University Grants Commission

Subject: Economics Code: 01

Unit-4: Mathematical Economics

Sub Unit 1: Introduction to Mathematical economics

Sl. No.	Topics: Introduction to Mathematical economics
1	4.1.1. Meaning and Importance
2	4.1.2. Overview of Mathematical Economics
3	4.1.3. Sequence
4	4.1.4. Introduction to Sequences
5	4.1.5. Concept of sequence
6	4.1.6. Fibonacci Sequence
7	4.1.7. Types of Sequences
8	4.1.8. Concept of Economic Function
9	4.1.9. Some Economic Functions
10	4.1.10. Marginal Concept and Economic Application
11	4.1.11. Application of Marginal Theory
12	4.1.12. Relationship between Average Revenue, Marginal revenue
13	4.1.13. Relationship between Average Cost and Marginal Cost
14	4.1.14. Concept of Elasticity
15	4.1.15. Different Types of Elasticity

Sub Unit – 2: Optimisation, Linear Programming and Input-Output Analysis:

Sl. No.	Topics: Optimisation, Linear Programming and Input-Output Analysis
1	4.2.1. Introduction to Optimization
2	4.2.2. solution of Optimization
3	4.2.3. Application of Optimization
4	4.2.4. Utility Maximization
5	4.2.5. Cost Minimization
6	4.2.6. Profit Maximization
7	4.2.7. Linear Programming Problem (LPP)
8	4.2.8. Terms of Linear programming Problem
9	4.2.9. Assumption of LPP
10	4.2.10. Application of LPP
11	4.2.11. Formulation of Linear Programming
12	4.2.12. Solution of LPP

13	4.2.13. Limitations of Linear Programming Technique
14	4.2.14. Input – Output Analysis
15	4.2.15. Assumptions of Input-Output Analysis
16	4.2.16. Input-Output Table
17	4.2.17. Hawkins – Simon Condition
18	4.2.18. Example of Input-Output Analysis
19	4.2.19 Limitation of Input-Output Analysis

Sub-Unit 3: Market Equilibrium Analysis

Sl. No.	Topics: Market Equilibrium Analysis
1	4.3.1. Introduction to Equilibrium Analysis
2	4.3.2. Equilibrium in Perfect Competitive Market
3	4.3.3 Equilibrium in the Monopoly
4	4.3.4. Consumer and Producer Surplus

Sub-Unit 1: Introduction to Mathematical Economics

4.1.1. Meaning and Importance:

Mathematical economics is the application of mathematical methods to represent economic theories and analyze problems posed in economics. It allows formulation and derivation of key relationships in a theory with clarity, generality, rigor, and simplicity. By convention, the methods refer to those beyond simple geometry, such as differential and integral calculus, difference and differential equations, matrix algebra, and mathematical programming and other computational methods.

Mathematics allows economists to form meaningful, testable propositions about many wide-ranging and complex subjects which could not be adequately expressed informally.

Further, the language of mathematics allows economists to make clear, specific, positive claims about controversial or contentious subjects that would be impossible without mathematics. Much of economic theory is currently presented in terms of mathematical economic models, a set of stylized and simplified mathematical relationships that clarify assumptions and implications.

Paul Samuelson argued that mathematics is a language. In economics, the language of mathematics is sometimes necessary for representing substantive problems. Moreover, mathematical economics has led to conceptual advances in economics.

4.1.2. Overview of Mathematical Economics:

Mathematical economics is a branch of economics that engages mathematical tools and methods to analyse economic theories. Mathematical economics is best defined as a sub-field of economics that examines the mathematical aspects of economies and economic theories. Or put into other words, mathematics such as calculus, matrix algebra, and differential equations are applied to illustrate economic theories and analyse economic hypotheses.

It may be interesting to begin the study of mathematical economics with an enquiry into the history of mathematical economics. It is generally believed that the use of mathematics as a tool of economics dates from the pioneering work of Cournot (1838). However there were many others who used mathematics in the analysis of economic ideas before Cournot. We shall make a quick survey of the most important contributors.

Sir William Petty is often regarded as the first economic statistician. In his Discourses on Political Arithmetic (1690), he declared that he wanted to reduce political and economic matters to terms of number, weight, and measure. The first person to apply mathematics to economics with any success was an Italian, Giovanni Ceva who in 1711 wrote a tract in which mathematical formulas were generously used. He is generally regarded as the first known writer to apply mathematical method to economic problems. The Swiss mathematician Daniel Bernoulli in 1738 for the first time used calculus in his analysis of a probability that would result from games of chance rather than from economic problems.

Among the early French writers who made some use of mathematics was Francois de Forbonnais who used mathematical symbols, especially for explaining the rate of exchange between two countries and how equilibrium is finally established between them. He is best known for his severe attack on Physiocracy.

While this is a long list to those who showed curiosity in the use of mathematics in economics, interestingly J. B. Say showed little or no interest in the use of mathematics. He did not favour the use of mathematics for explaining economic principles. A German who is reasonably well known, especially in location theory, is Johann Heinrich von Thunen. His first work, The Isolated State (1826), was an attempt to explain how transportation costs influence the location of agriculture and even the methods of cultivation.

The French engineer, A. J. E. Dupuit, used mathematical symbols to express his concepts of supply and demand. Even though he had no systematic theory he did develop the concepts of utility and diminishing utility, which were clearly stated and presented in graphical form.

He viewed the price of a good as dependent on the price of other goods. Another Lausanne economist, Vilfredo Pareto, ranks with the best in mathematical virtuosity. The Pareto optimum and the indifference curve analysis (along with Edgeworth in England) were clearly conceived in the mathematical frame work.

A much more profound understanding of the analytical power of mathematics was shown by Leon Walras. He is generally regarded as the founder of the mathematical school of economics. He set out to translate pure theory into pure mathematics. After the pioneering work of Jevons (1871) and Walras (1874), the use of mathematics in economics progressed at very slow pace for a number of years. The first of the latter group is F. Y. Edgeworth. His contribution to mathematical economics is found mainly in the 1881 publication in which he dealt with the theory of probability and statistical theory.

F. Y. Edgeworth's contribution to mathematical economics is found mainly in the 1881 publication in which he dealt with the theory of probability, statistical theory and the law of error. The indifference analysis was first propounded by Edgeworth in 1881 and restated in 1906 by Pareto and in 1915 by the Russian economist Slutsky, who used elaborate mathematical treatment of the topic. Alfred Marshall made extensive use of mathematics in his Principles of Economics (1890). His interest in mathematics dates from his early schooldays when his first love was for mathematics, not the classics." In the Principles he used mathematical techniques very effectively.

Another economist whose influence spans many years is the American, Irving Fisher. Fisher belongs to other groups as well as to the mathematical moderns. His life's work reveals him as a statistician, econometrician, mathematician, pure theorist, teacher, social crusader, inventor, businessman, and scientist. His contribution to statistical method, The Making of Index Numbers, was great. The well-known Fisher formula, $M V + M' V' = P T$, is an evidence of his contribution to quantitative economics.

A strong inducement to formulate economic models in mathematical terms has been the post-World War II development of the electronic computer. In the broad area of economics there has been a remarkable use of mathematical techniques. Economics, like several other disciplines, has always used quantification to some degree. Common terms such as wealth, income, margins, factor returns, diminishing returns, trade balances, balance of payments, and the many other familiar concepts have a quantitative connotation. All economic data have in some fashion been reduced to numbers which became generally known as economic statistics.

The most noteworthy developments, however, have come in the decades since about 1930. It was approximately this time that marked the ebb of neoclassicism, the rise of institutionalism, and the introduction of aggregate economics. Over a period of years scholars have developed new techniques designed to help in the explanation of economic behavior under different market situations using mathematics. Now we discuss some of the economic theories and the techniques of modern analysis. They are given largely on a chronological basis, and their significance is developed in the discussion.

The Theory of Games: The pioneering work was done by John von Neumann in 1928. The theory became popular with the publication of Theory of Games and Economic Behaviour in 1944. Basically, The Game Theory holds that the actions of players in gambling games are similar to situations that prevail in economic, political, and social life. The theory of games has many elements in common with real-life situations. Decisions must be made on the basis of available facts, and chances must be taken to win. Strategic moves must be concealed (or anticipated) by the contestants based on past knowledge and future estimates. Success or failure rests, in large measure, on the accuracy of the analysis of the elements. The theory of games introduced an interesting and challenging concept. Economists have made some use of it, and it has also been fitted into other social sciences, notably sociology and political science.

Linear Programming: Linear programming is a specific class of mathematical problems in which a linear function is maximized (or minimized) subject to given linear constraints. The founders of the subject are generally regarded as George B. Dantzig, who devised the simplex method in 1947, and John von Neumann, who established the theory of duality that same year. The scope of linear programming is very broad. It brings together both theoretical and practical problems in which some quantity is to be maximized or minimized. The data could be almost any fact such as profit, costs, output, and distance to or from given points, time, and so on. It also makes allowance for given technology and restraints that may occur in factor markets or in finance. Linear programming has been proven very useful in many areas. It is in common use in agriculture, where chemical combinations of proper foods for plants and animals have been worked out, and in the manufacture of many processed agricultural products. It is necessary in modem materials scheduling, in shipping, and in final production. The Nobel Prize in economics was awarded in 1975 to the mathematician Leonid Kantorovich (USSR) and the economist Tjalling Koopmans (USA) for their contributions to the theory of optimal allocation of resources, in which linear programming played a key role.

Input-Output Analysis: In terms of techniques, input-output analysis is a rather special case of linear programming. It was devised originally by Leontief and, in a sense, was a World War II-inspired analysis. Basically it was designed for presenting a general equilibrium theory suited for empirical study. The problem is to determine the interrelationship of sector inputs and outputs on other sectors or on all sectors which use the product. The rational for the term IOA can be explained like this. There is a close interdependence between different sectors of a modern economy. This interdependence arises out of the fact that the output of any given industry is utilized as an input by the other industries and often by the same industry itself. Thus the IOA analyses the interdependence between different sectors of an economy. The basis of IOA is the input - output table which can be expressed in the form of matrices.

4.13. Sequence:

4.1.4. Introduction to Sequences:

Imagine yourself at a pizza hut. You have placed an order and your order number is 282. So currently they are serving the order number 275. So how many orders do you think will be served before your number? Yes, six orders more because you are in a sequence. To understand this better, let us learn about the sequences and series. Let us do it right now.

4.1.5. Concept of sequence:

A sequence is a list of numbers in a special order. It is a string of numbers following a particular pattern, and all the elements of a sequence are called its terms. Let us consider a sequence,

[1, 3, 5, 7, 9, 11……]

We can say this is sequence because we know that they are the collection of odd natural numbers. Here the number of terms in the sequence will be infinite. Such a sequence which contains the infinite number of terms is known as an **infinite sequence.** But what if we put end to this.

[1, 3, 5, 7, 9, 11…..131]

If 131 is the last term of this sequence, we can say that the number of terms in this sequence is countable. So in such sequence in which the number of terms is countable, such sequences are called **finite sequences**. A finite sequence has the finite number of terms. So as discussed earlier here 1 is the term, 3 is the term so is 5, 7…..

4.1.6. Fibonacci Sequence:

The special thing about the Fibonacci sequence is that the first two terms are fixed. When we talk about the terms, there is a general representation of these terms in sequences and series. A term is usually denoted as an here ‘ n ‘ is the nth term of a sequence. For the Fibonacci sequence, the first two terms are fixed.

The first term is as a1= 1 and a2= 1. Now from the third term onwards, every term of this Fibonacci sequence will become the sum of the previous two terms. So a3will be given as a

1 + a2

Therefore, 1 + 1 = 2. Similarly, a4 = a2 + a3 ∴ 1 + 2 = 3 a5 = a3+ a4 ∴ 2 + 3 = 5

Therefore if we want to write the Fibonacci sequence, we will write it as, [1 1 2 3 5…]. So, in general, we can say,

an = an-1 + an-2

4.1.7. Types of Sequences

1. **Arithmetic sequence:** In an arithmetic (linear) sequence the difference between any two consecutive terms is constant.
2. **Quadratic Sequence:** A quadratic sequence is a sequence of numbers in which the second difference between any two consecutive terms is constant.
3. **Geometric Sequence:** A geometric sequence is a sequence of numbers where each term after the first is found by multiplying the previous one by a fixed, non-zero number called the common ratio.

4.1.8. Concept of Economic Function:

A variable represents a concept or an item whose magnitude can be represented by a number, i.e. measured quantitatively. Variables are called variables because they vary, i.e. they can have a variety of values. Thus a variable can be considered as a quantity which assumes a variety of values in a particular problem. Many items in economics can take on different values. Mathematics usually uses letters from the end of the alphabet to represent variables. Economics however often uses the first letter of the item which varies to represent variables. Thus p is used for the variable price and q is used for the variable quantity.

An expression such as $4x^3$ is a variable. It can assume different values because x can assume different values. In this expression x is the variable and 4 is the coefficient of x. Coefficient means 4 works together with x. Expressions such as $4x^3$ which consists of a coefficient times a variable raised to a power are called monomials. A monomial is an algebraic expression that is either a numeral, a variable, or the product of numerals and variables. (Monomial comes from the Greek word, *monos*, which means one.) Real numbers such as 5 which are not multiplied by a variable are also called monomials. Monomials may also have more than one variable. $4x^3y^2$ is such an example. In this expression both x and y are variables and 4 is their coefficient.

The following are examples of monomials: x, $4x^2$, $-6xy^2z$, 7

One or more monomials can be combined by addition or subtraction to form what are called polynomials. (Polynomial comes from the Greek word, poly, which means many.) A polynomial has two or more terms i.e. two or more monomials. If there are only two terms in the polynomial, the polynomial is called a binomial.

The expression $4x^3y^2 - 2xy^2 + 3$ is a polynomial with three terms.

These terms are $4x^3y^2$, $-2xy^2$, and 3. The coefficients of the terms are 4, −2, and 3.

The degree of a term or monomial is the sum of the exponents of the variables. The degree of a polynomial is the degree of the term of highest degree. In the above example the degrees of the terms are 5, 3, and 0. The degree of the polynomial is 5.

Remember that variables are items which can assume different values. A function tries to explain one variable in terms of another.

Independent variables are those which do not depend on other variables. Dependent variables are those which are changed by the independent variables. The change is caused by the independent variable. The independent variable is often designated by x. The dependent variable is often designated by y.

We say y is a function of x. This means y depends on or is determined by x.

mathematically we write y = f(x)

It means that mathematically y depends on x. If we know the value of x, then we can find the value of y.

A function is a mathematical relationship in which the values of a single dependent variable are determined by the values of one or more independent variables. Function means the dependent variable is determined by the independent variable(s). A function tries to define these relationships. It tries to give the relationship a mathematical form. An equation is a mathematical way of looking at the relationship between concepts or items. These concepts or items are represented by what are called variables. Economists are interested in examining types of relationships. For example an economist may look at the amount of money a person earns and the amount that person chooses to spend. This is a consumption relationship or function. As another example an economist may look at the amount of money a business firm has and the amount it chooses to spend on new equipment. This is an investment relationship or investment function. Functions with a single independent variable are called univariate functions. There is a one to one correspondence. Functions with more than one independent variable are called multivariate functions.

Example of use of functions:

y = f(x) = 3x + 4

This is a function that says that, y, a dependent variable, depends on x, an independent variable. The independent variable, x, can have different values. When x changes y also changes.

Find f(0). This means find the value of y when x equals 0.

f(0) = 3 times 0 plus 4

f(0) = 3(0) + 4 = 4

Find f(1). This means find the value of y when x equals 1.

f(1) = 3 times 1 plus 4

f(1) = 3(1) + 4 = 7

Find f(-1). This means find the value of y when x equals -1.

f(-1) = 3 times (-1) plus 4

f(1) = 3(-1) + 4 = 1

4.1.9. Some Economic Functions:

I. Demand function

Demand function express the relationship between the price of the commodity (independent variable) and quantity of the commodity demanded (dependent variable).It indicate how much quantity of a commodity will be purchased at its different prices. Hence, represent the quantity demanded of a commodity and Px is the price of that commodity. Then,

Demand function, dx = f (Px)

The basic determinants of demand function

= f (Px, Pr, Y, T, W, E)

Where Qx: quantity demanded of a commodity X, Px: price of commodity X, Pr: price of related good, Y: consumer's income, T: Consumer/s tastes and preferences, W: Consumer's wealth, E: Consumer's expectations.

Example: $Q_d = p^2 - 20p + 125$

This is a function that describes the demand for an item where p is the dollar price per item. It says that demand depends on price.

Find the demand when one item costs Rs. 2

$d(2) = 2^2 - 20(2) + 125 = 89$

Find the demand when one item costs Rs. 5

$d(5) = 5^2 - 20(5) + 125 = 50$

Notice that the demand decreases as the price increases which you know is the law of demand.

II. Supply function

The functional relationship between the quantity of commodities supplied and various determinants is known as supply function. It is the mathematical expression of the relationship between supply and factors that affect the ability and willingness of the producer to offer the product.

Mathematically, a supply function can be expressed as Supply $S_x= f (P_x)$

There are a number of factors and circumstances which can influence a producer's willingness to supply the commodity in the market. These factors are price of the commodity, price of the related goods, price of the factors of production, goal of producers, state of technology, miscellaneous factors (we can include factors such as means of transportation and communication, natural factors, taxation policy, expectations, agreement among the producers, etc.) Incorporating all such factors we may write the basic form of a supply function as;

Qs= f (Gf , P, I, T, , E,)

Where Qs: quantity supplied, Gf: Goal of the firm, P: Product's own price, I: Prices of inputs, T: Technology, Pr: Prices of related goods, E: Expectation of producer's, Gp: government policy.

Example: Given a supply function $Q_s = -20 + 3P$ and demand function $Q_d = 220 – 5P$, Find the equilibrium price and quantity.

Equilibrium in any market means equality of demand and supply.

Hence, at equilibrium $-20 + 3P = 220 – 5P$.

Solving the above we get, Equilibrium price P = 30, Equilibrium quantity $Q_s = Q_d = 70$

III. Utility function:

Utility function is a mathematical function which ranks alternatives according to their utility to an individual. The utility function measures welfare or satisfaction of a consumer as a function of consumption of real goods, such as food, clothing and composite goods rather than nominal goods measured in nominal terms. Thus the utility function shows the relation between utility derive from the quantity of different commodity consumed. A utility function for a consumer consuming three different goods may be represented:

$U = f(X_1, X_2, X_3 \ldots\ldots\ldots)$

Example: Given the utility unction of a consumer $U = 2x^2 + 5$, find the marginal utility.

Marginal utility is given by the first order derivative of the total utility function.

Marginal Utility = $\frac{du}{dx} = 4$

IV. Consumption Function

The consumption function refers to the relationship between income and consumption. It is a functional relationship between consumption and income. Symbolically, the relationship is represented as C= f(Y), where C is consumption, Y is income. Thus the consumption function indicates a functional relationship between C and Y, where C is the dependant variable and Y is the independent variable, i.e., C is determined by Y. In fact, propensity to consume or consumption function is a sketch of the various amounts of consumption expenditure corresponding to different levels of income.

In the Keynesian framework, the consumption function or propensity to consume, refers to a functional relationship between two aggregates, i.e., total consumption and gross national income. The **Keynesian Consumption function $C = a + bY_d$,** expresses the level of consumer spending depending on three factors as explained below.

Y_d = disposable income (income after government intervention – e.g. benefits, and taxes)

a = autonomous consumption (This is the level of consumption which does not depend on income. The argument is that even with zero income we still need to buy some food to eat, through borrowing or using our savings.)

b = marginal propensity to consume (also known as induced consumption).

The average propensity to consume is the ratio of consumption expenditure to any particular level of income. It is found by dividing consumption expenditure by income, or APC = C/Y. It is expressed as the percentage or proportion of income consumed. The marginal propensity to consume is the ratio of the change in consumption to the change in income. It can be found by dividing change in consumption by a change in income, or by finding the first derivative of the utility function. The MPC is constant at all levels of income.

The MPC is the rate of change in the APC. When income increases, the MPC falls but more than the APC. Contrariwise, when income falls, the MPC rises and the APC also rises but at a slower rate than the former. Such changes are only possible during cyclical fluctuations whereas in the short-run there is change in the MPC and MPC<APC.

V. Saving function

The relationship between disposable income and saving is called the saving function. The saving function can represent in a general from as S= (y), Where, S is savings, and y is income, f is the notation for a generic, unspecified functional for. Because savings is the difference between disposable income and consumption, the savings function is a complementary relation to the consumption function. It is assumed that whatever is not consumer is saved. So, MPC+MPS=1

Given a saving function S = 70 + 0.8Y, find MPS and MPC MPS = 0.8, MPC = 1 – MPS = 0.2

VI. Production function

In a crude sense, production is the transformation of inputs into output. In another way, production is the creation of utility. Production is possible only if inputs are available and used. There are different types of inputs and economists classify them into land, labour, capital and organization. In modern era, the meaning of these terms is redefined. Today, land covers all natural resources, labour covers human resources, capital is replaced by the term technology and finally, instead of organization, we use the term management.

We explain production as the transformation of inputs into output.

In a general form a production function can be written as $Q = f(x_1, x_2, x_3, \ldots\ldots\ldots\ldots x_n)$

Where, Q represents the quantity produced, $x_1 \ldots\ldots x_n$ are inputs. The general mathematical form of Production function is:

$$Q = f(L, K, R, S, v, e)$$

Where Q stands for the quantity of output, L is the labour, K is capital, R is raw material, S is the Land, v is the return to scale and e is efficiency parameters.

Example: Q = 42KL – 3K2 – 2L2, Q = K0.4L0.5

i. Linear, Homogeneous Production Function:

Production function can take several forms but a particular form of production function enjoys wide popularity among the economists. This is a linear homogeneous production function, that is, production function which is homogenous production function of the first degree. Homogeneous production function of the first degree implies that if all factors of production are increased in a given proportion, output also increased in a same proportion. Hence linear homogeneous production function represents the case of constant return to scales. If there are two factors X and Y, The production function and homogeneous production function of the first degree can be mathematically expressed as,

$Q = f(X, Y)$ Where Q stands for the total production, X and Y represent total inputs.

$mQ = f(mX, mY)$ m stands any real number

The above function means that if factors X and Y are increased by m-times, total production Q also increases by m-times. It is because of this that homogeneous function of the first degree yield constant return to scale.

More generally, a homogeneous production function can be expressed as

$Qmk = (mX,mY)$

Where m is any real number and k is constant. This function is homogeneous function of the k^{th} degree. If k is equal to one, then the above homogeneous function becomes homogeneous of the first degree. If k is equal to two, the function becomes homogeneous of the 2^{nd} degree.

If $k>1$, the production function will yield increasing return to scale.

If $k<1$, it will yield decreasing return to scale.

If $k=1$, it will yield constant return to scale.

ii. Cobb – Douglas Production Function

Many Economists have studied actual production function and have used statistical methods to find out relations between changes in physical inputs and physical outputs. A most familiar empirical production function found out by statistical methods is the Cobb – Douglas production function. Cobb – Douglas production function was developed by Charles Cobb and Paul Douglas. In C-D production function, there are two inputs, labour and capital, Cobb – Douglas production function takes the following mathematical form

$Q = AL^{\sigma}K^{\beta}$

Where Q is the manufacturing output, L is the quantity of labour employed, K is the quantity of capital employed, A is the total factor productivity or technology are assumed to be a constant. The α and β, output elasticity's of Labour and Capital and the A,α and β are positive constant.

Roughly speaking, Cobb –Douglas production function found that about 75% of the increasing in manufacturing production was due to the Labour input and the remaining 25 % was due to the Capital input.

- **Properties of Cobb – Douglas Production Function**

a). Average product of factors: The first important properties of C – D production function as well as of other linearly homogeneous production function is the average and marginal products of factors depend upon the ratio of factors are combined for the production of a commodity. Average product if Labour (APL) can be obtained by dividing the production function by the amount of Labour L. Thus,

Average Product Labour (Q/L)

$$Q = AL^{\alpha}K^{\beta}$$

$$\frac{Q}{L} = \frac{AL^{\alpha}K^{\beta}}{L} = \frac{AK^{\beta}}{L^{1-\alpha}} = A\ \left(\frac{K}{L}\right)^{\beta}$$

Thus Average Product of Labour depends on the ratio of the factors (K/L) and does not depend upon the absolute quantities of the factors used.

Average Product of Capital (Q/K)

$$Q = AL^{\alpha}K^{\beta}$$

$$\frac{Q}{K} = \frac{AL^{\alpha}K^{\beta}}{K} = \frac{AL^{\alpha}}{K^{1-\beta}} = A\ \left(\frac{L}{K}\right)^{\alpha}$$

So the average Product of capital depends on the ratio of the factors (L/K) and does nt0o depend upon the absolute quantities of the factors used.

b). Marginal Product of Factors: The marginal product of factors of a linear homogeneous production function also depends upon the ratio of the factors and is independent of the absolute quantities of the factors used. Note, that marginal product of factors, says Labour, is the derivative of the production function with respect to Labour.

$$Q = AL^{\alpha}K^{\beta}$$

$$MP_L = \frac{\partial Q}{\partial L} = \alpha AL^{\alpha-1}\, K^{\beta} = \frac{A\ \alpha K^{\beta}}{L^{1-\alpha}} = \alpha A \left(\frac{K}{L}\right)^{\beta}$$

$$MP_L = \alpha\ A\ P_L$$

It is thus clear that MP_L depends on capital – labour ratio, that is, Capital per worker and is independent of the magnitudes of the factors employed.

$$Q = AL^{\alpha}K^{\beta}$$

$$MP_K = \frac{\partial Q}{\partial K} = \beta K^{\beta-1}\, L^{\alpha} = \frac{\beta A\ L^{\alpha}}{K^{1-\beta}} = \beta A \left(\frac{L}{K}\right)^{\alpha}$$

$MP_L = \beta \; A \; P_K$

It is thus clear that MPL depends on capital – labour ratio, that is, capital per worker and is independent of the magnitudes of the factors employed.

$Q = AL^{\alpha}K^{\beta}$

$MP_K = \frac{\partial Q}{\partial K} = \beta K^{\beta-1} \, L^{\alpha} = \frac{\beta A \, L^{\alpha}}{K^{1-\beta}} = \beta A \left(\frac{L}{K}\right)^{\alpha} = \beta \; A \; P_K$

c). Marginal rate of substitution: Marginal rate of substitution between factors is equal to the ratio of the marginal physical products of the factors. Therefore, in order to derive MRS form Cobb – Douglas production function, we used to obtain the marginal physical products of the two factors from the C – D function.

$Q = AL^{\alpha}K^{\beta}$

Differentiating this with respect to L, we have

$MP_L = \frac{\partial Q}{\partial L} = \partial AL^{\alpha}K^{\beta} = \alpha AL^{\alpha-1}K^{\beta} = \frac{\alpha\left(AL^{\alpha}K^{\beta}\right)}{\mathrm{L}^{1}}$

Now, $Q = AL^{\alpha}K^{\beta}$,

Therefore, $= \alpha \left(\frac{Q}{L}\right)$

$\frac{\partial Q}{\partial L}$ Represents the marginal product of labour and $\frac{Q}{L}$ stands for the average of labour.

Thus, $MP_L = \alpha(AP_L)$

Similarly, by differentiating C – D production function with respect to capital, we can show that marginal product of capital.

$Q = AL^{\alpha}K^{\beta}$

$MP_K = \frac{\partial Q}{\partial K} = \beta \; AL^{\alpha}K^{\beta-1} = \frac{\beta\left(AL^{\alpha}K^{\beta}\right)}{K^{1}} = \beta \left(\frac{Q}{K}\right)$

$MRS_{LK} = \frac{MP_L}{MP_K} = \frac{\alpha\left(\frac{Q}{L}\right)}{\beta\left(\frac{Q}{K}\right)} = \frac{\alpha}{\beta} \times \frac{K}{L}$

d). C – D production function and Elasticity of substitution ($\boldsymbol{\ell s \; or \; \sigma}$) is equal to unity.

$$\ell s = \frac{Proportionate\ change\ in\ Capital-Labour\ ratio\ \left(\frac{K}{L}\right)}{Proportionate\ change\ in\ MRS_{LK}} = \frac{\frac{d\left(\frac{K}{L}\right)}{K\backslash L}}{d\left(\frac{MRS_{LK}}{MRS_{LK}}\right)}$$

Substituting the value of MRS obtain in above $= \frac{\frac{d\left(\frac{K}{L}\right)}{K\backslash L}}{\frac{\partial\left(\frac{\alpha}{\beta}\times\frac{K}{L}\right)}{\frac{\alpha}{\beta}\times\frac{K}{L}}} = 1$

e). Return to Scale: An important property of C – D production function is that the sum if its exponents measures returns to scale. That is, When the sum of exponents is not necessarily equal to zero is given below.

$Q = AL^{\alpha}K^{\beta}$

In this production function the sum of exponents $(\alpha + \beta)$ measures return to scale. Multiplying each input labour (L) and capital (K), by a constant factor g, we have

$Q = A(gL)^{\alpha}(gK)^{\beta} = \ Q = g^{\alpha}\ g^{\beta}\left(AL^{\alpha}K^{\beta}\right)$ i.e. $a^m \times a^n = a^{m+n}$

$= g^{\alpha+\beta}\left(AL^{\alpha}K^{\beta}\right)$

i.e. $Q' = g^{\alpha+\beta}\ Q$

This means that when each input is increased by a constant factor g, output Q increase by $g^{\alpha+\beta}$. Now, if $\alpha + \beta = 1$ then, in this production function.

$Q' = \ g'Q$

$Q' = \ g\ Q$

This is , when $\alpha + \beta = 1$, output (Q) also increases by the same factor g by which both inputs are increased. This implies that production function is homogeneous of first degree or, in other words, retune to scale are constant.

When $\alpha + \beta > 1$, say it is equal to 2, then, in this production function new output.

$Q' = g^{\alpha+\beta}\ A\ L^{\alpha}\ K^{\beta} = g^2\ Q.$

In this case, multiplying each input by constant g, then output (Q) increases by g^2.

Therefore, $\alpha + \beta > 1$.

C – D production function exhibits increasing return to scale. When $\alpha + \beta < 1$, say it is equal to 0.8, then in this production function, new output,

$$Q^{'} = g^{\alpha+\beta}\, A\, L^{\alpha}\, K^{\beta} = g^{0.8}\, Q.$$

That is increasing each input by constant factor g will cause output to increase by $g^{0.8}$, that is , less than g. Return to scale in this case are decreasing. Therefore $\alpha + \beta$ measures return to scale.

If $\alpha + \beta = 1$, return to scale are constant.

If $\alpha + \beta > 1$, return to scale are increasing.

If $\alpha + \beta < 1$, return to scale are decreasing.

f**) C – D Production Functions and Output Elasticity of Factors:**

The exponents of labour and capital in C – D production function measures output elasticity's of labour and capital. Output elasticity of a factor refers to the relative or percentage change in output caused by a given percentage change in a variable factor, other factors and inputs remaining constant. Thus,

$$O\ E = \frac{\partial Q}{\partial L} \times \frac{L}{Q} = a \times \frac{Q}{L} \times \frac{L}{Q} = a$$

Thus, exponent (a) of labour in C – D production function is equal to the output elasticity of labour.

Similarly, OE of Capital $= \frac{\partial Q}{\partial L} \times \frac{K}{Q}$

$$MP_K = b \cdot \frac{K}{Q} = b \cdot \frac{Q}{K} \times \frac{K}{Q} = b$$

Therefore, output elasticity of capital $= b \cdot \frac{Q}{K} \times \frac{K}{Q} = b$

g) C – D production Function and Euler's Theorem:

C – D production function $Q = A\, L^{\alpha}\, K^{\beta}$

Where $\alpha + \beta = 1$ helps to prove Euler theorem. According to Euler theorem, total output Q is exhausted by the distributive shares of all factors. When each factor is paid equal to its marginal physical product. As we know

$MP_L = A\,\alpha\left(\frac{K}{L}\right)^{\beta}$

$MP_K = A\,\beta\left(\frac{L}{K}\right)^{\alpha}$

According to Euler's theorem if production functions is homogenous of first degree then, Total output, $Q = L \times MP_L + K \times MP_K,$ substituting the values of MP_L and MP_K, we have

$Q = L \times A\,\alpha\left(\frac{K}{L}\right)^{\beta} + K \times A\,\beta\left(\frac{L}{K}\right)^{\alpha} = A\,\alpha\,L^{1-\beta}\,K^{\beta} + A\,\beta\,L^{\alpha}K^{1-\alpha}$

Now, in C – D production function with constant to scale $\alpha + \beta = 1$ and

Therefore: $\alpha = 1 - \beta$ *and* $\beta = 1 - \alpha$, we have

$Q = A\,\alpha\,L^{\alpha}\,K^{\beta} + A\,\beta\,L^{\alpha}K^{\beta} = (\alpha + \beta)A\,L^{\alpha}K^{\beta}$

Since, $\alpha + \beta = 1$ we have

$Q = A\,L^{\alpha}K^{\beta}$

$Q = Q$

Thus, in C – D production function with $\alpha + \beta = 1$ if $wage\ rate = MP_L$ and $rate\ of\ return\ on\ capital\ (K) = MP_K$, then total output will be exhausted.

h). C – D Production Function and Labour Share in National Income.

C – D production function has been used to explain labour share in national income (i.e., real national product). Let Y stand for real national product, L and K for inputs of labour and capital, then according to C – D production function as applied to the whole economy, we have

$Y = A\,L^{\alpha}K^{1-\alpha}$ … … … … … … … (1)

VII. Cost Function

Cost function expresses the relationship between cost and its determinants such as the level of output (Q), plant size (S), input prices (P), technology (T), and managerial efficiency (E) etc. Mathematically it can be expressed as

C = *f* (Q, S, P, T, E), where C is the unit cost or total cost

Example: Given the total cost function TC = Q^2 + 8Q + 90, find average cost (AC) and marginal cost (MC)

$AC = TC/Q = Q^2/Q + 8Q/Q + 90/Q = Q + 8 + 90/Q$

$MC = dTC/dQ = 2q + 8$

VIII. Revenue Function

If R is the total revenue of a firm, X is the quantity demanded or sold and P is the price per unit of output, we write the revenue function. Revenue function expresses revenue earned as a function of the price of good and quantity of goods sold. The revenue function is usually taken to be linear.

$R = P \times X$

Where R = revenue, P = price, X = quantity

If there are n products and $P_1, P_2 \ldots Pn$ are the prices and $X_1, X_2 \ldots Xn$ units of these products are sold then

$R = P_1X_1 + P_2X_2 + \ldots + PnXn$

Eg: $TR = 50 – 6Q^2$

Example: Given $P = Q^2 + 6Q + 5$, compute the TR function.

$TR = PQ = (Q^2 + 6Q + 5)Q = Q^3 + 6Q^2 + 5Q$. (Here quantity is assumed as Q)

Profit Function

Profit function as the difference between the total revenue and the total cost. If x is the quantity produced by a firm, R is the total revenue and C being the total cost then profit (π).

$P(x) = R(x) – C(x)$ or $\Pi = TR – TC$

Example: Given a $TR = 100Q – 5Q^2$ and $TC = Q^3 – 2Q^2 + 50Q$, find the profit function

$\Pi = TR – TC = (100Q – 5Q^2) – (Q^3 – 2Q^2 + 50Q)$

$= 100Q – 5Q^2 – Q^3 + 2Q^2 – 50Q = –Q^3 – 3Q^2 + 50Q$

IX. Investment function

The investment function explains how the changes in national income induce changes in investment patterns in the national economy. It shows the functional relation between investment and the rate if interest or income. So, the investment function can be written as
I = f (i), where, I is the investment and ‘i’ is the rate of interest.

4.1.10. Marginal Concept and Economic Application:

As you have already seen in your Microeconomics paper, marginal concepts relate to change in the total. For example, marginal utility is the change in total utility due to a change in the consumption. Marginal revenue is the change in total revenue due to a change in production. So marginal concepts refer to change and finding of a marginal function from a total function is basically a measurement of change. Since the derivative is a tool to measure change, we will see in this module how derivative is used to derive marginal functions from total functions. The simple rule to be followed is to find any marginal function from it total function, find the first order derivative of the total function.

4.1.11. Application of Marginal Theory:

I. Marginal Utility:

Marginal utility is the addition made to total utility by consuming one more unit of commodity.

Marginal utility can be measured with the help of the following equation:

$MU_n = TU_n - TU_{n-1}$

$MU = \frac{\Delta TU}{\Delta Q}$

Given a total utility function for an individual consuming one commodity, TU = f(x), in terms of derivatives, MU = dTU/dx

Example:

Given a total utility function $U = x^2 + 3x + 5$, find marginal utility

MU = dTU/dx = 2x + 3

II. Marginal Propensity to Consume

Marginal propensity to consume measures the change in consumption due to a change in income of the consumer. Mathematically, MPC is the first derivative of the consumption function.

Given a consumption function C = f(y), MPC= dC/dy —

Example: Given a consumption function C = 100 + 0.5 Y, find MPC and MPS

$$MPC = dC/d\cancel{y =}\ 0.5$$

MPS = 1−MPC = 1 − 0.5 = 0.5

III. Marginal Propensity to Save

Marginal propensity to save measures the change in saving due to a change in income of the consumer. Mathematically, MPS is the first derivative of the saving function.

Given a consumption function S = f(y), MPS = dS/dy

Example: Given a saving function S = 80 + 0.4 Y, find MPS and MPC

$$MPS = dS/dy = 0.4$$

MPC = 1−MPS = 1 − 0.4 = 0.6

IV. Marginal Product:

Marginal product of a factor of production refers to addition to total product due to the use of an additional unit of that factor.

The Marginal Product of Labour (MP_L) or Marginal Physical Product of Labour (MPP_L) is given by the change in TP due to a one unit change in the quantity of labour used. MP is derived by finding the derivative of the TP.

$MP_L = \frac{dTC}{dL}$, Where L is labour, $MP_k = \frac{dTC}{dK}$, where K is capital.

V. Marginal Cost

Marginal Cost (MC) refers to the change in total cost (TC) due to the production of an additional unit of output.

$MC = \frac{dTC}{dQ}$, Where Q is output

Example: Given the total cost function $TC = x^2 – 4xy – 2y^3$, of a firm producing two goods x and y, find the marginal cost of x and y.

$MCx = \frac{dTC}{dx} = 2x – 4y$

$MCy = \frac{dTC}{dy} = – 4x – 6y^2$

VI. Marginal Revenue

Marginal Revenue (MR) is the change in total revenue (TR) due to the production of an additional unit of output.

$MR = \frac{dTR}{dQ}$ Where Q is output

Example

Given the TR function of a firm producing two goods x and y, $TR = 5xy^3 + 3x^2y$, find the marginal revenue from good x and good y.

$MRx = \frac{dTR}{dx} = 5y^3 + 6xy$

$MRy = \frac{dTR}{dx} = 15xy^2 + 3x^2$

VII. Marginal Rate of Substitution (MRS)

As you have seen in the indifference curve analysis, marginal rate of substitution (MRS) for a consumer consuming two goods X and Y represents the rate at which the consumer is prepared to exchange goods X and Y. Thus for a consumer who uses two goods x and y, Marginal Rate of Substitution of x for y (MRSxy) is the amount of good y that the consumer is willing to give up to get one additional unit of good x. MRS is also referred to as RCS (Rate of Commodity Substitution). MRS thus refers to the change in the stock of good y due to one unit change in the stock of good X. Therefore, the derivatives dy/dx measures MRSxy.

MRSxy is also equal to marginal utility of good x divided by marginal utility of good y.

MRSxy = MUx/MUy

Example:

1. Find MRSxy for the function U = 12X+Y

MRSxy = MUx/MUy

MUx = dU/dX = 12

MUy = dU/dY = 1

MRSxy = MUx/MUy = 12/1 = 12

VIII. Marginal Rate of Technical Substitution (MRTS)

Like MRS in the indifference curve analysis, you have seen MRTS in the iso-quant analysis.

MRTS represents the amount of one input the producer is willing to give up for obtaining an additional unit of the other input. This exchange or trade off will help the producer to stay on the same isoquant. Thus for a producer who uses two inputs K and L, Marginal Rate of Technical Substitution of L for K ($MRTS_{LK}$) is the amount of input K that the producer is willing to give up to get one additional unit of input L. The $MRTS_{LK}$ is also equal to, the ratio of the marginal products of the two inputs. As the firm moves down an isoquant, $MRTS_{LK}$ diminishes. Note that MRTS is also equal to the negative of the slope of isoquant.

Thus given the equation of an isoquant $Q = f(L, K) = c$,

$MRTS_{LK} = -\frac{\partial k}{\partial L} = \frac{\partial Q/\partial L}{\partial Q/\partial L} = MP_L / MP_K$. Since $-\frac{\partial k}{\partial L}$ is the slope of an isoquant, MRTS is

also equal to the negative of slope of an isoquant.

Example:

1. Given a production function $Q = 6x^2 + 3xy + 2y^2$., find $MRTS_{xy}$ when y = 4 and x = 5

$MRTS_{xy} = MPx / MP_y$

$MP_L = \frac{\partial Q}{\partial x} = 12x + 3y$

$MP_k = \frac{\partial Q}{\partial y} = 3x + 4y$

at x = 5 and y = 4, MPx = 12(5) + 3(4) = 72
at x = 5 and y = 4, MPy = 3(5) + 4(4) = 31

Now, $MRTS_{xy} = MP_x / MP_y = 72/31 = 2.32$

4.1.12. Relationship between Average Revenue, Marginal revenue:

An important relationship between MR, AR (price) and price elasticity of demand, which is extensively used in making price decisions by firms. This relationship can be proved algebraically also.

MR = AR (1 – 1/e) = P (1 – 1/e)

Where, P = price and e = point elasticity of demand

4.1.13. Relationship between Average Cost and Marginal Cost:

Average cost (AC) or Average total cost (ATC) is the cost of producing one unit of output. It is obtained by dividing total cost (TC) by quantity of output (TC/Q). Marginal cost is the addition to total cost by the production of an additional unit of the commodity. Mc obtained by finding the first derivatives of the TC function (dTC/dQ). Since both AC and MC are derived from the TC function, they are closely related.

The following may be noted regarding the relation between AC and MC.

The slope of AC curve will be positive if and only if the marginal cost curve lies above the AC curve.

The slope of AC curve will be zero if and only if the marginal cost curve interests the AC curve.

The slope of AC curve will be negative if and only if the marginal cost curve lies below the AC curve.

4.1.14. Concept of Elasticity:

Elasticity in common language is a measure of a variable's sensitivity to a change in another variable. In economics, as you have seen in Microeconomics course, elasticity refers the degree to which individuals, consumers or producers change their demand or the amount supplied in response to price or income changes. Elasticity can be found for changes in price of a good and response to it in quantity demanded of the good (price elasticity of demand), changes in price of a good and response to it in quantity supplied of the good (price elasticity of supply), changes in income and response to it in quantity of goods demanded (income elasticity of demand), changes in price of one good and response to it in quantity demanded of some other good – like its substitutes or complements (cross elasticity of demand) and so on.

4.1.15. Different Types of Elasticity:

I. Price Elasticity of Demand

The concept of elasticity is commonly used to assess the change in consumer demand as a result of a change in a good or service's price which is called price elasticity. Price elasticity of demand express the response of quantity demanded of a good to change in its price, given the consumer's income, his tastes and prices of all other goods

Price elasticity = η = $\frac{Proportionate\ chnage\ in\ quantity\ demanded}{proportionate\ change\ in\ price} = \frac{dQ}{dP} * \frac{P}{Q}$,

Where η is the coefficient of price elasticity of demand.

When $\eta > 1$, then demand is price elastic When

$\eta < 1$, then demand is price inelastic When $\eta =$

0, demand is perfectly inelastic When $\eta =$

infinity, demand is perfectly elastic

Note that price elasticity of demand (Ep) is always negative, since the change in quantity demanded is in opposite direction to the change in price. But for the sake of convenience in understanding to the change in price, we ignore the negative sign and take in to account only the numerical value of the elasticity.

Example: Given the demand function $q = -5p + 100$, find price elasticity of demand when price is equal to 5.

$\eta = \frac{dQ}{dP} * \frac{P}{Q}$

Given, $q = -5p + 100$, $\frac{dQ}{dP} = -5$

When P = 5, Q = 75

So, η = -5*(5/75) = 0.33

II. Income Elasticity:

Income elasticity of demand shows the degree of responsiveness of quantity demanded of good to a small change in income of consumers. The degree of response of quantity demanded to a change in income is measured by dividing the proportionate change in quantity demanded by the proportionate change in income. Thus, more precisely, the income elasticity of demand may be defined as the ration of the proportionate change in quantity purchased of a good to the proportionate change in income which include the former.

Income elasticity = $e_y = \frac{Proportionate\ chnage\ in\ purchased\ of\ goods}{proportionate\ change\ in\ income} = \frac{dQ}{dy} * \frac{y}{Q}$

Based on the value of elasticity we can distinguish between different types of goods.

Normal Goods: Normal goods have a positive income elasticity of demand. So as income rise demand also rise for a normal good at each price level.

Necessary Goods: Necessities have an income elasticity of demand of between 0 and +1. Demand rises with income, but less than proportionately. Often this is because we have a limited need to consume additional quantities of necessary goods as our real living standards rise.

Luxuries: Luxuries have an income elasticity of demand greater than 1. (Demand rises more than proportionate to a change in income).

Inferior Goods: Inferior goods have a negative income elasticity of demand. Demand falls as income rises.

Example:

Given Q = 700 – 2P + 0.02Y, find income elasticity of demand when P = 25 and Y = 5000.

Income elasticity = $e_y = \frac{dQ}{dy} * \frac{y}{Q}$ = 0.02 (5000/750) = 0.133

III. Cross Elasticity of Demand:

Cross price elasticity (e_{xy}) measures the responsiveness of demand for good X due to a change in the price of good Y (a related good – could be a substitute or a complement or even an unrelated good).

e_{xy} = Proportionate change in the demand for good X divided by Proportionate change in the price for good Y.

In terms of calculus, for a consumer using two goods x and y, the cross elasticity of demand may be written as $e_{xy} = \frac{dQx}{dPy} * \frac{Py}{Qx}$

With cross price elasticity we can make an important distinction between substitute products and complementary goods and services.

Substitutes: For substitute goods such as tea and coffee an increase in the price of one good will lead to an increase in demand for the other good. Cross price elasticity for two substitutes will be positive.

Complements: for complementary goods the cross elasticity of demand will be negative. When there is no relationship between two goods, the cross price elasticity of demand is zero.

Example:

Given $Q_1 = 100 - P_1 + 0.75P_2 - 0.25P_3 + 0.0075Y$. At $P_1 = 10$, $P_2 = 20$, $P_3 = 40$ and Y = 10,000, find the cross elasticity of demand between goods 1 and goods 2.

Here the given function relates the demand for a goods (Q_1) and price of that goods (P_1) and prices of other two related goods (P_2 and P_3).

Cross price elasticity between goods 1 and goods 2 is given by $e_{12} = \frac{dQ1}{dP2} * \frac{P1}{Q2}$

$\frac{dQ1}{dP2} = 0.75$

Substituting, $e_{12} = 0.75\ (20/170) = 0.88$

Since, e_{12} is positive goods 1 and goods 2 are substitutes. An increase in P_2 leads to increase in Q_1.

Sub Unit – 2: Optimisation, Linear Programming and Input-Output Analysis

4.2.1. Introduction to Optimization:

Optimization is the process of finding relative maximum or minimum of a function.
There are two conditions for obtaining optimum value of a function.

1. The First order conditions: Set the first derivative equal to zero and solve for the critical values. It identifies all the points at which function is neither increasing nor decreasing, but at a plateau.
2. Take the second derivative, evaluate it at critical point(s) and check the sign.

f(a) < 0 for maximum
And f(a)>0 for minimum.

The problem of optimization of some quantity subject to certain restrictions or constraint is a common feature of economics, industry, defence, etc. The usual method of maximizing or minimizing a function involves constraints in the form if equations. Thus utility may be maximized subject to the budget constraint of fixed income, given in the form of equation. The minimization of cost is a familiar problem to be solved subject to some minimum standards. If the constraints are in the form of equations, methods of calculus can be useful. However, if the constraints are inequalities instead of equations and we have an objective function may be optimized subject to these inequalities, we use the method of mathematical programming.

4.2.2. Solution of Optimization:

1. Substitution Method

Another method of solving the objective function with subject to the constraint is substitution methods. In this method, substitute the values of x or y, and the substitute this value in the original problem, differentiate this with x and y.

Consider a utility function of a consumer

$$U = X^{0.3} + Y^{0.3}$$

The budget constraint 20x + 10y =200

Rewrite the above equation, $y = \frac{200-20x}{10} = 20\text{-}2x$

Then the original utility function $U = X^{0.3} + (20\text{-}2x)$

2. Lagrange Method

Constrained maxima and minima: in mathematical and economic problems, some relation or constraint sometimes restricts the variables in a function. When we wish to maximize or minimize $f(x_1, x_2 \ldots x_n)$ subject to the condition or constraint $g(x_1, x_2 \ldots x_n) = 0$, there exist a method known as the method of Lagrange Multiplier. For example utility function $U = u(x_1, x_2 \ldots x_n)$ may be subject to the budget constraint that income equals expenditure that is $Y = p_1x_1 + p_2x_2 + \ldots\ldots + p_nx_n$. We introduce a new variable ʎ called the Lagrange Multiplier and construct the function.

$$Z = f(x_1, x_2 \ldots x_n) + ʎ\, g(x_1, x_2 \ldots x_n)$$

This new function z is a function on n+1 variable $x_1, x_2 \ldots x_n$ and ʎ

4.2.3. Application of Optimization:

4.2.4. Utility Maximization

There are two approaches to study consumer behavior- the first approach is a classical one and is known as cardinal utility approach and the second approach is ordinal utility approach popularly known as indifference curve approach. In both the approaches, we assume that consumer always behaves in a rational manner, because he derives the maximum utility (satisfaction) out of his budget constraint.

Example:

The utility function of the consumer is given by $U = x_1{x_2}^2 - 10x_1$, where x_1 and x_2 are the qualities of two commodities consumed. Find the optimal utility value if his income 116 and product price are 2 and 8 respectively.

Solution:

We have utility function

$U = f(x_1x_2) = x_1{x_2}^2 - 10x_1$ -------------- (1)

Now, Budget constraint, $116 - 2x_1 - 8x_2 = 0$

$$or, x_1 = 58 - 4x_2 \text{ ------------ (2)}$$

By the substitution method, we have substitute the value of x_1 into eg equation (1)

We get, $U = (58 - 4x_2){x_2}^2 - 10(58 - 4x_2)$

$$= 58{x_2}^2 - 4{x_2}^3 - 580 + 40x_2$$

For utility maximization,

$\frac{du}{dx_2} = 0$

$or, 116x_2 - 12{x_2}^2 + 40$

$or, 3{x_2}^2 - 29x_2 - 10 = 0$

$or, (3x_2 + 1)(x_2 - 10) = 0$

So, $3x_2 + 1 = 0$ $\quad\quad$ $x_2 - 10 = 0$

$x_2 = -\frac{1}{3}$ $\quad\quad$ $x_2 = 10$

Now, x_2 cannot be negative

Hence, $x_2 = 10$ $\quad\quad$ $x_1 = 58 - 4x_2$

$= 58 - 40 = 18$

For 2nd order condition,

$\frac{d^2u}{dx_2} = 116 - 24x_2 = 116 - 240 = -124 < 0$

Therefore, the utility is maximum at $x_2 = 10$ and $x_1 = 18$

4.2.5. Cost Minimization

Cost minimization involves how a firm has to produce a given level of output with minimum cost. Consider a firm that uses labour (L) and capital (K) to produce output (Q). Let W is the price of labour, that is, wage rate and r is the price of capital and the cost (C) incurred to produce a level of output is given by

C = wL + rK

The objective of the form is to minimize cost for producing a given level of output. Let the production function is given by following.

Q = f (L, K)

In general there is several labour – capital combinations to produce a given level of output. Which combination of factors a firm should choose which will minimize its total cost of production. Thus, the problem of constrained minimization is

Minimize C = wL +rK

Subject to produce a given level of output, say Q_1 that satisfies the following production function

$$Q_1 = f(L, K)$$

The choice of an optimal factor combination can be obtained by using Lagrange method. Let

us first form the Lagrange function is given below

$$Z = wL + rK + \lambda(Q_1 - f(L, K))$$

Where, λ is the Lagrange multiplier

For minimization of cost it necessary that partial derivatives of Z with respect to L, K and be zero.

$\frac{\partial z}{\partial L} = w - \frac{\lambda \partial f(L,K)}{\partial L} = 0$ ---------(1)

$\frac{\partial z}{\partial K} = r - \frac{\lambda \partial f(L,K)}{\partial K} = 0$ ----------(2)

$\frac{\partial z}{\partial \lambda} = \theta_1 - f(L,K) = 0$ ---------(3)

Note that, $\frac{\partial f(L,K)}{\partial L}$ and $\frac{\partial f(L,K)}{\partial K}$ are the marginal products of labour and Capital respectively.

Rewriting the above equation we have

$w - \lambda MP_L = 0$ ---------(4)

$r - \lambda MP_K = 0$ ----------(5)

$\theta_1 = f(L,K)$ ------------(6)

By combining two equation, we have

$\frac{w}{r} = \frac{MP_L}{MP_K}$ -----------------(7)

The last equation shows that total cost is minimized when the factor price ratio $\frac{w}{r}$ equal the ration of MPP of labour and capital.

4.2.6. Profit maximization:

Maximizations of profit subject to the constraint can also used to identify the optimum solution for a function.

Assumes that TR = PQ

TC = wL +rK

Π = TR – TC

Π = PQ – (wL + rK)

Thus the objective function of the firm is to maximize the profit function

Π = PQ – (wL +rK)

The firm has to face a constraint Q = f (K, L)

From the Lagrange function,

Z = (PQ – (wL + rK)) + λ(F (K, L) – Q)

Now, the 1st order condition.

$\frac{\partial z}{\partial \theta} = P - \lambda = 0$ ---------------(1)

$\frac{\partial z}{\partial L} = -w + \lambda f'(L)$ -----------(2)

$\frac{\partial z}{\partial K} = -r + \lambda f'(K)$ -----------(3)

$\frac{\partial z}{\partial \theta} = f(K, L) - \theta = 0$ -------(4)

From equation 1, we get $P = \lambda$

Substitute this in (2) and (3)

From equation (2), we get,

$w = \lambda f(L) = Pf'(L)$

$or, P = \frac{w}{f'(L)}$ -----------(5)

Form equation (3), we get,

$r = \lambda f'(K) = Pf'(K)$

$or, P = \frac{r}{f'(K)}$ ---------(6)

From equation (5) and (6) we get

$\frac{w}{r} = \frac{f'(L)}{f'(K)} = MRTS_{LK}$ ----------(7)

The above condition is profit maximization.

4.2.7. Linear Programming Problem (LPP):

The term linear programming consists of two words, linear and programming. The linear programming considers only linear relationship between two or more variables. By linear relationship we mean that relations between the variable can be represented by straight lines. Programming means planning or decision- making in a systematic way. “Linear programming refers to a technique for the formulation and solution of problems in which some linear function of two or more variables is to be optimized subject to a set of linear constraints at least one of which must be expressed as inequality”. American mathematician George B. Danzig has invented the linear programming technique.

Linear programming is a practical tool of analysis which yields the optimum solution for the linear objective function subject to the constraints in the form of linear inequalities. Linear objective function and linear inequalities and the techniques, we use is called linear programming, a special case of mathematical programming.

4.2.8. Terms of Linear Programming

(1) Objective Function

Objective function, also called criterion function, describe the determinants of the quantity to be maximized or to be minimized. If the objective of a firm is to maximize output or profit, then this is the objective function of the firm. If the linear programming requires the minimization of cost, then this is the objective function of the firm. An objective function has two parts – the primal and dual. If the primal of the objective function is to maximize output then its dual will be the minimization of cost.

(2) Technical Constraints

The maximization of the objective function is subject to certain limitations, which are called constraints. Constraints are also called inequalities because they are generally expressed in the form of inequalities. Technical constraints are set by the state of technology and the availability of factors of production. The number of technical constraints in a linear programming problem is equal to the number of factors involved it.

(3) Non- Negativity Constraints

This express the level of production of the commodity cannot be negative, ie it is either positive or zero.

(4) Feasible Solutions

After knowing the constraints, feasible solutions of the problem for a consumer, a particular, a firm or an economy can be ascertained. Feasible solutions are those which meet or satisfy the constraints of the problem and therefore it is possible to attain them.

(5) Optimum Solution

The best of all feasible solutions is the optimum solution. In other words, of all the feasible solutions, the solution which maximizes or minimizes the objective function is the optimum solution. For instance, if the objective function is to maximize profits from the production of two goods, then the optimum solution will be that combination of two products that will maximizes the profits for the firm. Similarly, if the objective function is to minimize cost by the choice of a process or combination of processes, then the process or a combination of processes which actually minimizes the cost will represent the optimum solution. It is worthwhile to repeat that optimum solution must lie within the region of feasible solutions.

4.2.9. Assumptions of LPP

The LPP are solved on the basis of some assumptions which follow from the nature of the problem.

(a) Linearity

The objective function to be optimized and the constraints involve only linear relations. They should be linear in their variables. If they are not, alternative technique to solve the problem has to be found. Linearity implies proportionality between activity levels and resources. Constraints are rules governing the process.

(b) Non- negativity

The decision variable should necessarily be non –negative.

(c) Additive and divisibility

Resources and activities must be additive and divisible.

4.2.10. Application of linear programming

There is a wide variety of problem to which linear programming methods have been successfully applied.

I.Diet problems

To determine the minimum requirements of nutrients subjects to availability of foods and their prices.

II.Transportation problem

To decide the routes, number of units, the choice of factories, so that tha cost of operation is the minimum.

III.Manufacturing problems

To find the number of items of each type that should be made so as to maximize the profits.

IV.Production problems

To decide the production schedule to satisfy demand and minimize cost in the face of fluctuating rates and storage expenses.

V.Assembling problems

To have, the best combination of basic components to produce goods according to certain specifications.

VI.Purchasing problems

To have the least cost objective in, say, the processing of goods purchased from outside and varying in quantity, quality and prices.

VII.Job assigning problem

To assign jobs to workers for maximum effectiveness and optimum results subject to restrictions of wages and other costs.

4.2.11. Formulation of Linear Programming Problem:

STEPS

I. Identify decision variables

II. Write objective function

III. Formulate constraints

❖ Example 1:

A firm produces three products. These products are processed on three different machines. The time required to manufacture one unit of each of the three products and the daily capacity of the three machines are given in the table below:

Machine	Time per unit (Minutes)			Machine Capacity (minutes/day)
	Product 1	Product 2	Product 3	
M1	2	3	2	440
M2	4	-	3	470
M3	2	5	-	430

It is required to determine the daily number of units to be manufactured for each product. The profit per unit for product 1, 2 and 3 is Rs. 4, Rs.3 and Rs.6 respectively. It is assumed that all the amounts produced are consumed in the market. Formulate the mathematical (L.P.) model that will maximise the daily profit.

➢ Formulation of Linear Programming Model

Step 1

From the study of the situation find the key-decision to be made. In the given situation key decision is to decide the extent of products 1, 2 and 3, as the extents are permitted to vary.

Step 2

Assume symbols for variable quantities noticed in step 1. Let the extents (amounts) of products 1, 2 and 3 manufactured daily be x_1, x_2 and x_3 units respectively.

Step 3

Express the feasible alternatives mathematically in terms of variable. Feasible alternatives are those which are physically, economically and financially possible. In the given situation feasible alternatives are sets of values of $x1$, $x2$ and $x3$ units respectively. Where, $x1$, $x2$ and $x3 \geq 0$. Since negative production has no meaning and is not feasible.

Step 4

Mention the objective function quantitatively and express it as a linear function of variables. In the present situation, objective is to maximize the profit.

i.e., $Z = 4x1 + 3x2 + 6x3$

Step 5

Put into words the influencing factors or constraints. These occur generally because of constraints on availability (resources) or requirements (demands). Express these constraints also as linear equations/inequalities in terms of variables.

Here, constraints are on the machine capacities and can be mathematically expressed as

$$2x1 + 3x2 + 2x3 \leq 440$$

$$4x1 + 0x2 + 3x3 \leq 470$$

$$2x1 + 5x2 + 0x3 \leq 430$$

Examples: 2

A firm can produce a good either by (1) a labour intensive technique, using 8 units of labour and 1 unit of capital or (2) a capital intensive technique using 1 unit of labour and 2 unit of capital. The firm can arrange up to 200 units of labour and 100 units of capital. It can sell the good at a constant net price (P), ie P is obtained after subtracting costs. Obviously we have simplified the problem because in this 'P' become profit per unit. Let P = 1.

Let x_1 and x_2 be the quantities of the goods produced by the processes 1 and 2 respectively. To maximize the profit $P x_1 + P x_2$,

we write the objective function.

$\Pi = x_1 + x_2$ (since P = 1).

The problem becomes Max $\pi = x_1 + x_2$

Subject to: The labour constraint $8 x_1 + x_2 \leq 200$

The capital constraint $x_1 + x_2 \leq 100$

And the non- negativity conditions $x_1 \geq 0$, $x_2 \geq 0$

This is a problem in linear programming.

Example: 3

Two foods F_1, F_2 are available at the prices of Rs. 1 and Rs. 2 per unit respectively. N_1, N_2, N_3 are essential for an individual. The table gives these minimum requirements and nutrients available from one unit of each of F_1, F_2. The question is of minimizing cost (C), while satisfying these requirements.

Nutrients	**Minimum requirements**	**One units of F_1**	**One units of F_2**
N_1	17	9	2
N_2	19	3	4
N_3	15	2	5

Total Cost (TC) $c = P_1 x_1 + P_2 x_2$ (x_1, x_2 quantities of F_1, F_2) Where $P_1 = 1$, $P = 2$

We therefore have to Minimize $C = x_1 + 2x_2$

Subject to the minimum nutrient requirement constraints,

$$9x_1 + x_2 \geq 17$$

$$3x_1 + 4x_2 \geq 19$$

$$2x_1 + 5x_2 \geq 15$$

Non- negativity conditions $x_1 \geq 0$, $x_2 \geq 0$.

4.2.12. Solution of LPP:

There are two methods available to find optimal solution to a Linear Programming Problem. One is graphical method and the other is simplex method.

Graphical method can be used only for a two variables problem i.e. a problem which involves two decision variables. The two axes of the graph (X & Y axis) represent the two decision variables X1 & X2.

1. Graphical Solution

If the LPP consist of only two decision variable. We can apply the graphical method of solving the problem. It consists of seven steps, they are

1. Formulate the problem in to LPP.

2. Each inequality in the constraint may be treated as equality.

3. Draw the straight line corresponding to equation obtained steps (2) so there will be as many straight lines, as there are equations.

4. Identify the feasible region. This is the region which satisfies all the constraints in the problem.

5. The feasible region is a many sided figures. The corner point of the figure is to be located and they are coordinate to be measures.

6. Calculate the value of the objective function at each corner point.

7. The solution is given by the coordinate of the corner point which optimizes the objective function.

Example 1:

A company manufactures two products A and B. Both products are processed on two machines M_1 & M_2.

	M_1	M_2
A	6 Hrs/Unit	2 Hrs/Unit
B	4 Hrs/Unit	4 Hrs/Unit
Availability	7200 Hrs/month	4000 Hrs/month

Profit per unit for A is Rs. 100 and for B is Rs. 80. Find out the monthly production of A and B to maximise profit by graphical method.

Formulation of LPP

X_1 = No. of units of A/Month

X_2 = No. of units of B/Month

Max $Z = 100 X_1 + 80 X_2$

Subject to constraints:

$6 X_1 + 4 X_2 \leq 7200$

$2 X_1 + 4 X_2 \leq 4000$

$X_1, X_2 \geq 0$

Step 2: Determination of each axis

Horizontal (X) axis: Product A (X_1)

Vertical (Y) axis: Product B (X_2)

Step 3: Finding co-ordinates of constraint lines to represent constraint lines on the graph.

The constraints are presently in the form of inequality (≤). We should convert them into equality to obtain co-ordinates.

Constraint No. 1:

$6 X1 + 4 X2 \leq 7200$

Converting into equality:

$6 X_1 + 4 X_2 = 7200$

X_1 is the intercept on X axis and X_2 is the intercept on Y axis.

To find X_1, let $X_2 = 0$

$6 X_1 = 7200$

$\therefore X_1 = 1200; X_2 = 0$

(1200, 0)

To find X_2, let $X_1 = 0$

$4 X_2 = 7200$,

$X_2 = 1800$, $X_1 = 0$;

(0, 1800)

Hence the two points which make the constraint line are:

(1200, 0) and (0, 1800)

Note: When we write co-ordinates of any point, we always write (X_1, X_2). The value of X_1 is written first and then value of X_2. Hence, if for a point X_1 is 1200 and X_2 is zero, then its co-ordinates will be (1200, 0).

Similarly, for second point, X_1 is 0 and X_2 is 1800. Hence, its co-ordinates are (0, 1800).

Constraint No. 2:

$2 X_1 + 4 X_2 \leq 4000$

To find X_1, let $X_2 = 0$

$2 X_1 = 4000$

$\therefore X_1 = 2000; X_2 = 0$ (2000, 0)

To find X_2, let $X_1 = 0$

$4 X_2 = 4000$

$\therefore$ X_2 = 1000; X_1 = 0 (0, 1000)

Each constraint will be represented by a single straight line on the graph. There are two constraints, hence there will be two straight lines.

The co-ordinates of points are:

1. Constraint No. 1: (1200, 0) and (0, 1800)
2. Constraint No. 2: (2000, 0) and (0, 1000)

Step 4: Representing constraint lines on graph

To mark the points on the graph, we need to select appropriate scale. Which scale to take will depend on maximum value of X_1 & X_2 from coordinates.

For X_1, we have 2 values $\longrightarrow$ 1200 and 2000

$\therefore$Max. value for X_2 = 2000

For X_2, we have 2 values $\longrightarrow$ 1800 and 1000

$\therefore$Max. value for X_2 = 1800

Assuming that we have a graph paper 20 X 30 cm. We need to accommodate our lines such that for X-axis, maximum value of 2000 contains in 20 cm.

$\therefore$ Scale 1 cm = 200 units

$\therefore$ 2000 units = 10 cm (X-axis)

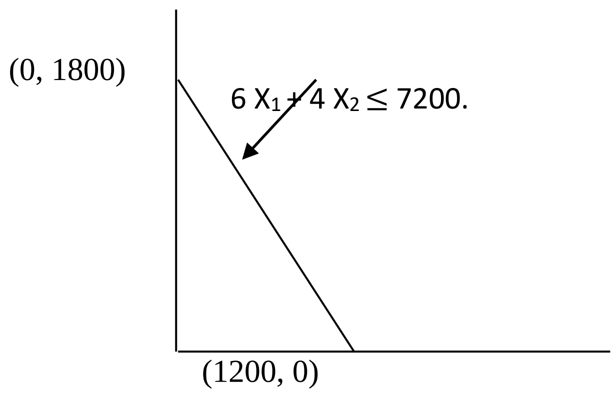
1800 units = 9 cm (Y-axis)

The scale should be such that the diagram should not be too small.

Constraint No. 1:

The line joining the two points (1200, 0) and (0, 1800) represents the constraint 6 X_1 + 4 $X_2 \le 7200$.

Fig 1.

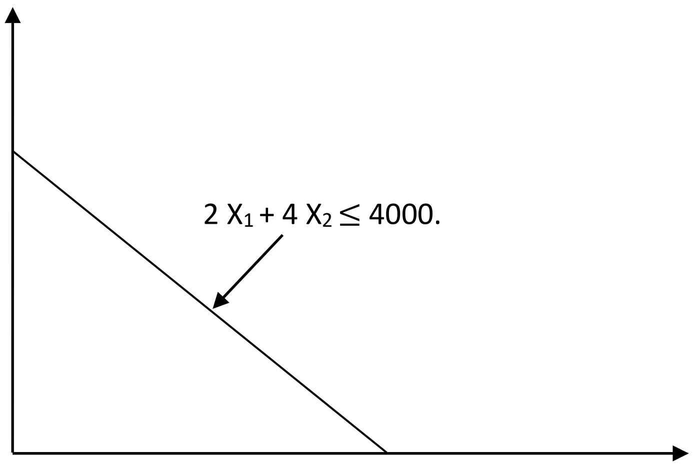


Every point on the line will satisfy the equation (equality) $6 X_1 + 4 X_2 \leq 7200$.
Every point below the line will satisfy the inequality (less than) $6 X_1 + 4 X_2 \leq 7200$.

Constraint No. 2:
The line joining the two points (2000, 0) and (0, 1000) represents the constraint $2 X_1 + 4 X_2 \leq 4000$
Every point on the line will satisfy the equation (equality) $2 X_1 + 4 X_2 \leq 4000$. Every point below the line will satisfy the inequality (less than) $2 X_1 + 4 X_2 \leq 4000$.

Fig 2

(0, 1000)



(2000, 0)

Now the final graph will look like this:

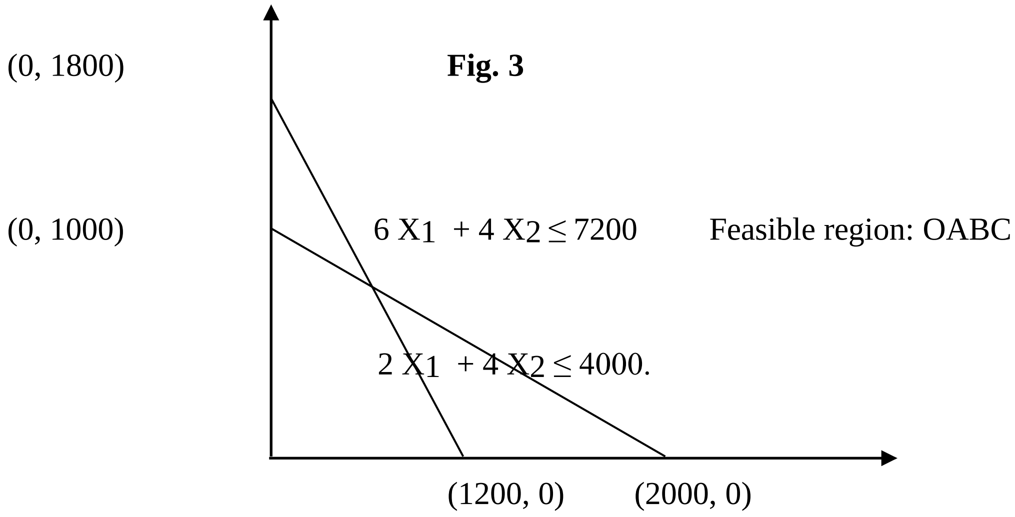


Fig. 3

Step 5: Identification of Feasible Region

The feasible region is the region bounded by constraint lines. All points inside the feasible region or on the boundary of the feasible region or at the corner of the feasible region satisfy all constraints.

Both the constraints are 'less than or equal to' ($\leq$) type. Hence, the feasible region should be inside both constraint lines.

Hence, the feasible region is the polygon OABC. 'O' is the origin whose coordinates are (0, 0). O, A, B and C are called vertices of the feasible region.

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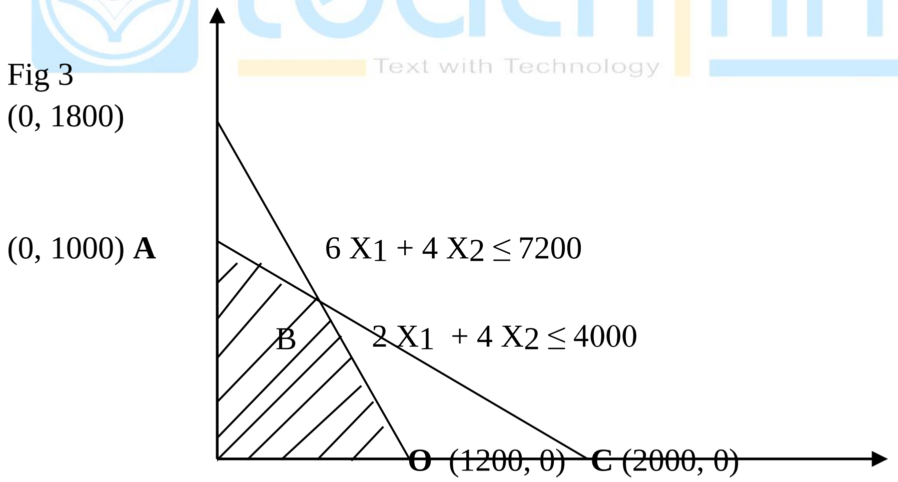
Fig 3



Step 6: Finding the optimal Solution

The optimal solution always lies at one of the vertices or corners of the feasible region. To find optimal solution:

We use corner point method. We find coordinates (X_1, X_2 Values) for each vertex or corner point. From this we fine 'Z' value for each corner point.

Vertex	**Co-ordinates**	**$Z = 100\ X_1 + 80\ X_2$**
O	$X_1 = 0,\ X_2 = 0$ From Graph	$Z = 0$
A	$X_1 = 0,\ X_2 = 1000$ From Graph	Z = Rs. 80, 000
B	$X_1 = 800,\ X_2 = 600$ From Simultaneous equations	Z = Rs. 1, 28, 000
C	$X_1 = 1200,\ X_2 = 0$ From Graph	Z = Rs. 1, 20, 000

Max. Z = Rs. 1, 28, 000 (At point B)

For B $\longrightarrow$ B is at the intersection of two constraint lines $6\ X_1 + 4\ X_2 \leq 7200$ and $2\ X_1 + 4\ X_2 \leq 4000$. Hence, values of X_1 and X_2 at B must satisfy both the equations. We have two equations and two unknowns, X_1 and X_2. Solving simultaneously. $6\ X_1 + 4\ X_2 \leq 7200$ (1)

$2\ X_1 + 4\ X_2 \leq 4000$ (2)

$4\ X_1 = 3200$ Subtracting (2) from (1)

$X_1 = 800$

Substituting value of X_1 in equation (1), we get $4\ X_2 = 2400$ $\therefore X_2 = 600$

Solution

Optimal Profit = Max Z = Rs. 1, 28, 000 Product Mix:

X_1 = No. of units of A / Month = 800 X_2 = No. of units of A / Month = 600

4.2.13. Limitations of Linear Programming Technique

Linear programming has turned out to be a highly useful tool of analysis for the business executive. It is being increasingly made use of in theory of the firm, in managerial economics, in interregional trade, in general equilibrium analysis, in welfare economics and in development planning. But it has its limitations.

1. It is not easy to define a specific objective function.
2. Even if a specific objective function is laid down, it may not be so easy to find out various technological, financial and other constraints which may be operative in pursuing the given objective.
3. Given a specific objective and a set of constraints, it is possible that the constraints may not be directly expressible as linear inequalities.
4. Even if the above problems are surmounted, a major problem is one of estimating relevant values of the various constant coefficients that enter into a linear programming model, i.e., prices, etc.

5. This technique is based on the assumption of linear relations between inputs and outputs. This means that inputs and outputs can be added, multiplied and divided. But the relations between inputs and outputs are not always linear. In real life, most of the relations are non-linear.

6. This technique assumes perfect competition in product and factor markets. But perfect competition is not a reality.

7. The LP technique is based on the assumption of constant returns. In reality, there are either diminishing or increasing returns which a firm experiences in production.

8. It is a highly mathematical and complicated technique. The solution of a problem with linear programming requires the maximization or minimization of a clearly specified variable. The solution of a linear programming problem is also arrived at with such complicated method as the 'simplex method' which involves a large number of mathematical calculations.

9. It requires a special computational technique, an electric computer or desk calculator. Mostly, linear programming models present trial- and-error solutions and it is difficult to find out really optimal solutions to the various economic problems.

4.2.14. Input – Output Analysis:

In a modern economy the production of one good requires the input of many other goods as intermediate goods in the production process. Leontief Input-output model is a technique to explain the general equilibrium of the economy. It is also known as "inter-industry analysis". Before analysing the input-output method, let us understand the meaning of the terms, "input" and "output". According to Professor J.R. Hicks, an input is "something which is bought for the enterprise" while an output is "something which is sold by it." An input is obtained but an output is produced. Thus input represents the expenditure of the firm, and output its receipts. The sum of the money values of inputs is the total cost of a firm and the sum of the money values of the output is its total revenue.

The input-output analysis tells us that there are industrial interrelationships and inter-dependence in the economic system as a whole. The inputs of one industry are the outputs of another industry and vice versa, so that ultimately their mutual relationships lead to equilibrium between supply and demand in the economy as a whole.

4.2.15. Assumptions of Input-Output Analysis:

This analysis is based on the following assumptions:

1. The whole economy is divided into two sectors—“inter-industry sectors” and “final-demand sectors,” both being capable of sub-sectoral division.
2. The total output of any inter-industry sector is generally capable of being used as inputs by other inter-industry sectors, by itself and by final demand sectors.
3. No two products are produced jointly. Each industry produces only one homogeneous product.
4. Prices, consumer demands and factor supplies are given.
5. There are constant returns to scale.
6. There are no external economies and diseconomies of production.
7. The combinations of inputs are employed in rigidly fixed proportions. The inputs remain in constant proportion to the level of output. It implies that there is no substitution between different materials and no technological progress. There are fixed input coefficients of production.

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4.2.16. Input – output Table:

Let us now consider a formalized input – output table in our simple model. An input – output table shows the disposition of the total products and total inputs among the different industries. Let x_1 and x_2 be the amounts of total outputs of industry I and industry II respectively. Let x_{ij} be the amount of output of the i-th industry going to the j-th industry. $(i, j = 1, 2)$. Further let c_1 and c_2 denote the final demands of the two industries. Let x_0 represents an input – output table for out simplified economy.

Formalized Input – Output Table

Industries	Inputs to Industry I	Inputs to Industry II	Final Consumption	Total Output
Industry I	x_{11}	x_{12}	c_1	x_1
Industry II	x_{21}	x_{22}	c_2	x_2
Lobour Services	x_{01}	x_{02}	… …	x_0

Each of the first two rows shows what happens to the total output of each industry. The first row shows that a part of the output is used in the first industry (x_{11}), a part is used in the second industry (x_{12}), and a part is used as final consumption (c_1). Similarly, the second r5ow shows the allocation of the total output of the second industry. It should be noted that all the entries in this table are flows, i.e., physical units per year. Further, the entries in any row are all measured in the same physical unit, so that it is possible to add across the rows. The total output column gives the output not each commodity and the available input of labour. Items in the same column are , however, not measured in the same units so that it is not possible to add down the columns. But each column can be thought of as a whole (i.e, as a vector). It does then have meaning. The first column describes the input – structure of the first industry. Thus the first industry used x_{11} units of its own output, x_{21} units of the output of the second industry and x_{01} units of labour as inputs. Similarly, the second column describes the input – structure of the second industry. In other words, a column gives one point on the production function of the corresponding industry. The final demand column shows the commodity breakdown of what is available for consumption and government expenditure. It is assured that labour is not directly consumed. The production functions for the two industries can be written as follows:

$$x_1 = f_1\,(x_{11}, x_{21}, x_{01})\ and\ \ x_2 = f_2\,(x_{12}, x_{22}, x_{02})$$

Further, we can always add across the rows, so we know that

$$x_{11} + x_{21} + c_1 = x_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$x_{21} + x_{22} + c_2 = x_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

and

$$x_{01} + x_{02} = x_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Leontief assumers fixed coefficients of production. This means that it takes a certain minimum input of each commodity per unit of output of each commodity. Let a_{ij} be the required minimum input of commodity i per unit of output of commodity j $(where\ i = 0, 1, 2\ and\ j = 1, 2)$. Then the two production functions can be written as follows :

$$x_1 = \min\left(\frac{x_{11}}{a_{11}}, \frac{x_{21}}{a_{21}}, \frac{x_{01}}{a_{01}}\right) \quad and \quad x_2 = \min\left(\frac{x_{12}}{a_{12}}, \frac{x_{22}}{a_{22}}, \frac{x_{02}}{a_{02}}\right)$$

In other words, this means that x_1 will be equal to the minimum of the three ratios within the bracket and x_2 will also be the minimum of the three ratios within the bracket.

It can be seen that if each of the x_{ij} is multiplied by a constant the corresponding x_j is multiplied by the same constant so that we have constant returns to scale. If any a_{ij} is zero we can either omit the corresponding term from the right hand side or we can think of $\frac{x_{ij}}{a_{ij}}$ as infinity in which case it will certainly never be the smallest ratio.

An alternative way of writing the production function is to note that since x_1 equals the smallest6 of $\frac{x_{11}}{a_{11}}, \frac{x_{21}}{a_{21}}, \frac{x_{01}}{a_{01}}$ it must be less than or equal to all three of the ratios, i.e., $\frac{x_{11}}{a_{11}} \geq x_1, i.e., x_{11} \geq a_{11} \cdot x_1$ Similarly, $x_{21} \geq a_{21} \cdot x_1$ and $x_{01} \geq a_{01} \cdot x_1$. Further, $x_{12} \geq a_{12} \cdot x_2$, $x_{22} \geq a_{22} \cdot x_2$ and $x_{02} \geq a_{02} \cdot x_2$

It should be noted that the equalit6y will hold at least once in each row. In fact, if none of the commodities concerned is free good, the equality will hold everywhere. Assuming that no goods are free, we can consider all the above qualities. Then putting the values of x_{ij} in equations (1) , (2) and (3), We get, $a_{11} \cdot x_1 + a_{12} \cdot x_2 + c_1 = x_1 \dots\dots\dots\dots\dots\dots\dots (4)$

$a_{21} \cdot x_1 + a_{22} \cdot x_2 + c_2 = x_2 \dots\dots\dots\dots\dots\dots\dots (5)$

$a_{01} \cdot x_1 + a_{02} \cdot x_2 = x_0 \dots\dots\dots\dots\dots\dots\dots (6)$

The problem in the Leontief model can now be stated as follows: Suppose the society specifies a set of final demands c_1 and c_2. In other words, assume that the final demands are given. What should be the gross output to meet these final demands? Further, given the supply of labour, is it possible to meet these final demands? These are the two main equations to be answered in the Leontief Static Open Model.

Determination of Gross Outputs

If c_1 and c_2 are given, the gross outputs x_1 and x_2 required to meet these final demands can be obtained by solving the two equations (4) and (5). Note that in (4) and (5), c_1 and c_2 are given. Further, the a_{ij} coefficients are also constants by assumption. Hence there are only two unknowns x_1 and x_2 in these two equations. Equations (4) and (5) can be written in matrix form.

Suppose X stands for5 the column matrix $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, C stands for the column matrix $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ and A stands for the matrix of a_{ij} coefficients, $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then equations (4) and (5) can be written as:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$or, X = AX + C$

It should be noted that the matrix A has some properties, Firstly, it is non-negative. This means that no element of A matrix can be negative. Secondly, it is a square matrix having equal number of rows and columns. Thirdly, it is indecomposable, which means that $a_{12} > 0$ and $a_{21} > 0$. If this condition is violated then we cannot get interdependence in our system. For example, if $a_{12} = 0$, this means that $x_{12} = 0$. In this case the production function of the second commodity becomes $x_2 = f_2\ (x_{22}, x_{02})$. The first commodity is not required at all to produce the second commodity. Similarly if $a_{21} = 0$ then the second commodity is not required at all to produce the first commodity. If both a_{12} *and* a_{21} are zero at the same time, then the two commodities becomes independent. On the other hand , if one of them is zero while the other is non zero, then one commodity is independent, though the other is not. To avoid both these cases and to maintain complete interdependence, it is assumed that both a_{12} *and* a_{21} are positive. Consider now the equation $X = AX + C\ \ or, IX = AX + C$, Where I is the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Then $IX - AX = C$ $or,$ $(I - A)X = C$

Pre-multiplying both sides by $(I - A)^{-1}$

$$(I - A)^{-1} \cdot (I - A) \cdot X = (I - A)^{-1} . C \quad or, X = (I - A)^{-1} \cdot C$$

This gives us the solutions of the two equation (4) and (5) in matrix form. The matrix $(I - A)^{-1}$ can be obtained from the matrix A. Thus given the C matrix and given the atrix, the X matrix can be determined. The matrix $(I - A)^{-1}$ can be evaluated as follows:

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 - a_{11} & -a_{12} \\ -a_{21} & 1 - a_{22} \end{bmatrix}$$

Matrix of co-factors $= \begin{bmatrix} 1 - a_{22} & -a_{21} \\ -a_{12} & 1 - a_{11} \end{bmatrix}$

Transpose of this matrix or, adjoint matrix of $(I - A) = \begin{bmatrix} 1 - a_{22} & -a_{12} \\ -a_{21} & 1 - a_{11} \end{bmatrix}$

Now let $\Delta = |I - A|$ = determinant of the $(I - A)$ matrix. Assume that $\Delta \neq 0$, i.e, $(I - A)$ is non-singular. Then the inverse matrix can be written as

$$(I - A)^{-1} = \begin{bmatrix} \frac{1-a_{22}}{\Delta} & \frac{a_{12}}{\Delta} \\ \frac{a_{21}}{\Delta} & \frac{1-a_{11}}{\Delta} \end{bmatrix} \quad or, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1-a_{22}}{\Delta} & \frac{a_{12}}{\Delta} \\ \frac{a_{21}}{\Delta} & \frac{1-a_{11}}{\Delta} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$or, \quad x_1 = \frac{1-a_{22}}{\Delta} \cdot c_1 + \frac{a_{12}}{\Delta} \cdot c_2 \quad \ldots\ldots\ldots(7)$$

$$x_1 = \frac{a_{21}}{\Delta} \cdot c_1 + \frac{1-a_{11}}{\Delta} \cdot c_2 \quad \ldots\ldots\ldots(8)$$

Where $\Delta = \begin{vmatrix} 1 - a_{11} & -a_{12} \\ -a_{21} & 1 - a_{22} \end{vmatrix}$. The solutions (7) and (8) can also be obtained without the use of the matrix method in this case where we have two industries. The matrix method is, however, more useful when we have a large number (say, n) of industries.

Let us see how (7) and (8) can be obtained solutions of simultaneous equations (4) and (5). Equations can also be solved by the use of Cramer's rule. Let Δ denote the determinant formed by coefficients of x_1 and x_2 in the above equations and let us assume that $\Delta \neq 0$. Then according to Cramer's rule

$$x_1 = \frac{\begin{vmatrix} c_1 & -a_{12} \\ c_2 & 1-a_{22} \end{vmatrix}}{\begin{vmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{vmatrix}} = \frac{c_1(1-a_{22})+c_2 a_{12}}{\Delta} = \frac{(1-a_{22})}{\Delta} c_1 + \frac{a_{12}}{\Delta} c_2$$

And $x_1 = \frac{\begin{vmatrix} 1-a_{11} & c_1 \\ -a_{21} & c_2 \end{vmatrix}}{\begin{vmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{vmatrix}} = \frac{c_2(1-a_{11})+c_1a_{21}}{\Delta} = \frac{a_{21}}{\Delta}c_1 + \frac{(1-a_{11})}{\Delta}c_2$

Thus the same solutions have been obtained with the use of Cramer's rule also. Equations (7) and (8) give us the solutions of equations (4) and (5). Thus given the a_{ij} coefficients and the final demands the gross outputs can be obtained from (7) and (8).

4.2.17. Hawkins – Simon Condition:

An economy in which the input requirements for production are directly proportional to the levels of production can be described by a set of linear equations.

The linear equations can be expressed in terms of a matrix, which is given above.

Suppose an economy has n industries each producing a single unique product. (There is a generalization of input output analysis, called activity analysis, in which an industry may produce more than one product, some of which could be pollutants.) Let the product input requirements per unit of product output be expressed as an nxn matrix A. Let X be the n dimensional vector of outputs and F the n dimensional vector of final demands. The amounts of production used up in producing output X is AX. This is called the intermediary demand. The total demand is thus AX+F. The supply of products is just the vector X. For equilibrium between supply and demand the following equations must be satisfied.

X = AX + F

The equilibrium production is then given by

$X = (I-A)^{-1}F$

A viable economy is one in which any vector of nonnegative final demand induces a vector of nonnegative industrial productions. In order for this to be true the elements of $(I-A)^{-1}$ must all be positive. For this to be true (I−A) has to satisfy certain conditions.

A minor of a matrix is the value of a determinant. The principal leading minors of an n∗n matrix are evaluated on what is left after the last m rows and columes are deleted, where m runs from (n-1) down to 0.

The condition for the n∗n matrix of (I−A) to have an inverse of nonnegative elements is that its principal leading minors be positive. This is known as the Hawkins-Simon conditions.

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4.2.18. Example of Input-Output Analysis:

Example 1.

From the following information find the final output goals of each industry to satisfy the specified bill of final consumption.

	X	Y	Z	Bill of final consumption
X	0.3	0.2	0.2	80
Y	0.2	0.1	0.5	30
Z	0.2	0.4	0.2	50
Labour	0.4	0.3	0.1	-

Find the total labour requirement also.

Solution:

Let x_1, x_2 *and* x_3 be the gross output levels of X, Y *and* Z respectively. Then we get the following 3 equations to equate supply and demand in the 3 industries:

$$x_1 = 0.3x_1 + 0.2x_2 + 0.2x_3 + 80 \; or, 0.7x_1 - 0.2x_2 - 0.2x_3 = 80$$

$$x_2 = 0.2x_1 + 0.1x_2 + 0.5x_3 + 30 \; or, -0.2x_1 + 0.9x_2 - 0.5x_3 = 30$$

$$x_3 = 0.2x_1 + 0.4x_2 + 0.2x_3 + 50 \; or, -0.2x_1 - 0.4x_2 + 0.8x_3 = 50$$

Multiplying the above 3 equations by 10 on both sides we get

$7x_1 - 2x_2 - 2x_3 = 800$ …………….. (1)

$-2x_1 + 9x_2 - 5x_3 = 300$ ………….. (2)

$-2x_1 - 4x_2 + 8x_3 = 500$ ………….. (3)

Let $\Delta = \begin{vmatrix} 7 & -2 & -2 \\ -2 & 9 & -5 \\ -2 & -4 & 8 \end{vmatrix} = 7\begin{vmatrix} 9 & -5 \\ -4 & 8 \end{vmatrix} + 2\begin{vmatrix} -2 & -5 \\ -2 & 8 \end{vmatrix} - 2\begin{vmatrix} -2 & 9 \\ -2 & -4 \end{vmatrix}$

$$= 7(72 - 20) + 2(-16 - 10) - 2(8 + 18)$$

$$= 7 \times 52 + 2(-26) - 2 \times 26 = 364 - 52 - 52 = 260$$

Let $\Delta_1 = \begin{vmatrix} 800 & -2 & -2 \\ 300 & 9 & -5 \\ 500 & -4 & 8 \end{vmatrix} = 800\begin{vmatrix} 9 & -5 \\ -4 & 8 \end{vmatrix} + 2\begin{vmatrix} 300 & -5 \\ 500 & 8 \end{vmatrix} - 2\begin{vmatrix} 300 & 9 \\ 500 & -4 \end{vmatrix}$

$$= 800(72 - 20) + 2(2400 + 2500) - 2(-1200 - 4500)$$

$$= 800 \times 52 + 2 \times 4900 + 2 \times 5700 = 62800$$

Let $\Delta_2 = \begin{vmatrix} 7 & 800 & -2 \\ -2 & 300 & -5 \\ -2 & 500 & 8 \end{vmatrix} = 7\begin{vmatrix} 300 & -5 \\ 500 & 8 \end{vmatrix} - 800\begin{vmatrix} -2 & -5 \\ -2 & 8 \end{vmatrix} - 2\begin{vmatrix} -2 & 300 \\ -2 & 500 \end{vmatrix}$

$$= 7(2400 + 2500) - 800(-16 - 10) - 2(-1000 + 600)$$

$$= 7 \times 4900 + 800 \times 26 + 2 \times 400 = 55900$$

Let $\Delta_3 = \begin{vmatrix} 7 & -2 & 800 \\ -2 & 9 & 300 \\ -2 & -4 & 500 \end{vmatrix} = 7\begin{vmatrix} 9 & 300 \\ -4 & 500 \end{vmatrix} + 2\begin{vmatrix} -2 & 300 \\ -2 & 500 \end{vmatrix} + 800\begin{vmatrix} -2 & 9 \\ -2 & -4 \end{vmatrix}$

$$= 7(4500 + 1200) + 2(-1000 + 600) + 800(8 + 18)$$

$$= 7 \times 5700 - 2 \times 400 + 800 \times 26 = 59900$$

Now from Cramer's rule of solving simultaneous equations we know that

$$x_1 = \frac{\Delta_1}{\Delta};\ x_2 = \frac{\Delta_2}{\Delta}\ and\ x_3 = \frac{\Delta_3}{\Delta}$$

Then $x_1 = \frac{62800}{260} = 241.5, x_2 = \frac{55900}{260} = 215, x_3 = \frac{59900}{260} = 230.4$

These are the required final output goals. Then total labour requirement is

$$0.4x_1 + 0.3x_2 + 0.1x_3 = 0.4 \times 241.5 + 0.3 \times 215 + 0.1 \times 230.4 = 184.14$$

Example 2.

Given the following information, estimate the gross levels of output of $X\ and\ Y$ required to satisfy the given bill of final consumption:

	X	Y	Bill of final consumption
X	0.4	0.5	Rs. 10 million
Y	0.4	0.3	Rs. 2 million
Labour	0.2	0.2	

Also indicate the labour requirement.

Solution:

Let $x_1\ and\ x_2$ be the gross output levels of $X\ and\ Y$ industry respectively. Then we get following two equations:

$x_1 = 0.4x_1 + 0.5x_2 + 10\ or, 0.6x_1 - 0.5x_2 = 10$ ……… (1)

$x_2 = 0.4x_1 + 0.3x_2 + 2\ or, -0.4x_1 + 0.7x_2 = 2$ ……….. (2)

Solving equations (1) & (2) we get

$$x_1 \left| \frac{\begin{matrix} 10 & -0.5 \\ 2 & 0.7 \end{matrix}}{\begin{matrix} 0.6 & -0.5 \\ -0.4 & 0.7 \end{matrix}} \right| = \frac{(7 + 1)}{(0.42 - 0.20)} = \frac{8}{0.22} = 36.4$$

$$x_2 = \frac{\begin{vmatrix} 0.6 & 10 \\ -0.4 & 2 \end{vmatrix}}{0.22} = \frac{(1.2 + 4)}{0.22} = \frac{5.2}{0.22} = 23.6$$

Total labour requirement $= 0.2x_1 + 0.2x_2 = Rs.(0.2 \times 36.4 + 0.2 \times 23.6)\ million = Rs.12\ million.$

Example .

In example 3 suppose that the bill of final consumption is not given. Instead it is given that the total available labour supply is 12 units. Determine the equation of the consumption possibility locus.

Solution:

Let c_1 *and* c_2 be the final demands for X *and* Y respectively. Then

$$x_1 = 0.4x_1 + 0.5x_2 + c_1\ or, 0.6x_1 - 0.5x_2 = c_1$$

$$x_2 = 0.4x_1 + 0.3x_2 + c_2\ or, -0.4x_1 + 0.7x_2 = c_2$$

Solving these equations for x_1 *and* x_2 with the help of Cramer's rule we get

$$x_1 = \frac{\begin{vmatrix} c_1 & -0.5 \\ c_2 & 0.7 \end{vmatrix}}{\begin{vmatrix} 0.6 & -0.5 \\ -0.4 & 0.7 \end{vmatrix}} = \frac{0.7c_1 + 0.5c_2}{0.22}$$

$$x_2 = \frac{\begin{vmatrix} 0.6 & c_1 \\ -0.4 & c_2 \end{vmatrix}}{0.22} = \frac{0.6c_2 + 0.4c_1}{0.22}$$

Putting these values of x_1 *and* x_2 in the labour supply equation we get

$$12 = 0.2x_1 + 0.2x_2$$

$$or, 12 = 0\left(\frac{0.7c_1 + 0.5c_2}{0.22}\right) + 0.2\left(\frac{0.4c_1 + 0.6c_2}{0.22}\right)$$

$$or, 12 \times 0.22 = (0.2 \times 0.7 + 0.2 \times 0.4)c_1 + (0.2 \times 0.5 + 0.2 \times 0.6)c_2$$

$$or, 2.64 = 0.22c_1 + 0.22c_2$$

This is the required equation of the consumption possibility locus.

4.2.19 Limitation of Input-Output Analysis:

In this framework, basic assumption of constancy of input co-efficient of production which was split up above as constant returns of scale and technique of production. The assumption of constant returns to scale holds good in a stationary economy, while that of constant technique of production in stationary technology.

1. Assumption of fixed co-efficient of production ignores the possibility of factor substitution. There is always the possibility of some substitutions even in a short period, while substitution possibilities are likely to be relatively greater over a longer period.

2. The assumption of linear equations, which relates outputs of one industry to inputs of others, appears to be unrealistic. Since factors are mostly indivisible, increases in outputs do not always require proportionate increases in inputs.

3. The rigidity of the input-output model cannot reflect such phenomena as bottlenecks, increasing costs, etc.

4. The input-output model is severely simplified and restricted as it lays exclusive emphasis on the production side for the economy. It does not tell us why the inputs and outputs are of a particular pattern in the economy.

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Sub Unit 3: Market Equilibrium Analysis

4.3.1. Introduction to Equilibrium Analysis

Equilibrium literally means balance. It means a position from which there is no tendency to change. The forces which determine it can said to be in balance at the equilibrium point. Unless these forces change, the equilibrium would not change. Equilibrium price, then, is the price at which the forces which determine the price are in balance. The two forces which determine the price of any commodity in the market are demand and supply. Geometrically, this is the price where the demand and the supply curves cross. If we let D (p) be the market demand curve and S (p) the market supply curve, the equilibrium price is the price p* that solves the equation.

D (p) = S (p)

The solution to this equation, p*, is the price where market demand equals market supply. When market is in equilibrium, then there is no excess demand and supply.

Assuming that both demand and supply curves are linear, demand – supply model can be stated in the form of the following equation.

Demand function, QD = a - bp… (1)

Supply function, QS = c + dp… (2)

Equilibrium Condition, QD = QS (3)

Where QS and QS are quantities demanded and supplied respectively, a and c are intercept coefficients of demand and supply curves respectively, b and d are the coefficients that measures the slop of these curves, equation (3) is the equilibrium condition.

a - bp = c + dp

a – c = dp + bp = p (d+b)

Dividing both sides by d + b we have

(a – c)/(d + b) = p

Or, equilibrium price,

P = (a – c)/(d + b) ………………… (4)

Substituting (4) into (1) we have equilibrium quantity

$$QD = (a - b)\frac{a-c}{d+b} = \frac{ad+bc}{b+d}$$

Equation (4) and (5) describe the qualitative results of the model. If the values of the parameters a, b, c and d are given we can obtain the equilibrium price and quantity by substituting the values of these parameters in the qualitative results of equation (4) and (5).

Example:

Suppose the following demand and supply functions of a commodity are given which is being produced under perfect competition. Find out the equilibrium price and quantity.

QD = 10 – 4p…………………… (1)

QS = -2 + 8p…………………… (2)

Solution: There are two alternatives ways of solving for equilibrium price and quantity.

First we can find out the equilibrium price and quantity by using the equilibrium condition, namely

QD =QS…………………… (3)

Second, we can obtain equilibrium price and quantity by using the qualitative results of the demand and supply model.

P= (a-c)/(d+b) and QD = (ac+bc)/(b+d)

Since in equilibrium QD=QS

10 -4p = -2 + 8p

-2 + 8p + 4p -10 = 0

12 = 12p

P= = 1

P=1 …………………… (4)

Substituting the equilibrium price of p* =1 either into demand equation (1) or supply equation

(2), we get the equilibrium quantity QD.

Now, QD = 10 – 4(1) = 6

QS = -2 + 8(1) = 6

So, the equilibrium level of quantity is 6 and price is 1 respectively.

4.3.2. Equilibrium in Perfect Competitive Market.

Perfect competition is a market situation where the market is automatically regulated by the forces of demand and supply over which individual sellers have no control. In the perfect competitive market the firms are in equilibrium when they maximize their profits (Π). The profit is the difference between the total cost and total revenue, i.e. Π = TR-TC

The conditions for equilibrium are

1. MC = MR
2. Slop of MC > slope of MR

Derivation of the equilibrium of the

firm

The firms aim at the maximization of its profit

Π = TR-TC
Where

Π =Profit

R =Total Revenue

C =Total cost

Clearly $R = f(x)$ and $c = f(x)$, given the price P

a) The first order condition for the maximization of a function is that us first derivative (with respect to x in this case) be equal to zero. Differentiating the totoal profit function and equating to zero, we obtain.

$$\frac{\partial \pi}{\partial x} = \frac{\partial R}{\partial x} - \frac{\partial c}{\partial x} = 0$$

$$or, \frac{\partial R}{\partial x} = \frac{\partial c}{\partial x}$$

$$or, MR = MC$$

Thus the first order condition for profit maximization is $MR = MC$

Given that $MR > 0$ must also be positive at equilibrium.

Since, $MR = P$ the first order condition may be written as $MC = P$

b) The second –order condition for a optimization requires that the second derivatives of the function be negative (implying that aster its highest point the curve turns downwards). The second derivatives of the total profit function is

$$\frac{d^2\pi}{dx^2} = \frac{d^2R}{dx^2} = \frac{d^2c}{dx^2}$$

This must be negative if the function has been maximized, that is

$$\frac{d^2R}{dx^2} - \frac{d^2c}{dx^2} < 0 \qquad or, \qquad \frac{d^2R}{dx^2} < \frac{d^2c}{dx^2}$$

or, $slope\ of\ MR < slope\ of\ MC$

It is implies that , the slope of MC curve must greater than slope of MR.

By simplification we get,

$$0 < \frac{d^2c}{dx^2}$$

That is the MC curve must have a positive slope, or the MC must be rising.

Example:

A perfectly Competitive market price $P = 4$ and $T = x^3 - 7x^2 + 12x + 5$. Find the best level of output of the firm. Also find the profit of the firm at this level of output.

Solution:

First order condition, $MR = MC$

Now, $TC = x^3 - 7x^2 + 12x + 5$

$MC = \frac{d}{dx}(TC) = 2x^2 - 14x + 12$

And $TR = Price\ \times quality$

$= 4 \times x = 4x$

$MR = \frac{d}{dx}(TR) = 4$

Setting $MR = MC$ and solving for x to find the critical value.

4 = 3X^2 – 14X + 12

Or, 3X^2 – 14X + 8 = 0

By factorization we have the values as (3X – 2) and (X – 4)

So, 3X – 2 = 0 or, X = 2/3 and X – 4 = 0 or, X = 4

This means that at equilibrium point MR = MC, X = 2/3, 4

The second condition requires that MC must be rising at this point of intersection. In other words, the slope of the MC curve should be positive at the point where MC = MR. the equation for the slope of the MC curve is to find its derivatives.

$\frac{\delta MC}{\delta X}$ = 6X – 14

Then substitute the two critical values X = 2/3 and X = 4 in the above equation to find out the point. Which maximize the profit.

When, X = 2/3, $\frac{\delta MC}{\delta X}$ = -10. It is not the profit maximizing output.

When, X = 4, $\frac{\delta MC}{\delta X} = 10$.

So, the profit maximized output is 4,

Now, the profit Π = TR – TC

$\Pi = 4X - (X^3 - 7X^2\ 12X + 5)$

$= X^3 - 7X^2 - 8X - 5$

So, the total profit of the firm at point X = 4 is

$\Pi = -65 + 112 - 32 - 5$

$= 11$

The firm maximizes its profit at the output level of 4 units and at this level its maximum profit is Rs.11.

4.3.3. Equilibrium in the Monopoly

Monopoly is the form of market organization in which a single firm sells a commodity for which there are no close substitutes. Thus, the monopolist represents the industry and faces the industry's negatively sloped demand curve for the commodity. As opposed to a perfectly competitive firm, a monopolist can earn profits in the long run because entry into the industry is blocked or very difficult. And the monopolist has complete control over price.

A. Short-run Equilibrium

The monopolist maximizes his short-run profit if the following two conditions are fulfilled:

The MC is equal to the MR. i,e, MC = MR

The slope of the MC is greater than the slope of the MR at the point of intersection. Mathematical derivation

The given demand function is X = g(P) Which may be solved for P, P = f(X)

The given cost function is C = f(X)

The monopolist aims at the maximization of his profit

Π = TR– TC

a) The first order condition for the maximization of a function is that us first derivative (with respect to x in this case) be equal to zero. Differentiating the totoal profit function and equating to zero, we obtain.

$$\frac{\partial \pi}{\partial x} = \frac{\partial R}{\partial x} - \frac{\partial c}{\partial x} = 0$$
$$or, \frac{\partial R}{\partial x} = \frac{\partial c}{\partial x}$$
$$or, MR = MC$$

b) The second –order condition for an optimization requires that the second derivatives of the function be negative (implying that aster its highest point the curve turns downwards). The second derivatives of the total profit function is

$$\frac{d^2\pi}{dx^2} = \frac{d^2R}{dx^2} = \frac{d^2c}{dx^2}$$

This must be negative if the function has been maximized, that is

$$\frac{d^2R}{dx^2} - \frac{d^2c}{dx^2} < 0 \qquad or, \qquad \frac{d^2R}{dx^2} < \frac{d^2c}{dx^2}$$

or, $slope\ of\ MR < slope\ of\ MC$

Equilibrium conditions are same for the monopoly and perfect competitive market, but monopolist has set the price by the demand function, which not equal to MC.

Example:

Given the demand curve of the monopolist X = 80 – 0.2P and cost function C= 50 + 20X. Find out the maximum profit.

Solution:

So, X = 80 – 0.2P

Or, P = 100 – 8X

Now, the total revenue R = X * P = X (100 - 8X) = $100X - 8X^2$

Then, MR = 100 – 16X

And MC = 20

Now, the equilibrium condition is MR =MC

So, 100 – 16X = 20

Or, X = 5

The monopolist's price is found by substituting x = 5 into the demand equation

P = 100 – 8X = 100 – 8*5 = 60

The profit is, $\Pi = R - C = 100X - 8X^2 - 50 + 20X$
$= 100 * 5 - 8 * 5^2 - 50 + 20 * 5$
=500 -200-50+100
=150
This profit is the maximum possible, since the second- order condition is satisfied

$\frac{dMR}{dX}$ = -16 and $\frac{dMC}{dX}$ = 0

So, $slope\ of\ MR < slope\ of\ MC$

So, the 2nd order condition is satisfied.

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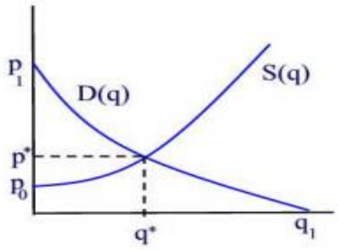
4.3.4. Consumer and Producer Surplus:

The definitions of demand and supply must be remembered: Demand tells us the price that consumers would be willing to pay for each different quantity. According to the law of demand, when the price increases the demand decreases and when the price decreases the demand increases. The graphical representation of the relationship between the quantity demanded of a good and the price of the good is known as the demand curve. Supply tells us the price that producers would be willing to charge in order to sell the different quantities. The law of supply asserts that as the price of a good rises, the quantity supplied rises, and as the price of a good falls the quantity supplied falls. The graphical representation of the relationship between the quantity supplied of a good and the price of the good is known as the supply curve. The demand and supply curve intersects at the point of equilibrium ($q*$, $p*$). We call $p*$ the equilibrium price and the $q*$ the equilibrium quantity. See below figure,



Consumers’ Surplus Consumers’ surplus is the economic gain accruing to a consumer (or consumers) when they engage in trade. The gain is the difference between the price they are willing to pay and the actual price. At the equilibrium level, the consumers’ surplus is the difference between what consumers are willing to pay and their actual expenditure: It therefore represents the total amount saved by consumers who were willing to pay more than p ∗ per unit.

Now, consumer surplus = $\int_0^{q*} D(q)dq$ −p*q*

Where, D(q) is demand function, P* is equilibrium price and q* is equilibrium quantity.

Producers’ Surplus

The producer surplus measures the suppliers’ gain from trade. It is the total amount gained by producers by selling at the current price, rather than at the price they would have been willing to accept. For example, a producer might be willing to sell at a price lower than p ∗ so that he can stay in business. However, by selling at the price of p ∗ he is making a benefit which is the producers’ surplus. Thus, the producers’ surplus can be defined as the extra amount earned by producers who were willing to charge less than the selling price of p ∗ per unit, and is given by

Producer surplus = p*q* - $\int_0^{q*} S(q)dq$

Where, D(q) is demand function, P* is equilibrium price and q* is equilibrium quantity.