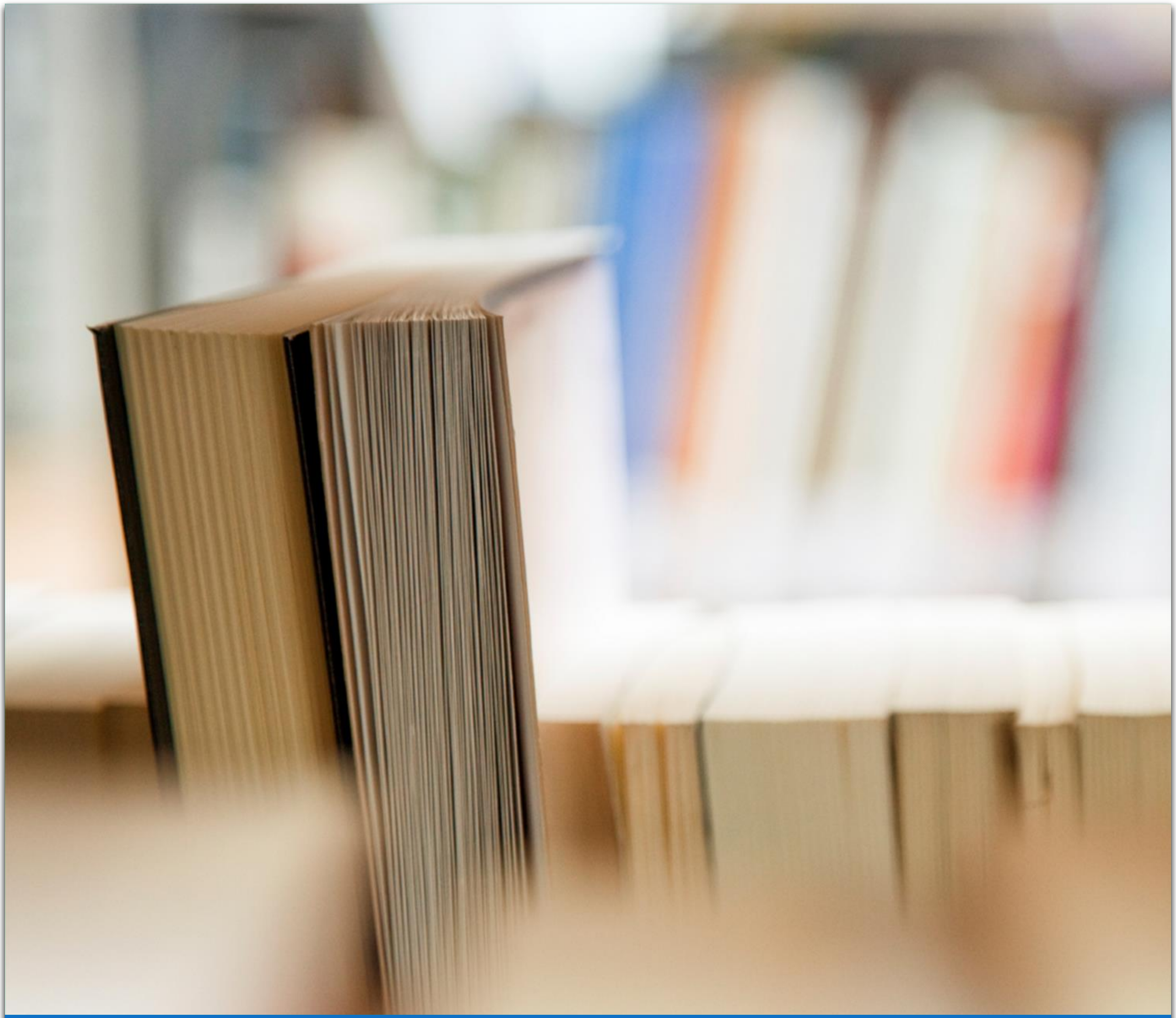

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Unit-1: Micro Economics

Demand Analysis:

Demand: - In economic analysis, the term 'demand' is not just the term to mean "need". It also means, a rational consumer must have the money to purchase that commodity to meet the need i.e., the ability to purchase. So, by the definition, demand of a product is the mutual coexistence of the "purchasing power" and "desired purchase", where the basic objective of a consumer is utility maximisation, obtained from the consumption of that product. So, 'demand', the term is used to represent the relationship between price of a commodity and the amount of that commodity, the consumer is Willing to purchase at that time period.

According to prof. Beinham, **"the demand for anything at a given price is the amount of it which will be bought per unit of time at that price".**

- The 'planned demand' of one consumer or family at a "given" or 'expected price' of a product is the 'individual demand' or 'family demand' where as the demand of all the consumers is a market in aggregate sense will be the 'market demand'.

Demand Function: -

The 'demand function' will represent the relation between quantity demanded or demand for a product and its determinants.

Thus, the aggregate demand function can be written as, from the previous discussion,

$Q_d^{\times} = f(p_{\times}, p_y, p_z, M, T, t, d, P, r, D)$————(i)

The 'reduced' from of the demand function can be developed considering 'other factors' remain constant, i.e., "ceteris peribus". Here, the other factors are:

$p_y, p_z, M, T, t, d, P, r, D.$

Hence the reduced form of the demand function can be written as:

$Q_d^{\times} = f(p_{\times})$————(ii)

So, the other factor in equation (ii) is called 'PARAMETER'.

Demand Law: -

The 'demand law' states the relationship between price of a product and quality demanded for that product. According to the demand law,

"Other factors remain constant, price and quantity demanded both are inversely related".

$$Q_d^{\times} \infty \frac{1}{p_{\times}}$$

$$\Rightarrow Q_d^{\times} = K.\frac{1}{p_{\times}} \; [K \; is \; a \; variation \; constant]$$

$$\Rightarrow p_d^{\times} \uparrow\downarrow \Rightarrow p_{\times} \downarrow\uparrow$$

Assumptions: -

i) No change in consumer's "taste and preference".

ii) Consumer's income remains constant According to Marshall 'money income' in constant whereas according to Friedman, 'real income is constant'.

iii) The prices of the related products like substitutes and complements are constant.

Demand Curve: -

i) If the demand function is 'linear', i.e. it follows the structure $y = mx + c$ than it can be depicted by a demand line.

ii) If the demand function is 'non-linear', i.e. it follows the structure y = $ax^2 + bx + c,$ than it can be depicted by a demand curve.

Whatever may be the shape, we do not have any confusion that the 'usual' slope of a demand line or demand curve will be negatively sloped and for the 'exception', it will be positively sloped.

Why the demand curve be negatively sloped?

According to R. G. Lipsey, "The curve, which shows the relation between the price of a commodity and the amount of that commodity the consumer wishes to purchase, is called the demand curve".

The following reasons are responsible for negatively sloped of a demand cure:

1. Law of Diminishing Marginal Utility:

$$\times \uparrow\downarrow \Rightarrow MU_{\times} \downarrow\uparrow$$

According to the Marshalling approach, the consumer's equilibrium condition for a single product, $\times$, can be written as $MU_{\times} = \lambda \, p_{\times}$; $p_{\times} \rightarrow price \; per \; unit$

$$\lambda \rightarrow MU\ of\ money\ (constant)$$

Since λ is constant therefore a fall in $p_{\times}$ means fall in

$MU_{\times}$ $and\ fall\ in\ MU_{\times}\ is\ possible\ omly\ by\ increase\ in\ \times$

$$\therefore\ p_{\times}\ \downarrow\uparrow\rightarrow\ MU_{\times}\ \downarrow\uparrow\rightarrow \times \uparrow\downarrow$$

$$\Rightarrow\ P_{\times}\ and\ \times\ are\ inversly\ related$$

$$\Rightarrow Demand\ cure\ is\ (-)vely\ sloped$$

2. Price Effect:

The concept is based on the 'ordinal utility approach'. It measures the effects on quantity demanded of a product due to a change in price of that product.

Price effect can be decomposed of a product due to a change in price of that product.

Price effect can be decomposed in two major components:

(a) Substitution effect

$$P_{\times} \downarrow\uparrow\rightarrow Y \downarrow\uparrow \times \downarrow\uparrow$$

$$\Rightarrow\ P_{\times}\ f\ \ \times\ are\ inversely\ related$$

$$\Rightarrow Demand\ curve\ is\ (-)sloped.$$

(b) Income effect

$$P_{\times}\ \downarrow\uparrow\ R.I.\uparrow\downarrow\rightarrow \times \uparrow\downarrow$$

$$\Rightarrow\ P_{\times}\ \&\times\ are\ inversely\ related$$

$$\Rightarrow Demand\ curve\ is\ (-)vely\ sloped.$$

[One should note that following the substation effect, $P_{\times}\ \downarrow\uparrow\rightarrow Y \downarrow\uparrow\rightarrow \times\uparrow\downarrow$, wich is satisfied for both the cases normal as well as inferior. Due to this inverse relation between

$P_{\times}\ and\ \ \times, the$ substation effect is always negative (-).

But the income effect may be positive or even negative.

for NORMAL: $P_{\times}\ \downarrow\rightarrow RI\ \ \uparrow\rightarrow \times\uparrow$

and for INFERLOR: $P_{\times}\ \downarrow\rightarrow RI\uparrow\rightarrow \times\downarrow$

3. Bandwagon Effect:

↳ This concept is developed by Prof. Samuelson. Following this approach, it there is a fall in price of a product then there will be two types of impacts simultaneously.

(a) Creation of new customer at a lower price.

(b) Increase in demand by the old customer.

⇒ So again, we have the inverse relationship and the demand curve is negatively sloped.

4. Alternative Use:

One product for example, 'STEEL' can be used for different purposes, So, a fall in price will lead to a rise in demand by all the sectors. Hence, we have the inverse relationship.

Exceptions to the Demand Law: -

If price of a product goes down (up) then the demand for that product also goes down (up), other factors remain constant.

Following are the examples relating to the exception:

1. **Veblen Effect:** There are some products, like diamond and jewellery represent the social status. So, the consumption of such products is the 'CONS PICUOUS CONSUMPTION', is explained by the American Economist Veblen. So increase in price of such product will tend to a further increase in demand for this product. Now price and quantity demand both are directly related rather than inverse relation.

2. **Speculative market:** $\frac{Pshare\ of}{a\ company} \uparrow \rightarrow \frac{Dshare\ of}{that\ company} \uparrow$

Since consumer prefers to invest assuming it profitable

⇒ Price & quality demanded both are $+vely$ related

⇒ Demand curve is upward sloping.

3. **Ignorance:** According to Prof. Benham, a lower priced commodity may be considered as inferior or a higher priced commodity as superior.

↳ Price & quantity demanded are $+vely$ related.

↳ Demand curve is upward sloping.

4. **Future Expectation:** If consumer expects a rise in price in near future due to the shortage in stock natural calamities, then they will rush to purchase more, at the present price.

↳ We have the direct relation.

5. **Giffen Good**: These goods are special type of interior, occupy the lowest place in the consumer's budget. Fall in price of such goods means reduction in demand for that product.

Determinants of Own Price Elasticity of Demand: -

1. **Natural of the product**: Necessary good → Elasticity is low.

   Luxury good → Elasticity is high.

2. Availability of Substitutes: If close substitutes of a good are available in the market ⇒ Demand for the commodity will be quite price elastic.

3. Number of Uses: Large number of alternatives uses ⇒ highly elastic.

4. Product durability: More the product is durable; higher will be the price elasticity.

5. Consumption Postponement: If consumption of a product can be postponed for a specific period of time, price elasticity will be high.

6. Habit: Those commodities, whose consumption is a habit like cigarettes, have a low-price elasticity.

7. Income distribution: ●High quality product in high income class

↳ Price elasticity is low

● For the poorer class it is very high

8. Income & Expenditure: Lesser (larger) is the proposition of expenditure on the commodity elasticity will be less (relatively elastic).

9. Price level: Highly priced product such as diamonds and law priced product like salt have low price elasticity because a change in their price has very little effect on demand.

10. Time period: In the short period price elasticity is low and for the long period it is high.

11. Joint demand: If the demand for car is relatively in elastic then the demand for petrol will also be inelastic.

12. Marginal utility: If diminishing rate of MU is high them a further fall in price will lead to a small rise in demand, i.e. relatively elastic.

1.1.12.4. Arc Elasticity:

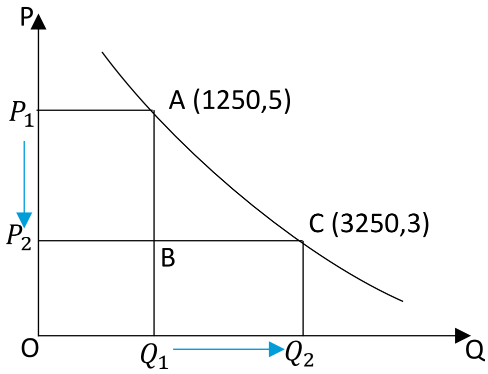
Elasticity of demand as measured above is relevant in a situation in which there is a small change in the price of a commodity. But often, business firms contemplate significant changes in the price of their products. They would be interested to estimate the possible response of the demand to significant price change.

In the type of situation, we need to calculate what is known as the are elasticity.

Let us consider one demand curve. Then one segment on this curve or range between two points is called as 'Arc'. It is to be noted that the arc elasticity is the 'AVERAGE MEASURE' of elasticity on the Arc of one demanded curve. Since Arc means the range between two points on a demand curve therefore as the gap between two points increases (decreases) the Arc will be large (small).

Now we have to measure the 'Arc elasticity' over the range Ac. By the Definition or formula of price elasticity.

$E_p = \frac{change\ in\ quantity\ demanded}{change\ in\ price} \times \frac{Initial\ price}{Initial\ quantity}.$



So, price elasticity is the product of two fractions. Now the basic problem is relating to the second fraction, i.e. what should be the value of the fraction. If the movement is restricted along the demand curve from A to C, then we have one fraction i.e. $\frac{0p_1}{0Q_1}$, but it the movement is from C to A, then the fraction differs, i.e. $\frac{0p_2}{0Q_2}$.

To overcome this problem one overage price and overage quantity can be considered as the initial price and initial quality.

Hence, initial price = $\frac{p_1+p_2}{2}$ and initial quality = $\frac{Q_1+Q_2}{2}$ Therefore, the arc elasticity can be defined as,

$$E_p = (-)\frac{\Delta Q}{\Delta p}.\frac{p}{Q}$$

$$= (-)\frac{Q_2 - Q_1}{p_2 - p_1}.\frac{\frac{p_1 + p_2}{2}}{\frac{Q_1 + Q_2}{2}} = (-)\frac{Q_2 - Q_1}{p_2 - p_1}.\frac{p_1 + p_2}{Q_1 + Q_2}$$

$$= (-)\left(-\frac{p_1 + p_2}{p_1 - p_2}\right).\left(-\frac{Q_1 - Q_2}{Q_2 + Q_1}\right) = (-)\frac{p_1 + p_2}{p_1 - p_2}.\frac{Q_1 - Q_2}{Q_1 + Q_2}$$

Unit – 3: Statistics and Econometrics

Event:

One or more outcomes together are called the event. Event are two types →

i) Elementary Event: Elementary event are those events which we cannot decompose into several events.

ii) Composite Event: Composite events are those events which we can decompose into several events.

e.g. → If we tossed a coin two times, HH and TT are elementary event. And HT, TH are the composite event.

Sample's Space:

The set of all the possible outcomes of a random experiment is called the sample's space which is denoted by S.

e.g., → If we tossed a coin are time the sample's space is $S = \{H, T\}$.

Classical definition of Probability:

The probability of an event A is denoted by $P(A)$ and according to classical definition.

$$P(A) = \frac{No\ of\ favourable\ cases\ to\ the\ event}{Total\ No.of\ mutually\ exclusive,exhaustive\ and\ equally\ likely\ set\ of\ outcome}$$

Mutually Exclusive event:

Mutually exclusive events are those events which cannot occur simultaneously. So, they are mutually exclusive events.

Exhaustive:

Two more events are said to be exhaustive, if at least one of them will occur.

e.g., → If we tossed a coin one time the events are H, T.

They are exhaustive, because at least one of them must occur.

Equally Likely:

Two or more events are said to equally likely if they have equal chances to occur.

e.g., → If we tossed a coin one time, the events are H, T.

$P(H) = \frac{1}{2},\ P(T) = \frac{1}{2}.$

So here the events are equally likely events.

Set Theory:

$U = Union\ (+)$

$\cap = Intersection\ (X).$

$P(A \cup B) = P(A) + P(B) - P(A \cap B).$

But when A and B are mutually exclusive events then $(A \cap B) = 0$

$\therefore P(A \cup B) = P(A) + P(B).$

$\therefore P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

3.1.11. Conditional Probability:

3.1.11.1. Independent event / with replacement / Unconditional probability

Event A and B are said to be independent if $P(AB) = P(A \cap B) = P(A) \cdot P(B)$

3.1.11.2. Dependent event / without replacement / Conditional probability

$P(A \cap B) = P(A) \cdot P(B/A)$ $\qquad [B\ given\ A = \frac{B}{A}]$

$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)\ P(B/A)}{P(B)}$

$P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)} = \frac{P(B)\ P(A/B)}{P(A)}$

3.1.12. Bayes Theorem (State and Proof):

Statement: An event A can occur only when one of the M.E, exhaustive and equally likely set of events $B_1, B_2, \ldots\ldots\ldots, B_n$ occur.

The unconditional probability $P(B_1), P(B_2), \dots\dots\dots, P(B_n)$, and the conditional probability $P(A/B_1), P(A/B_2), \dots\dots\dots\dots, P(A/B_n)$ are the given. Then the probability of event $B_i(i = 1,2,\dots\dots, n)$ when A has already occurred

i) $P(B_i/A) = \frac{P(B_i)\cdot P(A/B_i)}{\sum_{i=1}^{n} P(B_i)\cdot P(A/B_i)}$

Proof: Event A can occur with $B_i(i = 1,2,\dots\dots, n)$ in $B_1A, B_2A, \dots\dots\dots, B_nA$ ways.

So,

$P(A) = P(B_1A) + P(B_2A) + \dots\dots\dots + P(B_nA)$

$P(A) = P(B_1)\cdot P\left(\frac{A}{B_1}\right) + P(B_2)\cdot P\left(\frac{A}{B_2}\right) + \dots\dots\dots + P(B_n)\cdot P\left(\frac{A}{B_n}\right)$

$P(A) = \sum_{i=1}^{n} P(B_i)\cdot P(A/B_i) \dots\dots\dots\dots\dots (i)$ $[using\ multiplication\ theorem]$

Now,

$P(B_iA) = P(AB_i)$

$P(B_i)\cdot P\left(\frac{A}{B_i}\right) = P(A)\cdot P\left(\frac{B_i}{A}\right)$

$\therefore P\left(\frac{B_i}{A}\right) = \frac{P(B_i)\cdot P\left(\frac{A}{B_i}\right)}{P(A)} = \frac{P(B_i)\cdot P\left(\frac{A}{B_i}\right)}{\sum_{i=1}^{n} P(B_i)\cdot P(A/B_i)}$ [From (i)] (Proved)

- **Show that, $P(A\,B) \le P(A) \le P(A+B) \le P(A) + P(B)$**

Ans: Let us consider $P(AB) = P(A)\cdot P\,(B/A)$

Now,

$P(B/A)$ is a probability i.e., $P(B/A) \le 1$

When it is equal to 1 then $P(AB) = P(A) \dots\dots\dots\dots\dots\dots (1)$

But when it is less than 1 then, $P(AB) < P(A) \dots\dots\dots\dots\dots\dots (2)$

Combining (1) and (2) we have, $P(AB) \le P(A) \dots\dots\dots\dots\dots\dots (X)$

Now,

$P(A+B) = P(A) + P(B) - P(AB)$

$= P(A) + P(AB) + P(A^cB) - P(AB)$

$= P(A) + P(A^cB)$

Since, $P(A^cB)$ is a probability, i.e., $P(A^cB) \ge 0$

When it is $= 0$ then $P(A+B) = P(A) \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ldots (3)$

When it is > 0 then $P(A+B) > P(A) \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ldots (4)$

Combining (3) and (4) we get,

$P(A+B) \geq P(A) \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ldots (Y)$

Now, $P(A+B) = P(A) + P(B) - P(AB)$

Now, $P(AB)$ is a probability, i.e., $P(AB) \geq 0$

When it is equal to 0 then $P(A+B) = P(A) + P(B) \ldots\ldots \ldots \ldots\ldots \ldots\ldots (5)$

When it is equal to > 0 then $P(A+B) < P(A) + P(B) \ldots\ldots \ldots \ldots\ldots \ldots\ldots (6)$

Combining (5) and (6) we have,

$P(A+B) \leq P(A) + P(B) \ldots\ldots \ldots \ldots\ldots \ldots\ldots (Z)$

Now combining (X), (Y) and (Z) we get,

$P(A\,B) \leq P(A) \leq P(A+B) \leq P(A) + P(B)$ (Proved)

❖ **If event A and B are independent prove that A^c and B^c are also independent.**

Ans:- Since A and B are independent, we can write $P(AB) = P(A) \cdot P(B)$

Now, A^c, B^c will be independent if $P(A^c B^c) = P(A^c)\, P(B^c)$

Now,

$$
\begin{aligned}
P(A^c B^c) &= P(A \cup B)^c \\
&= 1 - P(A \cup B) \\
&= 1 - [P(A) + P(B) - P(A \cap B)] \\
&= 1 - P(A) - P(B) + P(A) \cdot P(B) \\
&= [1 - P(A)] - P(B)[1 - P(A)] \\
&= [1 - P(A)][1 - P(B)
\end{aligned}
$$

$P(A^c B^c) = P(A^c) \cdot P(B^c)$ (Proved)

3.1.13. Axiom of the probability:

Let S be a samples space of a Random experiment. If each event A of the set of all possible events of S, we associate or number $P(A)$, then $P(A)$ called the probability of event A, if the following axioms are satisfied.

i) $P(A) \geq 0$

ii) $P(S) = 1$

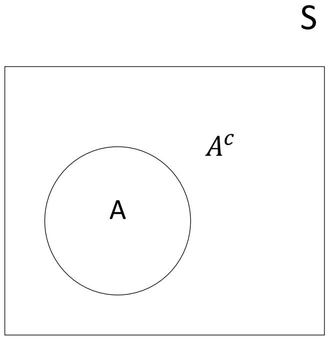
iii) For n mutually events of $(A_1, A_2, \ldots\ldots\ldots, A_n)$ then

$P(A_1 \cup A_2 \cup \ldots\ldots \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots\ldots \ldots + P(A_n)$.

❖ Show that $P(A^c) = 1 - P(A)$ with the help of axioms of probability:

A and A are mutually exclusive and

$P(S) = P(A \cup A^c) = P(A) + P(A^c)$ $[By\ axiom\ (3)]$



Now by axiom (2)

$P(S) = 1$

$or,\ P(A) + P(A^c) = 1$

$or,\ P(A^c) = 1 - P(A)$

❖ Show that $0 \leq P(A) \leq 1$

By axiom (1) we get, $P(A) \geq 0$

Now, we know that $P(A^c) = 1 - P(A)$

$or, P(A) = 1 - P(A^c)$

So, $P(A^c)$ is a probability then, $P(A^c) \geq 0$

When $P(A^c) = 0$ then $P(A) = 1 \ldots\ldots\ldots\ldots\ldots (i)$

When $P(A^c) > 0$ then $P(A) < 1 \ldots\ldots\ \ldots\ldots\ldots\ ..(ii)$

Combining (i) and (ii) we have, $P(A) \leq 1$

∴ This is proved that, $0 \leq P(A) \leq 1$

❖ When $A \subseteq B$ [$A\ is\ subset\ of\ B$] then prove that $P(A) \leq P(B)$.

A and $(A^c \cap B)$ are mutually exclusive.

From axiom (3) we have, $P(B) = P(A) + P(A^c \cap B)$

When $P(A^c \cap B)$ is a probability then $P(A^c \cap B) \geq 0$

Now, when it is > 0 then $P(B) > P(A)\ldots\ldots\ldots\ \ldots\ldots\ldots\ldots (i)$

and when it is $= 0$ then $P(B) = P(A)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$

From (i) and (ii) we get,

$P(B) \geq P(A)$

$\therefore P(A) \leq P(B)$ (proved)

3.1.14. Mutually independent event:

Several events $(A_1, A_2, \ldots\ldots\ldots, A_n)$ are said to be mutually independent event when the probability of the joint occurrence will be equal to the product of the probability.

e.g., → In cases of 3 events, we have, (A_1, A_2, A_3)

$P(A_1 \cap A_2) = P(A_1)\, P(A_2)$

$P(A_2 \cap A_3) = P(A_2)\, P(A_3)$

$P(A_1 \cap A_3) = P(A_1)\, P(A_3)$

$P(A_1 \cap A_2 \cap A_3) = P(A_1)\, P(A_2)\, P(A_3)$.

3.1.14.1. Pairwise Independent event:

Several events $(A_1, A_2, \ldots\ldots\ldots, A_n)$ are said to be pairwise independent event when every pair of this events are independent.

e.g.→ For three events, (A_1, A_2, A_3)

$P(A_1 \cap A_2) = P(A_1)\, P(A_2)$

$P(A_2 \cap A_3) = P(A_2)\, P(A_3)$

$P(A_1 \cap A_3) = P(A_1)\, P(A_3)$

State and proved the addition theorem / Probability Statement:

If the two events A and B are mutually exclusive, then the probability of the occurrence of either A or B is given by the sum of their probability, i.e.,

$P(A + B) = P(A) + P(B)$ this is known as addition theorem.

Proof: Let, a random experiment has a possible outcomes, which are mutually exclusive, exhaustive, equally likely and out of $n\ cases\ there\ are\ m$ favourable cases in the event $A\ and\ m$ favourable cases in the $event\ B$. Then,

$P(A) = \frac{m_1}{n} \left[\frac{favourable\ cases}{total\ cases}\right]$

$P(B) = \frac{m_2}{n}$

Now, the number of cases favourable to either $A\ or\ B$ is $(m_1 + m_2)$

$\therefore P(A + B) = P\left(\frac{m_1+m_2}{n}\right)$

But $\left(\frac{m_1+m_2}{n}\right) = \frac{m_1}{n} + \frac{m_2}{n} = P(A) + P(B)$

$\therefore$ This proves that $P(A + B) = P(A) + P(B)$ (Proved)

3.1.15. State and prove the theorem of compound probability or conditional probability or multiplication theorem:

Statement: The probability of occurrence of the event A as well as B, is given by the product of unconditional probability of A and conditional probability. i.e.,

$$P(AB) = P(A) \cdot P(B/A)$$

This is known as multiplication theorem.

Proof: Let, a random experiment has n possible outcomes, which are mutually exclusive, exhaustive and equally likely out of n cases let m cases are favourable to an event A. i.e., $P(A) = \frac{m}{n}$

Out of m cases let m_1 cases be favourable to another event B also,

i.e., $P(AB) = \frac{m_1}{n}$ and $P\left(\frac{B}{A}\right) = \frac{m_1}{m}$

We know, $\frac{m_1}{n} = \frac{m}{n} \cdot \frac{m_1}{n} = P(A) \cdot P\left(\frac{B}{A}\right)$

$\therefore P(AB) = P(A) \cdot P\left(\frac{B}{A}\right)$ (Proved.)

$P(A) = \frac{m}{n}$ $\quad P(B) = \frac{m_1}{n}$

$P(AB) = \frac{m}{n} \cdot \frac{m_1}{n} = \frac{m_1}{n}$

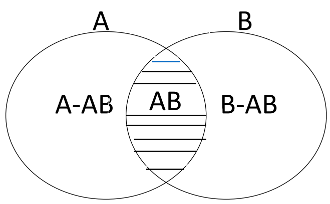
*** If A and B be any two events corresponding to a random experiment E, then**

$P(A + B) = P(A) + P(B) - P(AB)$

$or, P(A \cup B) = P(A) + P(B) - P(AB)$

Proof:- From the given Venn diagram, it is clear that the events

A-AB, AB and B-AB, are mutually exclusive and also



$A = (A - AB) + AB$

$B = AB + (B - AB) \quad and$

$A + B = (A - AB) + (AB) + (B - AB)$

Now since $(A - AB), AB\ and\ (B - AB)$ are mutually exclusive events; then

$P(A) = P(A - AB) + P(AB) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$

$P(B) = P(AB) + P(B - AB) \ldots\ldots\ldots\ldots\ldots\ldots (ii)$

and $P(A + B) = P(A - AB) + P(AB) + P(B - AB) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (iii)$

From (i) and (ii)

$P(A - AB) = P(A) - P(AB) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (iv)$

$P(B - AB) = P(B) - P(AB) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (v)$

From (iii), (iv) and (v), we get

$P(A + B) = P(A) - \cancel{P(AB)} + \cancel{P(AB)} + P(B) - P(AB)$

i.e., $P(A \cup B) = P(A) + P(B) - P(AB)$

4. If A, B, C be any three events, then
$P(A+B+C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(CA) + P(ABC)$

Proof:

$$
\begin{aligned}
P(A+B+C) &= P\{(A+B)+C\} \\
&= P(A+B) + P(C) - P\{(A+B)C\} \\
&= P(A) + P(B) - P(AB) + P(C) - P(AC+BC) \ \ [by\ distributive\ law] \\
&= P(A) + P(B) - P(AB) + P(C) - P(AC) - P(BC) + P(ACBC) \\
&= P(A) + P(B) + P(C) - P(AB) - P(BC) - P(AC) + P(ABC)
\end{aligned}
$$

Measure of Central Tendency

Mean

In mathematics, **mean** has several different definitions depending on the context.

In probability and statistics, **mean** and expected value are used synonymously to refer to one measure of the central tendency either of a probability distribution or of the random variable characterized by that distribution. In the case of a discrete probability distribution of a random variable 'X', the mean is equal to the sum over every possible value weighted by the probability of that value; that is, it is computed by taking the product of each possible value x of X and its probability P(x), and then adding all these products together, giving $\mu = \sum xP(x)$.

An analogous formula applies to the case of a continuous probability distribution. Not every probability distribution has a defined mean; see the Cauchy distribution for an example. Moreover, for some distributions the mean is infinite: for example, when the probability of the value 2^n is $\frac{1}{2^n}$ for n = 1, 2, 3,...

Arithmetic Mean

- **Concept:**

The **arithmetic mean** (or **mean** or **average**) is the most commonly used and readily understood measure of central tendency. In statistics, the term average refers to any of the measures of central tendency. The arithmetic mean is defined as being equal to the sum of the numerical values of each and every observation divided by the total number of observations.

- **Formula:**

1. **Simple Arithmetic Mean:** Symbolically, if we have a data set containing the values X_1, X_2, …, X_n, the arithmetic mean 'AM' is defined by the formula,

   $AM = \frac{1}{n}\sum_{i=1}^{n} Xi$

2. **Weighted Arithmetic Mean:** If we have a data set containing the values X_1, X_2, …, X_n have frequencies f_1, f_2, …, f_n then, $AM = \frac{1}{N}\sum_{i=1}^{n} fi\ Xi$; where, $N = \sum_{i=1}^{n} fi$

3. **Composite Mean:** If two groups contain n_1 and n_2 observations with means $\overline{x_1}$ and $\overline{x_2}$ respectively, then the mean ($\bar{x}$) of the composite group ($n_1 + n_2$) observations is given by the relation, $N\bar{x} = n_1\,\overline{x_1} + n_2\,\overline{x_2}$; where, $N = n_1 + n_2$

Characteristics	Groups I	Groups II	Composite Group
No. of observations	n_1	n_2	N
Mean	$\overline{x_1}$	$\overline{x_2}$	$\bar{x}$

- **Calculation (Application):**

1. **Example 5.1: Calculation of Simple Arithmetic Mean:**

Let us consider the monthly salary of 10 employees of a firm: 2500, 2700, 2400, 2300, 2550, 2650, 2750, 2450, 2600, 2400

The Simple Arithmetic Mean ($\bar{x}$) is

$$\frac{2500+2700+2400+2300+2550+2650+2750+2450+2600+2400}{10} = 2530.$$

2. **Example 5.2: Calculation of Weighted Arithmetic Mean:**

Table - 5.1: Calculation for Weighted Arithmetic Mean

Price(Rs) per table	Number of table sold(f)	fx
36	14	504
40	11	440
44	9	396
48	6	288
Total	40	1628

Weighted Arithmetic Mean ($\bar{x}$) = $\frac{\sum fx}{N}$ = 1628/40 = 40.70

Important Note: Although A.M. can be calculated directly, the calculations can be considerably simplified based on the following theory:

(i) If $y_1, y_2, \ldots, y_n$ represent the deviations of $x_1, x_2, \ldots, x_n$ from an arbitrary constant c, then
Mean of x = c + Mean of y [i.e., if y = x – c, then $\bar{x} = c + \bar{y}$]

(ii) If $y_1, y_2, \ldots, y_n$ represent the deviations of $x_1, x_2, \ldots, x_n$ from an arbitrary constant c, in units of another constant d, then
Mean of x = c + d (Mean of y)

i.e., if $y = \frac{x-c}{d}$ then, $\bar{x} = c + d\bar{y}$

3. Example 5.3:

Calculate the Arithmetic Mean from the following frequency distribution of earners by monthly income:

Income (Rs.):	Below 200	200-399	400-599	600-799	800-999	1000-1199
No. of earners:	25	72	47	22	13	7

Table - 5.2: Calculation for Weighted Arithmetic mean

Class Interval	Frequency	Mid Value (x)	$y = \frac{x-499.5}{200}$	f.y
0 – 199	25	99.5	-2	-50
200 – 399	72	299.5	-1	-72
400 – 599	47	499.5	0	0
600 – 799	22	699.5	1	22
800 – 999	13	899.5	2	26
1000 – 1199	7	1099.5	3	21
Total	186			-53

If y = (x – c) /d, then $\bar{x} = c + d\bar{y}$

Here, c = 499.5 and d = 200

Thus, $\bar{x} = 499.5 + 200\left(\frac{-53}{186}\right) = 499.5 - 56.99 = 442.51$Rs.

4. Example 5.4: Calculation of Composite Mean:

There are two branches of an establishment employing 100 and 80 persons respectively. If the arithmetic means of the monthly salaries paid by the two branches are Rs. 275 and Rs. 225 respectively, find the A.M. of the salaries of the employees of the establishment as a whole.

Table 5.3: Mean of Composite Group

Characteristics	Groups I	Groups II	Composite Group
No. of observations	$n_1 = 100$	$n_2 = 80$	$N = 180$
Mean salary (Rs.)	$\overline{x_1} = 275$	$\overline{x_2} = 225$	$\bar{x} = ?$

$N\bar{x} = n_1 \overline{x_1} + n_2 \overline{x_2}$

Or, $180\bar{x} = 100 * 275 + 80 * 225 = 45{,}500$

Or, $\bar{x} = 45500 / 180 = 252.78$ Rs.

- **Remarks:**

If the data set is a statistical population (i.e., consists of every possible observation and not just a subset of them), then the mean of that population is called the **population mean**.

On the other hand, if the data set is a statistical sample (a subset of the population), we call the statistic resulting from this calculation a **sample mean**.

The arithmetic mean of a variable is often denoted by a bar, for example as in $\bar{x}$ (read x bar), which is the mean of the n values $x_1, x_2, \ldots, x_n$.

- **Limitations of Arithmetic Mean:**

1. The strongest drawback of AM is that it is very much affected by extreme observations. Two or three very large values of the variable may unduly affect the value of AM.
2. AM cannot be used in the case of open end classes such as <10 or >70 etc.
3. It cannot be determined by inspection nor can it be located graphically.
4. AM cannot be used if we are dealing with qualitative characteristics which cannot be measured quantitatively such as intelligence, honesty, beauty etc.
5. In extremely asymmetrical (skewed) distribution, usually AM is not representative of the distribution and hence is not a suitable measure of locations.

Geometric Mean

- **Concept:**

In mathematics, the **geometric mean** is a type of mean or average, which indicates the central tendency or typical value of a set of numbers by using the product of their values (as opposed to the arithmetic mean which uses their sum).

- **Formula:**

The geometric mean is defined as the nth root of the product of n numbers, i.e., for a set of numbers $\{x_i\}_{i=1}^N$, the geometric mean is defined as $\left(\prod_{i=1}^N x_i\right)^{1/N}$.

For instance, the geometric mean of two numbers, say 2 and 8, is just the square root of their product; that is, $\sqrt{2 \cdot 8} = 4$. As another example, the geometric mean of the three numbers 4, 1, and 1/32 is the cube root of their product (1/8), which is 1/2; that is, $\sqrt[3]{4 \cdot 1 \cdot 1/32} = 1/2$

- **Calculation (Application):**

Example 5.5:

Table 5.4: Calculation for Geometric Mean

Group	Group Index (x)	Weight(f)	log x	f *(log x)
A	118	4	2.0719	8.2876
B	120	1	2.0792	2.0792
C	97	2	1.9868	3.9736
D	107	6	2.0294	12.1764
E	111	5	2.0453	10.2265
F	93	2	1.9685	3.9370
Total	-	20	-	40.6803

Log GM = $1/N \sum f_i (\log x_i)$ = 40.6803/20 =2.0340

GM= antilog 2.0340 = 108.1

G denotes geometric mean.

- **Remarks:**

A geometric mean is often used when comparing different items – finding a single "figure of merit" for these items – when each item has multiple properties that have different numeric ranges. For example, the geometric mean can give a meaningful "average" to compare two companies which are each rated at 0 to 5 for their environmental sustainability, and are rated at 0 to 100 for their financial viability. If an arithmetic mean were used instead of a geometric mean, the financial viability is given more weight because its numeric range is larger—so a small percentage change in the financial rating (e.g. going from 80 to 90) makes a much larger difference in the arithmetic mean than a large percentage change in environmental sustainability (e.g. going from 2 to 5). The use of a geometric mean "normalizes" the ranges being averaged, so that no range dominates the weighting, and a given percentage change in any of the properties has the same effect on the geometric mean. So, a 20% change in environmental sustainability from 4 to 4.8 has the same effect on the geometric mean as a 20% change in financial viability from 60 to 72.

- **Limitations of Geometric Mean:**

1. If any one of the observations is zero and if any one of the observations is negative, GM becomes imaginary regardless of the magnitude of other items.
2. It cannot be calculated if some of values in a series are not known.

Harmonic Mean

- **Concept:**

In mathematics, the **harmonic mean** (sometimes called the **sub-contrary mean**) is one of several kinds of average, and in particular one of the Pythagorean means. Typically, it is appropriate for situations when the average of rates is desired.

- **Formula:**

The harmonic mean H of the positive real numbers $x_1, x_2, \ldots, x_n$ is defined to be

$$H = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}} = \frac{n \cdot \prod_{j=1}^{n} x_j}{\sum_{i=1}^{n} \frac{\prod_{j=1}^{n} x_j}{x_i}}.$$

From the third formula in the above equation, it is more apparent that the harmonic mean is related to the arithmetic and geometric means.

- **Calculation (Application):**

Example 5.6:

Equivalently, the harmonic mean is the reciprocal of the arithmetic mean of the reciprocals. As a simple example, the harmonic mean of 1, 2, and 4 is

$$\frac{3}{\frac{1}{1} + \frac{1}{2} + \frac{1}{4}} = \frac{1}{\frac{1}{3}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{4}\right)} = \frac{12}{7}.$$

Median

- **Concept:**

In statistics and probability theory, the **median** is the number separating the higher half of a data sample, a population, or a probability distribution, from the lower half. If each group contains less than half the population, then some of the population is exactly equal to the median. For example, if a < b < c, then the median of the list {a, b, c} is b, and, if a < b < c < d, then the median of the list {a, b, c, d} is the mean of b and c; i.e., it is (b + c)/2.

- **Formula:**

In terms of notation, some authors represent the median of a variable x either as $\tilde{x}$ or as $\mu_{1/2}$, sometimes also M. There is no widely accepted standard notation for the median, so the use of these or other symbols for the median needs to be explicitly defined when they are introduced. The median is the 2nd quartile, 5th decile, and 50th percentile.

Median (M_d) = $l_1 + \{(N/2) - F\}/f_m \times c$

Where,

l_1 = lower boundary of the median class

N = total frequency

F = cumulative frequency corresponding to l_1

f_m = frequency of the median class

c = width of the median class

- **Calculation (Application):**

1. Example 5.7:

The median of a finite list of numbers can be found by arranging all the observations from lowest value to highest value and picking the middle one [e.g., the median of (3, 3, 5, 9, 11) is 5]. If there is an even number of observations, then there is no single middle value; the median is then usually defined to be the mean of the two middle values [the median of {3, 5, 7, 9} is (5 + 7) / 2 = 6], which corresponds to interpreting the median as the fully trimmed mid-range. The median is of central importance in robust statistics, as it is the most resistant statistic, having a breakdown point of 50%: so long as no more than half the data is contaminated, the median will not give an arbitrarily large result. A median is only defined on ordered one-dimensional data, and is independent of any distance metric.

2. Example 5.8:

Find the median and the median class of the data given below:

Class boundaries:	15-25	25-35	35-45	45-55	55-65	65-75
Frequency :	4	11	19	14	0	2

 # Method 1:

For a grouped frequency distribution,

Median = Value of the variable corresponding to cumulative frequency N/2

Table 5.5: Cumulative Frequency Distribution

Class Boundary	Cumulative Frequency (Less than)
15	0
25	4
35	15
Median	N / 2 = 25
45	34
55	48
65	48
75	50 = N

Since N / 2 = 25 lies between the cumulative frequencies 15 and 34, the corresponding value of the variable, viz., median, must lie in the interval between 35 and 45. The median class is, therefore, (35 – 45).

Now applying simple interpolation

$\frac{\text{Median}-35}{45-35} = \frac{25-15}{34-15}$

Or, $\frac{\text{Median}-35}{10} = \frac{10}{19}$

Or, Median – 35 = $\frac{10}{19} * 10 = \frac{100}{19}$ = 5.26

Or, Median = 35 + 5.26 = 40.26

 # Method 2: We first calculate the cumulative frequencies against each class interval as shown below:

Table 5.6: Calculation of Cumulative Frequency

Class Boundaries	Frequency	Cumulative Frequency
15-25	4	4
25-35	11	15 = F
(35-45)	19 = f_m	34
45-55	14	48
55-65	0	48
65-75	2	50 = N

From the last column of table 2.7 it is seen that, N/2 = 25 is more than cumulative frequency 15, but is less than the next cumulative frequency 34; hence the median class is (35 – 45). Hence l_1 = 35, N = 50, F = 15, f_m =19, c = 45 – 35 = 10.

Median = $35 + \frac{25-15}{19} * 10 = 35 + 5.26 = 40.26$

- **Remarks:**

In a sample of data, or a finite population, there may be no member of the sample whose value is identical to the median (in the case of an even sample size); if there is such a member, there may be more than one so that the median may not uniquely identify a sample member. Nonetheless, the value of the median is uniquely determined with the usual definition. At most, half the population has values strictly less than the median, and, at most, half have values strictly greater than the median.

The median can be used as a measure of location when a distribution is skewed, when end-values are not known, or when one requires reduced importance to be attached to outliers, e.g., because they may be measurement errors.

- **Limitations of Median:**

1. In case of even number of observations for an ungrouped data, median cannot be determined exactly. We merely estimate it as the arithmetic mean of the two middle terms.
2. Median, being a positional average, is not based on each and every item of the distribution. It depends on all the observations only to the extent whether they are smaller than or greater than it; the exact magnitude of the observations being immaterial.
3. It does not lead itself to algebraic treatment.
4. It is affected by fluctuations of items.
5. It is less stable measure of central tendency than mean.
6. In the case of continuous series, it cannot be calculated exactly.

Mode

- **Concept:**

The mode is the value that appears most often in a set of data. The mode of a discrete probability distribution is the value x at which its probability mass function takes its maximum value. In other words, it is the value that is most likely to be sampled. The mode of a continuous probability distribution is the value x at which its probability density function has its maximum value, so the mode is at the peak.

- **Formula:**

In symmetric uni-modal distributions, such as the normal distribution, the mean (if defined), median and mode all coincide. For samples, if it is known that they are drawn from a symmetric distribution, the sample mean can be used as an estimate of the population mode.

Mode = $l_1 + \{d_1/(d_1+d_2)\} \times c$

Where,

l_1 = lower boundary of the modal class

d_1 = difference of frequencies in the modal class and the preceding class

d_2 = difference of frequencies in the modal class and the following class

c = common width of the classes.

- **Calculation (Application):**

Example 5.9:

Let the monthly profits in rupees of 100 shops are distributed as follows:

Profits per shop (Rs.):	0-100	100-200	200-300	300-400	400-500	500-600
No of shops :	12	18	27	20	17	6

In order to calculate the Mode directly, we find that the largest class frequency, viz., 27, lies in the class 200-300 and hence this is the modal class.

Therefore, in formula, $l_1 = 200$, $d_1 = 27-18 = 9$, $d_2 = 27-20 = 7$, $c = 300-200 = 100$.

Mode = $200 + \frac{9}{9+7} \times 100 = 256.25$

- **Remarks:**

Like the statistical mean and median, the mode is a way of expressing, in a single number, important information about a random variable or a population. The numerical value of the mode is the same as that of the mean and median in a normal distribution, and it may be very different in highly skewed distributions.

The mode is not necessarily unique, since the probability mass function or probability density function may take the same maximum value at several points x_1, x_2, etc. The most extreme case occurs in uniform distributions, where all values occur equally frequently. When a probability density function has multiple local maxima it is common to refer to all of the local maxima as modes of the distribution. Such a continuous distribution is called multimodal (as opposed to uni-modal).

- **Limitations of Mode:**

1. Since mode is the value of X corresponding to the maximum frequency, it is not based on all the observations of the series.
2. Mode is not suitable for further mathematical treatment.
3. As compared with mean, mode is affected to a greater extent by the fluctuations of sampling.

Measure of Dispersion

In statistics, **dispersion** (also called **variability**, **scatter**, or **spread**) denotes how stretched or squeezed a distribution (theoretical or that underlying a statistical sample) is. Common examples of absolute measures of statistical dispersion are the range, quartile deviation, mean deviation, standard deviation and variance.

Range (Absolute measure of Dispersion)

- **Concept:**

The range is defined as the difference between the highest and the lowest values in the series. This is the simplest method of measuring dispersion.

- **Formula:**

Range = (H-L) ; where, H = Highest value and L = Lowest value

- **Calculation (Application):**

Example 5.10: The weights of 11 forty-year old men were 148, 154, 158, 160, 161, 162, 166, 170, 182, 195 and 236 pounds.

In the given data, maximum value = 236 and minimum value = 148.

Hence,

Range = 236 – 148 = 88 pounds.

- **Remarks:**

It is easy to understand. Range is the difference between the two extreme values and as such represents the maximum possible difference between any two observations.

- **Limitations of Range:**

1. It is not based on the entire set of data.
2. It is very much affected by fluctuations of sampling. Its value varies widely from sample to sample.
3. It is not possible to find out the range in open-end frequency distribution.
4. It does not present the accurate picture of the series.
5. It is affected by extreme values.

Co-efficient of Range (Relative measure of Dispersion)

- **Concept:**

Range as defined above is an absolute measure of dispersion and depends on the units of measurement. Thus if we want to compare the variability of two or more distributions with the same units of measurement, we may use the above formula. However, to compare the variability of the distribution given in different units of measurement we cannot use the above formula but we need a relative measure which is independent of the units of measurement.

- **Formula:**

This relative measure, called the coefficient of range, is defined as follows,

Coefficient of Range = $\frac{\mathbf{H-L}}{\mathbf{H+L}}$

- **Calculation (Application):**

Example 5.11:

Continuing the above example 8.1, we have in the given data, maximum value = 236 and minimum value = 148.

Hence,

Coefficient of Range = $\frac{\mathbf{H-L}}{\mathbf{H+L}} = \frac{\mathbf{236-148}}{\mathbf{236+148}} = \frac{\mathbf{88}}{\mathbf{384}} = 0.2292$

Quartile Deviation (Absolute measure of Dispersion)

- **Concept:**

Quartile Deviation is defined as half the difference between the upper and the lower quartiles.

- **Formula:**

Quartile deviation (Q.D.) is obtained by dividing the difference of the third quartile and the first quartile ($Q_3 - Q_1$) by 2.

Thus, Quartile Deviation = $\frac{Q3 - Q1}{2}$

Where, Q_3 = Third Quartile or Upper Quartile and Q_1 = First Quartile or Lower Quartile

- **Calculation (Application):**

1. Example 5.12:

Calculate the quartile deviation from the following:

Class interval:	10-15	15-20	20-25	25-30	30-40	40-50	50-60	60-70	Total
Frequency :	4	12	16	22	10	8	6	4	82

To calculate Quartile Deviation, we have to find Q_1 (first quartile) and Q_3 (third quartile), i.e., values of the variable corresponding to cumulative frequencies N/4 and 3N/4.

Here, total frequency = 82. Therefore, N/4 = 20.5 and 3N/4 = 61.5

Table 5.7: Cumulative Frequency Distribution

Class Boundary	Cumulative Frequency (Less than)
10	0
15	4
20	16
Q_1	N/4 = 20.5
25	32
30	54
Q_3	3N/4 = 61.5
40	64
50	72
60	78
70	82 = N

Applying simple interpolation,

$\frac{Q1 - 20}{25 - 20} = \frac{20.5 - 16}{32 - 16}$ Or, $\frac{Q1 - 20}{5} = \frac{4.5}{16}$

Or, $Q_1 - 20 = \frac{4.5}{16} * 5 = 1.4$

Or, $Q_1 = 20 + 1.4 = 21.4$

Similarly,

$\frac{Q3-30}{40-30} = \frac{61.5-54}{64-54}$ Or, $\frac{Q3-30}{10} = \frac{7.5}{10}$

Or, $Q_3 - 30 = \frac{7.5}{10} * 10 = 7.5$

Or, $Q_3 = 30 + 7.5 = 37.5$

Therefore,

Quartile Deviation = $\frac{Q3-Q1}{2} = \frac{37.5-21.4}{2} = 8.05$

2. Example 5.13:

Calculate the quartile deviation:

Wage per week (Rs.)	No. of wage earners
Less than 35	14
35-37	62
38-40	99
41-43	18
Over 43	7

To calculate quartile deviation at first, we have to determine the quartiles Q_1 and Q_3, which can be done from a cumulative frequency distribution using simple interpolation.

Table 5.8: Cumulative Frequency Distribution

Wages per week (Rs)	Cumulative frequency
34.5	14
Q_1	N/4 = 50
37.5	76
Q_3	3N/4 = 150
40.5	175
43.5	193
…	200 = N

Applying simple interpolation,

$(Q_1 - 34.5) / (37.5 - 34.5) = (50 - 14) / (76 - 14)$

$(Q_3 – 37.5) / (40.5 – 37.5) = (150 – 76) / (175 – 76)$

Solving, $Q_1 = 36.24$ and $Q_3 = 39.74$

Therefore, Quartile Deviation = (39.74 – 36.24) / 2 = 1.75 Rs.

- **Remarks:**

The quartile deviation gives the average amount by which the two quartiles differ from median. For symmetric distribution we have,

$(Q_3 – M_d) = (M_d - Q_1)$

Or, $M_d = \frac{Q3+Q1}{2}$

i.e., median lies half way on the scale from Q_1 to Q_3. Thus, for a symmetric distribution we have,

Q.D. + $Q_1 = \frac{Q3 - Q1}{2} + Q_1 = \frac{Q3+Q1}{2} = M_d$

And, Q_3 - Q.D. = $Q_3 - \frac{Q3 - Q1}{2} = \frac{Q3+Q1}{2} = M_d$

In other words, for a symmetric distribution we have,

$Q_1 = (M_d – Q.D.)$

And, $Q_3 = (M_d + Q.D.)$

Since in a distribution 25% of the observations lie between Q_1 and 25% observations lie above Q_3, 50% of the observations lie between Q_1 and Q_3. Therefore, we can conclude that for a symmetric distribution,

M_d ± Q.D. covers exactly 50% of the observations.

Unit-4: Mathematical Economics

Meaning and Importance:

Mathematical economics is the application of mathematical methods to represent economic theories and analyze problems posed in economics. It allows formulation and derivation of key relationships in a theory with clarity, generality, rigor, and simplicity. By convention, the methods refer to those beyond simple geometry, such as differential and integral calculus, difference and differential equations, matrix algebra, and mathematical programming and other computational methods.

Mathematics allows economists to form meaningful, testable propositions about many wide-ranging and complex subjects which could not be adequately expressed informally.

Further, the language of mathematics allows economists to make clear, specific, positive claims about controversial or contentious subjects that would be impossible without mathematics. Much of economic theory is currently presented in terms of mathematical economic models, a set of stylized and simplified mathematical relationships that clarify assumptions and implications.

Paul Samuelson argued that mathematics is a language. In economics, the language of mathematics is sometimes necessary for representing substantive problems. Moreover, mathematical economics has led to conceptual advances in economics.

Overview of Mathematical Economics:

Mathematical economics is a branch of economics that engages mathematical tools and methods to analyse economic theories. Mathematical economics is best defined as a sub-field of economics that examines the mathematical aspects of economies and economic theories. Or put into other words, mathematics such as calculus, matrix algebra, and differential equations are applied to illustrate economic theories and analyse economic hypotheses.

It may be interesting to begin the study of mathematical economics with an enquiry into the history of mathematical economics. It is generally believed that the use of mathematics as a tool of economics dates from the pioneering work of Cournot (1838). However there were many others who used mathematics in the analysis of economic ideas before Cournot. We shall make a quick survey of the most important contributors.

Sir William Petty is often regarded as the first economic statistician. In his Discourses on Political Arithmetic (1690), he declared that he wanted to reduce political and economic matters to terms of number, weight, and measure. The first person to apply mathematics to economics with any success was an Italian, Giovanni Ceva who in 1711 wrote a tract in which mathematical formulas were generously used. He is generally regarded as the first known writer to apply mathematical method to economic problems. The Swiss mathematician Daniel Bernoulli in 1738 for the first time used calculus in his analysis of a probability that would result from games of chance rather than from economic problems.

Among the early French writers who made some use of mathematics was Francois de Forbonnais who used mathematical symbols, especially for explaining the rate of exchange between two countries and how equilibrium is finally established between them. He is best known for his severe attack on Physiocracy.

While this is a long list to those who showed curiosity in the use of mathematics in economics, interestingly J. B. Say showed little or no interest in the use of mathematics. He did not favour the use of mathematics for explaining economic principles. A German who is reasonably well known, especially in location theory, is Johann Heinrich von Thunen. His first work, The Isolated State (1826), was an attempt to explain how transportation costs influence the location of agriculture and even the methods of cultivation.

The French engineer, A. J. E. Dupuit, used mathematical symbols to express his concepts of supply and demand. Even though he had no systematic theory he did develop the concepts of utility and diminishing utility, which were clearly stated and presented in graphical form.

He viewed the price of a good as dependent on the price of other goods. Another Lausanne economist, Vilfredo Pareto, ranks with the best in mathematical virtuosity. The Pareto optimum and the indifference curve analysis (along with Edgeworth in England) were clearly conceived in the mathematical frame work.

A much more profound understanding of the analytical power of mathematics was shown by Leon Walras. He is generally regarded as the founder of the mathematical school of economics. He set out to translate pure theory into pure mathematics. After the pioneering work of Jevons (1871) and Walras (1874), the use of mathematics in economics progressed at very slow pace for a number of years. The first of the latter group is F. Y. Edgeworth. His contribution to mathematical economics is found mainly in the 1881 publication in which he dealt with the theory of probability and statistical theory.

F. Y. Edgeworth’s contribution to mathematical economics is found mainly in the 1881 publication in which he dealt with the theory of probability, statistical theory and the law of error. The indifference analysis was first propounded by Edgeworth in 1881 and restated in 1906 by Pareto and in 1915 by the Russian economist Slutsky, who used elaborate mathematical treatment of the topic. Alfred Marshall made extensive use of mathematics in his Principles of Economics (1890). His interest in mathematics dates from his early schooldays when his first love was for mathematics, not the classics.” In the Principles he used mathematical techniques very effectively.

Another economist whose influence spans many years is the American, Irving Fisher. Fisher belongs to other groups as well as to the mathematical moderns. His life's work reveals him as a statistician, econometrician, mathematician, pure theorist, teacher, social crusader, inventor, businessman, and scientist. His contribution to statistical method, The Making of Index Numbers, was great. The well-known Fisher formula, $M V + M' V' = P T$, is an evidence of his contribution to quantitative economics.

A strong inducement to formulate economic models in mathematical terms has been the post-World War II development of the electronic computer. In the broad area of economics there has been a remarkable use of mathematical techniques. Economics, like several other disciplines, has always used quantification to some degree. Common terms such as wealth, income, margins, factor returns, diminishing returns, trade balances, balance of payments, and the many other familiar concepts have a quantitative connotation. All economic data have in some fashion been reduced to numbers which became generally known as economic statistics.

The most noteworthy developments, however, have come in the decades since about 1930. It was approximately this time that marked the ebb of neoclassicism, the rise of institutionalism, and the introduction of aggregate economics. Over a period of years scholars have developed new techniques designed to help in the explanation of economic behavior under different market situations using mathematics. Now we discuss some of the economic theories and the techniques of modern analysis. They are given largely on a chronological basis, and their significance is developed in the discussion.

The Theory of Games: The pioneering work was done by John von Neumann in 1928. The theory became popular with the publication of Theory of Games and Economic Behaviour in 1944. Basically, The Game Theory holds that the actions of players in gambling games are similar to situations that prevail in economic, political, and social life. The theory of games has many elements in common with real-life situations. Decisions must be made on the basis of available facts, and chances must be taken to win. Strategic moves must be concealed (or anticipated) by the contestants based on past knowledge and future estimates. Success or failure rests, in large measure, on the accuracy of the analysis of the elements. The theory of games introduced an interesting and challenging concept. Economists have made some use of it, and it has also been fitted into other social sciences, notably sociology and political science.

Linear Programming: Linear programming is a specific class of mathematical problems in which a linear function is maximized (or minimized) subject to given linear constraints. The founders of the subject are generally regarded as George B. Dantzig, who devised the simplex method in 1947, and John von Neumann, who established the theory of duality that same year. The scope of linear programming is very broad. It brings together both theoretical and practical problems in which some quantity is to be maximized or minimized. The data could be almost any fact such as profit, costs, output, and distance to or from given points, time, and so on. It also makes allowance for given technology and restraints that may occur in factor markets or in finance. Linear programming has been proven very useful in many areas. It is in common use in agriculture, where chemical combinations of proper foods for plants and animals have been worked out, and in the manufacture of many processed agricultural products. It is necessary in modem materials scheduling, in shipping, and in final production. The Nobel Prize in economics was awarded in 1975 to the mathematician Leonid Kantorovich (USSR) and the economist Tjalling Koopmans (USA) for their contributions to the theory of optimal allocation of resources, in which linear programming played a key role.

Input-Output Analysis: In terms of techniques, input-output analysis is a rather special case of linear programming. It was devised originally by Leontief and, in a sense, was a World War II-inspired analysis. Basically it was designed for presenting a general equilibrium theory suited for empirical study. The problem is to determine the interrelationship of sector inputs and outputs on other sectors or on all sectors which use the product. The rational for the term IOA can be explained like this. There is a close interdependence between different sectors of a modern economy. This interdependence arises out of the fact that the output of any given industry is utilized as an input by the other industries and often by the same industry itself. Thus the IOA analyses the interdependence between different sectors of an economy. The basis of IOA is the input - output table which can be expressed in the form of matrices.

Sequence:

Introduction to Sequences:

Imagine yourself at a pizza hut. You have placed an order and your order number is 282. So currently they are serving the order number 275. So how many orders do you think will be served before your number? Yes, six orders more because you are in a sequence. To understand this better, let us learn about the sequences and series. Let us do it right now.

Concept of sequence:

A sequence is a list of numbers in a special order. It is a string of numbers following a particular pattern, and all the elements of a sequence are called its terms. Let us consider a sequence,

[1, 3, 5, 7, 9, 11……]

We can say this is sequence because we know that they are the collection of odd natural numbers. Here the number of terms in the sequence will be infinite. Such a sequence which contains the infinite number of terms is known as an **infinite sequence.** But what if we put end to this.

[1, 3, 5, 7, 9, 11…..131]

If 131 is the last term of this sequence, we can say that the number of terms in this sequence is countable. So in such sequence in which the number of terms is countable, such sequences are called **finite sequences**. A finite sequence has the finite number of terms. So as discussed earlier here 1 is the term, 3 is the term so is 5, 7…..

Fibonacci Sequence:

The special thing about the Fibonacci sequence is that the first two terms are fixed. When we talk about the terms, there is a general representation of these terms in sequences and series. A term is usually denoted as an here ‘ n ‘

is the nth term of a sequence. For the Fibonacci sequence, the first two terms are fixed.

The first term is as a1= 1 and a2= 1. Now from the third term onwards, every term of this Fibonacci sequence will become the sum of the previous two terms. So a3will be given as a

1 + a2

Therefore, 1 + 1 = 2. Similarly, a4 = a2 + a3 ∴ 1 + 2 = 3 a5 = a3+ a4 ∴ 2 + 3 = 5

Therefore if we want to write the Fibonacci sequence, we will write it as, [1 1 2 3 5…]. So, in general, we can say,

an = an-1 + an-2

Types of Sequences

1. **Arithmetic sequence:** In an arithmetic (linear) sequence the difference between any two consecutive terms is constant.
2. **Quadratic Sequence:** A quadratic sequence is a sequence of numbers in which the second difference between any two consecutive terms is constant.
3. **Geometric Sequence:** A geometric sequence is a sequence of numbers where each term after the first is found by multiplying the previous one by a fixed, non-zero number called the common ratio.

Concept of Economic Function:

A variable represents a concept or an item whose magnitude can be represented by a number, i.e. measured quantitatively. Variables are called variables because they vary, i.e. they can have a variety of values. Thus a variable can be considered as a quantity which assumes a variety of values in a particular problem. Many items in economics can take on different values. Mathematics usually uses letters from the end of the alphabet to represent variables. Economics however often uses the first letter of the item which varies to represent variables. Thus p is used for the variable price and q is used for the variable quantity.

An expression such as $4x^3$ is a variable. It can assume different values because x can assume different values. In this expression x is the variable and 4 is the coefficient of x. Coefficient means 4 works together with x. Expressions such as $4x^3$ which consists of a coefficient times a variable raised to a power are called monomials. A monomial is an algebraic expression that is either a numeral, a variable, or the product of numerals and variables. (Monomial comes from the Greek word, *monos*, which means one.) Real numbers such as 5 which are not multiplied by a variable are also called monomials. Monomials may also have more than one variable. $4x^3y^2$ is such an example. In this expression both x and y are variables and 4 is their coefficient.

The following are examples of monomials: x, $4x^2$, $-6xy^2z$, 7

One or more monomials can be combined by addition or subtraction to form what are called polynomials. (Polynomial comes from the Greek word, poly, which means many.) A polynomial has two or more terms i.e. two or more monomials. If there are only two terms in the polynomial, the polynomial is called a binomial.

The expression $4x^3y^2 - 2xy^2 + 3$ is a polynomial with three terms.

These terms are $4x^3y^2$, $-2xy^2$, and 3. The coefficients of the terms are 4, −2, and 3.

The degree of a term or monomial is the sum of the exponents of the variables. The degree of a polynomial is the degree of the term of highest degree. In the above example the degrees of the terms are 5, 3, and 0. The degree of the polynomial is 5.

Remember that variables are items which can assume different values. A function tries to explain one variable in terms of another.

Independent variables are those which do not depend on other variables. Dependent variables are those which are changed by the independent variables. The change is caused by the independent variable. The independent variable is often designated by x. The dependent variable is often designated by y.

We say y is a function of x. This means y depends on or is determined

by x. mathematically we write $y = f(x)$

It means that mathematically y depends on x. If we know the value of x, then we can find the value of y.

A function is a mathematical relationship in which the values of a single dependent variable are determined by the values of one or more independent variables. Function means the dependent variable is determined by the independent variable(s). A function tries to define these relationships. It tries to give the relationship a mathematical form. An equation is a mathematical way of looking at the relationship between concepts or items. These concepts or items are represented by what are called variables. Economists are interested in examining types of relationships. For example an economist may look at the amount of money a person earns and the amount that person chooses to spend. This is a consumption relationship or function. As another example an economist may look at the amount of money a business firm has and the amount it chooses to spend on new equipment. This is an investment relationship or investment function. Functions with a single independent variable are called univariate functions. There is a one to one correspondence. Functions with more than one independent variable are called multivariate functions.

Example of use of functions:

$y = f(x) = 3x + 4$

This is a function that says that, y, a dependent variable, depends on x, an independent variable. The independent variable, x, can have different values. When x changes y also changes.

Find f(0). This means find the value of y when x

equals 0. f(0) = 3 times 0 plus 4

$f(0) = 3(0) + 4 = 4$

Find f(1). This means find the value of y when x

equals 1. f(1) = 3 times 1 plus 4

$f(1) = 3(1) + 4 = 7$

Find f(-1). This means find the value of y when x

equals -1. f(-1) = 3 times (-1) plus 4

$f(1) = 3(-1) + 4 = 1$

4.1.9. Some Economic Functions:

I. Demand function

Demand function express the relationship between the price of the commodity (independent variable) and quantity of the commodity demanded (dependent variable).It indicate how much quantity of a commodity will be purchased at its different prices. Hence, represent the quantity demanded of a commodity and Px is the price of that commodity. Then,

Demand function, dx = f (Px)

The basic determinants of demand function

= f (Px, Pr, Y, T, W, E)

Where Qx: quantity demanded of a commodity X, Px: price of commodity X, Pr: price of related good, Y: consumer's income, T: Consumer/s tastes and preferences, W: Consumer's wealth, E: Consumer's expectations.

Example: $Q_d = p^2 - 20p + 125$

This is a function that describes the demand for an item where p is the dollar price per item. It says that demand depends on price.

Find the demand when one item costs

Rs. 2 $d(2) = 2^2 - 20(2) + 125 = 89$

Find the demand when one item costs

Rs. 5 $d(5) = 5^2 - 20(5) + 125 = 50$

Notice that the demand decreases as the price increases which you know is the law of demand.

II. Supply function

The functional relationship between the quantity of commodities supplied and various determinants is known as supply function. It is the mathematical expression of the relationship between supply and factors that affect the ability and willingness of the producer to offer the product.

Mathematically, a supply function can be expressed as Supply S_x= f (P_x)

There are a number of factors and circumstances which can influence a producer's willingness to supply the commodity in the market. These factors are price of the commodity, price of the related goods, price of the factors of production, goal of producers, state of technology, miscellaneous factors (we can include factors such as means of transportation and communication, natural factors, taxation policy, expectations, agreement among the producers, etc.) Incorporating all such factors we may write the basic form of a supply function as;

Qs= f (Gf , P, I, T, , E,)

Where Qs: quantity supplied, Gf: Goal of the firm, P: Product's own price, I: Prices of inputs, T: Technology, Pr: Prices of related goods, E: Expectation of producer's, Gp: government policy.

Example: Given a supply function $Q_s = -20 + 3P$ and demand function $Q_d = 220 – 5P$, Find the equilibrium price and quantity.

Equilibrium in any market means equality of demand and supply.

Hence, at equilibrium $-20 + 3P = 220 – 5P$.

Solving the above we get, Equilibrium price P = 30, Equilibrium quantity $Q_s = Q_d = 70$

III. Utility function:

Utility function is a mathematical function which ranks alternatives according to their utility to an individual. The utility function measures welfare or satisfaction of a consumer as a function of consumption of real goods, such as food, clothing and composite goods rather than nominal goods measured in nominal terms. Thus the utility function shows the relation between utility derive from the quantity of different commodity consumed. A utility function for a consumer consuming three different goods may be represented:

$U = f (X_1, X_2, X_3………)$

Example: Given the utility unction of a consumer $U = 2x^2 +5$, find the marginal utility.

Marginal utility is given by the first order derivative of the total utility function.

Marginal Utility = $\frac{du}{dx}$ = 4

IV. Consumption Function

The consumption function refers to the relationship between income and consumption. It is a functional relationship between consumption and income. Symbolically, the relationship is represented as C= f(Y), where C is consumption, Y is income. Thus the consumption function indicates a functional relationship between C and Y, where C is the dependant variable and Y is the independent variable, i.e., C is determined by Y. In fact, propensity to consume or consumption function is a sketch of the various amounts of consumption expenditure corresponding to different levels of income.

In the Keynesian framework, the consumption function or propensity to consume, refers to a functional relationship between two aggregates, i.e., total consumption and gross national income. The **Keynesian Consumption function $C = a + bY_d$,** expresses the level of consumer spending depending on three factors as explained below.

Y_d = disposable income (income after government intervention – e.g. benefits, and taxes)

a = autonomous consumption (This is the level of consumption which does not depend on income. The argument is that even with zero income we still need to buy some food to eat, through borrowing or using our savings.)

b = marginal propensity to consume (also known as induced consumption).

The average propensity to consume is the ratio of consumption expenditure to any particular level of income. It is found by dividing consumption expenditure by income, or APC = C/Y. It is expressed as the percentage or proportion of income consumed. The marginal propensity to consume is the ratio of the change in consumption to the change in income. It can be found by dividing change in consumption by a change in income, or by finding the first derivative of the utility function. The MPC is constant at all levels of income.

The MPC is the rate of change in the APC. When income increases, the MPC falls but more than the APC. Contrariwise, when income falls, the MPC rises and the APC also rises but at a slower rate than the former. Such changes are only possible during cyclical fluctuations whereas in the short-run there is change in the MPC and MPC<APC.

V. Saving function

The relationship between disposable income and saving is called the saving function. The saving function can represent in a general from as S= (y), Where, S is savings, and y is income, f is the notation for a generic, unspecified functional for. Because savings is the difference between disposable income and consumption, the savings function is a complementary relation to the consumption function. It is assumed that whatever is not consumer is saved. So, MPC+MPS=1

Given a saving function S = 70 + 0.8Y, find MPS and MPC MPS = 0.8, MPC = 1 – MPS = 0.2

VI. Production function

In a crude sense, production is the transformation of inputs into output. In another way, production is the creation of utility. Production is possible only if inputs are available and used. There are different types of inputs and economists classify them into land, labour, capital and organization. In modern era, the meaning of these terms is redefined. Today, land covers all natural resources, labour covers human resources, capital is replaced by the term technology and finally, instead of organization, we use the term management.

We explain production as the transformation of inputs into output.

In a general form a production function can be written as $Q = f(x_1, x_2, x_3, \ldots\ldots x_n)$

Where, Q represents the quantity produced, $x_1 \ldots\ldots x_n$ are inputs. The general mathematical form of Production function is:

$$Q = f(L, K, R, S, v, e)$$

Where Q stands for the quantity of output, L is the labour, K is capital, R is raw material, S is the Land, v is the return to scale and e is efficiency parameters.

Example: Q = 42KL – 3K2 – 2L2, Q = K0.4L0.5

i. Linear, Homogeneous Production Function:

Production function can take several forms but a particular form of production function enjoys wide popularity among the economists. This is a linear homogeneous production function, that is, production function which is homogenous production function of the first degree. Homogeneous production function of the first degree implies that if all factors of production are increased in a given proportion, output also increased in a same proportion. Hence linear homogeneous production function represents the case of constant return to scales. If there are two factors X and Y, The production function and homogeneous production function of the first degree can be mathematically expressed as,

Q = f(X, Y) Where Q stands for the total production, X and Y represent total inputs.

mQ = f(mX, mY) m stands any real number

The above function means that if factors X and Y are increased by m-times, total production Q also increases by m-times. It is because of this that homogeneous function of the first degree yield constant return to scale.

More generally, a homogeneous production function can be expressed

as Qmk = (mX,mY)

Where m is any real number and k is constant. This function is homogeneous function of the k^{th} degree. If k is equal to one, then the above homogeneous function becomes homogeneous of the first degree. If k is equal to two, the function becomes homogeneous of the 2^{nd} degree.

If k>1, the production function will yield increasing return to scale.

If k<1, it will yield decreasing return to scale.

If k=1, it will yield constant return to scale.

ii. Cobb – Douglas Production Function

Many Economists have studied actual production function and have used statistical methods to find out relations between changes in physical inputs and physical outputs. A most familiar empirical production function found out by statistical methods is the Cobb – Douglas production function. Cobb – Douglas production function was developed by Charles Cobb and Paul Douglas. In C-D production function, there are two inputs, labour and capital, Cobb – Douglas production function takes the following mathematical form

$Q = AL^{\sigma}K^{\beta}$

Where Q is the manufacturing output, L is the quantity of labour employed, K is the quantity of capital employed, A is the total factor productivity or technology are assumed to be a constant. The α and β, output elasticity's of Labour and Capital and the A,α and β are positive constant.

Roughly speaking, Cobb –Douglas production function found that about 75% of the increasing in manufacturing production was due to the Labour input and the remaining 25 % was due to the Capital input.

❖ **Properties of Cobb – Douglas Production Function**

a). Average product of factors: The first important properties of C – D production function as well as of other linearly homogeneous production function is the average and marginal products of factors depend upon the ratio of factors are combined for the production of a commodity. Average product if Labour (APL) can be obtained by dividing the production function by the amount of Labour L. Thus,

Average Product Labour (Q/L)

$$Q = AL^{\alpha}K^{\beta}$$

$$\frac{Q}{L} = \frac{AL^{\alpha}K^{\beta}}{L} = \frac{AK^{\beta}}{L^{1-\alpha}} = A\left(\frac{K}{L}\right)^{\beta}$$

Thus Average Product of Labour depends on the ratio of the factors (K/L) and does not depend upon the absolute quantities of the factors used.

Average Product of Capital (Q/K)

$$Q = AL^{\alpha}K^{\beta}$$

$$\frac{Q}{K} = \frac{AL^{\alpha}K^{\beta}}{K} = \frac{AL^{\alpha}}{K^{1-\beta}} = A\left(\frac{L}{K}\right)^{\alpha}$$

So the average Product of capital depends on the ratio of the factors (L/K) and does nt0o depend upon the absolute quantities of the factors used.

b). Marginal Product of Factors: The marginal product of factors of a linear homogeneous production function also depends upon the ratio of the factors and is independent of the absolute quantities of the factors used. Note, that marginal product of factors, says Labour, is the derivative of the production function with respect to Labour.

$$Q = AL^{\alpha}K^{\beta}$$

$$MP_L = \frac{\partial Q}{\partial L} = \alpha AL^{\alpha-1}\, K^{\beta} = \frac{A\,\alpha K^{\beta}}{L^{1-\alpha}} = \alpha A\left(\frac{K}{L}\right)^{\beta}$$

$$MP_L = \alpha\ A\ P_L$$

It is thus clear that MP_L depends on capital – labour ratio, that is, Capital per worker and is independent of the magnitudes of the factors employed.

$$Q = AL^{\alpha}K^{\beta}$$

$$MP_K = \frac{\partial Q}{\partial K} = \beta K^{\beta-1}\, L^{\alpha} = \frac{\beta A\, L^{\alpha}}{K^{1-\beta}} = \beta A\left(\frac{L}{K}\right)^{\alpha}$$

$$MP_L = \beta\ A\ P_K$$

It is thus clear that MPL depends on capital – labour ratio, that is, capital per worker and is independent of the magnitudes of the factors employed.

$$Q = AL^{\alpha}K^{\beta}$$

$$MP_K = \frac{\partial Q}{\partial K} = \beta K^{\beta-1}\, L^{\alpha} = \frac{\beta A\, L^{\alpha}}{K^{1-\beta}} = \beta A\left(\frac{L}{K}\right)^{\alpha} = \beta\ A\ P_K$$

c). Marginal rate of substitution: Marginal rate of substitution between factors is equal to the ratio of the marginal physical products of the factors. Therefore, in order to derive MRS form Cobb – Douglas production function, we used to obtain the marginal physical products of the two factors from the C – D function.

$$Q = AL^{\alpha}K^{\beta}$$

Differentiating this with respect to L, we have

$$MP_L = \frac{\partial Q}{\partial L} = \partial AL^{\alpha}K^{\beta} = \alpha AL^{\alpha-1}K^{\beta} = \frac{\alpha\left(AL^{\alpha}K^{\beta}\right)}{L^{1}}$$

Now, $Q = AL^{\alpha}K^{\beta}$,

Therefore, $= \alpha\left(\frac{Q}{L}\right)$

$\frac{\partial Q}{\partial L}$ Represents the marginal product of labour and $\frac{Q}{L}$ stands for the average of labour.

Thus, $MP_L = \alpha(AP_L)$

Similarly, by differentiating C – D production function with respect to capital, we can show that marginal product of capital.

$$Q = AL^{\alpha}K^{\beta}$$

$$MP_K = \frac{\partial Q}{\partial K} = \beta\ AL^{\alpha}K^{\beta-1} = \frac{\beta(AL^{\alpha}K^{\beta})}{K^1} = \beta\left(\frac{Q}{K}\right)$$

$$MRS_{LK} = \frac{MP_L}{MP_K} = \frac{\alpha\left(\frac{Q}{L}\right)}{\beta\left(\frac{Q}{K}\right)} = \frac{\alpha}{\beta} \times \frac{K}{L}$$

d). C – D production function and Elasticity of substitution ($\boldsymbol{\ell s\ or\ \sigma}$) is equal to unity.

$$\ell s = \frac{Proportionate\ change\ in\ Capital-Labour\ ratio\ \left(\frac{K}{L}\right)}{Proportionate\ change\ in\ MRS_{LK}} = \frac{\frac{d\left(\frac{K}{L}\right)}{K\backslash L}}{d\left(\frac{MRS_{LK}}{MRS_{LK}}\right)}$$

Substituting the value of MRS obtain in above $= \frac{\frac{d\left(\frac{K}{L}\right)}{K\backslash L}}{\frac{\partial\left(\frac{\alpha}{\beta}\times\frac{K}{L}\right)}{\frac{\alpha}{\beta}\times\frac{K}{L}}} = 1$

e). Return to Scale: An important property of C – D production function is that the sum if its exponents measures returns to scale. That is, When the sum of exponents is not necessarily equal to zero is given below.

$$Q = AL^{\alpha}K^{\beta}$$

In this production function the sum of exponents $(\alpha + \beta)$ measures return to scale. Multiplying each input labour (L) and capital (K), by a constant factor g, we have

$Q = A(gL)^{\alpha}(gK)^{\beta} = Q = g^{\alpha}\ g^{\beta}\left(AL^{\alpha}K^{\beta}\right)$ i.e. $a^m \times a^n = a^{m+n}$

$= g^{\alpha+\beta}\left(AL^{\alpha}K^{\beta}\right)$

i.e. $Q' = g^{\alpha+\beta}\ Q$

This means that when each input is increased by a constant factor g, output Q increase by $g^{\alpha+\beta}$. Now, if $\alpha + \beta = 1$ then, in this production function.

$$Q' = g' Q$$

$$Q' = g\, Q$$

This is , when $\alpha + \beta = 1$, output (Q) also increases by the same factor g by which both inputs are increased. This implies that production function is homogeneous of first degree or, in other words, retune to scale are constant.

When $\alpha + \beta > 1$, say it is equal to 2, then, in this production function new output.

$$Q' = g^{\alpha+\beta}\, A\, L^{\alpha}\, K^{\beta} = g^{2}\, Q.$$

In this case, multiplying each input by constant g, then output (Q) increases by g^2.

Therefore, $\alpha + \beta > 1$.

C – D production function exhibits increasing return to scale. When $\alpha + \beta < 1$, say it is equal to 0.8, then in this production function, new output,

$$Q' = g^{\alpha+\beta}\, A\, L^{\alpha}\, K^{\beta} = g^{0.8}\, Q.$$

That is increasing each input by constant factor g will cause output to increase by $g^{0.8}$, that is , less than g. Return to scale in this case are decreasing. Therefore $\alpha + \beta$ measures return to scale.

If $\alpha + \beta = 1$, return to scale are constant.

If $\alpha + \beta > 1$, return to scale are increasing.

If $\alpha + \beta < 1$, return to scale are decreasing.

f) **C – D Production Functions and Output Elasticity of Factors:**

The exponents of labour and capital in C – D production function measures output elasticity's of labour and capital. Output elasticity of a factor refers to the relative or percentage change in output caused by a given percentage change in a variable factor, other factors and inputs remaining constant. Thus,

$$O\, E = \frac{\partial Q}{\partial L} \times \frac{L}{Q} = a \times \frac{Q}{L} \times \frac{L}{Q} = a$$

Thus, exponent (a) of labour in C – D production function is equal to the output elasticity of labour.

Similarly, OE of Capital $= \frac{\partial Q}{\partial L} \times \frac{K}{Q}$

$MP_K = b \cdot \frac{K}{Q} = b \cdot \frac{Q}{K} \times \frac{K}{Q} = b$

Therefore, output elasticity of capital $= b \cdot \frac{Q}{K} \times \frac{K}{Q} = b$

g) C – D production Function and Euler's Theorem:

C – D production function $Q = \ A\, L^{\alpha}\ K^{\beta}$

Where $\alpha + \beta = 1$ helps to prove Euler theorem. According to Euler theorem, total output Q is exhausted by the distributive shares of all factors. When each factor is paid equal to its marginal physical product. As we know

$MP_L = \ A\, \alpha \left(\frac{K}{L}\right)^{\beta}$

$MP_K = \ A\, \beta \left(\frac{L}{K}\right)^{\alpha}$

According to Euler's theorem if production functions is homogenous of first degree then, Total output, $Q = L \times MP_L + \ K \times MP_K$, substituting the values of MP_L and MP_K, we have

$Q = L \times A\, \alpha \left(\frac{K}{L}\right)^{\beta} + \ K \times A\, \beta \left(\frac{L}{K}\right)^{\alpha} = A\, \alpha\, L^{1-\beta}\, K^{\beta} + A\, \beta\, L^{\alpha} K^{1-\alpha}$

Now, in C – D production function with constant to scale $\alpha + \beta = 1$ and

Therefore: $\alpha = 1 - \beta \ and\ \beta \ = 1 - \alpha$, we have

$Q = \ A\, \alpha\, L^{\alpha}\ K^{\beta} + A\, \beta\, L^{\alpha} K^{\beta} = (\alpha + \beta) A\, L^{\alpha} K^{\beta}$

Since, $\alpha + \beta = 1$ we have

$Q = A\, L^{\alpha} K^{\beta}$

$Q = Q$

Thus, in C – D production function with $\alpha + \beta = 1$ if $wage\ rate \ = \ MP_L$ and $rate\ of\ return\ on\ capital\ (K) = \ MP_K$, then total output will be exhausted.

h). C – D Production Function and Labour Share in National Income.

C – D production function has been used to explain labour share in national income (i.e., real national product). Let Y stand for real national product, L and K for inputs of labour and capital, then according to C – D production function as applied to the whole economy, we have

$$Y = A\,L^{\alpha}K^{1-\alpha} \dots\dots\dots\dots\dots\dots\dots (1)$$

Unit-5: International Economics

International Trade Theory

International Trade theory has started with three basic questions, **1. What is the basic for trade? 2. What are the gains from trade?** Presumably, a nation will voluntarily engage in trade only if it benefits from trade. But how are gains from trade generated? How large are the gains and how are they divided among the trading nations? 3**. What is the pattern of trade?** That is, what commodities are traded and which commodities are exported and imported by each nation?

Theories of International trade enable us to find out the answer to the above mention questions. Trade Theory started by the Mercantilism.

Mercantilist Trade Theory:

Englishman Thomas Munn is his book *England Treasure by Foreign Trade* briefly discussed about the elementary trade theory. Specifically, during the seventeenth and eighteenth centuries a group of men (merchants, bankers, government officials, and even philosophers) wrote essays and pamphlets on international trade that advocated an economic philosophy known as mercantilism. Three basic characteristics have mentioned in this theory. (i) More export then imports; the mercantilists maintained that the way for a nation to become rich and powerful was to export more than it imported. (ii) Bullionism; export surplus would then be settled by an inflow of bullion, or precious metals, primarily gold and silver. The more gold and silver a nation had, the richer and more powerful it was. Thus, the government had to do all in its power to stimulate the nation’s exports and discourage and restrict imports. (iii) Zero-sum game; a country can gain by the trade if partner country will loss in that trade.

Classical Trade Theory:

The Classical theory of international Trade was formulated by Adam Smith (1723-1790) in terms of his "Absolute Advantage Model". David Ricardo (1722-1823) expanded it further into what is called Comparative Advantage model. The models of Smith and Ricardo together constitute what is sometimes referred to as "Supply version of Classical Theory of Trade. They treated supply or production costs the determining factor of trade and gains from trade. The modern version of Classical Theory of Trade however treats supply and demand with equal weight, John Stuart Mill being the first to indicate that demand considerations must be incorporated into the Comparative Advantage model.

Absolute advantage theory:

Adam Smith extolled the virtues of free trade. These are the result of the advantages of division of labour and specialization both at the national and international levels.

Assumption:

I. Labour is the only factor of production.
II. Full employment in the economy.
III. Labour is perfectly immobile between the countries.
IV. Law of constant returns operate in production process.
V. No technical change

Theory:

According to Smith, the division of labour at the international level requires the existence of absolute differences in costs. Every country should specialize in the production of that commodity which it can produce more cheaply than others and exchange it for the commodities which cost less in other countries.

Example:

	Commodity X	Commodity Y
Country A	10	5
Country B	5	10

In above table have shown that country A can produce 10X or 5Y with one unit of labour and country B can produce 5X or 10Y with one unit of labour.

In this case country A has an absolute advantage in the production of X and country B has an absolute advantage in the production of Y.

And, therefore Adam Smith maintained that country A should specialize in the production of commodity X and country B in the production of commodity Y. Both the countries would gain by this specialization.

Comparative Advantage Theory:

The principle of comparative cost is based on the difference in production costs of similar commodities in different countries. Production costs differ in countries because of geographical division of labour and specialization in production, due to differences in climate, natural resources, geographical situation and efficiency of labour. A country can produce one commodity at, lower cost than the other. In this way, each country specializes in the production of that commodity in which its comparative cost of production is the least. Therefore, when a country enters into trade with sonic other country, it will export those commodities in which its comparative cost of production is the least and will import those commodities in which its comparative production costs are high. This is the basis of international trade, according to Ricardo.

Assumption:

I. Model started with 2 countries, 2 commodities and 1 factor of production that is labour (2 *2 * 1)
II. The cost of production equals total labour.
III. All the units of labour are homogeneous and it used in the fixed proportion.
IV. Law of constant returns operate in production process.
V. Perfect competition in market system.
VI. No transportation cost.
VII. No barriers to trade. Three is free trade between the two countries, i.e., there are no restrictions in the movement of commodities.
VIII. No technical change.

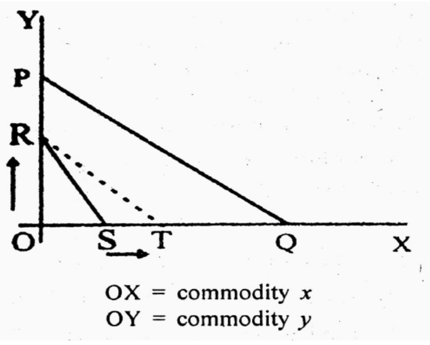
Theory: Comparative advantage occurs due to Comparative cost difference in production of different commodity. Comparative differences in cost occur when one country has an absolute advantage in the production of both commodities, but a comparative advantage in the production of one commodity than in the other.

Example:
Suppose country A can produce 10x or 1Oy and country B can produce 6x or 8y.

	Commodity X	Commodity Y
Country A	10	10
Country B	6	8

In this case country A has an absolute advantage on the production of both x and y, but a comparative advantage in the production of x. Country B is at an absolute disadvantage in the production of both commodities but it has comparatively less disadvantage in the production of y.

Thus, country A has a comparative advantage in the production of commodity x, and has least comparative disadvantage in the production of y. Thus trade is beneficial for both countries. The comparative advantage position of both countries is illustrated in the following figure.



In the figure, PQ is the production possibility curve of country A, and RS that of country B. Country A enjoys an absolute advantage in the production of both x and y over country B. It produces OQ units of x or OP units of y as against OS units of x or OR units y produced by country B. But the slope RT reveals that A has a comparative advantage in the production of commodity x only because if it gives up the resources required to produce OR of commodity, it can produce OT > OS of commodity x. On the other hand, if it gives up resources required to produce OS units of x, it would be able to produce commodity x by an amount less than OR. Thus country A has a comparative advantage in the production of commodity x and country B has a less comparative disadvantage in the production of commodity y.

Labour theory of Value: It is given by David Ricardo and further modified by Karl Marx. Ricardo used this theory extensively to explain the comparative advantage theory.
Under the labour theory of value, the value or price of a commodity depends exclusively on the amount of labour going into the production of the commodity. This implies (1) that either labor is the only factor of production or labour is used in the *same* fixed proportion in the production of all commodities and (2) that labour is homogeneous (i.e., of only one type)

for the other commodity. The term 'reciprocal demand' was introduced by J. S. Mill to explain the determination of the equilibrium terms of trade. It is used to indicate a country's demand for one commodity in terms of the quantities of the other commodities it is prepared to give up in exchange.

Assumption:

I. Model started with 2 countries, 2 commodities and 1 factor of production that is labour (2 *2 * 1)
II. Law of constant returns operate in production process.
III. Perfect competition in market system.
IV. No transportation cost.
V. Full employment
VI. There is free trade.

Theory:

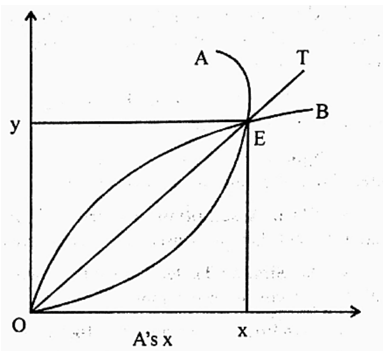
(i)The possible range of barter terms of trade is given by the respective domestic terms of trade as set by comparative efficiency in each country.

(ii) Within this range, the actual terms of trade will depend upon reciprocal demand, i.e., the strength and elasticity of each country demand for the other country's product.

(iii) Only those barter terms of trade will be stable at which the exports offered by each country just suffice to pay for the imports it desires.

Mill's theory of reciprocal demand is explained diagrammatically in terms of Marshall's offer curves.

In the below figure, country A's production of X is taken on the horizontal axis and country B producing only Y is taken on vertical axis. The curve OA is country A's offer curve. It shows how many units of X country A will give up for given quantity of Y. Similarly OB is the offer curve of country B which shows how many units of Y country B is prepared to give up in exchange for a given quantity of cloth. The point E where the two offer curves OA and OB intersect is the y equilibrium point at which OX of commodity X is traded by country A for OY of commodity Y of country B. The rate at which X is exchanged for Y is equivalent to the slope of the line OT.



Offer Curve: Offer curves were devised and introduced into international economics by Alfred Marshall and Ysidro Edgeworth, two British economists, at the turn of the twentieth century. The offer curve of a nation shows how much of its import commodity the nation demands for it to be willing to supply various amounts of its export commodity

Opportunity Cost Theory:

Classical theory of comparative cost advantage is based on the labour theory of value, where all factors of production can be converted into labour equivalent and prices of goods and services are determined by the amount of labour needed to produce these goods and services. However, labour theory of value is an unrealistic theory. Modern economists have discarded the labour theory of value. Haberler has presented an explanation of the Comparative cost theory in terms of opportunity cost.

The opportunity cost theory says that if a country can produce either commodity X or Y, the opportunity cost of commodity X is the amount of the other commodity Y that must be given up in order to get one additional unit of commodity X. Thus the exchange ratio between the two commodities is expressed in terms of their opportunity costs.

Assumption:

I. There is perfect competition in both the factor and commodity markets.
II. There is fixed supply of factors.
III. There is full employment.
IV. There is free trade between the countries. The price of each commodity equals its marginal money costs.
V. The price of each factor equals its marginal value productivity in each employment.

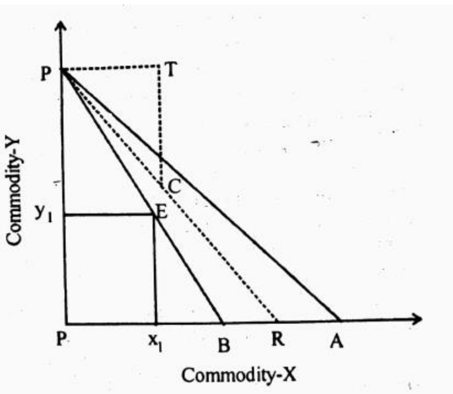
It is the shape of the production possibility curve under different cost conditions that determines the bases and the gains from international trade under the theory of opportunity cost.

(A) Trade under Constant Opportunity Costs: Under constant opportunity costs, the production possibility curve is a straight line.

In the figure, PA is the production possibility curve of count!) A, and P13 of country 13. As the relative prices differ in the two countries, trade is possible between the two.

Country A has a comparative advantage in the production of X and B has a comparative advantage (least disadvantage) in the production of V. under the circumstances, county A will specialize in the production of commodity X and export it to and count!), B will specialize in 1 and export it to A.

PR is the new international price line. Before trade, B was consuming and producing both the commodities at point E. After trade it specializes exclusively in the production of Y at point I and its consumption level shifts up from point E to C on the international price line PR. It will now export 'TC of Y to country international price line PR. It will now export 'IC of Y to country.

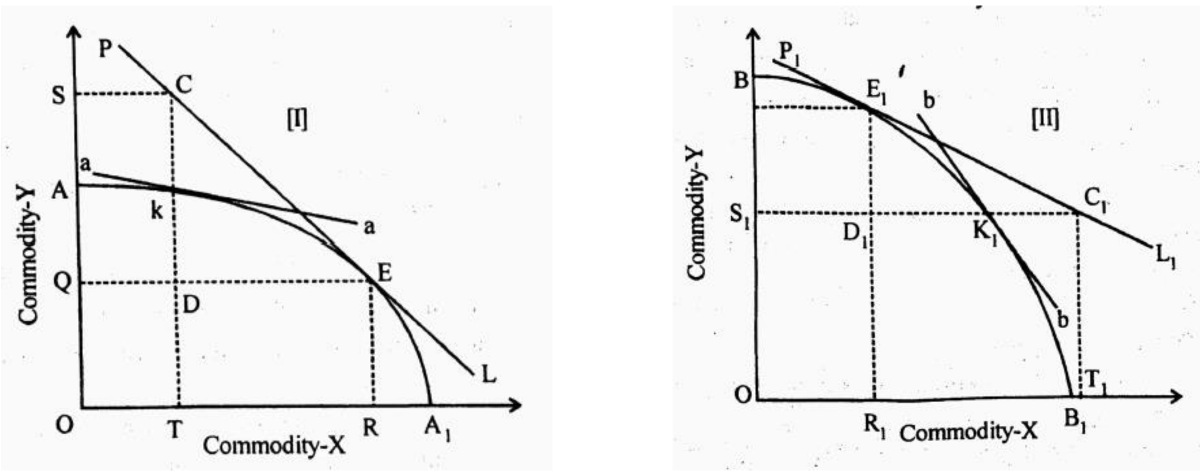


(B) Trade Under Increasing Opportunity Costs. The production possibility) curve under increasing opportunity costs is concave to the origin. In the figure (panel [I]), AA_I is the production possibility of country A which is concave to the origin. The slope of this curve shows that A will specialize in the production of X.

Panel [II] of the figure shows BB_1 is the production possibility curve of country B. The slope of this curve reveals that it will specialize in the production of commodity Y.

In the absence of foreign trade, country A produces and consumes some quantities of both X and Y at point K where the line aa is tangent to AA_1 curve. The line aa indicates the domestic relative commodity prices of X and Y. Similarly country B produces and consumes some quantities of the two commodities at point K1 where its price line bb is tangent to the production possibility curve $BB_{1.}$ International trade is possible only if the international price ratio of the two commodities differs from that prevailing in the domestic market of each country. The international price ratio is given by the PL line in country A and P1L1 in country B. The new equilibrium point as determined by the price line PLA is E. It will thus produce OR of X and OQ of Y. The consumption point for country A will be at C on the price line PL. It wills then exports TR of X and import QS of Y. The new equilibrium point for country B is point E_1. The consumption point of country B will at C_1 on the price line P_1L_1. It will import DA of X and export DIE, of Y on the trade triangle F_1 D_1 C_1.

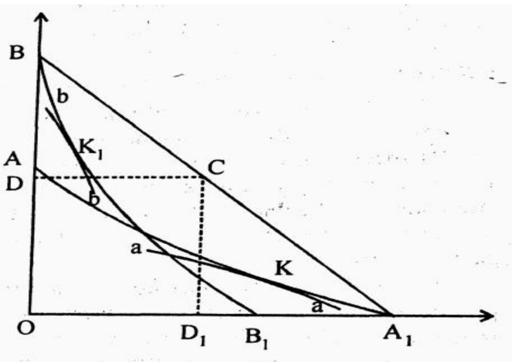
Under increasing opportunity costs specialization is always incomplete clue to decreasing returns. Hence the gains from trade arc less than that under complete specialization. Further. The law of comparative costs is valid only under increasing opportunity costs. In a two good world if one country is more efficient in producing both goods than another country, it profits by concentrating on the product in which it has a greater comparative advantage.



(C) Trade under Decreasing Opportunity Cost: When two countries experience decreasing opportunity costs, their production possibility curves are convex to the origin. Under decreasing opportunity costs, each country completely specializes in the production of only one commodity after trade because there are increasing returns based on international economies of production.

In the figure, AA_I is the production possibility curve of country A_l and BB_I is that of country B. The pre-trade consumption and production point of country A is K and that of country B is K. The slope of the domestic price line as of country a shows that its comparative advantage is greater in the production of commodity X. Similarly. The slope of country B's domestic price ratio bb shows its greater comparative advantage in the production of commodity Y.

If trade starts between the two countries, the international price line (ratio) will be BA_1. The country A will completely specialize in the production of X at point A_l, and country B will completely specialize in the production of Y at B. Now each will move along the international price line BA_1, country A from point A_1 upward and country B from point B downward, and reach point C in consumption. A will export D_IA_1 of X to country B and import D_1C of Y from it on the trade triangle CD_1A_1.



Production Possibility Frontier (PPF): PPF is a curve that shows the alternative combination of the two commodities that a nation can produce by fully utilizes all of its recourse with the best technology available to it. Slope of the PPF is Opportunity cost.

Modern Trade Theory: Heckscher-Ohlin Theory or H-O theory

The classical comparative cost theory did not satisfactorily explain why comparative costs of producing various commodities differ as between different countries. The new theory propounded by Heckscher and Ohlin went deeper into the underlying forces which cause differences in comparative costs.

Assumption:

The Heckscher–Ohlin theory is based on the following assumptions:

I. There are two nations (Nation 1 and Nation 2), two commodities (commodity X and commodity Y), and two factors of production (labour and capital).
II. Both nations use the same technology in production.
III. Commodity X is labour intensive, and commodity Y is capital intensive in both nations.
IV. Both commodities are produced under constant returns to scale in both nations.
V. There is incomplete specialization in production in both nations.
VI. Tastes are equal in both nations.
VII. There is perfect competition in both commodities and factor markets in both nations.
VIII. There is perfect factor mobility within each nation but no international factor mobility.
IX. There are no transportation costs, tariffs, or other obstructions to the free flow of international trade.

X. All resources are fully employed in both nations.
XI. International trade between the two nations is balanced.

Before started the H-O theory we need to have a clear knowledge of some basic concepts—

- Factor Intensity: In a world of two commodities (X and Y) and two factors (labour and capital), we say that commodity Y is capital intensive if the capital–labour ratio (K/L) used in the production of Y is greater than K/L used in the production of X.

  For example, if two units of capital (2*K*) and two units of labour (2*L*) are required to produce one unit of commodity Y, the capital–labour ratio is one. That is, 2/2 in the production of Y. If at the same time 1*K* and 4*L* are required to produce one unit of X, *K/L* = 1/4 for commodity X. Since *K/L* = 1 for Y and *K/L* = 1/4 for X, we say that Y is *K* intensive and X is *L* intensive.

- Factor Abundance: It can be defined by two ways (i) Physical Term (ii) Relative Factor Price. According to the definition in terms of physical units, Nation 2 is capital abundant if the ratio of the total amount of capital to the total amount of labour (TK/TL) available in Nation 2 is greater than that in Nation 1 (i.e., if TK/TL for Nation 2 exceeds TK/TL for Nation 1). According to the definition in terms of factor prices, Nation 2 is capital abundant if the ratio of the rental price of capital to the price of labour time (P_K $/P_L$) is *lower* in Nation 2 than in Nation 1 (i.e., if P_K/P_L in Nation 2 is smaller than P_K $/P_L$ in Nation 1).

Theory:

We can state the Heckscher– Ohlin theorem as follows: A nation will export the commodity whose production requires the intensive use of the nation's relatively abundant and cheap factor and import the commodity whose production requires the intensive use of the nation's relatively scarce and expensive factor. In short, the relatively labour-rich nation exports the relatively labour-intensive commodity and imports the relatively capital-intensive commodity. Now, Nation 1 exports commodity X because commodity X is the L-intensive commodity and L is the relatively abundant and cheap factor in Nation 1. Conversely, Nation 2 exports commodity Y because commodity Y is the K-intensive commodity and K is the relatively abundant and cheap factor in Nation 2.

Of all the possible reasons for differences in relative commodity prices and comparative advantage among nations, the H–O theorem isolates the difference in relative factor abundance, or factor endowments, among nations as the basic cause or determinant of comparative advantage and international trade. For this reason, the H–O model is often referred to as the factor-proportions or factor-endowment theory. That is, each nation specializes in the production and export of the commodity intensive in its relatively abundant and cheap factor and imports the commodity intensive in its relatively scarce and expensive factor.

H-O Theory has been criticised on the following grounds—

I. The assumption of two-by-two-by-two model is not realistic.
II. Like the classical theory the H-O model is static in nature.
III. Factors are not homogeneous.
IV. Production techniques cannot be homogeneous.

Empirical Evidence on H-O theory (Leontief Paradox):

The first empirical test of the Heckscher–Ohlin model was conducted by Wassily Leontief in 1951 using U.S. data for the year 1947. Since the United States was the most K-abundant nation in the world, Leontief expected to find that it exported K-intensive commodities and imported L-intensive commodities.

The results of Leontief's test were startling. U.S. import substitutes were about 30 percent more K intensive than U.S. exports. That is, the United States seemed to export L-intensive commodities and import K-intensive commodities. This was the opposite of what the H–O model predicted, and it became known as the Leontief paradox.

The Factor Price Equalization Theorem or Heckscher-Ohlin-Samuelson (H-O-S) Theorem:

H-O-S theorem states that International trade will bring about equalization in the relative and absolute returns to homogeneous factors across nations. As such, international trade is a substitute for the international mobility of factors.

What this means is that international trade will cause the wages of homogeneous labour (i.e., labour with the same level of training, skills, and productivity) to be the same in all trading nations (if all of the assumptions in H-O theory hold). Similarly, international trade will cause the return to homogeneous capital (i.e., capital of the same productivity and risk) to be the same in all trading nations. That is, international trade will make w the same in Nation 1 and Nation 2; similarly, it will cause r to be the same in both nations. Both relative and absolute factor prices will be equalized.

- **The Rybczynski Theorem**: This theorem postulates that at constant commodity prices, an increase in the endowment of one factor will increase by a greater proportion the output of the commodity intensive in that factor and will reduce the output of the other commodity. For example, if only L grows in Nation 1, then the output of commodity X (the L-intensive commodity) expands more than proportionately, while the output of commodity Y (the K-intensive commodity) declines at constant P_X and P_Y.

New trade theory:

H-O based on some unrealistic assumption which is hard to satisfy. That's why, in 1970's some economists, who have tried to explain the trade theory by relaxations of those assumptions. New trade theory (NTT) suggests that a critical factor in determining international patterns of trade are the very substantial economies of scale and network affects that can occur in key industries.

These economies of scale and network effects can be so significant that they outweigh the more traditional theory of comparative advantage. In some industries, two countries may have no discernible differences in opportunity cost at a particular point in time. But, if one country specializes in a particular industry then it may gain economies of scale and other network benefits from its specialization.

Another element of new trade theory is that firms who have the advantage of being an early entrant can become a dominant firm in the market. This is because the first firms gain substantial economies of scale meaning that new firms can't compete against the incumbent firms. This means that in these global industries with very large economies of scale, there is likely to be limited competition, with the market dominated by early firms who entered, leading to a form of monopolistic competition.

Monopolistic competition is an important element of New Trade Theory; it suggests that firms are often competing on branding, quality and not just simple price. It explains why countries can both export and import designer clothes.

This means that the most lucrative industries are often dominated in capital-intensive countries, who were the first to develop these industries. Therefore, being the first firm to reach industrial maturity gives a very strong competitive advantage. (Some may say unfair advantage)

New trade theory also becomes a factor in explaining the growth of globalisation.

It means that poorer, developing economies may struggle to ever develop certain industries because they lag too far behind the economies of scale enjoyed in the developed world. This is not due to any intrinsic comparative advantage, but more the economies of scale the developed firms already have.

Paul Krugman was a leading academic in developing New Trade Theory. He was awarded a Nobel Prize (2008) in economics for his contributions in modelling these ideas. "for his analysis of trade patterns and location of economic activity".

- **Some Model on New Trade Theory:**

Technology Gap Model:

According to the technological gap model sketched by Posner in 1961, a great deal of the trade among industrialized countries is based on the introduction of new products and new production processes. These give the innovating firm and nation a temporary monopoly in the world market. Such a temporary monopoly is often based on patents and copyrights, which are granted to stimulate the flow of inventions.

Product Cycle Model:

A generalization and extension of the technological gap model is the product cycle model, which was fully developed by Vernon in 1966. According to this model, when a new product is introduced, it usually requires highly skilled labour to produce. As the product matures and acquires mass acceptance, it becomes standardized; it can then be produced by mass production techniques and less skilled labour. Therefore, comparative advantage in the product shifts from the advanced nation that originally introduced it to less advanced nations, where labour is relatively cheaper. This may be accompanied by foreign direct investments from the innovating nation to nations with cheaper labour.

Unit-8: Growth and Development Economics

Introduction (Concepts of Economic Growth and Economic Development):

In recent years, there has come into existence a new branch of economics known as the "Economics of Development". It refers to the problems of the economic development of underdeveloped or backward countries. In addition to the illuminating reports of the U.N.O. on the subject, some top-ranking economists like Nurkse, Dobb, Staley, Buchanan, Rostow and Ellis have made some original contributions to the Economics of Development. The main reason for the growing popularity of "Economics of Development" as a separate branch of economic theory is the increasing tendency on the part of the newly independent countries of Asia and Africa to resort to developmental planning as a means to eliminate their age-old poverty and raise living standards.

Definitions of Economic Development:

The term 'economic development' is generally used in many other synonymous terms such as economic growth, economic welfare, secular change, social justice and economic progress. As such, it is not easy to give any precise and clear definition of economic development. But in view of its scientific study and its popularity, a working definition of the term seems to be quite essential. Economic development, as it is now generally understood, includes the development of agriculture, industry, trade, transport, means of irrigation, power resources, etc. It, thus, indicates a process of development. The sectoral improvement is the part of the process of development which refers to the economic development. Broadly speaking, economic development has been defined in different ways and as such it is difficult to locate any single definition which may be regarded entirely satisfactory.

Characteristics of a Developed Economy:

A developed economy is the characterised by increase in capital resources, improvement in efficiency of labour, better organisation of production in all spheres, development of means of transport and communication, growth of banks and other financial institutions, urbanisation and a rise in the level of living, improvement in the standards of education and expectation of life, greater leisure and more recreation facilities and the widening of the mental horizon of the people, and so on. In short, economic development must break the poverty barrier or the vicious circle and bring into being a self-generating economy so that economic growth becomes self-sustained.

Distinction between Developed and Underdeveloped Economies:

We may now distinguish between the features of an underdeveloped economy from that of developed one as follows:

i. Underdeveloped economies are distinguished from developed economies on the basis of per capita income. In general, those countries which have real per capita incomes less than a quarter of the per capita income of the United States, or roughly less than 5000 dollars per year, are categorized as under-developed countries.

ii. An underdeveloped economy, compared with an advanced economy, is underequipped with capital in relation to its population and natural resources. The rate of growth of employment and investment in such an economy lags behind the rate of growth of population. The resources are not only employed but also underemployed.

iii. High rate of growth of population is an important characteristic of most of the underdeveloped economies. Population growth in underdeveloped countries neutralises economic growth. In advanced economies, the case is different. As Prof. Hansen points out, one of the empirical tests of secular stagnation in advanced economies is the declining rate of population growth. The stagnation problem in a developed economy is a problem of population, natural resources and technology failing to keep pace with capital accumulation.

iv. The central problem of underdeveloped economies is the prevalence of mass poverty which is the cause as well as the consequence of their low level of development. Shortage and scarcity are the main economic problems in these economies, whereas the affluent societies of advanced countries have economic problems resulting from abundance.

Difference between Economic Growth and Economic Development:

Economic	Development	Economic Growth
Concept	Normative concept	Narrowed concept than economic development
Scope	Concerned with structural changes in the economy	Growth is concerned with increases in the economy's output
Growth	Development relates to growth of human capital indexes, a decrease in inequality figures, and structural changes that improve the general population's quality of life	It refers to an increase in the real output of goods and services in the country like increase the income in savings, in investment etc.
Implication	It implies changes in income, saving and investment along with progressive changes in socio-economic structure of country	It refers to an increase in the real output of goods and services in the country like increase the income in savings, in investment etc.
Measurement:	Qualitative. HDI (Human Development Index), gender-related index (GDI), Human poverty index (HPI), infant mortality, literacy rate etc.	Quantitative Increase in real GDP Shown in PPF.
Effect	Brings qualitative and quantitative	Brings quantitative changes the economy

Theories of Economic Development (Some theories)

Adam Smith

Introduction:

Adam Smith is regarded as the foremost classical economist. His monumental work, An Enquiry into the Nature and Causes of the Wealth of Nations published in 1776, was primarily concerned with the problem of economic development. Though he did not expound any systematic growth theory, yet a coherent theory has been constructed by later day economists which is explained below.

Assumptions

1. Population growth was taken as an endogenous variable. It was considered to be a function of subsistence available to accommodate increasing work force.
2. Investment was also taken as an endogenous variable and was considered to be a function of rate of savings.
3. Land growth could take place either by conquest of new land via colonization which prevailed then or improvement in the fertility of old lands.
4. Specialization increases the productivity and enhances the rate of growth.
5. Smith assumed that there existed perfect competition in the market.

1. Natural Law: Adam Smith strongly believed in the efficiency of laissez faire market system. He proposed maximization of self-interest automatically leads to maximization of social interest. When each individual tries to maximize his own individual interest, he is led by an 'invisible hand'. When each individual will maximize his own wealth in a free Laissez faire economy, then all individuals, if left free, will maximize aggregate wealth. He supported free trade and criticized any form of government intervention.

2. Division of Labour: Division of labour increases the specialization of a worker and thereby increases the overall productivity. Division of labour: (a) increases the dexterity of every worker; (b) saves time of producing goods; (c) leads to invention f large number of labour-saving machines. However, increase in productivity also stems from capital through improved technology which depends on the size of market.

3. Process of Capital Accumulation: Division of labour leads to capital accumulation and this capital accumulation leads to a higher rate of development. But it is capital accumulation which must precede division of labour because it will stimulate specialization. Smith assumed that only capitalists and landlords were capable of savings. Labourers could not save because of 'Iron Law of Wages' which states that at any point of time wages tend to equal to the amount necessary for subsistence of labourers. If it is more than this, then there will be increase in competition for employment and wages will decrease.

4. Investment is made to earn Profits: Classical economists stated that capitalists made investment in an expectation to earn profits on them and these expectations depended on the present climate for investment and actual profits in the present. Smith also proposed that profits tend to fall with increase in the rate of capital accumulation. As economy's capital stock grows, demand for labour force increases, it increases competition for getting labour which leads to increase in wage bill and thereby reduces profits.

5. Interest: Quantity of capital for lending will increase with the fall in interest rates and vice versa. Interest rates will fall with progress and prosperity and hence supply of capital will go up.

6. Agents of Growth: Smith believed that farmers, producers and business man are agents of economic growth. The functions performed by these agents of economic growth are interrelated.
Despite this, Smith's theory of economic development points toward certain factors that are helpful in the process of developing underdeveloped countries. Farmers, traders and producers, the three agents of growth mentioned by Smith, can help in developing the economy by raising productivity in their respective spheres. In the absence of a free market economy, the state can induce them to produce more, as is being done in India. Their interdependence also points toward the importance of balanced growth for such economies.

Ricardo

Like Smith, David Ricardo also presented his views on economic development in an unsystematic manner in his book The Principles of Political Economy and Taxation. This book was published in 1917. It was its third edition of 1921 and Ricardo's correspondence with a number of economists that contain his ideas on which his model of development has been built. Ricardo never propounded any theory of development. He simply discussed the theory of distribution. Therefore, Ricardo's analysis is a detour. The Ricardian theory is based on the marginal and the surplus principles. The marginal principle explains the share of rent in the national output, and the surplus principle explains the division of the remaining share between wages and profits.

Assumptions

1. All land is used for the production of one crop say, corn.
2. Land is subject to diminishing returns to a factor;
3. Supply of land is fixed; 4. Labour and capital both are variable inputs;
5. Technology is given and remains unchanged;
6. Wage rate is equal to subsistence level;
7. There exists perfect competition in the market;
8. Wage rate and quantity supplied of labour is given and constant;
9. Demand for labour is a function of accumulations;
10. Capital accumulations occur from profits

Important Features:-

1. **Rent, Profit and Wages:** Ricardo defined rent as that portion of the produce of the earth which is paid to the landlord for the use of original and indestructible powers of the soil. The wage rate is determined by wage fund divided by number of workers employed at subsistence level. Ricardo opined that in total produce of the corn, the first payment is made to landlord; the residual is distributed between wage and profits (interest is included in profits). Had land been in unlimited supply and uniform it would have earned no interest.

2. **Capital Accumulation:** According to Ricardo, capital accumulation depends on following factors:

(a) The Profit Rate: Profit divided by capital employed gives us the rate of profit. As long as, rate of profit is positive, capital accumulation will continue to take place. Since depend on wages which in turn depend on the price of corn and fertility of land. Hence, profits and wages are inversely related.

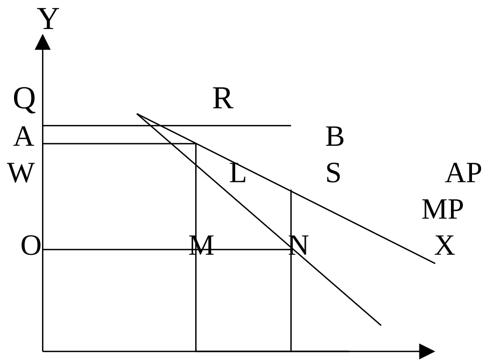
(b) Increase in wages: If the cost of subsistence increases, wage rate will increase. With the increase in demand for food more land will have to be brought under agriculture. It will increase the demand for labour and wages will rise. With rise in wages, price of crop will also rise and hence rent will increase. But profits will fall leading to a decline in capital accumulation.

(c) Declining Profits in Other Industries: Ricardo took agriculture as the determining sector. The profits in agriculture determine profits in other industries. Therefore, the profit rate in both agriculture and industrial sector must be same.

3. Other Sources of Capital Accumulation: Ricardo states that higher will be the difference between production and consumption, higher will the rate of profits. Hence, capital accumulation can be increased by increasing production or by decreasing unproductive consumption. Productivity of labour can increase through technological progress and better organization. However, use of capital intensive techniques will lead to unemployment. Ricardo considered following as additional sources of capital accumulation:

(a) **Taxes**: Ricardo suggested that taxes should be levied to reduce conspicuous consumption as it is unproductive consumption and no how increases the productivity of labour. These taxes can be used by government for capital accumulation. The taxes which affect incomes of landlords or labourers were not favoured by Ricardo.

(b) **Free Trade:** Ricardo is in favour of free trade as it promotes capital accumulation.
Stationary State: According to David Ricardo, in the long run, profits have a natural tendency to fall so that country ultimately reaches a stationary state. Rise in profits → Rise in Capital Accumulation → Rise in production → Increased wage fund → Increase in population → increase in demand for corn and price of corn goes up → Demand for land increases → rent increases and profits and wages decrease → wage become equal to subsistence level It is shown with the help of following figure. Labour is measured on x axis and AP and MP on y-axis. It is shown that as demand for labour rises, leading to-rise in wage bill from OWLM to OQRM, all profits disappear. Share of rent increased.



Marxian Theory of Economic Development:

Karl Marx's contribution in the theory of economic development is critical because he provided his famous reproduction schema in a multi-sector growth model and introduced the concept of "SteadyState" growth equilibrium. He also took labour as exogenous to wages. He proposed that wages are determined by the bargaining between capitalists and workers. However, the bargaining ability of workers depends on the number of unemployed labourers in the economy. He called it "reserve army of labour". He also advocated that savings and capital accumulation depends on profits.

Organic Composition of Capital and Surplus Value

Marx proposed that in the long run profits tend to fall due to "rising organic composition of capital".
Organic composition of capital is the ratio of constant capital to variable capital. Constant capital means circulating capital like raw material. Variable capital means advancement to labour i.e. total wage bill.
So, Marx gave total value of output as:
$y = c + v + s$ where y is output; c is constant capital; v is variable capital; s is surplus value.
Marx defined rate of profit is equal to

$$r = s/(v + c) \text{ and,}$$
$$s = y - (c + v)$$

Marx called s/v, the ratio of surplus value to variable capital as "rate of exploitation". Given $r = s/(v + c)$; if we divide this equation by v on both sides, it will be equal to:
$r = [(s/v)\ (v/(v + c)]$

Therefore, rate of profit is a positive function of exploitation rate (s/v) and a negative function of organic composition of capital (c/v).

Declining Rate of Profit

Marx claimed that rate of profit tends to fall because s/v tend to be same and c/v tends to fall. In a static economy, when the surplus accrues to capitalists, they reinvest it and output expands. It exerts a pressure on constant labour supply pushing wages upward therefore v i.e. variable capital rises and r i.e. rate of profits fall. Rise in wages motivate capitalists to introduce labor-saving machinery and profits increase and unemployment increases.

It will have two effects

(a) Variable capital will fall and constant capital will increase therefore, c/v remains constant.

(b) "Reserve army of labour" will affect wage rate and reduce it to subsistence level. Therefore, it declines further. It will increase c/v and the rate of profit will fall. However, introduction of labour saving machinery and laying off of labour would mean rise in c and fall in v i.e. organic composition of capital rises. Therefore r, rate of profit falls.

Increasing Rate of Exploitation Capitalists make an effort to compensate themselves for this declining rate of profit by increasing the rate of exploitation. The rate at which labour is released is higher than the rate at which it is reabsorbed. Therefore, it creates permanent technological unemployment. However, there is a limit to which this rate of exploitation can be increased. As large firms will buy small firms, there will be concentration of capital in fewer hands. This in combination with the misery of labour would create giant crisis leading to destruction of capitalism as a whole.

A Critical Appraisal Strengths

(a) Smith, Ricardo and Marx all concluded that increasing share of rent in total output leads to declining rate of profits and results in stationary state.
(b) It proved that capitalism can't sustain for long.
(c) He explained class struggle through his version of economic growth.

Weaknesses

(a) Fall in profit is possible but not inevitable.
(b) Marx's rate of exploitation is limited by the length of working day. It is not plausible.
(c) Technological progress may not necessarily increase organic composition of capital.

Important Notes

• *The classical school of economic thought was formally propounded by Adam Smith, who is called 'father of Economics', Malthus, David Ricardo, John Mill and J. B. Say. Each thinker has put forward a view which is different from others but still has some similarities.*
• *Adam Smith wrote a book 'An inquiry into the nature and causes of wealth of nations' which was published in 1776.*
• *Like Smith, Ricardo also never gave a systematic theory of development but gave his views in an unsystematic manner in his book 'The Principles of Political Economy and Taxation'.*
• *Malthus in his book 'The Progress of Wealth' gave a more systematic theory of growth. Malthus in his theory of population states that unchecked population growth always exceeds the growth and the means of subsistence which makes means more and more scarce.*
• *Karl Marx's contribution in the theory of economic development is critical because he provided his famous reproduction schema in a multi-sector growth model and introduced the concept of "steady-State" growth equilibrium.*

Schumpeter Model of Growth:

Joseph Alois Schumpeter first presented his theory of economic growth in "Theory of Economic Development" published in German in 1911 (its English edition appeared in 1934) which was elaborated and refined but in no way altered in any essential respect in his Business Cycles (1939) and Capitalism, Socialism and Democracy (1942).

Schumpeter's Theory of Capitalistic Development

All theories given by classical economists emphasized on the supply side of the production. They claimed that economic growth meant increase in productive capacity or supply of greater goods and services. Schumpeter was not different. But classical economists believed that output increases by increase in capital formulation; Schumpeter claimed it happens due to Innovations. According to Schumpeter, innovation means the ability of entrepreneurs to use the new ideas or invention to create a new combination of factor inputs which reduces cost and increases profits. In other words, innovation is the capability of organizers to use resources in a different combination which increases their efficiency. Schumpeter did not give so much importance to capital formation. He claimed that innovation in an economy is a continuous process. More effective innovation leads to more efficient utilization of resources and thereby higher profits. Innovations can take form of:

(a) Introduction of a new good
(b) Introduction of new method of production
(c) Innovating a new market
(d) Finding new source of supply of raw material
(e) Designing a new form of organization

Schumpeter called his theory 'a creative destruction' because every new innovation makes the old things obsolete. And this creative destruction leads to a process of incessant revolutionary change from within. For example, telecommunication from landline to wireless, then mobile and now mobile is a mini computer and every new creation in the industry makes the older one obsolete.

All classical economists took mobilization of savings as a source of capital formation and economic growth. However, Schumpeter felt that innovations involve risk. So, the funds for trying innovations come through credit. This credit is provided by the capital market. Hence, Schumpeter gave importance to existence of well-organized capital market in the economy to ensure innovations and introduction of newer products, better technology and thereby enhanced output and increased rate of economic growth.

Schumpeter's theory is an endogenous theory. It takes capital formation as a social process through the working out of a system in which there is win-lose competition. He did not agree to Neo-Classical economists who restored to perfect competition and competition without rivalry. The differences between views of Schumpeter and Neo-Classical Economists can be summarized as follows:

(a) Neo-Classical Economists claim that there is perfect competition prevailing in the market and hence no super normal profits exist in the long run; Schumpeter claims the existence of monopoly (through intellectual property rights) and Monopolistic competition and super normal profits do exist.

(b) Neo-Classical Economists discussed short run and a static model; Schumpeter discussed long rum and a dynamic model.

(c) Neo-Classical Economists claimed that there will more savings which will be better mobilized and bring about economic growth. However, Schumpeter explained two forces for economic growth; (a) availability of intellectual man power and organizers;
(b) innovations and technological progress.

Important

- J.A. Schumpeter is an economist who emphasized the role of innovations in economic growth and development.
- All theories given by classical economists emphasized on the supply side of the production.
- They claimed that economic growth meant increase in productive capacity or supply of greater goods and services.
- Schumpeter called his theory 'a creative destruction' because every new innovation makes the old things obsolete.
- All classical economists took mobilization of savings as a source of capital formation and
- economic growth.
- Schumpeter did not use mathematical tools to find the quantitative relationships between
- innovations and critical economic variables.
- Schumpeter's theory also gives an explanation for business cycles.
- Sometimes it is very much possible that new innovations do come but do not make the old ones obsolete.
- Schumpeter believed money to have a vital and role in the economic system here he agreed to Keynes but he criticized Keynes for not considering the basic structural change in the economy in his theories.
- There are some models in which research and development process has been taken as a force for economic growth and development.

Rostow Stages of Economics Growth theory

Prof. W.W. Rostow has sought an historical approach to the process of economic development. He distinguishes five stages of economic growth, viz., (1) the traditional society; (2) the pre-conditions for take-off; (3) the take- ff; (4) the drive to maturity; and (5) the age of high mass-consumption.

Traditional society: This is an agricultural economy of mainly subsistence farming, little of which is traded. The size of the capital stock is limited and of low quality resulting in very low labour productivity and little surplus output left to sell in domestic and overseas markets

Pre-conditions for take-off: Agriculture becomes more mechanized and more output is traded. Savings and investment grow although they are still a small percentage of national income (GDP). Some external funding is required - for example in the form of overseas aid or perhaps remittance incomes from migrant workers living overseas

Take-off: Manufacturing industry assumes greater importance, although the number of industries remains small. Political and social institutions start to develop - external finance may still be required. Savings and investment grow, perhaps to 15% of GDP. Agriculture assumes lesser importance in relative terms although the majority of people may remain employed in the farming sector. There is often a dual economy apparent with rising productivity and wealth in manufacturing and other industries contrasted with stubbornly low productivity and real incomes in rural agriculture.

Conditions for Take-off: The requirements of take-off are the following three related but necessary conditions:
(1) A rise in the rate of productive investment from, say, 5 per cent or less to over 10 per cent of national income or net national product;

(2) the development of one or more substantial manufacturing sectors with a high rate of growth;

(3) the existence or quick emergence of a political, social and institutional framework which exploits the impulses to expansion in the modern sector and gives to growth an outgoing character.

Drive to maturity: Industry becomes more diverse. Growth should spread to different parts of the country as the state of technology improves - the economy moves from being dependent on factor inputs for growth towards making better use of innovation to bring about increases in real per capita incomes

Age of mass consumption: Output levels grow, enabling increased consumer expenditure. There is a shift towards tertiary sector activity and the growth is sustained by the expansion of a middle class of consumers.

IMPORTANCE AND LIMITATIONS OF TAKE-OFF FOR UNDERDEVELOPED COUNTRIES:

The concept of take-off is ideally suited for the industrialization of underdeveloped countries. As Dasgupta has written, “The term lacks precision and yet it is suggestive and can be given interpretation which is useful for an understanding of the process of economic development of an underdeveloped country. It is indeed the vagueness of the term that gives it strength for one can put an interpretation upon it to suit the conditions of the economy in which one is interested Of the three necessary conditions for take-off, the first two, namely, capital formation over 10 per cent of national income and the development of one or more leading sectors, are helpful in the process of industrialization of underdeveloped countries. So far as the first condition is concerned, there can be little doubt about achieving that percentage. But the second condition can be moulded to suit a country’s environments.

For instance, the leading sectors can be in agriculture or in the production of primary products for exports. The last condition is more important in the context of underdeveloped countries where monetary and political institutions, and skills and technology are at a low level whereby they retard the expansion of the modern sector.

Balanced & Unbalanced growth:

A mechanism of endogenous growth suitable for investigation of sectoral or regional interaction is developed. It is shown how the high value placed on production linkages by economic historians might be reconciled with the high value placed on openness (often implying lack of linkages) by observers of contemporary less developed countries. When the output of one sector is traded and the output of the other is non-traded, it is shown how the traded goods sector acts as the 'engine of growth' in the sense that its profitability of knowledge acquisition primarily determines the steady state aggregate growth rate. It is also shown how sectors or regions interact out of steady state through product, labour, and capital markets, and in particular how if the former interaction dominates the growth of one sector 'pulls along' the growth of the other while if the latter two interactions dominate one sector or region booms while the other declines. The unit builds on these results to show why liberalization of foreign trade should lead to a transition from a lower to a higher steady state growth rate and why, during the course of this transition, growth might initially be even slower than before liberalization.

In macroeconomics, balanced-growth equilibrium means that the capital intensity of an economy, its capital stock divided by total output, remains constant. In the standard exogenous growth model, balanced growth is a basic assumption, while other variables like the capital stock, real GDP, and output per worker are growing. Developing economies may adopt a strategy of unbalanced growth to rectify previous investment decisions, as put forward by economist Albert O. Hirschman.

Balanced Growth:

Balanced growth has at least two different meanings in economics. In macroeconomics, balanced growth occurs when output and the capital stock grow at the same rate. This growth path can rationalize the long-run stability of real interest rates, but its existence requires strong assumptions. In development economics, balanced growth refers to the simultaneous, coordinated expansion of several sectors. The usual arguments for this development strategy rely on scale economies, so that the productivity and profitability of individual firms may depend on market size. In macroeconomics, balanced growth is usually associated with constant returns to scale. For most development economists, the term is more strongly associated with increasing returns, and a debate that began with Rosenstein-Rodan (1943). He argued that the post-war industrialization of Eastern and South-Eastern Europe would require coordinated investments across several industries. The idea is that expansion of different sectors is complementary, because an increase in the output of one sector increases the size of the market for others. A sector that expands on its own may make a loss, but if many sectors expand at once, they can each make a profit. This tends to imply the need for coordinated expansion, or a "Big Push", and potentially justifies a role for state intervention or development planning

Ragnar Nurkse's balanced growth theory:

The balanced growth theory is an economic theory pioneered by the economist Ragnar Nurkse (1907- 1959). The theory hypothesises that the government of any underdeveloped country needs to make large investments in a number of industries simultaneously. This will enlarge the market size, increase productivity, and provide an incentive for the private sector to invest. Nurkse was in favour of attaining balanced growth in both the industrial and agricultural sectors of the economy. He recognised that the expansion and inter-sectoral balance between agriculture and manufacturing is necessary so that each of these sectors provides a market for the products of the other and in turn, supplies the necessary raw materials for the development and growth of the other.

Nurkse's theory discusses how the poor size of the market in underdeveloped countries perpetuates its underdeveloped state. Nurkse has also clarified the various determinants of the market size and puts primary focus on productivity. According to him, if the productivity levels rise in a less developed country, its market size will expand and thus it can eventually become a developed economy. Apart from this, Nurkse has been nicknamed an export pessimist, as he feels that the finances to make investments in underdeveloped countries must arise from their own domestic territory. No importance should be given to promoting exports

Size of market and inducement to invest:

The size of a market assumes primary importance in the study of what induces investment in a country. Ragnar Nurkse referenced the work of Allyn A. Young to assert that inducement to invest is limited by the size of the market. The original idea behind this was put forward by Adam Smith, who stated that division of labour (as against inducement to invest) is limited by the extent of the market.

According to Nurkse, underdeveloped countries lack adequate purchasing power. Low purchasing power means that the real income of the people is low, although in monetary terms it may be high. If the money income were low, the problem could easily be overcome by expanding the money supply; however, since the meaning in this context is real income, expanding the supply of money will only generate inflationary pressure. Neither real output nor real investment will rise. It is to be noted that a low purchasing power means that domestic demand for commodities is low. Apart from encompassing consumer goods and services, this includes the demand for capital as well.
Ragnar Nurkse concluded,

"The limited size of the domestic market in a low-income country can thus constitute an obstacle to the application of capital by any individual firm or industry working for the market. In this sense the small domestic market is an obstacle to development generally."