\ increase\ by\ 1.$$

We can conclude that, if we change one element of A, either rank of A will be same or decreased by 1 or increased by 1.

So, options (3) and (4) are incorrect.

Given $R(A) = r\ \ and\ \ R(B) = s, then\ s = r\ or\ r - 1\ or\ r + 1\ \Rightarrow In\ all\ cases, s \leq r + 1$

$$\&\ r - 1\ \leq s.$$

So, options (1) and (2) are correct.

13. Let $M_n(k)$ denote the space of all $n \times n$ matrices with entries in a field k. Fix a non – singular matrix $A = (A_{ij}) \in M_n(k)$ and consider the linear map $T : M_n(k)$ given by $T(X) = AX$.Then
1) $Trace\ (T) = n\ \sum_{i=1}^{n} A_{ij}$
2) $\sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} = Trace\ (T)$
3) $Rank\ (T) = n^2$
4) T is non – singular.

Sol:-

$$Fix\ \ A = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}_{n\times n} = I_{n\times n}\,(non - singular)$$

Given that, $T : M_n(k) \to M_n(k)$ defined by $T(X) = AX = IX = X, i.e.,$ the identity transformation.
So, matrix of T relative to any basis B of $M_n(k)$ is the identity matrix of order n^2

i.e., $[T]_B = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}_{n^2\times n^2}$

$$R(T) = n^2\ and\ T\ is\ non - singular.$$

$$Trace\ (T) = n \sum_{i=1}^{n} A_{ij} = n^2$$

But $Trace\ (T) \neq \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} = n$
So, option (2) is incorrect and options (1), (3), (4) are correct.

14. For arbitrary subspaces $U, V\ and\ W$ of a finite dimensional vector space, which of the following hold?
1) $U \cap (V + W) \subset U \cap V + U \cap W$
2) $U \cap (V + W) \supset U \cap V + U \cap W$
3) $(U \cap V) + W \subset (U + W) \cap (V + W)$
4) $(U \cap V) + W \supset (U + W) \cap (V + W)$

Sol:-

Given that $U, V\ and\ W$ are arbitrary subspaces of finite dimensional vector spaces.
Let $U = \{(x, y) : y = x\}$, $V = \{(x, y): x = 0\}\ and\ W = \{(x, y): y = 0\}$ be a subspaces of $\mathbb{R}^2(\mathbb{R})$ $\Rightarrow U \cap V = \{(0,0)\}\ and\ U \cap W = \{(0,0)\}$
Also $V + W = \{(x, y): x, y \in \mathbb{R}\} = \mathbb{R}^2$

$$\Rightarrow U \cap (V + W) = \mathbb{R}^2 \cap U = U, (U \cap V) + (U \cap W) = \{(0,0)\}$$

$$\therefore U \cap (V + W) \not\subset (U \cap V) + (U \cap W)$$

So, option (1) is incorrect.
Also, $U + W = \{(x, x)\} + \{(x, 0)\} = \{(2x, x)\} = \left\{(x, y) \in \mathbb{R}^2 \middle| y = \frac{x}{2}\right\}\ and\ V + W = \mathbb{R}^2$

$$\Rightarrow (U+W)\cap(V+W) = \left\{(x,y)\in\mathbb{R}^2 : y = \frac{x}{2}\right\} = \left\{(x,y)\in\mathbb{R}^2 : y = \frac{x}{2}\right\}$$

But $U \cap V = \{(0,0)\}$ *and* $W = \{(x,y)\in\mathbb{R}^2 : y = 0\}$.

$$(U\cap V)+W = \{(0,0)\}+W = W = \{(x,y)\in\ \mathbb{R}^2 : y = 0\}$$

$$\Rightarrow (U+W)\cap(V+W)\ \not\subset (U\cap V)+W$$

So, option (4) is incorrect.

For option (2)

Let $x \in U\cap V + U\cap W$

$$x = x_1 + x_2, where\ x_1 \in U\cap V\ \ and\ x_2 \in U\cap W.$$

$$\Rightarrow x_1, x_2 \in U, \qquad x_1 \in V, \qquad x_2 \in W\ \Rightarrow\ x_1 + x_2 \in U\ and\ x_1 + x_2 \in V + W.$$

$$\Rightarrow\ x_1 + x_2 \in U\cap(V+W) \Rightarrow x \in U\cap(V+W)$$

$$\Rightarrow U\cap V + U\cap W \subset U\cap(V+W)$$

So, option (2) is correct.

Similarly, we also can show that option (3) is correct.

15. Let A be a 4×7 real matrix and B be a 7×4 matrix such that $AB = I_4$, where I_4 is the 4×4 identity matrix. Which of the following is/are always true?

1) $rank(A) = 4$
2) $Rank(B) = 7$
3) $Nullity\ (B) = 0$
4) $BA = I_7, where\ I_7 is\ the\ 7\times 7\ aidentity\ matrix.$

Sol:-

A is a 4×7 matrix and $B$is a 7×4 matrix such that $AB = I_4$.

We know that $R(AB)\le R(A)$& $R(AB)\le R(B)$

Since, $AB = I_4$,

$$Rank\ (AB) = R(I_4) = 4\ \Rightarrow R(A)\ge 4\ \&\ R(B)\ge 4. \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Also, A is a 4×7 matrix.

$$\Rightarrow R(A) \le \min\{4,7\} = 4\ and\ B\ i\ s\ 7\times 4\ matrix.$$

$$\Rightarrow R(B)\le \min\{7,4\} = 4$$

$$\therefore R(A)\le 4\ \&\ R(B)\le 4.\ldots\ldots\ldots\ldots\ldots(2)$$

From (1) and (2) we get $R(A) = 4\ \&\ R(B) = 4$.

So, option (1) is correct and (2) is incorrect.

Using rank – nullity theorem, $R(B) + N(B) = number\ of\ columns\ of\ B = 4$

$$\Rightarrow N(B) = 4 - R(B) = 4 - 4 = 0$$

So, option (3) is correct.

If $BA = I_7\ \ \Rightarrow R(BA) = R(I_7) = 7$.

But we know that $R(BA)\le R(A) = 4 \Rightarrow R(BA)\ cannot\ be\ equal\ to\ 7$.

BA cannot be equal to I_7.

So, option (4) is correct.

16. Let $\mathbb{R}[x]$ denote the vector space of all real polynomials. Let $D: \mathbb{R}[x]$ denote the map $Df = \frac{df}{dx}, for\ all\ f$. Then

1) D is one to one
2) D is onto
3) There exists $E: \mathbb{R}[x]\ so\ that\ D\ \left(E(f)\right) = f, \forall\ f.$
4) There exists $E: \mathbb{R}\ [x] \to \mathbb{R}[x]\ so, that\ E\left(D(f)\right) = f, \forall f$

Sol:-

As $D(x + 1) = 1$ & $D(x + 2) = 1\ \Rightarrow D$ is not one to one.

So, option (1) is incorrect.

$$\forall\ f(x) =\ a_0 + a_1x + a_2x^2 + \ldots\ldots \ldots\ .. \in \mathbb{R}\ [x]$$

$$\exists\ g(x) = a_0x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} + \ldots\ldots \ldots\ .. \in \mathbb{R}\ [x] such\ that\ D\left(g(x)\right) = f(x).$$

So, D is onto map.

This option (2) is correct.

There exists $E: \mathbb{R}[x] \to \mathbb{R}[x]$ defined by $E\left(f(x)\right) =\ \int_0^x f(x)\, dx$ such that

$$D(E\left(f(x)\right) = D\ \left(\int_0^x f(x)\, dx\ \right) = f(x).$$

So, option (3) is correct.

But $E\left(D(f)\right) \neq f, \forall\ f$ ($\because\ it\ may\ happens\ that\ E\left(D(1)\right) = E(0) =\ \int_0^x 0\, dx = 0 \neq 1$)

So, option (4) is incorrect.

17. Which of the following are eigen values of the matrix $\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$?

1) +1
2) -1
3) +i
4) -i

Sol:-

A is a real symmetric matrix (Let A be the given matrix) and we know that real symmetric matrix has real eigen values.

So, $i, -i$ cannot be the eigen values of A.

So, option (3) and (4) are incorrect.

If $'\lambda'$ is an eigen value of matrix A, then it will satisfy $\det(A - \lambda I) = 0$

Just need to check $\det(A - (I)I) = 0$ and

$$A - I = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}$$

First and fourth column are scalar multiple of each other $\Rightarrow \det(A - I) = 0$.

$$\Rightarrow 1 \textit{ is an eigen values of } A.$$

Similarly, $(A - (-1)I) = 0 \Rightarrow -1$ *is also an eigen values of* A.

So, option (1) and (2) are correct.

18. Let $A = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$, where $x, y \in \mathbb{R}$, such that $x^2 + y^2 = 1$. Then we must have

1) For any $n \geq 1, A^n = \begin{pmatrix} cos\theta & sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, where\ x = \cos\left(\frac{\theta}{n}\right), y = \sin\left(\frac{\theta}{n}\right)$
2) $t_r(A) \neq 0$
3) $A^t = A^{-1}$
4) A is similar to a diagonal matrix $\mathbb{C}$.

Sol.

For option (1)

$$x = \cos\left(\frac{\theta}{n}\right),\ y = \sin\left(\frac{\theta}{n}\right)$$

$$A = \begin{pmatrix} \cos\left(\frac{\theta}{n}\right) & \sin\left(\frac{\theta}{n}\right) \\ -\sin\left(\frac{\theta}{n}\right) & \cos\left(\frac{\theta}{n}\right) \end{pmatrix} \Rightarrow A^n = \begin{pmatrix} \cos\left(\frac{n\theta}{n}\right) & \sin\left(\frac{n\theta}{n}\right) \\ -\sin\left(\frac{n\theta}{n}\right) & \cos\left(\frac{n\theta}{n}\right) \end{pmatrix}$$

$$\Rightarrow A^n = \begin{pmatrix} cos\theta & sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

So, option (1) is correct.

For option (2)

Take $x = 0, y = 1 \Rightarrow A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow trace\ (A) = 0$

So, option (2) is incorrect.

For option (3)

$$A = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = A^t = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

$$A^tA = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}\begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \begin{pmatrix} x^2 + y^2 & 0 \\ 0 & x^2 + y^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; (\because x^2 + y^2 = 1)$$

So, $AA^t = I \Rightarrow A$ is orthogonal matrix $\Rightarrow A^t = A^{-1}$

So, option (3) is correct.

Since, A is orthogonal matrix and every orthogonal matrix is diagonalizable over $\mathbb{C}$.

$\Rightarrow A$ *is* similar to diagonal matrix over $\mathbb{C}$.

So, option (4) is correct.

Answer

Sl.No	Answer	Sl. No	Answer	Sl. No	Answer
1.	2	**2.**	1	**3.**	2
4.	1	**5.**	3	**6.**	4
7.	1 & 3	**8.**	1, 2 & 3	**9.**	2,3& 4
10.	1,2,3 &4	**11.**	3 &4	**12.**	1 &2
13.	1, 3 & 4	**14.**	2 &3	**15.**	1 &3
16.	2 &3	**17.**	1 &2	**18.**	1, 3 & 4

Previous Year Questions & Solution

Linear Algebra

June – 2015

Part – B

1. Let V be the space of twice differentiable functions on $\mathbb{R}$ satisfying $f'' - 2f' + f = 0$. Define $T: V \to \mathbb{R}^2$ by

$T(f) = \big(f'(0), f(0)\big)$. Then T is

1) One to one and onto

2) one to one but not onto

3) onto but not one to one

4) neither one to one onto

Sol:-

$$f'' - 2f' + f = 0$$

Auxiliary equation is $(D^2 - 2D + 1) = 0 \Rightarrow (D-1)^2 = 0$.
It is solution is $f = (c_1 + c_2 x)e^x = c_1 e^x + c_2 x\, e^x$
Consider $B = \{(1,0), (0,1)\}$ be the standard basis of $\mathbb{R}^2$

$$T(e^x) = (e^x|_{x=0}, xe^x|_{x=0}) = (1,1) = 1(1,0) + 1(0,1)$$

$$T(xe^x) = (xe^x + e^x|_{x=0}, xe^x|_{x=0}) = (1,0) = 1(1,0) + 0(0,1)$$

$$\therefore T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$Rank\ T = 2, \therefore T\ is\ onto.$

Nullity $T = 0,\ \therefore T\ is\ one - one.$

$$\Rightarrow T\ is\ one - one\ and\ onto$$

So option (1) is correct.

2. The row space of a 20×50 matrix A has dimension 13. What is the dimension of the space of solutions of $Ax = 0$?

Sol:

A is a 20×50 matrix

Dimension of row space of $A\ =\ 13 \Rightarrow row\ rank = 13$
Rank of $A = row\ rank\ of\ A = 13$.
Dimension of space of solutions of

$$Ax = 0 = nullity\ of\ A = No. of\ columns\ of\ A - Rank\ of\ A = 50 - 13 = 37$$

So option (2) is correct.

3. Let A, B be $n \times n$ matrices. Which of the following equals to $t_r\ (A^2B^2)$?

1)$(t_r(AB))^2$

2)$t_r\ (AB^2A)$

3)$t_r((AB)^2)$

4)$t_r\ (BABA)$

Sol.

We know, $t_r(AB) = t_r\ (BA)$

So, $t_r(A^2B^2) = t_r\big(A(ABB)\big) = t_r(ABBA) = t_r\ (AB^2A)$

$\therefore$ Option (2) is correct.

Take, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$to rule out options (1), (3) and (4).

4. Given a 4×4 real matrix A, let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation defined by $Tv = Av$, where we think of $\mathbb{R}^4$ as the set of real 4×1 matrices. For which choices of A given below, do $Image\ (T)\ and\ Image(T^2)$ have respectively the dimensions 2 and 1? ($*$ $denotes\ a\ non-\ zero\ entry$).

1)$A = \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$

2)$A = \begin{pmatrix} 0 & 0 & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{pmatrix}$

3)$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \end{pmatrix}$

4)$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$

Sol:

Only we need to check from options that matrix 'A' has $R(A) = 2\ and\ R(A^2) = 1.$

For option (1)

$A = \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}_{4\times 4}$ Where $'*'$ is a non – zero real entry.

Clearly, $R(A) = 2$ ($\because$ $two\ independent\ columns$)

$$A^2 = A \cdot A = \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & *\cdot* \\ 0 & 0 & 0 & *\cdot* \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{4\times 4}$$

$$(non - \ zero\ entry\ x\ non - \ zero\ entry\ = \ non - \ zero\ entry)$$

$$\Rightarrow Rank(A^2) = 1\ (One\ idependent\ column)$$

So, option (1) is correct.

For option (2)

$A = \begin{pmatrix} 0 & 0 & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{pmatrix}$ Clearly, $R(A) = 2\ (\because two\ independent\ column)$.

$$A^2 = A \cdot A = \begin{pmatrix} 0 & 0 & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{pmatrix} \begin{pmatrix} 0 & 0 & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & *\cdot* \\ 0 & 0 & 0 & *\cdot* \\ 0 & 0 & 0 & *\cdot* \\ 0 & 0 & 0 & *\cdot* \end{pmatrix}$$

$$\Rightarrow R(A^2) = 1\ (One\ indepenndent\ column)$$

So, option (2) correct.

For option (3)

Take $A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ then $A^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Therefore $R(A) = 2$ & $R(A^2) = 2$
So, option (3) is incorrect.
For option (4)

Take $A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

Clearly $R(A) = 1,$ but we have to need $R(A) = 2$

So, option (4) is incorrect.

5. Let $T\ be\ a\ 4 \times 4$ real matrix such that $T^4 = 0. Let\ k_1 = \ dim\ kerT^i$ for $1 \le i \le 4$. Which of the following is not a possibility for the sequences $k_1 \le k_2 \le k_3 \le k_4$?

1)$3 \le 4 \le 4 \le 4$

2)$1 \le 3 \le 4 \le 4$

3) $2 \le 4 \le 4 \le 4$

4)$2 \le 3 \le 4 \le 4$

Sol:

For option (1)

Take $T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$T^2 = T^3 = T^4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow\ k_1 = \dim kerT = 3,\ k_2 = \dim kerT^2 = 4, k_3 = 4\ and\ k_4 = 4.$

$\Rightarrow$ option (1) is incorrect as $3 \le 4 \le 4 \le 4$ is possible sequence.

For option (3)

Take $T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$T^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, T^3 = T^4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow\ k_1 = 2, k_2 = 4, \qquad k_3 = k_4 = 4$$

$\Rightarrow$ option (3) is incorrect as $2 \le 4 \le 4 \le 4$ possible sequence.

For option (4),

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, T^3 = T^4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow\ k_1 = 2, k_2 = 3, \qquad k_3 = k_4 = 4$$

$\Rightarrow$option (4) is incorrect $2 \le 3 \le 4 \le 4$ is possible sequence.

So, option (2) is correct as rest of the options are incorrect.

6. Which of the following is a linear transformation from $\mathbb{R}^3\ to\ \mathbb{R}^2$?

(a) $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ x+y \end{pmatrix}$ (b) $g\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} xy \\ x+y \end{pmatrix}$ (c) $h\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z-x \\ x+y \end{pmatrix}$

1)$only\ f$
2) $only\ g$
3)$only\ h$
4)$all\ the\ transformations\ f, g, and\ h$

Sol:-

For a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$

$T(\alpha x + \beta y) = \alpha t(x) + \beta T(y); \forall\, x, y \in \mathbb{R}^3\,,\ \alpha, \beta \in \mathbb{R}$.

Let $x', x'' \in \mathbb{R}^3\,; x' = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, x'' = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$

For option (1)

$$f\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} = \begin{pmatrix} 4 \\ x_1 + x_2 + y_1 + y_2 \end{pmatrix} \neq f\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + f\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

So, f is not a linear transformation.

For option (2)

$$f\begin{pmatrix} x_2 + x_1 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} = \begin{pmatrix} (x_1 + x_2)(y_1 + y_2) \\ x_1 + x_2 + y_1 + y_2 \end{pmatrix} \neq g\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + g\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

So, g is not a linear transformation.

For option (3)

$$h\begin{pmatrix} \alpha x_1 + \beta x_2 \\ \alpha y_1 + \beta y_2 \\ \alpha z_1 + \beta z_2 \end{pmatrix} = \begin{pmatrix} \alpha z_1 + \beta z_2 - \alpha x_1 - \beta x_2 \\ \alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2 \end{pmatrix} \neq \alpha h\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \beta h\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

So, h is a linear transformation.

$\therefore$ (3) is correct.

7. Let A be an $n \times n$ matrix of rank n with real entries. Choose the correct statement.

1)$AX = b$ has a solution for any b.

2)$AX = 0$ does not have a solution.

3) if $AX = b$ has a solution, then it is unique.

4)$y'A = 0$ for some non – zero $y, where\ y'$ denotes the transpose of the vector y.

Sol: - For option (1)

$AX = b$ has a solution if $R(A) = R[A\!:b]$, where A is $m \times n$ matrix of $rank\ n$.

If $R(A) = R[A\!:b] = number\ of\ columns\ of\ A, then\ AX = b\ has\ a\ unique\ solution.$

If A is 4×3 matrix of $rank$ 3 and if $R[A\!:b] = 4$ for some b, then $Ax = b$ has no solution for that b,

So, option (1) is incorrect.

For e.g.,

Take $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}_{4\times3}$, $b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}_{4\times1}$

$[A:b] = \begin{pmatrix} 1 & 0 & 00 \\ 0 & 1 & 00 \\ 0 & 0 & 10 \\ 0 & 0 & 01 \end{pmatrix}$ has $rank$ 4 whereas $rank\ A = 3$

So, $R[A:b] \neq R(A)$

For option (2),

In this case $R(A) = R(A:0)$

So, $Ax = 0$ always has a solution.

For option (3)

If $Ax = b$ has a solution then $R(A) = R(A:b) = n = number\ of\ columns\ in\ A,$ so it has unique solution. So, (3) is correct.

For option (4)

Let A be $n \times n (m = n)$, the identity matrix $\Rightarrow y^t A = y^t I = 0 \ \Rightarrow y = 0$.

So, option (4) is incorrect.

Part – C

8. Let $F: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the function $F(x,y) = \langle Ax, y \rangle$, where $\langle , \rangle$ is the standard inner product of $\mathbb{R}^n$ and A is a $n \times n$ real matrix. Here D denotes the total derivative. Which of the following statements are correct?

1)$\big(DF(x,y)\big)(u,v) = \langle Au, y \rangle + \langle Ax, v \rangle$

2)$\big(Df(x,y)\big)(0,0) = 0$

3)$DF(x,y)$ may not exist for some $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$

4)$Df(x,y)$ does not exist at $(x,y) = (0,0)$.

Sol:-

9. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function such that $\int_{\mathbb{R}^n} |f(x)|\, dx < \infty$.
Let A be a real $n \times n$ invertable matrix and for $x, y \in \mathbb{R}^n$, let $\langle x, y \rangle$ denote the standard inner product in $\mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} f(Ax)\, e^{i\langle y,x \rangle} dx =$$

1)$\int_{\mathbb{R}^n} f(Ax)\, e^{i\langle (A^{-1})^T y, x \rangle} \frac{dx}{|detA|}$

2)$\int_{\mathbb{R}^n} f(Ax)\, e^{i\langle (A)^T y, x \rangle} \frac{dx}{|detA|}$

3)$\int_{\mathbb{R}^n} f(Ax)\, e^{i\langle (A^T)^{-1} y, x \rangle} \frac{dx}{|detA|}$

4)$\int_{\mathbb{R}^n} f(Ax)\, e^{i\langle (A)^{-1} y, x \rangle} \frac{dx}{|detA|}$

Sol:-

10. Let S be the set of 3×3 real matrices A with

$A^T A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then the set S contains

1) a nilpotent matrix

2) a matrix of rank one.

3) a matrix of rank two

4) a non – zero skew – symmetric matrix.

Sol:-

For option (1)

Take $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$A^T A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A is nilpotent as its eigen values are zero.

So, S contains a nilpotent matrix.

If A is a real matrix, then $rank\ (A^T A) = R(A)$

So, option (2) is correct.

Clearly, $rank\ (A^T A) = 1\ \Rightarrow S$ contains a matrix of $rank\ one$, but it does not contain a matrix of $rank\ two$.

So, option (3) is incorrect.

Rank of a non – zero skew – symmetric matrix cannot be one.

So, S does not contain a non – zero skew symmetric matrix.

So, option (4) is incorrect.

11. An $n \times n$ complex matrix A satisfies $A^k = I_n$, the $n \times n$ identity matrix, where $k\ is\ a\ positive\ integer > 1$. Suppose 1 is not an eigen value of A. Then which of the following statements are necessarily true?

1)$A\ is\ diagonalizable.$

2)$A + A^2 + \ldots\ldots\ .. + A^{k-1} = 0, the\ n \times n\ zero\ matrix.$

3)$t_r(A) + t_r(A^2) + \ldots\ldots\ .. + t_r(\ A^{k-1}) = -n$

4)$A^{-1} + A^{-2} + \ldots\ldots\ .. + A^{-(k-1)} = -I_n$

Sol:-

Given that, A is $n \times n$ matrix (complex) such that $A^k = I_n$

$$\Rightarrow A\ satisfies\ \ x^k - 1 = (x-1)(x^{k-1} + x^{k-2} + \ldots\ldots + x + 1)$$

Since, $'1'$ is not an eigen value of A, which implies that minimal polynomial of A cannot have a factor $'x-1'$.

We can say, $m_A\ (x) | (x^{k-1} + x^{k-2} + \ldots\ldots + x + 1) \ldots\ldots\ldots\ldots\ldots (i)$

$\Rightarrow\ x^{k-1} + x^{k-2} + \ldots\ldots + x + 1 = \ m_A(x) \cdot P(x)$, for some $P(x)$.

We know, every matrix satisfies its minimal polynomial,

i.e., $m_A(A) = 0\ \Rightarrow A^{k-1} + A^{k-2} + \ldots\ldots + A + I = 0 \ldots\ldots\ldots\ldots (ii)$

So, option (2) is incorrect.

From option (1), $m_A\ (x) | (x^{k-1} + x^{k-2} + \ldots\ldots + x + 1)$

$\Rightarrow m_A\ (x) | (x - \xi_1)(x - \xi_2) \ldots \ldots \ldots . (x - \xi_{k-1})$, where ξ_i are distinct roots of $(x^{k-1} + x^{k-2} + \ldots \ldots + x + 1) \quad \Rightarrow m_A(x)$has distinct linear factors $\Rightarrow A$ is diagonalizable.

So, option (1) is correct.

From option (2), we have $A^{k-1} + A^{k-2} + \ldots \ldots + A + I = 0$

Take 'trace' on both sides, we get $\Rightarrow t_r(A^{k-1} + A^{k-2} + \ldots \ldots + A + I) = 0$

$$\Rightarrow \ t_r(A^{k-1}) + t_r(A^{k-2}) + \ldots \ldots + t_r(A) + t_r(I)$$

$$\Rightarrow \ t_r(A) + \ t_r(A^2) + \ldots \ldots .. + t_r(A^{k-1}) = \ -n.$$

So, option (3) is correct.

Multiplying both sides by A^{-k} of equation (i) we get

$$A^{-k}(A^{k-1} + A^{k-2} + \ldots \ldots + A^2 + A + I) = 0$$

$$\Rightarrow A^{-1} + A^{-2} + \ldots \ldots + A^{-(k-1)} = -I_n$$

So, option (4) is correct.

12. Let $S: \mathbb{R}^n \ \to \mathbb{R}^n$ be given by $S(v) = \alpha v$ for a fixed $\alpha \in \mathbb{R}, \alpha \neq 0$. Let $T: \mathbb{R}^n \ \to \mathbb{R}^n$ be L.T. such that $B = \{v_1, v_2, \ldots \ldots \ldots , \ v_n\}$ is a set of linearly independent eigen values of T. Then

1) The matrix of T with respect to B is diagonal.

2) The matrix of $(T - S)$ with respect to B is diagonal.

3) The matrix of T with respect to B is not necessarily diagonal but is upper triangular.

4) The matrix of T with respect to B is diagonal but the matrix of $(T - S)$ with respect to B is not diagonal.

Sol:-

The matrix of T with respect to 'B' is given by

$$[T]_B = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_n \end{pmatrix}$$

So, $[T]_B$ is diagonal matrix.

Again, $[S]_B = \begin{pmatrix} \alpha & & & \\ & \alpha & & \\ & & \ddots & \\ & & & \alpha \end{pmatrix}$

Now the matrix $T - S$ with respect to B is given by

$$[T - S]_B = \begin{pmatrix} \alpha_1 - \alpha & & & \\ & \alpha_2 - \alpha & & \\ & & \ddots & \\ & & & \alpha_n - \ \alpha \end{pmatrix}$$

So, $[T-S]_B$ is also a diagonal matrix.

Hence, option (1), (2) are correct and (3), (4) are incorrect.

13. Let $P_n(x) = x^n$ for $x \in \mathbb{R}$ *and* let $\wp = span\{P_0, P_1, P_2, \ldots\ldots\}$. Then

1)$\wp$ is the vector space of all real valued continuous functions on $\mathbb{R}$.

2)$\{P_0, P_1, P_2, \ldots\ldots\}$ is a linearly independent set in the vector space of all continuous functions on $\mathbb{R}$.

3)$\{P_0, P_1, P_2, \ldots\ldots\}$ is a linearly independent set in the vector space of all continuous functions on $\mathbb{R}$,

4) Trigonometric functions belong to Q.

Sol:-

Since $\wp = span\{1, x, x^2, \ldots\ldots\} = set$ of all polynomials over real numbers.

$\Rightarrow \wp \underset{\neq}{\subset} V = \{f | f: R \to R \text{ and } f \text{ is continuous}\}$ over $\mathbb{R}$ (i.e., the set of all real valued continuous functions)

Since $\wp$ is the vector space and also V is vector space over $\mathbb{R}$ *and* $\wp \underset{\neq}{\subset} V$.

$\Rightarrow \wp$ is a subspace of real valued continuous function on $\mathbb{R}$.

So, option (1) is incorrect and option (2) is correct.

$\{P_0, P_1, P_2, \ldots\ldots\} = \{1, x, x^2, \ldots\ldots\}$ is clearly linearly independent set ($\because$ none of the elements of above set can be written as linear combination of other elements)

So, option (3) is correct.

For option (4)

$\sin x$ *and* $\cos x$ are trigonometric function $\wp$ ($\because \sin x \,\&\, \cos x$ cannot be written as a linear combination of finite elements of $\wp$) .

So, option (4) is incorrect.

14. Let $A = \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}$ be a 3×3 matrix, where a, b, c, d are integers. Then we must have

1) If $a \neq 0$, there is a polynomial $P \in Q[x]$ *such that* $P(A)$ is the inverse of A.

2) For each polynomial $q \in \mathbb{Z}[x]$, the matrix $q(A) = \begin{pmatrix} q(a) & q(b) & q(c) \\ 0 & q(a) & q(d) \\ 0 & 0 & q(a) \end{pmatrix}$

3) If $A^n = 0$ for some positive integer n, then $A^3 = 0$.

4)Acommute with every matrix of the form $\begin{pmatrix} a' & 0 & c' \\ 0 & a' & 0 \\ 0 & 0 & a' \end{pmatrix}$

Sol:-

For option (1)

Let $A = \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}$ be a 3×3 matrix, where a, b, c, d are integers.

$C_A(x) = (a - \lambda)^3$.

A satisfies its characteristic polynomial, i.e., $C_A(x) = (a - A)^3 = 0$

$$a^3 + 3a^2A + 3a\,A^2 - \,A^3 = 0$$

If $a \neq 0$, then A^{-1} exist.

$$a^3A^{-1} = \,A^2 - 3a^2A + \,3a^2I \;\Rightarrow\; A^{-1} = \frac{A^2}{a^3} - \frac{3A}{a^2} + \frac{3I}{a}$$

So, if $a \neq 0$, there is a polynomial $P(x) \in Q[x]\ such\ that\ P(A)$ is the inverse of A.

So, option (1) is correct.

For option (2)

Take $q(x) = x^2 + 1 \in \mathbb{Z}[x]$

$$q(A) = A^2 + I = \begin{pmatrix} a^2+1 & 2ab & 2ac+bd \\ 0 & a^2+1 & 2ad \\ 0 & 0 & a^2+1 \end{pmatrix} \neq \begin{pmatrix} q(a) & q(b) & q(c) \\ 0 & q(a) & q(d) \\ 0 & 0 & q(a) \end{pmatrix}$$

So, option (2) is incorrect.

For option (3)

If $A^n = 0$ for some positive integer n.

$\Rightarrow$ 0 is the only eigen value of A with multiplicity 3 (as A is a nilpotent matrix)

$\Rightarrow\; C_A(x) = x^3 \;\Rightarrow A$ satisfies the characteristic polynomial $\Rightarrow x^3 = 0$.

So, option (3) is correct.

Let $B = \begin{pmatrix} a' & 0 & c' \\ 0 & a' & 0 \\ 0 & 0 & a' \end{pmatrix}, BA = \begin{pmatrix} a' & 0 & c' \\ 0 & a' & 0 \\ 0 & 0 & a' \end{pmatrix}\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} = \begin{pmatrix} aa' & a'b & 2ac' \\ 0 & aa' & a'd \\ 0 & 0 & aa' \end{pmatrix}$

$$AB = \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}\begin{pmatrix} a' & 0 & c' \\ 0 & a' & 0 \\ 0 & 0 & a' \end{pmatrix} = \begin{pmatrix} aa' & a'b & 2ac' \\ 0 & aa' & a'd \\ 0 & 0 & aa' \end{pmatrix}$$

$$\therefore AB = BA$$

$\Rightarrow Option$ (4) is correct.

15. Which of the following are subspaces of the vector space $\mathbb{R}^3$?

1)$\{(x,y,z): x+y=0\}$

2)$\{(x,y,z): x-y=0\}$

3)$\{(x,y,z): x+y=1\}$

4)$\{(x,y,z): x-y=1\}$

Sol:-

Clearly, $S_1 = \{(x,y,z): x+y=0\}$ is a subspace of $\mathbb{R}^3$.

$\because$ $x+y=0$ is a linear condition and (0,0,0) passes through it.

So, option (1) is correct.

By similar arguments option (2) is correct.

Let $S_3 = \{(x,y,z): x+y=1\}$; as $(0,0,0) \notin S_3$. So, it is not a subspace of $\mathbb{R}^3$.

So, option (3) is incorrect.

Similarly, we can say option (4) is incorrect.

16. Consider the non – zero vector spaces v_1, v_2, v_3, v_4 and L.T. $Q_1: V_1 \to V_2$, $\phi_2: V_2 \to V_3$, $\phi_3 : V_3 \to V_4$ such that $\ker(\phi_1) = \{0\}$, $Range(\phi_1) = \ker(\phi_2)$, $Range\ (\phi_2) = \ker(\phi_3)$, $Range\ (\phi_3) = \ker(\phi_4)$. Then

1)$\sum_{i=1}^{4}(-1)^i\, dimV_i = 0$

2)$\sum_{i=2}^{4}(-1)^i\, dimV_i > 0$

3)$\sum_{i=1}^{4}(-1)^i\, dimV_i < 0$

4)$\sum_{i=1}^{4}(-1)^i\, dimV_i \neq 0$

Sol:-

We have $nullity\ (\phi_1) = 0, Rank(\phi_1) = nullity\ (\phi_2), as\ range\ (\phi_1) = rank(\phi_2) = rank(\phi_3) = nullity\ (\phi_3), as\ range\ (\phi_2) = \ker(\phi_3)\ , rank(\phi_3) = dim(V_4)$, $as\ range(\phi_3) = V_4$.

By Rank – Nullity theorem, we have

$$\dim V_1 = nullity\ (\phi_1) + rank\ (\phi_1)$$

$$\dim V_1 = nullity\ (\phi_2) \ldots\ldots\ldots\ldots\ldots (i)$$

$$\dim V_2 = nullity\ (\phi_2) + rank\ (\phi_2)$$

$$\dim V_2 = nullity\ (\phi_2) + nullity\ (\phi_3) \ldots\ldots\ldots (ii)$$

$$\dim V_3 = nullity\ (\phi_3) + rank\ (\phi_3)$$

$$\dim V_3 = nullity\ (\phi_3) + dim\ (V_4)............(iii)$$

From equation (i), (ii) & (iii), we get $\sum_{i=1}^{4}(-1)^i\, V_i = 0$

So, option (1) is correct.

$$\sum_{i=2}^{4}(-1)^i\, V_i = \dim V_2 - \dim V_3 + \ \dim V_4 = \ \dim V_1 > 0.$$

as dimension of non – zero vector space is always positive.
⇒ option (2) is correct.

17. Let A be an invertable 4×4 real matrix. Which of the following is not true?

1)$Rank(A) = 4$
2) For every vector $b \in \mathbb{R}^4, Ax = b$ has exactly one solution.
3)$\dim(null\ space\ A) \geq 1.$
4) 0 is an eigen values of A.
Sol:-

Since A is an invertible matrix so $|A| \neq 0 \ \Rightarrow Rank(A) = 4$

So, option (1) is incorrect.
For every vector $b \in \ \mathbb{R}^4, Rank(A) = Rank(A{:}\,b) = 4 = number\ of\ variables.$
$\Rightarrow Ax = 0$ has exactly one solution.
So, option (2) is incorrect.
By Sylvester's theorem

$$R(A) + N(A) = 4 \ \ \Rightarrow N(A) = 4 - 4 = 0 \Rightarrow \dim(null\ space\ of\ A) = 0$$

So, option (3) is correct.
We have $|A| \neq 0.$
Since $|A|$ is the product of the eigen values so 0 is not an eigen value of A.
So, option (4) is correct.

18. Let$\underline{u}$ be a real $n \times 1$ vector satisfying $\underline{u'u} = 1$, where u is the transpose of $\underline{u'}$. Define $A = I - 2\underline{uu'}$, where I is the $n - th$ order identity matrix. Which of the following statements are true?

1)$A\ is\ singular.$
2)$A^2 = A$
3)$Trace(A) \ = \ n - 2$
4)$A^2 = I$
Sol:- Given that, $A = I - 2\underline{uu'}$ such that $\underline{u'u} = 1$, where u is a real $n \times 1$ vector.

Take $u = \begin{pmatrix}1\\0\end{pmatrix} \ \Rightarrow \underline{u'u} = (1{,}0)\begin{pmatrix}1\\0\end{pmatrix} = (1 + 0) = (1)$

So, $A = I - 2\underline{uu'} = I - 2\ \begin{pmatrix}1\\0\end{pmatrix}(1{,}0) = \ \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - \begin{pmatrix}2 & 0\\0 & 0\end{pmatrix} = \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$
A is non – singular as $|A| = -1 \neq 0$ and $A^2 \neq A$
So, option (1) and (b) are incorrect.

$$A = I - 2\underline{uu'}$$

$A^2 = (I - 2uu')^2 = I^2 + 4(uu')^2 - 4uu' = I + 4(uu')(uu') - 4uu' = I + 4uu' - 4uu'$
$\Rightarrow A^2 = I.$
So, option (4) is correct.

Since, $A = I - 2uu' \Rightarrow traceA = trace\ (I - 2uu') = trace(I) - trace(2uu')$

$$= trace(I) - 2trace(uu') = n - 2$$

So, option (3) is correct.

Answer

Sl. NO	Answer	Sl. No	Answer	Sl. No	Answer
1.	1	2.	4	3.	2
4.	1 & 2	5.	2	6.	3
7.	3	8.	-	9.	-
10.	1 & 2	11.	1, 3 & 4	12.	1 & 2
13.	2 & 3	14.	1, 3 & 4	15.	1 & 2
16.	1 & 2	17.	3 & 4	18.	3 & 4

Previous Year Questions & Solution

Linear Algebra

December – 2015

Part – B

1. Let S denote the set of all the prime numbers P with the property that the matrix $\begin{pmatrix} 91 & 31 & 0 \\ 29 & 31 & 0 \\ 79 & 23 & 59 \end{pmatrix}$ has an inverse in the field $\frac{\mathbb{Z}}{P\mathbb{Z}}$. Then

1) A is singular.

2)$S = \{31,59\}$

3)$S = \{7, 13, 59\}$

4)S is infinite.

Sol:-

Let $A = \begin{pmatrix} 91 & 31 & 0 \\ 29 & 31 & 0 \\ 79 & 23 & 59 \end{pmatrix} \Rightarrow |A| = 2 \cdot (31)^2 \cdot 59$

Thus, the matrix 'A' has an inverse in the field $\frac{\mathbb{Z}}{p\mathbb{Z}}$ for all primes 'P' such that $P \neq 2, 31\ and\ 59$.

Since, option (1), (2) and (3) are not correct as they continuous '31' & '59'.

So, only option (4) is correct.

2. For a positive integer n, let p_n denote the vector space of polynomials in one variable x with real co – efficient and with $degree \leq n$. Consider the map $T: p_2 \to p_4$ defined by

$T\big(p(x)\big) = p(x^2)$. Then

1) T is a linear transformation and dim $range\ (T) = 5$.

2) T is a linear transformation and dim $range\ (T) = 3$

3) T is a linear transformation and dim $range(T) = 2$.

4) T is a not a linear transformation.

Sol:-

$$\forall\ \alpha, \beta \in \mathbb{R}$$

$$T\big((\alpha p + \beta q)(x)\big) = (\alpha p + \beta q)(x^2) = \alpha p(x^2) + \beta q(x^2) = \alpha T\big(p(x)\big) + \beta T\big(q(x)\big)$$

$$\Rightarrow T\ is\ a\ L.T.$$

Let $B = \{1, x, x^2\}$ and $B = \{1, x,\ x^2, x^3, x^4\}$ are basis of p_2 and p_4 respectively.

$$T(1) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot\ x^2 + 0 \cdot x^3 + 0 \cdot x^4$$

$$T(x) =\ x^2 = 0 \cdot 1 + 0 \cdot x + 1 \cdot\ x^2 + 0 \cdot x^3 + 0 \cdot x^4$$

$$T(\ x^2) =\ x^4 = 0 \cdot 1 + 0 \cdot x + 0 \cdot\ x^2 + 0 \cdot x^3 + 0 \cdot x^4$$

The matrix T is given by, $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow Rank(T) = 3$

$\Rightarrow T$ is a linear transformation and $\dim(rangeT) = 3$.

Hence, option (2) is correct.

3. Let $\boldsymbol{A}$ be a real 3×4 matrix of $rank$ 2. Then the rank of A^tA, where A denotes the transpose of A^t, is

1) exactly 2

2) exactly 3.

3) exactly 4

4) at most 2 but not necessarily 2.

Sol:-

$$R(A) = R(A^tA)$$

Let, $x \in N(A), the\ null\ space\ of\ A$.

So, $Ax = 0\ \Rightarrow A^TAx = 0 \Rightarrow x \in N(A^TA)$

Hence, $N(A) \subseteq N(A^TA)$

Again Let $x \in N(A^TA)$, So, $A^TAx = 0$

$$\Rightarrow\ x^TA^TAx = 0 \Rightarrow\ (Ax)^T\ Ax = 0 \Rightarrow ||Ax||^2 = 0$$

$$\Rightarrow Ax = 0\ \ \Rightarrow x \in N(A) \Rightarrow N(A^TA) \subseteq N(A)$$

Hence, $N(A^TA) = N(A)$

$$\Rightarrow \dim\big(N(A^TA)\big) = \dim\big(N(A)\big) \Rightarrow R(A^TA) = R(A) = 2$$

So, option (1) is correct.

4. Consider the quadratic form $Q(v) = v^t\, Av$, where

$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, v = (x, y, z, w)$. Then

1)Q has rank 3.

2)$xy + z^2 = Q(pv)$ for some invertable 4×4 real matrix p.

3) $xy + y^2 + z^2 = Q(pv)$ for some invertable 4×4 real matrix p.

4)$y^2 + z^2 - zw = Q(pv)$ for some invertable 4×4 real matrix p.

Sol:-

$$Q(v) = v^t\, Av = (x \quad y \quad z \quad w) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = x^2 + z^2 + 2zw$$

For option (1)

As the matrix associated with quadratic from is A, so Q has rank 4.

Hence, option (1) is correct.

For option (2)

$xy + z^2 = Q(pv) = (pv)^t\, A(pv) = v^t p^t\, A\, pv$, where p is invertable.

$$\Rightarrow (x \quad y \quad z \quad w) \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = (x \quad y \quad z \quad w) p^t\, A\, p \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

$$\Rightarrow p^t\, A\, p = \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Comparing determinants on both sides $|p||A||p| = 0$ ($\because\ |p^T||p|$) $\Rightarrow\ |p|^2(-1) = 0$

$$\Rightarrow\ |p|^2 = 0\ \Rightarrow |p| = 0\ \Rightarrow p \text{ is not invertable}.$$

So, option (2) is incorrect.

For option (3)

$xy + y^2 + z^2 = Q(pv) = (pv)^t\, A(pv) = v^t p^t\, A\, pv$, where p is an invertable matrix.

$$\Rightarrow (x \quad y \quad z \quad w) \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = (x \quad y \quad z \quad w) p^t\, A\, p \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

$$\Rightarrow p^t\, A\, p = \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Taking determinants

$\Rightarrow\ |p||A||p| = 0\ \ \Rightarrow\ -|p|^2 = 0 \Rightarrow\ |p| = 0,$ a contradiction to the fact that p is invertable, option (3) is correct.

As all other options are incorrect.

∴ option (4) is correct.

5. If A is a 5×5 real matrix with trace 15 and if $2\ and\ 3$ are eigen values of A, each with algebraic multiplicity 2, then the determinant of A is equal to

1) 0

2) 24

3) 120

4) 180

Sol:-

Since $2\ and\ 3$ are eigen values of A, each with algebraic multiplicity 2, so the characteristic polynomial of A is of the type $C_A(x) = (x-\alpha)(x-2)^2(x-3)^2$

Also $tr(A) = 15\ \Rightarrow sum\ of\ the\ eigen\ values = 15\ \Rightarrow \alpha + 2 + 2 + 3 + 3 = 15\ \Rightarrow \alpha = 5$

As $|A| = product\ of\ all\ the\ eigen\ values\ of\ A\ \Rightarrow\ |A| = 180$

So, option (4) is correct.

6. Let $A \neq I_n$ be an $n \times n$ matrix such that $A = A$, whereI_n is the identity matrix of order n. Which of the following statements is false?

1)$(I_n - A)^2 = (I_n - A)$

2)$Trace(A) = Rank(A)$

3)$Rank(A) + Rank\ (I_n - A) = n$

4) The eigen values of A are each equal to 1.

Sol:-

As $A^2 = A \Rightarrow A(A - I) = 0 \Rightarrow$ The minimal polynomial of A,

$$m_A(x)|x(x-1)\ \Rightarrow m_A(x) = x, (x-1)\ \ or\ x(x-1)$$

If $m_A(x) = x - 1 \Rightarrow A\ satisfies\ \ A - I = 0\ \Rightarrow A = I,$ which is a contradiction, as it is given that $A \neq I_n\ \Rightarrow\ m_A(x) = x\ or\ x(x-1) \Rightarrow '0'$is an eigen value of A.

$\Rightarrow$ option (4) is false statement and hence is the correct answer.

Part – C

7. Let $A\ and\ B$ be $n \times n$ matrices over $\mathbb{C}$. Then

1)$AB\ and\ BA$ always have the same set of eigen values.

2) if $AB\ and\ BA$ have the same set of eigen values then $AB = BA$.

3) If A^{-1} exists the $AB\ and\ BA$ are similar.

4) The rank of AB is always the same as the rank of BA.

Sol:-

Case – I:-A is non – singular matrix. The $AB = A^{-1}(AB)A$.

$\Rightarrow BA\ and\ AB$ are similar matrices $\Rightarrow AB\ and\ BA$ have the same set of eigen values.

Similar is case, when B is non – singular.

Case – II:-A is singular.

$\Rightarrow AB\ and\ BA$ both are singular matrices $\Rightarrow$ both $AB\ and\ BA$ have zero as an eigen values. We have to show that non – zero eigen values are co-inside.

Let $\lambda \neq 0$ be an eigen value of AB.

$\exists$ an eigen vector $x \neq 0$ such that $ABx = \lambda x$

Pre-multiply B on both sides, we get $BABx = B\lambda x = \lambda Bx$

$\Rightarrow BA\ (Bx) = \lambda(Bx)\ \Rightarrow BAy = \lambda y, where\ y = Bx \neq 0\ \Rightarrow \lambda\ is\ an\ eigen\ value\ of\ BA.$

The converse can be argued similarly.

Similarly, the case when B is singular matrix.

So, option (1) is correct.

Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix},\ B = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$

Here, $AB\ and\ BA$ have same set of eigen values but $AB \neq BA$.

So, option (2) is incorrect.

If A^{-1} exists, we can write $BA = A^{-1}(AB)A\ \Rightarrow AB\ \ and\ \ BA$ are similar.

So, option (3) is correct.

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$Rank(AB) = 1\ \&\ Rank(BA) = 0$$

So, option (4) is incorrect.

8. Let A be an $m \times n$ real matrix and $b \in \mathbb{R}^n$ with $b \neq D$.

1) The set of all real solutions of $Ax = b$ is a vector space.

2) If $u\ and\ v$ are two solutions of $Ax = b,$ then $\lambda u + (1 - \lambda)v$ is also a solution of $Ax = b,$ for any $\lambda \in \mathbb{R}$

3) For any two solutions u and v of $Ax = b$, the linear combination $\lambda u + (1 - \lambda)v$ is also a solution $Ax = b$. Only when $0 \leq \lambda \leq 1$.

4) If $rank\ of\ A$ is n , then $Ax = b$ has at most one solution.

Sol:-

If x_1, x_2 are the solutions of $Ax = b$, then $x_1 + x_2$ is not the solution.

$\Rightarrow$ set of all real solutions of $Ax = b$ is not a vector space.

So, option (1) is incorrect.

Given that $Au = b,\ Av = b$

$$A(\lambda u + (1 - \lambda)v) = \lambda Au + (1 - \lambda)Av = \lambda b + (1 - \lambda)b = b \in for\ any\ \lambda \in \mathbb{R}.$$

So, option (2) is correct and option (3) is ruled out.

For option (4)

If $R(A) \neq R(A{:}\,b),\ then\ Ax = b$ has no solution.

But if $R(A) = R(A{:}\,b) = number\ of\ variables = n, then\ Ax = b$ has unique solution.

So, $Ax = b$ has at most one solution.

So, option (4) is correct.

9. Let A be an $n \times n$ matrix over $\mathbb{C}$ such that every non – zero vector of $\mathbb{C}^n$ is an eigen vector of A. Then

1) All eigen values of a are equal.

2) All eigen values of A are distinct.

3)$A = \lambda I$ for some $\lambda \in \mathbb{C},$ where I is the $n \times n$ identity matrix.

4) If $\chi_A$and m_A denote the characteristic polynomial and minimal polynomial respectively, then $\chi_A = m_A$

Sol:-

Since every vector is an eigen vector of

$$A \Rightarrow Ae_1 = \lambda_1 e_1$$

$$Ae_2 = \lambda_2 e_2$$

$$Ae_3 = \lambda_3 e_3$$

. . ..

$$Ae_n = \lambda_n e_n$$

$(e_1 + e_2 + \ldots\ldots + e_n)$ is also an eigen vector of A.

$$\Rightarrow A\,(e_1 + e_2 + \ldots\ldots + e_n) = \lambda\,(e_1 + e_2 + \ldots\ldots + e_n), for\ some\ \lambda \in \mathbb{R}.$$

$$\Rightarrow\ Ae_1 + Ae_2 + \ldots\ldots + Ae_n = \lambda\,(e_1 + e_2 + \ldots\ldots + e_n)$$

$$\Rightarrow (\lambda_1 - \lambda)e_1 + (\lambda_2 - \lambda)e_2 + \ldots\ldots + (\lambda_n - \lambda)e_n = 0$$

Since $e_1, e_2, \ldots\ldots, e_n$ are linearly independent.

$$\Rightarrow\ \lambda_i - \lambda = 0\ \forall\ i\ \Rightarrow\ \lambda_i = \lambda\ \forall\ i\ \Rightarrow All\ the\ eigen\ values\ are\ equal.$$

$$\Rightarrow Ax = \lambda x\ \forall\ x \in \mathbb{C}^n\ \Rightarrow (A - \lambda I)x = 0\ \forall\ x \in \mathbb{C}^n\ \Rightarrow \ker(A - \lambda I) = \mathbb{C}^n$$

$$\Rightarrow range\ (A - \lambda I) = \{0\}\ \Rightarrow rank\ (A - \lambda I) = 0\ \Rightarrow A - \lambda I = 0$$

$\Rightarrow A = \lambda I\ \Rightarrow A$ is scalar multiple of identity matrix.

So, option (1), (3) are correct & option (2) is incorrect.

Since, $A = \lambda I \Rightarrow A$ is diagonalizable (every scalar matrix is diagonalizable)

$$\Rightarrow\ \chi_A(x) = (x - \lambda)^n\ and\ m_A(x) = x - \lambda \Rightarrow\ \chi_A(x) \neq m_A(x)$$

So, option (4) is incorrect.

10. Consider the matrices $A = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}\ and\ B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Then

1)$A\ and\ B$ are similar over the field of rational numbers Q

2)A is diagonalizable over the field of rational numbers Q.

3) B is the Jordan canonical form of A.

4) The minimal polynomial and the characteristic polynomial of A are the same.

Sol:-

The minimal polynomials of the matrices are same and they are not square free so A is not diagonalizable.

So, option (2) is incorrect.

Clearly, B is the Jordan canonical form of A as highest order block corresponding to 2 is the Jordan canonical form is of order 2.

Thus, $A\ and\ B$ are similar over Q.

∴ options (1) and (3) are correct.

Also, similar matrices have same characteristic polynomial and minimal polynomial.

Hence, option (4) is also correct.

11. Let V be a finite dimensional vector space over $\mathbb{R}$. Let $T: V \to V$ be a L.T. Such that $rank(T^2) = rank\ (T)$. Then,

1)$Kennel\ (T^2)\ =\ Kennel\ (T)$

2)$Range\ (T^2)\ =\ Range\ (T)$

3)$Kennel\ (T) \cap\ Range\ (T) = \{0\}$

4)$Kennel\ (T^2) \cap Range\ (T^2) = \{0\}$

Sol:-

Given that, $T: V \to V$ is a linear Transformation such that $Rank\ (T) = Rank(T^2) \ldots\ldots\ldots (i)$

$$\Rightarrow \dim(Range\ T) = \dim(Range T^2)$$

Since $Range(T^2) \subseteq Range\ (T)\ \Rightarrow\ Range(T^2) = Range\ (T)$

So, option (2) is correct.

From equation (i), $Rank(T^2) = Rank\ (T) \Rightarrow Nullity(T^2) = Nullity\ (T)$

$$\Rightarrow \dim\{ker(T^2)\} = \dim(kerT), (by\ R - N\ theorem)$$

So, option (1) is correct.

Let $x \in \ker(T) \cap$ Range (T) $\Rightarrow x \in \ker(T)\ and\ x \in Range(T)$

$$\Rightarrow T(x) = 0\ \ and\ x = T(y), for\ some\ y \in V.$$

Since, $T(x) = 0\ \Rightarrow T\big(T(y)\big) = 0, y \in V\ \Rightarrow\ T^2(y) = 0 \Rightarrow y \in \ker(T^2) = \ker\ (T)$

$$\Rightarrow T(x) = 0\ and\ x = 0\ \Rightarrow \ker(T) \cap Range\ (T) = \{0\}.$$

So, option (3) is correct.

Let $x \in \ker(T^2) \cap\ Range\ (T^2) =\ \ker\ (T) \cap\ Range(T)$

So, we get $\ker(T^2) \cap Range\ (T^2) = \{0\}$

So, option (4) is correct.

12. Let V be the vector space of polynomials over $\mathbb{R}$ of degree less than or equal to n. For $P(x) = a_0 +\ a_1\,x +\ a_2x^2 + \ldots\ldots\ldots + a_nx^n$ in V, define a linear transformation $T: V \to V$ by $(TP)(x) = a_n +\ a_{n-1}\,x +\ \ldots\ldots\ldots + a_0x^n$. Then

1) T is one to one.

2) T is onto

3) T is invertable.

4)$\det T =\ \pm 1$

Sol:-

$$1 \to 0 + 0 \cdot x + 0 \cdot x^2 + \ldots\ldots + 1 \cdot x^n$$

$$x \to 0 + 0 \cdot x + 0 \cdot x^2 + \ldots\ldots + 1 \cdot x^{n-1} + 0 \cdot x^n$$

. . . .

$$x^n \to 1 + 0 \cdot x + 0 \cdot x^2 + \ldots\ldots\ldots\ldots + 0 \cdot x^n.$$

The matrix corresponding to T is $\begin{pmatrix} 0 & 0 & \ldots & 0\ 1 \\ 0 & 0 & \ldots & 1\ 0 \\ \ldots & \ldots & \ldots & \ldots\ldots \\ 0 & 1 & 0 \ldots & 0\ 0 \\ 1 & 0 & \ldots \ldots & 0\ 0 \end{pmatrix}$

$$\Rightarrow \det T = \pm 1 \Rightarrow \det T \neq 0 \Rightarrow T\ is\ invertable.$$

$$\Rightarrow T\ is\ one - one\ \ \Rightarrow T is\ onto$$

So, options (1), (2), (3) and (4) are correct.

Answer

Sl. No	Answer	Sl. No	Answer	Sl. No	Answer
1.	4	2.	2	3.	1
4.	4	5.	4	6.	4
7.	1 & 3	8.	2 & 4	9.	1 & 3
10.	1, 3 & 4	11.	1, 2, 3 & 4	12.	1, 2, 3 & 4

Previous Year Questions & Solution

Linear Algebra

June– 2016

Part – B

1. Given a $n \times n$ matrix B define e^B by $e^B = \sum_{j=0}^{\infty} \frac{B^j}{J}$. Let P be the characteristic polynomial of B. Then the matrix $e^{P(B)}$ is

1)$I_{n\times n}$
2)$Q_{n\times n}$
3)$eI_{n\times n}$
4)$\pi I_{n\times n}$

Sol:- Given, $n \times n$ matrix B and $e^B = \sum_{j=0}^{\infty} \frac{B^j}{J} = I + \frac{B}{1!} + \frac{B^2}{2!} + \frac{B^3}{3!} + \ldots\ldots\ldots\ldots..$

Now, $e^B = I + \frac{P(B)}{1!} + \frac{\left(P(B)\right)^2}{2!} + \frac{\left(P(B)\right)^3}{3!} + \ldots\ldots\ldots\ldots..$

$\because$ $P(x)$ is the characteristic polynomial of B.

$\therefore$ $P(B) = 0$ $(By\ Cayley\ Hamilton\ Theorem)$ $\Rightarrow e^{P(B)} = I_{n\times n}$.

$\therefore$ option (1) is correct.

2. Let A be a matrix and b be a $n \times 1$ vector (with real entries). Suppose the equation $Ax = b, x \in \mathbb{R}^m$ admits a unique solution. Then we can conclude that

1)$m \geq n$
2) $n \geq m$
3)$n = m$
4)$n > m$

Sol:-

Let A be a $n \times n$ matrix and b be a $n \times 1$ vector.

Then the number of variables in the matrix $A\ are\ m$ and the number of equations are n.

$\therefore$ For unique solution, number of variables $\leq$ no. of equation. $\Rightarrow m \leq n$.

$\therefore$ option (2) is correct.

3. Let V be the vector space of all real polynomials of degree $\leq 10.$ Let $T\big(P(x)\big) = P'(x)\ for\ P \in V$ be a L.T. from V to V.

Consider the basis $\{1, x, x^2, \ldots\ldots\ldots, x^{10}\}$ of V. Let A be the matrix of T with respect to this basis. Then

1) Trace A=1
2)$detA = 0$
3) There is no $m \in \mathbb{N}$ such that $A^m = 0$
4) A has a non – zero eigen value.

Sol:-

Let V be the vector space of all real polynomials ≤ 10 and T be the linear transformation de4fined by $T\big(P(x)\big) = P'(x)$

Let the basis of V is $B = \{1, x, x^2, \dots\dots\dots, x^{10}\}$

Now, all constant polynomials are generated by 1 and the image of all constant polynomials is 0.

$\therefore$ $'0'$ must be an eigen value of T $\Rightarrow \det T = 0$.

Hence, option (2) is correct.

4.Let $x = (x_1,\ x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$ be linearly independent.

Let $\delta_1 =\ x_2y_3 - y_2x_3,\ \delta_2 =\ x_1y_3 - y_1x_3, \delta_3 =\ x_1y_2 - y_1x_2$. If V is the span of x, y, then

1)$V = \{(u, v, w): \delta_1 u - \delta_2 v + \delta_3 w = 0\}$

2)$V = \{(u, v, w): \delta_1 u + \delta_2 v + \delta_3 w = 0\}$

3)$V = \{(u, v, w): \delta_1 u + \delta_2 v - \delta_3 w = 0\}$

4)$V = \{(u, v, w): \delta_1 u + \delta_2 v + \delta_3 w = 0\}$

Sol:- Given that, $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$ are lin3aearly independent in $\mathbb{R}^3$.

$\therefore Matrix\ \ A = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$ has Rank 2.

Now, if $(u, v, w) \in span$ of $\{x, y\}$, then $\begin{vmatrix} u & v & w \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = 0$

$$\Rightarrow u(x_2y_3 - y_2x_3) - v(x_1y_3 - y_1x_3) + w(x_1y_2 - y_2x_1) = 0$$

$$\Rightarrow \delta_1 u - \delta_2 v + \delta_3 w = 0$$

So, option (1) is correct.

5. Let A be a $n \times n$ real symmetric non – singular matrix. Suppose there exists $x \in \mathbb{R}^n$ such that $x'Ax < 0$. Then we can conclude that

1)$\det A < 0$

2)$B = -A\ is\ positive\ definite.$

3)$\exists\ y \in \mathbb{R}^3; y^1 A^{-1}\ \ y < 0$

4)$\forall\ y \in \mathbb{R}: y^1 A^{-1}\ \ y < 0$

Sol:-

Consider the matrix $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\ and\ x = (1,0)$

$$\therefore x^t A x = (1 \quad 0)\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \ (1 \quad 0)\begin{pmatrix} -1 \\ 0 \end{pmatrix} = -1 < 0 \quad and\ \det A = 1.$$

$\therefore$ option (1) is incorrect.

Again, if we take $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\ and\ x = (-1,0)$, then $x^t A x < 0$ and $A^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Take $y = (0,1)$, then $y^1 A^{-1}\ \ y = (0 \quad 1)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \ (0 \quad 1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 > 0$

So, option (4) is incorrect.

Also, $B = -A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Take, $x = (0 \quad 1) \therefore x^t Bx = (0 \quad -1)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\ -1\end{pmatrix} = (0 \quad -1)\begin{pmatrix}0\\ 1\end{pmatrix} = -1 < 0.$
B is not positive definite matrix (for positive definite matrix 'B', we must have

$$x^t Bx > 0 \; \forall x \neq 0)$$

Option (2) is also incorrect.
Hence option (3) is only correct option (as all other options are eliminated).

6. Let $A = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. Let $f: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(v, w) = w^T\, Av$.
Pick the correct statement from below.
1) There exists an eigen vector $v \; of \; A$ such that Av is perpendicular to v.
2) The set $\{v \in \mathbb{R}^2 | f(v, v) = 0\}$ is an non – zero subspace of $\mathbb{R}^2$.
3) if $v, w \in \mathbb{R}^2$ are non – zero vectors such that $f(v, v) = 0 = f(w, w)$, then v is a scalar multiple of w.
4) For every $v \in \mathbb{R}^2$, there exists a non – zero $w \in \mathbb{R}^2$ such that $f(v, w) = 0$.

Sol:-
Given, $A = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and $f: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(v, w) = w^T\, Av$.
For option (1)
Let $v = \begin{pmatrix}x\\ y\end{pmatrix} \neq \begin{pmatrix}0\\ 0\end{pmatrix}$be eigen vector of A such that Av is perpendicular to v. $\Rightarrow \langle Av, v\rangle = 0$
$\Rightarrow v^t Av = 0 \Rightarrow (x \quad y)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow x^2 - y^2 = 0 \Rightarrow v = \begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}\pm y\\ y\end{pmatrix}$, but $Av = A\begin{pmatrix}\pm y\\ y\end{pmatrix} \neq \begin{pmatrix}\pm y\\ y\end{pmatrix}$
Which is contraction to the fact that $'v'$ is an eigen vector of A.
So, option (1) is correct.
Let $V = \{ v \in \mathbb{R}^2, f(v, v) = 0\} \Rightarrow (x \quad y)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = 0 \Rightarrow (x \quad y)\begin{pmatrix}x\\ -y\end{pmatrix} = 0$

$$\Rightarrow x^2 - y^2 = 0 \Rightarrow x = \pm y$$

$\Rightarrow$ the set contains the vectors (a, a) and $(a, -a)$, but $(a, a) + (a, -a) = (2a, 0) \notin V$ for some $a \neq 0, a \in \mathbb{R}$,
$\therefore$ option (2) is correct.
Clearly, $(1,1), (2, -2) \in \mathbb{R}^2$ such that $f(1,1) = 0 = f(2,2), but \; (1,1) \neq \alpha(2, -2)$ for any scalar α.
So, (3) is incorrect.
Let $v = (x, y) \in \mathbb{R}^2$.
$\therefore$ By option (2) $\forall (xy) \in \mathbb{R}^2$

$$\therefore f(v, w) = (y \quad x)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = (y \quad x)\begin{pmatrix}x\\ -y\end{pmatrix} = xy - xy = 0.$$

So, option (4) is correct.

Part – C

7. Let V be the vector space of all complex polynomials P with $\deg P \leq n$. Let $T : V \to V$ be the map $(TP)(x) = P'(1), x \in \mathbb{C}$.
Which of the following are correct?
1)$dim\ ker\ T = n$
2)$dim\ range\ T = 1$
3) $dim\ ker\ T = 1$
4)$dim\ range\ T = n + 1$

Sol:-

$$\ker T = \left\{P(x) \in V : T\big(P(x)\big) = 0\right\} = \{P(x) \in V\ |P'(1) = 0\}$$
$$\Rightarrow nullity\ T = \dim(kerT) = n + 1 - 1 = n.$$
Using rank nullity theorem $R(T) + N(T) = \dim(V) = n + 1$
$$\Rightarrow R(T) = n + 1 - n = 1 \Rightarrow \dim range\ T = 1$$
So, options (1) and (b) are correct.

8. Let A be an $n \times n$ real matrix. Pick the correct answers from the following
1) A has at least one real eigen value.

2) For all non – zero vectors $v, w \in \mathbb{R}^2, (Aw)^T\ Av > 0$.

3) Every eigen value of $A^T A$ is a non – negative real number.

4)$1 + A^T A$ is invertable.

Sol:-
Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow A$ has no real eigen value.
$\therefore$ option (1) is incorrect.
Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, w = (-1 \quad 1)^T,\ v = (1 \quad 1)^T$
$$Aw = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad Av = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$(Aw)^T(Av) = (-1 \quad 1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$
So, option (2) is incorrect.
Take $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^T A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix}$
$$tr\ (A^T A) = a^2 + c^2 + b^2 + d^2 \geq 0; \forall\ a, b, c, d \in \mathbb{R}$$
Also,
$$\det(A^T A) = a^2b^2 + a^2d^2 + b^2c^2 + c^2d^2 - a^2b^2 - c^2d^2 - 2abcd = (ad - bc)^2 \geq 0$$
$\therefore$ if λ_1, λ_2 are eigen values of $(A^T A)$, then $\lambda_1, \lambda_2 \geq (\lambda_1 + \lambda_2 \geq 0, \lambda_1 \cdot \lambda_2 \geq 0)$
So, option (3) is correct.
Further, eigen values of $A^T A + I$ are $1 + \lambda_1, 1 + \lambda_2;\ where\ \lambda_1, \lambda_2$ are eigen values of $A^T A$.
Clearly; $1 + \lambda_1, 1 + \lambda_2 > 0\ \Rightarrow I + A^T A\ \ is\ invertable.$

So, option (4) is correct.

9. Let T be a $n \times n$ matrix with the property $T^n = 0$. Which of the following is are true?
1) T has n distinct eigen values.
2)T has one eigen value of multiplicity n.
3)0 is an eigen value of T.
4)T is similar to a diagonal matrix.

Sol:-
Given that, T is a $n \times n$ matrix with the property $T^n = 0$.
$\Rightarrow T\ is\ nilpotent\ \Rightarrow$ The eigen value of $T\ is\ 0$ only with multiplicity n.
Hence option (1) is incorrect and (2) and (3) are correct.
Since non – zero nilpotent matrix is never diagonalizable.
So, option (4) is incorrect.

10. Let $\{f : [0,1] \to \mathbb{R} |\ f\ is\ a\ polynomial\ of\ degree\ has\ than\ or\ equal\ to\ n\}$. Let $f_j(x) = x^j$ for $0 \leq j \leq n$ and let A be the $(n+1) \times (n+1)$ matrix given by $a_{ij} = \int_0^1 f_i(x)\, f_j(x) dx$. Then which of the following is /are true?
1)$\dim V = n$
2)$\dim V > n$
3)A is non – negative definite, i.e., for all $v \in \mathbb{R}^n, \langle Av, v \rangle \geq 0$.
4)$\det A > 0$

Sol:-

$$\dim V = n + 1 > n$$

So, option (1) is incorrect.
Take $n = 2$. $f_0(x) = x^0 = 1, f_1(x) = x;\ f_2(x) = x^2$

$$a_{00} = \int_0^1 1x\, dx = \frac{1}{2}$$

$$a_{02} = \int_0^1 1x^2\, dx = \frac{1}{3}\ , \qquad a_{10} = \int_0^1 x \cdot 1\, dx = \frac{1}{2}$$

$$a_{11} = \int_0^1 x \cdot x\, dx = \frac{1}{3}\ , \qquad a_{12} = \int_0^1 x \cdot x^2\, dx = \frac{1}{4}\ , a_{20} = \int_0^1 x^2 \cdot 1\, dx = \frac{1}{3}$$

$$a_{21} = \int_0^1 x^2 \cdot x\, dx = \frac{1}{4}\ , \qquad a_{22} = \int_0^1 x^2 \cdot x^2\, dx = \frac{1}{5}$$

$$\therefore\ A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}$$

$$A_{11} = 1 > 0, \qquad A_{22} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} > 0$$

$$A_{33} = 1\left(\frac{1}{15} - \frac{1}{16}\right) - \frac{1}{2}\left(\frac{1}{10} - \frac{1}{12}\right) + \frac{1}{3}\left(\frac{1}{8} - \frac{1}{9}\right) = \frac{1}{15 \times 16} - \frac{2}{2 \times 10 \times 12} + \frac{1}{3 \times 8 \times 9} > 0$$

So, A is non – negative definite and $|A| > 0$.
So, option (3) and (4) are correct.

11. Consider the real vector space v of polynomials of degree less than or equal to d. For $p \in V$ define $\|P\|_k = \max\{|P(0)|, |P^{(1)}(0)|, \ldots\ldots, |P^{(k)}(0)|\}$, where $P^{(i)}(0)$ is the i-th derivative of P evaluated at 0. Then $\|P\|_k$ defines a norm on v if and only if

1)$k \geq d - 1$
2)$k < d$
3)$k \geq d$
4)$k < d - 1$

Sol:-
Let $d = 3, \Rightarrow V = \{P(x) = a_0 + a_1x + a_2x^2 + a_3x^3, where\ a_0, a_1, a_2, a_3 \in \mathbb{R}\}$
Let $k = 2$.

$$\|P(x)\|_2 = \max\{|P(0)|, |P^{(1)}(0)|, |P^{(2)}(0)|\}$$
$$\|x^3\| = \{|x^3|_{x=0}, |3x^2|_{x=0}, |6x|_{x=0}\} = \max\{0,0,0\} = 0$$

Which is wrong (norm of non – zero element is non – zero)
For $k = 2\ and\ d = 3, \|P\|$ does not defined a norm on V.
So, options (1) and (2) are incorrect.
Take $k = 1\ and\ d = 3$
$\|x^3\| = \{|x^3|_{x=0}, |3x^2|_{x=0}\} = \max\{0,0\}$, which is wrong.
So, option (4) is incorrect.
So, option (3) is correct (as other options have been eliminated).

12. Let A, B be $n \times n$ real matrices such that $\det A > 0$ and $\det B < 0$. For $0 \leq t \leq t$, consider $C(t) = tA +)1 - t)B$. Then

1)$C(t)$ is invertable for each $t \in [0,1]$
2) There is a $t_0 \in (0,1)$ such that $C(t_0)$ is not invertable.
3)$C(t)$ is not invertable for each $t \in [0,1]$.
4)$C(t)$ is invertable for only finitely many $t \in [0,1]$.

Sol:-
Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Then $\det A > 0\ and \det B < 0$.

$$\therefore C(g) = tA + (1-t)B = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} + \begin{pmatrix} 1-t & 0 \\ 0 & t-1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2t-1 \end{pmatrix}$$

Clearly, $C(t)$ is not invertable only if $2t - 1 = 0\ i.e., t = \frac{1}{2}$
So, options $(1), (3) \& (4)\ incorrect.$
13. Let A be an $n \times n$ matrix with real entries. Define $\langle x, y \rangle_A = \langle Ax, Ay \rangle, x, y \in \mathbb{R}^n$. Then $\langle x, y \rangle_A$ defines an inner product if and only if

1)$\ker A = \{0\}$
2)$Rank\ A = n$
3) All eigen values of a are positive.
4) All eigen values of A are non- negative.

Sol:-
Since $\langle Ax, Ay\rangle = \langle x,\ A^*Ay\rangle$
Thus, $\langle Ax, Ay\rangle = \langle x, y\rangle$ if only if $A^*A = I$
$\Rightarrow A$ is an orthogonal matrix.
We know that eigen values of real orthogonal matrix are of unit modulus and as A is an $n \times n$ real matrix

$$\Rightarrow |A| = \pm 1 (\neq 0) \Rightarrow R(A) = n \Rightarrow ket\ (A) = \{0\}$$

So, option (1) and (2) are correct.
Take, $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ which satisfies $\langle x, y\rangle = \langle Ax, Ay\rangle\ for\ \ x, y \in R^2$.
But eigen values of A are negative.
So, option (3) and (4) are incorrect.

14. Let $\{a_1, a_2, \ldots\ldots., a_n\}\ and\ \{b_1, b_2, \ldots\ldots., b_n\}$ be two basis of $\mathbb{R}^n$. Let P be $n \times n$ matrix with real entries such that $P_{ai} = b_i,\ i = 1, 2, \ldots.\ \ldots, n$. Suppose that every eigen value of P is either -1 or 1. Let $Q = I + 2P$. Then which of the following statements are true?
1)$\{a_i + 2b_i | i = 1, 2, \ldots\ldots, n\}$ is also a basis of V.
2)Q is invertable.
3) Every eigen value of Q is either 3 0r -1.
4)$\det Q > 0$ if $\det P > 0$.

Sol:-
$\because \{a_1, a_2, \ldots\ldots., a_n\}\ and\ \{b_1, b_2, \ldots\ldots., b_n\}$ are two basis of.
Both the set of basis are linearly independent.
Also, we have $P_{ai} = b_i\ \forall\ \ i = 1, 2, \ldots.\ \ldots, n.$
Also, P is invertable as eigen values of P are 1 or -1.
$\therefore \exists$ a Transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(a_i) = b_i \ \forall\ i \in \mathbb{N}$.

$$\therefore T(a_i) = -a_i\ or\ T(a_i) = a_i (\because eigen\ values\ of\ T\ are\ 1\ \&\ -1)$$

Since,$a_i + 2b_i = a_i + 2T(a_i) \Rightarrow a_i + 2b_i$ is either $-a_i\ or\ 3a_i$
In both cases $\left\{a_i + 2b_{ij}, i = 1, 2, \ldots\ldots. n\right\}$ is a basis of $\mathbb{R}^n$.
Hence, option (1) is correct.
Since, eigen values of P is either 1 or -1 and $Q = I + 2P$.
Then eigen values of

$$Q = eigen\ values\ of\ I + eigen\ values\ of\ 2P\ so\ the\ eigen\ values\ of\ Q\ are\ 3\ or\ -1$$

$$\Rightarrow Q\ is\ invertable.$$

Also, if $\det P > 0,\ then\ detQ > 0$.
So, option (2), (3), (4) are correct.

15. Suppose $\{v_1, v_2, \ldots\ldots\ldots, v_n)$ are unit vectors in $\mathbb{R}^n$ such that $\|v\|^2 = \sum_{i=1}^{n} |\langle v_i, v\rangle|^2, \forall\ v \in\ \mathbb{R}^n$. Then decide the correct of the following

1)$v_1,\ v_2, \ldots\ldots\ldots, v_n$ are mutually orthogonal.

2)$\{v_1,\ v_2, \ldots\ldots\ldots, v_n\}$ is a basis of $\mathbb{R}^n$.

3)$v_1,\ v_2, \ldots\ldots\ldots, v_n$ are not mutually orthogonal.

4) At most $n-1$ of the elements is the set $\{v_1,\ v_2, \ldots\ldots\ldots, v_n\}$ can be orthogonal.

Sol:-

Given $S = \{v_1,\ v_2, \ldots\ldots\ldots, v_n\}$ are unit vectors in $\mathbb{R}^n$ such that

$$\|v\|^2 = \sum_{i=1}^{n} |\langle v_i, v\rangle|^2, \forall\ v \in\ \mathbb{R}^n$$

Put $v = v_k$ for some k,

$$\Rightarrow \|v_k\|^2 = |\langle v_1, v_k\rangle|^2 + |\langle v_2, v_k\rangle|^2 + \ldots\ldots\ldots + |\langle v_k, v_k\rangle|^2 + \ldots\ldots\ldots\ldots\ldots + |\langle v_n, v_k\rangle|^2$$

Since, $v_i{}'s$ are unit vectors $\forall\ i = 1, 2, \ldots\ldots\ldots, n$.

$$\Rightarrow\ \|v_i\| = 1\ for\ all\ i = 1, 2, \ldots\ldots., n.$$

$$\Rightarrow\ |\langle v_1, v_k\rangle|^2 + |\langle v_2, v_k\rangle|^2 + \ldots\ldots\ldots + |\langle v_{k-1}, v_k\rangle|^2 + |\langle v_{k+1}, v_k\rangle|^2 + \ldots\ldots\ldots\ldots + |\langle v_n, v_k\rangle|^2 = 0$$

$$\Rightarrow |\langle v_i, v_k\rangle|^2 = 0 \Rightarrow\ \forall\, i \neq k\ \Rightarrow\ \langle v_i, v_k\rangle = 0\ \forall\ i \neq k.$$

$\Rightarrow S = \{v_1,\ v_2, \ldots\ldots\ldots, v_n\}$ is orthogonal set in $\mathbb{R}^n$

$\Rightarrow v_1,\ v_2, \ldots\ldots\ldots, v_n\ are$ mutually orthogonal.

So, option (1) is correct and option (3) and (4) are incorrect.

Consider $\alpha_1 v_1 + \alpha_2\ v_2, \ldots\ldots\ldots + \alpha_n v_n = 0$

$$\Rightarrow \langle \alpha_1 v_1 + \alpha_2\ v_2, \ldots\ldots\ldots + \alpha_n v_n,\qquad v_i\rangle = 0\ \forall\ \ i$$

$$\Rightarrow \alpha_1\langle v_1, v_i\rangle + \alpha_2\langle v_2, v_i\rangle + \ldots\ldots\ldots + \alpha_n\langle v_n, v_i\rangle = 0$$

$$\Rightarrow \alpha_1\langle v_i, v_i\rangle = 0\ \Rightarrow \alpha_i\|v_i\|^2 = 0\ \ \forall\ i$$

$$\Rightarrow \alpha_i = 0\ \ \forall\ \ i\ (\because\ \ \|v_i\| = 1\ \forall\, i\ )$$

$\Rightarrow S = \{v_1,\ v_2, \ldots\ldots\ldots, v_n\}$ is linearly independent subset of $\mathbb{R}^n$ and also $|S| = n = \dim\big(\mathbb{R}^n(\mathbb{R})\big) \Rightarrow S\ is\ a\ basis\ of\ \mathbb{R}^n$.

So, option (2) is correct.

Answer

Sl. No	Answer	Sl. No	Answer	Sl. No	Answer
1.	1	2.	2	3.	2
4.	1	5.	3	6.	4
7.	1 & 2	8.	3 & 4	9.	2 & 3
10.	2, 3 $ 4	11.	3	12.	2
13.	1 & 2	14.	1,2,3 & 4	15.	1 & 2

Previous Year Questions & Solution

Linear Algebra

June– 2016

Part – B

1. Given a $n \times n$ matrix B define e^B by $e^B = \sum_{j=0}^{\infty} \frac{B^j}{J}$. Let P be the characteristic polynomial of B. Then the matrix $e^{P(B)}$ is

1)$I_{n\times n}$
2)$Q_{n\times n}$
3)$eI_{n\times n}$
4)$\pi I_{n\times n}$

Sol:- Given, $n \times n$ matrix B and $e^B = \sum_{j=0}^{\infty} \frac{B^j}{J} = I + \frac{B}{1!} + \frac{B^2}{2!} + \frac{B^3}{3!} + \ldots\ldots\ldots\ldots..$
Now, $e^B = I + \frac{P(B)}{1!} + \frac{(P(B))^2}{2!} + \frac{(P(B))^3}{3!} + \ldots\ldots\ldots\ldots..$
$\because P(x)$ is the characteristic polynomial of B.
$\therefore P(B) = 0 \ (By\ Cayley\ Hamilton\ Theorem) \Rightarrow e^{P(B)} = I_{n\times n}$.
$\therefore$ option (1) is correct.

2. Let A be a matrix and b be a $n \times 1$ vector (with real entries). Suppose the equation $Ax = b, x \in \mathbb{R}^m$ admits a unique solution. Then we can conclude that

1)$m \geq n$
2) $n \geq m$
3)$n = m$
4)$n > m$

Sol:-
Let A be a $n \times n$ matrix and b be a $n \times 1$ vector.
Then the number of variables in the matrix $A\ are\ m$ and the number of equations are n.
$\therefore$ For unique solution, number of variables $\leq$ no. of equation. $\Rightarrow m \leq n$.
$\therefore$ option (2) is correct.

3. Let V be the vector space of all real polynomials of degree ≤ 10. Let $T(P(x)) = P'(x)\ for\ P \in V$ be a L.T. from $V\ to\ V$.
Consider the basis $\{1, x, x^2, \ldots\ldots\ldots, x^{10}\}$ of V. Let A be the matrix of T with respect to this basis. Then

1)$Trace\ A = 1$
2)$detA = 0$
3) There is no $m \in \mathbb{N}$ such that $A^m = 0$
4) A has a non – zero eigen value.

Sol:-

Let V be the vector space of all real polynomials ≤ 10 and T be the linear transformation de4fined by $T\big(P(x)\big) = P'(x)$

Let the basis of V is $B = \{1, x, x^2, \ldots\ldots\ldots, x^{10}\}$

Now, all constant polynomials are generated by 1 and the image of all constant polynomials is 0.

$\therefore$ $'0'$ must be an eigen value of $T \Rightarrow \det T = 0$.

Hence, option (2) is correct.

4.Let $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$ be linearly independent.

Let $\delta_1 = x_2y_3 - y_2x_3,\ \delta_2 = x_1y_3 - y_1x_3, \delta_3 = x_1y_2 - y_1x_2$. If V is the span of x, y, then

1)$V = \{(u, v, w): \delta_1u - \delta_2v + \delta_3w = 0\}$

2)$V = \{(u, v, w): \delta_1u + \delta_2v + \delta_3w = 0\}$

3)$V = \{(u, v, w): \delta_1u + \delta_2v - \delta_3w = 0\}$

4)$V = \{(u, v, w): \delta_1u + \delta_2v + \delta_3w = 0\}$

Sol:-

Given that, $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$ are lin3aearly independent in $\mathbb{R}^3$.

$\therefore Matrix\ \ A = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$ has Rank 2.

Now, if $(u, v, w) \in span$ of $\{x, y\}$, then $\begin{vmatrix} u & v & w \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = 0$

$$\Rightarrow u(x_2y_3 - y_2x_3) - v(x_1y_3 - y_1x_3) + w(x_1y_2 - y_2x_1) = 0$$

$$\Rightarrow \delta_1u - \delta_2v + \delta_3w = 0$$

So, option (1) is correct.

5. Let A be a $n \times n$ real symmetric non – singular matrix. Suppose there exists $x \in \mathbb{R}^n$ such that $x'Ax < 0$. Then we can conclude that

1)$\det A < 0$

2)$B = -A\ is\ positive\ definite.$

3)$\exists\ y \in \mathbb{R}^3; y^1A^{-1}\ y < 0$

4)$\forall\ y \in \mathbb{R}: y^1A^{-1}\ y < 0$

Sol:-

Consider the matrix $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $and\ x = (1,0)$

$$\therefore x^tAx = (1 \quad 0)\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1 \quad 0)\begin{pmatrix} -1 \\ 0 \end{pmatrix} = -1 < 0 \quad and\ \det A = 1.$$

$\therefore$ option (1) is incorrect.

Again, if we take $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $and\ x = (-1,0)$, then $x^tAx < 0$ and $A^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Take $y = (0,1)$, then $y^1A^{-1}\ y = (0 \quad 1)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = (0 \quad 1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 > 0$

So, option (4) is incorrect.

Also, $B = -A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Take, $x = (0 \quad 1) \therefore x^t B x = (0 \quad -1)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = (0 \quad -1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1 < 0.$

B is not positive definite matrix (for positive definite matrix 'B', we must have

$$x^t B x > 0 \ \forall x \neq 0)$$

Option (2) is also incorrect.

Hence option (3) is only correct option (as all other options are eliminated).

6. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Let $f: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(v, w) = w^T Av$.

Pick the correct statement from below.

1) There exists an eigen vector $v \ of \ A$ such that Av is perpendicular to v.

2) The set $\{v \in \mathbb{R}^2 | f(v, v) = 0\}$ is an non – zero subspace of $\mathbb{R}^2$.

3) if $v, w \in \mathbb{R}^2$ are non – zero vectors such that $f(v, v) = 0 = f(w, w)$, then v is a scalar multiple of w.

4) For every $v \in \mathbb{R}^2$, there exists a non – zero $w \in \mathbb{R}^2$ such that $f(v, w) = 0$.

Sol:-

Given, $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $f: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(v, w) = w^T Av$.

For option (1)

Let $v = \begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$be eigen vector of A such that Av is perpendicular to v. $\Rightarrow \langle Av, v \rangle = 0$

$\Rightarrow v^t Av = 0 \Rightarrow (x \quad y)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x^2 - y^2 = 0 \Rightarrow v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \pm y \\ y \end{pmatrix}$, but $Av = A\begin{pmatrix} \pm y \\ y \end{pmatrix} \neq \begin{pmatrix} \pm y \\ y \end{pmatrix}$

Which is contraction to the fact that $'v'$ is an eigen vector of A.

So, option (1) is correct.

Let $V = \{ v \in \mathbb{R}^2, f(v, v) = 0\} \Rightarrow (x \quad y)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow (x \quad y)\begin{pmatrix} x \\ -y \end{pmatrix} = 0$

$$\Rightarrow x^2 - y^2 = 0 \Rightarrow x = \pm y$$

$\Rightarrow$ the set contains the vectors (a, a) and $(a, -a)$, but $(a, a) + (a, -a) = (2a, 0) \notin V$ for some $a \neq 0, a \in \mathbb{R}$,

$\therefore$ option (2) is correct.

Clearly, $(1,1), (2, -2) \in \mathbb{R}^2$ such that $f(1,1) = 0 = f(2,2), but \ (1,1) \neq \alpha(2, -2)$ for any scalar α.

So, (3) is incorrect.

Let $v = (x, y) \in \mathbb{R}^2$.

$\therefore$ By option (2) $\forall (xy) \in \mathbb{R}^2$

$$\therefore f(v, w) = (y \quad x)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (y \quad x)\begin{pmatrix} x \\ -y \end{pmatrix} = xy - xy = 0.$$

So, option (4) is correct.

Part – C

7. Let V be the vector space of all complex polynomials P with $\deg P \le n$. Let $T : V \to V$ be the map $(TP)(x) = P'(1), x \in \mathbb{C}$.
Which of the following are correct?
1)$dim\ ker\ T = n$
2)$dim\ range\ T = 1$
3) $dim\ ker\ T = 1$
4)$dim\ range\ T = n + 1$

Sol:-

$$\ker T = \{P(x) \in V : T(P(x)) = 0\} = \{P(x) \in V \mid P'(1) = 0\}$$
$$\Rightarrow nullity\ T = \dim(kerT) = n + 1 - 1 = n.$$
Using rank nullity theorem $R(T) + N(T) = \dim(V) = n + 1$
$$\Rightarrow R(T) = n + 1 - n = 1 \Rightarrow \dim range\ T = 1$$
So, options (1) and (b) are correct.

8. Let A be an $n \times n$ real matrix. Pick the correct answers from the following
1) A has at least one real eigen value.
2) For all non – zero vectors $v, w \in \mathbb{R}^2, (Aw)^T\ Av > 0$.
3) Every eigen value of $A^T A$ is a non – negative real number.
4)$1 + A^T A$ is invertable.

Sol:-
Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow A$ has no real eigen value.
$\therefore$ option (1) is incorrect.
Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, w = (-1 \quad 1)^T,\ v = (1 \quad 1)^T$
$$Aw = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad Av = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$(Aw)^T(Av) = (-1 \quad 1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$
So, option (2) is incorrect.
Take $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^T A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix}$
$$tr\ (A^T A) = a^2 + c^2 + b^2 + d^2 \ge 0; \forall\ a, b, c, d \in \mathbb{R}$$
Also,
$$\det(A^T A) = a^2b^2 + a^2d^2 + b^2c^2 + c^2d^2 - a^2b^2 - c^2d^2 - 2abcd = (ad - bc)^2 \ge 0$$
$\therefore$ if λ_1, λ_2 are eigen values of $(A^T A)$, then $\lambda_1, \lambda_2 \ge (\lambda_1 + \lambda_2 \ge 0, \lambda_1 \cdot \lambda_2 \ge 0)$
So, option (3) is correct.
Further, eigen values of $A^T A + I$ are $1 + \lambda_1, 1 + \lambda_2;\ where\ \lambda_1, \lambda_2$ are eigen values of $A^T A$.
Clearly; $1 + \lambda_1, 1 + \lambda_2 > 0 \Rightarrow I + A^T A\ \ is\ invertable.$
So, option (4) is correct.

9. Let T be a $n \times n$ matrix with the property $T^n = 0$. Which of the following is are true?
1) T has n distinct eigen values.
2)T has one eigen value of multiplicity n.
3)0 is an eigen value of T.
4)T is similar to a diagonal matrix.

Sol:-
Given that, T is a $n \times n$ matrix with the property $T^n = 0$.
$\Rightarrow T\ is\ nilpotent\ \Rightarrow$ The eigen value of $T\ is\ 0$ only with multiplicity n.
Hence option (1) is incorrect and (2) and (3) are correct.
Since non – zero nilpotent matrix is never diagonalizable.
So, option (4) is incorrect.

10. Let $\{f : [0,1] \to \mathbb{R} |\ f\ is\ a\ polynomial\ of\ degree\ has\ than\ or\ equal\ to\ n\}$. Let $f_j(x) = x^j$ for $0 \le j \le n$ and let A be the $(n+1) \times (n+1)$ matrix given by
$a_{ij} = \int_0^1 f_i(x)\, f_j(x) dx$. Then which of the following is /are true?
1)$\dim V = n$
2)$\dim V > n$
3)A is non – negative definite, i.e., for all $v \in \mathbb{R}^n, \langle Av, v\rangle \ge 0$.
4)$\det A > 0$

Sol:-

$$\dim V = n + 1 > n$$

So, option (1) is incorrect.
Take $n = 2$. $\quad f_0(x) = x^0 = 1, f_1(x) = x;\ f_2(x) = x^2$

$$a_{00} = \int_0^1 1x\, dx = \frac{1}{2}$$

$$a_{02} = \int_0^1 1x^2\, dx = \frac{1}{3}\ , \qquad a_{10} = \int_0^1 x \cdot 1\, dx = \frac{1}{2}$$

$$a_{11} = \int_0^1 x \cdot x\, dx = \frac{1}{3}\ , \qquad a_{12} = \int_0^1 x \cdot x^2\, dx = \frac{1}{4}\ , a_{20} = \int_0^1 x^2 \cdot 1\, dx = \frac{1}{3}$$

$$a_{21} = \int_0^1 x^2 \cdot x\, dx = \frac{1}{4}\ , \qquad a_{22} = \int_0^1 x^2 \cdot x^2\, dx = \frac{1}{5}$$

$$\therefore \quad A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}$$

$$A_{11} = 1 > 0, \qquad A_{22} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} > 0$$

$$A_{33} = 1\left(\frac{1}{15} - \frac{1}{16}\right) - \frac{1}{2}\left(\frac{1}{10} - \frac{1}{12}\right) + \frac{1}{3}\left(\frac{1}{8} - \frac{1}{9}\right) = \frac{1}{15 \times 16} - \frac{2}{2 \times 10 \times 12} + \frac{1}{3 \times 8 \times 9} > 0$$

So, A is non – negative definite and $|A| > 0$.

So, option (3) and (4) are correct.

11. Consider the real vector space v of polynomials of degree less than or equal to d. For $p \in V$ define $\|P\|_k = \max\{|P(0)|, |P^{(1)}(0)|, \ldots\ldots, |P^{(k)}(0)|\}$, where $P^{(i)}(0)$ is the i-th derivative of P evaluated at 0. Then $\|P\|_k$ defines a norm on v if and only if

1)$k \geq d - 1$

2)$k < d$

3)$k \geq d$

4)$k < d - 1$

Sol:-

Let $d = 3, \Rightarrow V = \{P(x) = a_0 + a_1x + a_2x^2 + a_3x^3, where\ a_0, a_1, a_2, a_3 \in \mathbb{R}\}$

Let $k = 2$.

$$\|P(x)\|_2 = \max\{|P(0)|, |P^{(1)}(0)|, |P^{(2)}(0)|\}$$

$$\|x^3\| = \{|x^3|_{x=0}, |3x^2|_{x=0}, |6x|_{x=0}\} = \max\{0,0,0\} = 0$$

Which is wrong (norm of non – zero element is non – zero)

For $k = 2\ and\ d = 3, \|P\|$ does not defined a norm on V.

So, options (1) and (2) are incorrect.

Take $k = 1\ and\ d = 3$

$\|x^3\| = \{|x^3|_{x=0}, |3x^2|_{x=0}\} = \max\{0,0\}$, which is wrong.

So, option (4) is incorrect.

So, option (3) is correct (as other options have been eliminated).

12. Let A, B be $n \times n$ real matrices such that $\det A > 0$ and $\det B < 0$. For $0 \leq t \leq t$, consider $C(t) = tA +)1 - t)B$. Then

1)$C(t)$ is invertable for each $t \in [0,1]$

2) There is a $t_0 \in (0,1)$ such that $C(t_0)$ is not invertable.

3)$C(t)$ is not invertable for each $t \in [0,1]$.

4)$C(t)$ is invertable for only finitely many $t \in [0,1]$.

Sol:-

Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Then $\det A > 0\ and \det B < 0$.

$$\therefore C(g) = tA + (1-t)B = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} + \begin{pmatrix} 1-t & 0 \\ 0 & t-1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2t-1 \end{pmatrix}$$

Clearly, $C(t)$ is not invertable only if $2t - 1 = 0\ i.e., t = \frac{1}{2}$

So, options $(1), (3)\&(4)\ incorrect.$

13. Let A be an $n \times n$ matrix with real entries. Define $\langle x, y\rangle_A = \langle Ax, Ay\rangle, x, y \in \mathbb{R}^n$. Then $\langle x, y\rangle_A$ defines an inner product if and only if
1)$\ker A = \{0\}$
2)$Rank\ A = n$
3) All eigen values of a are positive.
4) All eigen values of A are non- negative.
Sol:-
Since $\langle Ax, Ay\rangle = \langle x, A^*Ay\rangle$
Thus, $\langle Ax, Ay\rangle = \langle x, y\rangle$ if only if $A^*A = I$
$\Rightarrow A$ is an orthogonal matrix.
We know that eigen values of real orthogonal matrix are of unit modulus and as A is an $n \times n$ real matrix

$$\Rightarrow |A| = \pm 1 (\neq 0) \Rightarrow R(A) = n \Rightarrow ket\ (A) = \{0\}$$

So, option (1) and (2) are correct.
Take, $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ which satisfies $\langle x, y\rangle = \langle Ax, Ay\rangle\ for\ \ x, y \in R^2$.
But eigen values of A are negative.
So, option (3) and (4) are incorrect.

14. Let $\{a_1, a_2, \ldots\ldots, a_n\}\ and\ \{b_1, b_2, \ldots\ldots, b_n\}$ be two basis of $\mathbb{R}^n$. Let P be $n \times n$ matrix with real entries such that $P_{ai} = b_i,\ i = 1, 2, \ldots\ \ldots, n$. Suppose that every eigen value of P is either -1 or 1. Let $Q = I + 2P$. Then which of the following statements are true?
1)$\{a_i + 2b_i | i = 1, 2, \ldots\ldots, n\}$ is also a basis of V.
2)Q is invertable.
3) Every eigen value of Q is either 3 0r -1.
4)$\det Q > 0$ if $\det P > 0$.

Sol:-
$\because \{a_1, a_2, \ldots\ldots, a_n\}\ and\ \{b_1, b_2, \ldots\ldots, b_n\}$ are two basis of.
Both the set of basis are linearly independent.
Also, we have $P_{ai} = b_i\ \forall\ \ i = 1, 2, \ldots\ \ldots, n$.
Also, P is invertable as eigen values of P are 1 or -1.
$\therefore \exists$ a Transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(a_i) = b_i\ \forall\ i \in \mathbb{N}$.

$$\therefore T(a_i) = -a_i\ or\ T(a_i) = a_i (\because eigen\ values\ of\ T\ are\ 1\ \&\ -1)$$

Since,$a_i + 2b_i = a_i + 2T(a_i) \Rightarrow a_i + 2b_i$ is either $-a_i\ or\ 3a_i$
In both cases $\{a_i + 2b_{ij}, i = 1, 2, \ldots\ldots n\}$ is a basis of $\mathbb{R}^n$.
Hence, option (1) is correct.
Since, eigen values of P is either 1 or -1 and $Q = I + 2P$.
Then eigen values of

$$Q = eigen\ values\ of\ I + eigen\ values\ of\ 2P\ so\ the\ eigen\ values\ of\ Q\ are\ 3\ or - 1$$

$$\Rightarrow Q \ is \ invertable.$$

Also, if $\det P > 0, \ then \ detQ > 0$.
So, option (2), (3), (4) are correct.

15. Suppose $\{v_1, \ v_2, \ldots\ldots, v_n)$ are unit vectors in $\mathbb{R}^n$ such that $\|v\|^2 = \sum_{i=1}^{n} |\langle v_i, v\rangle|^2 , \forall \ v \in \ \mathbb{R}^n$. Then decide the correct of the following

1)$v_1, \ v_2, \ldots\ldots, v_n$ are mutually orthogonal.
2)$\{v_1, \ v_2, \ldots\ldots, v_n\}$ is a basis of $\mathbb{R}^n$.
3)$v_1, \ v_2, \ldots\ldots, v_n$ are not mutually orthogonal.
4) At most $n-1$ of the elements is the set $\{v_1, \ v_2, \ldots\ldots, v_n\}$ can be orthogonal.

Sol:-
Given $S = \{v_1, \ v_2, \ldots\ldots, v_n\}$ are unit vectors in $\mathbb{R}^n$ such that

$$\|v\|^2 = \sum_{i=1}^{n} |\langle v_i, v\rangle|^2 , \forall \ v \in \ \mathbb{R}^n$$

Put $v = v_k$ for some k,

$$\Rightarrow \|v_k\|^2 = \ |\langle v_1, v_k\rangle|^2 + |\langle v_2, v_k\rangle|^2 + \ldots\ldots + |\langle v_k, v_k\rangle|^2 + \ldots\ldots\ldots + |\langle v_n, v_k\rangle|^2$$

Since, $v_i's$ are unit vectors $\forall \ i = 1, 2, \ldots\ldots, n$.

$$\Rightarrow \ \|v_i\| = 1 \ for \ all \ i = 1, 2, \ldots\ldots, n.$$

$$\Rightarrow \ |\langle v_1, v_k\rangle|^2 + |\langle v_2, v_k\rangle|^2 + \ldots\ldots + |\langle v_{k-1}, v_k\rangle|^2 + |\langle v_{k+1}, v_k\rangle|^2 + \ldots\ldots\ldots + |\langle v_n, v_k\rangle|^2 = 0$$

$$\Rightarrow |\langle v_i, v_k\rangle|^2 = 0 \Rightarrow \ \forall \ i \neq k \ \Rightarrow \ \langle v_i, v_k\rangle = 0 \ \forall \ \ i \neq k.$$

$\Rightarrow S = \{v_1, \ v_2, \ldots\ldots, v_n\}$ is orthogonal set in $\mathbb{R}^n$
$\Rightarrow v_1, \ v_2, \ldots\ldots, v_n \ are$ mutually orthogonal.
So, option (1) is correct and option (3) and (4) are incorrect.
Consider $\alpha_1 v_1 + \alpha_2 \ v_2, \ldots\ldots + \alpha_n v_n = 0$

$$\Rightarrow \langle \alpha_1 v_1 + \alpha_2 \ v_2, \ldots\ldots + \alpha_n v_n, \qquad v_i\rangle = 0 \ \ \forall \ \ i$$

$$\Rightarrow \alpha_1\langle v_1, v_i\rangle + \alpha_2\langle v_2, v_i\rangle + \ldots\ldots + \alpha_n\langle v_n, v_i\rangle = 0$$

$$\Rightarrow \alpha_1\langle v_i, v_i\rangle = 0 \ \Rightarrow \alpha_i\|v_i\|^2 = 0 \ \ \forall \ \ i$$

$$\Rightarrow \alpha_i = 0 \ \ \forall \ \ i \ (\because \ \ \|v_i\| = 1 \ \forall \ i \ )$$

$\Rightarrow S = \{v_1, \ v_2, \ldots\ldots, v_n\}$ is linearly independent subset of $\mathbb{R}^n$ and also $|S| = n = \dim\big(\mathbb{R}^n(\mathbb{R})\big) \Rightarrow S \ is \ a \ basis \ of \ \mathbb{R}^n$.
So, option (2) is correct.

Answer

Sl. No	Answer	Sl. No	Answer	Sl. No	Answer
1.	1	2.	2	3.	2
4.	1	5.	3	6.	4
7.	1 & 2	8.	3 & 4	9.	2 & 3
10.	2, 3 $ 4	11.	3	12.	2
13.	1 & 2	14.	1,2,3 & 4	15.	1 & 2

Previous Year Questions & Solution

Linear Algebra

December– 2016

Part – B

1. The matrix $\begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$ is

1) Positive definite

2) Non – negative definite but not positive definite.

3) Negative definite.

4) Neither negative definite nor positive definite.

Sol:-

The matrix $A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

All principle minors $A_1 = 3 > 0$

$$A_2 = \begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} = 6 - 1 = 5 > 0$$

$$A_3 = \begin{vmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{vmatrix} = 3(6-1) + 1(-3) = 12 > 0$$

So, all principle minors $A_i > 0 \Rightarrow A$ is positive definite matrix.

So, option (1) is correct.

2. Which of the following subsets of $\mathbb{R}^4$ is basis of $\mathbb{R}^4$?

$$B_1 = \{(1,0,0,0), (1,1,0,0), (1,1,1,0), (1,1,1,1)\},$$

$$B_2 = \{(1,0,0,0), (1,2,0,0), (1,2,3,0), (1,2,3,4)\}$$

$$B_3 = \{(1,2,0,0), (0,0,1,1), (2,1,0,0), (-5,5,0,0)\}$$

1)B_1 *and* B_2 *but not* B_3

2)B_1, B_2 *and* B_3

3)B_1 *and* B_3 *but not* B_2

4) only B_1

Sol:-

We know if V is a vector space of dimension $'n'$ and $S \subseteq V$ such that $|S| = n \; and \; S$ is linearly independent.

Then S is a basis of V.

So, only we need to check B_i is linearly independent or not for $i = 1, 2 \; and \; 3$.

Clearly, $B_1 \; and \; B_2$ are linearly independent sets as determinant of matrix formed by vectors of $B_1 \; and \; B_2$ is non – zero. $\Rightarrow B_1 \; and \; B_2$ are basis, but set B_3 is linearly dependent set as $5(1,2,0,0) + 0(0,0,1,1) - 5(2,1,0,0) - 1(-5,5,0,0) = (0,0,0,0)$

So, option (1) is correct.

3. Let $D_1 = \det\begin{pmatrix} a & b & c \\ x & y & z \\ p & q & r \end{pmatrix} and\ D_2 = \det\begin{pmatrix} -x & a & -p \\ y & -b & q \\ z & -c & r \end{pmatrix}$

Then

1)$D_1 = D_2$

2)$D_1 = 2D_2$

3)$D_1 = -D_2$

4)$2D_1 = D_2$

Sol:-

Take $a = 1, y = 1, r = 1\ and\ x = p = q = b = c = z = 0$

Then $D_1 = det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\quad D_2 = det\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$D_1 = 1, \qquad D_2 = -1$$

So, options (1), (2), (4) are incorrect.

Hence, option (3) is correct.

4. Consider the matrix $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, where\ \theta = \frac{2\pi}{31}$. Then A^{2015} equals

1)A

2)I

3)$\begin{pmatrix} \cos 13\theta & \sin 13\theta \\ -\sin 13\theta & \cos 13\theta \end{pmatrix}$

4)$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Sol:-

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$A^n = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^n = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}$$

$$A^{2015} = \begin{pmatrix} \cos\frac{2\pi}{31}\cdot 2015 & \sin\frac{2\pi}{31}\cdot 2015 \\ -\sin\frac{2\pi}{31}\cdot 2015 & \cos\frac{2\pi}{31}\cdot 2015 \end{pmatrix} = \begin{pmatrix} \cos 130\pi & \sin 130\pi \\ -\sin 130\pi & \cos 130\pi \end{pmatrix}$$

$$A^{2015} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

So, option (2) is correct.

5. Let J denote the matrix of order $n \times n$ with all entries. $1\ and\ B$ be a $(3n) \times (3n)$ matrix given by $B = \begin{pmatrix} 0 & 0 & J \\ 0 & J & 0 \\ J & 0 & 0 \end{pmatrix}$

Then the rank of B is

1) 2n
2) 3n – 1
3) 2
4) 3

Sol:-
Let J be the 1×1 matrix with all entries '1' $j = [1]$
B is 3×3 given by $B = \begin{pmatrix} 0 & 0 & J \\ 0 & J & 0 \\ J & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$$R(B) = 3$$

So, option (1), (2) and (3) are incorrect.
Hence, option (4) is correct.

6. Which of the following sets of functions from ℝ to ℝ is a vector space over ℝ?

$$S_1 = \left\{ f \middle| \lim_{x \to 3} f(x) = 0 \right\}$$
$$S_2 = \left\{ f \middle| \lim_{x \to 3} g(x) = 1 \right\}$$
$$S_3 = \left\{ f \middle| \lim_{x \to 3} h(x)\ exists \right\}$$

1) Only S_1
2) Only S_2
3)S_1 and S_3 but not S_2
4) All the three are vector spaces.

Sol:-

$$S_1 = \left\{ f \middle| \lim_{x \to 3} f(x) = 0 \right\}$$

Let $f, g \in\ S_1\ \ and\ \alpha, \beta \in \mathbb{R}$.

$$\lim_{x \to 3} (\alpha f + \beta g)(x) = \alpha \left(\lim_{x \to 3} f(x) \right) + \beta \left(\lim_{x \to 3} g(x) \right) = \alpha \cdot 0 + \beta \cdot 0 = 0\ [\because f_1\ g \in S_1]$$

$\Rightarrow \alpha f + \beta\ g \in S_1\ \ \forall\ f_1\ g \in S_1, \forall\ \alpha, \beta \in \mathbb{R}\ \Rightarrow\ S_1$ is a subspace of V.

$$S_2 = \left\{ f \middle| \lim_{x \to 3} g(x) = 1 \right\}$$

Let $f, g \in\ S_2$

$$\lim_{x \to 3} (f + g)\,(x) = \lim_{x \to 3} f(x)\ + \lim_{x \to 3} g(x) = 1 + 1 = 2$$

$\Rightarrow f + g \notin\ S_2\ \Rightarrow\ S_2$ is not closed with respect to addition
$\Rightarrow\ S_2$is not a subspace of V.

$$S_3 = \left\{ f \middle| \lim_{x \to 3} h(x)\ exists \right\}$$

If $f, g \in S_3$

$$\lim_{x\to 3}(\alpha f + \beta g)(x) = \alpha \left(\lim_{x\to 3} f(x)\right) + \beta \left(\lim_{x\to 3} g(x)\right) \quad exists.$$

$\Rightarrow \alpha f + \beta g \in S_3 \ \forall \ f, g \in S_3, \forall \alpha, \beta \in \mathbb{R} \Rightarrow S_3$ is a subspace of V.

So, option (3) is true.

7. Let A be an $n \times n$ matrix with each entry equal to +1, -1 or 0 such that every column has exactly one +1 and exactly one – 1. We can conclude that

1)$Rank\ A \leq n - 1$

2)$Rank\ A = m$

3)$n \leq m$

4)$n - 1 \leq m$.

Sol:-

Consider that matrix $\begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}_{4\times 2}$

Here, $n = 4\ and\ m = 2$.

∴ option (3) and (4) are incorrect.

Also, $R(A) = 1$.

So, option (2) is incorrect.

Hence, option (1) is correct.

8. What is the number of non – singular 3×3 matrices over F_2, the finite field with two elements?

1) 168

2) 384

3)2^3

4)3^2

Sol:-

The number of non – singular $n \times n$ matrices over F_P is given by $= |GL_n(F_P)| = (P^n - 1)(P^n - P)\ldots\ldots\ldots.(P^n - P^{n-1})$.

The number of non – singular 3×3 matrices over F_2 is

$$(2^3 - 1)(2^3 - 2)(2^3 - 2^2) = \ 7 \times 6 \times 4 = 168$$

So, option (1) is correct.

Part – C

9. Let $A = [a_{ij}]$ be an $n \times n$ matrix. Such that a_{ij} is an integer for all i, j. Let $AB = I\ with\ B = [b_{ij}]$ (Where $I\ is\ the\ identity\ matrix$). For a $square\ matrix\ C, det\ C$ denotes its determinant. Which of the following statements is true?

1) If $\det A = 1\ then \det B = 1$.

2)A sufficient condition for each b_{ij} to be an integer is that $\det A$is an integer.

3)B is always an integer matrix.

4)A necessary condition for each b_{ij} to be an integer is $\det A\ \in \{-1, +1\}$.

Sol:

$A = [a_{ij}]_{n\times n}$ such that $a_{ij} \in \mathbb{Z}\ \forall\ i, j\ and\ AB = I$.
For option (1)
Since, $AB = 1\ \Rightarrow \det(AB) = 1 \Rightarrow (\det A) \cdot (\det B) = 1$
If $\det A = 1\ \Rightarrow \det B = 1$.
For option (2) and (3)
Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \det A = 4 \in \mathbb{Z}$
So, $B = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, so that $AB = I$.
Here, $\det A \in \mathbb{Z}$, but B is not integer matrix.
$\therefore$ option (2) is incorrect.
Also, B is not always on integer matrix by taking some example as $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
So, option (3) is in
Since $AB = I\ \Rightarrow B$ is inverse of A.
If $\det A =\ \pm 1\ \Rightarrow\ A^{-1}$has integer entries $\Rightarrow B$ has integer entries.
So, option (4) is true.

10. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and let $\alpha_n\ and\ \beta_n$ denote the two eigen values of A such that $|\alpha_n| \geq |\beta_n|$. Then

1)$\alpha_n \to \infty\ as\ n \to \infty$

2)$\beta_n \to \infty\ as\ n \to \infty$

3)$\beta_n\ is\ positive\ if\ n\ is\ even$

4)β_n is negative if n is odd.

Sol:-

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

The characteristics polynomial of $A = C_A(x) = x^2 - x - 1 = 0$.

$$x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$|\alpha_1| \geq |\beta_1| \Rightarrow \alpha_1 > 1, -1 < \beta_1 < 0$$

Then eigen values of $A^n$are $\alpha_1{}^n \; and \; \beta_1{}^n$.

$$(\alpha_1)^n = \alpha_n = \left(\frac{1+\sqrt{5}}{2}\right)^n and \;\; (\beta_1)^n = \beta_n = \left(\frac{1+\sqrt{5}}{2}\right)^n$$

As $\alpha_1 > 1 \Rightarrow \alpha_n = \;\; (\alpha_1)^n \to \infty \;\; as \;\; n \to \infty \;\; and \;\; -1 < \beta_1 < 0 \;\; \Rightarrow \beta_n \to 0 \;\; as\, n \to \infty$.

As, $\beta_n = (\beta_1)^n \Rightarrow \; \beta_n$ is positive if n is even and β_n is negative fn is odd.

So, option (1), (2), (3) and (4) true.

11. Let M_n denote the vector space of all $n \times n$ real matrices. Among the following subsets of M_n, decide which are linear subspaces.

1)$V_1 = \{A \in M_n: A \; is \; non-singular\}$

2)$V_2 = \{A \in M_n: \det(A) = 0\}$

3)$V_3 = \{A \in M_n: trace(A) = 0\}$

4)$V_4 = \{BA: A \in M_n\} \; where \; B \; is \; some \; fixed \; matrix \; in \; M_n.$

Sol:-

$$M_n = Vector \; space \; of \; all \; n \times n \;\; matrices.$$

$$V_1 = \{A \in M_n: A \; is \; non-singular\}$$

For $n = 2, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in V_1, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in V_1 \;\Rightarrow\; \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin \; V_1$

V_1 is not closed under addition.

So, option (1) is incorrect.

$$V_2 = \{A \in M_n: \det(A) = 0\}$$

For $n = 2, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in V_2, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in V_2 \;\Rightarrow\; \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin \; V_2$

V_2 is not closed under addition.

So, option (2) is incorrect.

$$V_3 = \{A \in M_n: trace(A) = 0\}$$

Let $A, B \in V_3 \;\Rightarrow trace(A) = 0, trace\,(B) = 0$

$\alpha A + \beta B$is also $n \times n$ matrix for all $\alpha, \beta \in \mathbb{R}$ and $trace\;(\alpha A + \beta B)$

$$= \alpha \; trace(A) + \beta \; trace\;(B) = \alpha \cdot 0 + \beta \cdot 0 = 0 \;\Rightarrow \alpha\, A + \beta\, B \in \; V_3 \;\; \forall \;\; \alpha, \beta \in \mathbb{R}, \forall \, A, B \in V_3 \Rightarrow V_3 \; is \; a \; subspace.$$

$V_4 = \{BA: A \in M_n\} \; where \; B \; is \; some \; fixed \; matrix \; in \; M_n.$

Let $A_1, A_2 \in M_n \;\Rightarrow\; A_1 + A_2 \in M_n \;\Rightarrow\; BA_1, BA_2 \in V_4 \;\; and \; B\;(A_1 + A_2) \in V_4$

i.e., $BA_1 + BA_2 \in V_4$.

Also, $B(\alpha A) \in \; V_4, \;\forall\; \alpha \in \mathbb{R} \;(\because \alpha A \in M_n \; for \; A \in M_n, \alpha \in \mathbb{R})$

Thus V_4 is closed under addition and scalar multiplication.

$\therefore \; V_4$ is a subspace of V_4

Thus, option (4) is correct.

12. If $P\ and\ Q$ are invertable matrices such that $PQ = -QP$, then we can conclude that

1)$T_r(P) = t_r(Q) = 0$

2)$T_r(P) = T_r(Q) = 0$

3)$T_r(P) = -T_r(Q)$

4)$T_r(P) \neq T_r(Q)$

Sol:-

Given that $P\ and\ Q$ are invertable matrices such that

$$PQ = -PQ \Rightarrow PQP^{-1} = -Q \Rightarrow Q\ and - Q\ are\ similar\ matrices.$$
$$\Rightarrow T_r(Q) = T_r(-Q) \Rightarrow T_r(Q) = -T_r(Q) \Rightarrow T_r(Q) = 0$$

Similarly, we can write, $PQ = -QP \Rightarrow P = -QPQ^{-1} \Rightarrow QPQ^{-1} = -P$

$\Rightarrow P\ and - P$ are similar matrices $\Rightarrow T_r(P) = -T_r(P) \Rightarrow T_r(P) = 0$

Hence, we get $T_r(P) = T_r(Q) = 0$ &$T_r(P) = -T_r(Q)$.

So, option (1) and (3) are correct.

13. Let n be an odd number≥ 7. Let $A = [a_{ij}]$ be an $n \times n$ matrix with $a_{i,i+1} = 1$ for all $i = 1, 2, \ldots\ldots, n-1\ and\ a_{n,1} = 1.$ Let $a_{ij} = 0$ for all the other pairs (i, j). Then we can conclude that

1) A has 1 as an eigenvalue.

2) A has -1 as an eigenvalue.

3) A has at least one eigen value with multiplicity ≥ 2.

4) A has no real eigenvalues.

Sol:-

Let n be an odd number ≥ 7.

$$A = [a_{ij}] \text{ is the } n \times n \text{ matrix with } a_{i,i+1} = 1 \text{ for all } i = 1, 2, \ldots\ldots, n-1\ and\ a_{n,1} = 1. \text{ Let } a_{ij} = 0 \text{ for all the other pairs}$$

So, $A = \begin{pmatrix} 0 & 1 & 0\,0 \ldots 0 \\ 0 & 0 & 1\,0 \ldots 0 \\ 0 & 0 & 0\,1 \cdots 0 \\ \ldots & \ldots & \ldots\ldots \cdots 1 \\ 1 & 0 & 0\,0 \cdots 0 \end{pmatrix}$

Above matrix is a permutation matrix.

A permutation matrix is a square that has exactly are entry of '1' in each column and 0's elsewhere. We can say that $\exists$ a permutation $\sigma \in S_n$ such that $i-th$ column of A is $\sigma(i)^{th}$column of $n \times n$ identity matrix $\Rightarrow \sigma = (1 \quad n \quad n-1 n-2 \quad \ldots\ldots \quad 2) \Rightarrow \sigma^n = I$

$\Rightarrow minimal\ polynomial\ of\ A = m_A(x) = x^2 - 1,\ n$ is odd.

Since, minimal polynomial of $A\ has\ degree\ = n\ and\ n =\ order\ of\ the\ matrix.$

$$\Rightarrow characteristic\ polynomial\ of\ A = C_A(x) = x^2 - 1, \qquad n\ is\ odd.$$

So, all eigen values are just $n-th$ roots of unity and also are distinct.

Since, n is odd.

$\Rightarrow 1\ is\ an\ eigen\ value\ but -1\ is\ not\ the\ eigenvalues\ of\ A.$

So, option (1) is correct and option (3), (2), (4) are incorrect.

14. Let w_1, w_2, w_3 be three distinct subspaces of $\mathbb{R}^{10}$ such that each w_i has dimension 9. Let $w = w_1 \cap w_2 \cap w_3$. Then we can conclude that

1)w may not be a subspace of $\mathbb{R}^{10}$

2)$\dim w \leq 8$

3)$\dim w \geq 7$

4)$\dim w \leq 3$.

Sol:-

Given that, w_1, w_2, w_3 are three distinct subspaces of $\mathbb{R}^{10}$ such that each w_i has dimension 9.

$$\Rightarrow w_1 \cap w_2 \cap w_3 \text{ is also a subspace of } \mathbb{R}^{10} \Rightarrow \text{option } (1) \text{is incorrect.}$$

Any '9' dimensional vector subspace of $\mathbb{R}^{10}$ is isomorphic to $\{(x_1, x_2, \dots\dots\dots, x_{10}) \in \mathbb{R}^{10} | x_i = 0 \text{ for some fixed } i\}$.

Since w_1, w_2, w_3 are distinct subspaces.

Let $w_1 = \{(x_1, x_2, \dots\dots\dots, x_{10}) \in \mathbb{R}^{10} : x_1 = 0\}$

$w_2 = \{(x_1, x_2, \dots\dots\dots, x_{10}) \in \mathbb{R}^{10} : x_2 = 0\}$

$w_3 = \{(x_1, x_2, \dots\dots\dots, x_{10}) \in \mathbb{R}^{10} : x_3 = 0\}$

$$w = w_1 \cap w_2 \cap w_3 = \{(x_1, x_2, \dots\dots\dots, x_{10}) \in \mathbb{R}^{10} : x_1 = x_2 = x_3 = 0\}$$

$$\dim w = \dim(w_1 \cap w_2 \cap w_3) = 7$$

So, option (2) and (3) are correct and option (4) is incorrect.

15. Let A be a real symmetric matrix. Then we can conclude that

1)A does not have 0 as an eigen value.

2) All eigen values of A are real.

3) If A^{-1} exists, then A^{-1} is real and symmetric.

4)A has at least one positive eigen value.

Sol:- Take $A = Zero\ matrix \Rightarrow '0'$ is an eigen value of A.

So, option (1) is incorrect.

Take $A = -I_n$, I is the identity matrix.

All eigen values of A are '-1' ⇒ so, option (4) is incorrect.

We eigenvalues of a real symmetric matrix is real.

So, option (2) is correct.

As, A is symmetric matrix and if A^{-1} exists ⇒ $A^{-1}$is a real matrix.

Also, $(A^{-1})^t = (A^t)^{-1} = A^{-1} \Rightarrow A$ is a symmetric matrix.

So, option (3) is correct.

Answer

Sl. No	Answer	Sl. No	Answer	Sl. No	Answer
1.	1	2.	1	3.	3
4.	2	5.	4	6.	3
7.	1	8.	1	9.	1 & 4
10.	1, 2, 3 & 4	11.	3 & 4	12.	1 & 3
13.	1	14.	2 & 3	15.	2 & 3

Previous Year Questions & Solution

Linear Algebra

June - 2017

Part – B

1. Let A be a 4×4 matrix. Suppose that the null space $N(A)$ of A is $\{(x,y,z,w) \in \mathbb{R}^4 : x + y + z = 0, x + y + w = 0\}$. Then
1)$\dim(Column\ space\ (A)) = 1$
2)$\dim(column\ space\ (A)) = 2$.
3)$rank(A) = 1$
4)$S = \{(1,1,1,0),(1,1,0,1)\}\ is\ a\ basis\ of\ N(A)$.

Sol:-
Given that, A is 4×4 and null space $N(A)$ of A is

$$\{(x,y,z,w) \in \mathbb{R}^4 : x + y + z = 0, x + y + w = 0\} \Rightarrow \dim(N(A)) = 4 - 2 = 2$$
$$\Rightarrow Nullity\ (A) = 2$$
$$\Rightarrow Rank(A) = 2$$

Thus, option (3) is incorrect.
Also, $column\ (rank\ A)\ =\ row\ (rank\ A)\ =\ rank\ (A)\ =\ 2$

$$\therefore \dim(column\ space(A)) = 2$$

So, option (2) is correct and option (1) is incorrect.
For option (4)

$$S = \{(1,1,1,0),(1,1,0,1)\}$$

It cannot be a basis of $N(A)$.
If it is a basis of $N(A) then\ \ S \subseteq N(A) = \{(x,y,z,w) : x + y + z = 0, x + y + w = 0\}$.
But, $(1,1,1,0) \notin S, (1,1,0,1) \notin S\ as\ 1 + 1 + 1 \neq 0\ and\ 1 + 1 + 0 \neq 0$.
So, S cannot be a basis of $N(A)$.
So, option (D) is incorrect.

2. Let A and B be real matrices such that $AB = BA$. Then
1)$Trace(A) = trace\ (B) = 0$
2)$Trace\ (A) = trace\ (B) = 1$
3)$Trace(A) = 0, trace(B) = 1$
4)$Trace\ (A) = 1, trace(B) = 0$

Sol:-
Given that, $A\ and\ B$ invertible matrices such that

$$AB = -BA \Rightarrow B^{-1}AB = -A(\ as\ B^{-1}\ exists)$$
$$\Rightarrow A\ and - A\ are\ similar\ matrices.$$
$$\Rightarrow Trace\ (A) = Trace(-A)$$

$$\Rightarrow Trace(A) - Trace(-A) = 0 \ \Rightarrow Trace(A) = 0$$

So, option (2) and (4) are incorrect.

Parallelly, we can write $AB = -BA \text{ as } ABA^{-1} = -B \ \Rightarrow B \text{ and } -B \text{ are similar matrices}$

$$\Rightarrow Trace\ (B) = 0$$

So, option (3) is incorrect and option (1) is correct.

3. Let A be an $n \times n$ self – adjoint matrix with eigen values $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$.

Let $\|X\|_2 = \sqrt{|X_1|^2 + |X_2|^2 + \ldots\ldots + |X_n|^2}$ for $X = (x_1, x_2, \ldots\ldots, x_n) \in \mathbb{C}^n$.

If $P(A) = a_0\, I + \ a_1\, A + \ldots\ldots + a_n A^n$ then $Sup_{\|X\|_2} \|P(A)X\|_2$ is equal to

1) $\max\{a_0 \ + \ a_1 \lambda_j \ + \ a_1 \lambda^2{}_j + \ldots\ldots + a_n \lambda^n{}_j : 1 \ \leq j \leq n\}$

2) $\max\{|a_0 \ + \ a_1 \lambda_j \ + \ a_1 \lambda^2{}_j + \ldots\ldots + a_n \lambda^n{}_j| : 1 \ \leq j \leq n\}$

3) $\min\{a_0 \ + \ a_1 \lambda_j \ + \ a_1 \lambda^2{}_j + \ldots\ldots + a_n \lambda^n{}_j : 1 \ \leq j \leq n\}$

4) $\min\{|a_0 \ + \ a_1 \lambda_j \ + \ a_1 \lambda^2{}_j + \ldots\ldots + a_n \lambda^n{}_j| : 1 \ \leq j \leq n\}$

Sol:-

$$P(A)X = (a_0\, I + \ a_1\, A + \ldots\ldots + a_n A^n)X$$

$$= (a_0\, X + \ a_1\, AX + \ldots\ldots + a_n A^n X)$$

$$= a_0\, X + \ a_1\, \lambda\, X + \ldots\ldots + a_n \lambda^n X \qquad (\because AX = \lambda X,\ A^n X = \lambda^n X)$$

$$\|P(A)X\|_2 = \ \|a_0\, X + \ a_1\, \lambda\, X + \ldots\ldots + a_n \lambda^n X\ \|_2$$

$$= \|a_0 \ + \ a_1\, \lambda \ + \ldots\ldots + a_n \lambda^n\|_2 \|X\ \|_2$$

$$Sup_{\|X\|_2} \|P(A)X\|_2 = \max\{|a_0 \ + \ a_1 \lambda_j + \ldots\ldots + a_n \lambda^n{}_j| : 1 \ \leq j \leq n\}$$

So, option (2) is correct and options (1), (3) and (4) are incorrect.

4. Let $P(x) = \alpha x^2 + \beta x + \gamma$ be a polynomial, where $\alpha, \beta, \gamma \in \mathbb{R}$. $Fix\ x_0 \in \mathbb{R}$.

Let $S = \{(a, b, c) \in \mathbb{R}^3 \ : P(x) = a(x - x_0)^2 + b\ (x - x_0) + c \ for\ all\ x \in \mathbb{R}$. Then the number of elements of S is

1) 0

2) 1

3) Strictly greater than 1 but finite

4) infinite.

Sol:-

Take $\alpha = 1, \beta = 0, \gamma = -1 \ \Rightarrow P(x) = \ x^2 - 1$

$Fix\ x_0 \in \mathbb{R}$.

$$S = \ \{(a, b, c) \in \mathbb{R}^3 | P(x) = ax^2 + bx + c \quad \forall\, x \in \mathbb{R}\}$$

$$\Rightarrow S = \ \{(a, b, c) \in \mathbb{R}^3 | x^2 - 1 = ax^2 + bx + c \quad \forall\, x \in \mathbb{R}\}$$

$$\Rightarrow S = \{(1, 0, -1)\}.$$

Thus, there is only one element in the sets.

∴ options (1), (3), (4) are incorrect.

So, option (2) is correct.

5. Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$ and I be the 3×3 identity matrix.

If $6\ A = aA^2 + bA + cI\ \ for\ a, b, c \in \mathbb{R}\ \ then\ (a, b, c)\ equals.$

1)$(1,2,1)$

2)$(1, -1, 2)$

3)$(4,1,1)$

4)$(1,4,1)$

Sol:- $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

The eigen values of A are 1, -2, -3.

$$\Rightarrow C_A(x) = (x-1)(x-2)(x-3) = (x-1)(x^2+5x+6)$$
$$= x^3 + 5x^2 + 6x - x^2 - 5x - 6$$
$$C_A(x) = x^3 + 4x^2 + x - 6$$

Using, Cayley's Hamilton theorem, we have

$$A^3 + 4A^2 + A - 6I = 0$$
$$\Rightarrow 6I = A^3 + 4A^2 + A$$

Multiply A^{-1} on both sides, we get $6A^{-1} = A^2 + 4A + I$

If $6A^{-1} = aA^2 + bA + c; for\ a, b, c \in \mathbb{R}$

Compare with above equation (1), we get $a = 1, b = 4, c = 1$

So, $(a, b, c) = (1,4,1)$

Thus, option (4) correct.

6. Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -2 & 5 \\ 2 & 5 & -3 \end{pmatrix}$. Then the eigen values of A are

1)$-4, 3, -3$

2)$4, 3, 1$

3) $4, -4 \pm \sqrt{13}$

4)$4, -2 \pm 2\sqrt{7}$

Sol:- $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

As, sum of entries of each column of A is 4 ⇒ 4 is one of the eigen values of A.

So, option (1) is incorrect.

Since, we know that $trace\ of\ A =\ sum\ of\ the\ eigen\ values\ of\ A.$

So, eigen values of A cannot be 4,3,1 as $4 + 3 + 1 \neq trace\ A = -4.$

So, option (2) is incorrect.

Also, eigen values of A cannot be

$$4, -2 \pm 2\sqrt{7}\ as\ 4 + \left(-2 + 2\sqrt{7}\right) + \left(-2 - 2\sqrt{7}\right) \neq trace\ A = -4.$$

So, option (4) is incorrect.

Hence, option (3) is correct.

Part – C

7. Consider the vector space V of real polynomials of degree less than or equal to n. Fix distinct real numbers $a_0,\ a_1, \dots\dots\dots, a_k$, for $P \in V$, $\max\{|P(a_j)|: 0 \leq j \leq k\}$ defines a norm of V.

1)$only\ if\ k < n$
2)$only\ if\ k \geq n$
3)$if\ k + 1 \leq n$
4)$if\ k \geq n + 1$

Sol:-

8. Let V be the vector space of polynomials of degree at most 3 in a variable x with co – efficient in $\mathbb{R}$. Let $T = \frac{d}{dx}$ be the L.T. on V to itself given by differentiation. Which of the following is correct?

1) T is invertible.
2)0 is an eigen value of T.
3) There is a basis with respect to which the matrix of T is nilpotent.
4) The matrix of T with respect to the basis $\{1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3\}$ is diagonal.

Sol-
Given that, V is vector space of polynomials of degree at most 3 in variable x with real co-efficient.
$T: V \to V$ be the L. T. defined as $T(P(x)) = \frac{d}{dx}(P(x))$.
Let $B = \{1, x, x^2,\ x^3\}$ be an ordered basis of V.

$$\Rightarrow T(1) = 0, T(x) = 1, T(x^2) = 2x, T(\ x^3) = 3\ x^2$$

Matrix of T with respect to basis 'B' is given by

$$[T]_B = \begin{pmatrix} 0 & 1 & 00 \\ 0 & 0 & 20 \\ 0 & 0 & 03 \\ 0 & 0 & 00 \end{pmatrix}$$

For option (1) and (2) as '0' is eigen value of $T \Rightarrow T$ is not invertable. So, option (1) incorrect and option (2) is correct.
For option (3)
For basis $B = \{1, x, x^2,\ x^3\}$ of V the matrix T with respect to B is nilpotent matrix.
So, option (3) is correct.
For option (4)

$$B' = \{1, 1 + x, 1 + x + x^2, 1 + x + x^2 +\ x^3\}$$

$$T(1) = 0 = 0 \cdot 1 + 0 \cdot (1 + x) + 0 \cdot (1 + x + x^2) +\ 0 \cdot (1 + x + x^2 +\ x^3)$$

$$T(1+x) = 1 = 1\cdot 1 + 0\cdot(1+x) + 0\cdot(1+x+x^2) + \ 0\cdot(1+x+x^2+\ x^3)$$
$$T(1+x+x^2) = 1+2x$$
$$= -1\cdot 1 + 2\cdot(1+x) + 0\cdot(1+x+x^2) + \ 0\cdot(1+x+x^2+\ x^3)$$
$$T(1+x+x^2+\ x^3) = 1+2x+3x^2$$
$$= 1\cdot 1 + (-1)\cdot(1+x) + 3\cdot(1+x+x^2) + \ 0\cdot(1+x+x^2+\ x^3)$$

Then matrix of T with respect to basis B is given by

$$[T]_B = \begin{pmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and above matrix is not diagonal.
So, option (4) is incorrect.

9. Let m, n, r be natural numbers. Let A be $m \times n$ matrix with real entries such that $(AA^t)^r = 1$, where I is the $m \times n$ identity matrix and A^t is the transpose of the matrix. We can conclude that

1)$m = n$
2)$AA^t\ is\ invertible.$
3)$A^tA\ is\ invertible.$
4)$if\ m = n, then\ A\ is\ \ invertible.$

Sol:
Given that, A is $m \times n$ matrix with real entries such that $(AA^t)^r = 1$.
Take $m = 2$ & $n = 3$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, A^t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \Rightarrow AA^t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow (AA^t) = I \Rightarrow (AA^t)^r = I.$$

But $m \neq n$
So, option (1) is incorrect.
For option (2)
If possible, let $(AA^t)^r$ is not invertable.
$\Rightarrow \det(AA^t) = 0 \ \Rightarrow \det((AA^t)^r) = 0 \ \Rightarrow \ (AA^t)^r$ can never be invertible.
$\Rightarrow (AA^t)^r$ can never be an identity matrix, which is contradiction to the given condition.
So, option (2) is correct.

Take $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, A^t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \ A^tA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

And $AA^t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \det(A^tA) = 0 \Rightarrow AA^t\ \ is\ not\ invertible.$
So, option (3) is incorrect.
For option (4)
If $m = n,\ as\ (AA^t)^r = I\ \ \Rightarrow \det((AA^t)^r) \Rightarrow \det I = I \Rightarrow (\det(AA^t))^r = I$

$$\Rightarrow (\det(A)\det(A^t))^{\,r} = I \ \Rightarrow \ (\det(A)\cdot(\det(A)))^r = 1 \ \Rightarrow (\det A)^{2r} = 1 \Rightarrow \det A \neq 0$$
$$\Rightarrow A\ is\ invertable.$$

So, option (4) is correct.

10. Let A be an $n \times n$ real matrix with $A^2 = A$. Then

1) The eigen values of A are either $0\ or\ 1$.

2)A is a diagonal matrix with diagonal entries $0\ or\ 1$.

3)$rank\ (A)\ =\ trace\ (A)$

4)$rank\ (I - A)\ =\ trace\ (I - A)$

Sol:-

Since $A^2 = A \Rightarrow$ minimal polynomial of A will divide $(x^2 - x)$,

i.e., $m_A(x)|(x^2 - x)\ \Rightarrow m_A(x)|\ x(x-1) \Rightarrow m_A(x) = x, x-1\ or\ x(x-1)$

So, eigen values of A are either 0 or 1.

So, option (1) is correct.

For option (2), Take $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \Rightarrow\ A^2 = A$

But, A is not diagonal matrix with diagonal entries $0\ or\ 1$.

So, option (2) is incorrect.

Since A is idempotent matrix and we know, idempotent matrix are always diagonalizable.

So, Jordan canonical form of A is diagonal matrix with diagonal entries $0\ or\ 1$.

We know $R(A)\ =\ R(Jordan\ form\ of\ A)\ and\ trace(A) = trace(Jordan\ form\ of\ A)$

$$\Rightarrow R(A) = number\ of\ 1\ on\ diagonals\ in\ Jordan\ form\ of\ A = tr(A).$$

So, option (3) is correct.

Since $A^2 = A$

So, $(I - A)^2 = I^2 + A^2 - 2A = I - A \Rightarrow I - A$ is also an idempotent matrix.

Using same procedure as in option (3) we get,

$$R(I - A) = tr(I - A)$$

So, option (4) is correct.

11. For any $n \times n$ matrix B, let $N(B) = \{X \in \mathbb{R}^n : BX = 0\}$ be the null space of B. Let A be a 4×4 matrix with $\dim\big(N(A - 2I)\big) = 2$, $\dim\big(N(A - 4I)\big) = 1\ and\ R(A) = 3$. Then

1)$0, 2\ and\ 4\ are\ eigen\ values\ of\ A$

2)$determinant\ (A) = 0$

3)$A\ is\ not\ diagonalizable.$

4)$trace(A) = 8$

Sol-

Given that, A is 4×4 matrix such that $\dim\big(N(A - 2I)\big) = 2\ \ and\ \dim\big(N(A - 4I)\big) = 1$ and $R(A) = 3$.

Using $R - N\ theorem,\quad R(A) + N(A) = no\ of\ columns\ of A.$

We get, $R(A - \lambda I) = 2, R(A - 4I) = 3\ and\ N(A - 0 \cdot I) = 1.$

Note:- If $R(A - \lambda I) < n$, then λ must be an eigen value of A.

So, $0, 2, 4$ are eigen values of A.

Since, nullity $(A - 0I) = 1 \Rightarrow Geometric\ multiplicity\ of\ '0'\ = 1$
nullity $(A - 2I) = 2 \Rightarrow Geometric\ multiplicity\ of\ '2'\ = 2$
and nullity $(A - 4I) = 1 \Rightarrow Geometric\ multiplicity\ of\ '4'\ = 1$

We know A.M. of $\lambda \geq G.M.of\ \lambda$

So, algebraic multiplicity of '0' ≥ 1

Algebraic multiplicity of '2' ≥ 2

Algebraic multiplicity of '4' ≥ 1

Since A is 4×4 matrix, only possibility we have

Algebraic multiplicity of '0' $= 2$

Algebraic multiplicity of '2' $= 2$

Algebraic multiplicity of '4' $= 1$

So, eigen values of A with multiplicity are $2, 2, 4, 0$.

For option (1)

Clearly $0, 2, 4$ are the only eigen values of A.

So, option (1) is correct.

Since '0' is one of the eigen values of A.

$$\Rightarrow \det A = product\ of\ eigen\ values\ of\ A = 0$$

So, option (2) is correct.

Since geometric multiplicity and geometric multiplicity of each eigen values are equal.

$$\Rightarrow A\ is\ diagonalizable.$$

So, option (3) is incorrect.

$$Trace\ (A) = Sum\ of\ the\ eigen\ values = 0 + 2 + 2 + 4 = 8$$

$$\Rightarrow option\ (4)\ is\ \ correct.$$

12. Which of the following 3×3 matrices are diagonalizable over $\mathbb{R}$?

1) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$

2) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{pmatrix}$

4) $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Sol:

Since, A has distinct real values, i.e., $1,4,6$ are the $eigen\ values\ of\ A \Rightarrow A\ is\ diagonalizable\ over\ \mathbb{R}$.

So, option (1) is correct.

For option (2), the eigen values are $1, i, -i$.

A has distinct eigen values but it has imaginary eigen values.

So, it is diagonalizable over $\mathbb{C}\ \ not\ over\ \mathbb{R}$.

⇒ option (3) is correct.

For option (4).

A is non – zero nilpotent matrix as all eigen values of A are '0'. We know, non – zero nilpotent matrices are not diagonalizable.

So, option (4) is incorrect.

13. Let H be a real Hilbert space and $M \subseteq H$ be a closed linear subspace. Let $x_0 \in H \backslash M$. Let $y_0 \in M$ be such that $\|x_0 - y_0\| = \inf\{\|x_0 - y\|: y \in M\}$. Then

1) Such a y_0 is unique.

2)$x_0 \perp M$

3)$y_0 \perp M$

4)$x_0 - y_0 \perp M$

Sol:

14. Let $A = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and $Q(X) = X^t AX$ for $X \in \mathbb{R}^3$. Then

1)A has exactly two positive eigenvalues.

2) all the eigenvalues of A are positive.

3)$Q(X) \geq 0\ \ \forall\ X \in \mathbb{R}^3$

4) $Q(X) < 0\ \ for\ some\ X \in \mathbb{R}^3$

Sol:

Let $A = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and $Q(X) = X^t AX$ for $X \in \mathbb{R}^3$

For option (1) and (2)

Since, $A = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

⇒ '6' is one of the eigen values of A and also $\det A = negative\ real\ number\ and\ Trace\ A = 6$

⇒ The eigen values of A is of the type $6, \lambda, -\lambda$ where $(\lambda \neq 0)$ real number.

So, A has two positive eigen values and one negative eigenvalues.

So, option (1) is correct and option (2) is incorrect.

For option (3)
If $Q(X) \geq 0 \; \forall \; X \in \mathbb{R}^3$
Then A is positive semi – definite, that means all the eigenvalues of A are non – negative, which is absurd,
So, option (3) is incorrect.
For option (4)
Since one of the eigen values of A is negative real number (say $'\mu'$)
Then, $\exists X(\neq 0) \in \mathbb{R}^3$(eigen values corresponding to $'\mu'$.)

$$\Rightarrow AX = \mu X \; (\, \mu \; is \; negative \; real \; number)$$
$$\Rightarrow X^t AX = \; X^t \mu \, X = \mu \, X^t X < 0$$
$$\Rightarrow Q(X) = X^t \mu \, X < 0 \;\; for \; some \; X \in \mathbb{R}^3$$

So, option (4) is correct.

15. Consider the matrix $A(x) = \begin{pmatrix} 1+x^2 & 7 & 11 \\ 3x & 2x & 4 \\ 8x & 17 & 13 \end{pmatrix}$; $x \in \mathbb{R}$. Then

1)$A(x)$ has eigen value 0 for some $x \in \mathbb{R}$.
2) 0 is not an eigen value of $A(x)$ for any $x \in \mathbb{R}$.
3)$A(x)$ has eigen value 0 for all $x \in \mathbb{R}$.
4)$A(x)$ is invertible for every $x \in \mathbb{R}$.

Sol:

$$\det\big(A(x)\big) = (1+x^2)(26x-68) - 7(39x-32x) + 11(51x-16x^2)$$
$$= 26x^3 + x^2(-68-176) + x(26-49+561) + (-68)$$
$$\det\big(A(x)\big) = 26x^3 - 244x^2 + 538x - 68$$

$\Rightarrow \det\big(A(x)\big)$ is cubic polynomial in 'x' with real co – efficient.
We know that every cubic polynomial with real co – efficient has at least one root in real number $\Rightarrow \exists \; x_0 \in \mathbb{R}$, such that $\det\big(A(x_0)\big) = 0$
$\Rightarrow A(x)$ has '0' as an eigen value for some $x = x_0 \in \mathbb{R}$
So, option (1) is correct and option (2) and (4) are incorrect.
Since, cubic polynomial has at most three real roots, i.e., $\det A(x) = 0$ for at most three values of $x \in \mathbb{R}$
$\Rightarrow A(x)$ has eigen values '0' for at most three values of $x \in \mathbb{R}$.
So, option (3) is incorrect.

Answer

Sl. No	Answer	Sl. No	Answer	Sl. No	Answer
1.	2	2.	1	3.	2
4.	2	5.	4	6.	3
7.	2 & 4	8.	2 & 3	9.	2 & 4
10.	1, 3 & 4	11.	1, 2 & 4	12.	1 & 3
13.		14.	1 & 4	15.	1

Previous Year Questions & Solution

Linear Algebra

December - 2017

Part – B

1. Let A be a real symmetric matrix and $B = I + iA, where\ i^2 = -1$. Then

1)B is invertible if and only if A is invertible.

2) All eigen values of B are necessary real.

3)$B - I$ is necessarily invertible.

4)B is necessarily invertible.

Sol:
Take $A = 0\ (zero\ matrix)$
Then $B = I + i \cdot 0 = I\ \Rightarrow B - I = 0$
$\Rightarrow$ B is invertible but A is not invertible and $B - I = 0$ is not invertible.
So, option (1) and (3) are incorrect.
Take, $A = I\ (identity\ matrix)$
Then $B = (1 + i)I$.
$\Rightarrow$ eigen values of B are non – zero complex numbers.
So, option (2) is incorrect.
Hence, option (4) is correct.

2. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$. Then the smallest positive integer n such that $A^n = I$ is

1) 1

2) 2

3) 4

4) 9

Sol:

$$A^2 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$
$$A^4 = A^2 \cdot A^2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$
$$A^6 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\therefore$ The least positive integer for which $A^n = I\ is\ n = 6$

3. Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & \alpha \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 3 \\ \beta \end{pmatrix}$. Then the system $AX = b$ over the real numbers has

1) No solution whenever $\beta \neq 7$.

2) An infinite solution whenever $\alpha \neq 2$.

3) An infinite number of solution if $\alpha = 2$ and $\beta \neq 7$

4)An unique solution if $\alpha \neq 2$.

Sol:

$$[A\!:b] = \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 2 & 3 & \alpha & \beta \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & +2 & 0 & 2 \\ 0 & 5 & \alpha - 2\beta - 2 \end{pmatrix}$$

$$\xrightarrow{R_2 = \frac{1}{2}R_2} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 5 & \alpha - 2\beta - 2 \end{pmatrix} \xrightarrow{R_3 \to R_3 - 5R_2} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \alpha - 2\beta - 7 \end{pmatrix}$$

Given system has

(i) Unique solution if $\alpha - 2 \neq 0$ $i.e., \alpha \neq 2$.

(ii) Infinite solution if $\alpha - 2 \neq 0$ and $\beta - 7 = 0 \Rightarrow b = 7$.

(iii) No solution if $\alpha - 2 = 0, but\ \beta - 7 \neq 0, i.e., if\ \alpha = 2, but\ \beta \neq 7$.

Clearly, option (4) is correct.

4. Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \in M_2(\mathbb{R})$ and $\phi: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be the bilinear map defined by $\phi(v, w) = v^T A\, w$.

Choose the correct statement from below.

1)$\phi(v, w) = \phi(w, v)\ \ \forall\, v, w \in \mathbb{R}^2$

2) There exists non – zero $v \in \mathbb{R}^2$ such that $\phi(v, w) = 0$ for all $w \in \mathbb{R}^2$.

3) There exists a 2×2 symmetric matrix B such that $\phi(v, v) = v^T\, B\, v\ for\ all\ v \in \mathbb{R}^2$

4) The map $\psi : \mathbb{R}^4 \to \mathbb{R}$ defined by $\psi\left(\begin{bmatrix} v_1 \\ v_2 \\ w_1 \\ w_2 \end{bmatrix}\right) = \phi\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right)$ is linear.

Sol:

Part – C

5. Let $M = \left\{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z} \; and \; the \; eigenvalues \; of \; A \; are \; in \; Q\right\}$. Then

1) M is empty.

2) $M = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z}\right\}$

3) If $A \in M$, then the eigen values of A are in $\mathbb{Z}$.

4) If $A, B \in M$ are such that $AB = I$ then $\det A \in (+1, -1)$

Sol:

Let, $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \in M$

$\therefore$ M is non – empty.

So, option (1) is incorrect.

Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \notin M \quad (\because eigen \; values \; of \; A \; are \; imaginary)$.

$\therefore$ option (2) is incorrect.

If $A \in M,$ then eigen values of A are in $\mathbb{Z}$. $[\because if \; \alpha_1, \; \alpha_2 \in Q - \mathbb{Z} \;\; are \; eigen \; values \; of \; A$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M \;\; then \; a,b,c,d \notin M$ which is a contradiction]

$$\therefore option \; (3) \; is \; correct.$$

Let $A, B \in M$ such that $AB = I \Rightarrow \; |AB| = I \; \Rightarrow \; |A| \cdot |B| = I \; \Rightarrow \; |A|, |B| \in \{-1,1\}$.

Thus, option (4) is correct.

6. Let A be a 3×3 matrix with real entries. Identify the correct statements.

1) A is necessarily diagonalizable over $\mathbb{R}$.

2) If A has distinct real eigenvalues then it is diagonalizable over $\mathbb{R}$.

3) If A has distinct eigen values then it is diagonalizable over $\mathbb{C}$.

4) If all eigenvalues of A are non – zero then it is diagonalizable over $\mathbb{C}$.

Sol:

Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is not diagonalizable.

$\therefore$ option (1) is not correct.

Let $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

All the eigenvalues of A are non – zero but A is not diagonalizable, since minimal polynomial of A is not linear.

So, option (4) is incorrect.

If a matrix 'A' has distinct real eigenvalues, then it is diagonalizable over $\mathbb{R}$.

If a matrix 'A' has distinct eigenvalues, then it is diagonalizable over $\mathbb{C}$.

7. Let V be the vector space over $\mathbb{C}$ of all polynomials in a variable X of degree at most 3. Let $D: V \to V$ be the linear operator given by differentiation with respect to X. Let A be the matrix of D with respect to some basis for V which of the following are true?

1) A is a nilpotent matrix.

2) A is a diagonalizable matrix.

3) The rank of A is 2.

4) The Jordan canonical form of A is $\begin{pmatrix} 0 & 1 & 00 \\ 0 & 0 & 10 \\ 0 & 0 & 01 \\ 0 & 0 & 00 \end{pmatrix}$

Sol:

Let $B = \{1, x,\ x^2, x^3\}$ be the standard basis of V.

$$\therefore D(1) = 0; D(x) = 1, D(x^2) = 2x, D(x^3) = 3x^2$$

$\therefore$ Matrix of D with respect to 'B' is given by $A = \begin{pmatrix} 0 & 1 & 00 \\ 0 & 0 & 20 \\ 0 & 0 & 03 \\ 0 & 0 & 00 \end{pmatrix}$

Clearly, A is nilpotent [$\because$ all eigenvalues of A are zero]

$\therefore$ option (1) is correct.

Also, A nilpotent matrix is diagonalizable if only if it is zero matrix.

Option (2) is incorrect.

Clearly, $P(A) = 3$

$\therefore$ option (3) is incorrect.

Further, only eigenvalues of $A = 0 with\ algebraic\ multiplicity = 4$.

And $geometric\ multiplicity\ = 4 - P(A - 0 \cdot I) = 4 - 3 = 1$.

$\therefore$ Jordan canonical form of A is $\begin{pmatrix} 0 & 1 & 00 \\ 0 & 0 & 10 \\ 0 & 0 & 01 \\ 0 & 0 & 00 \end{pmatrix}$

So, option (4) is correct.

8. For every 4×4 real symmetric non – singular matrix A, there exists a positive integer P such that

1)$PI + A$ is positive definite.

2)A^P is positive definite.

3)A^{-P} is positive definite.

4)$\exp(PA) - I$ is positive definite.

Sol:

Let A be a 4×4 real symmetric non – singular matrix and $\alpha_1,\ \alpha_2, \alpha_3, \alpha_4$ are non – zero and purely real.

Choose $P > |\alpha_i|\ \ \forall\ \ i = 1, 2, 3, 4$

$$\therefore\ \ eigenvalues\ of\ PI + A = eigenvalues\ of\ PI + eigenvalues\ of\ A.$$

$\Rightarrow P + \alpha_1, P + \alpha_2, P + \alpha_3, P + \alpha_4$ are eigenvalues of $PI + A$.

Clearly, $P + \alpha_i > 0\ \ \forall\, i = 1, 2, 3, 4$

Thus, $PI + A$ is positive definite
So, option (1) is correct.
As $\alpha_1,\ \alpha_2, \alpha_3, \alpha_4$ are eigenvalues of A.
$\Rightarrow\ \alpha_1{}^2,\ \alpha_2{}^2, \alpha_3{}^2, \alpha_4{}^2$ are eigenvalues of A^2.
Clearly, $\alpha_i{}^2 > 0\ \forall\ i = 1, 2, 3, 4$
Thus, A^2 is positive definite.
Moreover, A^P is positive definite for $P =\ even\ integer$.
So, option (2) is correct.
As $\alpha_1,\ \alpha_2, \alpha_3, \alpha_4$ are eigenvalues of A.
$\Rightarrow \frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3}, \frac{1}{\alpha_4}$ are eigenvalues of A^{-1} [$\because\ |A| \neq 0 \Rightarrow A^{-1}$ exists]
$\Rightarrow \frac{1}{\alpha_1{}^2}, \frac{1}{\alpha_2{}^2}, \frac{1}{\alpha_3{}^2}, \frac{1}{\alpha_4{}^2}$ are eigenvalues of A^2.
Clearly, A^{-P} is positive definite for $P = even\ integer$.
So, option (3) is correct.

For $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

$\exp(PA) - I$ cannot be positive definite for any positive integer P.
So, option (4) is correct.

9. Let A be an $m \times n$ matrix of $rank\ m$ with $n > m$. If for some real number $'\alpha'$, we have $x^t\ A\ A^t\ x = \alpha\ x^t\ x, for\ all\ x \in \mathbb{R}^m\ then\ A^t\ A$ has

1) Exactly two distinct eigen values.
2)0 is an eigen values of with multiplicity $n - m$.
3)α as a non – zero eigenvalue.
4) Exactly two non – zero distinct eigenvalues.

Sol:
Let A be an $m \times n$ matrix of rank m with $n > m$.
Given that, for some non – zero real number $'\alpha'$

$$x^t\ A\ A^t\ x = \alpha\ x^t\ x, \forall\ x \in \mathbb{R}^m$$
$$\Rightarrow x^t(A\ A^t - \alpha I)\ x = 0\ \ \forall\ x \in \mathbb{R}^m$$
$$\Rightarrow\ A\ A^t\ = \alpha\ I = \begin{pmatrix} \alpha & & & \\ & \alpha & & \\ & & \ddots & \\ & & & \alpha \end{pmatrix}_{m\times n}$$

Since, A is $m \times n$ matrix where $n > m$.
Using the fact that, if A is $m \times n$ matrix and B is $n \times m$ matrix where $n > m$, then AB is $m \times n$ matrix and BA is $n \times n$ matrix.
If $\lambda_1,\ \lambda_2, \dots\ \dots, \lambda_m$ be the eigenvalues of AB then $\lambda_1,\ \lambda_2, \dots\ \dots, \lambda_m, \underbrace{0, 0, \dots\dots\dots, 0}_{n-m\ times}$ be the eigenvalues of BA.

i.e., out of n $eigenvalues\ of\ BA$, m $eigenvalues\ of\ BA$ are same as AB and rest of $(n - m)$ $eigenvalues\ of\ BA\ are$ ‘0’.

Since, $\underbrace{\alpha, \alpha, \ldots\ldots\ldots, \alpha}_{m\ times}$ are eigenvalues of AA^t.

$\Rightarrow$ $\underbrace{\alpha, \alpha, \ldots\ldots\ldots, \alpha}_{m\ times}$, $\underbrace{0, 0, \ldots\ldots\ldots, 0}_{n-m\ times}$ are eigenvalues of AA^t.

So, AA^t has exactly two distinct eigenvalues $'\alpha \neq 0'$ and ‘0’.

So, option (1) is correct and option (4) is incorrect.

AA^t has ‘0’ as an eigenvalue with multiplicity ‘$n - m$’.

So, option (2) is correct.

AA^t has only $'\alpha'$ as a non – zero eigenvalue.

So, option (3) is correct and (4) is incorrect.

Answer

Sl. No	Answer	Sl. No	Answer	Sl. No	Answer
1.	4	2.	4	3.	4
4.	3	5.	3 & 4	6.	2 & 3
7.	1 & 4	8.	1, 2 & 3	9.	1, 2 & 3

Previous Year Questions & Solution

Linear Algebra

December - 2018

Part – B

1. Let $\mathbb{R}^n, n \geq 2$, be equipped with standard inner product. Let $\{v_1, v_2, \ldots\ldots, v_n\}$ be n column vectors forming an orthogonal basis of $\mathbb{R}^n$. Let A be the $n \times n$ matrix formed by the column vectors $v_1, \ldots\ldots\ldots, v_n$. Then

1. $A = A^{-1}$
2. $A = A^T$
3. $Det(A) = 1$
4. $A^{-1} = A^T$

Sol:-
Since, $A = (v_1, v_2, \ldots\ldots\ldots, v_n)$, where v_i represent the i^{th} column of A.
Take, $n = 2$
Let $v_1 = (0,1), v_2 = (-1,0)$ be forming an orthogonal basis of $\mathbb{R}^2 \Rightarrow A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
Neither $A = A^T\ nor\ A^{-1} = A$
So, options (1) and (2) are incorrect.
Let $v_1 = (0,1), v_2 = (1,0)$ be forming an orthogonal basis of $\mathbb{R}^2$

$$\Rightarrow A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \det(A) = -1$$

So, option (3) is incorrect.
In conclusion, option (4) is correct.

2. Let A be a $(m \times n)$ matrtix and B a $(n \times m)$ matrix over real numbers with $m < n$. Then

1. AB is always non-singular
2. AB is always singular
3. BA is always non-singular
4. BA is always singular

Sol.
Let A be $m \times n$ matrix and B be $n \times m$ matrix over real number with $m < n$.
$\Rightarrow AB$ is $m \times m$ matrix and BA is $n \times n$ matrix.
$\Rightarrow$Since, r$ank(BA) \leq \min\big(R(A), R(B)\big) \leq m < n$
$\Rightarrow BA$ is $n \times n$ matrix such that $rank(BA) < n$
$\Rightarrow \det(BA) = 0 \Rightarrow BA$ is always singular
So, option (4) is correct.

3. If A is a (2×2) matrix over $\mathbb{R}$ with $\det(A + I) = I + \det(A)$, than we can conclude that

1. $\det(A) = 0$
2. $A = 0$
3. $Tr(A) = 0$
4. A is singular

Sol. A is 2×2 matrix over $\mathbb{R}$ with $\det(A+I) = I + \det(A)$
Let λ_1, λ_2 be eigenvalue of $A \Rightarrow \lambda_1 + 1, \lambda_2 + 1$ be eigenvalues of $A + I$.

$$\therefore \det(A+I) = I + \det A$$

$$\Rightarrow (\lambda_1 + 1)(\lambda_2 + 1) = 1 + \lambda_1\lambda_2 \Rightarrow \lambda_1 + \lambda_2 = 0 \Rightarrow Tr(A) = 0$$

So, option (3) is correct.

4. The system of equations

$$1.x + 2.x^2 + 3.xy + 0.y = 6$$
$$2.x + 1.x^2 + 3.xy + 1.y = 5$$
$$1.x - 1.x^2 + 0.xy + 1.y = 7$$

1. Has solution in national numbers.
2. Has solution in real numbers.
3. Has solutions in complex numbers.
4. Has no solution.

Sol.
Subtraction 1st equation from 2nd equation we get $1.x - 1.x^2 + 0.xy + 1.y = 7$
This shows that the given equations are linearly dependent. So, the given system has no solution.

5. The trace of the matrix $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^{20}$ is

1. 7^{20}
2. $2^{20} + 3^{20}$
3. $2.2^{20} + 3^{20}$
4. $2^{20} + 3^{20} + 1$

Sol.
Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$
Given values of A are $2, 2, 3$.

$$\Rightarrow Tr(A^{20}) = 2^{20} + 2^{20} + 3^{20} = 2.2^0 + 3^{20}$$

$\therefore$ option (3) is correct

Part-C

6. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}$ and define for $x, y, z \in \mathbb{R}, Q(x, y, z) = (x\ y\ z)A\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ which of the following statements are true ?

1. The matrix of second order partial derivatives of the quadratic form Q is $2A$.
2. The rank of the quadratic form of Q is 2.
3. The signature of the quadratic form of Q is $(+ + 0)$
4. The quadratic form Q takes the value 0 for some non-zero vector (x, y, z)

Sol:

7. Let $M_n(\mathbb{R})$ denote the space of all $n \times n$ real matrices define with the Euclidean space $\mathbb{R}^{n^2}$ fix a column vector $x \neq 0$ in $\mathbb{R}^n$. Define $f; M_n(\mathbb{R}) \to \mathbb{R}$ by $f(A) = \langle A^2x, x\rangle$ then.

1. f is linear.
2. f is differentiable.
3. f is continuous but not differentiable.
4. f is unbounded.

Sol.

Option (1) is incorrect, as fix $x = 2, n = 1$.
$f(A) = \langle A^2 . 2, 2\rangle = 4A^2$, which is clearly not linear in A.
Option (2) is correct, for any $A \epsilon M_n(\mathbb{R}); f(A)$ comes out be a polynomial.
Option (3) is incorrect as option (2) is correct
Option (4) is correct as for any $A \epsilon M_n(\mathbb{R}), f(A)$ comes out to a non-constant polynomial.

8. Let V denote the vector space of all sequences $a = (a_1, a_2, \dots.)$ of real numbers such that $\sum 2^n|a_n|$ converges. Define $\|\cdot\|: V \to \mathbb{R}$ by $\|a\| = \sum 2^n|a_n|$ which of the following are true ?

1. V contains only the sequence $(0,0, \dots \dots)$
2. V is finite dimensional.
3. V has a countable liner basis
4. V is a complete normal space.

Sol:

9. Let V be a vector space over $\mathbb{C}$ with dimension n. Let $T: V \to V$ be a linear transformation with only 1 as eigenvalue. Then which of the following must be true ?

1. $T - I = 0$
2. $(T - I)^{n-1} = 0$
3. $(T - I)^{n} = 0$
4. $(T - I)^{2n} = 0$

Sol.
$T: V \to V$ be a linear transformation, where V is vector space of dimension n.
In other words, T is $n \times n$ matrix with only 1 as an eigenvalue.
Take, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}_{2\times 2}$ (here $n = 2$)
$\Rightarrow T - I \neq 0$, but $(T - I)^2 = 0$
So, option (1) and (2) are incorrect.
Since 1 is only eigenvalue of T.
$\Rightarrow$ Characteristic polynomial of $T = (x-1)^n \Rightarrow (T-2)^n = 0$ and also $(T - I)^{2n} = 0$
$(\because if\ f(t) = 0 \Rightarrow (f(T))^k = 0\ for\ k \geq 1)$
So, option (3) and (4) are correct.

10. If A is a (5×5) matrix and the dimension of the solution space of $Ax = 0$ at least two, then

1. $Rank(A^2) \leq 3$
2. $Rank(A^2) \geq 3$
3. $Rank(A^2) = 3$
4. $\det(A^2) = 0$

Sol.
Given A is 5×5 matrix such that $dim\{X: Ax = 0\} \geq 2$

$$\Rightarrow \dim(kerA) \geq 2 \Rightarrow nullity(A) \geq 2$$

Consider $A = 0$ (Zero matrix)$\Rightarrow$ A^2=0

$$Rank(A^2) = 0$$

So, options (2) and (3) are incorrect

Since, $nullity(A) \geq 2$

$\Rightarrow Rank(A) \leq 3 < 5$ (using R-N theorem)$\Rightarrow \det A = 0$

$$\Rightarrow \det(A^2) = 0 \Rightarrow (\det A)^2 = 0$$

Also $rank(A^2) \leq Rank(A) \leq 3$
So, option (1) and (4) are correct.

11. Let $A \epsilon M_3(\mathbb{R})$ be such that $A^8 = I_{3\times 3}$. Then

1. Minimal polynomial of A can only be of degree 2.
2. Minimal polynomial of A can only be of degree 3.
3. Either $A = I_{3\times 3}\ or\ A = -I_{3\times 3}$.
4. There are uncountably many A satisfying the above.

Sol.

Given that $A\epsilon M_{3\times3}(\mathbb{R})$ such that $A^8 = I_{3\times3}$

Take, $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow C_A(x) = (x-1)(x+1)^2$ and $m_A(x) = (x-1)(x+1)$

Clearly, A satisfy the relation $A^8 = I$.

By above example, options (2) and (3) are incorrect.

Take, $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow C_A = (x-1)^3$ and $m_A = (x-1)$ and also satisfy the relation $A^8 = I$.

By above example, option (1) is incorrect

Let $A = \begin{pmatrix} 1 & 0 & 0 & \dots\dots & 0 & 0 \\ 0 & 1 & 0 & \dots\dots & 0 & 0 \\ 0 & 0 & 1 & \dots\dots & 0 & 0 \\ \dots & \dots & \dots & \dots\dots & \dots & \dots \\ 0 & 0 & 0 & \dots\dots & 0 & \lambda \\ 0 & 0 & 0 & \dots\dots & 1/\lambda & 0 \end{pmatrix}$, Where λ is non-zero real number.

Above matrix satisfy $A^2 = I \Rightarrow A^8 = I$.

So, there are uncountably many $A(as\ \lambda\epsilon\mathbb{R}\{0\})$ which satisfy $A^8 = I$. So, option (4) is correct.

12. Let A be an $n \times n$ matrix (with $n > 1$) satisfying $\quad - 7A + 12I_{n\times n} = 0_{n\times n}$, where $I_{n\times n}$ and $0_{n\times n}$ denote the identity matrix and zero matrix of order n respectively. Then which of the following statements are true?

1. A is invertable.
2. $t^2 - 7t + 12n = 0\ where\ t = Tr(A)$.
3. $d^2 - 7d + 12 = 0\ where\ d = \det(A)$.
4. $\lambda^2 - 7\lambda + 12 = 0\ where\ \lambda is\ an\ eigenvalue\ of\ A$.

Sol.

Take, $n = 2$

$\therefore\ A^2 - 7A + 12\ I_{n\times n} = 0_{2\times2} \Rightarrow$ minimal polynomial of A divides $x^2 - 7x + 12 = 0$

$\therefore$ minimal polynomial of A can be $(x-3)\ or\ (x-4)\ or\ (x-3)(x-4)$

For $m_A(x) = (x-3) \Rightarrow C_A(x) = (x-3)^2$

In this case, $t = Tr(A)\ and\ d = \det(A) = 9$

$$\therefore\ t^2 - 7t + 12n = 6^2 - 7(6) + 12(2) = 36 - 42 + 24 \neq 0$$

And $d^2 - 7d + 12 = 9^2 - 7(9) + 12 = 81 - 63 + 12 \neq 0$

Thus, options (2) and (3) are incorrect

Now, for general also, minimal polynomial of A divides $x^2 - 7x + 12 = 0$

$\therefore$ Minimal polynomial of A can be $(x-3)\ or\ (x-4)\ or\ (x-3)\ (x-4)$

$\Rightarrow$ eigenvalue of A is either 3 or 4 or 3 and 4 In all case λ satisfies $\lambda^2 - 7\lambda + 12 = 0$

Also, A is invertable.

Thus $(1)\ and\ (4)$ are correct.

13. Let A be a 6×6 matrix over $\mathbb{R}$ with characteristic polynomial $=(x-3)^2(x-2)^4$ and minimal polynomial$=(x-3)(x-2)^2$. Then Jordan canonical form of A can be

1. $\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$
2. $\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$
3. $\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$
4. $\begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

Sol.

$A_{6\times 6}$ be a matrix over $\mathbb{R}$ with $C_A(x) = (x-3)^2(x-2)^4 \; and \; m_A(x) = (x-3)(x-2)^2$
Since, algebraic multiplicity of $'3' = 2$ and degree of $(x-3)$ in $m_A(x)$ is one.
So, highest order Jordan block corresponding to eigenvalue $\lambda = 3$ is one.
Since, algebraic multiplicity of $'2' = 4$ and degree of $(x-2)$ in $m_A(x)$ is two.
So, highest order Jordon block corresponding to eigenvalue $\lambda = 2$ is two.
Jordan form of A is given by

$$\begin{bmatrix} J_1(3) & & & \\ & J_1(3) & & \\ & & J_2(2) & \\ & & & J_2(2) \end{bmatrix} or \begin{pmatrix} J_1(3) & & & & \\ & J_1(3) & & & \\ & & J_2(2) & & \\ & & & J_1(2) & \\ & & & & J_1(2) \end{pmatrix}$$

i.e., $\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$ or $\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

So, Jordon form of A given by options (2), (3)
So, options (2) and (3) are correct.

14. Let V be an inner product space and S be a subset of V. Let $\overline{S}$ denote the closure of S in V with respect to the topology include by the metric given by the inner product. Which of the following statements are true?

1. $S = (S^1)^1$
2. $\overline{S} = (S^1)^1$
3. $\overline{span(S)} = (S^1)^1$
4. $S^1 = ((S^1)^1)^1$

Sol:

Answer

Sl. No	Answer	Sl. No	Answer	Sl. No	Answer
1.	4	2.	4	3.	3
4.	4	5.	3	6.	4
7.	2& 4	8.	4	9.	3&4
10.	1 & 4	11.	4	12.	1 & 4
13.	2 & 3	14.	3 & 4		