COUNCILE OF SCIENTIFIC & INDUSTRIAL RESEARCH

Mathematical Science Code : 04

Unit – 3 :

Syllabus

Sub Unit – 1: Ordinary Differential Equation:

Sl. No	Topic
1.	1.1. Differential Equations
2.	1.2. Ordinary Differential Equation
3.	1.3. Order of a differential equation
4.	1.4. Degree of a differential Equation
5.	1.5. Linear and non-linear differential equation
6.	1.6. Solved Problems
7.	1.7. Solution of a differential equation 1.7.(i) General Solution 1.7.(ii) Particular Solution 1.7. (iii) Singular Solution
8.	1.8. Existence and Uniqueness theorem for solution of initial value Problems for first order ordinary differential equation
9.	1.9. Exact and Homogeneous differential equations 1.9.1. Exact equation 1.9.2. Homogeneous Equation
10.	1.10. Solution of first order but not first degree differential equation 1.10.1. Clairaut's form 1.10.2. P – Discriminant relation 1.10.3. C – Discriminant relation
11.	1.11. Linear Differential Equation with constant coefficient

Ordinary Differential Equations

1.1. Differential Equation:

Definition: An equation involving derivatives or differential of one or more dependent variables with respect to one or more independent variables is called a differential equation.

Examples- 1.1.

$dy = (e^x + x)dx$, $\frac{dy}{dx} = \sin x$, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = xe^x$, etc.

1.2. Ordinary Differential Equation:

Definition: A differential equation involving derivatives with respect to a single independent variable is called an ordinary differential equation.

Examples – 1.2.

$xdy + ydy = 0,\ \frac{dy}{dx} = (x+y)^2,\ \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0,\ etc.$

1.3. Order of a differential equation

Definition: The order of a differential equation is the order of the highest order derivative involved in a differential equation.

Examples – 1.3.

$\frac{dy}{dx} = \sin x$ is a first order differential equation and $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ is a second order differential equation.

1.4. Degree of a differential Equation

Definition: The degree of a differential equation is the degree of the highest order derivative, after the differential equation has been made free from radicals and fractions of concerned derivative.

Examples – 1.4.

The degree of $x\frac{dy}{dx} + y = 0$ is 1

The degree of $x\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 - y = \cos x$ is 1

The degree of $\left(\frac{dy}{dx}\right)^{\frac{3}{2}} + y = x^2$ is 3, etc.

1.5. Linear and non-linear differential equation.

Definition: A differential equation is said to be linear if every dependent variable and every derivative involved occurs in the first degree only, and no products of dependent variable and derivatives occur.

Example – 1.5

$dy = (x + \cos x)dx$, $3x^2 \frac{dy}{dx} + y = x$ are linear.

A differential equation which is not linear is called non-linear differential equation.

Examples – 1.6

$\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^3 - y = x,\ y\frac{dy}{dx} + y = e^x$ are non – linear equation.

1.6. Solved Problems:

Example – 1.7

Find the order and degree of the following differential equations. Also classify them as linear and non – linear.

(a) $y = \sqrt{x}\left(\frac{dy}{dx} + \frac{5}{\frac{dy}{dx}}\right)$

(b) $y = x\left(\frac{dy}{dx}\right) + c\left\{1 + \left(\frac{d^2y}{dx^2}\right)^2\right\}^{\frac{3}{2}}$

(c) $\frac{dy}{dx} = y + \sin x$

(d) $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \sin x$

Solution:

(a) $y = \sqrt{x}\left(\frac{dy}{dx} + \frac{5}{\frac{dy}{dx}}\right)$

$or,\ y\frac{dy}{dx} = \sqrt{x}\left\{\left(\frac{dy}{dx}\right)^2 + 5\right\}$

Which is a first order and second degree equation. Here $y \cdot \frac{dy}{dx}$ is present. So it is a non – linear differential equation.

(b) $y = x\left(\frac{dy}{dx}\right) + c\left\{1 + \left(\frac{d^2y}{dx^2}\right)^2\right\}^{\frac{3}{2}}$

$or,\ y - x\frac{dy}{dx} = c\left\{1 + \left(\frac{d^2y}{dx^2}\right)^2\right\}^{\frac{3}{2}}$

$or,\ \left(y - x\frac{dy}{dx}\right)^2 = c^2\left\{1 + \left(\frac{d^2y}{dx^2}\right)^2\right\}^3$

It is a second order differential equation and degree is 6. Also $y \cdot \frac{dy}{dx}$ is present, so it is non-linear.

(c) $\frac{dy}{dx} = y + \sin x$

This is a first order and first degree linear differential equation.

(d) $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \sin x$

This is a second order and first degree linear differential equation.

1.7. Solution of a differential equation.

(i) General Solution:

A solution of a n^{th} order ordinary differential equation containing n independent arbitrary constants is called a general solution.

(ii) Particular Solution:

A solution obtained from a general solution by giving particular values to one or more of the n independent arbitrary constants is called a Particular solution.

(iii) Singular Solution:

A solution which cannot be obtained from any general solution by any choice of the n independent arbitrary constants is called a singular solution.

1.8. Existence and Uniqueness theorem for solution of initial value Problems for first order ordinary differential equation:

We consider the first order initial value problem $\frac{dy}{dx} = f(x, y)\ with\ y(x_0) = y_0$ where $f(x, y)$ is subject to the following conditions:

(i) $f(x, y)$ is continuous in a given region R,

(ii) $|f(x, y)| \leq M$, a fixed real number in R.

(iii) Lipschitz condition is satisfied i.e.,$|f(x, y_1) - f(x, y_2)| \leq k|y_1 - y_2|$, k being a fixed quantity for any two points $(x, y_1)\ and\ (x, y_2)$ in the region R.

If now (x_0, y_0) be any point in R such that the rectangle R as given by $|x - x_0| \leq a$, $|y - y_0| \leq b$, where $b > aM$, then there exists one and only one solution $y = \phi(x)$ having continuous derivatives in $|x - x_0| \leq a$ which satisfies the differential equation $\frac{dy}{dx} = f(x, y)$ and $\phi(x_0) = y_0$.

Examples – 1.8

Show that the function $y = cx^2 + x + 3$ is a solution, though not unique, of the initial value problem $x^2y'' - 2xy' + 2y = 6$ with $y(0) = 3,\ y'(0) = 1$ on $(-\infty, \infty)$.

Solution:-

$y = cx^2 + x + 3$

$y' = 2cx + 1,\ y'' = 2c$

Now,

$$x^2y'' - 2xy' + 2y = \cancel{2}cx^2 - \cancel{4}cx^2 - \cancel{2}x + \cancel{2}cx^2 + \cancel{2}x + 6 = 6$$

So, $y = cx^2 + x + 3$ is a solution but it is not unique. For different c we get different solution.

Examples – 1.9

Check the existence and uniqueness the solution of the IVP $\frac{dy}{dx} = y^{\frac{1}{3}}$ with $y(0) = 0$ in the neighbourhood of 0.

Solution:- Here $f(x,y) = y^{\frac{1}{3}}$, which is continuous in the neighbourhood of 0 but $\frac{\partial f}{\partial y}$ does not exist in the neighbourhood of 0. So the solution exists but not unique.

Examples – 1.10

Check the existence and uniqueness of the solution of the problem $\frac{dy}{dx} = \frac{y^2}{2\sqrt{x}}$ with $y(1) = -1$.

Solution:

Here $f(x,y) = \frac{y^2}{2\sqrt{x}}$, $f(x,y)$ is bounded and continuous in the neighbourhood of $x = 1$. Also $\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x}}$ is continuous and bounded in neighbourhood of $x = 1$ So, the solution of the problem exists and unique.

1.9. Exact and Homogeneous differential equations

1.9.1. Exact equation: A differential equation $M(x,y)dx + N(x,y)dy = 0$ is said to be exact if there exists a function $u(x,y)$ such that $M\,dx + N\,dy = du$.

Theorem: The necessary and sufficient condition that $M\,dx + N\,dy = 0$ be exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

1.9.2. Homogeneous Equation:

It $M(x,y) = x^n\,\phi\left(\frac{y}{x}\right)$ or $y^n\,\phi\left(\frac{x}{y}\right)$ and

$N(x,y) = x^n\,\psi\left(\frac{y}{x}\right)$ or $y^n\,\psi\left(\frac{x}{y}\right)$

i.e., M and N are homogeneous functions of degree n then $M\,dx + N dy = 0$ is said to be a homogeneous differential equation of degree n.

Note:- A differential equation which is not homogeneous is called non-homogeneous differential equation.

Examples – 1.11 $\frac{dy}{dx} = \frac{2xy}{x^2+y^2} = \frac{2\frac{y}{x}}{1+\left(\frac{y}{x}\right)^2} = f\left(\frac{y}{x}\right)$

This is a homogeneous differential equation of degree 0.

- **Working rule for solving an exact differential equation:-**

Let $M\,dx + N\,dy = 0$ be an exact differential equation, then the solution is

$$\underset{[y\ as\ constant]}{\int M\,dx} + \underset{[x\ as\ constant]}{\int N\,dy} = c$$

No term is repeated in both the integral.

Example – 1.12

Solve $(2x - y + 1)dx + (2y - x - 1)dy = 0$

Solution:- Here, $M = 2x - y + 1,\ N = 2y - x - 1$

$$\frac{\partial M}{\partial y} = -1, \qquad \frac{\partial N}{\partial x} = -1, \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

So it is an exact equation.

$$\int M\,dx = \int (2x - y + 1)\,dx = x^2 - xy + x$$

$$\int N\,dy = \int (2y - x - 1)\,dy = y^2 - xy - y$$

So the solution is $x^2 + y^2 - xy + x - y = c$

- **Integrating Factors:**

A function $\mu(x,y)$ is said to be an integrating factor (I.F) of the equation $M\,dx + N\,dy = 0$ if there exists a function $u(xy)$ such that $\mu(x,y)[M\,dx + N\,dy] = du$.

- **Theorem:** The differential equation $M\,dx + N\,dy = 0$ possess an infinite number of integrating factors.

- **Rules for finding Integrating Factors:-**

Rule – 1:- If $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function of x only say $f(x)$, then $e^{\int f(x)dx}$ is an integrating factor of $M\,dx+N\,dy=0$

Rule – 2:- If $\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function of y only say $\phi(y)$, then $e^{-\int \phi(y)dx}$ is an integrating factor of $M\,dx+N\,dy=0$.

Rule – 3:- If $M\,dx+N\,dy=0$ to be a homogeneous equation and if $Mx+Ny\neq 0$, then $\frac{1}{Mx+Ny}$ is an integrating factor of the equation $M\,dx+N\,dy=0$

Rule – 4:- If $Mx-Ny\neq 0$ and $M=yf_1(xy),\ N=xf_2(xy)$, then $\frac{1}{Mx-Ny}$ is an integrating factor of $M\,dx+N\,dy=0$.

Examples – 1.13

Solve: $(x^2y-2xy^2)\,dx-(x^3-3x^2y)\,dy=0$

Solution: Here, $M=x^2y-2xy^2,\quad N=-x^3+3x^2y$

$$\therefore Mx+Ny=\ x^3y-2x^2y^2-x^3y+3x^2y^2=x^2y^2\neq 0$$

So, $\frac{1}{Mx+Ny}=\frac{1}{x^2y^2}$ is an integrating factor.

On multiplying by $\frac{1}{x^2y^2}$

$\left(\frac{1}{y}-\frac{2}{x}\right)dx-\left(\frac{x}{y^2}-\frac{3}{y}\right)dy=0$, which is exact.

Now, $\int\left(\frac{1}{y}-\frac{2}{x}\right)dx=\frac{x}{y}-2\log|x|$

$\int\left(\frac{-x}{y^2}+\frac{3}{y}\right)dy=\frac{x}{y}+3\log|y|$

So the solution is $\frac{x}{y}-2\log|x|+3\log|y|=c$

Examples – 1.14

Solve: $y(1+xy)dx+x(1-xy)dy=0$

Solution: Here, $M=y(1+xy)=y\,f_1(xy)$

$N=x(1-xy)=x\,f_2(xy)$

Again $Mx-Ny=xy(1+xy)-xy(1-xy)=2x^2y^2\neq 0$

So, $\frac{1}{Mx-Ny}=\frac{1}{2x^2y^2}$ is an integrating factor.

On multiplying by $\frac{1}{2x^2y^2}$

$\frac{1}{2}\left(\frac{1}{x^2y}+\frac{1}{x}\right)dx+\frac{1}{2}\left(\frac{1}{xy^2}-\frac{1}{y}\right)dy=0$

Now,

$\int \frac{1}{2}\left(\frac{1}{x^2y}+\frac{1}{x}\right) dx = -\frac{1}{2xy}+\frac{1}{2}\log|x|$

$\int \frac{1}{2}\left(\frac{1}{xy^2}-\frac{1}{y}\right) dy = -\frac{1}{2xy}-\frac{1}{2}\log|y|$

Hence the solution is $-\frac{1}{2xy}+\frac{1}{2}\log|x|-\frac{1}{2}\log|y| = c$

Examples – 1.15

Solve: $(x^2+y^2+1)dx - 2xy\, dy = 0$

Solution:- Here, $M = x^2+y^2+1,\ N = -2xy$

$\frac{\partial M}{\partial y} = 2y,\ \frac{\partial N}{\partial x} = -2y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, it is not an exact equation.

Now, $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) = \frac{1}{-2xy}(2y+2y) = \frac{4y}{-2xy} = -\frac{2}{x} = f(x)$

So, integrating factor $= e^{\int f(x)dx} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = \frac{1}{x^2}$

On multiplying by $\frac{1}{x^2}$ the equation reduces to $\left(1+\frac{y^2}{x^2}+\frac{1}{x^2}\right)dx - \frac{2y}{x}\, dy = 0$

Now,

$\int\left(1+\frac{y^2}{x^2}+\frac{1}{x^2}\right) dx = x-\frac{y^2}{x}-\frac{1}{x}$

$\int -\frac{2y}{x}\, dy = -\frac{y^2}{x}$

Hence, the solution is $x-\frac{y^2}{x}-\frac{1}{x} = c$

$or, x^2-y^2-1 = cx$

Examples – 1.16

Solve:- $(2xy^2-2y)dx + (3x^2y-4x)dy = 0$

Solution:-

Here, $M = 2xy^2-2y,\ N = 3x^2y-4x$

$\frac{\partial M}{\partial y} = 4xy-2,\ \frac{\partial N}{\partial x} = 6xy-4$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, so it is not an exact equation.

Now,

$\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) = \frac{1}{2xy^2-2y}(4xy-2-6xy+4) = \frac{-2xy+2}{2xy^2-2y} = \frac{1-xy}{-y(1-xy)} = -\frac{1}{y} = \phi(y)$

Integrating factor $= e^{-\int \phi(y)dy} = e^{\int \frac{1}{y}dy} = y$

On multiply by y we get $(2xy^3-2y^2)dx + (3x^2y^2-4xy)dy$

Now, $\int(2xy^3 - 2y^2)dx = x^2y^3 - 2y^2x$

$\int(3x^2y^2 - 4xy)dy = x^2y^3 - 2xy^2$

So the solution is $x^2y^3 - 2xy^2 = c$

1.10. Solution of first order but not first degree differential equation.

1.10.1. Clairaut's form:

A differential equation of the form $y = px + f(p)$ is called clairaut's equation where $p = \frac{dy}{dx}$.

Now

$\frac{dy}{dx} = p.1 + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$

$or,\ p = p + \frac{dp}{dx}(x + f'(p))$

$or,\ \{x + f'(p)\}\frac{dp}{dx} = 0$

$\Rightarrow \frac{dp}{dx} = 0$ or $x + f'(p) = 0$

$\Rightarrow p = c$

So general solution is $y = cx + f(c)$ and the singular solution is obtained by eliminating p between $x + f'(p) = 0$ and the given equation.

1.10.2. P – Discriminant relation:-

Let $f(x,y,p) = 0$ be the given differential equation. The $p - discriminant$ relation is obtain by eliminating p between $f(x,y,p) = 0\ and\ \frac{\partial f}{\partial p} = 0$.

1.10.3. C – Discriminant relation:-

Let $\phi(x,y,c) = 0$ be the general solution of the given differential equation. Then $c - discriminant\ relation$ is obtained by eliminating c between $\phi(x,y,c) = 0\ and\ \frac{\partial\phi}{\partial c} = 0$.

Examples – 1.17

Find the general and singular solution of $8ap^3 = 27y$.

Solution: Solving for p, $p = \frac{dy}{dx} = \frac{3}{2}\frac{y^{\frac{1}{3}}}{a^{\frac{1}{3}}}$

$or,\ dx = \frac{2}{3}a^{\frac{1}{3}}y^{-\frac{1}{3}}\,dy$

Integrating $x + c = a^{\frac{1}{3}}\,y^{\frac{2}{3}}$ $or, (x + c)^3 = ay^2$

This is the general solution.

Now differentiating partially with respect to c

$3(x + c)^2 = 0\ \ or,\ x + c = 0\ \ or,\ \ c = -x$

Eliminating c we get $y = 0$.

Hence $y = 0$ is the singular solution.

Examples – 1.18

Find the general and singular solution of equation $p = log\,(px - y)$

Solution:- $px - y = e^p$

$y = px - e^p$

This is in clairaut's form.

Its general solution is $y = cx - e^c$

Differentiating partially with respect to c, we get

$0 = x - e^c \quad or,\ c = \log x$

eliminating c the $c - discriminant$ is given by $y = xlogx - e^{\log x} = x\log x - x$.

Now, $y = x\log x - x$ gives $p = \log x + 1 - 1 = \log x$

These values of $p\ and\ y$ satisfy the given equation.

So the singular solution is $y = x\log x - x$

1.11. Linear Differential Equation with constant coefficient:-

The general form of the equation is

$\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots\ \ldots\ldots + a_ny = X,$ where X is a function of x only and $a_1, a_2, \ldots\ldots, a_n$ are constants.

The general solution is $y = complementary\ function(C.F) + Particular\ integral\ (P.I)$

where $C.F.$ involves n arbitrary constants and $P.I.$ does not contain any arbitrary constants.

- **Rules for finding complementary functions (C.F.)**

Let $y = c\,e^{mx}(\neq 0)$ be a trial solution of the equation

$\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots\ \ldots\ldots + a_ny = 0$

Then the auxiliary equation (A.E.) is $m^n + a_1m^{n-1} + a_2m^{n-2} + \ldots\ldots\ldots\ \ldots + a_n = 0$

Case – 1: When all the roots of the A.E. are real and distinct. Then the complementary function is $y = c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x} + \ldots\ldots\ldots\ \ldots + c_ne^{m_nx}$

Where $m_1, m_2, \ldots\ldots, m_n$ be the n real distinct roots of A.E. and $c_1, c_2, \ldots\ldots, c_n$ are arbitrary constants.

Case – 2: When the auxiliary equation has equal roots.

Let the roots $m_1\ and\ m_2$ be equal and other roots are distinct.

Then the complementary function is

$y = (c_1 + c_2x)\, e^{m_1x} + c_3e^{m_3x} + c_4e^{m_4x} + \ldots\ldots\ldots\, .. + c_ne^{m_nx}$

Case – 3: When the A.E. has complex roots. Let the two roots be complex say $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$

Then the corresponding part of the $C.F. = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$.

Similarly if the complex roots are repeated twice then the corresponding part of the C.F. is

$e^{\alpha x}\,[(c_1 + c_2x)\cos\beta x + (c_3 + c_4x)\sin\beta x]$

Examples – 1.19

Solve $(D^4 - 5D^2 + 4)y = 0, \qquad D = \frac{d}{dx}$

Solution: A.E. is $m^4 - 5m^2 + 4 = 0 \Rightarrow m = 2, -2,\ 1, -1$

So, C. F. is $y = c_1e^{2x} + c_2e^{-2x} + c_3e^{x} + c_4e^{-x}$

Examples – 1.20

Solve $(D^3 - 8)y = 0, \qquad D = \frac{d}{dx}$

Solution: A.E. is $m^3 - 8 = 0 \Rightarrow m = 2, -1 \pm i\sqrt{3}$

So C.E. is $y = c_1e^{2x} + e^{-2x}(c_2 \cos\sqrt{3}\,x + c_3 \sin\sqrt{3}x$

Examples – 1.21

Solve: $(D-2)^3y = 0, \qquad D = \frac{d}{dx}$

Solution: A.E. is $(m-2)^3 = 0 \Rightarrow m = 2, 2, 2$

So, C.F. is $y = (c_1 + c_2\,x + c_3x^2)e^{2x}$

- **Working rule for finding P.I.**

Rule – 1: If $X = e^{ax}$

Then $P.I. = y_p = \frac{1}{F(D)}e^{ax} = \frac{e^{ax}}{F(D)} \qquad provided\ F(a) \neq 0.$

Rule – 2: If $X = e^{ax}$ then

$$P.I. = y_p = \frac{1}{F(D)}e^{ax} = e^{ax}\frac{1}{F(D+a)}\,1\ if\ \ F(a) = 0.$$

Rule – 3: If $X = e^{ax} \cdot V,\ \ V\ is\ a\ function\ of\ x$ then

$P.I. = y_p = \frac{1}{F(D)}e^{ax}\,V = e^{ax}\frac{1}{F(D+a)}\,V$

Rule – 4: If $X = \sin ax\ \ or \cos ax$ and

Let $F(D) = H(D^2)$, then

$P.I. = y_p = \frac{1}{F(D)}\,X = \frac{1}{H(D^2)}\,X = X \cdot \frac{1}{H(-a^2)}, provided\ H(-a^2) \neq 0$

Rule – 5: If $X = \sin ax \ or \cos ax$ and

Let $F(D) = H(D^2)$, then

$P.I. = y_p = \frac{1}{F(D)} X$

$= \frac{1}{H(D^2)} X = Imaginary\ part\ of\ \left\{\frac{1}{H(D^2)} e^{iax}\right\} If\ H(-a^2) = 0\ and\ X = \sin ax.$

$= Real\ part\ of \left\{\frac{1}{H(D^2)} e^{iax}\right\}\ \ if\ H(-a^2) = 0\ \ and\ X = \cos ax.$

Examples – 1.22

Find Particular integral of $(D^2 - 3D + 2)y = e^{5x}$

Solution:-

$P.I. = \frac{1}{D^2-3D+2} e^{5x} = \frac{e^{5x}}{5^2-3\times5+2} = \frac{e^{5x}}{12}$

Examples – 1.23

Find P.I. of $(D^2 + 4D + 4)y = e^{-2x}$

Solution:-

$P.I. = \frac{1}{D^2+4D+4} e^{-2x} = \frac{1}{(D+2)^2} e^{-2x} = e^{-2x} \frac{1}{(D-2+2)^2} 1 = e^{-2x} \frac{x^2}{2} = \frac{x^2 e^{-2x}}{2}$

Examples – 1.24

Solve $(D + 2)(D - 3)^3 y = e^x$, Find P.I.

Solution:-

$P.I. = \frac{1}{(D+2)(D-1)^3} e^x = \frac{1}{(D-1)^3} \frac{e^x}{(1+2)} = \frac{1}{3} e^x \frac{1}{(D+1-1)^3} 1 = \frac{1}{3} e^x \frac{x^3}{6} = \frac{x^3 e^x}{18}$

Examples – 1.25

Solve: $(D^2 + 1)y = \cos 2x$

A.E. is $m^2 + 1 = 0 \ \Rightarrow m = \pm i$

$\therefore C.F = c_1 \cos x + c_2 \sin x$

$P.I. = \frac{1}{D^2+1} \cos 2x = \frac{\cos 2x}{-2^2+1} = -\frac{1}{3} \cos 2x$

So, the general solution is $y = C.F. + P.I. = c_1 \cos x + c_1 \sin x - \frac{1}{3} \cos 2x$

Examples – 1.26

Solve - $(D^2 - 3D + 2)y = \sin 3x$

A.E. is $m^2 - 3m + 2 = 0\ \ or, m = 1, 2$

$C.F. = c_1 e^x + c_2 e^{2x}$

$P.I. = \frac{1}{D^2-3D+2} \sin 3x = \frac{1}{-3^2-3D+2} \sin 3x = \frac{1}{-(3D+7)} \sin 3x = \frac{-(3D-7)}{9D^2-49} \sin 3x$

$= -(3D - 7) \frac{\sin 3x}{9(-3^2)-49} = \frac{1}{130} (3D - 7) \sin 3x = \frac{1}{130} (9 \cos 3x - 7 \sin 3x)$

General solution is $C.F. + P.I$

$y = c_1 e^x + c_2 e^{2x} + \frac{1}{130}(9\cos 3x - 7\sin 3x)$

Examples – 1.27

Find P.I of $(D^2 + a^2)y = \sin ax$ and $(D^2 + a^2)y = \cos ax$.

Solution:-

Now $\frac{1}{D^2+a^2} e^{iax} = \frac{e^{iax}}{(D+ia)^2+a^2} 1 = \frac{e^{iax}}{D^2+2Dia} 1 = \frac{e^{iax}}{2Dia}\left(1+\frac{D}{2ia}\right)^{-1} 1$

$= \frac{e^{iax}}{2Dia}\left(1 - \frac{D}{2ia} + \ldots\ldots \ldots\right) 1 = \frac{e^{iax}}{2ia} \cdot x = \frac{(\cos ax + i\sin ax)x}{2ia} = \frac{x}{2a}\sin ax - i\frac{x}{2a}\cos ax$

$\therefore \frac{1}{D^2+a^2}\cos ax = Real\ part\ of\ \frac{1}{D^2+a^2} e^{iax} = \frac{x}{2a}\sin ax$ and

$\frac{1}{D^2+a^2}\sin ax = Imaginary\ Part\ of\ \frac{1}{D^2+a^2} e^{iax} = -\frac{x}{2a}\cos ax$

1.12. Wronskian, linearly dependent and independent set of functions.

1.12.1. Wronskian: The Wronskian of n functions $y_1(x), y_2(x), \ldots\ldots.., y_n(x)$ is denoted by $W(y_1, y_2, \ldots\ldots.., y_n)$ and is defined by

$$W(y_1, y_2, \ldots\ldots.., y_n) = \begin{vmatrix} y_1 y_2 & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & y_n \\ y'_1 y'_2 & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & y'_n \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ y_1^{n-1} y_2^{n-1} & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & y_n^{n-1} \end{vmatrix}$$

1.12.2. Linearly dependent and independent set of function:

n functions $y_1(x), y_2(x), \ldots\ldots.., y_n(x)$ are linearly dependent if there exist constants $c_1, c_2, \ldots\ldots.., c_n$ (not all zero), such that $c_1 y_1 + c_2 y_2 + \ldots\ldots. + c_n y_n = 0$.

If $c_1 y_1 + c_2 y_2 + \ldots\ldots\ldots. + c_n y_n = 0\ implies\ c_1 = c_2 = \ldots\ldots\ldots\ldots = c_n = 0$ then $y_1, y_2, \ldots\ldots.., y_n$ are said to be linearly independent.

1.12.3. Some Important Theorem:

(i) Theorem:

If $y_1(x)\ and\ y_2(x)$ are any two solutions of $a_0(x)y'' + a_1(x)y' + a_2(x)\,y = 0$, then the linear combination $c_1 y_1 + c_2 y_2$, where $c_1\ and\ c_2$ are constants, is also a solution of the given equation.

(ii) Theorem: Two solutions $y_1(x)\ and\ y_2(x)$ of the equation $a_0(x)y'' + a_1(x)y' + a_2(x)\,y = 0,\ a_0(x) \neq 0$ are linearly dependent if and only if their Wroskian is identically zero. Two solutions are linearly independent if and only if their Wronskian is not zero.

(iii)Theorem: The n^{th} order homogeneous linear equation $a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_n(x)\, y = 0$ always possesses n independent solutions $y_1(x), y_2(x), \ldots\ldots\ldots, y_n(x)$ and its general solution is given by $y = c_1y_1 + c_2y_2 + \ldots\ldots\ldots + c_ny_n$ where $c_1, c_2, \ldots\ldots\ldots, c_n$ are n arbitrary constants.

Examples – 1.28

Show that e^{2x} *and* e^{3x} are linearly independent solutions of $y'' - 5y' + 6y = 0$. Find the solution $y(x)$ with $y(0) = 0, y'(0) = 1$.

Solution: $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = e^{5x} \neq 0$ showing that e^{2x}, e^{3x} are linearly independent solution. The general solution is given by

$y(x) = c_1e^{2x} + c_2e^{3x}$

$y'(x) = 2c_1e^{2x} + 3c_2e^{3x}$

$y(0) = 0 \text{ and } y'(0) = 1 \Rightarrow c_1 + c_2 = 0 \text{ and } 2c_1 + 3c_2 = 1$

$c_1 = -1,\ c_2 = 1$

So, $y(x) = e^{3x} - e^{2x}$.

(v) Theorem: Abel's Formula:-

Let functions p_1 *and* p_2 in $y''(x) + p_1(x)y'(x) + p_2(x)\, y(x) = 0,\ x \in I$ be defined and continuous on an interval I. Let ϕ_1 *and* ϕ_2 be two linearly independent solutions existing on I containing a point x_0 . Then $W(\phi_1, \phi_2)(x) = \exp\left(-\int_{x_0}^{x} p_1(x)\, dx\right) W(\phi_1, \phi_2)\,(x_0)$.

Examples – 1.29

Let $Y_1(x)$ *and* $Y_2(x)$ defined on $[0,1]$ be twice continuously differentiable functions satisfying $Y''(x) + Y'(x) + Y(x) = 0$. Let $W(x)$ be the Wronskian of Y_1 *and* Y_2 and satisfy $W\left(\frac{1}{2}\right) = 0$. Then which is the correct option.

$(a)\ W(x) = 0 \text{ for } x \in [0,1]$

$(b)\ W(x) \neq 0 \text{ for } x \in \left[0, \frac{1}{2}\right] \cup \left[\frac{1}{2}, 1\right]$

$(c)\ W(x) > 0 \text{ for } x \in \left(\frac{1}{2}, 1\right)$

$(d)\ W(x) < 0 \text{ for } x \in \left(0, \frac{1}{2}\right)$

Ans:- Abel's formula is

$W(Y_1, Y_2)(x) = EXP\left(-\int_{x_0}^{x} p_1(x)\, dx\right) W(Y_1, Y_2)\,(x_0)$ Here $p_1 = 1, x_0 = \frac{1}{2}$

$\therefore W(x) = EXP\left(-\int_{\frac{1}{2}}^{x} dx\right) \times W\left(\frac{1}{2}\right)\left[\because W\left(\frac{1}{2}\right) = 0\right]$

$= 0 for\ x \in [0, 1]$

$\therefore$ (a) is correct.

1.13. Method of variation of Parameters:

- Working rule for solving $\frac{dy}{dx} + Py = Q$ by variation of Parameters, where $P\ and\ Q$ are functions of x or constants:

First we take corresponding homogeneous equation as $\frac{dy}{dx} + py = 0$

Solution is $y = au, a$ being an arbitrary constants.

General solution of the given equation is $y = au(x) + u(x)\int\left(\frac{Q}{u}\right)dx.$

Examples – 1.30

Solve $(x+y)\frac{dy}{dx} + 3y = 3$ here $p = \frac{3}{x+4}$, $Q = \frac{3}{x+4} = 3\,(x+4)^{-1}$

If $Q = 0 \Rightarrow \frac{dy}{dx} + \frac{3}{x+4}\ y = 0\ or, \frac{dy}{y} + \frac{3}{x+4}\ \ dx = 0$

Integrating, $\log y + 3\log(x+4) = \log a \Rightarrow y = a\,(x+4)^{-3}$

We take $u = (x+4)^{-3}$

General solution is

$y = c\,u(x) + u(x)\int\left(\frac{Q}{u}\right)dx = c\,(x+4)^{-3} + 3\,(x+4)^{-3}\int(x+4)^2\,dx$

$= c\,(x+4)^{-3} + 1$

- Method of variation of parameters for solving $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R, where\ P, Q, R$ functions of x or constants.

Let $y_1(x)\ and\ y_1\,(x)$ be two linearly independent solutions.

Then the general solution will be of the form

$y = c_1y_1(x) + c_2y_2(x) + u(x)\cdot y_1(x) + v(x)\cdot y_1(x)$

Where $u(x) = \int\frac{-y_2\,R(x)}{W(y_1,y_2)}dx$ and $v(x) = \int\frac{y_1\cdot R(x)}{W(y_1,y_2)}\ dx$, provided $W(y_1, y_2) \neq 0$.

Examples – 1.31

Solve:- $\frac{d^2y}{dx^2} + n^2y = \sec nx$

Solution:-

General solution is $y = c_1\cos nx + c_2\sin nx + u(x)\cdot\cos nx + v(x)\cdot\sin nx$

Now,

$W(y_1, y_2) = \begin{vmatrix} \cos nx & \sin ax \\ -n\sin ax & n\cos ax \end{vmatrix} = n \neq 0$

$u(x) = \int \frac{-\sin nx \times \sec nx}{n}\, dx = -\frac{1}{n^2}\log\sec nx = \frac{1}{n^2}\log\cos nx$

$v(x) = \int \frac{\cos nx \times \sec nx}{n}\, dx = \frac{x}{n}$

So the general solution is $y = c_1\cos nx + c_2\sin nx + \frac{1}{n^2}\cos nx \,.\log\cos nx + \frac{x}{n}\cdot\sin nx$

1.14. Sturm – Liouville boundary value problem:-

A Sturm – Liouville problem of second order is a homogeneous boundary value problem of the form $\frac{d}{dx}\left(P_1(x)\cdot\frac{dy}{dx}\right) + \left(P_2(x) + \lambda\, P_3(x)\right)y = 0$, in which λ is a parameter; $P_1(x), P_2(x), P_3(x)$ are real valued continuous functions of $x\ on\ [a, b]$ with the boundary conditions $a_1\, y(a) + a_2 y'(a) = 0$ and $b_1\, y(b) + b_2 y'(b) = 0$ where $a_1\ and\ a_2$and likewise $b_1\ and\ b_2$ are not both zero.

Examples – 1.32

Find the eigen values and eigen functions for the equation $\frac{d^2y}{dx^2} + \lambda\, y = 0$ which satisfies the boundary conditions $y(0) = 0\ \ and\ \ y(\pi) = 0$.

Solution: If $\lambda = 0$, The general solution is $y = c_1 + c_2 x$

$y(0) = 0,\ y(\pi) = 0 \ \Rightarrow\ c_1 = c_2 = 0$

The solution becomes trivial.

If $\lambda < 0$. The general solution is $y = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}\,x}$

$y(0) = 0,\ y(\pi) = 0 \ \Rightarrow\ c_1 + c_2 = 0\ \ and\ c_1 e^{\sqrt{-\lambda}\pi} + c_2 e^{-\sqrt{-\lambda}\pi} = 0 \Rightarrow\ c_1 = c_2 = 0$

This solution is also trivial.

If $\lambda > 0$. The general solution is $y = c_1\cos\sqrt{\lambda}\,x + c_2\sin\sqrt{\lambda}\,x$

$y(0) = 0,\ y(\pi) = 0 \ \Rightarrow\ c_1 = 0\ and\ \ c_2\sin\sqrt{\lambda}\pi = 0$

Now, $c_2 \neq 0$

$\Rightarrow\ \sin\sqrt{\lambda}\pi = 0 = \sin n\pi$

$\Rightarrow \sqrt{\lambda}\,\pi = n\pi\ \Rightarrow \lambda_n = n^2\ ,\ n = 1\,, 2, 3, \ldots\ldots..\ .$

So the eigen values are $\lambda_n = n^2$ and eigen functions are $\phi_n(x) = A_n\sin nx$ where $n = 1\,, 2, 3, \ldots ...$

1.15. System of first order ordinary differential equation.

Let us consider the system of equations

$\frac{dx}{dt} = a_1\, x + b_1\, y$

$\frac{dy}{dt} = a_1\,x + b_2\,y$ where $a_1\,, a_2, b_1, b_2$ real constants.

i.e.,$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = A\,\begin{bmatrix}x\\ y\end{bmatrix}$, $where\ \dot{x} = \frac{dx}{dt},\ \dot{y} = \frac{dy}{dt}\ \ and\ \ A = \begin{bmatrix}a_1 & b_1\\ a_2 & b_2\end{bmatrix}$

We assume $|A| = \begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix} \neq 0$, so that (0,0) is the only critical point.

The characteristics equation of A is

$\begin{vmatrix}a_1-\lambda & b_1\\ a_2 & b_2-\lambda\end{vmatrix} = 0\ \ or,\ \lambda^2 - (a_1+b_2)\lambda + (a_1b_2 - a_2b_1) = 0$

Let $\lambda_1\ and\ \lambda_2$ be the roots of the equation. The eigen values of A are $\lambda_1\ and\ \lambda_2$.

(i) If $\lambda_1\ and\ \lambda_2$ be real, distinct and of the same sign, then it is a node. The node will be stable or unstable according as $\lambda_1\ and\ \lambda_2$ are both negative or both positive.

(ii) If $\lambda_1\ and\ \lambda_2$be real, distinct and of opposite signs, then it is a saddle point.

(iii) If $\lambda_1\ and\ \lambda_2$ be conjugate complex but not purely imaginary, then it is a spiral point. The spiral point is stable or unstable according as the real point is negative or positive.

(iv) If $\lambda_1\ and\ \lambda_2$ be purely imaginary, then it is a centre.

Examples – 1.33

Determine the nature of the equilibrium point of the system $\frac{dx}{dt} = y,\ \frac{dy}{dt} = -5x + 2y.$ Find the general solution of the system.

Solution:- $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -5 & 2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$

So the characteristic equation is

$\begin{bmatrix}-\lambda & 1\\ -5 & 2-\lambda\end{bmatrix} = 0\ \ i.e.,\ \lambda^2 - 2\lambda + 5 = 0\ \Rightarrow \lambda = 1 \pm 2i$

which are complex conjugate with Real $\lambda > 0$

Hence the equilibrium point is an unstable spiral Point.

Now,

$\frac{d^2x}{dt^2} = \frac{dy}{dt} = -5x + 2y = -5x + 2\frac{dx}{dt}$

$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 5x = 0 \Rightarrow m = \frac{2\pm\sqrt{-16}}{2} = 1 \pm 2i$

So $x = e^{\,t}(A\cos 2t + B\sin 2t)$

$y = \frac{dx}{dt} = e^{\,t}(-2A\sin 2t + 2B\cos 2t) + e^{\,t}(A\cos 2t + B\sin 2t)$

This is the general solution.

1.16. Green's function

Consider a second order differential equations

$[P(x)y'(x)]' + q(x)\,y(x) + r\,(x) = 0.$

Green's function is constructed only when trivial solution $y(x) = 0$ exist for the given differential equation with the help of given boundary conditions and the Green's function is given by – $G(x,t) = \begin{cases} -\frac{y_1(x)y_2(t)}{c}, & x < t \\ -\frac{y_2(x)y_1(t)}{c}, & x > t \end{cases}$

Where $c = P(t).W\big(y_1(t)y_2(t)\big)$ here $y_1(x)$ *and* $y_2(x)$ are two independent solutions of $[y(x)]'' = 0$ by using given boundary conditions.

Examples – 1.34

Construct the Green's Function of BVP

$y'' + \lambda\, y = x\,,\ y(0) = y\left(\frac{\pi}{2}\right) = 0$

Solution:- $y'' + \lambda\, y - x = 0$ *or,* $y'' + \phi(x) = 0$ *where* $\phi(x) = \lambda\, y - x$

Now, $[y(x)]'' = 0 \Rightarrow y(x) = ax + b$ $[\,a, b\ are\ arbitrary\ constants]$

When $y(0) = 0, y\left(\frac{\pi}{2}\right) = 0$

$\therefore\ b = 0\,, again\ u\left(\frac{\pi}{2}\right) = a\cdot\frac{\pi}{2} + 0\ or, 0 = a\cdot\frac{\pi}{2} \Rightarrow a = 0.$

$\therefore\ y(x) = 0$ for $a = 1, y_1(x) = x$

For $y\left(\frac{\pi}{2}\right) = 0 \Rightarrow y\left(\frac{\pi}{2}\right) = a\frac{\pi}{2} + b \Rightarrow 0 = a\frac{\pi}{2} + b \Rightarrow a = -\frac{2b}{\pi}$

$y(x) = a\,x + b = -\frac{2b}{\pi}\,x + b = \left(1 - \frac{2}{\pi}x\right)b$

For $b = 1$

$y_2(x) = \left(\frac{\pi - 2x}{\pi}\right)\cdot 1 = 1 - \frac{2x}{\pi}$

Now,

$$W\big(y_1(t), y_2(t)\big) = \begin{vmatrix} y_1(t) & y_2(t) \\ y'_1(t) & y'_2(t) \end{vmatrix} = \begin{vmatrix} t & 1 - \frac{2}{\pi}t \\ 1 & -\frac{2}{\pi} \end{vmatrix} = -1$$

$P(t) = 1 \Rightarrow c = -1$

$$\therefore\ G(x,t) = \begin{cases} -\frac{x\left(1 - \frac{2t}{\pi}\right)}{-1}, & 0 \le x < t \\ -\frac{\left(1 - \frac{2x}{\pi}\right)t}{-1}, & t < x < \frac{\pi}{2} \end{cases}$$

i.e., $G(x,t) = \begin{cases} x\left(1-\frac{2t}{\pi}\right), & 0 \le x < t \\ \left(1-\frac{2x}{\pi}\right)t, & t < x < \frac{\pi}{2} \end{cases}$

Examples – 1.35

Construct Green's function for the BVP $y'' - y = x, y(0) = y(1) = 0$

Solution:-

$y'' - y + \phi(x) = 0\,, where\ \phi(x) = -x$

$y'' - y = 0,\ A.E. is\ \ m^2 - 1 = 0 \ \ \Rightarrow m = \pm 1$

General solution is $y(x) = a\cosh x + b\sinh x$

Now using boundary conditions $y(0) = 0, y(1) = 0$

$$0 = a \cdot 1 + b \cdot 0 \ \Rightarrow a = 0$$

Again, $y(1) = a\cosh 1 + b\sinh 1 \ \Rightarrow 0 = b\sinh 1 \Rightarrow b = 0$ $\qquad [\because \sinh 1 \neq 0]$

$\therefore$ Only trivial solution exists.

$y(0) = 0 \ \Rightarrow a = 0 \ \Rightarrow y(x) = b\sinh x$

For $b = 1 \qquad y_1(x) = \sinh x$

$b = \frac{-a\cosh 1}{\sinh 1}$

Now,

$y(x) = a\cosh x - \frac{a\cosh 1}{\sinh 1}\sinh x = \frac{-a\sinh(x-1)}{\sinh 1}$

For $a = -\sinh 1$

$y_2(x) = \sinh(x-1)$

$W\big(y_1(t)y_2(t)\big) = \begin{vmatrix} \sinh t & \sinh(t-1) \\ \cosh t & \cosh(t-1) \end{vmatrix} = \sinh 1 \ \Rightarrow c = \sinh 1$

Green's function is

$\therefore \ G(x,t) = \begin{cases} \frac{-\sinh x\sinh(t-1)}{\sinh 1}, & 0 \le x < t \\ \frac{-\sinh(x-1)\sinh t}{\sinh 1}, & t < x \le 1 \end{cases}$