COUNCIL OF SCIENTIFIC & INDUSTRIAL RESEARCH

Unit – 1 : REAL ANALYSIS

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1. Real Analysis

1.1. Natural Number:

1.1.1 Well Ordering Property: Every non-empty subset of $\mathbb{N}$ has a least element.

1.1.2. Principle of Induction:

i). **First principle of Induction**: Let S be a subset of $\mathbb{N}$ such that
(a). $1 \in S$ and
(b). if $k \in S$ $then$ $k+1 \in S$
Then $S = \mathbb{N}$.

ii). **Second Principle of induction:** Let S be a subset of $\mathbb{N}$ such that
(a). $1 \in S$ and
(b). if $\{1,2, \ldots \ldots \ldots .. k\} \subset then\ k+1 \in S.\ Then\ \ S = \mathbb{N}$

Example (1.1):

(i). First Principle of Induction: For each $n \in \mathbb{N}$, $1+2+\ldots\ldots\ldots\ldots+n = \frac{n(n+1)}{2}$.

(ii). For each $n \in \mathbb{N}, \left(3+\sqrt{5}\right)^n + \left(3-\sqrt{5}\right)^n$ is an integer. (2nd principle of induction). [Hints $= 3+\sqrt{5}\ , b = 3-\sqrt{5},\ a^{k+1}+b^{k+1} = (a^k+b^k)(a+b)-(a^{k-1}+b^{k-1})$]

1.2. Rational Number:

1.2.1. Density Property of $\mathbb{Q}$: If $x, y \in \mathbb{Q}\ with\ x < y, then\ \exists\ r \in \mathbb{Q}\ such\ that\ x < r < y.$

(i). There does not exists $r \in \mathbb{Q}\ such\ that\ r^2 = 2.$

Ans. Suppose$r = \frac{p}{q} \in \mathbb{Q}\ such\ that\ \left(\frac{p}{q}\right)^2 = 2, gc\,d(p,q) = 1\ \Rightarrow p^2 = 2q^2 \Rightarrow p\ is\ even.$

$\Longrightarrow p = 2m, \quad m \in z\ \Rightarrow\ 2m^2 = q^2 \Rightarrow q\ is\ even\ i.e, \gcd(p,q) \neq 1, contradiction.$

(ii). Let m be a non-square positive integer. There does not exist a$r \in \mathbb{Q}\ such\ that\ r^2 = m.$

1.3. Real Numbers:

1.3.1. Order Properties of $\mathbb{R}$:

(i). If $a, b \in \mathbb{R}, then\ exactly\ one\ of\ the\ following\ statements\ holds\ a < b, or\ a = b, a > b$ (low of trichotomy).

(ii). $a < b\ and\ b < c\ \Rightarrow a < c\ for\ a, b, c \in \mathbb{R}$ (transitivity).

(iii). $a < b\ \Rightarrow a + c < b + c\ for\ a, b, c \in \mathbb{R}.$

(iv). $a < b\ and\ 0 < c\ \Rightarrow ac < bc\ for\ a, b, c \in \mathbb{R}.$

Note: The field $\mathbb{R}$ together with the relation defined on$\mathbb{R}$ satisfying (i) to (iv) becomes an ordered field.

$* \, If \quad a, b \in \mathbb{R}, \max\{a, b\} = \frac{1}{2}\{a + b + |a - b|\}$

$$\min\{a, b\} = \frac{1}{2}\{a + b - |a - b|\}$$

1.3.2. Completeness Property of $\mathbb{R}$: Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound (or a supremum).

$\Rightarrow$ Let S be a non-empty subset of $\mathbb{R}$, bounded below. Then S has an infimum.

1.3.3. Definition (Complete Ordered Field): An ordered field is said to be a compute ordered field if the completeness property (i.e. the supremum and infimum property) holds, in it. Thus $\mathbb{R}$ is a complete ordered field.

Note: $\mathbb{Q}$ is not complete ordered field. Let $S : \{1, \frac{1}{1!} + 1, 1 + \frac{1}{1!}, \frac{1}{2!}, \ldots\ldots\}$ which is bounded above, but has no supremum.

1.3.4. Properties of the supremum and the infimum: Let S be a non-empty subset of $\mathbb{R}$, bounded above. Then $\sup S$ exist. Let $M = \sup S$. $then\ M \in \mathbb{R}\ and\ M$ satisfies the followings:

(i). $x \in S \ \Rightarrow x \leq M$

(ii). for each $\varepsilon > 0$, $\exists \ y(\varepsilon) \in S \ such\ that \ M - \varepsilon < y \leq M$

$Let\ S \subset \mathbb{R}$ be bounded below. Then $inf\ S$ exists and let $m = inf\ S$. Then $m \in \mathbb{R}$ and m satisfies the followings:

(i). $x \in S \ \Rightarrow x \geq m.$

(ii). for each $\varepsilon > 0, \exists \ y(\varepsilon) \in S \ such\ that\ m < y < m + \varepsilon$

1.3.5. Archimedean Property of $\mathbb{R}$: If $x, y \in \mathbb{R}\ and\ x > 0, y > 0, then \exists\ on\ x \in \mathbb{N}, such\ that\ ny > x.$

$\Rightarrow If\ x \in \mathbb{R},\ then \exists\ n \in \mathbb{N}\ such\ that\ n > x.$

1.3.6. Density Property of $\mathbb{R}$:

(i). If $x, y \in \mathbb{R}\ with\ x < y\ then \exists \ r \in \mathbb{Q}\ such\ that\ x < r < y.$

(ii). If $x, y \in \mathbb{R}\ with\ x < y\ then \exists \ s \in \mathbb{R}\backslash\mathbb{Q},\ irrational\ number, such\ that\ x < s < y$

Proof:

(i). $y > x \Rightarrow y - x > 0 \Rightarrow$(by Archimedean property) $\exists \ n \in \mathbb{N}$ such that

$0 < \frac{1}{n} < y - x \ \Rightarrow ny - nx > 1 \ \Rightarrow nx + 1 < ny$ ....(a)

Again, $nx \in \mathbb{R}$ (by Archimedean property) $\exists\ m \in \mathbb{N}\ such\ that\ m - 1 \ \leq nx < m$.......(b)

$\Rightarrow m \leq nx + 1 < m + 1 \ \Rightarrow \big(by\ (a)\big)\ m \leq nx + 1 < ny \ \Rightarrow nx < m < xy\ by\ (b)$

$\Rightarrow x < \frac{m}{n} < y$

(ii). Let $\sqrt{2}x < \sqrt{2}y \Rightarrow \exists\, r \in \mathbb{Q} \Rightarrow \sqrt{2}x < r < \sqrt{2}y \Rightarrow x < \frac{r}{\sqrt{2}} < y$

1.3.7. Extended set of real numbers: The set of real numbers with the addition of two elements ∞ *and* $-\infty$. We defined the followings –

(i). $\forall\, x \in \mathbb{R},\ x + \infty = \infty = \infty + x$

(ii). $\forall\, x > 0,\ x \cdot \infty = \infty = \infty \cdot x$

(iii). $\forall\, x < 0,\ x \cdot \infty = -\infty = \infty \cdot x$

(iv). $\infty + \infty = \infty, (-\infty) + (-\infty) = -\infty, (\infty)\cdot(\infty) = \infty, (-\infty)(-\infty) = \infty, (\infty)(-\infty) = (-\infty)(\infty) = -\infty.$

(v). [Undefined]: $(\infty) + (-\infty), (-\infty) + (\infty), 0 \cdot \infty, \infty \cdot 0, 0 \cdot (-\infty), (-\infty) \cdot 0$

(vi). $\forall\, x \in \mathbb{R}, -\infty < x < \infty$

1.4. Sets in $\mathbb{R}$:

Let, $a, b \in \mathbb{R}$ with $a < b$. Then $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ is called an open interval of $\mathbb{R}$ *and* $[a, b] = \{x \in \mathbb{R} : a \le x \le b\}$ is called a closed interval of $\mathbb{R}$ *and* $[a, b) = \{x \in \mathbb{R} : a \le x < b\}$ is called a semi open interval of $\mathbb{R}$.

Example (1.2): $[a, a] = \{a\}, (-\infty, \infty) = \mathbb{R}$.

1.4.1. Definition (Neighbourhood): Let $c \in \mathbb{R}$. A subset $S \subseteq \mathbb{R}$ is said to be a neighbourhood of c *if* $\exists$ an open interval (a, b) such that $c \in (a, b) \subset S$.

(i). Infinite number, union and finite number of intersection of neighbourhood of $c \in \mathbb{R}$ is also a neighbourhood of c. But infinite number of intersection of neighbourhood may not be a neighbourhood of c.

Example (1.3): For each $n \in \mathbb{N}, \left(-\frac{1}{n}, \frac{1}{n}\right)$, is a neighbourhood of O. But $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$ is not a neighbourhood of 0.

1.5. Definition (Interior Point): Let S be a subset of $\mathbb{R}$. A point $x \in S$ is said to be an interior point of S if $\exists$ a neighbourhood $N(x)$ of x such that $N(x) \subset S. intS = S^0$ denote the set of all interior points of S.

Example (1.4):

(i). $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots \ldots\ldots\right\} \Rightarrow int\ S = S^0 = \phi$

(ii). $S = \mathbb{N}, \mathbb{Q},\ \mathbb{Z},\ int\ S = \phi$

(iii). $S = (1,3),\ int\ S = S'$

1.5.1. Definition (Open Set): Let $S \subseteq \mathbb{R}. S$ is said to be open set if each point of S is an interior point of S. i.e, $int\ S = S'$

Example(1.5):

(i). $S = (1,3),\ int\ S = S\ \Rightarrow S\ is\ an\ open\ set.$

(ii). $S = [1,3]$ is not an open set as $1,3 \notin int\ S$.

(iii). $S = \phi$ contains no point. Therefore requirement in the definition is vacuously satisfied. $S = \phi$ is open.

1.5.2. Theorem:

1. (i) The arbitrary union of open set is open.

(ii)The finite number of intersection of open set in open. But infinite number of intersection of open set may not be open.

Example (1.6):

(i). Let $I_n = \left(-\frac{1}{n}, \frac{1}{n}\right), n \in \mathbb{N}$. Then each I_n is open. But $\bigcap_{n=1}^{\infty} I_n = \{0\}$ is not open.

(ii). $I_n = \left(a - \frac{1}{n}, b + \frac{1}{n}\right),\ a, b \in \mathbb{Z}\ with\ a < b\ \Rightarrow \bigcap_{n=1}^{\infty} I_n = [a, b]\ closed.$

2. Let $S \subseteq \mathbb{R}$. Then $int\ \mathbb{S}$ in an open set in $\mathbb{R}$ and it is the largest open set contained in S.

3. Let $S = [0,1]\ and\ T = \{\frac{1}{n} : n \in \mathbb{N}\}$.
Then $S - T = \left(\frac{1}{2}, 1\right) \cup \left(\frac{1}{3}, \frac{1}{2}\right), \cup \left(\frac{1}{4}, \frac{1}{3}\right), \cup \ldots \ldots \Rightarrow S - T\ is\ open.$

4. A non – empty bounded open set in $\mathbb{R}$ is the union of a countable collection of disjoint open intervals.

1.6. Limit Point:

1.6.1. Definition (Limit Point): Let $S \subseteq \mathbb{R}$. A point $p \in \mathbb{R}$ is said to be a limit point (or accumulation point, cluster point) of S if every neighbourhood of p contains a point of S other then $p. \Rightarrow p \in \mathbb{R}$ is a limit point of$S$ if for each $\varepsilon > 0, [N(p, \varepsilon) - \{p\}] \cap S \neq \phi$
$i.e, N'(p, \varepsilon) \cap S \neq \phi$ where $N'(p, \varepsilon) = N(p, \varepsilon) - \{p\} = (p - \varepsilon, p + \varepsilon) \setminus \{p\}$.

1.6.2. Definition (Isolated Point): Let $S \subseteq \mathbb{R}$. A point $x \in S$ is said to be an isolated point of S if x is not a limit point of S.
$\Rightarrow \exists\ some\ \varepsilon > 0\ such\ that\ N'_{\varepsilon}(x) \cap S = \phi\ \ i.e, N_{\varepsilon}(x)$ contains no point of S other than x.

Example (1.7):

(i). $S = \mathbb{Z}, \mathbb{N}$ every point of S is isolated.

(ii). $S = \mathbb{Q}, \mathbb{R}$ has no isolated point.

(iii). Let $S \subseteq \mathbb{R}\ and\ p$ be a limit point of S. Then every neighbourhood of p contains infinitely many points of S.

(iv). Interior point of S is a limit point of S.

1.6.3. Bolzano-Weierstrass Theorem: Every bounded infinite subset of $\mathbb{R}$ has at least one limit point ($in\ \mathbb{R}$).

1.6.4. Definition (Derived Set). Let $S \subseteq \mathbb{R}$. The set of all limit points of S is said to be the derived set of S and is denoted by S'.

Example (1.8):

(i). Let S be finite. Then $S' = \phi$.

(ii). $S = \mathbb{N}, \mathbb{Z}, then\ S' = \phi$

(iii). $S = \mathbb{Q}, \mathbb{R}, then\ S' = \mathbb{R}$.

(iv). $S = \phi, then\ S' = \phi$.

1.6.5. Let $A, B \subseteq \mathbb{R}$ then –

(i). $If\ A \subset B\ \Rightarrow A' \subset B'$

(ii). $(A')' \subset A'$

(iii). $(A \cap B)' = A' \cap B'\ and\ (A_1 \cap A_2 \cap \ldots . \cap A_m)' \cap\ A'_1 \cap A'_2 \cap \ldots . \cap A'_m$

(iv). $(A \cup B)' = A' \cup B'\ and\ (A_1 \cup A_2 \cup \ldots . \cup A_n)' =\ A'_1 \cup A'_2 \cup \ldots . \cup A'_n$

1.6.6. Definition (Closed Set): Let $S \subset \mathbb{R}$. S is said to be closed if $S' \subset S$ (i.e, if S contains all its limit points.).

Example (1.9):

(i). Let $S = \mathbb{N}, \mathbb{Z}$. $Then\ S' = \phi \Rightarrow S' \subset S \Rightarrow S$ is closed.

(ii). $S = \{a_1, a_2, \ldots \ldots . ., a_m\}\ Then\ S' = \phi \Rightarrow S' \subset S \Rightarrow S$ is closed.

(iii). $S = [a, b]$ is closed.

(iv). $S = (a, b)$ is not closed as $a, b \in S'\ and\ S' = [a, b] \not\subset S = (a, b)$

1.6.7. Theorem:

(i) The intersection of arbitrary collection of closed set in $\mathbb{R}$ is closed.

(ii). The union of finite number of closed set in $\mathbb{R}$ is closed. But infinite number of union of closed set may not be closed.

Example (1.10): Let $a, b \in \mathbb{Z}$ with $a < b - 1$.
Let $I_n = \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$ Then each I_n is closed. But $\cup_{n=1}^{\infty} I_n = (a, b)$ is open.

1.6.8. Theorem:

(i). The complement of open set (closed set) in $\mathbb{R}$ is closed set (respectively open set) in $\mathbb{R}$.

(ii). If S be a non-empty closed and bounded subset of $\mathbb{R}$, Then $sup\ S\ and\ inf\ S$ belongs to S.

1.7. Adherent Point: Let $S \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is said to be an adherent point of S if every neighbourhood of x contains a point of S.

$\Rightarrow x \in \mathbb{R}$ is a adherent points if $N_\varepsilon(x) \cap S \neq \phi$ for all $\varepsilon > 0$.

The set of all adherent points of S is said to be the closure of S and is denoted by $\bar{S} = S \cup S'$.

1.7.1. Theorem:

(i) Let $S \subseteq \mathbb{R}$. Then S is closed $\Leftrightarrow S = \bar{S}$

(ii). $\bar{S}$ is the smallest closed containing S.

1.7.2. Theorem: Let $A, B \subseteq \mathbb{R}$. Then –

(i). $A \subset B \Rightarrow \bar{A} \cup \bar{B}$

(ii). $\overline{A \cup B} = \bar{A} \cup \bar{B}$ and $\overline{A_1 \cup A_2 \cup \ldots .\cup A_n} = \overline{A_1} \cup \overline{A_2} \cup \ldots ..\cup \overline{A_n}$

(iii). $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$

1.7.3. $\mathbb{R}$ *and* $\mathbb{Q}$ being both closed and open in $\mathbb{R}$ and no non-empty proper subset of $\mathbb{R}$ is both open and closed.

1.7.4. Definition (Dense Set): Let $S \subseteq \mathbb{R}$. A subset $T \subset \mathbb{S}$ is said to be dense in S if $S \subset T'$.

Example (1.11)

(i). $\mathbb{Q}$ is dense in $\mathbb{R}$, since $\mathbb{Q} \subset \mathbb{Q}' = \mathbb{R}$

(ii). $S = (1,2)$ *Then* $S \subset S' = [1,2]$. S is dense in itself.

1.7.5. Definition (Perfect Set): Let $S \subseteq \mathbb{R}$. S is said to be perfect set if $S = S'$ i.e, S is dense in itself.

Example (1.12.): (i) $S = [1,2]$.Then $S = S' = [1,2]$. S is a perfect set.

1.7.6. Definition (Nested Intervals): If $\{I_n : x \in \mathbb{N}\}$ be a family of intervals such that

$I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$, then the family $\{I_n\}$ is said to be a family of nested intervals.

Example (1.13):

(i) $I_n = \left\{x \in \mathbb{R} : 0 < x < \frac{1}{n}\right\},\ n \in \mathbb{N}$

(ii). $I_n = \{x \in \mathbb{R} : x > n\},\ n \in \mathbb{N}$

(iii). $I_n = \left\{x \in \mathbb{R} : -\frac{1}{n} < x < \frac{1}{n}\right\}, n \in \mathbb{N}$

1.7.7. Theorem (Nested Intervals):

If $\{[a_n, b_n] : n \in \mathbb{N}\}$ be a family of nested closed and bounded intervals then $\cap_{n=1}^{\infty}[a_n, b_n]$ is non – empty.

Further more, if inf $\{(b_n - a_n) : n \in \mathbb{N}\} = 0$, Here is one and only one point x such that $x \in \cap_{n=1}^{\infty}[a_n, b_n]$.

Note – I: If $\{I_n : n \in \mathbb{N}\}$ be a family of nested open bounded intervals, then $\bigcap_{n=1}^{\infty} I_n$ may not be non-empty.

Example (1.14.): If $I_n = \left(0, \frac{1}{n}\right), then \ \bigcap_{n=1}^{\infty} I_n = \phi$

Note – II: If $\{I_n : n \in \mathbb{N}\}$ be a family if nested closed unbounded intervals then $\bigcap_{n=1}^{\infty} I_n$ may not be non-empty.

Example (1.15): If $I_n = (n, \infty), then \ \bigcap_{n=1}^{\infty} I_n = \phi.$

1.7.8. Cantor's Intersection Theorem: Let $F_1, F_2, \dots\dots$ be a countable collection of non-empty closed and bounded subsets of $\mathbb{R}$ such that $F_1 \supset F_2 \supset F_3 \supset \dots\dots\dots$Then the intersection $\bigcap_{i=1}^{\infty} F_i$ is non-empty.

1.7.9. Decimal representation of a real number: If $x \in \mathbb{R}$, it has two decimal representations –

(i). $x = p \cdot a_1\, a_2\, a_3 \dots\dots a_k 0\ 0\ 0 \dots\dots$ where $a_i \in \{0,1,2, \dots\dots, 9\}$ and

$p \leq x < p + 1 \ \exists\, p \in \mathbb{Z},$

(ii). $x = \ p \cdot a_1\, a_2\, a_3 \dots.(a_k - 1)999 \dots\dots\ where\ a_n \in \{0,1,2, \dots, 9\}\ and$

$p \leq x < p + 1, \exists\, p \in \mathbb{Z}$

Example (1.16): $If\ x = \frac{437}{1000} \Rightarrow x = 0.437000 \dots\dots or,\ \ x = 0.436999 \dots\dots$

1.7.10. Definition (Periodic Decimal): A decimal $p \cdot a_1 a_2 a_3 \dots\dots$ is said to be a periodic decimal (or recurring decimal) if $\exists\, m, k \in \mathbb{N}$ such that $a_n = a_{n+m}\ \forall\, n \geq k$. The smallest $m \in \mathbb{N}$ with this property is called the period of the decimal. In this case the block of digits $a_k a_{k+1} a_{k+2} \dots\dots a_{k+m-1}$ is respected once the k^{th} digit is reached.

Example (1.17.):0.235636363 .... is periodic with repeating block 63.

1.7.11. Definition (Terminating decimal): A decimal $p \cdot a_1 a_2 a_3 \dots\dots$ is said to be a terminating decimal if $\exists\, k \in \mathbb{N}\ such\ that\ a_n = 0\ \forall\, n \geq k$

1.7.12. A positive real number is rational $\Leftrightarrow$ its decimal representation is periodic.

1.7.13. A non-terminating non – recurring decimal represents a positive irrational number.

1.8. Enumerable and Uncountable Set:

1.8.1. Definition (Enumerable Set): Let $S \subset \mathbb{R}$. S is said to be enumerable (or denumerable) if $\exists$ a bijective mapping $f : \mathbb{N} \to S$ i.e, $S\ and\ \mathbb{N}$ are equipotent sets.

1.8.2. Definition (Countable Set): A set which is either finite or enumerable is said to be a countable set.
An enumerable set is also called a countable infinite set.

1.8.3. The union of an enumerable number of enumerable sets is enumerable.

Example (1.18): $\mathbb{Q}$ is enumerable. Let $\mathbb{Q} = P \cup P' \cup \{0\}$ where P is the set of all positive rational numbers and P' is the set of all negative rational numbers $P\ and\ P'$ are equipotent. Since $f : P \to P'$ defined by $f(x) = x, x \in P$ is bijective.

Now, $P = \cup_{k-1}^{\infty} A_k$ where $A_k = \left\{\frac{1}{k}, \frac{2}{k}, \frac{3}{k}, \frac{4}{k} \ldots \ldots \right\}, k = 1,2,3, \ldots \ldots$, is enumerable set.

(ii). $\mathbb{N} \times \mathbb{N}$ is enumerable and $P(\mathbb{N})$ is uncountable. $\mathbb{N} \times \mathbb{N} = \cup_{k=2}^{\infty} A_k, A_k = \{(n,m): m + n = k\}$

(iii).The set $\mathbb{R}$ is not enumerable. Since any closed interval $[a,b]\ of\ \mathbb{R}$ is not enumerable. [Follows from theorem on nested intervals].

(Another proof: Let $I = [0,1]$ be countable. Then $I = \{x_1, x_2, x_3, \ldots \ldots \ldots\}$
Let $I_1 \subset I$ be a closed interval such that $x_1 \notin I_1$. Again consider $I_2 \subset I_1$ be a closed interval such that $x_2 \notin I_2$. Proceeding in this way, let $x_n \notin I_n$ where $I_n \subset I_{n-1} \subset I_{n-2} \subset \ldots \ldots \subset I_1$. By nested intervals theorem $\cap_{n=1}^{\infty} I_n$ is non-empty but for any $n \in \mathbb{N}, x_n \notin \cap_{n=1}^{\infty} I_n - contradiction \Rightarrow I = [0,1]$ is uncountable and hence $\mathbb{R}$ in uncountable.
$\Rightarrow \mathbb{R} \backslash \mathbb{Q}$ is uncountable as $\mathbb{Q}$ is countable and $\mathbb{R}$ is uncountable.)

(iv). Let A_k be countable. Then $\prod_{k=1}^{\infty} A_k$ is uncountable.

(v).The set $S = \mathbb{R} \backslash \mathbb{Q}$ of all irrational numbers in non – enumerable.

(vi). $\mathbb{Z} = \mathbb{Z}^+ \cup \{0\} \cup \mathbb{Z}^-\ is\ enumerable. (Since\ f: \mathbb{N} \to \mathbb{Z}$ defined by $f(1) = 0, f(2n) = n, f(2n-1) = -n)$.

1.8.4. Definition (Equipotent Sets and Cardinal Number): Let A, B be subsets of a universal set $P(x)$, the power set of a non-empty set X. A is said to be equipotent with $B\ if\ \exists$ bijective mapping $f: A \to B$. We write $A \sim B$. The relation of equipotence on the set $P(x)$ is equivalence. The set belonging to the same equipotence class are said to have the same potency or the same cardinal number.

The cardinal number assigned to the equipotence class of finite sets each with n elements is n. The cardinal number to the null set ϕ is 0. The cardinal number of an infinite set is said to be a transfinite cardinal number. The cardinal number of the set $\mathbb{N}$ is denoted by d and of $\mathbb{R}$ denoted by c.

1.8.5. Definition (Point of Condensation): Let S be a subset of $\mathbb{R}$. A point $x \in \mathbb{R}$ is said to be a pint of condensation of S if every neighbourhood of x contains uncountably many points of S.

Example (1.19.):

(i). Every limit point of S may not be point of condensation but, every point of condensation of$S$ is a limit point of S. For $S = \mathbb{Q}$. Then S_c set of all point of condensation of S is ϕ as S is countable.(By differentiable countable set do not have any point of condensation).

(ii) $S = \{x \in \mathbb{R} : 1 < x \leq 3\} \Rightarrow S_c = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$

1.8.6. Theorem:

(i). Every uncountable subset $S \subseteq \mathbb{R}$ has at least one point of condensation.

(ii). If no point of $S \subseteq \mathbb{R}$ is a condensation point of S then S is a countable set.

(iii). For every set $S \subseteq \mathbb{R}$, $S - S_c$ is countable set. Further more, $S = (S - S_c) \cup (S \cap S_c)$ it follows that if S is uncountable, then $S \cap S_c$ is countable $\Rightarrow$ if S be uncountable set, it contains uncountable many points of condensation of S.

(iv). For every $S \subset \mathbb{R}, S_c$ *is a closed set*.

1.8.7. Definition (F_σ, G_δ set): The union of an enumerable collection of closed sets in $\mathbb{R}$ is said to be an F_σ set and the intersection of collection of open sets in $\mathbb{R}$ is said to be a G_δ set.

Example (1.20):

(i). $\mathbb{Q}$ is an F_σ set, as $\mathbb{Q} = \cup_i\{x_i\}$ and $\{x_i\}$ is closed.

(ii). An open bounded interval (a, b) is an F_σ set. Since $(a, b) = \cup_{n=1}^{\infty}(a + \frac{1}{n}, b - \frac{1}{n}), x \in \mathbb{N}$.

(iii). A closed bounded interval (a, b) is a G_δ set. Since $(a, b) = \cap_{n=1}^{\infty}(a - \frac{1}{n}, b + \frac{1}{n}), x \in \mathbb{N}$.

1.8.8. Theorem:

(i). The union of countable collection of F_σ sets is an $F_\sigma$set.

(ii). The intersection of countable collection of G_δ sets is a G_δ set.

(iii). The complement of F_σ (or G_δ) set in $\mathbb{R}$ is a G_δ (respectively F_σ) set.

1.8.9. Definition (Borel Set): A set that can be obtained as the union and intersection of an enumerable collection of closed sets and open sets in $\mathbb{R}$ is said to be a Borel set.

Example (1.21.):

(i). F_σ and G_δ are Borel set.

(ii). The union of a countable collection of G_δ sets in a Borel set and it is denoted by $G_{\delta\sigma}$.

(iii). The intersection of a countable collection of F_σ sets is a Borel set and it is denoted by $F_{\sigma\delta}$.

1.9. Cover and Open Cover:

1.9.1. Definition (Cover and Open Cover): Let S be a subset of $\mathbb{R}$. A collection C of sets $\{A_\alpha = \alpha \in \Lambda\}$, Λ being the index set, is said to be a cover (or a covering) of S if $S \subset \cup_{\alpha\in\Lambda} A_\alpha$.

If g be a collection of open sets $\{G_\alpha : \alpha \in \Lambda\}$ such that $S \subset \cup_{\alpha\in\Lambda} G_\alpha, g$ is said to be an open cover of S.

Example (1.22):

$\left\{I_n = \left\{x \in \mathbb{R} : \frac{1}{2^n} < x < 2\right\}, n\right\}$ is an open cover of the set $S = \{x \in \mathbb{R} : 0 < x < 1\}$

1.9.2. Definition (Sub Cover): Let $S \subseteq \mathbb{R}$ *and* S a cover of S. If C' be a sub collection of C such that C' also covers S then C' is said to be a sub cover of$C$. If C' contains finite number of sets of C, then C' is called a finite sub cover of C.

Example (1.23):$g = \{I_n : n \in \mathbb{N}\}$ *where* $I_n = \{x \in \mathbb{R} : -n < x < n\}$. Then g is and open cover of $\mathbb{Q}$. Now $g' = \{I_{2n} : x \in \mathbb{N}\}$ is a sub cover of g.

1.9.3. Heine –Borel Theorem: Let S be a closed and bdd subset of $\mathbb{R}$.Then every open cover of$S$ has a finite sub cover.

1.9.4. Definition (Compact Set): Let S be a subset of $\mathbb{R}$. S is said to be a compact set if every open cover of S has a finite sub-cover.

(Converse of Heine Borel Theorem): A compact subset of $\mathbb{R}$ is closed and bdd in $\mathbb{R}$.

1.9.5. If K be a compact set in $\mathbb{R}$, every finite subset of K has a limit point in k and conversely.
$\Rightarrow$ $\mathbb{R}$ is not compact, since $\mathbb{Z}$ is an finite subset of $\mathbb{R}$ having no limit point in $\mathbb{R}$.

1.9.6. Corollary:

(i). Every finite subset of $\mathbb{R}$ is compact in $\mathbb{R}$.

(ii). Let K be a compact subset of$\mathbb{R}$ and $F \subset K$ be a closed subset in $\mathbb{R}$. Then F is compact in $\mathbb{R}$.

(iii). Let $K \neq \phi$ be a compact set in $\mathbb{R}$. Then K has a least element.

1.9.7. Lindelof's Theorem: If $S \subset \mathbb{R}$, every open cover of S has a countable sub-cover.

1.10. Sequence:

1.10.1. Definition (Real Sequence): A mapping $f : \mathbb{N} \to \mathbb{R}$ is said to be a sequence in $\mathbb{R}$ or a real sequence and it is denoted by $\{f(n)\} = \{f(1), f(2), \ldots\ldots..\} = \{f(n): n \in \mathbb{N}\}$

$$= \{u_n : n \in \mathbb{N}\}, f(n) = U_n \ \forall\, n \in \mathbb{N}.$$

Examples (1.24):

(i) $f: \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = n, n \in \mathbb{N}$. Then the sequence is $\{n\} = \{0,1,2,3, \ldots\ldots.\}$

(ii) $f: \mathbb{N} \to \mathbb{R}$defined by $f(n) = (1)^n$. Then the sequence is $\{(-1)^n\} = \{-1,1,-1,1, \ldots\ldots\ldots.\}$

1.10.2. Definition (Bounded Sequence): A real sequence $\{f(n)\}$ is said to *bounded* above if $\exists$ a real number $M \in \mathbb{R}\, such\ that\ f(n) \leq M \ \forall\, n \in \mathbb{N}$. M is called an upper *bound* of the sequence. Bounded below if $\exists\, m \in \mathbb{R}\, such\ that\ f(n) \geq m \forall\, n \in \mathbb{N}$, m is called a lower *bound* of the sequence. The sequence $\{f(n)\}$ is said to be *bounded* if $\exists\, m, M \in \mathbb{R}\, such\ that$
$m \leq f(n) \leq M \ \forall\, n \in \mathbb{N}$ and

$$\sup f(n) = \min\{M: M\ is\ an\ upper\ bounded\ of\ \{f(n)\}\}$$

$$inf\ f(n) = \max\{m: m\ is\ a\ lower\ bounded\ of\ \{f(n)\}\}$$

Example (1.25):

(i). The sequence $\left\{\frac{1}{n}\right\}$ is *bounded* and its *supremum* $= 1\ and\ infimum = 0$.

(ii). The sequence $\{n^2\}$ is *bounded* below and its *infimum* $= 1\ and\ supremum = \infty$.

1.10.3. Definition (Limit of a sequence): Let $\{f(n)\}$ be a real sequence. A real number $l \in \mathbb{R}$ is said to be a limit of the sequence $\{f(n)\}$ if corresponding to a pre-assigned positive $\varepsilon, \exists\, k \in \mathbb{N}\,(k - depending\ on\ \varepsilon\ i.e.\ k(\varepsilon))$ such that

$$|f(n) - l| < \varepsilon\ \forall\, n \geq k \quad i.e, l - \varepsilon < f(n) < l + \varepsilon\ \forall\, n \geq k$$

Note: A sequence can have at most one limit.

1.10.4. Definition (Convergent Sequence): A real sequence $\{f(n)\}$ is said to be a convergent sequence if it has a limit $l \in \mathbb{R}$. In this case the sequence is said to converge to l. We write $\lim\limits_{n\to\infty} f(n) = l$.

Example (1.26):

(i) The sequence $\left\{\frac{1}{n}\right\}$ converge to 0.
(ii). The sequence $\{2\}$ converges to 2.
(iii). The sequence $\{n\}$ diverges.
(iv). Convergent sequence is bounded. Converge is not true [example $\{(-1)^n\}$]
(v). An unbounded sequence is not convergent.

1.10.5. Let $\{u_n\}$ and $\{v_n\}$ be two convergent sequence that converges to $u\ and\ v$ respectively. Then –
(i). $\lim(u_n + v_n) = u + v$.
(ii). $If\ c \in \mathbb{R}, \lim(c\, u_n) = cu$.
(iii). $\lim(u_n\, v_n) = uv$.
(iv). $lim \frac{u_n}{v_n} = \frac{u}{v}, provided\ \{v_n\}\ is\ sequence\ of\ non-zero\ real\ numbers\ and\ v \neq 0$.

1.10.6. Let $\{u_n\}$ be a convergent sequence of real numbers converging to u. Then the sequence $\{|u_n|\}$ converges to $|u|\left(Hint: \big||u_n| + |u|\big| \leq |u_n - u|\right)$. Convergent is not true.

Example (1.27): Let $u_n = (-1)^n$. Then $\{|u_n|\}$ converges to 1 but $\{u_n\}$ divergent

$\lim|u_n| = |\lim u_n|$, provide RHS exists.

1.10.7. Sandwich Theorem: Let $\{u_n\}, \{v_n\}$ and $\{w_n\}$ be three sequence of real numbers and there is a $m \in \mathbb{N}$ such that $u_n < v_n < w_n\ \forall\, n \geq m$.

If $\lim u_n = \lim w_n = l, then\ \{w_n\}$ is convergent and $\lim v_n = l$.

Example (1.28):

(i). $\lim\limits_{n\to\infty} \frac{3n^2+2n+1}{n^2+1} = \lim\limits_{n\to\infty} \frac{3+\frac{2}{n}+\frac{1}{n^2}}{1+\frac{1}{n^2}} \left(form\ \frac{u_n}{v_n}\right) = \frac{3}{1} = 3.$

(ii). $\lim\limits_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) = \lim\limits_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = \lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} \left(\frac{1}{1+\sqrt{1+\frac{1}{n}}}\right) \left(form\ \frac{u_n}{v_n}\right) = 0$

(iii). **[Sandwich Theorem]**: $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\ldots\ldots+\frac{1}{\sqrt{n^2+n}}\right)=1$

Let $u_n=\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\ldots\ldots+\frac{1}{\sqrt{n^2+n}}$

Now, $\frac{1}{\sqrt{n^2+i}}\le\frac{1}{\sqrt{n^2+1}},\ i=2,3,\ldots\ldots,n\ \Rightarrow\ \frac{1}{\sqrt{n^2+n}}<\frac{1}{\sqrt{n^2+i}}, i=1,2,\ldots\ldots(n-1)$

$$\therefore \frac{n}{\sqrt{n^2+n}}<u_n<\frac{n}{\sqrt{n^2+1}}\ \Rightarrow \lim\frac{1}{\sqrt{1+\frac{1}{n}}}<\lim u_n<\lim\frac{1}{\sqrt{1+\frac{1}{n^2}}}$$

$$\Rightarrow 1<\lim u_n<1\ \Rightarrow\ \lim u_n=1$$

1.10.8. Definition (Null Sequence): A sequence $\{u_n\}$ is said to be a null sequence if $\lim u_n=0$.

1.10.9. If $\{u_n\}$ be null sequence then $\{|u_n|\}$ is a null sequence and conversely.

1.10.10. Definition (Divergent Sequence): A real sequence $\{f(n)\}$ is said to diverge to ∞ if corresponding to a pre-assigned $M>0\ (however\ large)\ \exists\ k\in\mathbb{N}\ such\ that\ f(n)>M\ \ \forall\ n\ge k$ and in this case $\lim f(n)=\infty$.

A real sequence $\{f(n)\}$ is said to diverge to $-\infty$ if corresponding to a pre – assigned

$M>0\ (however\ large)\exists\ m\in\mathbb{N}\ such\ that\ f(n)<-M\ \ \forall\ \ n\ge m$ and we write $\lim f(n)=-\infty$.

1.10.11. Definition (Oscillatory Sequence): A bounded sequence that is not convergent is said to be an oscillatory sequence of finite oscillation.
An unbounded sequence that is not properly divergent $(i.e, divergent\ to\ \infty\ and-\infty\ both)$ is said to be an oscillatory sequence of infinite oscillation.
Examples (1.29):
(i). The sequence $\{(-1)^n\}$ is bounded but not convergent. It is an oscillatory sequence of finite oscillation.
(ii). The sequence $\{n(-1)^n\}$ is unbounded and it is not properly divergent. It is an oscillatory sequence of infinite oscillation.

1.10.12. Some important limits:

(i). $\lim_{n\to\infty} r^n=0\ if\ |r|<1$

(ii). $\lim_{n\to\infty} a^{\frac{1}{n}}=1\ if\ a>0$

(iii). If $\lim_{n\to\infty} x_n=0\ and\ \ a>0\ then\ \lim_{n\to\infty} a^{x_n}=1$

a) $If\ \ \lim_{n\to\infty} x_n=l\ and\ \ a>0\ then\ \lim_{n\to\infty} a^{x_n}=a^l$

(iv). If $\lim_{n\to\infty} x_n = 0\ then\ \lim_{n\to\infty} \log(1+x_n) = 0$

a). If $\lim_{n\to\infty} x_n = c > 0\ and\ x_n > 0\ \forall\ x_n \in \mathbb{N}, Then\ \lim_{n\to\infty}(\log x_n - \log c) = \lim_{n\to\infty} \log\frac{x_n}{c} = \lim_{n\to\infty} \log\left(1+\frac{x_n - c}{c}\right) = 0 \Rightarrow \lim_{n\to\infty} \log x_n = \log c$

(v). If $u_n > 0$ and $\lim x_n = u > 0\ \ \forall n \in \mathbb{N}\ and\ \lim_{n\to\infty} v_n = v$

$then\ \lim_{n\to\infty}(u_n)^{v_n} = u^v$

(vi). $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

(vii). Behaviour of the sequence $\{r^n\}$ for different real values of r.

Case –I: $r > 1, \lim r^n = \infty$

Case – II: $r = 1, \lim r^n = 1$

Case – III: $|r| < 1, \lim r^n = 0$

Case – IV: $r = -1,\ r^n = \{-1,1,-1,\dots\dots\}: finite\ oscillatory.$

Case – V: $r < -1,\ r^n = \{(-1)^n s^n\},\ s > 1 : infinite\ oscillatory.$

1.11. Convergence Test of Sequence:

1.11.1. Ratio Test: Let $\{u_n\}$ be a sequence of positive real numbers such that $\lim_{n\to\infty}\frac{u_n+1}{u_n} = l.$

(i). If $0 \le l < 1\ then\ \lim_{n\to\infty} u_n = 0,$

(ii). If $l > 1\ then\ \lim_{n\to\infty} u_n = \infty$

No conclusion for $l = 1$ For example –

(a) If $u_n = \frac{n+1}{n}, Then\ \lim_{n\to\infty}\frac{u_n+1}{u_n} = \lim_{n\to\infty}\frac{n+2}{n+1}\times\frac{n+1}{n} = 1\ and\ \lim_{n\to\infty} u_n = 1$

(b) $u_n = \frac{1}{n}, Then\ \lim_{n\to\infty}\frac{u_n+1}{u_n} = \lim_{n\to\infty}\frac{n}{n+1} = 1\ but\ \lim_{n\to\infty} u_n = 0$

1.11.2. Root Test: Let $\{u_n\}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} {u_n}^{\frac{1}{n}} = l$

Then –

(i). if $0 \le l < 1, then\ \lim u_n = 0$

(ii). if $l > 1, then\ \lim u_n = \infty$

No conclusion for $l = 1$.

Example (1.30):

(a) If $u_n = \frac{n+1}{n}$, $then \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{\frac{1}{n}} = 1\ and \lim_{n\to\infty} u_n = 1$

(b) If $u_n = \frac{n+1}{2n}$, $then \lim_{n\to\infty}\left(\frac{n+1}{2n}\right)^{\frac{1}{n}} = 1\ and \lim_{n\to\infty} u_n = \frac{1}{2}$

1.11.3. Definition (Monotone Sequence): A real sequence $\{f(n)\}$ is said to be a monotone increasing (or decreasing) sequence if $f(n+1) \geq f(n)\ \forall\ n \in \mathbb{N}(respectively\ f(n+1) \leq f(n)\ \forall\ n \in \mathbb{N})$.

Example (1.31):

(i). $\{2^n\}$ is monotone increasing.

(ii) $\left\{\frac{1}{n}\right\}$ is monotone decreasing.

(iii). $\{(-1)^n\}$ is either monotone increasing nor monotone decreasing.

1.11.4: (i) A monotone increasing (or decreasing) sequence, if bounded above (or below) is convergent and it converges to the least upper bounded (respectively greatest lower bounded).

(ii) A monotone increasing (or decreasing) sequence that is unbounded above (below) diverges to $\infty (respectively - \infty)$.

1.11.5. Some Important Sequence:

(i). $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is monotone increasing and bounded sequence and $2 < \left(\frac{1}{n}\right)^n < 3\ \forall\ n \geq 2$ and the limit is $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\ (Hints\ AM \geq GM, 1+\frac{1}{n}, \ldots\ldots, 1+\frac{1}{n}, 1\ (n+1\ terms))$.

(ii). The sequence $\{x_n\}$ where $x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\ldots + \frac{1}{n!}$ is monotone increasing sequence, bounded above and $\lim_{n\to\infty} x_n = e$.

$$Since\ \left(1+\frac{1}{n}\right)^n = 1 + \frac{1}{1!} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \ldots\ldots + \frac{n(n-1)\ldots\ldots 2\cdot 1}{n!}\cdot\frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \ldots\ldots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\frac{2}{n}\cdot\frac{1}{n} < 1 + 1 + \frac{1}{2!} + \ldots\ldots + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{2} + \ldots\ldots + \frac{1}{2^{n-1}} = 1 + 2\left[1 - \left(\frac{1}{2}\right)^n\right] < 3,\ \forall\ n > 2$$

(iii). The sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is a monotone decreasing sequence with limit e.

Example (1.32)

(iii). The sequence $\{u_n\}$ defined by $u_n = \sqrt{2u_n}, n \geq 1\ and\ u_1 = \sqrt{2}$ is mi and bounded above it is convergent. Let e be the limit $\Rightarrow e^2 = 2e \Rightarrow e = 2$.

(iv). The sequence $\{u_n\}$ defined by $u_n + 1 = \sqrt{7 + u_n}\,, n \geq 1$ and $u_1 = \sqrt{7}$ is monotone increasing and bounded above. Hence it is convergent. Let e be its limit $\Rightarrow e^2 = 7 + e \Rightarrow e = \frac{1+\sqrt{29}}{2}$ as $u_n > 0$.

(v). Let $u_n = 1 + \frac{1}{2} + \ldots\ldots. + \frac{1}{n} - \log n\,, v_n = 1 + \frac{1}{2} + \ldots\ldots. + \frac{1}{n-1} - \log n\,,\ n \geq 2$. Then

$u_{n+1} - u_n = \frac{1}{n+1} - \log\,(n+1) + \log n = \frac{1}{n+1} - \log\left(1 + \frac{1}{n}\right)$

Since $\left(1 + \frac{1}{n}\right)^{n+1}$ is monotone decreasing and e its limit $\left(1 + \frac{1}{n}\right)^{n+1} > e$

$\Rightarrow n + 1 \log\left(1 + \frac{1}{n}\right) < 0 \Rightarrow \frac{1}{n+1} - \log\left(1 + \frac{1}{n}\right) < 0 \Rightarrow \{u_n\}$is monotone decreasing.

Now, $v_{n+1} - v_n = \frac{1}{n} - \log\,(n+1) + \log n = \frac{1}{n} - \log\left(1 + \frac{1}{n}\right)$ since $\left(1 + \frac{1}{n}\right)^n$is monotone increasing and its limit is $e \Rightarrow \left(1 + \frac{1}{n}\right)^n < e \Rightarrow n \log\left(1 + \frac{1}{n}\right) < 1 \Rightarrow \frac{1}{n} - \log\left(1 + \frac{1}{n}\right) > 0$

$\Rightarrow \{v_n\}$is monotone increasing. Let e be the limit of $\{u_n\}$ since $u_n - v_n = \frac{1}{n}$ and

$\lim(u_n - v_n) = 0 \Rightarrow \lim u_n = e \Rightarrow \{u_n\}$ and $\{v_n\}$ convergence to the same limit.

Note: $u_n = 1 + \frac{1}{2} + \ldots\ldots. + \frac{1}{n} - \log n$ convergent and let $\gamma_n = \lim u_n . \gamma$ is called Euler's constant and $0.3 < \gamma < 1$. Let$\gamma_n = 1 + \frac{1}{2} + \ldots\ldots. + \frac{1}{n} - \log n \Rightarrow 1 + \frac{1}{2} + \ldots\ldots. + \frac{1}{n}$

$= \gamma_n + \log n$. If $S_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots. + \frac{1}{2n}$, Then $-$

$\lim_{n\to\alpha} S_n = \lim_{n\to\alpha}\left[\left(1 + \frac{1}{2} + \ldots\ldots. + \frac{1}{2n}\right) - \left(1 + \frac{1}{2} + \ldots\ldots. + \frac{1}{n}\right)\right]$

$= \lim_{n\to\alpha}[(\gamma_{2n} + \log 2n) - (\gamma_{2n} + \log n)] = \lim_{n\to\alpha}[\gamma_{2n} - \gamma_n - \log 2] = \log 2$

(vi). Two sequence $\{x_n\}$ and $\{y_n\}$ are defined by $-$

$x_{n+1} = \frac{1}{2}(x_n + y_n),\ \ y_{n+1} = \sqrt{x_n y_n}\,, x \geq 1$ and $x_1 > 0, y_1 > 0$

both have same limit. Since$y_1 < y_2 < y_3 < \ldots\ldots\ldots\ldots < x_4 < x_3 < x_2 < x_1$.

1.12. Sub Sequence:

Definition (Sub Sequence): Let $\{u_n\}$ be a real sequence and $\{x_k\}$ be a strictly increasing sequence of natural numbers i.e., $x_1 < x_2 < x_3 \ldots.$ .. Then the sequence $\{u_{nk}\}$ is called a subsequence of $\{u_n\}$.

Example (1.33): Let $u_n = \frac{1}{n}$, then $\{u_{2n}\} = \left\{\frac{1}{2}, \frac{1}{4}, \ldots\ldots\right\}$ is a sub sequence of $\{u_n\}$.

1.12.1. If a sequence $\{u_n\}$ converges to l then every subsequence of $\{u_n\}$ also converse to l.

But the converges in not true.

Example (1.34): Let $u_n = (-1)^n$. Then $\{v_n\} = 1,1,\ldots..,1,\ldots\}$ and $w_n = \{-1,-1,\ldots.\}$ are both two subsequence of $\{u_n\}$ are both convergent. $v_n \to 1$ and $w_n \to -1$ as $n \to \infty$, but $\{u_n\}$ is not convergent.

1.12.2. If the sub sequences $\{u_n\}$ and $\{u_{2n-1}\}$ of a sequence $\{u_n\}$ converges to the same limit l then the sequence $\{u_n\}$ is convergent and $\lim_{n\to\infty} u_n = l$. But if any two sub sequences converges to the same limit l, then the sequence may not be convergent.

Example (1.35.): Let $u_n = \sin\frac{n\pi}{4}$. Then the sub sequences $\{u_{4n} - 3\}$ is $\{\sin\frac{\pi}{4}, \sin\frac{9\pi}{4}, \sin\frac{17\pi}{4},\ldots..\}$ converges to $\frac{1}{\sqrt{2}}$ and the sub sequence $\{u_{8n} - 5\}$ is $\{\sin\frac{3\pi}{4}, \sin\frac{11\pi}{4},\ldots..\}$ converges $\frac{1}{\sqrt{2}}$ .But $\{u_n\}$ is not convergent.

Note: If $k \in \mathbb{N}$ and k sub sequences $\{u_{kn}\}$, $\{u_{k\,n-1}\},\ldots\ldots.\{u_{k\,n-k+1}\}$ converge to the same limit l then $\{u_n\}$ is convergent and $lim\, u_n = l$.

Example (1.36.): Let $\{u_n\}$ be a sequence defined by $0 < u_1 < u_2$ and $u_{n+2} = \frac{1}{2}(u_n + u_{n+1})$

$\Rightarrow$ $u_1 < u_3 < u_2$ and $u_3 < u_4 < u_2$, $u_3 < u_5 < u_4$, $u_5 < u_6 < u_4,\ldots\ldots\ldots\ldots\ldots\ldots$

$\Rightarrow$ $u_1 < u_3 < u_5 <\ldots\ldots.< u_6 < u_4 < u_2 \Rightarrow$ $\{u_{2n-1}\}$ is monotone increasing and bounded above and hence convergent and $\{u_{2n}\}$ is monotone decreasing and bounded below and hence convergent.

Let $l = lim\, u_{2n} + lim\, u_{2n-1} \Rightarrow 2l = l + m \Rightarrow l = m$. Since both $\{u_{2n}\}$ and $\{u_{2n-1}\}$ converges to the same limit l, $\{u_n\}$ converges.

1.12.3. Every sub sequence of a monotone increasing (or monotone decreasing) sequence of real numbers is monotone increasing (respectively monotone decreasing).

1.12.4. A monotone sequence of real numbers having a convergent sub sequence with limit l, in convergent with limit l.

Example (1.37.): The sequence $\{u_n\}$ where $u_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots. + \frac{1}{n}$, is divergent. Science $u_{n+1} - u_n = \frac{1}{n+1} \Rightarrow \{u_n\}$ is monotone increasing. Let $\{u_{2n}\}$ be a sub sequence of $\{u_n\}$. Then $-$

$$u_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots. + \frac{1}{2^n}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \ldots\ldots + \frac{1}{8}\right) + \ldots\ldots\ldots + \left(\frac{1}{2^{n-1}+1} + \ldots\ldots. + \frac{1}{2^n}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \ldots\ldots + \frac{1}{8}\right) + \ldots\ldots\ldots + \left(\frac{1}{2^n} + \ldots\ldots. + \frac{1}{2^n}\right)$$

$$= 1 + \frac{1}{2} + 2\cdot\frac{1}{2^2} + 2^2\cdot\frac{1}{2^3} + \ldots\ldots\ldots + 2^{n-1}\cdot\frac{1}{2^n} = 1 + \frac{n}{2} \to \infty \text{ as } n \to \infty$$

$\Rightarrow \{u_n\}$ is divergent.

1.12.5. Every sequence of real numbers has a monotone sub sequence.

1.12.6. Sub sequential limit: Let $\{u_n\}$ be a real sequence. A real number l is said to be a sub sequential limit of the sequence $\{u_n\}$ if $\exists$ a sub sequence $\{u_n\}$ that converges to l.

1.12.7. A real number l is said to be a sub sequential limit of the sequence $\{u_n\} \Leftrightarrow$ every neighbourhood of$l$ contains infinitely many elements of $\{u_n\}$.

Note: The limit of a sequence, if it exists, is also a sub sequential limit of a sequence.

1.12.8. Bolzano–Weierstrass theorem: Every bounded sequence of real numbers has a convergent sub sequence.

Example (1.38.): The sequence $\{u_n\}$ where $u_n = \sin\frac{n\pi}{2}, n \geq 1$ is bounded, since $|u_n| \leq 1\ \forall\ n \geq 1$.

(i) $\{u_{4n-3}\} = \{u_1, u_5, \ldots\ldots\ldots\} = \{1,1, \ldots\ldots\ldots.\}$ converges to 1.

(ii). $\{u_{2n}\} = \{u_2, u_4 \ldots\ldots\ldots\} = \{0,0, \ldots \ldots..\}$ converges to 0.

1.12.9. Characterisation of a compact set: Let $\phi = K \subseteq \mathbb{R}$. Then K is compact $\Leftrightarrow$ every sequence in K has a convergent sub sequence in K.

1.12.10. Limit Superior and Lint Inferior: Let $\{u_n\}$ be a bounded sequence in $\mathbb{R}$. The greatest sub sequential limit of $\{u_n\}$ is said to be the limit superior of $\{u_n\}$ and is denoted by $\overline{lim}\, u_n$ or $lim\ sup\ u_n$.

The least sub sequential limit of $\{u_n\}$ is said to be the limit inferior of $\{u_n\}$ and is denoted by $\underline{lim}\, u_n$ or $\lim\inf u_n$.

If $\{u_n\}$ is unbounded above the m we define $\overline{lim}\, u_n = \infty$.

If $\{u_n\}$ is unbounded below then we define $\underline{lim}\, u_n = \infty$.

If $\{u_n\}$ is unbounded above (or below) but bounded below (or bounded above) then $\underline{lim}\, u_n$ is defined to be the least sub sequential limit (respectively greatest sub sequential limit).

Example (1.39):

(i) Let $u_n = (-1)^n\left(1 + \frac{1}{n}\right), n \geq 1$. Then $\{u_n\}$ is bounded and $\overline{lim}\, u_n = 1$ and $\underline{lim}\, u_n = -1$.

(ii). Let $u_n = \frac{1}{n}, n \geq 1$ then $\overline{lim}\, u_n = \underline{lim}\, u_n = 0$

(iii). $u_n = (-1)n^2, n \geq 1$ then $\{u_n\}$ is unbounded above and unbounded below. Hence $\overline{lim}\, u_n = \infty\ and\ \underline{lim}\, u_n = -\infty$

1.12.11. (i). A bounded sequence $\{u_n\}$ is convergent $\Leftrightarrow \overline{lim}\, u_n = \underline{lim}\, u_n$ i.e., it has only one limit.

(ii). Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences. Then –

(a). $\overline{lim}\, u_n + \overline{lim}\, v_n \geq \overline{lim}(u_n + v_n)$

(b). $\underline{lim}\, u_n + \underline{lim}\, v_n \leq \underline{lim}(u_n + v_n)$

1.12.12. Cauchy's General Principle of Convergence: A necessary and sufficient condition for the convergence of a sequence $\{u_n\}$ is that for a pre – assigned $\varepsilon > 0, \exists\ a\ m \in \mathbb{N}$, such that $|u_{n+p} - u_n| < \varepsilon\ \forall\ n \geq m\ and\ p = 1, 2, 3 \ldots \ldots \ldots$ [this is known as Cauchy condition of convergent].

Example (1.40): Consider the sequence $\{u_n\}$ where $u_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n}$. Let $p \in \mathbb{N}$.

Then $|u_{n+p} - u_n| = \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots + \frac{1}{n+p}$ $\quad Choose\ p = n = m$

Then $|u_{2m} - u_m| = \frac{1}{m+1} + \frac{1}{m+2} + \ldots\ldots + \frac{1}{2m} > \frac{1}{2m} + \frac{1}{2m} + \ldots\ldots + \frac{1}{2m} = \frac{1}{2} = \varepsilon, say.$

i.e., if we choose $\varepsilon = \frac{1}{2}$ $then\ k \in \mathbb{N}$ can be found such that $|u_{n+p} - u_n| < 2\ \forall\ n \geq k \Leftrightarrow$ $\{u_n\}$ is not convergent.

1.12.13. Definition (Cauchy Sequence): A sequence $\{u_n\}$ is said to Cauchy sequence if for pre – assigned $\varepsilon > 0, \exists\ k \in \mathbb{N}$ such that $|u_m - u_n| < \varepsilon\ \forall\ m, n \geq k$.

1.12.14. A real sequence is a convergent $\Leftrightarrow$ it is Cauchy sequence.

Example (1.41): $\left\{\frac{1}{n}\right\}$ be a Cauchy sequence. Let $\varepsilon > 0$ be arbitrary. Then $\exists\ k \in \mathbb{N}$ such that $\frac{2}{k} < \varepsilon$. Then $\left|\frac{1}{m} - \frac{1}{n}\right| < \frac{1}{m} + \frac{1}{n} < \frac{2}{k} < \varepsilon\ \forall\ m,\ n > k \Rightarrow \left\{\frac{1}{n}\right\}$ is Cauchy and hence convergent.

1.12.15. Cauchy's Theorem on limits:

(i). If $lim\, u_n = l$, then $lim \frac{u_1 + u_2 + \cdots \ldots\ldots + u_n}{n} = l.$

(ii). If $lim\, u_n = l, u_n > 0\ \forall\ n\ and\ l \neq 0, then\ lim(u_1 u_2 \ldots . u_n)^{\frac{1}{n}} = l$

1.12.16. Let $u_n > 0$ for all $n \in \mathbb{N}$ and $lim \frac{u_n+1}{u_n} = l\ (limit\ or\ infinite)$. Then $\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = l$ But converge is not true.

Example (1.42):$u_n = \frac{3+(-1)^n}{2}$. Then$lim(u_n)^{\frac{1}{n}} = 1. But\ \lim_{n\to\infty} \frac{u_n+1}{u_n}$does not exist.

1.12.17. If $u_n > 0$ for all $n \in \mathbb{N}$, then$-\ \underline{lim} \frac{u_n+1}{u_n} \leq \underline{lim}(u_n)^{\frac{1}{n}} \leq \overline{lim}(u_n)^{\frac{1}{n}} \leq \overline{lim} \frac{u_n+1}{u_n}$

Example (1.43):

(i). $\lim n^{\frac{1}{n}} = 1. Let\ u_n = n, \forall\ n \in \mathbb{N}, then\ \lim \frac{u_n+1}{u_n} = \lim_{n\to\infty} \frac{n+1}{n} = 1 \Rightarrow \lim(u_n)^{\frac{1}{n}} = 1$

(ii). Proved that $\frac{(n!)^{\frac{1}{n}}}{n} = \frac{1}{e}. Let\ u_n = \frac{n!}{n^k}. Then\ u_n > 0\ \forall\ n \in \mathbb{N}\ and\ \lim \frac{u_n+1}{u_n} = \frac{1}{e} > 0 \Rightarrow$ $\lim(u_n)^{\frac{1}{n}} = \frac{1}{e}.$

1.12.18. $\overline{\lim}\, a_n = \lim\limits_{n\to\infty} \inf a_n = \sup\limits_{n} \ \inf\{a_n, a_{n+1}, a_{n+2}, \ldots\ldots\}$

$\underline{lim}\, a_n = \lim\limits_{n\to\infty} \sup a_n = \inf\limits_{n} \sup\ \{a_n, a_{n+1}, a_{n+2}, \ldots\ldots\}$

1.12.19. Some important limits (Continued): For $\alpha > 0, \beta > 0$ *and* x *be real, we have* $-$

(i). $\lim\limits_{x\to\infty} \frac{(\ln x)^\alpha}{x^\beta} = 0$ ($Hint$: $f(x) = \ln x,\ g(x) = x^{\frac{\beta}{\alpha}}$ and apply L' Hospital rule and then $\lim\limits_{x\to\infty} \left(\frac{f(x)}{g(x)}\right)^\alpha = \left[\frac{f(x)}{g(x)}\right]^\alpha$)

(ii). $\lim\limits_{x\to\infty} \frac{x^\alpha}{e^{\beta x}} = 0$

(iii). $\lim\limits_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = \lim\limits_{n\to\infty} \left(\left(1 + \frac{1}{\left(\frac{n}{a}\right)}\right)^{\frac{n}{a}}\right)^a = \left[\lim\, (1 + \frac{1}{\left(\frac{n}{a}\right)})^{\frac{n}{a}}\right]^a = e^a,\ a > 0.$

(iv). $\lim\limits_{n\to\infty} \left\{\left(1 + \frac{1}{n^2}\right)\left(1 + \frac{2}{n^2}\right)\left(1 + \frac{3}{n^2}\right)\right\}^{n^2} = e \cdot e^2 \cdot e^3 = e^6.$

(v). $\lim\limits_{n\to\infty} (1 + \frac{1}{3n+1})^n = \lim\limits_{n\to\infty} \left[(1 + \frac{1}{3n+1})^{3n+1}\right]^{\frac{1}{3}} \left(1 + \frac{1}{3n+1}\right)^{-\frac{1}{3}} = e^{\frac{1}{3}} \cdot 1 = e^{\frac{1}{3}}$

1.13. Definition (Series): Let $\{u_n\}$ be a sequence. Then the sequence $\{S_n\}$ defined by $S_1 = u_1, S_2 = u_1 + u_2, \ldots\ldots, S_n = u_1 + u_2 + \ldots\ldots + u_n, \ldots\ldots\ldots$ is represented by the symbol $u_1 + u_2 + \ldots\ldots$, which is said to be an infinite series (or a series) generated by the sequence $\{u_n\}$. The series is denoted by the symbol $\sum_{n=1}^{\infty} u_n$ *or by* $\sum u_n$. The element S_k of the sequence $\{S_n\}$ are called the partial sums of the series $\sum u_n$.

The infinite series $\sum u_n$ is said to be convergent or divergent according as the sequence $\{S_n\}$ is convergent or divergent.

In case of convergence, if $\lim S_n = S$ then S is said to be the sum of the series $\sum u_n$.

If, however, $\lim S_n = \infty(-\infty)$the series $\sum u_n$ is said to diverge to $\infty(or, -\infty)$.

Example (1.44):

(i). Consider the series $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots\ldots\ldots.$

Then $u_n = \frac{1}{n(n+1)}$ Let $S_n = u_1 + u_2 + \ldots\ldots + u_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots\ldots + \frac{1}{n(n+1)}$

$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots\ldots\ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$

and $\lim S_n = 1 \Rightarrow \sum u_n$ is convergent and $sum = 1$.

(ii). $\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 \ldots\ldots\ldots\ldots$

Let $S_n = -1 + 1 - 1 + \ldots\ldots\ldots\ldots + (-1)^n$ $\quad\therefore \lim\limits_{n\to\infty} S_n = \begin{cases} 0, if\ n\ is\ even \\ 1,\ if\ n\ is\ odd \end{cases}$

$\therefore\ \sum(-1)^n$ is divergent.

(iii). **Geometric Series:** Consider $1 + a + a^2 + \ldots\ldots \quad where\ |a| < 1.$

Let $S_n = 1 + a + a^2 + \ldots\ldots + a^{n-1} \quad Then -$

$S_n = \frac{1-a^n}{1-a} = \frac{1}{1-a} - \frac{a^n}{1-a} \Rightarrow \lim_{n\to\infty} S_n = \frac{1}{1-a} (\because |a| < 1)$

(iv). **Harmonic Series:** $1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots$

$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots + \frac{1}{n}$

$S_2 = 1 + \frac{1}{2}$

$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{2}{2}$

$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots + \frac{1}{8} = S_n = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$

$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \ldots\ldots\ldots + \frac{1}{8}\right) = 1 + \frac{1}{2} + \frac{2}{4} + \frac{2^2}{2^3} = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2}$

$S_{2^n} > 1 + \frac{n}{2}$ and $\lim_{n\to\infty} S_{2^n} = \infty \Rightarrow \sum u_n$ is divergent.

1.13.1. Let $m \in \mathbb{N}$. Then two series $u_1 + u_2 + \ldots\ldots$ and $u_{m+1} + u_{m+2} + \ldots\ldots$ converge or diverge together.

Note: We can remove from the beginning a finite number of term from a given series or add to be beginning a finite number of terms to a given series without changing its behaviour regarding convergence or divergence.

1.13.2. Cauchy's Principle of Convergence: A necessary and sufficient condition for the convergence of a series $\sum u_n$ is that corresponding to a pre-assigned $\varepsilon > 0, \exists\ a\ \ m \in \mathbb{N}$ such that $|u_{n+1} + u_{n+2} + \ldots\ldots + u_{n+p}| < \varepsilon\ \forall\ n \geq m, p \in \mathbb{N}$.

1.13.3. A necessary condition for the convergence of a series $\sum u_n$ is $\lim u_n = 0$. The converse is not true.

Example (1.45):

(i). The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}, \lim_{n\to\infty} \frac{1}{n} = 0\ but\ \sum \frac{1}{x}$ is not convergent.

(ii). $\sum_{n=1}^{\infty} \frac{n}{n+1}$ is not convergent, since $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

1.13.4. (i). A series of positive terms $\sum u_n$ is convergent if only if the sequence $\{S_n\}$ of partial sums is bounded above.

(ii). Let $\sum u_n$ be a series of positive real numbers and $\sum v_n$ is obtained from $\sum u_n$ by grouping its terms. Then –

(a) if $\sum u_n$ converges to the sum s, so does $\sum v_n$;

(b) if $\sum v_n$ converges to the sum t, so does $\sum u_n$

Note: The theorem does not hold if $\sum u_n$ be the series of arbitrary terms.

Example (1.46): Consider $\sum(-1)^{n+1}$. Then

$$(1-1)+(1-1)+\ldots\ldots\ldots\ldots=0 \; and \; 1-(1-1)-(1-1)-\ldots\ldots.=1$$

(iii). Let $\sum u_n$ be a convergent series of positive real numbers. Then any re-arrangement of $\sum u_n$ is convergent and the sum remain unaltered.

1.14. Convergence Test of Series:

1.14.1. Let $\sum u_n$ and $\sum v_n$ be two series of positive real numbers and there is a number $m \in \mathbb{N}$ such that $u_n \leq k v_n \; \forall \, n \geq m$, k being a fixed positive number. Then $\sum u_n$ converges if $\sum v_n$ converges.

1.14.2. (Limit Form): Let $\sum u_n$ and $\sum v_n$ be two series of positive real number and $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = l$ *where* l is a non-zero finite number. Then the two series $\sum u_n$ and $\sum v_n$ converge or diverge together.

1.14.3. The series $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots.$ converges for $p>1$ and diverges for $p \leq 1$.

Case – I: $p>1 \; \frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots.<1+\left(\frac{1}{2^p}+\frac{1}{2^p}\right)+\left(\frac{1}{4^p}+\ldots\ldots\ldots+\frac{1}{4^p}\right)+\ldots\ldots..$

$=1+\frac{2}{2^p}+\frac{2^2}{2^{2p}}+\ldots\ldots=\; 1+\frac{2}{2^{p-1}}+\frac{2^2}{2^{2(p-1)}}+\ldots.$ convergent as $\frac{1}{2^{p-1}}<1$

Case – II: $p=1$, $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ \ldots\ldots..$ divergent.

Case – III: $0<p<1,\ 1+\frac{1}{2^p}+\frac{1}{3^p}+\ldots\ldots>1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots..$ divergent.

Case – IV: $p \leq 0$, $\lim\limits_{n\to\infty} \frac{1}{n^p} \neq 0$, divergent.

Example (1.47):

(i). Consider $\frac{1+2}{2^3}+\frac{1+2+3}{3^3}+\frac{1+2+3+4}{4^3}+\ldots\ldots..$

Here $u_n=\frac{1+2+\ldots\ldots+n+1}{(n+1)^3}=\frac{(n+1)(n+2)}{2(n+1)^3}=\frac{n+2}{2(n+1)^2}$.

Let $v_n=\frac{1}{n}$. Then $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=\lim\limits_{n\to\infty}\frac{n(n+2)}{2(n+1)^2}=\frac{1}{2}$

Since $\sum v_n$ is divergent, $\sum u_n$ is also divergent.

(ii). Consider the series $\frac{1}{1\cdot 2^2}+\frac{1}{2\cdot 3^2}+\frac{1}{3\cdot 4^2}+\ldots\ldots\ldots..$

Here $u_n=\frac{1}{n(n+1)^2}$ Let $v_n=\frac{1}{n^3}$. Then $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=1$

But $\sum v_n$ converges $\Rightarrow \sum u_n$ is convergent.

(iii). Consider $\sum u_n$ where $u_n=\sqrt{n^4+1}-\sqrt{n^4-1}=\frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$

Let $v_n = \frac{1}{n^2}$ Then $\lim_{n\to\infty} \frac{u_n}{v_n} = 1$, Since $\sum v_n$ is convergent, $\sum u_n$ is also convergent.

1.14.4. Comparison test (2nd type): Let $\sum u_n$ and $\sum v_n$ be two series of positive real numbers and there is a $m \in \mathbb{N}$ such that $\frac{u_n+1}{u_n} \le \frac{v_n+1}{v_n}$ $\forall\, n \ge m$. Then –

(i). $\sum u_n$ is convergent, if $\sum v_n$ is convergent.

(ii). $\sum u_n$ is divergent, if $\sum v_n$ is divergent.

1.14.5. D' Alembert's Ratio Test: Let $\sum u_n$ be a series of positive real numbers and let $\lim_{n\to\infty} \frac{u_n+1}{u_n} = l$. Then –

(i). $\sum u_n$ is convergent if $l < 1$.

(ii). $\sum u_n$ is divergent if $l > 1$.

(iii). No conclusion for $l = 1$.

Example (1.48): Let $u_n = \frac{1}{n}$ Then $\lim_{n\to\infty} \frac{u_n+1}{u_n} = 1$.But $\sum \left\{\frac{1}{n}\right\}$ is divergent. Let $u_n = \frac{1}{n^2}$ then $\lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$ But $\sum \frac{1}{n^2}$ is convergent.

1.14.6. Cauchy's Root Test: Let $\sum u_n$ be a series of positive real numbers and let $\lim_{n\to\infty} u_n^{\frac{1}{n}} = l$. Then–

(i). $\sum u_n$ is convergent, if $l < 1$.

(ii). $\sum u_n$ is divergent, if $l > 1$

(iii). No conclusion for $l = 1$.

Example (1.49): Let $u_n = \frac{1}{n}$. Then $\lim_{n\to\infty} u^{\frac{1}{n}}{}_n = 1$. But $\sum u_n$ is divergent.

Let $v_n = \frac{1}{n^2}$ Then $\lim_{n\to\infty} \left(\frac{1}{n^2}\right)^{\frac{1}{n}} = 1$.But $\sum v_n$ is convergent.

Example (1.50):

(i) $1 + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!} + \ldots\ldots$.. Let $\sum u_n$ be the given series. Then $u_n = \frac{2n-1}{2!}$ and

$\lim_{n\to\infty} \frac{u_n+1}{u_n} = \lim \frac{2(n+1)-1}{(n+1)!} \times \frac{n!}{2n-1} = \lim \frac{2n+1}{(n+1)(2n-1)} = 0 < 1$. Hence $\sum u_n$ is convergent.

(ii). $1 + \frac{1}{2^3} + \frac{1}{2^2} + \frac{1}{2^5} + \frac{1}{2^4} + \ldots\ldots\ldots$.. Here $u_n = \{2^n + (-1)^n\}^{-1}$ and

$\lim u_n^{\frac{1}{n}} = \lim \left\{2^{1+\frac{(-1)^n}{n}}\right\}^{-1} = \frac{1}{2} < 1$. Hence convergent. But $\lim_{n\to\infty} \frac{u_n+1}{u_n} =$

$\begin{cases} \frac{1}{8} & if\ n\ be\ odd \Rightarrow Ratio\ test\ fails \\ 2 & if\ n\ be\ even\ \Rightarrow Root\ test\ is\ more\ powerful\ than\ ratio\ test. \end{cases}$

1.14.7. General Form of Ratio test: Let $\sum u_n$ be a series of positive real numbers and let $\overline{\lim}\frac{u_n+1}{u_n} = R$ and $\underline{\lim}\frac{u_n+1}{u_n} = r$ Then –

(i). $\sum u_n$ is convergent if $R < 1$,

(ii). $\sum u_n$ is divergent if $r > 1$.

1.14.8. General Form of Root Test: Let $\sum u_n$ be a series of positive real numbers and let $\overline{\lim}\, u_n^{\frac{1}{n}} = r$. Then–

(i). $\sum u_n$ is convergent if $r < 1$.

(ii). $\sum u_n$ is divergent if $r > 1$.

Example (1.51):

(i). $\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \ldots\ldots\ldots.$ Let $\sum u_n$ be the given series. Then –

$u_{2n} = \frac{1}{3^n},\ u_{2n+1} = \frac{1}{2^{n+1}},\ u_{2n-1} = \frac{1}{2^n}$

$$\lim_{n\to\infty}\frac{u_{2n}}{u_{2n-1}} = \lim_{n\to\infty}\left(\frac{2}{3}\right)^n = 0$$

$$\lim_{n\to\infty}\frac{u_{2n+1}}{u_{2n}} = \lim_{n\to\infty}\frac{1}{3}\left(\frac{2}{3}\right)^{n+1} = \infty$$

$\therefore\ \overline{\lim}\frac{u_{n+1}}{u_n} = \infty$ *and* $\underline{\lim}\frac{u_{n+1}}{u_n} = 0$

Ratio test gives no conclusion.

But, $\lim_{n\to\infty}(u_{2n})^{\frac{1}{2n}} = \lim_{n\to\infty}\left(\frac{1}{3n}\right)^{\frac{1}{2n}} = \frac{1}{\sqrt{3}}$ and $\lim_{n\to\infty}(u_{2n+1})^{\frac{1}{2n+1}} = \lim_{n\to\infty}\left(\frac{1}{2^{n+1}}\right)^{\frac{1}{2n+1}} =$ $\lim_{n\to\infty}\frac{1}{2^{\frac{n+1}{2n+1}}} = \frac{1}{\sqrt{2}}$

$\therefore\ \overline{\lim}(u_n)^{\frac{1}{n}} = \frac{1}{\sqrt{2}} < 1 \Rightarrow \sum u_n$ is convergent.

1.14.9. Cauchy's Condensation Test: Let $\{f(x)\}$ be a monotone decreasing sequence of positive real numbers and let a be positive integer greater than 1. Then the series $\sum_1^\infty f(x)$ and $\sum_1^\infty a^n f(a^n)$ converge or diverge together.

Example (1.52): Consider the series $\sum_1^\infty \frac{1}{n^p},\ p > 0$. Let $f(x) = \frac{1}{n^p}$. As $p > 0$, the sequence $\{f(x)\}$ is monotone decreasing By Cauchy condensation test, the two series $\sum f(x)$ and $\sum_1^\infty 2^n f(2^n)$ converge or diverge together. But $2^n f(2^n) = \frac{2^n}{2^{np}} = \frac{1}{2^{n(p-1)}}$ But $\sum\left\{\frac{1}{2^{p-1}}\right\}^n$ is a geometric series and converges if$p > 1$ and diverges if$p < 1$.

1.14.10. Raabe's Test: Let $\sum u_n$ be a series of positive real numbers and let $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$. Then–

(i) $\sum u_n$ is convergent, if $l > 1$.

(ii). $\sum u_n$ is divergent, if $l < 1$.

(iii). No conclusion for $l = 1$.

Example (1.53): Consider the series $\sum \frac{1}{n}$. Then $u_n = \frac{1}{n}$ *and* $\lim_{n\to\infty} n(\frac{u_n}{u_{n+1}} - 1)$

$\lim_{n\to\infty} n\,(\frac{n+1}{n} - 1) = 1, But\ \sum \frac{1}{n}$ is divergent.

Let $\sum v_n$ be a series, where $v_n > 0\ \forall\, n \in \mathbb{N}$ *and* $\frac{v_n}{v_{n+1}} = 1 + \frac{1}{n} + \frac{2}{n\log n}$, $n \geq 2$ Then $\lim_{n\to\infty} n(\frac{v_n}{v_{n+1}} - 1) = 1\ but\ \sum v_n$ is convergent.

1.14.11. General Form of Raabe's Test: Let $\sum u_n$ be a series of positive real numbers and $\overline{\lim}\ n(\frac{u_n}{u_{n+1}} - 1) = R$ and $\underline{\lim}\ n(\frac{u_n}{u_{n+1}} - 1) = r$ Then –

(i).$\sum u_n$ is convergent if $r > 1$,

(ii). $\sum u_n$ is divergent, if $R < 1$.

Example (1.54):$1 + \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{7} + \ldots\ldots\ldots.$

$u_n = \frac{1\cdot3\cdot5\cdots\cdots(2n-3)}{2\cdot4\cdot6\cdots\cdots(2n-2)}\cdot\frac{1}{2n-1}\ \forall\, n \geq 2.$

Therefore, $\frac{u_{n+1}}{u_n} = \frac{(2n-1)^2}{2n(2n+1)}$ and $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = 1(ratio\ test\ fails)$

Now, $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty}\frac{n^2-n}{(2n-1)^2} = \frac{3}{2} > 1 \Rightarrow \sum u_n$ is convergent.

1.14.12. Logarithmic Test: Let $\sum u_n$ be a series of positive real numbers and $\lim_{n\to\infty} n\log\left(\frac{u_n}{u_{n+1}}\right) = l$. Then –

(i). $\sum u_n$ is convergent if $l > 1$

(ii). $\sum u_n$ is divergent if $l < 1$

Example (1.55):$1 + \frac{x}{1!} + \frac{2^2x^2}{2!} + \frac{3^3\cdot x^3}{3!} + \ldots\ldots., x > 0$

$u_n = \frac{n^n.\,x^n}{n!}$(ignoring 1st term).

$\frac{u_{n+1}}{u_n} = \left(1 + \frac{1}{n}\right)^n x$ and $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = ex$

By ratio test, convergent if $0 < x < \frac{1}{e}$. $\sum u_n$ is divergent if $x > \frac{1}{e}$ $for\ x =$
$\frac{1}{e}$, $\lim_{n\to\infty} n \log\left\{\left(1+\frac{1}{n}\right)^n \frac{1}{e}\right\}^{-1} = \lim_{n\to\infty} n \log\left[e\left(\frac{n}{n+1}\right)^n\right]$

$= \lim_{n\to\infty} n\left[1 + n\log\left(\frac{n}{n+1}\right)\right] = \lim_{n\to\infty}\left[n + n^2\log\left(\frac{n}{n+1}\right)\right] = \frac{1}{2}$

$\left(As \lim_{n\to\infty}\left[n - n^2\log\left(1+\frac{1}{n}\right)\right] = \lim_{n\to\infty}\left[n - n^2\left\{\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3}\right\}\right] = \frac{1}{2}\right)$

$\Rightarrow \sum u_n$ is divergent.

1.15. Series of Arbitrary Terms:

1.15.1. Definition (Absolutely Convergent): Let $\sum u_n$ be a series of positive terms and negative terms. Then $\sum u_n$ is said to be absolutely convergent if $\sum |u_n|$ is convergent.

1.15.2. An absolutely convergent series is convergent. Converse is not true.

1.15.3. Leibnitz's Test: If $\{u_n\}$ be a monotone decreasing sequence of positive real numbers and $lim\, u_n = 0$ then the alternating series $u_1 - u_2 + u_3 - u_4 + \ldots\ldots$ is convergent.

Example (1.56): (i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots$ is convergent but not absolutely.

1.15.4. Abel's Test: It the sequence $\{b_n\}$ is monotone bounded sequence and $\sum a_n$ is a convergent series then the series $\sum a_n b_n$ is convergent.

Example (1.57):

(i). $\sum_r^\infty \frac{(-1)^{n+1}}{n\log n}$. Here $a_n = \frac{(-1)^{n+1}}{n}$ and $b_n = \frac{1}{\log n}$, $n \ge 2$. Then $\sum a_n$ is convergent and $\{b_n\}$ is bounded and monotone. Hence $\sum a_n b_n$ is convergent.

(ii). $\sum_1^\infty \frac{(-1)^{n+1} n^n}{(n+1)^{n+1}}$, $a_n = \frac{(-1)^{n+1}}{n+1}$, $b_n = \left(1+\frac{1}{n}\right)^{-n}$

1.15.5. Dirichlet's Test: If the sequence $\{b_n\}$ is a monotone sequence converging to 0 and he sequence of partial sums $\{S_n\}$ of the series $\sum a_n$ is bounded, then the series $\sum a_n b_n$ is convergent.

Example (1.58): (i) $\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}$, $a_n = (-1)^{n+1}\ and\ b_n = \frac{1}{n}$Then $\{b_n\}$ is a monotone decreasing sequence converges to 0 and partial sum of $\sum a_n$ is bounded. Hence $\sum a_n b_n$ is convergent.

(ii) $\sum_1^\infty \frac{(-1)^{n+1}}{\log(n+1)}$, $a_n = (-1)^{n+1}$, $b_n = \frac{1}{\log(n+1)}$

1.15.6. Definition (Conditionally Convergent): A series $\sum u_n$ is called conditionally convergent if $\sum u_n$ is convergent but $\sum |u_n|$ is not convergent.

Example (1.59): $\sum \frac{(-1)^{n+1}}{n}$

1.15.7. If the terms if an absolutely convergent series be rearranged the series remains convergent and its sum remains unaltered.

1.15.8. Riemann Theorem: By approximate re-arrangement of terms, a conditionally convergent series $\sum u_n$ can be made

(i) to converge to any number l.

or, (ii) to diverge to $+\infty$ *or* $-\infty$.

or, (iii) to oscillate finitely.

or, (iv) to oscillate infinitely.

1.16. Limit:

Definition (***Limit of f at c***): Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Let c be a limit point of D. A real number $l \in \mathbb{R}$ is said to be a limit point of f *at* c if corresponding to a pre-assigned $\varepsilon > 0$, $\exists\ \delta = \delta(\varepsilon)$ such that

$|f(x) - l| < \varepsilon\ \forall\ x \in\ N'_\delta(c) \cap D, where\ N'_\delta(c) = \{x \in \mathbb{R} : 0 < |x - c| < \delta\}$

$= (c - \delta, c + \delta)\backslash\{c\}$

This expressed by symbol $\lim_{x \to c} f(x) = l$.

1.16.1. Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Let $c \in D'$. Then f can have at most one limit at c.

Example (1.60): (i) $\lim_{x \to 2} f(x) = 4, where\ f(x) = \frac{x^2-4}{x-2}, x \neq 2 \Rightarrow l = f(c)$

(ii)) $\lim_{x \to 2} f(x) = 4, where\ f(x) = \begin{cases} \frac{x^2-4}{x-2}, x \neq 2 \Rightarrow l = f(c) \\ 10, \qquad x = 2 \end{cases}$

1.16.2. Sequential Criterion: Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $c \in D'$ *and* $l \in \mathbb{R}$. Then $\lim_{x \to c} f(x) = l \Leftrightarrow$ for every sequence $\{x_n\}$ in $D - \{c\}$ converging to c, the sequence $\{f(x_n)\}$ converges to l.

Example(1.61):

(i). $\lim_{x \to 0} f(x)$ does not exists, where $f(x) = \sin\frac{1}{x}, x \neq 0$. Here $D = \mathbb{R}\backslash\{0\}$ and $0 \in D'$. Let us consider the sequence $\{x_n\}$ in D where $x_n = \frac{2}{(4n-3)\pi}, n \in \mathbb{N}\ i.e, \{\sin\frac{2}{\pi}, \sin\frac{2}{5\pi}, \sin\frac{2}{9\pi}, \ldots..\}$ converges to 0. But the sequence $\{f(x_n)\}$ is $\{\sin\frac{\pi}{2}, \sin\frac{5\pi}{2}, \sin\frac{9\pi}{2}, \ldots..\}$ i.e., $\{1,1,1, \ldots..\}$ converge to 1.

Consider another sequence $\{y_n\}$ in D by $y_n = \frac{1}{n\pi}, n \in \mathbb{N}$. Then $\{y_n\}$ converge to 0 but $\{f(y_n)\}\ i.e., \{0,0, \ldots.\}$ converges to 0. Thus $\lim_{x \to 0} \sin\frac{1}{x}$ does not exists.

(ii). $\lim_{x \to 0}[x]$ *does* not exists where $[x] = \begin{cases} -1, & -1 \leq x < o \\ 0\ , & 0 \leq x < 1 \end{cases}$

(a) Let $\{x_n\}$ in (-1, 0), let $x_n = \frac{1}{n+1}$ Then $\{x_n\}$ converges to 0 but $[x_n] = 0 \Rightarrow \{[x_n]\} = \{0,0,0, \ldots.\}$ converge to 0.

(b) Let $\{y_n\}$ in $(-1, 0)$, let $y_n = -\frac{1}{n+1}$ Then $\{y_n\}$ converges to 0 but $[y_n] = -1 \Rightarrow \{[y_n]\}$

converges to -1.

1.16.3. Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $c \in D'$. If f has a limit l at then f is bounded on some neighbourhood $N(c) \cap D\ of\ c$.

Example (1.62): $\lim_{x \to 0} \frac{1}{x}$ does not exist in $\mathbb{R}$ as $\frac{1}{x}$ is not bounded at on a neighbourhood on 0.

1.16.4. Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $c \in D'\ and\ \lim_{x \to c} f(x) = l$.

(i) If$l > 0\ then\ \exists\ a\ \ \delta > 0$ such that $f(x) > 0\ \forall\ x \in\ N'_\delta(c) \cap D$.

$[Hint: choose\ \varepsilon > 0\ such\ that\ \ l - \varepsilon > 0]$

(ii). If $l < 0\ then\ \exists\ a\ \delta > 0$ such that $f(x) < 0\ \forall\ x \in\ N'_\delta(c) \cap D$.

Note: If limit of$f(x) =\ l,\ g(x) =\ m$ exist, then limit of $\ f + g,\ f \cdot g\ ,\frac{f}{g}$ exist and will be

$$l + m, \qquad lm, \qquad \frac{l}{m}, (m \neq 0)$$

1.16.5. Let $f, g : D \subseteq \mathbb{R} \to \mathbb{R}$ be two functions and $c \in D'$. If f is bounded on $N'(c) \cap D$ and $\lim_{x \to c} g(x) = 0\ \ then\ \lim_{x \to c}(f \cdot g)(x) = 0\ [(f \cdot g)(x) = f(x) \cdot g(x)]$.

Example (1.63)

(i). $\lim_{x \to 0} x \sin\frac{1}{x^2} = 0,\ \ let\ f(x) = \sin\frac{1}{x^2}\ ,\ \ g(x) = x.$

(ii). $\lim_{x \to 0} \sqrt{x} \sin\frac{1}{x} = 0,\ g(x) = \sqrt{x}\ ,\ f(x) =\ \sin\frac{1}{x}$

1.16.6. Sandwich Theorem: Let $D \subseteq \mathbb{R}\ and\ f, g, h : D \to \mathbb{R}$ be functions and let $c \in D'$. If $f(x) \leq g(x) \leq h(x)\ \forall\ \ x \in D - \{c\}$ and if $\lim_{x \to c}\ f(x) = \lim_{x \to c}\ h(x) = l$, then $\lim_{x \to c} g(x) = l$.

Example (1.64): $\lim_{x \to 0}\ x \cos\frac{1}{x} = 0.$

Let $f(x) =\ \cos\frac{1}{x}\ ,\ x \in D = \{x \in \mathbb{R} : x \neq 0\} \Rightarrow -1\ \ \leq f(x) \leq 1\ \forall\ x \in D.$

$Hence \qquad -x\ \ \leq x\ f(x) \leq x\ \ \forall\ \ x > 0$ and

$$x \leq x\ f(x) \leq -x\ \forall\ x < 0$$

$\Rightarrow -|x| \leq x\, f(x) \leq |x| \;\; \forall\, x \neq 0, and \;\lim_{x\to 0} -|x| = 0 = \lim_{x\to 0}|x|$

$\Rightarrow \lim_{x\to 0} x\, f(x) = \lim_{x\to 0} x\cos\frac{1}{x} = 0$

1.16.7. Cauchy's Principle:

Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $c \in D'$. A necessary and sufficient condition for the existence of $\lim_{x\to c} f(x)$is that for a pre-assigned $\varepsilon > 0\; \exists\; a\; \delta > 0$ such that $|f(x') - f(x'')| < \varepsilon$ for every pair of points $x',\; x'' \in\; N'_\delta(c) \cap D$.

Example (1.65):

(i) $\lim_{x\to a} f(x)$ does not exist where $f(x) = \begin{cases} 1, if\; x \text{ is rational in } (0,1) \\ -1, if\; x \text{ is irrational in } (0,1) \end{cases}$ and

$a \in [0,1], D = (0,1)$

Let $\varepsilon = 1$. Then $\exists\; \delta > 0$ such that for every pair of points$x',\; x'' \in\; N'_\delta(a) \cap D$.

$|f(x') - f(x'')| < 1$. Let x' be rational and x'' be irrational in $N'_\delta(a) \cap D$. Then $f(x') = 1, f(x'') = -1$ and hence $|f(x') - f(x'')| = 2 \nless \varepsilon$ for some pair $x; x''$ in $N'_\delta(a) \cap D$.

(ii). $\lim_{x\to 0} \cos\frac{1}{x}$ does not exist.

Let $f(x) = \cos\frac{1}{x}\,,\;\; x \neq 0\; and\; \varepsilon = \frac{1}{2}\; and\; x; x'' \in\; N'_\delta(0) \cap D\; and\; \left|\frac{1}{2n\pi} - \frac{2}{(4n+1)\pi}\right| < \delta$

$x' = \frac{1}{2n\pi}\; and\; x'' = \frac{2}{(4n+1)\pi} \Rightarrow f(x') = 1\;\; and\;\; f(x'') = 0.$

$\Rightarrow\; |f(x') - f(x'')| = 1\;\; \not\subset\; \varepsilon = \frac{1}{2}$for some pair of points$x', x'' \in N'_\delta(0) \cap D$.

1.16.8. Definition (Right Hand Limit (R.H.L)): Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$ be a function, let $c \in D'_1$where$D' = D \cap (c, \infty) = \{x \in D : x > c\}$ f is said to have a right hand limit $l(\in \mathbb{R})$ at c if corresponding to a $\varepsilon > 0, \exists\; a\; \delta > 0\; such\; that$

$|f(x) - l| < \varepsilon\;\; \forall\, x \in\; N'_\delta(c) \cap D_1$

i.e., $l - \varepsilon < f(x) < l + \varepsilon\;\; \forall\, x \in\;\; D\;\; satisfying\; c < x < c + \delta.$

we write $\lim_{x\to c+} f(x) = l.$

Similarly left hand limit, $\quad l - \varepsilon < f(x) < l + \varepsilon\, \forall\;\; x \in D\;\; satisfying\; c - \delta < x < c$
and we write $\lim_{x\to c-} f(x) = l.$

Example (1.66):

(i). (Both exist but not equal)

$$f(x) = \operatorname{sgn} x = \begin{cases} 1, & x > 0 \quad then \lim\limits_{n\to 0+} f(x) = 1 \\ 0, & x = 0 \\ -1, & x < 0 \quad \lim\limits_{n\to 0-} f(x) = -1 \end{cases}$$

(ii). [L.H.L exist but not R.H.L]

$$f(x) = e^{\frac{1}{x}} \qquad \lim_{x\to 0+} e^{\frac{1}{x}} = \infty \qquad \text{but } \lim_{x\to 0-} e^{\frac{1}{x}} = 0$$

(iii). [Both RHL and LHL may not exist]:

$f(x) = \sin\frac{1}{x}$, $x \neq 0$. Then $\lim\limits_{x\to 0+} f(x)$ *and* $\lim\limits_{x\to 0-} f(x)$ do not exist.

1.16.9. Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $c \in D'$. Then $\lim\limits_{x\to c} f(x) = l \in \mathbb{R} \Leftrightarrow \lim\limits_{x\to c+} f(x) = l = \lim\limits_{x\to c-} f(x)$

Example (1.67.)

(i). $\lim\limits_{x\to 0+} \sin x = 0 = \lim\limits_{x\to 0-} \sin x = \lim\limits_{x\to 0} \sin x$

(ii). $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$.

Since, in $0 < x < \frac{\pi}{2}$, $\sin x < x < \tan x \Rightarrow 1 < \frac{x}{\sin x} < \frac{1}{\tan x}$ *in* $0 < x < \frac{\pi}{2}$

As, $\lim\limits_{x\to 0+} \left(\frac{1}{\cos x}\right) = 1 \Rightarrow \lim\limits_{x\to 0+} \frac{\sin x}{x} = 1$

$In\ -\frac{\pi}{2} < x < 0, \quad \tan x < x < \sin x$

$\Rightarrow \frac{1}{\cos x} < \frac{x}{\sin x} < 1 \quad in\ -\frac{\pi}{2} < x < 0$

$\lim\limits_{x\to 0-} \frac{1}{\cos x} = 1 \Rightarrow \lim\limits_{x\to 0-} \frac{\sin x}{x} = 1$

(iii). $\lim\limits_{x\to 0} [x]$ does not exist. $[x] = \begin{cases} -1, & -1 \le x < 0 \\ 0, & 0 \le x < 1 \end{cases}$

$\lim\limits_{x\to 0-} [x] = -1,$ *and* $\lim\limits_{x\to 0+} [x] = 0$

(iv). $\lim\limits_{x\to 3} \left([x] - \left[\frac{x}{3}\right]\right) = 2$

$[x] = \begin{cases} 2, & 2 \le x < 3 \\ 3, & 3 \le x < 4 \end{cases}$; $\left[\frac{x}{3}\right] = \begin{cases} 0, & 0 \le x < 3 \\ 1, & 3 \le x < 6 \end{cases}$

$\therefore f(x) = [x] - \left[\frac{x}{3}\right] = \begin{cases} 2, & 2 \le x < 3 \\ 2, & 3 \le x < 4 \end{cases}$

$\therefore \lim_{x \to 3-} f(x) = 2 = \lim_{x \to 3+} f(x) = \lim_{x \to 3} f(x)$

1.16.10. Definition (Infinite Limit): Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $c \in D'$. If corresponding to a pre-assigned $M > 0, \exists\, \delta > 0$ such that $f(x) > M \ \forall\ x \in N'_{\delta}(c) \cap D$ then we say that f tends to ∞ as $x \to c$ and we write $\lim_{x \to c} f(x) = \infty$.

and if $f(x) < -M \ \forall\ x \in N'_{\delta}(c) \cap D$ we say that f tends to $-\infty$ as $x \to c$ and we write $\lim_{x \to c} f(x) = -\infty$.

Example (1.68):

(i). $\lim_{x \to 0} \frac{1}{x^4} = \infty$.

(ii). $\lim_{x \to 0} \left(-\frac{1}{x^2}\right) = -\infty$.

(iii). $\lim_{x \to 0} \frac{1}{x}$ does not exist, since $\lim_{x \to 0-} \frac{1}{x} = -\infty$ *and* $\lim_{x \to 0+} \frac{1}{x} = \infty$

(iv). $\lim_{x \to \frac{\pi}{2}+} \tan x = \infty$ and $\lim_{x \to \frac{\pi}{2}-} \tan x = -\infty \Rightarrow \lim_{x \to \frac{\pi}{2}} \tan x$ does not exist.

1.16.11. Definition (Limit at Infinity): Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Let $(c, \infty) \subset D$ for some $c \in \mathbb{R}$. We say f tends to $l(\in \mathbb{R})$ as $x \to \infty$ if corresponding to a pre-assigned $\varepsilon > 0, \exists\ a\ M > c$ such that $|f(x) - l| < \varepsilon \ \forall\ x > M$.

In this case we write $\lim_{x \to \infty} f(x) = l$.

If $|f(x) - l| < \varepsilon \ \forall\ x < -M$ then we say that $f(x)$ tends to l as $x \to -\infty$ and we write $\lim_{x \to -\infty} f(x) = l$.

Examples (1.69): $\lim_{x \to \infty} \frac{1}{x} = 0$.

1.16.12.

(i). Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $(c, \infty) \subset D$ for some $c \in \mathbb{R}$. Then –

$\lim_{x \to \infty} f(x) = l(\in \mathbb{R}) \Leftrightarrow \lim_{x \to 0+} f\left(\frac{1}{x}\right) = l$

(ii) Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $(-\infty, c) \subset D$. For some $c \in \mathbb{R}$. Then –

$\lim_{x \to -\infty} f(x) = l(\in \mathbb{R}) \Leftrightarrow \lim_{x \to 0-} f\left(\frac{1}{x}\right) = l$

Example (1.70). $\lim\limits_{x\to\infty}\frac{\sin x}{x} = \lim\limits_{y\to 0+}\frac{\sin\left(\frac{1}{y}\right)}{\left(\frac{1}{y}\right)} = \lim\limits_{y\to 0+} y\ \sin\frac{1}{y} = 0$

1.16.13. Some Important Limits :

(i) $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

(ii). $\lim\limits_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e$ since $\lim\limits_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = \lim\limits_{y\to\infty}\left(1-\frac{1}{y}\right)^{-y}$

$= \lim\limits_{y\to\infty}\left(\frac{y}{y-1}\right)^y = \lim\limits_{t\to\infty}\left(\frac{t+1}{t}\right)^{t+1} = \lim\limits_{t\to\infty}\left(1+\frac{1}{t}\right)^t\left(1+\frac{1}{t}\right) = e$

(iii). $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e$

1.17. Continuity:

Definition (Continuous at a point): Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $c \in D$ f is said to be continuous at c if for a pre – assigned $\varepsilon > 0$, $\exists$ $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon \ \forall \ x \in (c-\delta, c+\delta) \cap D$$

We write $\lim\limits_{x\to c} f(x) = f(c)$

1.17.1 Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function. If c be an isolated point of D then f is continuous at c.

1.17.2 [Sequential Criterion]: Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $c \in D \cap D'$. f is continuous at $c \Leftrightarrow$ for every sequence $\{x_n\}$ in D converging to c, the sequence $\{f(x_n)\}$ converges to $f(c)$.

Example (1.71):

(i) $f(x) = k(\in \mathbb{R})\ \forall\ x \in \mathbb{R}$ is continuous.

(ii) $f(x) = x\ \forall\ x \in \mathbb{R}$ is continuous.

(iii) $f(x) = \begin{cases} \cos\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is not continuous at $x = 0$.

Let $x_n = \frac{1}{2n\pi}$ then $\{x_n\}$ converges to 0 but $f(x_n) = 1 \Longrightarrow \{f(x_n)\}$ converges to $1 \neq 0 = f(0)$

(iv) $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$ is not continuous at any point $a \in \mathbb{R}$.

Case – 1. Let $a \in \mathbb{Q}$, $f(a) = 1$ but we can find a sequence $\{x_n\}$ of irrational number which converges to a and $f(x_n) = 0 \Rightarrow \{f(x_n)\}$ converges to $0 \neq 1 = f(a)$.

Case – 2. Similarly for $a \in \mathbb{R}\setminus\mathbb{Q}$

Note: This function $f(x)$ is called Dirichlet's function which is every where discontinuous on $\mathbb{R}$.

1.17.3 Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be continuous at $a \in D$ (or on D) then $|f|$ is continuous at $a \in D$ (or on D). But converges is not true.

Example (1.72):

$$f(x) = \begin{cases} 1 , & x \in \mathbb{Q} \\ -1 , & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

1.17.4 Let $f, g : D \subseteq \mathbb{R} \to \mathbb{R}$be two functions. We define the functions –

$\sup(f,g);\ inf(f,g) : D \to \mathbb{R}$ by

$\sup(f,g)(x) = sup\{f(x),\ g(x)\},\ x \in D$

$\inf(f,g)(x) = inf\{f(x),\ g(x)\},\ x \in D$

1.17.5 Let $f, g : D \subseteq \mathbb{R} \to \mathbb{R}$ be continuous at $c \in D$. Then $sup(f,g)$ and $inf(f,g)$ are continuous at c.

Since,

$$\sup(f,g)(x) = \sup\{f(x), g(x)\} = \tfrac{1}{2}\big(f(x) + g(x)\big) + \tfrac{1}{2}|f(x) - g(x)|$$

$$= \tfrac{1}{2}(f+g)(x) + \tfrac{1}{2}|f-g|(x),\ x \in D$$

$$\inf(f,g)(x) = \inf\{f(x), g(x)\} = \tfrac{1}{2}\big(f(x) + g(x)\big) - \tfrac{1}{2}|f(x) - g(x)|$$

$$= \tfrac{1}{2}(f+g)(x) - \tfrac{1}{2}|f-g|(x),\ x \in D$$

1.17.6. Continuity of some important function:

(i) Polynomial Function:
$f(x) = a_0 x^n + a_1 x^{n-1} + \ldots\ldots\ldots\ldots + a_{n-1} x + a_n\ \forall\ x \in \mathbb{R}$ continuous in $\mathbb{R}$.

(ii) Rational Function:
$f(x) = \frac{p(x)}{q(x)}$, $p(x),\ q(x)$ be polynomial in $\mathbb{R}$ and $x \neq \alpha_1, \ldots\ldots\ldots, \alpha_r$ where $\alpha_i'\, s$ are root of $q(x)$. Then $f(x)$ is continuous $\forall\, x \in \mathbb{R}$for which $f(x)$ is defined.

(iii) Trigonometric Function:

(a.) $\sin x , \cos x$ continuous on $\mathbb{R}$.

(b.) $\tan x$ is continuous on $\mathbb{R}$ except $x = (2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$.

(c.) $\cot x$, $\cos x$, $\sec x$ are continuous on their respective domains.

(iv) $f(x) = a^x , a > 0 , x \in \mathbb{R}$ is continuous on $\mathbb{R} \Rightarrow e^x$ is continuous on $\mathbb{R}$.

(v) Logarithmic Function :
$f(x) = \log x,\ x > 0$ $\quad f$ is continuous on $(0, \infty)$

(vi) Square root Function :
$f(x) = \sqrt{x}$, $x \geq 0$ $\quad f$ is continuous $(0, \infty)$

(vii) (a.) $f : D \subseteq \mathbb{R} \to \mathbb{R}$ such that $f(x) \geq 0 \quad \forall\ x \in D$ and f is continuous on D Then $\sqrt{f}$ is continuous on D.

Example (1.73): $f(x) = \sqrt{\sin x}\ , x \in [0\ , \pi]$ is continuous

(b.) $f : D \subseteq \mathbb{R} \to \mathbb{R}$ such that $f(x) > 0$ and continuous then $log\, f$ is continuous on D.

(c.) If $f: D \subseteq \mathbb{R} \to \mathbb{R}$ is continuous on D, then e^f is continuous on D.

1.17.7 Some important limits:

(i) $\lim\limits_{x \to 0} \frac{\log(1+x)}{x} = 1$

(ii) $\lim\limits_{x \to 0} \frac{e^x - 1}{x} = 1$

(iii) $\lim\limits_{x \to 0} \frac{a^x - 1}{x} = log_e a\ , a > 0$

1.18 Properties of continuous functions:

1.18.1 Neighborhood properties: Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$ be continuous on D and $c \in D$. If $f(c) \neq 0$ then $\exists$ a suitable $\delta > 0$ such that $\forall\ x \in N_\delta(c) \cap D$, $f(x)$ keeps the same sign as $f(c)$.

Note: This is a local property of continuous function and is known as sign preserving property of continuous function.

Cor – 1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. Then $S = \{x \in \mathbb{R} : f(x) > 0\}$ and $T = \{x \in \mathbb{R} : f(x) < 0\}$ are open sets in $\mathbb{R}$.

Cor – 2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. Then $S = \{x \in \mathbb{R} :\ f(x) \neq 0\}$ is an open set is $\mathbb{R}$ and $T = \{x \in \mathbb{R} : f(x) = 0\}$ is a closed set in $\mathbb{R}$.

1.18.2 Let $I = [a, b]$ be a closed, bounded interval and $f : I \to \mathbb{R}$ be continuous on $\mathbb{R}$ then f is bounded on I and $\exists\ c, d \in I$ such that $f(c) = \sup\limits_{x \in I} f(x)$ and $f(d) = \inf\limits_{x \in I} f(x)$

But this is not true for open interval $I = (a\ , b)$ which is bounded.

Example (1.74):

(i) $f : I = (2\ , 3) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$, $x \in (0\ , 1)$ Then f is continuous on I but not bounded.

(ii) $f : I = (2\ , 3) \to \mathbb{R}$ defined by $f(x) = x^2$. Then $\sup\limits_{x \in I} f(x) = 9$ and $\underset{x \in I}{int}\, f(x) = 4$. But $\nexists\ x_0\ c,\ d \in I$ such that $f(c) = 9$ and $f(d) = 4$, $x \in I$.

(iii) A function f continuous as a closed interval I may not be bounded as I.

Example (1.75): $f : [0\ , \infty] \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}$, $x \geq 0$. $f$is continuous on $[0\ , \infty]$ but f is not bounded on $[0\ , \infty]$.

1.18.3 Bolzano Theorem: Let $I = [a\,,b]$ be a closed and bounded interval and $f : I \to \mathbb{R}$ be continuous on I. If $f(a)$ and $f(b)$ one of opposite signs, then $\exists$ at least one $c \in (a\,,b)$ such that $f(c) = 0$.

1.18.4 Intermediate Value Theorem: Let $I = [a\,,b]$ be a closed, bounded interval and $f : [a\,,b] \to \mathbb{R}$ be continuous on I. If $f(a) \neq f(b)$ then f attains every value between $f(a)$ and $f(b)$ at least once in the open interval $(a\,,b)$ converse is not true.

Example (1.76): Let $f : [0\,,2] \to \mathbb{R}$ be defined by -$f(x) = \begin{cases} 0 & , \quad x = 0 \\ x & , \quad 0 < x \leq 1 \\ 3 - x & , \quad 1 < x < 2 \\ 2 & , \quad x = 2 \end{cases}$

f assume every value between 0 and 2 on $[0\,,2]$. But f is not continuous at$x = 1\,,2$.

1.18.5. Let $I = [a\,,b]$ be a closed and bounded interval and $f : I \to \mathbb{R}$ be continuous on I. If $M = \sup_{x \in I} f(x) \neq m = \inf_{x \in I} f(x)$ and $m < \mu < M$ then $\exists$ $p \in (a\,,b)$ such that $f(p) = \mu$.

1.18.6. Let $f : I = [a\,,b] \to \mathbb{R}$ be continuous on I.Then $f(I) = \{f(x) : x \in I\}$ in a closed and bounded interval.

Note:

(i) The continuous image of a closed and bounded interval $[a, b]$ is a closed and bounded interval $[m\,,M]$. If particular, if f is constant on $[a, b]$, the image reduces to a point.

(ii) The continuous image of an open interval may not be open.

Example (1.77): $f : (-1\,,1) \to \mathbb{R}$ defined by $f(x) = x^2\,, \forall\, x \in I = (0\,,1)$ then

$f(I) = [0\,,1)$which is not open.

1.18.7. Let I be an interval and $f = I \to \mathbb{R}$ be continuous (non-constant) in I.Then $f(I)$ is an interval.

Examples (1.78):

(i) If f: $[0\,,1] \to [0\,,1]$ is continuous on$[0\,,1]$, then $\exists$ a point $c \in [0\,,1]$ such that $f(c) = c[c\ is\ called\ a\ fixed\ point\ of\ f]$.
[Hint: if $(0) = 0$ or $f(1) = 1$, done. Let $f(0) \neq 0\,,\ f(1) \neq 1$.
Define $g(x) = f(x) - x$. Then g is continuous on $[0\,,1]$ and $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 < 0 \Rightarrow$ by Bolzano then, $\exists\, c \in (0\,,1)$ such that $g(c) = 0 \Rightarrow f(c) = c$]

(ii) If f: $[0\,,1] \to \mathbb{R}$ is continuous on $[0\,,1]$ and assumed only rational values and $f\left(\frac{1}{2}\right) = \frac{1}{2}$, then $f(x) = \frac{1}{2}\ \forall\, x \in [0\,,1]$
[Hint: Let $x_1 \in [0\,,\ \frac{1}{2}]$ and consider on $x_1\,,\frac{1}{2}$ let $f(x_1) = p \neq \frac{1}{2}$,p is rational. Let $\mu \in (p\,,\frac{1}{2})$ irrational by intermediate theorem, $\exists\, c \in (x_1\,,\frac{1}{2})$ such that
$f(c) = \mu$, contradiction hence $f(x_1) = \frac{1}{2}$]

1.18.8. Let $f: \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. Then for every open subset G of $\mathbb{R}$, $f^{-1}(G)$ is open in $\mathbb{R}$. Conversely, if $f^{-1}(G)$ in open in $\mathbb{R}$ for every open set Gin $\mathbb{R}$. Then f is continuous on $\mathbb{R}$.

But if f is continuous then image of open set may not open.

Example (1.79): $f : (0\,,1) \to \mathbb{R}$ defined by $f(x) = 2 \quad \forall\, x \in (0\,,1)$.

1.18.9. Function $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R} \Leftrightarrow f^{-1}(F)$ is closed in $\mathbb{R}$ whenever F is closed in $\mathbb{R}$.

1.18.10 The functions $f\,,\ g : \mathbb{R} \to \mathbb{R}$ are both continuous on $\mathbb{R}$. Then the

Lets,

(i) $s = \{x \in \mathbb{R} : f(x) < g(x)\}$ is open set in $\mathbb{R}$.
(ii) $T = \{x \in \mathbb{R} : f(x) \neq g(x)\}$ is open set in $\mathbb{R}$.
(iii) $P = \{x \in \mathbb{R} : f(x) = g(x)\}$ is closed set in $\mathbb{R}$.
(iv) If $\{f(x) = g(x)\}$ at all $x \in \mathbb{Q}$, then $f(x) = g(x)\ \ \forall\, x \in \mathbb{R}$
[Hint: $\mathbb{Q} \subseteq P \subseteq \mathbb{R}$ and P is closed $\Rightarrow P = \bar{P} = \mathbb{R}$]
(v) If $f(x) = k$, constant $\forall\, x \in \mathbb{Q}$, then $f(x) = k \quad \forall\, x \in \mathbb{R}$.

[Hint: let$g(x) = k,\ \forall\ x \in \mathbb{R} \ \Rightarrow f(x) = g(x)\,\forall\ x \in \mathbb{Q}\ \Rightarrow f(x) = g(x) = k\ \ \forall\, x \in \mathbb{R}$]

1.18.11 Let $f : I = (a,b) \to \mathbb{R}$ be monotone increasing on I. Then at any point $c \in I$,

(i) $f(c-0) = \sup\limits_{x\in(a,c)} f(x)$
(ii) $f(c+0) = \inf\limits_{x\in(c,b)} f(x)$
(iii) $f(c-0) \le f(c) \le f(c+0)$

1.18.12 Discontinuity of first kind: Let $c \in (a\,,b) \in I$ and$f$ be continuous on (a,c) and $(c\,,b)$, but discontinuous at $c \in (a,b) \in I$ and $\lim\limits_{x\to c-} f(x)$ and $\lim\limits_{x\to c+} f(x)$ both exist. –

(i) $\lim\limits_{x\to c-} f(x) = \lim\limits_{x\to c+} f(x)$
  a. f is not defined at$c$, f is discontinuous at c.
  b. f is defined at$c$, but$f(c) \neq \lim\limits_{x\to c} f(x)$
(ii) $\lim\limits_{x\to c-} f(x) \neq \lim\limits_{x\to c+} f(x)$. In this case f is discontinuous at c. Whether f is defined at c or not. This type of discontinuity is called jump discontinuity.
  <u>Right jump</u> :$f(c+0) - f(c)$
  <u>Left jump</u> :$f(c) - f(c-0)$

1.18.13. Discontinuity of second kind: If at least one of $\lim\limits_{x\to c-} f(x)$ and $\lim\limits_{x\to c+} f(x)$ does not exist. But$f$ is bounded in some bounded $N'\delta(c)$ of. In this case$f$ is discontinuous at c whether f is defined at c or not. This type of discontinuity is called oscillatory discontinuity.

1.18.14. If$f : (a,b) \to \mathbb{R}$ be monotone on (a,b) , then at every point $c \in (a,b)$, $f(c-0)\ and\ f(c+0)$both exist. Monotone function$f$ can not have discontinuity of second kind.

1.18.15. If $f : [a, b] \to \mathbb{R}$ be monotone on $[a, b]$, then the set of points at discontinuities of f in $[a, b]$ is a countable set.

$\Rightarrow$ If $f = \mathbb{R} \to \mathbb{R}$ be monotone on $\mathbb{R}$, then the set of points of discontinuities is a countable set.

[Hint: $\mathbb{R} = (\cup_{n=0}^{\infty}[n-1\,,\ n+1]) \cup (\cup_{n=0}^{\infty}[-(n+1)\,,\ -(n-1)])$]

1.18.16. If a function $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and injective on $[a, b]$ then f is strictly monotone on $[a, b]$.

$\Rightarrow$ Let I be an interval and $f : I \to \mathbb{R}$ is continuous and injective on I. Then f is strictly monotone on I.

1.18.17. If $f : [a, b] \to \mathbb{R}$ satisfies intermediate value property on $[a, b]$ and f is injective on $[a, b]$ then-

(i) f is strictly monotone on $[a, b]$
(ii) f is continuous on $[a, b]$

1.19. Uniform continuity:

Definition: A function $f : I \to \mathbb{R}$ is said to uniformly continuous on I if corresponding to a pre – assigned $\varepsilon > 0$, $\exists\ \ \delta > 0$ such that for any pair of point $x_1\ x_2 \in I$,

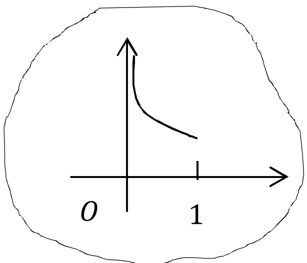
$|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$

Note: Uniform continuity is a global property.

Example (1.80):

(i) $f(x) = \frac{1}{x}\,,\ x \in [1, \infty]$ is uniformly continuous on $[1, \infty]$
Since $|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{x-y}{xy}\right| \le |x - y| < \varepsilon, say\ (\because x, y \ge 1)$ then
$\forall\ x, y \in [1, \infty]$ with $|x - y| < \varepsilon \Rightarrow\ |f(x) - f(y)| < \varepsilon$.
(ii) $f(x) = sinx,\ x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.
Since $x, y \in \mathbb{R}, |\sin x - \sin y| = 2\left|\sin\frac{x-y}{2}\right|\left|\cos\frac{x+y}{2}\right| \le 2\left|\sin\frac{x-y}{2}\right| \le 2.|x - y| < \varepsilon, say|x - y| < \varepsilon \Rightarrow |\sin x - \sin y| < \varepsilon$

1.19.1. Let I be an interval and a function $f : I \to \mathbb{R}$ be uniformly continuous on I. Then f is continuous on I. But not conversely.



Example (1.81): $f(x) = \frac{1}{x}\,, 0 < x \le 1$ is continuous but not uniformly.

1.19.2. Let $I = [a, b]$ be a closed and bdd interval and $f : I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I.

1.19.3. Let $f : D \subseteq \mathbb{R}$ be uniformly continuous on D. If $\{x_n\}$ be a Cauchy sequence in D, then $\{f(x_n)\}$ a Cauchy sequence in $\mathbb{R}$.

Example (1.82): $f(x) = \frac{1}{x}$, $x \in [0,1]$ is not uniformly continuous in $[0,1]$. Since $\left\{\frac{1}{n}\right\}$ in a Cauchy sequence in $[0,1]$ but $\left\{f\left(\frac{1}{n}\right) = n\right\}$ is not Cauchy in $\mathbb{R}$.

1.19.4. Let I be a bounded interval and a function $I \to \mathbb{R}$ be uniformly continuous on I. Then f is bounded on I converse is not true.

Example (1.83): $f(x) = \sin\frac{1}{x}$, $x \in (0,1)$.Then $f(x)$ is continuous on bdd interval $(0,1)$ and $|f(x)| \leq 1$ but $f(x)$ is not uniformly continuous. Since $\left\{\frac{2}{(2n+1)\pi}\right\}$ is Cauchy in $(0,1)$ but $\left\{f\left(\frac{2}{(2n+1)\pi}\right\}$ is not Cauchy in $\mathbb{R}$.

1.19.5. Let f be continuous on an open bdd interval (a,b). Then f is uniformly continuous on $(a,b) \Leftrightarrow \lim\limits_{x \to a+} f(x)$ and $\lim\limits_{x \to b-} f(x)$ both exist finitely.

1.19.6 Continuous Extension: Let f be continuous on an interval I. A function g is said to be a continuous extension of$f$ to $\mathbb{R}$ if g be continuous on $\mathbb{R}$ and $g(x) = f(x) \quad \forall\, x \in I$

Example (1.84): Let $f : [a,b]$be continuous and$g : \mathbb{R} \to \mathbb{R}$ be defined by-

$$g(x) = \begin{cases} f(a) , & x < a \\ f(x) , & x \in [a,b] \\ f(b) , & x > b \end{cases}$$

Then g is continuous extension of f.

Let f be continuous on an bdd open interval (a,b). Then$f$ admits of a continuous extension to $\mathbb{R} \Leftrightarrow f$ be uniformly continuous on (a,b).

1.19.7. Definition (Lipschitz function): Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is said to satisfy a Lipschitz condition on I if $\exists \ 0 < M \in \mathbb{R}$ such that $|f(x_1) - f(x_2)| \leq M\,|x_1 - x_2|$ for any two points $x_1, x_2, \in I$. In this case f is said to be a Lipschitz function on I.

Example (1.85):

Let $f(x) = x^2,\ x \in [0,2]$. Then

$|f(x_1) - f(x_2)| = |{x_1}^2 - {x_2}^2| \leq 4|x_1 - x_2| \ \forall\, x_1 x_2 \in [0,2]$

1.19.8. Let $f : I \to \mathbb{R}$ be a Lipschitz function on I. Then f is uniformly continuous on I.

Example (1.86):

$f(k) = \sin x , x \in \mathbb{R}$

$|\sin x - \sin y| \leq |x - y|$

1.19.9. Continuity on a compact set: Let $D \subseteq \mathbb{R}$ be a compact set and a function $f : D \to \mathbb{R}$ be continuous on D. Then $f(D)$ is a compact set in $\mathbb{R}$.

1.19.10. Let $D \subseteq \mathbb{R}$ be a compact set and $f : D \to \mathbb{R}$ is continuous D. Then f is uniformly continuous on D.

Converse of (1.19.8) is not true.

Example (1.87): $f(x) = \sqrt{x}, x \quad x \in [0, a],\ a > 0$.

But $f(x) = \sqrt{x}$ is satisfies Lipschitz condition on $[1, a],\ \forall a > 1$

1.19.11. Some special uniform continuous functions:

(i) **Periodic function:** If f be a continuous function such that $f(x+p) = f(x)$ for some $P \in \mathbb{R}$, then f is uniformly continuous on $\mathbb{R}$.

(ii) If $f(x+y) = f(x) + f(y) \ \forall\ x, y \in \mathbb{R}$ be continuous at a point$c \in \mathbb{R}$, then f is uniformly continuous on $\mathbb{R}$.

(iii) Let $\phi \neq A \in \mathbb{R}$ *and* $f_A(x) = inf\,\{|x-a| : a \in A\}\ \forall\, x \in \mathbb{R}$. $f$is uniformly continuous on $\mathbb{R}$.

(iv) If $f'(x)$ exists and bdd then f satisfice Lipschitz condition and hence it is uniformly continuous.

1.20. Differentiation:

Definition (Differentiability and derivative): Let $I = [a, b]$ be an interval and $f : I \to \mathbb{R}$ be a function. f is said to be differentiable at $c \in I$ if $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists. If l be its limit, then l is said to be the derivative of f at$c$and is denoted by $f'(c)$.

(i) If c be an interior point of the domain, then $\lim_{x \to c-} \frac{f(x)-f(c)}{x-c}$ and $\lim_{x \to c+} \frac{f(x)-f(c)}{x-c}$ should exist and they are equal in order to $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exist.

(ii). If $c = a$, then $\lim_{x \to a+} \frac{f(x)-f(a)}{x-a}$ exists and the limit is called derivative of f at a and is denoted by $f'(a)$.

(iii). If $c = b$, then $\lim_{x \to b-} \frac{f(b)-f(x)}{b-x}$ exists and limit is called derivative of f at b and is denoted by $f'(b)$.

1.20.1. Definition (Right and left hand derivative): Let I be an interval and$f : I \to \mathbb{R}$ and $c \in I$. If $\lim_{x \to c+} \frac{f(x)-f(c)}{x-c}$ exists the limit is called the right hand derivative of *at* c and is denoted by$R\ f'(c)$.

If $\lim_{x \to c-} \frac{f(x)-f(c)}{x-c}$ exists, the limit is called left hand limit derivative of f and is denoted by $Lf'(c)$.

1.20.2. Let $f : I \to \mathbb{R}$ be differentiable at a point $c \in I$. Then f is continuous at c. But converse is not true.

Example (1.88): $f(x) = |x|,\ x \in \mathbb{R}$. At $x = 0$, $f(x)$ is continuous but

$$\lim_{x \to 0+} \frac{|x|-|0|}{x-0} = \lim_{x \to 0+} \frac{x}{x} = 1 = Rf'(0)$$

$$\lim_{x \to 0-} \frac{|x|}{x} = \lim_{x \to 0-} \frac{-x}{x} = -1 = Lf'(0)$$

As $Rf'(0) \neq Lf'(0) \Rightarrow f$ is not differentiable at 0.

Note: Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ it is possible to define differentiability of f at $c \in D$, provided $c \in D'$ also i.e., if $c \in D \cap D'$, then f is said to be differentiable at c if $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists and the limit is called derivative of f at c and is denoted by $f'(c)$.

1.20.3. Let$f, g : I \to \mathbb{R}$ be differentiable at $c \in I$. Then –

(i). $(f+g)'(c) = f'(c) + g'(g)$

(ii) if $k \in \mathbb{R}, (k\, f)'(c) = k\, f'(c)$

(iii) $(f \cdot g)'(c) = f'(c)\, g(c) + f(c)\, g'(c)$

(iv) $\left(\frac{f}{g}\right)'(c) = \frac{g(c)\, f'(c) - f(c)\, g'(c)}{\{g\ (c)\}^2}$, proved $g(c) \neq 0$

1.20.4. Let I and J be intervals. Let $f\ : I \to \mathbb{R};\ \ g : J \to \mathbb{R}$ and $f(I) \in J$. Let $c \in I$ and f is differentiable at c and$g$ is differentiable at e $and (g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.

Example (1.89): Let $f(x) = x^{\alpha}$, $x > 0$ and $d \in \mathbb{R} \Rightarrow f(x) = e^{\alpha \log x}$

Let $g(x) = \alpha\ \log x$, $x > 0$ and $h(x) = e^{x}$, $x \in \mathbb{R}$

Then $f(x) = (h \circ g)(x) = h(g(x)) = e^{\alpha \log x} = x^{\alpha}$

$\Rightarrow f'(x) = h'(g(x)) \cdot g'(x) = e^{\alpha \log x} \cdot \frac{\alpha}{x} = x^{\alpha} \cdot \frac{\alpha}{x} = \alpha\, x^{\alpha - 1}, x > 0.$

1.20.5. Let $I \subseteq \mathbb{R}$ be an interval and a function $f\ : I \to \mathbb{R}$ be strictly monotone and continuous on I. Let $J = f(I)$ and Let $g : J \to \mathbb{R}$ be the inverse of$f$. If f is differentiable at $c \in I$ and $f'(c) \neq 0$. then$g$ is differentiable at $d = f(c)$ and $g'(d) = \frac{1}{f'(c)}$.

Example (1.90):

(i). $f(x) = x^2$, $x \in [0,\infty]$. f is strictly increasing and continuous on $[0,\infty]$. Let $I = [0,\infty]$ then $f(I) = [0,\infty]$. The inverse function g is defined by $g(y) = \sqrt{y}$, $y \in [0,\infty]$ is continuous on $[0,\infty]$ f is differentiable on $[0,\infty]$ and $f'(x) = 2x$, $x \in [0,\infty]$, $f'(x) \neq 0 \ on \ (0,\infty]$.

Let $I_1 = (0,\infty)$. Then $f(I_1) = (0,\infty)$. Hence $g'(y)$ exists $\forall\, y \in (0,\infty)$ and $g'(y) = \frac{1}{f'(x)} = \frac{1}{2x} = \frac{1}{2g(y)} = \frac{1}{2\sqrt{y}}$, $y \in (0,\infty)$.

(ii). $f(x) = e^x$, $x \in \mathbb{R}$. Then $f(\mathbb{R}) = (0,\infty)$. Inverse of f is g be field by $g(y) = \log y$, $y \in (0,\infty)$ since f is strictly increasing and monotone on $(0,\infty) f'(x) \neq 0 \ on \ \mathbb{R}$. So $g'(y) = \frac{1}{f'(x)} = \frac{1}{e^x} = \frac{1}{e^{\log y}} = \frac{1}{y}$, $y \in (0,\in \infty)$.

(iii) $f(x) = \sin x$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $f(I) = [-1,1]$. The inverse of g is defined by $g(y) = \sin^{-1} y$, $y \in [-1,1]$, $f'(x) \neq 0 \ on \ \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$\therefore g'(y) = \frac{1}{f'(x)} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-y^2}}$, $y \in (-1,1)$.

Thus $\frac{d}{dx} \sin^{-1} y = \frac{1}{\sqrt{1-y^2}}$, $y \in (-1,1)$

(iv) $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$

Then $\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0} \frac{x^2 \sin\frac{1}{x}}{x} = \lim\limits_{x\to 0} x \sin\frac{1}{x} = 0 \Rightarrow f'(0) = 0$

$$\therefore f'(x) = \begin{cases} 2x \ \sin\frac{1}{x} - \cos\frac{1}{x}, & x \neq 0 \\ 0\,, & x = 0 \end{cases}$$

$\lim\limits_{x\to 0} \cos\frac{1}{x}$ does not exist (by Cauchy principle) $\Rightarrow f'(x)$ is continuous at 0.

(v) $f : \mathbb{R} \to \mathbb{R}$defined by

$$f(x) = \begin{cases} 0, & x < 0 \\ 0, & x \in \mathbb{R}|\mathbb{Q} \\ \frac{1}{q} \ if \ x = \frac{p}{q}, & p, q \in \mathbb{Z}, q \neq x \ and \ g \subset d\ (p,q) = 1 \end{cases}$$

$\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0} \frac{f(x)}{x}$ Let $x_n = \frac{1}{n}$ Then $f(x_n) = \frac{1}{n}$

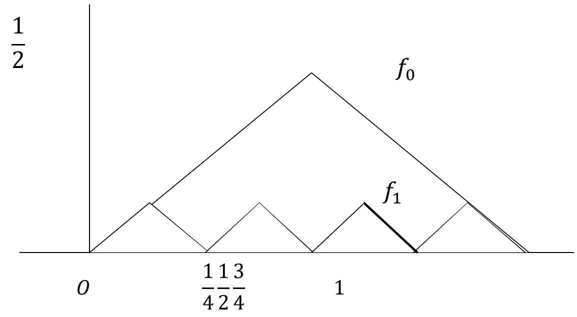
Hence $\lim\limits_{x_n\to 0} \frac{f(x_n)}{x_n} = 1$ and let $\{x_n\}$ be a sequence of irrational numbers converging to 0. $\Rightarrow \lim\limits_{x_n\to 0} \frac{0}{x_n} = 0$. Hence f is not differentiable at 0.

(vi) Give an example of continuous function which is nowhere differentiable.

$f_0(x) = d(x,\mathbb{Z}) = \inf\{|x-k| : k \in \mathbb{Z}\}$

$f_m(x) = \frac{1}{4m} f_0(4^m x)$

$f = \lim_{m\to\infty} f_m(x)$ is everywhere continuous but nowhere differentiable.



1.20.6. Definition (Higher Order Derivatives): Let I be an interval and $f : I \to \mathbb{R}$ be differentiable at $c \in I$. If f be differentiable at every point of some sub interval $I_1(c)$ such that $c \in I_1(c) \subset I$, then $f' : I_1(c) \to \mathbb{R}$ is a function on $I_1(c)$. If f' be differentiable at c then the derivative of f' *at* c is called second order derivative of f *at* c and is denoted by $f''(c)$ *or* $f^{(2)}(c)$.

1.20.7. Let $I \subset \mathbb{R}$ be a interval and $f : I \to \mathbb{R}$ be differentiable at $c \in I$

(i) If $f'(c) > 0$ then f is increasing at c.

(ii) If $f'(c) < 0$ then f is decreasing at c.

Example (1.91):

(i) Let $f(x) = \begin{cases} x, & x < 1 \\ 2x-1, & x \geq 1 \end{cases}$ Then f *is increasing at* 1 *but not differentiable at* 1.

(ii) $f(x) = \begin{cases} 1-x, & x < 0 \\ 1-2x, & x \geq 0 \end{cases}$ Then f *is increasing at* 0 *but not differentiable* .

(iii) If f is increasing at c then $f'(c)$ may not be positive.

Example (1.92): $f(x) = x^3, x \in \mathbb{R}$ $\quad$ f *is increasing at* 0, *but* $f'(0) = 0$.

(iv) If f is decreasing atcthen $f'(c)$ may not be negative. $f(x) = -x^3, x \in \mathbb{R}$, f is decreasing at 0 but $f'(c) = 0$.

(v) $f'(c) > 0$ does not imply that f is monotone in a neighbourhood of c.

Example (1.93): $f(x) = \begin{cases} \frac{x}{2} - 1\, x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Then $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}(\frac{1}{2} + x\sin\frac{1}{2}) = \frac{1}{2} > 0$ But is a neighbourhood of 0 $\quad f$ takes both positive and negative values.

1.20.8. Darboux: Let $f: I = [a, b] \to \mathbb{R}$ be differentiable on I. Let $f'(a) \neq f'(b)$. If $k$be a real number lying between $f'(a)\ and\ f'(b) then\ \exists\ c \in (a, b) such\ thta\ f'(c) = k$. [similar results as for continuous function].

Example (1.94): Let $f: [-1, 1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 0, & x \in [-1,0] \\ 1, & x \in (0,1) \end{cases}$ Does $\exists$ a function g such that $g'(x) = f(x),\ x \in [-1,1]$?

If possible, let $g: [-1,1] \to \mathbb{R}$ such that $g'(x) = f(x)\ in\ [-1,1]$.

Then$g'(-1) = 0\ \Rightarrow 1 = g'(1)$ by Dorboux theorem for every real number $\mu \in (g'(-1), g(1)) = (0,1), \exists\ c \in [-1,1]\ such\ that\ g'(c) = \mu - a$ contradiction.

1.20.9. Let I be an interval and $f: I \to \mathbb{R}$ be differentiable on$I$. Then $f'(I)$ is an interval.

1.20.10. If $f: [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ then f' can not have a jump discontinuity on$[a, b]$.

1.21. Mean Value Theorem (MVT):

1.21.1. Rolle's Theorem: Let $f: [a, b] \to \mathbb{R}$ be a function such that

(i) f is continuous on $[a, b]$

(ii) f is differentiable in (a, b) and

(iii) $f(a) = f(b)$

Then $\exists$ at least one $c \in (a, b) such\ that\ f'(c) = 0$

1.21.2. Lagrange Mean Value Theorem (MVG): Let $f: [a, b] \to \mathbb{R}$ be a function such that

(i) f is continuous on $[a, b]$ and

(ii) f is differentiable in (a, b)

(iii) $f(a) \neq f(b)$

Then$\exists$at least one point $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

1.21.3. Let $f : [a, b] \to \mathbb{R}$ satisfies (i) and (ii) of (iv) and $f'(x) = 0\ \forall\ x \in (a, b)$ then $f(x)$ is constant on $[a, b]$.

1.21.4. Let $f, g : [a, b] \to \mathbb{R}$ satisfies (i) and (ii) of (10) and $f'(x) = g'(x) \forall\ x \in (a, b)$, then $f = g + c\ (constant)$.

Example (1.95):$\frac{x}{1+x} < \log(1 + x) < x\ \forall\ x > 0$

Let $f(x) = \log(1 + x) - \frac{x}{1+x}, x \geq 0$

$\Rightarrow f'(x) = \frac{x}{(1+x)^2} > 0\ \forall\ x > 0\ \Rightarrow f$ is strictly increasing

$\Rightarrow f(x) > f(0) \Rightarrow \log(1 + x) > \frac{x}{1+x}$

Let $g(x) = x - \log(1 + x)$, $x > 0$

$\Rightarrow g'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0$ strictly increasing

$\Rightarrow g(x) > g(0) \Rightarrow x > \log(1 + x)$, $x > 0$

Hence $\frac{x}{1+x} < \log(1 + x) < x \;\; or\; x > 0$

1.21.5. Let I be an interval. If a function$f : I \to \mathbb{R}$ be such that f' exists and is bounded on I then f is uniformly continuous on I.

[Since: $|f'(x)| \le k \;\Rightarrow \left|\frac{f(x_2)-f(x_1)}{x_2-x_2}\right| \le k \Rightarrow |f(x_2) - f(x_1)| \; \le k|x_2 - x_2|$,

$lipschitz\ condition\ satisfy$.

Example (1.96):

$f(x) = \frac{1}{x^2+1}, x \in \mathbb{R}$. Then $f'(x) = -\frac{2x}{(x^2+1)^2}$, $x \in \mathbb{R}\ and\ |f'(x)| < 2\ \forall\ x \in \mathbb{R} \Rightarrow$ $f\ is\ uniformly\ continuous\ on\ \mathbb{R}$.

1.21.6. Generalised MVT: Let $f, g: [a, b] \to \mathbb{R}$ such that

(i) $f\ and\ g$ are both continuous on $[a, b]$ and

(ii) $f\ and\ g$ are both differentiable in (a, b)

Then $\exists$ a point $c \in (a, b)$ such that $[g(b) - g(a)]f'(c) = [f(b) - f(a)]g'(c)$.

1.21.7. Cauchy's MVT: Let $f, g : [a, b] \to \mathbb{R}$ be such that satisfy (i), (ii) of (12) and (iii) $g'(x) \neq 0\ \forall\ x \in (a, b)$. Then $\exists\ c \in (a, b)\ such\ that\ \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$.

1.21.8. Leibnitz's Theorem: Let $f\ and\ g$ be two functions each differentiable $n\ times\ at\ a$, then the n^{th} derivative of the product$fg\ at$ a given by –

$(fg)^{(n)}(a) = \sum_{r=0}^{n} \binom{n}{r} D^{n-r}f(a)D^r g(a)$where $D^r(a) = f^r(a), r \ge 1\ and\ Df(a) = f(a)$.

1.21.9. Taylor's Theorem: Let $f : [a, a + h] \to \mathbb{R}$ be such that

(i) f^{n-1} is continuous on $[a, a + h]$, and

(ii) f^{n-1} is differentiable in $(a, a + h)$.

Then $\exists\ \theta\ (0 < \theta < 1)$ such that –

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a + \theta h)$$

where$p$ is a positive integer $\le n$.

Note: The last term $\frac{h^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a + \theta h)$ is called the remainder after $n\ terms$ and it is denoted by R_n.

Cauchy's Form: If $p-1, R_n = \frac{h^n(1-\theta)^{n-p}}{(n-1)!} f^n(a+\theta h)$

Lagrange's Form: If $p = n, R_n = \frac{h^n}{n!} f^n(a+\theta h)$

1.21.10. Maclaurin's Theorem: Let $f : [0,h] \to \mathbb{R}$ be such that

(i) f^{n-1} is continuous on $[0,h]$ and

(ii) f^{n-1} is differentiable in $(0,h)$.

Then for $x \in (0,h], \exists\, \theta\ (0<\theta<1) such\ that$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots\cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n(1-\theta)^{n-p}}{p(n-1)!}f^n(\theta x)$$

where p is a positive integer $\leq n$. For $p = 1, cauchy\ form\ and\ p = n\ lagrenge's\ form.$

Examples (1.97):

(i) Let $c \in \mathbb{R}\ and\ f : \mathbb{R} \to \mathbb{R}$ be such that f'' is continuous on some neighbourhood of c. Then $\lim_{h\to 0} \frac{f(x+h)-2f(c)+f(c-h)}{h^2} = f''(c)$.

Since f'' is continuous on $(c-\delta, c+\delta)$ for some $\delta > 0$. By Taylors theorem with Lagrange's form after remainder $(after\ 2\ terms)\ for\ any\ h\ with\ 0<h<\delta,$

$$f(c+h) = f(c) + hf'(c) + \frac{h^2}{2!}f''(c+\theta h), 0<\theta<1$$

$$f(c-h) = f(c) - hf'(c) + \frac{h^2}{2!}f''(c+\theta h), 0<\theta'<1$$

$$\therefore\ f(c+h) - 2f(c) + f(c-h) = \frac{h^2}{2!}[f''(c+\theta h) + f''(c+\theta' h)]$$

$$\Rightarrow \frac{f(c+h)-2f(c)+f(c-h)}{h^2} = \frac{1}{2}[f''(c+\theta h) + f''(c+\theta' h)]$$

$$\Rightarrow \lim_{h\to 0}\frac{f(c+h)-2f(c)+f(c-h)}{h^2} = f''(c)\ [\because f''is\ continuous\ at\ c]$$

(ii) Use Taylor's Theorem, $1+\frac{x}{2}-\frac{x^2}{8} < \sqrt{1+x} < 1+\frac{x}{2}, if\ x>0$

Let $f(x) = \sqrt{1+x}, x \geq 0$Then –

$$f'(x) = \frac{1}{2\sqrt{1+x}}, f''(x) = -\frac{1}{4(1+x)^{\frac{3}{2}}}, f'''(x) = \frac{3}{8(1+x)^{\frac{5}{2}}}$$

By Taylor's theorem with Lagrange's form of remainder $(after\ 3\ terms)$ for any$x>0$.

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(c)\ for\ some\ c \in (0,x)$$

$$or, \sqrt{1+x} = 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16(1+c)^{\frac{5}{2}}} \Rightarrow \sqrt{1+x} > 1+\frac{x}{2}-\frac{x^2}{8}\ (\because x>0)$$

By Taylor's theorem with Lagrange's form of remainder $(after\ 2terms)\ for\ any\ x>0.$

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(d)\ \ for\ some\ d \in (0, x)$

$or, \sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8(1+d)^{\frac{3}{2}}} \Rightarrow \sqrt{1+x} < 1 + \frac{x}{2}\ (\because x > 0)$

1.21.11. Taylor's Infinite Series: Let $a \in \mathbb{R}\ and\ f$ defined on some neighbourhood $N(a)$ of a such that f^{n-1} is differentiable on $N(a)$. Then for any $x \in N(a) - \{a\}, f(x) = P_n(x) + R_n(x)$, where $R_n(x)$ is the remainder after $n\ terms$ and $P_n(x) = f(a) + (x-a)f'(a) + \cdots \cdot + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a)$. $P_n(x)$ is a polynomial of degree $n-1$ and $P_n(x)$ is such that –

$P_n(a) = f(a),\ P'_n(a) = f'(a),\ P_n''(a) = f''(a), \ldots\ldots\ldots\ldots\ P_n{}^{n-1}(a) = f^{n-1}(a). P_n(x)$ is called the nth Taylor Polynomial of f about the point a. If for all n, f^n exists on $N(a)$, then $P_n(x)$ be an infinite series $f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots\ldots\ldots$ which is convergent if $\{P_n(x)\}$ is convergent and if $\lim\limits_{n\to\infty} R_n(x) = 0$ then we have –

$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots\ldots\ldots\ldots$

If $a = 0$, we have Maclaurin's infinite series

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots\ldots\ldots$

1.21.12. Expansion of some functions:

(i) Let $f(x) = e^x,\ x \in \mathbb{R}$. Then $f^n(x) = e^x, \forall\ x \in \mathbb{N}$. By Taylor's theorem with Lagrange's form of remainder after $n\ terms \forall\ 0 \neq x \in \mathbb{R}$,

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots\ldots\ .+ \frac{x^{n-1}}{(x-1)!}f^{n-1}(0) + R_n(x)$.where

$R_n(x) = \frac{x^n}{n!}f^n(\theta x), 0 < \theta < 1.$

$= \frac{x^n}{n!}e^{\theta x}$. Let $u_n(x) = \frac{x^n}{n!}e^{\theta x}, \Rightarrow \lim\limits_{x\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim\limits_{x\to\infty}\frac{|x|}{n+1} = 0$

$\Rightarrow \lim\limits_{x\to\infty}|R_n(n)| = 0$

$\therefore\ e^x = 1 + x + \frac{x^2}{2!} + \cdots\ \forall\ x \in \mathbb{R}.$

(ii) $f(x) = \sin x, x \in \mathbb{R}$. Then $f^n(x) = \sin\left(\frac{n\pi}{2} + x\right)$,

$\therefore f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots\ldots\ldots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + R_n(x), (n \neq 0)$

Where $R_n(x) = \frac{x^n}{n!}f^n(\theta x) = \frac{x^n}{n!}\sin\left(\frac{n\pi}{2} + \theta x\right), 0 < \theta < 1$

$\lim\limits_{n\to\infty}|R_n(x)| = \lim\limits_{n\to\infty}\frac{|x|^n}{n!}\left|\sin\left(\frac{n\pi}{2} + \theta x\right)\right| \leq \lim\limits_{n\to\infty}\frac{|x|^n}{n!} = 0\ \left(\because \frac{u_{n+1}}{u_n} = \frac{|x|}{n+1}\right)$

$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\ldots, \forall\ x \in \mathbb{R}.$

(iii) $f(x) = \log(1+x)\,, x > -1, f^n(x) = \frac{(-1)^{n-1}(n-1)^1}{(1+x)^n}\ \ for\ \ x > -1$

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots.\ for\ x \in (-1,1)$

(iv) e is irrational :

$e^x = 1 + x + \frac{x^2}{2!} + \ldots\ldots. + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}e^\theta, 0 < \theta < 1\ [by(i)]$

$\Rightarrow e = 1 + 1 + \frac{1}{2!} + \ldots\ldots.. + \frac{1}{(n-1)!} + \frac{1}{n!}e^\theta$

$\Rightarrow e > 2$ and $0 < e^\theta < e < 3\ (\because 0 < \theta < 1)$

Let e be rational, then $\exists\ p, q \in \mathbb{Z}\ with\ \gcd(p,q) = 1\ and\ p, q > 0$ such that $e = \frac{p}{q}$

Let $n > q\ , then$

$\frac{p(n-1)!}{q} - (n-1)!\left\{1 + 1 + \frac{1}{2!} + \ldots\ldots + \frac{1}{(n-1)!}\right\} = \frac{e^\theta}{n}$

$(integer)$ $[integer]$

$\Rightarrow \frac{e^\theta}{n}$ is an integer.

But$0 < e^\theta < e < 3 < n\ \Rightarrow 0 < \frac{e^\theta}{n} < 1\ (Proper\ fraction),$

$\Rightarrow e$ is irrational $[e = 2.7182818284$ (correct upto 10 decimal places.)]

1.22. Maximum and Minimum:

1.22.1. Global maximum and global minimum:

Let I be an interval and $f : I \to \mathbb{R}$ be a function$f$is said to have a global maximum (or minimum) on $I\ if\ \exists$ a point$c \in I$ such that $f(c) \geq f(x)[respectively\ minimum)$ point for $f\ on\ I$.

f is said to have a local maximum (or minimum) at a point $c \in I$ if $\exists$ a neighbourhood $N_\delta(c)$ ofcsuch that $f(c) \geq f(x)[respectively\ \ f(c) \leq f(x)]\ \forall\ x \in N_\delta(c) \cap I$.

1.22.2. Let $f: I \to \mathbb{R}$ be such that of has a local extremum at an interior point $c \in I$. If $f'(c)$ exists then $f'(c) = 0$. Converse is not true.

1.22.3. Corollary: Let $f: I \to \mathbb{R}$ and$c \in I$, where f has local minimum. Then either $f'(c)$ does not exists or $f'(c) = 0$.

Example (1.98.):

(i) $f(x) = |x|,\ x \in \mathbb{R}$ has local minimum at $x = 0, but\ f'(0)$ does not exist.

(ii) Let $f(x) = x^3,\ x \in \mathbb{R}$. Then $f'(0) = 0\ but\ 0\ is\ not\ an\ extremum\ point.$

(iii) (interior condition of$c$ is in necessary). Let $f(x) = x, x \in [0,1]$. f has minimum at 0 and maximum at 1 but$f'(0) = 1 = f'(1) \neq 0$.

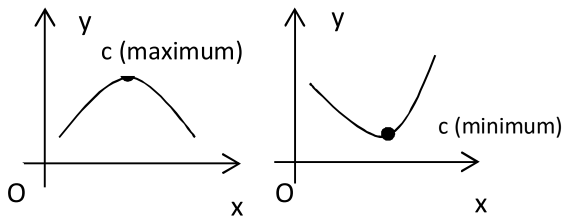
1.22.4. [First derivative Test for extreme a]

Let $f: I = [a, b] \to \mathbb{R}$ continuous and c be and interior point of I and let f be differentiable on (a, c) and (c, b). Then $-$

(i) If $\exists$ a neighbourhood $(c-\delta, c+\delta) \subset I$ such that for $x \in (c, \delta, c), f'(x) \geq 0$

$(or, f'(x) \leq 0)$ $and\ for$ $x \in (c, c+\delta)$, $f'(x) \leq 0$ $(respectively\ f'(x) \geq 0)$ the f has local maximum (respectively local minimum) at c.

(ii) If $f'(x)$ keeps the same sign on $(c-\delta, c)$ and $(c, c+\delta)$ then f has no extremum at c. Converse is not true.



Example (1.99):

let $f(x) = \begin{cases} 2x^2 + x^2 \sin\frac{1}{x}, x \neq 0 \\ \quad 0 \quad , \ x = 0 \end{cases}$

Then f has local minimum at 0.

$$f'(x) = \begin{cases} 4x + 2x\sin\frac{1}{x} - \cos\frac{1}{x}, \ \ x \neq 0 \\ \quad 0 \quad , x = 0 \end{cases}$$

f' takes both positive and negative values on both since of 0 (in any neighbourhood of 0).

1.22.5. Higher Order Derivative test for extreme:

Let $f: I \to \mathbb{R}$ and $c$be an interior point of I.

If $f'(c) = f''(c) = f'''(c) = \ldots . = f^{n-1}(c)$ and $f^n(c) \neq 0$, then f has

(i) no extremum at c if n be odd, and

(ii) a local extremum at c if n be even;

a local maximum if $f''(c) < 0$, a local minimum if $f^n(c) > 0$.

Example (2.00):

$f(x) = x^5 - 5x^4 + 5x^3 + 10$

$f'(x) = 5x^4 - 20x^3 + 15x^2 = 0 \Rightarrow x = 0, 1, 3$

$f''(x) = 20x^3 - 60x^2 + 30x$

$f'''(x) = 60x^2 - 120x$

$f^{iv}(x) = 120x$, $f^v(x) = 120$

Now, $At\ x = 0, f'(0) = 0, f''(0) = 0, f'''(0) = 0, f^{iv}(0) = 0, f^v(x) \neq 0$, so no extremum.

$At\ x = 1, f(1) = 0, f''(1) < 0, f\ has\ maximum\ at\ x = 1$

$At\ x = 3, f'(3) = 0, f''(3) > 0, f\ has\ minimum\ at\ x = 3.$

1.22.6. Indeterminate forms: Let $\lim_{x \to c} f(x) = l\ and\ \lim_{x \to c} g(x) = m \neq 0,$ then –

$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{l}{m}$. But if $l = m = 0,$ in this case the limit of quotient $\frac{f}{g}$ is said to take the indeterminate form $\frac{0}{0}$.

Note: Other indeterminate forms are $\frac{\infty}{\infty},\ \infty - \infty,\ 0 \cdot \infty,\ 0^0, 1^\infty, 1^{-\infty},\ \infty^0$

1.22.7. Let $c \in \mathbb{R}\ and\ f, g : \mathbb{R} \to \mathbb{R}$ be two functions such that

$f(c) = g(c) = 0\ and\ g(x) \neq 0$ in some deleted neighbourhood $N'_\delta(c) of\ c\ and\ f, g$ are differentiable at c and $g'(c) \neq 0$. Then $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f\prime(c)}{g\prime(c)}$.

1.22.8. If $f, g : [a, b] \to \mathbb{R}\ and\ f(a) = g(a) = 0,\ g(x) \neq 0\ on\ (a, b)\ and\ f, g$ are differentiable at $a\ and\ g'(a) \neq 0$. Then $\lim_{n \to a+} \frac{f(x)}{g(x)} = \frac{f\prime(a)}{g\prime(a)}$

Example (2.01): $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, x \neq 0\ and\ g(x) = \sin x, x \in \mathbb{R} \\ 0 \qquad , \ x = 0 \end{cases}$

Then $f(0) = 0 = g(0),\ g(x) \neq 0$ is some deleted neighbourhood of $0\ and\ f'(0)\ and\ g'(0)$ both exist and $g'(0) = 1 \neq 0$. So, $\lim_{n \to a} \frac{f(x)}{g(x)} = \frac{f\prime(0)}{g\prime(0)} = 0$

1.22.9. L' Hospital Rule: Let $c \in \mathbb{R}\ and\ f, g : \mathbb{R} \to \mathbb{R}$ be such that $f^n(x), g^n(x)$ exist a_n some neighbourhood of $N'_\delta(c) and\ g^n(x) \neq 0$ on $N'_\delta(c)$ and

$$\lim_{x \to c} f(x) = \lim_{x \to c} f'(x) = \ldots\ldots\ldots \lim_{x \to c} f^{x-1}(x) = 0$$

$$\lim_{x \to c} g(x) = \lim_{x \to c} g'(x) = \ldots\ldots\ldots \lim_{x \to c} g^{x-1}(x) = 0$$

Then if $\lim_{x \to c} \frac{f^n(x)}{g\prime(x)}$ exists in $\mathbb{R}$, then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f^n(x)}{g^n(x)}$.

Example (2.01):

$$\lim_{x \to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x \sin x} \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x} - \frac{2}{1+x}}{x \cos x + \sin x} \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x} + \frac{2}{(1+x)^2}}{-x \sin x + 2\cos x} \quad = 1$$

1.23. Functions of Bounded Variation:

Definition: Let $[a,b]$be a closed and bounded interval and$f:[a,b]\to\mathbb{R}$be a function. Let $P=\{x_0,x_1,\dots\dots\dots,x_n\}\,where\ a=x_0<x_1<x_2<\dots\dots\dots<x_n=b,$ be a partition of $[a,b]$. Let us consider the sum

$$V(P,f)=\ |f(x_1)-f(x_0)|+\ |f(x_2)-f(x_1)|+\dots\ \dots\dots+|f(x_n)-f(x_{n-1})|$$

$=\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|$

For different partitions $P\in\wp[a,b],V(P,f)$ given a set of non-negative numbers. If the set $\{V(P,f):P\in\wp[a,b]\}$ be bounded above, then f is said to be a function of bounded variation on $[a,b]$.

The supremum of the set $\{V(P,f):P\in\wp[a,b]\}$ is said to be the total variation of $f\ \ on\,[a,b]$ and is denoted by $V_f[a,b]$.

Example (2.02):

(i) Let$k\in\mathbb{R}\,,f(x)=k\ \forall\,x\in[a,b]\ \Rightarrow V(P,f)=0\ \forall\ P\in\wp[a,b]\ \Rightarrow\ V_f[a,b]=0\ \Rightarrow f$ is a function of bounded variation on $[a,b]$.

(ii) $f(x)=x,x\in[a,b]\ \Rightarrow\ V_f[a,b]=b-a<\infty$

(iii) $f(x)=\ \sin x\,,x\in[a,b],\ V_f[a,b]\le(b-a)\ \ (\because\ \ |\sin x_2-\sin x_1|\le|x_2-x_1|)$

(iv) **Not a Function of bounded variation:**

Let $f:[0,1]\to\mathbb{R}$ be defined by $f(x)=\begin{cases}1\ \ if\ x\in\mathbb{Q}\\0\ if\ x\in\mathbb{R}|\mathbb{Q}\end{cases}$

Let $P=\{x_0,x_1,\dots\dots\dots,x_{2n}\}$ be a partition of $[a,b]$ such that $x_0,x_2,\dots\dots\dots,x_{2n}$ are all rational and $x_0,x_3,\dots\dots\dots,x_{2n-1}$ are all irrational. Then

$$V(P,f)=\ |f(x_1)-f(x_0)|+\dots\ \dots\dots+|f(x_{2n})-f(x_{2n-1})|=2n\to\infty\ as\ x\to\infty$$

1.23.1. Let$f:[a,b]\to\mathbb{R}$ be a function of bounded variation on $[a,b]$. Then f is bounded on $[a,b]$. converse is not true.

Example (2.03):

(i) $f(x)=\begin{cases}1\ ,x\in\mathbb{Q}\quad ,x\in[0,1]\\ \qquad 0\,,x\in\mathbb{R}|\mathbb{Q}\end{cases}$

(ii) $f:[0,1]\to\mathbb{R}$ defined by$f(x)=\begin{cases}x\ \cos\frac{\pi}{2k},x\neq 0\\ \ \ 0\ ,\ x=0\end{cases}$

Then $|f(x)|\ \le 1\ \ \forall\,x\in[0,1]$

Let $P=\{0,\frac{1}{2n},\frac{1}{2n-1},\dots\dots\dots\frac{.1}{2},1\}$ be a partition of $[0,1]$

Then $f\left(\frac{1}{2r}\right)=\frac{1}{2r}\cos\left(\frac{r\pi}{2}\right)=\frac{1}{2r}(-1)^r$ for $r=1,2,\dots\dots\dots,n$

and $f\left(\frac{1}{2r-1}\right)=\frac{1}{2r-1}\cos\frac{(2r-1)\pi}{2}=0\ for\ r=1,2,\dots\dots,n$

$\therefore V(p,f) = \left|f\left(\frac{1}{2n}\right) - f(0)\right| + \ldots\ldots.+ \left|f(1) - f(\frac{1}{2})\right|$

$= \frac{1}{2n} + \frac{1}{2n} + \frac{1}{2n-2} + \frac{1}{2n-2} + \ldots\ldots + \frac{1}{2} + \frac{1}{2} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n} \to \infty \; as \; x \to \infty$

1.23.2. Let $f : [a,b] \to \mathbb{R}$ be monotone on $[a,b]$. Then f is a function of bounded variation on $[a,b]$. Converse is not true.

Example (2.04): $f(x) = \sin x, x \in [a,b]$.

1.23.3. Let $f : [a,b] \to \mathbb{R}$ be a Lipschitz function on $[a,b]$. Then f is a function of bounded variation on $[a,b]$. Converse is not true.

Example (2.05): $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = \sqrt{x},\; x \in [0,1]$.

$\Rightarrow f$ is monotone increasing on $[0,1]$ $\Rightarrow f$ is a function of bounded variation on $[0,1]$ but f is not Lipschitz function on $[0,1]$. If since, for $x_1 = 0$, there is no $M \in \mathbb{R}$ such that

$$|f(x_2) - f(x_1)| \leq M|x_2 - x_1| \forall\, x_2[0,1].$$

1.23.4. Let $f : [a,b] \to \mathbb{R}$ continuous on $[a,b]$, f' exists and be bounded on (a,b).Then f is a function of bounded variation on $[a,b]$.

Note-I : Boundedness of f' on $[a,b]$ is not necessary.

Example (2.06): $f(x) = \sqrt{x}, x \in [0,1]$ is a function of bounded variation on $[0,1]$ as it is monotonic increasing but $f'(x) = \frac{1}{2\sqrt{x}}, x \in (0,1]$ is not abounded on $(0,1)$.

Note-II: A function f continuous and bounded on a closed interval $[a,b]$ may not be a function of bounded variation on $[a,b]$

Example (2.07): $f(x) = \begin{cases} x\,\cos\frac{\pi}{2x}, & x \in (0,1) \\ 0, & x = 0 \end{cases}$

1.23.5. Let $f, g : [a,b] \to \mathbb{R}$ be functions of bounded variation on $[a,b]$. Then–

(i) $f + g$ is also so and $V_{f+g} \leq V_f + V_g$

(ii) $f - g$ is also so and $V_{f-g} \leq V_f + V_g$

(iii) cf $(c \in \mathbb{R})$ is also so.

(iv) fg is also so and $V_{fg} \leq A\, V_f + B\, V_g$, $A = \sup\{|g(x)| = x \in [a,b]\}$,

$$B = \sup\{|f(x)| : x \in [a,b]\}$$

(**Note:** The close S of all $BV - functions$ on $[a,b]$ form a real vector space)

(v) If $\exists$ a $k \in \mathbb{R}$ such that $0 < k \leq f(x) \;\forall\; x \in [a,b]$, then $\frac{1}{f}$ is a $BV - function$ on $[a,b]$ and $V_{\frac{1}{f}} \leq \frac{v_f}{k^2}$

(vi) $|f|$ is also so.

1.23.6. Definition (Refinement of partition): Let $P = \{x_0, x_1, \ldots\ldots\ldots, x_n\}$ be a partition of $[a,b]$. A partition $\mathbb{Q}$ of $[a,b]$ is said to be a refinement of $P.P$ is a proper subset of $\mathbb{Q}$.

Example (2.08): $P = \{0, 1.4, \frac{1}{2}, \frac{3}{4}, 1\}$ is a partition of $[a,1] and$ $\mathbb{Q} = \left\{0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{3}{4}, 1\right\}$ then $Q\ is\ a\ refinement\ of\ P.$

1.23.7. Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation a $[a,b] and\ P$ be a partition of $[a,b]$. If Q be a refinement of P then $V(Q,f) \geq V(P,f)$

1.23.8. Let $f : [a,b] \to \mathbb{R}$ be a function on $[a,b]$ $and\ c \in (a,b)$ then –

(i) f is bounded variation on $[a,c] and\ on\ [c,b]$

(ii) $V_f[a,b] = V_f[a,c] + V_f\,[c,b]$

1.23.9. Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,c]\ and\ on\ [c,b]\ where\ c \in (a,b). Then -$

(i) f is of bounded variation on $[a,b]$

(ii) $V_f[a,c] + V_f[c,b] = V_f[a,b]$

Example (2.09): Let $f: [0,3] \to \mathbb{R}$ be defined by $f(x) = x^2 - 4x + 3,\ x \in [0,3]$.

$f'(x) = 2x - 4$. So $f'(x) < 0\ for\ x \in [0,2]\ and\ f'(x) > 0\ for\ x \in [2,3] \Rightarrow f$ is decreasing on $[0,2]$and increasing on $[2,3] \Rightarrow f$ is a $BV - function\ on\ [0,]$.

$V_f[0,2] = f(0) - f(2) = 4\ and\ V_f[2,3] = f(3) - f(2) = 1$

$\therefore\ V_f[0,3] = V_f[0,2] + V_f[2,3] = 5$

1.23.10. Let $f : [a,b] \to \mathbb{R}$ be a $BV - function$ on $[a,b]$ and $\phi : [a,b] \to \mathbb{R}$ be such that ϕ is bounded on $[a,b]$ and $\phi(x) = f(x)$ except at a finite number of points in $[a,b]$, then ϕ is a $BV - function\ in\ [a,b]$.

Example (2.10): Let $f: [0,3] \to \mathbb{R}$ be defined by $f(x) = x - [x], x \in [1,3]$

$$f(x) = \begin{cases} x - 1, & 1 \leq x < 2 \\ x - 2, & 2 \leq x < 3 \\ 0, & x = 3 \end{cases}$$

Let $\phi_1: [1,2] \to \mathbb{R}$ be defined by $\phi_1(x) = x - 1,\ x \in [1,2]$

$\phi_2 : [2,3] \to \mathbb{R}$ be defined by $\phi_2(x) = x - 2,\ x \in [2,3]\ S$

Then ϕ_1 is increasing on $[1,2]\ and\ \phi_2$ is function of bounded variation on $[2,3]$.

Hence $f(x) = \phi_1(x) + \phi_2(x)\ \ x \in [1,3]\ except\ x = 2,3.$ Hence $f(x)$ is a function of bounded variation on $[1,3]$.

1.23.11. Definition (Variation Function): Let $f : [a,b] \to \mathbb{R}$ be function of bounded variation on $[a,b]\ and\ x \in (a,b]$. Then $V_f\,[a,x]$ is a function of $x\ \forall\ x \in [a,b]$. Let $V : [a,b] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} V_f[a,b], a < x \leq b \\ 0 \quad , \ x = a \end{cases}$$

V is called the variation function of f on $[a,b]$

Note: (i) V is monotone increasing on $[a,b]$

(ii) $V + f$ and $V - f$ are also monotone increasing on $[a,b]$,

1.23.12. Let $f : [a,b] \to \mathbb{R}$ be a function. Then f is a function of bounded variation on $[a,b] \Leftrightarrow f$ can be expressed as the difference of two monotone increasing functions on $[a,b]$.

Example (2.11): Let $f : [-1,1] \to \mathbb{R}$ be defined by $f(x) = x^2, x \in [-1,1]$.

Then $f'(x) = 2x$ *and so* $f'(x) < 0, x \in [-1,0]$ *and* $f'(x) > 0, x \in \{0,1]$ $\Rightarrow f$ decreasing on $[-1,0]$ *and* increasing on $[0,1]$ $\Rightarrow f$ *is BV* $-$ *function on* $[-1,0]$ *and* $[0,1]$ hence on $[-1,1]$.

$V(-1) = 0$

If $-1 < x \leq 0$, *then* $V(x) = V_f[-1, x] = f(-1) - f(x) = 1 - x^2$. Since f is decreasing on $(-1,0)$.

1.23.12. If $0 < x \leq 1$, *then* $V(x) = V[-1,x] = V_f[-1,0] + V_f[0,x]$

$= f(-1) - f(0) + f(x) - f(0)$, *since f is increasing on* $[0,1]$

$= 1 + x^2$

Therefore, $V(x) = \begin{cases} 1 - x^2, & -1 \leq x \leq 0 \\ 1 + x^2, & 0 < x \leq 1 \end{cases}$ and $V(x)$ is increasing on $[-1,1]$.

$(V + f)(x) = \begin{cases} 1 \ , & -1 \leq x \leq 0 \\ 1 + 2x^2, & 0 < x \leq 1 \end{cases}$ $\Rightarrow V + f$ is a monotone increasing on $[-1,1]$.

$\therefore f = (V + f) - V$, the difference of two monotone increasing functions.

1.23.13. Let $f : [a,b] \to \mathbb{R}$ be a $BV - function$ on $[a,b]$ then f can have only discontinuity of first kind and the points of discontinuity of f form a countable set.

1.23.14. Let $f: [a,b] \to \mathbb{R}$ be a $BV - function\ on\ [a,b]$ and let V be the variation function on $[a,b]$. If f be continuous at a point $c \in [a,b]$ then V is continuous at c and conversely.

1.23.15. Corollary: If $f : [a,b] \to \mathbb{R}$ be continuous and be of bounded variation on $[a,b]$ then f can be expressed as the difference of two monotone and continuous functions on $[a,b]$ and conversely.

1.23.16. Definition (Positive Variation and Negative Variation): Let $f : [a,b] \to \mathbb{R}$ be a $BV - function$ on $[a,b]$ *and* $P = \{x_0, x_1, \ldots\ldots, x_n\}$ be a partition of $[a,b]$.

$V(P,t) = |\Delta f_1| + \ldots\ldots + |\Delta f_n|$ *where* $\Delta f_r = f(x_r) - f(x_{r-1}), r = 1,2,\ldots\ldots,n$

Let $V_+(P,f) = \sum_{\Delta f_i > 0} |\Delta f_i|$ *and* $V_-(P,f) = \sum_{\Delta f_i < 0} |\Delta f_i|$ Then$-$

$V_+(P,f) - V_-(P,f) = f(b) - f(a)$

$V_+(P,f) + V_-(P,f) = V(P,f)$

and $\sup_p \{V_+(P,f) : P \in p[a,b]\} = P_f[a,b]$ *or* $(V_+)_f[a,b]$ is called positive variation of f on $[a,b]$ and $\sup_p \{V_-(P,f) : P \in p[a,b]\} = n_f[a,b]$ *or* $(V_-)_f[a,b]$ is called negative variation of f on $[a,b]$.

Positive variation function V_+ or $p(x)$

$$p(x) = V_+(x) = \begin{cases} P_f[a,x], & x \in [a,b] \\ 0, & x = 0 \end{cases}$$

Negative variation function V_- or $n(x)$:

$$n(x) = V_-(x) = \begin{cases} n_f[a,x], & x \in [a,b] \\ 0, & x = 0 \end{cases}$$

Note: $p(x)$ *and* $n(x)$ are monotone increasing on $[a,b]$.

1.23.17. Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$. Then –

(i) $p(x) + n(x) = V(x) \ \forall \ x \in [a,b]$

(ii) $p(x) - n(x) = f(x) - f(a) \ \forall \ x \in [a,b]$.

$\Rightarrow p(x) = \frac{1}{2}[V(x) + f(x) - f(a)$

$n(x) = \frac{1}{2}[V(x) - f(x) + f(a)$

Example (2.12): Let $f : [-1,1] \to \mathbb{R}$ be defined by $f(x) = x^2, \ \forall \ x \in [-1,1]$.

Then f is $BV - function$ on $[-1,1]$ and $V(x) = \begin{cases} 1 - x^2, & -1 \le x \le 0 \\ 1 + x^2, & 0 < x \le 1 \end{cases}$

$$\therefore p(x) = \begin{cases} \frac{1}{2}[1 - x^2 + x^2 - 1] = 0, & -1 \le x \le 0 \\ \frac{1}{2}[1 + x^2 + x^2 - 1] = x^2, & 0 < x \le 1 \end{cases}$$

$$\therefore n(x) = \begin{cases} \frac{1}{2}[1 - x^2 - x^2 + 1] = 1 - x^2, & -1 \le x \le 0 \\ \frac{1}{2}[1 + x^2 - x^2 + 1] = 1, & 0 < x \le 1 \end{cases}$$

1.24 Riemann Integral :

Let $[a,b]$ be a closed bounded interval and $f : [a,b] \to \mathbb{R}$ be a bounded function on $[a,b]$. Let $P = \{x_1, x_2 \ldots\ldots x_n\}$ be a partition of $[a,b]$. Then f is bounded on each $Ir = [x_{r-1}, x_r]$ for $r = 1, 2, \ldots\ldots n$

Let $M_r = \sup_{x \in Ir} f(x), m_r = \inf_{x \in Ir} f(x), \ M = \sup_{x \in [a,b]} f(x), \ m = \inf_{x \in [a,b]} f(x)$

Then $m \le m_r \le M_r \le M$ for $r = 1, 2, \ldots\ldots n$ …….. (i)

$U(P,f) = \sum_{r=1}^{n} M_r(x_r - x_{r-1})$ = Upper Darbou x $sum\ of\ f\ corresponding\ to\ P$......(ii)

$L(P,f) = \sum_{r-1}^{n} m_r(x_r - x_{r-1})$ = Lower Darbou x $sum\ of\ f\ corresponding\ to\ P$.

Now,

(i) $\Rightarrow\ \ m(x_r - x_{r-1}) \le m_r(x_r - x_{r-1}) \le M_r(x_r - x_{r-1}) \le M(x_r - x_{r-1})$

$\Rightarrow\ \ m\sum_{r-1}^{n}(x_r - x_{r-1}) \le \ \sum_{r-1}^{n} m_r(x_r - x_{r-1}) \le \sum_{r-1}^{n} M_r(x_r - x_{r-1}) \le \sum_{r-1}^{n} M\ (x_r - x_{r-1})$

$\Rightarrow\ \ m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)\ldots\ldots\ldots(b)$ (ii)

If $sup\{L(P,f) : P \in \wp[a,b]$ exists, it is called the lower integral of f on $[a,b]$ and is denoted by $\int_{\underline{a}}^{b} f dx = \int_{\underline{a}}^{b} f$

And if $inf\{U(P,f)\ :\ P \in \wp[a,b]\}$ exists, it is called the upper integral of f on $[a,b]$ and is denoted by $\int_{a}^{\overline{b}} f dx = \int_{a}^{\overline{b}} f$.

f is said to be Riemann integrable on $[a,b]$ if $\int_{\underline{a}}^{b} f = \int_{a}^{\overline{b}} f$ and the common value $\int_{\underline{a}}^{b} f$ or $\int_{a}^{\overline{b}} f$ is called the Reimann integral of f on $[a,b]$ and is denoted by $\int_{a}^{b} f(x)dx$ or $\int_{a}^{b} f$

We also define $\int_{a}^{a} f = 0$ and $\int_{a}^{b} f = -\int_{b}^{a} f$

Note-1: $m(b-a) \le \int_{\underline{a}}^{b} f \le M(b-a),\ \ m(b-a) \le \int_{a}^{\overline{b}} f \le M(b-a)$

Note-2: The class of all Riemann integrable function on $[a,b]$ is denoted by $R[a,b]$ and $R[a,b]\ \subset\ B[a,b]$. The class of functions of bounded variation on $[a,b]$.

Example (2.13):

(i) Let $f: [a,b] \to \mathbb{R}$ be defined by $f(x) = c,\ \ x \in [a,b]$
Take$p = \{x_0,\ x_1, \ldots\ldots x_n\}$be a partition of $[a,b]$. Then $Mr = c = mr$
$\Rightarrow U(P,f) = c(x_1 - x_0) + c(x_2 - x_1) + \ldots\ldots + c(x_n - x_{n-1}) = c(b-a)$
$L(P,f) = c(b-a)$
$\Rightarrow inf\{U(P,f)\ :\ P \in P[a,b]\} = (b-a) = sup\{L(P,f) : P \in P[a,b]\}$
$\Rightarrow f$ is Riemann integrable on $[a,b]$ and $\int_{a}^{b} f(x)dx = c(b-a)$.

(ii) Let $f: [0,1] \to \mathbb{R}$ be define by $f(x) = \begin{cases} 1 & , \quad x \in \mathbb{Q} \\ 0 & , \quad x \in \mathbb{R}|\mathbb{Q} \end{cases}$
Let $P = \{x_0,\ x_1, \ldots\ldots\ldots x_n\}$ be a partition of $[0,1]$ Then $M_r = 1, m_r = 0$
$\therefore U(P,f) = \ \sum_{r=1}^{n} M_r(x_r - x_{r-1}) = 1(1-0) = 1\ \ L(P,f)\ =\ 0$
$\therefore inf\{U(P,f)\ : P \in \wp[a,b]\ \} = 1 \ne 0 = \sup\{L(P,f) : P \in \wp[a,b]\}$
Hence f is not Riemann integrable on $[0,1]$.

1.24.1 Let $f: [a,b] \to \mathbb{R}$ be bounded on $[a,b]$ and P be a partition of $[a,b]$.

If Q be a refinement of P, then

$U(P,f) \ge U(Q,f)$ and $L(P,f) \le L(Q,f)$

$\Rightarrow L(P,f) \leq L(Q,f) \leq U(Q,f) \leq U(P,f)$.

1.24.2Definition (Noun of a Partition) : Let $[a,b]$ be a closed and bounded interval and $P = \{x_0, x_1, \ldots\ldots x_n\}$ be a partition of $[a,b]$. The norm of P is denoted by $\|P\|$ and in defined by $\|P\| = \max\{(x_1 - x_0), (x_2 - x_1), \ldots\ldots, (x_n - x_{n-1})\}$

Note: If Q be a refinement of P. Then $\|Q\| \leq \|P\|$

1.24.3 Let $f: [a,b] \to \mathbb{R}$ be bounded on $[a,b]$ and P a partition of $[a,b]$ with $\|P\| = \delta$ if P_k be a refinement of P with k additional Point of Partition, then

$$0 \leq U(P,f) - U(P_k,f) \leq (M-m)k\delta,$$

$0 \leq L(P_k,f) - L(P,f) \leq (M-m)k\delta$

1.24.4 Let $f: [a,b] \to \mathbb{R}$ be bounded on $[a,b]$ and $p, \mathbb{Q}$ be any two partitions of $[a,b]$. Then $L(P,f) \leq U(Q,f); L(Q,f) \leq U(P,f)$

$$\Rightarrow \int_{\underline{a}}^{b} f \leq \int_{a}^{\overline{b}} f \Rightarrow m(b-a) \leq \int_{\underline{a}}^{b} f \leq \int_{a}^{\overline{b}} f \leq M(b-a)$$

Example (2.14):

Let $f : [a,b]$ be defined by $f(x) = x, x \in [a,b]$ consider $P_n = \{a, a+h, a+2h, \ldots\ldots, a+nh\}$ be a partition of$[a,b]$ here $h = \frac{b-a}{n}$

$\therefore M_r = \sup_{x\in[a+(r-1)h,\ a+rh]} f(x) = a + rh,\ m_r = \sup_{x\in[a+(r-1)h,\ a+rh]} f(x) = a + (r-1)h$

$\therefore U(P_n,f) = h[(a+h) + (a+2h) + \ldots\ldots. + (a+2h)]$

$= h\,[na + h(1 + 2 + \ldots\ldots + n)]$

$= nha + \frac{nh(nh+a)}{2}$

$= a(b-a) + \frac{(b-a)}{2}(b - a + \frac{b-a}{n})$

$= ab - a^2 + \frac{1}{2}(b-a)^2\left[1 + \frac{1}{n}\right] \to ab - a^2 + \frac{1}{2}(b-a)^2 = \frac{b^2-a^2}{2}$asn $\to \alpha$

$L(P_n,f) = h[a + (a+h) + \ldots\ldots + a + (n-1)h$

$= h[na + h(1 + 2 + \ldots.. + (n-1)]$

$= nah + h\frac{(n-1)nh}{2}$

$= a(b-a) + \frac{(b-a)}{2}\left(b - a - \frac{b-a}{n}\right) \to ab - a^2 + \frac{1}{2}(b-a)^2 = \frac{r-a^2}{2}$ as $n \to \alpha$

$\therefore \int_a^b f = \int_{\underline{a}}^b f = \int_a^{\overline{b}} f = \frac{b^2-a^2}{2}$

1.24.5Condition for integrability: Let $f: [a,b] \to \mathbb{R}$ be bounded on $[a,b]$. Then f is integrable on $[a,b] \Leftrightarrow$ for each $\varepsilon > 0, \exists$ a partition P of $[a,b]$ such that

$$U(P,f) - L(P,f) < \varepsilon$$

1.24.6 Darboux Theorem: Let $[a,b]$ be a closed and bounded interval and $f: [a,b] \to \mathbb{R}$ be bounded on $[a,b]$. Then-

To each pre-assigned $\varepsilon > 0 \quad \exists \quad \delta > 0$ such that

$U(P,f) < \overline{\int_a^b} f + \varepsilon \quad \forall P$ of $[a,b]$ with $\|P\| \le \delta$ and

$L(P,f) > \underline{\int_a^b} f - \varepsilon \quad \forall P$ of $[a,b]$ with $\|P\| \le \delta$

1.24.7 Let $f:[a,b] \to \mathbb{R}$ be monotone on $[a,b]$. If $\{P_n\}$ be a sequence of partitions of $[a,b]$ such that the sequence $\{\|P_n\|\}$ converge to 0, then –

(i) $\lim_{n\to\infty} U(P_n, f) = \overline{\int_a^b} f$ and
(ii) $\lim_{n\to\infty} L(P_n, f) = \underline{\int_a^b} f$

1.24.8 Some Riemannintegrable functions:

(i) Let $f : [a,b] \to \mathbb{R}$ be monotone on $[a,b]$. Then f is integrableon $[a,b]$.
(ii) Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. Then f is integrable on $[a,b]$.
(Note: $C[a,b]$ denote the class of all continuous function on $[a,b]$ and $C[a,b] \subset R[a,b]$)
(iii) Let $f : [a,b] \to \mathbb{R}$ be bounded on $[a,b]$ and let f be continuous on $[a,b]$ except for a finite number points is $[a,b]$. Then f is integrable on $[a,b]$.
$\Rightarrow$ If $f : [a,b] \to \mathbb{R}$ be piecewise continuous on $[a,b]$ then f is integrable on $[a,b]$.
(iv) Let $f : [a,b] \to \mathbb{R}$ be bounded on $[a,b]$ and let f be continuous on $[a,b]$ except on a infinite Subset $S \subset [a,b]$ such that the number of limit points of S is finite. Then f is integrable on $[a,b]$.

Example (2.15):

a) $f : [0,1] \to \mathbb{R}, \quad f(x) = \begin{cases} 0 & , \quad x = 0 \\ (-1)^{r-1} & , \quad \frac{1}{r+1} < x \le \frac{1}{r} \quad , \quad r = 1,2,3,\ldots. \end{cases}$

F is continuous on $[0,1]$ except at the points $0, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\ldots$ Then set of points of discontinuity of f has only the limit point 0 and f is bounded on $[0,1] \Rightarrow f \in R\,[0,1]$

b) **Converse of (iv) is not true :**

Example (2.16.):

$$f:[0,1] \to \mathbb{R}, \ f(x) = \begin{cases} 0 & , \quad x = 0 \\ 0 & , \quad x \in \mathbb{R}|\mathbb{Q} \\ \frac{1}{q} & , \quad x = \frac{p}{q} \quad , \quad p,q > 0 \ with \ gcd(p,q) = 1 \end{cases}$$

f is bounded on $[0,1]$ and f is continuous at 0 and every irrational number and discontinuous at non-zero rational number is $[0,1]$ so, the set S of points of discontinuity have infinite number of limit point. But f is Riemann integrable on $[0,1]$.

1.24.9. Lebesgue: A necessary and sufficient condition for a bounded function on $[a,b]$ to be Riemann integrable on $[a,b]$ is that the points of discontinuity of f is a set of measure zero.

1.24.10 Definition (Set of Measure Zero): A set $S \subset \mathbb{R}$ is said to the a set of measure zero if for each $\varepsilon > 0$ there is a countable collection of open intervals $\{I_n\}$ such that $S \subseteq \bigcup_{n=1}^{\infty} I_n \; and \; \sum_{n=1}^{\infty} |I_n| < \varepsilon$

Example (2.17):

(a) A finite set $S \subseteq \mathbb{R}$ is a set of measure zero.

[**Hint**: $I_r = \left(x_r - \frac{\varepsilon}{2(m+1)}, x_r + \frac{\varepsilon}{2(m+1)}\right) \; for \; r = 1,2, \dots\dots\dots, m.$]

(b) An enumerable subset $S \; of \; \mathbb{R}$ is a set of measure zero

[**Hint**: $I_r = \left(x_r - \frac{\varepsilon}{2^{r+2}}, x_r + \frac{\varepsilon}{2^{r+2}}\right)$]

$\Rightarrow \mathbb{Q}$ is a set of meausre zero.

(c) Let S be a bounded infinite subset of $\mathbb{R}$ having finite (countable) number of limit points. Then S is a set of measure zero.

[Hint: Let $x_1, x_2, \dots\dots\dots, x_m$ be the limit points of S condition $I_r = \left(x_r - \frac{\delta_r}{2}, x_r + \frac{\delta_r}{2}\right)$ open interval containing x_r and let $\delta_1 + \delta_2 + \dots\dots + \delta_m < \frac{\varepsilon}{2}$. Then there are finite number of points out side $\bigcup_{r=1}^{m} I_r$. So we can cover these points by open interval whose sum of length is $< \frac{\varepsilon}{2}$.]

1.25. Properties of Riemann Integrable Function :

1.25.1. Let $f, g : [a,b] \to \mathbb{R}$ be two Riemann integrable functions on $[a,b]$. Then –

(i) $f + g \in R\,[a,b] \; and \; \int_a^b f + g = \int_a^b f + \int_a^b g$

(ii) $cf \in R\,[a,b] \; and \; \int_a^b cf = c\int_a^b f \quad, c \in \mathbb{R}$

(iii) $|f| \in R[a,b]$, but converse is not true.

[**Example (2.18):** $f : [a,b] \to \mathbb{R}, \; f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [a,b] \\ -1, & x \in (\mathbb{R}|\mathbb{Q}) \cap [a,b] \end{cases}, |f(x)| = 1, x \in [a,b] \; but \; f \notin R\,[a,b]$]

(iv) $f^2 \in R\,[a,b]$

(v) $fg \in R\,[a,b] \left(\because fg = \frac{1}{2}(f+g)^2 - \frac{1}{2}f^2 - \frac{1}{2}g^2\right)$

(vi) $\frac{1}{f} \in R\,[a,b] \; provide \; f(x) \geq k > 0 \;\; \forall\, x \in [a,b]$.

($\boldsymbol{Note}$: $f(x) > 0 \;\; \forall \; x \in [a,b], then \; f(x) \; may \; not \; belong \; to \; R\,[a,b]$.

Example(2.19) : $f : [0,1] \to \mathbb{R}, f(x) = \begin{cases} x, & 0 < x \leq 1 \\ 1, & x = 0 \end{cases} \quad Then$

$f \in R\,[0,1] \; as \; it \; is \; continuous \; on \; [0,1] \; except \; x = 0.$

$But \frac{1}{f}$ $is\ not\ bounded\ on$ $[0,1] \Rightarrow \frac{1}{f} \notin R\ [0,1])$

(vii) If $c \in (a,b),\ then\ f \in R\ [a,b] and\ f \in R\ [c,b]\ and\ \int_a^b f = \int_a^c f + \int_c^b f$ converse is also true i.e., if $f \in R\ [c,b]\ ,\ then\ f \in R\ [a,b]\ and\ \int_a^c f + \int_c^b f = \int_a^b f$

1.25.2. Let $I = [a,b] \subset \mathbb{R}\ and\ f : I \to \mathbb{R}$be integrable on$I\ and\ J = [c,d] \subset \mathbb{R}$ such that $f\ (I) \subset J\ and\ \phi : [c,d] \to \mathbb{R}$ be continuous on$[c,d]$. Then the composition function $\phi \circ f \in R\ [a,b]$.

Note: Continuity of ϕ is necessary.

Example (2.20):$f : [0,1] \to \mathbb{R}\ ,\ f(x) = \begin{cases} 0\ , & x \in \ \mathbb{R} |\ \mathbb{Q} \\ \frac{1}{n}\ , & x = \frac{m}{n}\ ,\gcd(m,n),\ \ m,n \in \ \mathbb{Z}^* \end{cases}$

$\phi : [0,1] \to \mathbb{R},\ \phi(x) = \begin{cases} 1, & if\ x \neq 0 \\ 0, & if\ x = 0 \end{cases}$ Then -

$\phi \circ f : [0,1] \to \mathbb{R}\ ,\ (\phi \circ f)(x) = \begin{cases} 0, & x \in \mathbb{R}|\mathbb{Q} \\ 1, & x\ \in \mathbb{Q} \end{cases} \ \Rightarrow\ Q \circ f \notin\ R\ [0,1]$

1.25.3. Let $f,\phi : [a,b] \to \mathbb{R}$be both bounded on $[a,b]\ and\ f(x) = \phi\ (x)$ except for a finite number of points in $[a,b]$. Iff be integrable on $[a,b]$then$\phi \in R\ [a,b]\ and\ \ \int_a^b f = \int_a^b \phi$.

Note: If $f(x) = \phi(x)$ enumerable number of points, thenϕmay not belong to$R\ [a,b]$.

Example (2.21): $f,\phi : [0,1] \to \mathbb{R}$ be defined by $f(x) = 1, x \in [0,1]\ \Rightarrow f \in R\ [0,1]$.

$\phi\ (x) = \begin{cases} 0,\ x \in [0,1] \cap \mathbb{Q} \\ 1,\ x \in [0,1] \cap (\mathbb{R}|\mathbb{Q}) \end{cases} \Rightarrow \phi\ (x) \neq f(x), x \in [0,1] \cap \mathbb{Q}$

$\phi \notin R\ [0,1]$

1.25.4. Definition (Piecewise Continuous Function): A function $f\colon [a,b] \to \mathbb{R}$ is said to be a piecewise continuous function on $[a,b]\ if\ \exists$ a partition $P = \{x_0, x_1, \ldots\ldots\ldots, x_n\}\ of\ [a,b]$ such that f is continuous on the open interval $(x_{k-1}, x_k)\ for\ 1\ \leq k \leq n$ and each of $f(a+0), f(b-0),\ f(x_k+0),\ f(x_k-0)$ exist for $1\ \leq k \leq n-1$. Clearly, a piecewise continuous function on $[a,b]$ is continuous on $[a,b]$ except for a finite number of points of jump discontinuity.

Example (2.22): A step function on $[a,b]$

1.25.5. Let $f : [a,b] \to \mathbb{R}$ be bounded on $[a,b]$ and for every $c \in (a,b), f \in R\ [c,b]$. Then $f \in R\ [a,b]$.

[$\boldsymbol{Hint}\colon let\ M = \sup_{x \in [a,b]} f(x),\ m = \inf_{x \in [a,b]} f(x)\ and \{c_n\}\ such\ that c_n \to a\ \ as\ n \to \infty.$ Then $\varepsilon > 0\ \exists\ \ k \in \mathbb{N}\ such\ that\ |c_n - a| < \frac{\varepsilon}{2(M-m)}\ \ \forall\ n \geq k\ \ \Rightarrow\ |c_k - a| < \frac{\varepsilon}{2(M-m)}\ \ and$

$f \in R\ [c_k, b] \Rightarrow \exists\ \ partition\ Q\ of\ [c_k, b]\ such\ that\ U(Q_k, f) - L(Q_k, f) < \frac{\varepsilon}{2}.\ LetP = \{a\}.\cup \mathbb{Q}.\ Then\ \ U(P,f) - L(P,f) < (M,m)(c_k - a) + \big(U(Q,f) - L(Q,f)\big) < \frac{\varepsilon}{1} + \frac{\varepsilon}{2} = \varepsilon]$

1.25.6. Corollary – I: Let $f:[a,b]$ be bounded on $[a,b]$ and for every $d \in (a,b), f \in R[a,d]$. Then $f \in R[a,b]$.

1.25.7. Corollary – II: Let $f:[a,b] \to \mathbb{R}$ be bounded on $[a,b]$ and for every c, d saftisfying $a < c < d < b \;\; f \in R[c,d].\; Then\; f \in R[a,b]$.

1.25.8. Inequalities: Let $f:[a,b] \to \mathbb{R}$ be integrable on $[a,b]$. If $M\, and\, m$ be the supremum of f and infimum of f on $[a,b]$ respectively, then $m(b-a) \leq \int_a^b f \leq M(b-a)$

1.25.9. Corollary – (a): Let $f:[a,b] \to \mathbb{R}$ be ingegrable on $[a,b]$. Then $\exists\, \mu \in \mathbb{R}$ satisfying $m \leq \mu \leq M$ such that $\int_a^b f = \mu(b-a)$.

1.25.10. Corollary – (b) Let $f:[a,b] \to \mathbb{R}$ be continuous on $[a,b]$. Then $\exists\, a\, point\, c \in [a,b]$ such that $\int_a^b f = f(c)(b-a)$.

1.25.11. Let $f:[a,b] \to \mathbb{R}$ be integrable on $[a,b]\; and\; f(x) \geq 0\; \forall\, x \in [a,b]$ such that $\int_a^b f \geq 0$.

1.25.12. Let $f, g:[a,b] \to \mathbb{R}$ be both integrable on $[a,b]\; and\; f(x) \geq g(x)\; \forall\, x \in [a,b]$.Then $\int_a^b f \geq \int_a^b g$.

1.25.13. Let $f:[a,b] \to \mathbb{R}$ be integrable on $[a,b]$ and $f(x) \geq 0\; \forall\, x \in [a,b]$. Let $\exists\, c \in [a,b]$ such that f is continuous at c and $f(c) > 0, then\; \int_a^b f > 0$.

Note – (a) If f is continuous on $[a,b]\; and\; f(x) > 0\; on\; [a,b]\; then\; \int_a^b f > 0$.

Note – (b) if $f \in R\,[a,b]\; and\; f(x) > 0\; on\; [a,b] then\; also\; \int_a^b f > 0\; becuase$ $\exists$ at least a point of discontinuity $c \in [a,b]$ of f.

1.25.14. Let $f:[a,b] \to \mathbb{R}$ be integrable on $[a,b]$. Then $\left|\int_a^b f\right| \leq \int_a^b |f|$.

Examples (2.23):

(a) If f be continuous on $[a,b]$ and $f(x) \geq 0$ on $[a,b]$ and $\int_a^b f = 0$ then $f = 0\; on\; [a,b]$ identically.
[**Hint**: If $\exists\; c \in [a,b]\;\; such\; that\; f(c) > 0 \;\Rightarrow\; \int_a^b f > 0$]

(b) $\frac{\pi^2}{9} < \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{x}{\sin x}\, dx < \frac{2\pi^2}{9}$

[**Hint** :$1 \leq \frac{1}{\sin x} \leq 2\;\; , x \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right] \Rightarrow x \leq \frac{x}{\sin x} \leq 2x\; , x \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]\; and\; at\, \frac{\pi}{3},\;\; \frac{\pi}{3} < \frac{1}{\sin\left(\frac{\pi}{3}\right)} < \frac{2\pi}{3}$]

1.25.15. Let $f:[a,b] \to \mathbb{R}$ be integrable on $[a,b]$ then the function $F(x)$ defined by $F(x) = \int_a^x f(t)\, dt,\;\; x \in [a,b]$ is continuous on $[a,b]$.

Note: $F(x)$ always continuous even if $f(x)$ may not continuous on $[a,b]$and also $F(x)$ is uniform continuous on $[a,b]$.

Example (2.24): Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 0 & ,-1 \le x \le 0 \\ 1 & , \quad 0 < x \le 1 \end{cases}$

$-1 \le x \le 0, F(x) = \int_{-1}^{x} f(t)\,dt = 0$

$0 < x \le 1, F(x) = \int_{-1}^{x} f(t)\,dt = \int_{-1}^{0} f(t)\,dt + \int_{0}^{x} f(t)\,dt = 0 + \int_{-1}^{x} dx = x$

We have $F(x) = \begin{cases} 0 & ,-1 \le x \le 0 \\ x & , \quad 0 < x \le 1 \end{cases} \Rightarrow F$ is continuous on $[-1,1]$.

1.25.16. If $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ then the function $F(x) = \int_{a}^{x} f(t)\,dt,\ x \in [a,b]$ is differentiable at any point $c \in [a,b]$ at which f is continuous and$F'(c) = f(c)$.

1.25.17. Corollary: If $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b] then\ F$ is differentiable on $[a,b]\ and\ F'(x) = f(x)\ \forall\, x \in [a,b]$.

1.26. Fundamental Theorem of Integral Calculus :

1.26.1. Definition (Anti-derivative or Primitive): A function ϕ is called an anti-derivative or a primitive of a function f on an interval I if $\phi'(x) = f(x)\ \forall\, x \in I$.

1.26.2. If $f: [a,b] \to \mathbb{R}$ be continuous on $[a,b]\ and\ \phi : [a,b] \to \mathbb{R}$ be an anti-derivative of $f$on $[a,b], then\ \int_{a}^{b} f = \phi(b) - \phi(a)$.

1.26.3. Fundamental Theorem of Integral Calculus:

(i) $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b] and$

(ii) f possesses an anti $-$ derivative $\phi\ on\ [a,b], then$

$\int_{a}^{b} f = \phi(b) - \phi(a)$

1.26.4. Note –I: (Integrability$\not\Rightarrow$ existence of anti-derivative) :

Example (2.25):$f: [-1,1] \to \mathbb{R}, f(x) = \begin{cases} 0 & ,-1 \le x < 0 \\ 1 & , \quad 0 \le x \le 1 \end{cases} \Rightarrow f \in R[-1,1]\ on\ f$ is continuous on$[-1,1]$except at 0.

Let ϕ be anti-derivative of $f\ \ on\ [-1,1]$. Then $\phi'(x) = \begin{cases} 0 & ,-1 \le x < 0 \\ 1 & , \quad 0 \le x \le 1 \end{cases}$

Since $\phi'(-1) \ne \phi'(1), by$ Darboux theorem ϕ' must assume every real number lying between $\phi'(-1)\ and\ \phi'(1)\ i.e., 0\ \ and\ \ 1$. But it does not do so.

1.26.5. Note –II: (Existence of anti-derivative $\not\Rightarrow$ Integrability):

Example (2.26): Let $f: [-1,1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 2x \sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2} & , \ x \ne 0 \\ 0 & , \ x = 0 \end{cases}$

$f \notin R\ [-1,1] as\ f$ is unbounded on every neighbourhood of 0.

Now, $\phi : [-1,1] \to \mathbb{R}$ defined by $\phi(x) = \begin{cases} x^2 \sin\frac{1}{x^2}, & x \ne 0 \\ 0 \quad , & x = 0 \end{cases}$

Then $\phi'(x) = f(x)\ \ on\ \ [-1,1]$. So, ϕ is anti-derivative of f on $[-1,1]$.

1.26.6. If

(i) $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ and

(ii) $\exists\ \phi : [a,b] \to \mathbb{R}$ such that ϕ is continuous on $[a,b]$ and

$\phi'(x) = f(x)\ \forall\ x \in [a,b],\ then\ \int_a^b f = \phi(b) - \phi(a)$.

1.26.7. Corollary: If

(i) $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ and

(ii) $\exists\ \phi : [a,b] \to \mathbb{R}$ such that ϕ is continuous on $[a,b]$ and

$\phi'(x) = f(x)\ \forall\ x \in [a,b]\backslash \mathrm{E},\ where$ E is a finite set $\subset$ [a, b],

then $\int_a^b f = \phi(b) - \phi(a)$.

1.27. Riemann Sum and another Definition of Integration:

1.27.1. Riemann Sum: Let $f : [a,b] \to \mathbb{R}\ and\ \ P = \{x_0, x_1, x_2, \ldots\ldots\ldots, x_n\}$ be a partition of $[a,b]\ and \xi_0, \xi_1, \xi_2, \ldots\ldots\ldots, \xi_n$ are arbitrarily chosen points such that $x_{r-1} \le \xi_r \le x_r\ \ for\ r = 1,2,3\ldots\ldots\ldots, n$. Then the sum $\sum_{r=1}^{n} f(\xi_r)(x_r - x_{r-1})$ is called a Reimann sum for f corresponding to the partition P and choose intermediate points ξ_r. This is denoted by $S(P,f)$.

1.27.2. Definition (Another Definition for Riemann Integrable): A function $f : [a,b] \to \mathbb{R}$ is said to be Riemann integrable on $[a,b]\ if \exists\ B > 0$ such that for each $\varepsilon > 0, \exists\ a\ \delta = \delta(\varepsilon) > 0$ satisfying $|S(P,f) - B| < \varepsilon\ \forall\ partition\ P\ \ of\ [a,b] with\ \|P\| < \delta\ where\ S(P,f)$ is a Riemann sum for f corresponding to the partition P and to any choice of intermediate points. In this case $B = \int_a^b f$.

This condition is expressed by the symbol $\lim_{\|P\| \to 0} S(P,f) = B$.

1.27.3. If $f : [a,b] \to \mathbb{R}$ be such that $\lim_{\|P\| \to 0} S(P,f) = B\ , then\ B$ is unique.

1.27.4. If $f : [a,b] \to \mathbb{R}$ be such that $\lim_{\|P\| \to 0} S(P,f)$ exists, then f is bounded on $[a,b]$.

1.27.5. (Integration by Substitution): Let $I = [\alpha, \beta]$ be a closed and bounded interval and a function $\phi : I \to \mathbb{R}$ be such that ϕ' is continuous and $\neq 0\ on\ I$. Let $\phi(\alpha) = a, \phi(\beta) = b$ and a function f be continuous on $\phi([\alpha, \beta])$. Then –

$$\int_\alpha^\beta f(\phi(t))\ \phi'(t)\, dt = \int_a^b f(x)\, dx$$

1.27.6. Integration by parts: Let $f, g : [a,b] \to \mathbb{R}$ be both differentiable on $[a,b] and\ f', g'$ are both intergable on $[a,b]$. Then -

$$\int_a^b f(x)\, g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)\, g(x)dx$$

1.28. Mean Value Theorem:

1.28.1 First Mean Value Theorem:

If (i) $f, g : [a, b] \to \mathbb{R}$ be both integrable on $[a, b]$, and

(ii) $g(x)$ has the same sign $\forall\, x \in [a, b]$

then there is a no μ *such that* $\int_a^b f(x)\, g(x)dx = \mu \int_a^b g(x)dx$

where $m < \mu \leq M$ *and* $m = \inf_{x \in [a,b]} f(x)$, $M = \sup_{x \in [a,b]} f(x)$. Further, f is continuous on $[a, b]$ there is a point $c \in [a, b]$ such that $\int_a^b f(x)\, g(x)dx = f(c) \int_a^b g(x)dx.$

1.28.2. Note:

(i) If $g(x) = 1,$ *then* $\int_a^b f(x)\, dx = \mu \int_a^b dx = \mu(b - a),$ *where* $m \leq \mu \leq M.$

(ii) If f is continuous ojn $[a, b]$ *and* $g(x) = 1,$ *then* $\exists\, c \in [a, b]$ *such that*

$$\int_a^b f(x)\, dx = f(c)(b - a).$$

Since $c \in [a, b],\ c = a + \theta\,(b - a)$ *for some* θ *satisfying* $0 \leq \theta \leq 1.$

$$\therefore \int_a^b f(x)\, dx = (b - a)f(a + \theta(b - a)), 0 \leq \theta \leq 1.$$

Example (2.27): Use first mean value theorem prove that

$$\frac{\pi}{6} \leq \int_0^{\frac{1}{2}} \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}dx \leq \frac{\pi}{6} \cdot \frac{1}{\sqrt{1-\frac{k^2}{4}}}, k^2 < 1$$

Let $(x) = \frac{1}{\sqrt{1-k^2x^2}}$, $g(x) = \frac{1}{\sqrt{(1-x^2)}}$, $x \in [0, \frac{1}{2}]$

Then $f, g \in R\left[0, \frac{1}{2}\right]$ *and* $g(x) > 0,\ \forall\, x \in \left[0, \frac{1}{2}\right]$

By first Mean Value Theorem $\exists$ *a* $c \in \left[0, \frac{1}{2}\right]$ *such that*

$$\int_0^{\frac{1}{2}} f(x)\, g(x)dx = f(x) \int_0^{\frac{1}{3}} g(x)dx = \frac{1}{\sqrt{1-k^2x^2}} \cdot \frac{\pi}{6}$$

Since $0 \leq c \leq \frac{1}{2}$, $1 \leq \frac{1}{\sqrt{1-k^2x^2}} \leq \frac{1}{\sqrt{1-\frac{k^2}{4}}} \Rightarrow \frac{\pi}{6} \leq \int_0^{\frac{1}{2}} f(x)\, g(x)dx \leq \frac{\pi}{6} \cdot \frac{1}{\sqrt{1-\frac{k^2}{4}}}$

1.28.3 Second Mean Value Theorem (Bonnets Form):

If (i) $f, g : [a, b] \to \mathbb{R}$ be both integrable on $[a, b]$, *and*

(ii) f is monotone decreasing and non-negative on $[a,b]$, $then \exists\ a\ point\ c \in [a,b]$ such that $\int_a^b f(x)\,g(x)dx = f(a)\int_a^c g(x)dx$

1.28.4. Second MVT, Weierstrass' form:

If (i) $f,g : [a,b] \to \mathbb{R}$ be both integrable on $[a,b]$, and

(ii) f is monotonic on $[a,b]$

$$then \exists\ a\ point\ c \in [a,b] such\ that \int_a^b f(x)\,g(x)dx = f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx$$

Example (2.28):

(i) Prove that $\left|\int_a^b \frac{\sin x}{x}dx\right| \le \frac{2}{a}$, $0 < a < b < \infty (Bonnets\ form)$.

(ii) Prove that $\left|\int_a^b \frac{\sin x}{x}dx\right| \le \frac{4}{a}$, $0 < a < b < \infty (Weierstrass\ form)$.

(i) Let $f(x) = \frac{1}{x}$, $g(x) = \sin x$, $\forall\, x \in [a,b]$. Since $f,g \in R\,[a,b]$ and f is monotone decreasing on $[a,b]$, by second mean value theorem (Bonnets form) $\exists\, c \in [a,b]$ such that $\int_a^b f(x)\,g(x)dx = f(a)\int_a^c g(x)dx = \frac{1}{a}\int_a^c \sin x\,dx = \frac{1}{a}[-\cos c + \cos a]$

$\Rightarrow \left|\int_a^b \frac{\sin x}{x}\,dx\right| \le \frac{2}{a}$

(ii) Since f is monotone on $[a,b]$, by second mean value theorem (Weierstrass form) $\exists\, c \in [a,b]$ $such\ that$ $\int_a^b f(x)\,g(x)dx = \frac{1}{a}\int_a^c g(x)dx + \frac{1}{b}\int_c^b g(x)dx$

$= \frac{1}{a}[-\cos c + \cos a] + \frac{1}{b}\,[-\cos b + \cos c]$

$\therefore \left|\int_a^b \frac{\sin x}{x}\,dx\right| \le \frac{4}{a}$

1.28.5. Definition (Logarithmic Function): The logarithmic function $L(or\ log)$ is defined by $L(x) = \log x = \int_1^x \frac{dt}{t}$, $x > 0$.

1.28.6. Definition (e): Then unique real number x satisfying $L(x) = 1$ is denoted by e i.e., $L(e) = 1$. Thereforee is denoted by $1 = \int_1^e \frac{1}{t}\,dt$.

1.29. Sequence of functions:

1.29.1. Definition: Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}$, let $f_n : D \to \mathbb{R}$ be a function, Then $\{f_n\}$ is a sequence of functions on $D\ to\ \mathbb{R}$. D may be $[a,b], [a,\infty]$ $\to$ closed intervals $(a,b), (a,\infty)$ $\to$ open intervals.

1.29.2. Definition (Pointwise Convergent): The sequence of functions $\{f_n\}$ on $D\ to\ \mathbb{R}$ is said to be pointwise convergent if for each $x \in D, \{f_n(x)\}$ converges.

Let for each $x \in D, \{f_n(x)\} \to l_x \ as\ n \to \infty$. Define $f : D \to \mathbb{R}$ by $f(x) = l_x$ for each $x \in D$, Then $f(x)$ is said to be the limit function of $\{f_n(x)\}\ on\ D$. Write $\lim_{x\to\infty} f_n(x) = f(x)\ on\ D$.

Examples (2.29): $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = x^n, x \in \mathbb{R}, \forall n \in \mathbb{N}$ then $f_n(x)$ is a sequence of functions on $\mathbb{R}$. For each $x \in (-1,1)$ $\{f_n(x)\}$ converges to 0 and for $x = 1$, $\{f_n(x)\}$ converges to 1. For all other $x \in \mathbb{R}$, the sequence $\{f_n(x)\}$is divergent. So, the sequence $\{f_n\}$ is pointwise convergent on $[-1,1]$ and the limit function f is defined by $f(x) = \begin{cases} 0, -1 < x < 1 \\ 1, \qquad x = 1 \end{cases}$

ii. $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{x}{n}$, $x \in \mathbb{R}, \forall\, n \in \mathbb{N}$. Then $f_n(x)$ converges to $0\ \forall\, n \in \mathbb{N}$. So its limit function is $f(x) = 0,\ x \in \mathbb{R}$.

iii. $f_n(x) = \tan^{-1}(nx),\ x \in \mathbb{R},\ x \in\ \mathbb{N}$

Then $\lim_{x\to\infty} f_n(x) = \begin{cases} \frac{\pi}{2}, x > 0 \\ 0, x = 0 \\ \frac{-\pi}{2},\ x < 0 \end{cases}$

So, the sequence $\{f_n\}$ is pointwise convergent on $\mathbb{R}$ and the limit function $(x) = \frac{\pi}{2}\ sin\ \ x$, $x \in \mathbb{R}$.

iv. $f_n(x) = \frac{sin\ nx}{n},\ x \in \mathbb{R}\ \ \lim_{x\to\infty} f_n(x) = 0 = f(x),\ x \in \mathbb{R}$

v. Let $f_n(x) = ne^{-nx}, x \geq 0, n \in \mathbb{N}$

For all $x \geq 0, 0 \leq\ ne^{-nx} \leq \frac{1}{n}, (since\ e^{nx} > nx,\ x > 0)$

$\therefore\ \lim_{x\to\infty} f_n(x) = 0 = f(x)$

1.29.3 Definition (Uniform Convergent): Let $D \subseteq \mathbb{R}$ and for each $n \in \mathbb{N}\ f_n = D \to \mathbb{R}$, be a function. The sequence $\{f_n(x)\}$ is said to be uniformly convergent on D to a function f if corresponding to a pre-assigned $\varepsilon > 0\ \exists\ k(\varepsilon) \in \mathbb{N}$ such that for all $n \in D$, $|f_n(x) - f(x)| < \varepsilon\ \forall\ n \geq k$.

We write $\lim_{x\to\infty} f_n = f$ uniformly on D or $f_n \to f$ uniformly on D.

f is said to be the uniform limit of $\{f_n\}$ on D.

If $\{f_n(x)\}$ is uniformly convergent on D to the function $f(x)$ then the sequence $\{f_n(x)\}$also converges pointwise on D to f. But the converges is not true.

Example (2.30): Let $f_n(x) = x^n,\ x \in \mathbb{R},\ x \in \mathbb{N}$. Then $\{f_n(x)\}$ converges on $[-1, 1]$ to the function f where $f(x) = \begin{cases} 0,\ -1 < x < 1 \\ 1, \qquad\ x = 1 \end{cases}$

Let $c \in (0,1)$. Then $|f_n(c) - f(c)| = c^n$ and let $0 < \epsilon < 1$. Then $|f_n(c) - f(c)| < \epsilon$ if $c^n < \epsilon$

as whenever $n\log\left(\frac{1}{c}\right) > \log\left(\frac{1}{\epsilon}\right)$

as whenever $n > \log\left(\frac{1}{\epsilon}\right) / \log\left(\frac{1}{c}\right)$.

Let $k = \left[\log\left(\frac{1}{\epsilon}\right) / \log\left(\frac{1}{c}\right)\right] + 1$ Then $|f_n(c) - f(c)| < \epsilon \; \forall \; n \geq k$.

$\therefore \; \forall \; n \in (0,1), |f_n(x) - f(x)| < \epsilon \;\; \forall \; n \geq k, k = \left[\log\left(\frac{1}{\epsilon}\right) / \log\left(\frac{1}{x}\right)\right] + 1$

This k depends on $\epsilon \; and \; x$. As $x \to 1, \; k \to \infty$

$\Rightarrow \quad \nexists \; k \in \mathbb{N}$ such that $x \in (0,1), \; |f_n(x) - f(x)| < \epsilon \;\; \forall \; n \geq k$.

Consequently $\{f_n\}$ is not uniformly convergent on $(0,1)$.

But $\{f_n\}$ is uniformly convergent on $[0, \; a], 0 < a < 1$ since, in $[0, \; a]$, the greatest value of $\log\left(\frac{1}{\epsilon}\right) / \log\left(\frac{1}{x}\right)$ is $\log\left(\frac{1}{\epsilon}\right) / \log\left(\frac{1}{a}\right)$

1.30. Uniform Convergent Criteria:

1.30.1 (Cauchy Criteria): Let $D \subseteq \mathbb{R}$ and $\{f_n\}$ be a sequence of functions on D to $\mathbb{R}$. A necessary and sufficient condition for uniform convergence of the sequence $\{f_n\}$ $on \; D$ is that for a pre-assigned $\epsilon > 0, \; \exists \; k \in \mathbb{N}$ such that for all $x \in D, \left|f_{n+p}(x) - f_n(x)\right| < \epsilon \; \forall \; n \geq k \; and \; p = 1,2, ...$

Or, $|f_m(x) - f_n(x)| < \epsilon \; \forall \; m,n \; \geq k$

Example (2.31): A sequence of functions $\{f_n\}$ is defined on $[0,a], 0 < a < 1$ by $f_n(x) = x^3$, $x \in [0,a]$. Choose $0 < \epsilon < 2$. $\quad \forall \; x \in [0,a] \quad and \quad \forall \; m,n \in \mathbb{N}$

$|f_m(x) - f_n(x)| = |x^m - x^n| \leq |x|^m + |x|^n \leq \; a^m + a^n \leq 2a^m \quad if \quad m \leq n.$

Now, $|f_m(x) - f_n(x)| < \epsilon$ if $a^m < {}^{\epsilon}/_{2}$ i.e, $m\log a < \log{}^{\epsilon}/_{2}$

i.e, $m > \frac{\log({}^{\epsilon}/_{2})}{\log a} (\therefore \; \log a < 0)$

Let $k = \left[\log({}^{\epsilon}/_{2}) / \log a\right] + 1$. Then $|f_m(x) - f_n(x)| < \epsilon \quad m,n \geq k \quad [n \geq m]$

$\Rightarrow \{f_n\}$ is uniformly convergent on $[0,a] \; for \; 0 < a < 1$.

1.30.2

Let $D \subseteq \mathbb{R}$ and $\{f_n\}$ be sequence of functions pointwise convergent on D to a function f.

Let $M_n = \sup_{x\in D}|f_n(x) - f(x)|$ Then $\{f_n\}$ is uniformly convergent on $D \Leftrightarrow \lim_{n\to\infty} m_n = U$

Examples (2.32): (i) Let $f_n(x) = x^n, \; x \in [0,1]$

For all $x \in [0,1], \; \lim_{n\to\infty} x^n = 0$ for $x = 1$. $\{f_n(x)\}$ converges to 1

$\therefore \; f(x) = \begin{cases} 0, 0 < x < 1 \\ 1, \qquad x = 1 \end{cases}$ Let $M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)| = 1 \nrightarrow 0 \; as \; n \to \infty$

$\Rightarrow \{f_n(x)\}$ is not uniform convergent on $[0,1]$.

(ii) Let $f_n(x) = \frac{x}{n+x^2}, x \in [0,1],\ n \in \mathbb{N}$ Then $\lim_{n\to\infty} \frac{x}{n+x^2} = 0 = f(x),\ x \in [0,1]$

Let $M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} \frac{x}{n+x^2}$

For $x > 1,\ u_n(x) = \frac{x}{n-x^2} u'_n(x) = \frac{n+x^2-2x^2}{(n+x^2)^2} = \frac{n-x^2}{(n+x^2)^2} > 0,\ \ n > 1$

$\Rightarrow$for $x > 1,\quad u_n(x)$ is a strictly increasing function of x on $[0,1]$

$\Rightarrow \sup_{x\in[0,1]} u_n(x) = \frac{1}{n+1}$ i.e, $x > 1,\ M_n = \frac{1}{1+n} \to 0\ as\ n \to \infty$

$\Rightarrow \{f_n\}$ is uniformly convergent on $[0,1]$

(iii)

For each $n \in \mathbb{N}$ let $f_n(x) = \frac{x}{1+nx^2},\ x \in [0,1]$. Then $f_n(x) \to f(x) = 0,\ x \in [0,1]$

Let $M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} \frac{x}{1+nx^2}$, $\quad x \in [0,1]$

For $x > 0,\ \frac{\frac{1}{x}+nx}{2} \geq \sqrt{\frac{1}{x}\cdot nx} = \sqrt{n} \quad \Rightarrow \quad \frac{x}{1+nx^2} \leq \frac{1}{2\sqrt{n}}$ equality

occurs when $x = \frac{1}{\sqrt{n}}$. i.e., $\frac{x}{1+nx^2} = \frac{1}{2\sqrt{n}}$ at $x = \frac{1}{\sqrt{n}}$

$\therefore\ M_n = \sup_{x\in[0,1]} \frac{x}{1+nx^2} = \frac{1}{2\sqrt{n}} \to 0\ as\ n \to \infty.$ Hence $\{f_n\}$ is uniformly convergent on $[0,1]$.

(iv) Let $f_n(x) = \frac{nx}{1+n^2x^2},\ \ x \in [0,1],\ x \in \mathbb{N}$. Then $f_n(x) \to f(x) = 0\ \ \forall\, x \in [0,1]$

Let $M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} \frac{nx}{1+n^2x^2}$

For $x > 0,\ \frac{\frac{1}{nx}+nx}{2} \geq \sqrt{\frac{1}{nx}\cdot nx} = 1$ equality occurs at $= \frac{1}{n}$.

$\Rightarrow \frac{nx}{1+n^2x^2} \nleq \frac{1}{2} \quad \Rightarrow \quad \frac{nx}{1+n^2x^2} = \frac{1}{2}$ where $x = \frac{1}{n}$

$\therefore\ M_n = \frac{1}{2} \nrightarrow 0$ as $n \to \infty$. Hence $\{f_n(x)\}$ is not uniformly convergent on $[0,1]$.

1.30.3.

Let $D \subseteq \mathbb{R}$ and $\{f_n\}$ be a sequence of functions on D to $\mathbb{R}$ and $x_0 \in D'$ and uniformly convergent on D. Then $\lim_{x\to x_0} \lim_{x\to\infty} f_n(x) = \lim_{x\to\infty} \lim_{x\to x_0} f_n(x)$.

Note: In this case interchange of limits is permissible.

1.30.4.

Let I be an interval and $\{f_n\}$ be uniformly convergent on I to a function f. Let $c \in I$ and each f_n be continuous at c, then f is continuous at c.

Example (2.33): Let $f_n(x) = x^n, x \in [0,1]$, $f_n(x) \to 0 = f(x)$ as $n \to \infty$, $\forall x \in [0,1]$. 1 is a limit point of $[0,1]$

$$\lim_{n\to\infty}\lim_{x\to 1} f_n(x) = \lim_{n\to\infty}\lim_{x\to 1} x^n = \lim_{n\to\infty} 1 = 1 \neq 0 = \lim_{x\to 1}\lim_{n\to\infty} x^n = \lim_{x\to 1}\lim_{n\to\infty} f_n(x)$$

$\Rightarrow \{f_n\}$ is not uniformly convergent on $[0,1]$.

1.30.5.

Let $D \subseteq \mathbb{R}$ and for each $n \in \mathbb{N}, f_n: D \to \mathbb{R}$ is bounded on D If the sequence $\{f_n\}$ be uniformly convergent on D, then the limit function f is bounded on D. But the converse is not true.

Example (2.34): $f_n(x) = \frac{nx}{1+n^2x^2}, x \in [0,1]$ Then $f(x) = 0$ and $\sup_{x\in[0,1]} f_n(x) = \frac{1}{2}$

$\Rightarrow f, f_n$ bounded on $[0,1]$ but $\{f_n\}$ is not uniformly convergent on $[0,1]$ as $M_n = \frac{1}{2} \nrightarrow 0 \quad as \quad n \to \infty$

1.30.6.

Let $D \subseteq \mathbb{R}$ and for each $n \in \mathbb{N}$, $f_n: D \to \mathbb{R}$ is continuous on D. If the sequence $\{f_n\}$ be uniformly convergent on D to a function f then f is continuous on D. But the converse is not true. (Previous Example)

Example (2.35): $f_n(x) = \tan^{-1}(nx)$, $x \in [0,1]$. Then $f_n(x)$ is continuous on $[0,1]$ for each $n \in \mathbb{N}$. But $\lim_{x\to\infty} f_n(x) = f(x) = \begin{cases} \frac{\pi}{2}, 1 \geq x \geq 0 \\ 0, \quad x = 0 \end{cases}$

And $f(x)$ is continuous on $[0,1] \Rightarrow \{f_n\}$ is not uniformly convergent on $[0,1]$.

1.30.7.[Dini's Theorem]: Let D be a compact subset of $\mathbb{R}$ and $f_n: D \to \mathbb{R}$ be a sequence of continuous functions on D that converges pointwise to a continuous function f. If the sequence $\{f_n\}$ be a monotone sequence on D i.e, either $f_{n+1}(x) \geq f_n(x)$ for each $n \in \mathbb{N}$, each $x \in D$ or $f_{n+1}(x) \leq f_n(x)$ for each $n \in \mathbb{N}$ and each $x \in D$, then the converges of the sequence $\{f_n\}$ is uniform on D.

Examples (2.36): $f_n(x) = x^n, x \in [0,1]$. Then $\lim_{x\to\infty} f_n(x) = 0 = f(x), x \in [0,1]$ and $f_n(x)$is continuous and monotone decreasing on $[0,1]$ and $f(x)$ continuous but $[0,1]$ is not impact in $\mathbb{R}$. Hence $\{f_n\}$is not uniformly convergent on $[0,1]$

1.30.3. Let $I = [a, b]$ be a closed and bounded interval and for each $n \in \mathbb{N}, f_n : I \to \mathbb{R}$ be R integrable on I. If the sequence $\{f_n\}$ converges uniformly to a function f on I then f is R-integrable on I and moreover, the sequence $\{\int_a^b f_n\}$converges to $\int_a^b f$. But converse is not true i.e., $\lim_{x\to\infty} \int_a^b f_n(x)\, dx = \int_a^b \lim_{x\to\infty} f_n(x)\, dx$.

Example (2.37):

(i) $f_n(x) = \frac{nx}{1+n^2x^2}, x \in \mathbb{N}, x \in [0,1]. Then\ f_n(x) \to f(x) = 0, x \in [0,1]$.

$\int_0^1 f_n(x)\,dx = \int_0^1 \frac{nx}{1+n^2x^2}dx = \frac{1}{2n}\log(1+n^2x^2)]_0^1 = \frac{1}{2n}\log(1+x^2) \to 0\; as\; n \to \infty$

and $\int_0^1 f_n\,(x)dx = 0.$ *But* $\{f_n\}$ is not uniformly convergent on $[0,1]$ as $M_n = \sup_{x\in[0,1]}(f_n(x) - f(x)) = \frac{1}{2} \nrightarrow 0\; as\; x \to \infty.$

(ii) Let $f_n(x) = nxe^{-nx^2},\; x \in [0,1],\; x \in \mathbb{N}$

For $x = 0, f_n(x) = 0 \to 0\; as\; x \to \infty$

For $x \in [0,1],\; u_n(x) = nxe^{-nx^2} > 0.\; \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{(x+1)}{n}e^{-x^2} = e^{-x^2} < 1$

$\Rightarrow \lim_{n\to\infty} u_n(x) = 0, \Rightarrow f(x) = 0, x \in [0,1] \Rightarrow \int_0^1 f(x)\;dx = 0$

Now,

$\int_0^1 f_n(x)\;dx = \int_0^1 nxe^{-nx^2}\,dx = \int_0^1 e^{-nx^2}\,d(nx^2) = \frac{1}{2}\,[e^{-nx^2}]_0^1$

$= \frac{1}{2}[1 - e^{-n}] \to \frac{1}{2} as\; x \to \infty$

$\therefore \lim_{x\to\infty}\int_0^1 f_n(x)\;dx = \frac{1}{2} \neq 0 = \int_0^1 f(x)\;dx = \int_0^1 \lim_{x\to\infty} f_n(x)\,dx$

$\Rightarrow \{f_n\}$ is not uniformly convergent on $[0,1]$.

1.30.9. Let $\{f_n\}$ be a sequence of function on $[a,b]$ such that for each $x \in \mathbb{N},\; f'_n(x)$ exist for all $x \in [a,b]$. If the sequence of derivatives $\{f'_n\}$ converges uniformly on $[a,b]$ to a function g and the sequence $\{f_n\}$ converges at least at one point $x_0 \in [a,b]$ then the sequence $\{f_n\}$ is uniformly convergent on $[a,b]$ and if the limit function be f then $f'(x) = g(x)\;\forall\, x \in [a,b]$.

Note: converse of (1.30.9.) is not true.

Example (2.38): let $f_n(x) = x - \frac{x^n}{n},\; x \in [0,1].\; \lim_{x\to\infty} f_n(x) = x, x \in [0,1].$

Let $M_n = \sup_{x\in[0,1]}|f_n(x) - f(x)| = \sup_{x\in[0,1]} = \left|\frac{x^n}{n}\right| = \frac{1}{n} \to 0\; as\;\; x \to \infty$

$\Rightarrow \{f_n\}$ is uniformly convergent on $[0,1]$.

Now, $f'_n(x)1 - x^{n-1}, x \in [0,1]\; \lim_{x\to\infty} f'_n(x) = \begin{cases} 1, 0 \le x < 1 \\ 0, x = 1 \end{cases}$

$\Rightarrow \{f'_n\}$ is not uniformly convergent on $[0,1]$ as the limit function is not continuous.

(ii) For sequence of function $\{f_n\}$ *where* f_n is differentiable on $[a,b]$ is not enough to ensure uniform convergent of the sequence $\{f_n\}$ on $[a,b]$.

Example (2.39) : Let $f_n(x) = \log(n + x^2)\,, x +\; [0,1]$

Then $f'_n(x) = \frac{2x}{n+x^2}\,, x \in [0,1]\,, f'_n(x) \to 0 = g(x)\; as\; n \to \infty$

Now, $M_n = \sup_{x\in[0,1]}\left|f'_n(x) - g(x)\right| = \sup_{x\in[0,1]}\frac{2x}{n+x^2}$

Let $U_n(x) = \frac{2x}{n+x^2}, x \in [0,1], n \in \mathbb{N} \Rightarrow U'_n(x) = \frac{2(n-x^2)}{(n+x^2)^2} > 0\ n > 1\ \forall\ u \in [0,1]$

$\therefore M_n = \frac{2}{1+n} \to 0\ as\ n \to \infty.\ Hence$ $\{f_n\}$ is not point wise convergent on $[0,1]$

1.30.10. Let $\{f_n\}$ be uniformly convergent on an interval. I and each f_n bounded on I. Then $\{f_n\}$ is uniformly bounded on.

Hints: Let $\lim_{n+\infty} f_n(x) = f(x)$ uniformly $\Rightarrow f(x)\ is\ bounded\ on\ I. \Rightarrow k_1 \in \mathbb{N}$ such that $|f(x)| < k_1. Now\ M_n = \sup_{n \in I} |f_n(x) - f(x)| \to 0 \Rightarrow \{M_n\}\ is\ bounded \Rightarrow M_n < k_2\ for\ some\ k_2 \in \mathbb{N} \Rightarrow |f_n(x) - f(x)| < k_2\ \forall\ n \in \mathbb{N} \Rightarrow |f_n(x)| \leq k_1 + k_2\ \forall\ n \in \mathbb{N}\ \forall\ x \in I.$

1.30.11. Let $f\ \mathbb{R} \to \mathbb{R}$ be uniformly continuous or $\mathbb{R}$. For each $n \in \mathbb{N}$, Let $f_n(x) = f\left(x + \frac{1}{n}\right), x \in \mathbb{R}\ .Then$ $\{f_n\}$ is uniformly convergent on $\mathbb{R}$.

[**Hints:** $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f\left(x + \frac{1}{n}\right) = f(x), as\ f_n$ continuous. Since f is uniformly convergent on $\mathbb{R}, given\ \epsilon > 0, \exists\ \delta > 0\ such\ that\ |x-u|\delta < \Rightarrow |f(x) - f(u)| < \epsilon\ \delta > 0, \exists\ k \in \mathbb{N}\ such\ that\ 0 < \frac{1}{n} < \delta\ \forall\ n \geq k\ . \Rightarrow \left|f\left(x + \frac{1}{n}\right) - f\ (x)\right| < \epsilon\ \forall\ n \geq k$

$\Rightarrow |f_n - f(x)| < \epsilon\ \forall\ n \geq k, \forall\ n \in \mathbb{R}$

$\Rightarrow \{f_n\}\ is$ uniformly convergent on $\mathbb{R}$].

1.31 Series of functions:

1.31.1 Let $D \leq \mathbb{R}\ and\ \{f_n\}$ be a sequence of functions on $D\ to\ \mathbb{R}\ then\ f_1 + f_2 + f_3 + \ldots\ldots$ is said to be a series of functions on$D$. The infinite series is denoted by $\sum f_n$. Let $S_n(x) = f_1(x) + \ldots\ldots + f_n(x)\ \forall\ x \in D, x \in \mathbb{N}\ .The\ sequence\ \{S_n(x)\}$ is said to be a sequence of partial sums of the infinite series $\sum f_n(x)$.

1.31.2 Definition (Point wise convergent and uniformly convergent) of the sequence $\{S_n(x)\}$ is point wise (or uniformly) convergent on $D\ then\ \sum f_n(x)n$ said to be point wise convergent (respectively uniformly convergent) on D.

If the series $\sum|f_n(x)|$ converges for each $n \in D, then\ the\ series\ \sum f_n(x)$ is said to be absolutely convergent on D.

Example (2.40):

(i) the series of function $1 + x + x^2 + \ldots\ldots, 0 \leq x < 1\ is\ convergent\ on$ $[0,1]$ since $S_n(x) = 1 + x + \ldots\ldots + x^{n-1} = \frac{1-x^2}{1-x}, x \in [0,1]. \lim_{n+\infty} S_n(x) = \frac{1}{1-x}, x \in [0,1]$ Now each $S_n(x)$is bounded on$[0,1]$ but S(x) is not bounded in [0,1]. Hence the series is not uniformly convergent.

(ii) $\frac{x}{1+x} + \frac{x}{(n+1)(2n+1)} + \frac{x}{(2n+x)(3x+1)} + \ldots\ldots, x \geq 0$

Let $S_n(x) = \frac{x}{1+x} + \frac{x}{(x+1)(2n+1)} + \ldots\ldots + \frac{x}{[(n-1)x+1](nx+1)}$

$= \left(1 - \frac{1}{1+x}\right) + \left(\frac{1}{x+1} - \frac{1}{2n+1}\right) + \ldots\ldots + \left(\frac{1}{(n-1)x+1} - \frac{1}{nx+1}\right)$

$$= 1 - \frac{1}{nx+1} = \frac{nx}{nx+1}$$

$\lim_{n\to\infty} S_n(x) = S(x)\begin{cases}0\,, x = 0\\ 1\,, 0 < x\end{cases} \Rightarrow The\ series\ is\ convergent\,.$ But the converges is not uniforms as each $S_n(x)\ is\ continuous$ but the sum function $S(x)$ is not continuous on $[0\,,\infty]$.

1.32 Uniform Convergent Criteria of Series for Functions:

1.32.1. Cauchy principal of convergence: Let $D \leq \mathbb{R}\ and\ \sum f_n\ be\ a$ series of functions on o $\mathbb{R}$. The series $\sum f_n$is uniformly convergent on $D\ \Leftrightarrow for\ a$ pre – assigned $\epsilon > 0\,, \exists\, k(\varepsilon) \in \mathbb{N}$ such that $\forall\, x\ \in D, \left|f_{n+1}(x) + f_{n+2}(x) + \ldots\ldots + f_{n+p}(x)\right| < \epsilon\, \forall\, n\ \geq k\,, p = 1,2, \ldots..$

1.32.2. Weierstrass M-test: Let $D \subseteq \mathbb{R}\ and\ \sum f_n\ be$ a series of functions on $D\ to\ \mathbb{R}$. Let $\{M_n\}$ be a sequence of positive real no. such that $\forall\, x\ \in D\,, |f_n(x)|\ \leq\ M_n\ \forall\, x\ \in \mathbb{N}$. If the series $\sum M_n\ be\ convergent\ then\ the\ series\ \sum f_n$ is uniformly and absolutely convergent on D.

Examples (2.41): The series $\sum \frac{x}{n+n^2x^2}$ is uniformly convergent $\forall\, x\ \in \mathbb{R}$. Let $f_n(x) = \frac{x}{n+n^2x^2}$ If $x = 0,\ f_n(x) = 0, where\ x \neq 0,\ \frac{n}{|x|} + n^2|x| \geq 2\, n^{2\frac{3}{2}}$ Equality occurs where $x = \frac{1}{\sqrt{n}}\,. \Rightarrow$ $|f_n(x)|\ \leq\ \frac{1}{2n^{\frac{3}{2}}}\ \forall\, n\ \in \mathbb{N}, \forall\, x\ \in \mathbb{R}$.

Since $\sum \frac{1}{2n^{\frac{3}{2}}}$ is Convergent, $\sum \frac{x}{n+n^2x^2}$ converges uniformly on $\mathbb{R}$.

1.32.3. Let $\sum f_n(x)$ be uniformly convergent a D and $x_0\ \in D'$. Then $\lim_{x\to x_0} \sum f_n(x) = \sum \lim_{x\to x_0} f_n(x)$. The interchange of the symbol $\sum$ and $lim_{x\to x_0}$ is permissible.

1.32.4. Let I be an interval and a series of functions $\sum f_n$ be uniformly convergent on I to a function $f(x)$. Let $c\ \in I$ and each f_n be continuous at c. Then f is continuous at c.

Example (2.42): Find $\lim_{x\to 0} \sum_{n=1}^{\infty} \frac{\cos nx}{n(n+1)}$ .Let $f_n \cos = \frac{\cos nx}{n(n+1)}$, $\forall\, x\ \in \mathbb{R}\ and\ x\ \in \mathbb{N}$. Then $|f_n\,(x)|\ \leq\ \frac{1}{n(n+1)} =\ M_n\ and\ \sum M_n\ is\ convergent\ \Rightarrow \sum f_n\ (x)$is uniformly convergent $\Rightarrow \lim_{x\to 0} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \lim_{x\to 0} f_n(x) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}\,.\ Let\ f_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots\ldots + \frac{1}{n(n+1)} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots\ldots \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1} \to 1\ as\ x \to \infty$

$\therefore \lim_{x\to 0} \sum_{n=1}^{\infty} \frac{\cos nx}{n(n+1)} = 1$

1.32.5. Let $D\ \subseteq \mathbb{R}$ and $f_n\ each\ n\ \in \mathbb{N}\,,\ f_n\ D \to \mathbb{R}$ is a continuous function on D. If the series $\sum f_n$ isuniformly convergent on D then the sum function S is continuous on. But the converse is not true.

Example (2.43): (i) Let $\sum f_n(x)$ be in series functions

Where$f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}$ $, x \in \mathbb{R}$

Let$S_n(x) = f_1(x) + \ldots\ldots f_n(x)$ $, x \in \mathbb{R}$

Then $S_n(x) = \frac{nx}{1+n^2x^2}$ $and \lim_{n+\infty} S_n(x) = 0 = S(x) \forall x \in \mathbb{R}$

$\ni S(x)$ is continuous .But$S_n(x)$ is not uniformly convergent on $\mathbb{R}$ as $M_n = \sup_{n\in\mathbb{R}} |S_n(x) - S(x)| = \frac{1}{2} \nrightarrow 0$ $as\ n \to \infty$.

(ii) $(1+x) + x(1-x) + x^2(1-x) + \ldots$.. is not uniformly convergent on [0,1].

$S_n(x) = (1-x) + x(1-x) + x^2(1-x) + \ldots\ldots + x^{n-1}(1-x)$

$= (1-x)[1 + x + \ldots\ldots + x^{n-1}]$

$= (1-x)\frac{1-x^n}{1-x} = 1 - x^n, x \in [0,1]$

$\therefore \lim_{x\to\infty} S_n(x) = S(x) = \begin{cases} 1, 0 \le x < 1 \\ 0, \quad x = 1 \end{cases} \Rightarrow S(x) is\ not\ continuous\ on\ [0,1]$

But $S_n(x)$ $is\ continuous\ on$ $[0,1]$ $\forall n \in \mathbb{N}$

$\Rightarrow$ the given series not uniformly convergent.

1.32.6. Let $I = [a, b]$ be a closed, bounded interval and each for each $x \in \mathbb{N}$, $f_n : I\ \mathbb{R}$ be integrable on I. If the series $\sum f_n$ be uniformly convergent on I to the function$S$ then –

(i) S is integrable on I.
(ii) $\sum \int_a^b f_n(x)dx = \int_a^b f(x)dx = \int_a^b (\sum f_n(x)dx)$
Converse is not true.

Example (2.44.):$f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}$ $, x \in [0, 1]$

Then, $S_n(x) = \frac{nx}{1+n^2x^2} \to 0 = S(x) \quad \forall x \in [0, 1] \Rightarrow \int_0^1 S(x)\, dx = 0$

And $\lim_{n\to\infty} \int_0^1 S_n(x)dx = \lim_{n\to\infty} \int_0^1 \frac{nx}{1+n^2x^2} dx = \lim_{n\to\infty} \frac{\log(1+n^2)}{2n} = 0 = \int_0^1 S(x)dx$

But $\{S_n\}$ is not uniformly convergent on $[0, 1]$.

1.32.7. Let $[a, b]$ be a closed and bounded interval and for each $x \in \mathbb{N}$ let f_n be differentiable on $[a, b]$. If the series of functions $f'_1 + f'_2 + \ldots\ldots$, converges uniformly on $[a, b]$ to a function g and the series $f_1 + f_2 + \ldots$.. converges uniformly on $[a, b]$ to a function s such that $s'(x) = g(x)\ \forall x \in [a, b]$.

1.32.8Abel's Test :

Let,

(i) The series of functions $\sum u_n(x)v_n(x)$ is uniformly convergent on $[a, b]$ and

(ii) The sequence of function $\{v_n(x)\}$ be monotonic for every $x \in [a,b]$ and uniformly bounded on $[a,b]$.
Then the series $\sum_{n=1}^{\infty} u_n(x) v_n(x)$ in uniformly convergent on $[a,b]$.

Examples (2.45): $e^{-x} - \frac{e^{-2x}}{2} + \frac{e^{-3x}}{3} - \frac{e^{-4x}}{4} + \ldots\ldots$ is uniformly convergent on $[0,1]$.

Let $u_n(x) = \frac{(-1)^{n+1}}{n}$, $u_n(x) = e^{-nx}$. Then $\sum u_n(x) = 1 - \frac{1}{2} + \frac{1}{3} + \ldots\ldots$ is convergent and $|v_n(x)| \leq |\ \forall\ x \in \mathbb{N}$ and $u_n(x)$ is monotonic decreasing on $[0,1]$ then $\sum \frac{(-1)^{n+1}}{n} e^{-nx}$ is uniformly convergent on $[0,1]$.

1.32.9 Dirichlet'sTest:

Let,

(i) The sequence of partial sums $\{s_n\}$ of the series if functions $u_1(x) + u_2(x) + \ldots\ldots$ be uniformly bounded on $[a,b]$.

(ii) The sequence of functions $\{v_n(x)\}$ be monotonic for every $x \in [a,b]$.

Then the series of functions $\sum_{n=1}^{\infty} u_n(x)\, v_n(x)$ is uniformly convergent on $[a,b]$.

Example (2.46):

(i) The series of functions $\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots\ldots$ is uniformly convergent on $[a,b] \subset (0,2\pi)$
Let $u_n = \sin nx$, $x \in [a,b] \subset (0,2\pi)$ and $v_n = \frac{1}{n}$
Then $\{v_n\}$ is monotonic decreasing sequence converging to 0

Let $s_n(x) = u_1(x) + u_2(x) + \ldots\ldots + u_n(x)$

$$= \sin x + \sin 2x + \ldots\ldots + \sin nx$$

$$= \frac{\sin\left(\frac{nx}{2}\right)\sin\frac{(n+1)x}{2}}{\sin\frac{x}{2}}$$

For each $n \in \mathbb{N}$, $|\sin(x)| \leq \left|\frac{1}{\sin\frac{x}{2}}\right|$

Since, $\sin\frac{x}{2} \neq 0$, $x \in [a,b] \subset (0,2\pi)$ and $f(x) = \frac{1}{\sin\frac{x}{2}}$ is continuous on $[a,b] \subset (0,2\pi)$, it is bounded on $[a,b]$, $\exists\ k > 0$ such that $|f(x)| \leq k$
$\Rightarrow$ $\{S_n\}$ is uniformly bounded on $[a,b]$.
Hence $\sum u_n v_n$ is uniformly convergent on $[a,b](0,2\pi)$.

(ii) $\sum (-1)^n \frac{x^2+n}{n^2}$, $x \in [a,b] \subseteq \mathbb{R}$ is uniformly convergent.
Let $u_n = (-1)^n$, $v_n = \frac{x^2+n}{n^2}$, $x \in [a,b]$,
Then $S_n = u_1 + u_2 + \ldots\ldots + u_n \Rightarrow$ $\{S_n\}$ is bounded

$$v_{n+1} - v_n = \frac{x^2 + (n+1)}{(n+1)^2} - \frac{x^2+n}{n^2} = x^2\left[\frac{1}{(n+1)^2} - \frac{1}{n^2}\right] + \left[\frac{1}{n+1} - \frac{1}{n}\right] < 0 \quad x \in [a,b]$$

$\Rightarrow \{v_n\}$ is monotonic decreasing and $\lim_{x\to\infty} v_n(x) = 0 \quad \forall\ x \in [a,b]$.
By Diri's theorem $\{v_n(x)\}$ is uniformly convergent on $[a,b] \subseteq \mathbb{R}$.

Hence $\sum u_n v_n = \sum (-1)^n \frac{x^2+n}{n^2}$is uniformly convergent on $[a,b] \subseteq \mathbb{R}$.

1.33. Power Series:

1.33.1 Definition: A series of the form $a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots\ldots$ where $a_1, a_2, \ldots\ldots. x \in \mathbb{R}$is called a power series about the point x_0.

This form reduces to the form $a_0 + a_1 x + a_2 x^2 + \ldots\ldots$ (which is a power series about0) by substituting $x' = x - x_0$

If a power series converges $\forall\, x \in \mathbb{R}$, it is called everywhere convergent. If a power series converges only at $x = 0$ $(\sum_0^\infty a_n x^n)$ it is called nowhere convergent.

Example (2.47):

(i) [Everywhere Convergent] : $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots, x \in \mathbb{R}$.
(ii) [Nowhere Convergent]: $1 + x + 2!\,x^2 + 3!\,x^2 + \ldots\ldots$, only $x = 0$.

1.33.2.

(i) If a power series $\sum_0^\infty a_n x^n$ converges for $x = x_1$, then the series converges absolutely $\forall\, x$ satisfying $|x| < |x_1|$.
(ii) If a power series $\sum_0^\infty a_n x^n$ diverges for $x = x_1$, then the series diverges for all x satisfying $|x| > |x_1|$.

1.33.3. Radius of Convergence:

If a power series $\sum_0^\infty a_n x^n$ be neither nowhere Convergent nor everywhere convergent, then $\exists$ a real no. $R > 0$ such that the series converges absolutely $\forall\, x$ satisfying $|x| < R$ and diverges $\forall\, x$ satisfying $|x| > R$. R is called radius of convergence of the power series $\sum_0^\infty a_n x^n$. The open interval $(-R, R)$ is called the interval of convergence of the series.

Note – I: We define $R = 0$ for a power series which is nowhere convergent and $R = \infty$ for a power series which is everywhere convergent

Note – II: The convergence of the power series at $x = R$ or $-R$ depend on the nature of$\{a_n\}$. It may converge at$x = +R$, or $-R$ or may not converges at$x = R$, or $-R$.

1.34. Determination of The Radius of Convergence:

1.34.1 Cauchy – Hadamard:

Let $\sum_0^\infty a_n x^n$ be a power series and let $\overline{lim}|a_n|^{\frac{1}{n}} = \mu$. Then if

(i) $\mu = 0$, the series is everywhere convergent.
(ii) $0 < \mu < \infty$ the series absolutely is convergent $\forall\, x$ satisfying $|x| < \frac{1}{\mu}$ and is divergent $\forall\, x$ satisfying $|x| > \frac{1}{\mu}$.
(iii) $\mu = \infty$, the series is nowhere convergent

1.34.2. Ratio Test:

Let $\sum_0^\infty a_n x^n$ be a power series and let $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \mu$. Then if

(i) $\mu = 0$ the series is everywhere convergent.

(ii) $0 < \mu < \infty$ the series is absolutely convergent $\forall\, x$ satisfying $|x| < \frac{1}{\mu}$ and the series is divergent $\forall\, x$ satisfying $|x| > \frac{1}{\mu}$

(iii) $\mu = \infty$, the series is nowhere convergent.

1.34.3.

Note: We have $\underline{\lim}\left|\frac{a_{n+1}}{a_n}\right| \le \underline{lim}|a_n|^{\frac{1}{n}} \le \overline{lim}|a_n|^{\frac{1}{n}} \le \overline{lim}\left|\frac{a_{n+1}}{a_n}\right|$ therefore, if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ exists then $\lim_{n\to\infty}|a_n|^{\frac{1}{n}}$ exists, but the converges it not true. Cauchy – Hadamard test is more powerful than the ratio for determination of the nature of a power series.

Example (2.48):

(i) $x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \ldots\ldots$ here$a_0 = 0, a_n = \frac{n^n}{n!}\ \forall\, n \in \mathbb{N}$

$\therefore \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^{n+1}}{(n+1)!} \times \frac{n!}{n^n}\right| = \left(1 + \frac{1}{n}\right)^n, n \in \mathbb{N}$

$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = e \,.\therefore R = \frac{1}{e}$

(ii) $\frac{1}{3} - x + \frac{x^2}{3^2} - x^3 + \frac{x^4}{3^4} - x^5 + \ldots\ldots$

Here $a_0 = \frac{1}{3}, a_1 = -1, a_2 = \frac{1}{3^2}, a_3 = -1, a_4 = \frac{1}{3^4}\ \ldots\ldots\ \ldots.$

$\therefore \overline{lim}|a_n|^{\frac{1}{n}} = 1 \therefore R = 1$

(iii) $x + \frac{(2!)^2}{4!}x^2 + \frac{(3!)^2}{6!}x^3 + \ldots\ldots + \frac{(n!)^2}{(2n)!}x^n + \ldots$

Here $a_0 = 0, a_1 = 1, a_n = \frac{(n!)^2}{(2n)!}\quad \forall\, n \ge 2$

$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\{(n+1)!\}^2}{(2n+2)!} \times \frac{(2n)!}{(n!)^2} = \lim_{n\to\infty}\frac{(n+1)^2}{(2n+2)(2n+1)} = \lim_{n\to\infty}\frac{(n+1)}{2(2n+1)} = \frac{1}{4}$

$\therefore R = 4$

1.35. Properties of Power Series:

1.35.1. Let $\sum_0^\infty a_n x^n$ be a power series with radius of convergence $R(>0)$. Then the series is uniformly convergent on $[-s, s]$, where$0 < s < R$.

1.35.2. Let $R(>0)$ be the radius of convergence of the power series $\sum_0^\infty a_n x^n$. Then $\sum_0^\infty a_n x^n$ is uniformly convergent on $[a, b] \subset (-R, R)$.

1.35.3. Let $\sum_0^\infty a_n x^n$ be a power series with radius of convergence $R(>0)$. Let $f(x)$ be the sum of the series on $(-R, R)$. Then f is continuous on $(-R, R)$.

1.35.4. A power series can be integrated term-by-term on any closed and bounded interval contained within the interval of convergence.

1.35.5. Let $R(>0)$ be the radius of convergence of the power series $\sum_0^\infty a_n x^n$ then the radius of convergence of the power series $a_0\,x + \frac{a_1}{2}x^2 + \frac{a_2}{3}\,x^3 + \ldots\ldots$ obtained by term-by-term integration is also R.

1.35.6. Let $R(>0)$ be the radius of convergence of the power series $\sum_0^\infty a_n x^n$ then the radius of convergence of the power series $a_1 + 2a_2\,x + 3a_3\,x^2 + \ldots\ldots$ obtained by term-by term differentiation is also R.

1.35.7. A power series can be differentiated term-by-term within the interval of convergence.

1.35.8. Let $\sum_0^\infty a_n x^n$ be a power series with radius of convergence $R(>0)$ and $f(x)$ be the sum of the series on $(-R,R)$. Then $a_k = \frac{fk(0)}{k!}$, $k = 0,1,2,\ldots\ldots$

$\left[\text{Note}: \sum_0^\infty \mathrm{a_n x^n} = \sum_0^\infty \frac{\mathrm{f^n(0)}}{\mathrm{n!}}\mathrm{x^n}\right]$

1.35.9.Taylor Series:

If a function f efined on some bounded $N(0)$ of 0 has derivation of all orders on $N(0)$, then the series $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots\ldots$ is called the Taylor's series of f about the point 0.

Thus every power series $\sum_0^\infty a_n x^n$ with radius of convergence $R(>0)$ is the Taylor's series about 0 of its sum function$f$. But the converse is not true. i.e, for a function f having derivatives of all orders on some and $N(0)$ of 0 be choses first and the Taylor's series $\sum_0^\infty \frac{f^n(0)}{n!}x^n$ be constructed, this may no be sum function as $f(x)$.

Example (2.49):$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$ we have $f^n(0) = 0$ for $n = 0,1,2,\ldots\ldots$

The Taylor's series of f about 0 is $0 + 0 + 0 + \ldots\ldots$ this converges to 0 not to $f(x)$.

1.35.10. Abel's Theorem:

Let $\sum_0^\infty a_n x^n$ be a power series with radius of convergence $R(>0)$. If the series converges at $x = R(or\ x = -R)$ then the series uniformly convergent on $[0,R]$ respectively$[a,b]$) and $\sum_0^\infty a_n R^n = \lim_{x\to R+} f(x) \left(\text{respectively} \lim_{x\to R+} f(x) = \sum a_n(-R)^n\right)$ but the converse is not true.

Example (2.50): The series $1 - x + x^2 - x^3 + \ldots\ldots$ on $(-1,1)$ can sum function $\frac{1}{1+x}$,1 being its radius of convergence.

Now, $\lim_{x\to 1-} \frac{1}{1+x} = 2$ but the series $1 - x + x^2 - x^3 + \ldots\ldots$ is not convergent at $x = 1$.

1.35.11. Uniqueness Theorem:

If two power series $\sum_0^\infty a_n x^n$ and $\sum_0^\infty b_n x^n$ converges on the same interval $(-R,R), R > 0$ to the same sum function f then $a_n = b_n$ for $n = 0,1,2,\ldots\ldots$

1.35.12. If R_1, R_2 be the radius of convergence of the power series $\sum_0^\infty a_n x^n$ and $\sum_0^\infty b_n x^n$ respectively and $\sum_0^\infty a_n x^n = f(x)$ for $|x| < R|$ and $\sum_0^\infty b_n x^n = g(x)$ for $|x| < R_2$ then the radius of convergence of the series $\sum_0^\infty (a_n + b_n)x^n$ is $R = min\{R_1, R_2\}$ and the sum of the series is $f(x) + g(x)$ on $(-R, R)$.

Examples (2.51):

(i) Let $f(x) = \sum_0^\infty a_n x^n$ on $(-R, R)$ for some $R > 0$. If $f(x) = f(-x) \forall\, x \in (-R, R)$. Then $a_n = 0$, odd n. Since $f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots.$ on $(-R, R)$ and $f(-x) = a_0 + a_1x + a_2x^2 - a_3x^3 + \ldots\ldots.$ on $(-R, R)$ and $f(x) = f(-x)$ on $(-R, R)$.
$\Rightarrow a_0 + a_1x + a_0x^2 + \ldots\ldots.$and $a_0 - a_1x + a_2x^2 - a_3x^3 + \ldots\ldots$ have the same some function $f(x)$ on $(-R, R)$. By uniqueness theorem $a_n = -a_n$ for all odd n.$\Rightarrow$ $a_n = 0$ for all odd n.

(ii) Assuming the power series expansion for $\frac{1}{\sqrt{1-x^2}}$ as $\frac{1}{\sqrt{1-x^2}} = 1 + \frac{x^2}{2} + \frac{1}{2} \cdot \frac{3}{4}x^4 + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\, x^6 + \ldots$(a) obtain the power series expansion for $sin^{-1}x$.
Deduce that $1 + \frac{1}{2\cdot3} + \frac{1\cdot3}{2\cdot4\cdot5} + \ldots\ldots. = \frac{\pi}{2}$.

Let $y = x^2$ then the series become $1 + \frac{y}{2} + \frac{1\cdot3}{2\cdot4}\, y^2 + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\, y^3 + \ldots\ldots(b)$

$a_0 = 1, a_n = \frac{1\cdot3\cdot5\ldots\ldots(2n-1)}{2\cdot4\cdot6\ldots\ldots2n}, n \geq 1$

$\lim_{n\to\infty} \left(\frac{a_n+1}{a_n}\right) = \lim_{n\to\infty} \left(\frac{2n+1}{2n+2}\right) = 1$. Hence the interval of converges of the series (b) is $(-1, 1)$ and hence of (a) is $(-1, 1)$.

Now, integrating term-by-term on $[o, x]$, $|x| < 1$ we have –

$$x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1\cdot3}{2\cdot4} \cdot \frac{x^5}{5} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6} \cdot \frac{x^7}{7} + \ldots\ldots. = sin^{-1}x, \ -1 < x < 1$$

At $x = 1$, the series become $1 + \frac{1}{2\cdot3} + \frac{1\cdot3}{1\cdot4\cdot5} + \ldots\ldots.$ (c)

$$a_0 = 1, \quad \therefore a_n = \frac{1\cdot3\ldots\ldots(2n-1)}{2\cdot4\ldots\ldots2n(2n+1)}, \quad n \geq 1$$

$$\frac{a_n}{a_n+1} = \frac{1\cdot3\ldots\ldots(2n-1)}{2\cdot4\ldots\ldots2n(2n+1)} \times \frac{2\cdot4\ldots\ldots2n(2n+2)(2n+3)}{1\cdot3\ldots\ldots(2n-1)(2n+1)}$$

$$= \frac{2(n+1)(2n+3)}{(2n+1)^2} = \frac{4n^2+10n+6}{(2n+1)^2}$$

$$\therefore n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\left(\frac{4n^2 + 10n + 6}{(2n+1)^2} - 1\right) = n\,.\frac{6n+5}{(2n+1)^2} = \frac{6 + \frac{5}{n}}{\left(2 + \frac{1}{n}\right)^2}$$

$\therefore \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{3}{2} > 1$ $\qquad$ $\therefore$ (c) is convergent.

At $x = -1, -1 - \frac{1}{2\cdot3} - \frac{1\cdot3}{2\cdot4\cdot5}\ldots\ldots.$ is also convergent

Hence$sin^{-1}x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1\cdot3}{2\cdot4}\frac{x^5}{5} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{x^7}{7} + \ldots\ldots$ for $-1 \leq x \leq 1$

And $\frac{\pi}{2} = sin^{-1}1 = 1 + \frac{1}{2\cdot 3} + \frac{1\cdot 3}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7} + \ldots\ldots..$