Ordinary Differential Equation

June – 2014

(1) Let $Y_1(x)\, and\, Y_2(x)$ defined on $[0,1]$ be twice continuously differentiable functions satisfying $Y''(x)+Y'(x)=0$. Let $w(x)$ be the Wronskian of $Y_1\, and\, Y_2$ and satisfy $W\left(\frac{1}{2}\right)=0$. Then

(a) $W(x)=0$ for $x\epsilon[0,1]$

(b) $W(x)\neq 0$ for $x\epsilon\left[0,\frac{1}{2}\right]\cup\left[\frac{1}{2},1\right]$

(c) $W(x)>0$ for $x\epsilon\left[\frac{1}{2},1\right]$

(d) $W(x)<0$ for $x\epsilon\left[0,\frac{1}{2}\right]$

Answer: (a)

Solution: By Abel's theorem

$W(x)=c\,e^{-\int p\,dx}=c\,e^{-\int 1\,dx}=c\,e^{-x}$

$W\left(\frac{1}{2}\right)=0 \Rightarrow c\,e^{-\frac{1}{2}}=0$

$or, c=0$

$\therefore W(x)=0$

So, the option (a) is correct.

(2) Consider the initial value problem in $R^2. Y'(t)=AY+BY; Y(0)=Y_0$, where $A=\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}, B=\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}$. Then $Y(t)$ is given by

(a) $e^{tA}e^{tB}Y_0$

(b) $e^{tB}e^{tA}Y_0$

(c) $e^{t(A+B)}Y_0$

(d) $e^{-t(A+B)}Y_0$

Answer: (c)

Solution: $\frac{dY}{dt}=(A+B)Y$

$\frac{dY}{Y}=(A+B)dt$

Integrating, $\log Y=(A+B)t+\log c \Rightarrow Y=c\,e^{(A+B)t}$

Given $Y(0)=Y_0$

$\therefore c=Y_0$

$\therefore Y(t)=e^{t(A+B)}Y_0$.

So, the option (c) is correct.

(3) Let $y_1(x)$ and $y_2(x)$ from a complete set of solutions to the differential equation $y'' - 2xy' + \sin\left(e^{2x^2}\right) y = 0, x \,\epsilon [0,1]$ with $y_1(0), y'_1(0) = 1, y_2(0) = 1, y'_2(0) = 1.$ Then the Wronskian $W(x)$ of $y_1(x)\ and\ y_2(x)$ at $x = 1$ is

(a) e^2

(b) $-e$

(c) $-e^2$

(d) e

Answer: (b)

Solution: $W(y_1, y_2)(x) = e \times \rho\left(-\int_0^x p_1\, dx\right) W(y_1, y_2)(0) = e \times \rho\left(-\int_0^x -2x\, dx\right) \cdot (-1)$

$= -e^{x^2}\ \left[\because W(y_1, y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y'_1(0) & y'_2(0) \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1\right]$

$\therefore W(y_1, y_2)(1) = -e$

So, the option (b) is correct.

(4) Consider the boundary value problem $-u''(x) = \pi^2 u(x), x\,\epsilon(0,1), u(0) = u(1) = 0$ if u and u' are continuous on $[0, 1]$ then

(a) $\int_0^1 u^3\,(x) dx = 0$

(b) $u'^2(x) + \pi^2 u^2(x) = u'^2(0)$

(c) $u'^2(x) + \pi^2 u^2(x) = u'^2(1)$

(d) $\int_0^1 u^2\,(x) dx = \frac{1}{\pi^2}\int_0^1 u^{12}\,(x) dx$

Answer: (b), (c) and (d)

Solution: $-u'(x) = \pi^2 u(x)$

i.e., $u'' + \pi^2 u = 0$

Auxiliary equation is $m^2 + \pi^2 = 0$

$or, m = \pm\, i\pi$

$\therefore u(x) = C_1 \cos \pi x + C_2 \sin \pi x$

$u(0) = 0 \Rightarrow C_1 = 0$

Also, $u(1) = 0 \Rightarrow 0 = C_1 \cos \pi + C_2 \sin \pi \Rightarrow C_2 \sin \pi = 0 \Rightarrow C_2 \neq 0$

$\therefore u(x) = C_2 \sin \pi x$

Let, $C_2 = 1$, $\therefore u(x) = \sin \pi x$

$u'(x) = \pi \cos \pi x$

$\therefore \int_0^1 \sin^3 \pi x\, dx \neq 0.$

$u'^2 + \pi^2 u^2 = \pi^2 = u'^2(0)$

$u'^2 + \pi^2 u^2 = \pi^2 = u'^2(1)$

$\int_0^1 \sin^2 \pi x\ \ dx = \int_0^1 \frac{(1-\cos 2\pi x)}{2} dx$

$= \left[\frac{x}{2} - \frac{\sin 2\pi x}{4\pi}\right]_0^1 = \frac{1}{2}$

Also, $\frac{1}{\pi^2}\int_0^1 u'^2\,(x) dx = \frac{1}{\pi^2}\int_0^1 \pi^2 \cos^2 \pi x\, dx = \int_0^1 \cos^2 \pi x\, dx = \frac{1}{2}$

$\therefore \int_0^1 u^2\,(x) dx = \frac{1}{\pi^2}\int_0^1 u'^2\,(x) dx$

So, the options (b), (c) and (d) are correct.

(5) Let $u(t)$ be a continuously differentiable function taking non-negative values for $t > 0$ satisfing $u'(t) = 3u(t)^{\frac{2}{3}}$ and $u(0) = 0$. Which of the following are possible solutions of the above equation?

(a) $u(t) = 0$

(b) $u(t) = t^3$

(c) $u(t) = \begin{cases} 0 \ \ for\ 0 < t < 1 \\ (t-1)^3\ for\ t \geq 1 \end{cases}$

(d) $u(t) = \begin{cases} 0\ for\ 0 < t < 3 \\ (t-1)^3\ for\ t \geq 3 \end{cases}$

Answer: (a), (b), (c) and (d)

Solution: $u'(t) = 3\,u(t)^{\frac{2}{3}}, u(0) = 0$

$\Rightarrow u^{-\frac{2}{3}}\,du = 3\,dt$

Integrating, $\frac{1}{3} \cdot \frac{u^{\frac{2}{3}+1}}{-\frac{2}{3}+1} = t + c$

$or, u^{\frac{1}{3}} = t + c$

$u(0) = 0 \Rightarrow c = 0$

$\therefore u(t) = t^3$

Also, $u(t) = (t-\gamma)^3\ for\ t \geq \gamma$

So, $u(t) = \begin{cases} 0\ for\ 0 < t < 1 \\ (t-1)^3\ for\ t \geq 1 \end{cases}$

and $u(t) = \begin{cases} 0\ for\ 0 < t < 3 \\ (t-3)^3\ for\ t \geq 3 \end{cases}$

Hence, all the options (a), (b), (c) and (d) are correct.

December – 2014

(1) Let $y: \mathbb{R} \to \mathbb{R}$ be differentiable and satisfy the ODE:

$$\left.\begin{aligned} \frac{dy}{dx} &= f(y)\,, x \in \mathbb{R} \\ y(0) &= y(1) = 0 \end{aligned}\right\}$$

Where $f: \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function.
Then
(a) $y(x) = 0$ if and only if $x \in \{0, 1\}$
(b) y is bounded
(c) y is strictly increasing
(d) $\frac{dy}{dx}$ is unbounded
Answer: (b)

Solution: $\frac{dy}{dx} = f(y) = y(say)$ (Lipschitz function)
$\Rightarrow \frac{dy}{y} = dx$
$\log y = x + \log c$
$\Rightarrow y = c\, e^x$
$y(0) = 0 \Rightarrow c = 0$
$\therefore y(x) = 0, \forall\, x$
So, option (a) is not possible.
Clearly y is constant, so it cannot be strictly increasing.
y is bounded, so $\frac{dy}{dx}$ cannot be unbounded.
Hence the option (b) is correct.

(2) For $\lambda \in \mathbb{R}$, consider the boundary value problem.

$$x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + \lambda y = 0, x \in [1, 2]\{-(P_\lambda)\; y(1) = y(2) = 0$$

Which of the following statement is true?
(a) There exists a $\lambda_0 \in \mathbb{R}$ such that (P_λ) has a nontrivial solution for any $\lambda > \lambda_0$.
(b) $\{\lambda \in \mathbb{R}: (P_\lambda) has\ a\ non-trivial\ solution\}$ is a dense subset of $\mathbb{R}$.
(c) For any continuous function $f: [1, 2] \to \mathbb{R}$ with $f(x) \neq 0$ for some $x \in [1, 2]$, there exists a solution u of (P_λ) for some $\lambda \in \mathbb{R}$ such that $\int_1^2 f\, u\, dx \neq 0$.
(d) There exists a $\lambda \in \mathbb{R}$ such that (P_λ) has two linearly independent solutions.
Answer: (c)

Solution: Set of all eigen value is countable.
So, the option (a) is not correct.
Also, It has no limit point. So, this set is never a dense subset of $\mathbb{R}$.
Option (b) is not correct.
For $\lambda \in \mathbb{R}$ there does not exist two linearly independent solutions.
Option (d) is not correct.
So, the option (c) is correct.

(3) The system of ODE
$\frac{dx}{dt} = (1 + x^2)y \,, t \in \mathbb{R}$
$\frac{dy}{dt} = -(1 + x^2)x \,, t \in \mathbb{R}$
$\left(x(0), y(0)\right) = (a, b)$
has a solution:
(a) Only if $(a, b) = (0\,, 0)$
(b) For any $(a, b) \in \mathbb{R} \times \mathbb{R}$
(c) Such that $x^2(t) + y^2(t) = a^2 + b^2$ for all $t \in \mathbb{R}$
(d) Such that $x^2(t) + y^2(t) \to \infty$ as $t \to \infty$ if $a > 0$ *and* $b > 0$.

Answer: (b), (c)

Solution: $\frac{dx}{dt} = (1 + x^2)y$
$\frac{dx}{y} = (1 + x^2)dt$ ------------------ (1)
$\frac{dy}{dt} = -(1 + x^2)x$
$\frac{dy}{dx} = -(1 + x^2)dt$ ----------------- (2)
From (a) and (b), $\frac{dy}{x} = -\frac{dx}{y}$
$x\,dx + y\,dy = 0$
Integrating, $x^2 + y^2 = c$
$x(0) = a, y(0) = b$
$\therefore c = a^2 + b^2$
$\therefore x^2(t) + y^2(t) = a^2 + b^2, \forall\, t \; and$
$=$ constant, any $(a, b) \in \mathbb{R}$
So, the option (b) and (c) are correct.

(4) Let $y: \mathbb{R} \to \mathbb{R}$ be a solution of the ODE $\frac{d^2y}{dx^2} - y = e^{-x} \,, x \in \mathbb{R}$
$y(0) = \frac{dy}{dx}(0) = 0$ then
(a) y attains its minimum on $\mathbb{R}$.
(b) y is bounded on $\mathbb{R}$.
(c) $\lim\limits_{x \to \infty} e^{-x}\; y(x) = \frac{1}{4}$
(d) $\lim\limits_{x \to -\infty} e^{x}\; y(x) = \frac{1}{4}$
Answer: (a), (c)
Solution: $A\,.E$ is $m^2 - 1 = 0$
$m = \pm 1$
So, $y(x) = c_1 e^x + c_2 e^{-x}$
$y(0) = 0, y'(0) = 0 \Rightarrow c_1 + c_2 = 0$
$c_1 - c_2 = 0$
$\therefore c_1 = c_2 = 0$
$\therefore P.I = \frac{1}{D^2-1} e^{-x} = e^{-x} \frac{1}{(D-1)^2-1} 1 = e^{-x} \frac{1}{D^2-2D} 1 = e^{-x} \frac{1}{-2D} \left(1 - \frac{D}{2}\right)^{-1} 1$
$= e^{-x} \frac{1}{-2D} 1 = -\frac{e^{-x}}{2} x$

$\therefore y = c_1 e^x + c_2 e^{-x} - \frac{e^{-x}.x}{2} = -\frac{x\,e^{-x}}{2}$
$\frac{dy}{dx} = -\frac{1}{2}[e^{-x} - x\,e^{-x}]$
$\frac{d^2y}{dx^2} = -\frac{1}{2}[-2e^{-x} + x\,e^{-x}]$
For max-min $\frac{dy}{dx} = 0 \Rightarrow x = 1$
$\therefore \frac{d^2y}{dx^2}\Big]_{x=1} = \dfrac{1}{2e} > 0$
So, y attains its minimum on $\mathbb{R}$.
y is not bounded.
So, option (b) is not correct.
Now, $\lim\limits_{x\to\infty} e^{-x}\ y(x) = \lim\limits_{x\to\infty} e^{-x}\cdot\left(-\frac{x}{2}e^{-x}\right)$
$= \lim\limits_{x\to\infty} \frac{-x\,e^{-2x}}{2} = \lim\limits_{x\to\infty} \frac{-x}{2e^{-2x}} = \frac{1}{4}$
$\lim\limits_{x\to-\infty} e^{x}\ y(x) = \lim\limits_{x\to-\infty} e^{x}\cdot\left(\frac{-x}{2}e^{x}\right) = \lim\limits_{x\to-\infty}\left(\frac{-x}{2}\right) \neq \frac{1}{4}$
So, the options (a) and (c) are correct.

(5) Let P, Q be continuous real valued functions defined on$[-1, 1]$ and $u_i: [-1, 1] \to \mathbb{R}, i = 1, 2$ be solutions of the ODE:

$$\frac{d^2u}{dx^2} + P(x)\frac{du}{dx} + Q(x)u = 0, x\,\epsilon[-1, 1]$$

Satisfying $u_1 \geq 0, u_2 \leq 0$ and
$u_1(0) = u_2(0) = 0$. Let w denote the Wronskian of $u_1\ and\ u_2$, then
(a) $u_1\ and\ u_2$ are linearly independent.
(b)$u_1 and u_2$ are linearly dependent.
(c) $w(x) = 0\ for\ all\ x\ \epsilon\ [-1, 1]$
(d) $w(x) \neq 0\ for\ some\ x\ \epsilon\ [-1, 1]$

Answer: (b) & (c)

Solution: $w(u_1\,, u_2)(0) = \begin{vmatrix} u_1(0) & u_2(0) \\ u'_1(0) & u'_2(0) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ u'_1(0) & u'_2(0) \end{vmatrix} = 0$
$\therefore w(u_1\,, u_2)(x) = \exp\left(-\int_0^x p\,(x)dx\right) w(u_1\,, u_2)(0) = 0\,, \forall\ x\ \epsilon[-1, 1]$
$\Rightarrow u_1, u_2$ are linearly dependent.
So, the options (b) and (c) are correct.

June – 2015
Part – B

1. Let $y(x)$ be a continuous solution of the initial value problem $y' + 2y = f(x), y(0) = 0$, where

$$f(x) = \begin{cases} 1\,, 0 \le x \le 1 \\ 0\ \ , x > 1 \end{cases}$$

Then $y\left(\frac{3}{2}\right)$ is equal to

(a) $\frac{\sin h(1)}{e^3}$

(b) $\frac{\cos h(1)}{e^3}$

(c) $\frac{\sin h(1)}{e^2}$

(d) $\frac{\cos h(1)}{e^2}$

Answer: (c)

Solution: when $0 \le x \le 1$

$\frac{dy}{dx} + 2y = 1 \Rightarrow y\, e^{2x} = \int 1. e^2\, dx + c,$

$y\, e^{2x} = \frac{e^{2x}}{2} + c,$

$y(0) = 0 \Rightarrow c_1 = -\frac{1}{2} \therefore y\, e^{2x} = \frac{1}{2}(e^{2x} - 1)$

$or, y = \frac{1}{2} - \frac{e^{-2x}}{2}$

When $x > 1$

$\frac{dy}{dx} + 2y = 0 \Rightarrow y\, e^{2x} = c_2$

$y = c_2 e^{-2x}$

For continuous solution $\lim_{x \to 1^-} y\,(x) = \lim_{x \to 1^+} y\,(x)$

$\Rightarrow \frac{1}{2} - \frac{e^{-2}}{2} = c_2 e^{-2}\ or, c_2 = \frac{e^2}{2} - \frac{1}{2}$

$\therefore y(x) = \frac{e^2 - 1}{2} e^{-2x}$

$\therefore y\left(\frac{3}{2}\right) = \frac{e^2 - 1}{2} e^{-3} = \frac{e - \frac{1}{e}}{2e^2} = \frac{\sin h(1)}{e^2}$

So, option (c) is correct.

2. The singular integral of the $ODE (x\, y' - y)^2 = x^2(x^2 - y^2)$ is.

(a) $y = x \sin x$

(b) $y = x \sin\left(x + \frac{\pi}{4}\right)$

(c) $y = x$

(d) $y = x + \frac{\pi}{4}$

Answer: (c)

Solution: $(xy' - y)^2 = x^2(x^2 - y^2)$

$or, (px - y)^2 = x^2(x^2 - y^2)\left[\frac{dy}{dx} = p\right]$

$or, x^2p^2 + y^2 - 2pxy - x^2(x^2 - y^2) = 0$
Here, $p -$ discriminant is $= 0$
$4x^2y^2 - 4x^2 \cdot \left(y^2 - x^2(x^2 - y^2)\right) = 0$
$or, 4x^2y^2 - 4x^2y^2 + 4x^4(x^2 - y^2) = 0$
$or, 4x^4(x^2 - y^2) = 0$
$\Rightarrow$ either $x = 0$ or $y = x \rightarrow$ Singular solution.
So, the option (c) is correct.

3. The initial value problem $y' = 2\sqrt{y}, y(0) = a$, has
(a) A unique solution is $a < 0$
(b) No solution if $a > 0$
(c) Infinitely many solutions is $a = 0$
(d) A unique solution if $a \geq 0$
Answer: (c)

Solution: $y' = 2\sqrt{y} or, \frac{dy}{\sqrt{y}} = 2\,dx$
Integrating, $\frac{1}{2} \cdot \frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = x + c$
$or, y^{\frac{1}{2}} = x + c\ or, y = (x + c)^2$
$y(0) =$ a gives $c^2 = a$.
$\Rightarrow c = \sqrt{a}$
If $a = 0$ then there exist infinitely many solutions.
So, the option (c) is correct.

Part – C
4. For the initial value problem $\frac{dy}{dx} = y^2 + \cos^2 x\,, x > 0, y(0) = 0$
The largest interval of existence of the solution predicted by Picard's theorem is
(a) $[0,1]$
(b) $\left[0, \frac{1}{2}\right]$
(c) $\left[0, \frac{1}{3}\right]$
(d) $\left[0, \frac{1}{4}\right]$

Answer: (b)

Solution: $R = \{(x, y) \epsilon \mathbb{R}^2 : |x - x_0| \leq a, |y - y_0| \leq b\}$
$h = \min\left\{a, \frac{b}{m}\right\}$
$|x - x_0| \leq h$
$m = \max f\,(x) = max(y^2 + \cos^2 x)$
$= 1 + b^2$
$h = \min\left\{a, \frac{b}{1+b^2}\right\} = \min\left\{a, \frac{1}{2}\right\}$
$= \frac{1}{2}\ if\ a \geq \frac{1}{2}$
$0 \leq |x - x_0| \leq a\ and\ |x - x_0| \leq h$

$\therefore |x| \le \frac{1}{2}$
$\therefore x \,\epsilon \left[0, \frac{1}{2}\right]$
So, option (b) is correct.

5. Let P be a continuous function on $\mathbb{R}$ and W the Wronskian of two linearly independent solutions y_1 and y_2 of the ODE:

$$\frac{d^2y}{dx^2} + (1+x^2)\frac{dy}{dx} + P(x)y = 0, x \,\epsilon\, R.$$

Let $w(1) = a, w(2) = b$ and $w(3) = c$, then
(a) $a < 0 \; and \; b > 0$
(b) $a < b < c \; or \; a > b > c$
(c) $\frac{a}{|a|} = \frac{b}{|b|} = \frac{c}{|c|}$
(d) $0 < a < b \; and \; b > c > c$

Answer: (b), (c)

Solution: $y'' + (1+x^2)y' + p(x)y = 0$
$W(x) = c \cdot e^{\int -(1+x^2)dx} = c\, e^{-x-\frac{x^3}{3}}$
$W(1) = a \Rightarrow c\, e^{\frac{-4}{3}} = a$
$W(2) = b \Rightarrow c\, e^{\frac{-14}{3}} = b$
$W(3) = c \Rightarrow c\, e^{-12} = c$
If $c > 0$ $a > b > c$ and if $c < 0$ then $a < b < c$.
Also, $\frac{a}{|a|} = \frac{b}{|b|} = \frac{c}{|c|} = \pm 1$
So, the option (b) and (c) are correct.

6. The critical point of the system $\frac{dx}{dt} = -4x - y, \frac{dy}{dt} = x - 2y$ is an
(a) Asymptotically stable node
(b) Unstable node
(c) Asymptotically stable spiral
(d) Unstable spiral

Answer: (a)

Solution: Characteristic equation for the matrix
$\begin{pmatrix} -4 & -1 \\ 1 & -2 \end{pmatrix}$ is $\lambda^2 - (-4-2)x + (8+1) = 0 \Rightarrow \lambda^2 + 6x + 9 = 0$
$\lambda = -3, -3$
$x = e^{-3t}, y = t\, e^{-3t}$
Distance function $D(t) = \lim_{t\to\infty} \sqrt{x^2+y^2} = \lim_{t\to\infty} e^{-3t}\sqrt{1+t^2} \to 0$
So, the critical point is asymptotically stable also it is a node. (eigen values are equal and same sign)
So, the option (a) is correct.

7. The function $G(x,\dot{\xi}) = \begin{cases} a + b\log\xi, 0 < x \le \xi \\ c + d\log x, \xi \le x \le 1 \end{cases}$ is a Green's function for $xy'' + y' = 0$, subject to y being bounded as $x \to 0$ and $y(1) = y'(1)$, if

(a) $a = 1, b = 1, c = 1, d = 1$
(b) $a = 1, b = 0, c = 1, d = 0$
(c) $a = 0, b = 1, c = 0, d = 1$
(d) $a = 0, b = 0, c = 0, d = 0$

Answer: Option are not correct.

Solution: $x\,y'' + y' = 0 \Rightarrow \frac{d}{dx}(xy') = 0$
Here, $P(x) = x$
$\therefore \left.\frac{da}{dx}\right]_{x=\xi^-}^{x=\xi^+} = -\frac{1}{P(\xi)}$
$\frac{d}{\xi} - 0 = -\frac{1}{\xi} \Rightarrow d = -1$
So, wrong question (Options are not correct)

December – 2015

1. Consider the system of ODE in $\mathbb{R}^2, \frac{dy}{dt} = AY, Y(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, t > 0$ where $A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ and $Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$.Then

(a) $y_1(t)\ and\ y_2(t)$ are monotonically increasing for $t > 0$
(b) $y_1(t)\ and\ y_2(t)$ are monotonically decreasing for $t > 1$
(c) $y_1(t)\ and\ y_2(t)$ are monotonically decreasing for $t > 0$
(d) $y_1(t)\ and\ y_2(t)$ are monotonically decreasing for $t > 1$

Answer: (d)

Solution:
$\begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow y'_1 = -y_1 + y_2\ and\ y'_2 = -y_2 \Rightarrow y_2 = c\,e^{-t}$
$y_2(0) = 1 \Rightarrow c = 1$
$\therefore y_2 = e^{-t}$
$\therefore y'_1 + y_1 = e^{-t}$
$\Rightarrow y_1 e^t = \int e^{-t}\,e^t dt + c' = t + c'$
Also, $y_1(0) = 0 \Rightarrow c' = 0$
$\therefore y_1 = t\,e^{-t}$
Now, $y'_1 = -y_1 + y_2 = -t\,e^{-t} + e^{-t} = -e^{-t(t-1)} \Rightarrow y'_1 < 0\ for\ t > 1$
So, $y_1\ and\ y_2$ are monotonically decreasing for $t > 1$.
$\therefore$ The option (d) is correct.

2. Consider the ODE on $\mathbb{R} y'(x) = f\big(y(x)\big)$. If f is an even function and y is an odd function, then
(a) $-y(-x)$ is also a solution.
(b) $y(-x)$ is also a solution.
(c) $-y(x)$ is also a solution.
(d) $y(x)\ y(-x)$ is also solution.

Answer: (a)

Solution: f is even $\Rightarrow f(-x) = f(x)\ \forall\ x \epsilon \mathbb{R}$
y is odd$\Rightarrow y(-x) = -y(x)\ \forall\ x \epsilon \mathbb{R}$
Let $g(x) = -y(-x)$
$g'(x) = y'(-x) = f\big(y(-x)\big) = f\big(-y(x)\big) = f\big(y(x)\big) = f\big(-y(-x)\big) = f\big(g(x)\big)$
So, $-y(-x)$ is a solution.
Let $g(x) = y(-x)$
$\therefore g'(x) = -y'(-x) = -f\big(y(-x)\big) = -f\big(-y(x)\big)$
$= -f\big(y(x)\big) = -f\big(-y(-x)\big) = -f\big(y(-x)\big) = -f\big(g(x)\big)$
So, $y(-x)$ is not a solution.
Similarly, $-y(x)$ and $y(x) \cdot y(-x)$ are not solutions (check).
So, the option (a) is correct.

3. Consider the boundary value problem $-u''(x) = \pi^2 u(x);\ x \epsilon (0,1)$,
$u(0) = u(1) = 0$. If $u\ and\ u'$ are considers on $[0,1]$, then
(a) $u^{12}(x) + \pi^2 u^2(x) = u^{12}(0)$
(b)$\int_0^1 u^{12}\,(x)dx - \pi^2 \int_0^1 u^2\,(x)dx = 0$
(c) $u^{12}(x) + \pi^2 u^2(x) = 0$
(d) $\int_0^1 u^{12}\,(x)dx - \pi^2 \int_0^1 u^2\,(x)dx = u^{12}(0)$

Answer: (a), (b)

Solution: $u'' + \pi^2 u = 0$
$u = c_1 \cos \pi x + c_2 \sin \pi x$
$u(0) = 0 \Rightarrow c_1 = 0$
$\therefore u = c_2 \sin \pi x$
$u(1) = 0 \Rightarrow 0 = c_2 \sin \pi$
For non-trivial solution $c_2 \neq 0$
$\therefore u = c_2 \sin \pi x$
$u' = \pi c_2 \cos \pi x$
$u'' = \pi^2 c_2 \cos \pi x$
$\therefore u^{12} + \pi^2 u^2$
$= \pi^2 c_2^2 \sin^2 \pi x + \pi^2 c_2^2 \sin^2 \pi x = c_2^2 \pi^2 = u^{12}(0)$
Now $\int_0^1 u^{12}\, dx - \pi^2 \int_0^1 u^2\, dx = \int_0^1 c_2^2\, \pi^2 \cos^2 \pi x\, dx - \pi^2 \int_0^1 c_2^2 \sin^2 \pi x\ \ dx$
$= c_2^2 \pi^2 \int_0^1 \cos 2\pi x\, dx = c_2^2 \pi^2 \left[\frac{\sin 2\pi x}{2\pi}\right]_0^1 = 0$
So, the options (a) and (b) are correct.

4. Let $u(t)$ be a continuously differentiable function taking nonnegative values for $t > 0$ and satisfying $u'(t) = 4u^{\frac{3}{4}}(t); u(0) = 0$. Then

(a) $u(t) = 0$

(b) $u(t) = t^4$

(c) $u(t) = \begin{cases} 0 \; for \; 0 < t < 1 \\ (t-1)^4 \; for \; t \geq 1 \end{cases}$

(d) $u(t) = \begin{cases} 0 \; for \; 0 < t < 10 \\ (t-10)^4 \; for \; t \geq 10 \end{cases}$

Answer: (a), (b), (c) and (d)

Solution: $u'(t) = 4u^{\frac{3}{4}}(t), u(0) = 0 \Rightarrow u^{-\frac{3}{4}}\, du = 4dt$

Integrating, $\frac{1}{4}\frac{u^{-\frac{3}{4}+1}}{-\frac{3}{4}+1} = t + c \Rightarrow u^{\frac{1}{4}} = t + c$

$u(0) = 0 \Rightarrow c = 0$

$\therefore u = t^4$

Also, $u(t) = (t-\gamma)^4 \; for \; t \geq \gamma$

So, all the options (a), (b), (c) and (d) are correct.

June – 2016

(1) Let $A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}, x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$ and $|x(t)| = \left(x_1^2(t) + x_2^2(t) + x_3^2(t)\right)^{\frac{1}{2}}$. Then any solution of the first order system of the ordinary differential equation

$x'(t) = A\, x(t)$

$x(0) = x_0$

Satisfies

(a) $\lim_{t\to\infty} |x(t)| = 0$

(b) $\lim_{t\to\infty} |x(t)| = \infty$

(c) $\lim_{t\to\infty} |x(t)| = 2$

(d) $\lim_{t\to\infty} |x(t)| = 12$

Answer: (a)

Solution: $x'(t) = \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\Rightarrow x'_1 = -2x_1 + x_2$

$x'_2 = -2x_2 + x_3$

$x'_3 = -2x_3$

Eigen values are $-2, -2, -2$

So, general solution is $x(t) = c_1x_1 + c_2x_2 + c_3x_3$

$= c_1u_1e^{-2t} + c_2(u_1 + u_2.t)e^{-2t} + c_3\left(u_1 + u_2.t + u_3.\frac{t^2}{2}\right)e^{-2t}$

$\therefore \lim_{t\to\infty} |x(t)| = 0$

$\therefore$ The option (a) is correct.

(2) Let y_1 and y_2 be two solutions of the problem.

$$y''(t) + a\,y'(t) + b\,y(t) = 0, t \in \mathbb{R}$$
$$y(0) = 0.$$

Where a and b are real constants. Let w be the Wronskian of y_1 and y_2.
Then,
(a) $w(t) = 0, \forall\, t \in \mathbb{R}$
(b) $w(t) = c, \forall\, t \in R$ for some positive constant c.
(c) w is a nonconstant positive function.
(d) There exists $t_1, t_2 \in \mathbb{R}$ such that $w(t_1) < 0 < w(t_2)$.

Answer: (a)
Solution: $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$.
$\therefore W(y_1, y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y'_1(0) & y'_2(0) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ y'_1(0) & y'_2(0) \end{vmatrix} = 0$
$W(y_1, y_2)(t) = exp\left(-\int_0^x a\,dx\right) W(y_1, y_2)(0) = 0, \forall\, t \in \mathbb{R}$
So, the option (a) is correct.

(3) Let $y: \mathbb{R} \to \mathbb{R}$ be a solution of the ordinary differential equation,

$$2y'' + 3y' + y = e^{-3x}, x \in \mathbb{R}$$

Satisfying, $\lim_{x\to\infty} e^x\, y(x) = 0$. Then
(a) $\lim_{x\to\infty} e^{2x}\; y(x) = 0$
(b) $y(0) = \frac{1}{10}$
(c) y is a bounded function on $\mathbb{R}$.
(d) $y(1) = 0$.

Answer: (a), (b)

Solution: $A.E$ is $2m^2 + 3m + 1 = 0$
$or, 2m^2 + 2m + m + 1 = 0$
$or, 2m(m+1) + 1(m+1) = 0$
$or, (2m+1)(m+1) = 0$
$m = -1, -\frac{1}{2}$.
$\therefore y_c(x) = c_1 e^{-x} + c_2 e^{-\frac{x}{2}}$
$P.I = \frac{1}{2D^2+3D+1} e^{-3x} = \frac{e^{-3x}}{2\cdot 9 - 9 + 1} = \frac{e^{-3x}}{10}$
$\therefore y(x) = c_1 e^{-x} + c_2 e^{-\frac{x}{2}} + \frac{e^{-3x}}{10}$
Given, $\lim_{x\to\infty} e^x \cdot \left[c_1 e^{-x} + c_2 e^{-\frac{x}{2}} + \frac{e^{-3x}}{10}\right] = 0 \Rightarrow c_1 = 0, c_2 = 0$
$\therefore y(x) = \frac{e^{-3x}}{10}$
$\therefore \lim_{x\to\infty} e^{2x}\; y(x) = 0$
Also, $y(0) = \frac{1}{10}$
So, the options (a) and (b) are correct.

(4) For $\lambda \epsilon \mathbb{R}$, consider the differential equation $y'(x) = \lambda \sin\big(x + y(x)\big), y(0) = 1$. Then this initial value problem has

(a) No solution in any neighborhood of 0.

(b) A solution in $\mathbb{R}$ if $|\lambda| < 1$.

(c) A solution in a neighborhood of 0.

(d) A solution in $\mathbb{R}$ only if $|\lambda| > 1$.

Answer: (b), (c)

Solution: $y'(x) = \lambda \sin\big(x + y(n)\big), y(0) = 1$

$\frac{dy}{dx} = f(x, y)$

$\therefore f(x, y) = \lambda \sin\big(x + y(x)\big)$

$\frac{\partial f}{\partial y} = \lambda \cos\big(x + y(x)\big) \leq \lambda$

$\therefore \frac{\partial f}{\partial y}$ is bounded

So, f satisfies Lipschitz condition. So, there exists a solution in a nbd of 0.

Also, a solution in$\mathbb{R}$ if $|\lambda| < 1$

Hence, the options (b) and (c) are correct.

(5) The problem

$$\left.\begin{array}{r} -y'' + (1 + x)y = \lambda y, x \epsilon (0,1) \\ y(0) = y(1) = 0 \end{array}\right\}$$

Has a non-zero solution

(a) For all $\lambda < 0$

(b) For all $\lambda \epsilon [0, 1]$

(c) For some $\lambda \epsilon (2, \alpha)$

(d) For a countable number of $\lambda' S$.

Answer: (c) & (d)

Solution:

$-y'' + (1 + x)y = \lambda y$

$y(0) = y(1) = 0$

$or, y'' = (1 + x - \lambda)y$

$or, y'' \cdot y = (1 + x - \lambda)y^2$

$\int_0^1 y''\, y\, dx = \int_0^1 (1 + x - \lambda)\, y^2 dx$

$[y'y]_0^1 - \int_0^1 y'\, y'\, dx = \int_0^1 (1 + x - \lambda)\, y^2 dx$

$-\int_0^1 y'^2\, dx = \int_0^1 (1 + x - \lambda)\, y^2 dx \Rightarrow 1 + x - \lambda < 0$

$or, \lambda > 1 + x$

$or, \lambda > 2$

So, the options (c) and (d) are correct.

December – 2016

(1) Let $(x(t), y(t))$ satisfy the system of $ODEs$ $\frac{dx}{dt} = -x + ty, \frac{dy}{dt} = tx - y$.
If $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are two solutions and $\Phi(t) = x_1(t)y_2(t) - x_2(t)y_1(t)$ then $\frac{d\Phi}{dt}$ is equal to

(a) -2Φ
(b) 2Φ
(c) $-\Phi$
(d) Φ

Answer: (a)

Solution: $x'(t) = -x + ty$
$y'(t) = tx - y$
$\Phi(t) = x_1(t)y_2(t) - x_2(t)y_1(t)$
$\Rightarrow \Phi'(t) = x'_1(t)y_2(t) + x_1(t)y'_2(t) - x'_2(t)\cdot y_1(t) - x_2(t)y'_1(t)$
$= (-x_1 + ty_1)y_2 + x_1(tx_2 - y_2) - y_1\cdot(-x_2 + ty_2) - x_2(tx_1 - y_1)$
$= -x_1y_2 + ty_1y_2 + t\,x_1x_2 - x_1y_2 + x_2y_1 - ty_1y_2 - t\,x_1x_2 + x_2y_1$
$= -2(x_1y_2 - x_2y_1)$
$= -2\Phi$
So, option (a) is correct.

(2) The boundary value problem $x^2y'' - 2xy' + 2y = 0$, subject to the boundary conditions.
$y(1) + \alpha\, y'(1) = 1, y(2) + \beta\, y'(2) = 2$ has a unique solution if

(a) $\alpha = -1, \beta = 2$
(b) $\alpha = -1, \beta = -2$
(c) $\alpha = -2, \beta = 2$
(d) $\alpha = -3, \beta = \frac{2}{3}$

Answer: (a)

Solution: Let $x = e^z$
Then $x^2y'' = \theta(\theta - 1)y$
$x\,y' = \theta\,y$ $\quad\left[\theta \equiv \frac{d}{dz}\right]$
$\therefore (\theta(\theta - 1) - 2\theta + 2)y = 0$
$(\theta^2 - 3\theta + 2)y = 0$
$A.E$ is $m^2 - 3m + 2 = 0 \;\; \Rightarrow m = 1, 2$
$C.F = c_1e^z + c_2e^{2z} \Rightarrow y(x) = c_1x + c_2x^2$
$y'(x) = c_1 + 2c_2x$
$\therefore y(1) + \alpha y'(1) = 1 \Rightarrow c_1 + c_2 + \alpha\,(c_1 + 2c_2) = 1$
$or, c_1(1 + \alpha) + c_2(1 + 2\alpha) = 1$ ------------------ (1)
Also, $y(2) + \beta\, y'(2) = 2 \Rightarrow 2c_1 + 4c_2 + \beta(c_1 + 4c_2) = 2$
$or, c_1(2 + \beta) + c_2(4 + 4\beta) = 2$ ------------------- (2)
From (a) and (b)
$\begin{pmatrix} 1+\alpha & 1+2\alpha \\ 2+\beta & 4+4\beta \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
For unique solution $\begin{vmatrix} 1+\alpha & 1+2\alpha \\ 2+\beta & 4+4\beta \end{vmatrix} \neq 0$
$or, 4 + 4\alpha + 4\beta + 4\alpha\beta - 2 - 4\alpha - \beta - 2\alpha\beta \neq 0$
$or, 2 + 3\beta + 2\alpha\beta \neq 0$

If $\alpha = -1, \beta = 2$
$2 + 3 \cdot 2 + 2 \cdot (-1) \cdot 2 = 4 \neq 0$
Option (a) is correct.
If $\alpha = -1, \beta = -2$
$2 + 3 \cdot (-2) + 2 \cdot (-1) \cdot (-2) = 0$
Option (b) is not correct.
If $\alpha = -2 \ \ \beta = 2$
$2 + 3 \cdot 2 + 2 \cdot (-2) \cdot 2 = 0$
Option (3) is not correct.
Similarly, the option (d) is not correct.
Hence the option (a) is correct.

(3) Let $x: [0, 3\pi] \to \mathbb{R}$ be a non-zero solution of the ODE
$x''(t) + e^{t^2} x(t) = 0, for\ t\ \epsilon [0, 3\pi]$.
Then the cardinality of the set
$\{t\ \epsilon [0, 3\pi]: x(t) = 0\}$ is
(a) Equal to 1
(b) Greater than or equal to 2
(c) Equal to 2
(d) Greater than or equal to 3
Answer: (b), (d)
Solution: $x(t)$ be a non-zero solution of the $ODE, x'' + e^{t^2} \cdot x = 0$
Then $e^{t^2} > 0\ and\ \int_1^{\infty} e^{t^2}\, dt = \alpha$
$\Rightarrow x(t)$ has infinite number of zeros.
So, the options (b) and (d) are correct.

(4) Consider the initial value problem
$y'(t) = f\big(y(t)\big), y(0) = a\ \epsilon\ \mathbb{R}$ where $f: \mathbb{R} \to \mathbb{R}$
Which of the following statements are necessarily true?
(a) There exists a continuous function $f: \mathbb{R} \to \mathbb{R}$ and $a\ \epsilon\ \mathbb{R}$ such that the above problem does not have a solution in any neighborhood of 0.
(b) The problem has a unique solution for every $a\ \epsilon\ \mathbb{R}$ when f is Lipschitz continuous.
(c) When f is twice continuously differentiable, the maximal interval of existence for the above initial value problem is $\mathbb{R}$.
(d) The maximal interval of existence for the above problem is $\mathbb{R}$ when f is bounded and continuously differentiable.
Answer: (b) & (d)
Solution: If f is Lipschitz continuous, then the IVP has a unique solution.
If f is bounded and continuously differentiable then there exists maximal interval.
So, the options (b) and (d) are correct.

(5) Let $\left(x(t), y(t)\right)$ satisfy for $t > 0\, \frac{dx}{dt} = -x + y, \frac{dy}{dt} = -y, x(0) = y(0) = 1.$
Then $x(t)$ is equal to
(a) $e^{-t} + t\, y(t)$
(b) $y(t)$
(c) $e^{-t}(1+t)$
(d) $-y(t)$
Answer: (a) & (c)
Solution: $\frac{dx}{dt} = -x + y$
$\frac{dy}{dt} = -y$
$or, \frac{dy}{y} = -dt$
Integrating, $y = c_1 e^{-t}$
$y(0) = 1 \Rightarrow 1 = c_1$
$\therefore y = e^{-t}$
$\therefore \frac{dx}{dt} = -x + e^{-t}$
$or, \frac{dx}{dt} + x = e^{-t}$
$I.F = e^{\int 1\, dt} = e^{t}$
$\therefore \frac{d}{dt}(x \cdot e^{t}) = e^{-t} \cdot e^{t} = 1$
Integrating, $x \cdot e^{t} = t + D$
$x(0) = 1 \Rightarrow 1 = D$
$\therefore x = (1+t)e^{-t}$
Clearly, $x(t) = (1+t)e^{-t}$ and also $x(t) = e^{-t} + ty(t)$
Hence, the options (a) and (c) are correct.

June – 2017

(1) Consider the solution of the ordinary differential equation $y'(t) = -y^3 + y^2 + 2y$
Subject to $y(0) = y_0 \epsilon (0, 2)$ Then
$\lim\limits_{t \to \infty} y(t)$ belongs to
(a) $(-1, 0)$
(b) $(-1, 2)$
(c) $(0, 2)$
(d) $(0, \alpha)$
Answer: (a), (b) and (c)
Solution: $y'(t) = -y^3 + y^2 + 2y$
$or, \frac{dy}{y^3-y^2-2y} = -dt$
$or, \frac{dy}{y(y^2-y-2)} = -dt$
$or, \int \frac{dy}{y(y-2)(y+1)} = \int -dt$
$\Big[let\ \frac{1}{y(y-2)(y+1)} = \frac{A}{y} + \frac{B}{y+1} + \frac{C}{y-2} = \frac{A(y^2-y-2)+B(y^2-2y)+C(y^2+y)}{y(y+1)(y-2)}$
$\Rightarrow A + B + C = 0$
$-A - 2B + C = 0$
$-2A = 1\ or, A = -\frac{1}{2}$
$\therefore B+C=\frac{1}{2}$
$2B-C=\frac{1}{2}$
$3B=1$
$B=\frac{1}{3}$
$\therefore C = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}\Big]$
$= \int -\frac{1}{2}\cdot\frac{1}{y}dy + \int \frac{1}{3}\cdot\frac{1}{y+1}dy + \int \frac{1}{6}\cdot\frac{dy}{y-2} = -t + D$
$or, -\frac{1}{2}\log|y| + \frac{1}{3}\log|y+1| + \frac{1}{6}\log|y-2| = -t + D$
$\Rightarrow \frac{(y-2)^{\frac{1}{6}}\cdot(y+1)^{\frac{1}{3}}}{y^{\frac{1}{2}}} = e^{-t}e^{D}$
As $t \to \infty$
$\frac{(y-2)^{\frac{1}{6}}(y+1)^{\frac{1}{3}}}{\sqrt{y}} = 0 \quad or, (y-2)^{\frac{1}{6}}(y+1)^{\frac{1}{3}} = 0$
$\Rightarrow y = 2\ or, -1$
Clearly the options (a), (b) and (c) are correct.

December – 2017

(1) Consider the differential equation $(x-1)y'' + xy' + \frac{1}{x}y = 0$. Then

(a) $x = 1$ is the only singular point.
(b) $x = 0$ is the only singular point.
(c) Both $x = 0$ and $x = 1$ are singular points.
(d) Neither $x = 0$ nor $x = 1$ are singular points.

Answer: (c)

Solution: $y'' + \frac{x}{x-1}y' + \frac{1}{x(x-1)}\ y = 0$. Clearly $x = 0$ and $x = 1$ are both singular points.
So, the option (c) is correct.

(2) The set of real numbers λ for which the boundary value problem.
$\frac{d^2y}{dx^2} + \lambda y = 0, y(0) = 0, y(\pi) = 0$ has non-trivial solutions in

(a) $(-\infty, 0)$
(b) $\{\sqrt{n}\ |n\ is\ a\ positive\ integer\}$
(c) $\{n^2\ |n\ is\ a\ possitive\ integer\}$
(d) $\mathbb{R}$

Answer: (c)

Solution: For $\lambda = 0, y = c_1x + c_2$
$y(0) = 0 \Rightarrow c_2 = 0$
$y(\pi) = 0 \Rightarrow c_1\pi = 0 \Rightarrow c_1 = 0$
$\therefore y = 0$
This is a trivial solution
When $\lambda > 0\ , let\ \lambda = k^2, k\epsilon\mathbb{R}$
$y'' + k^2\ y = 0$
$\Rightarrow y = c_1 \cos kx + c_2 \sin kx$
For $\lambda < 0\ , Let\ \lambda = -k^2, k\ \epsilon\mathbb{R}$
$y'' - k^2y = 0$
$y = c_1e^{kx} + c_2e^{-kx}$
$y(0) = 0 \Rightarrow c_1 + c_2 = 0$
$y(\pi) = 0 \Rightarrow c_1e^{k\pi} + c_2e^{-k\pi} = 0$
$or, -c_2e^{k\pi} + c_2e^{-k\pi} = 0$
$or, c_2(e^{-k\pi} - e^{k\pi}) = 0$
$\Rightarrow c_2 = 0$
$\therefore c_1 = 0$
$\therefore y = 0$
This is also a trivial solution.
When $\lambda > 0$. We get the non-trivial solution as $y = c_1 \cos\sqrt{\lambda}\ x + c_2 \sin\sqrt{\lambda}\ x$
$y(0) = 0 \Rightarrow c_1 = 0$
$y(\pi) = 0 \Rightarrow c_2 \sin\sqrt{\lambda}\ \pi = 0 = \sin n\ \pi$
$\Rightarrow \sqrt{\lambda}\,\pi = n\pi$
$\lambda = n^2, n\ \epsilon\mathbb{N}$
So, the option (c) is correct.

(3) Consider a system of first order differential equations $\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x(t)+y(t) \\ -y(t) \end{bmatrix}$.

The solution space is spanned by

(a) $\begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$ and $\begin{bmatrix} e^t \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} e^t \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \cos h\, t \\ e^{-t} \end{bmatrix}$

(c) $\begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix}$ and $\begin{bmatrix} \sin h\, t \\ e^{-t} \end{bmatrix}$

(d) $\begin{bmatrix} e^t \\ 0 \end{bmatrix}$ and $\begin{bmatrix} e^t - \frac{1}{2}e^{-t} \\ e^{-t} \end{bmatrix}$

Answer: (c), (d)

Solution: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Eigen values are $1, -1$

Eigen vector for $\lambda = -1$

$(A - \lambda I)y = 0$

$\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$2y_1 + y_2 = 0$

$\lambda = 1$

$(A - \lambda I)x = 0$

$\begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x_2 = 0$

General solution is

$y = c_1 u_1 e^{\lambda_1 t} + c_2 u_2 e^{\lambda_2 t}$

$= c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$= c_1 \begin{pmatrix} e^t \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$

So, the options (c) and (d) are correct.

(4) Consider the differential equation $\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} - y = 0$ defined on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Which among the following are true?

(a) There is exactly one solution $y = y(x)$ with $y(0) = y'(0) = 1$ and $y\left(\frac{\pi}{3}\right) = 2\left(1 + \frac{\pi}{3}\right)$.

(b) There is exactly one solution $y = y(x)$ with $y(0) = 1, y'(0) = -1$ and $y\left(-\frac{\pi}{3}\right) = 2\left(1 + \frac{\pi}{3}\right)$.

(c) Any solution $y = y(x)$ satisfies $y''(0) = y(0)$

(d) If y_1 and y_2 are any two solutions then $(ax + b)y_1 = (cx + d)y_2$ for some $a, b, c, d \,\epsilon \mathbb{R}$.

Answer: (a), (b), (c) and (d)

Solution: $y'' - 2\tan x\, y' - y = 0$

$\cos x\, y'' - 2\sin x\, y' - y\cos x = 0$

$(y'\cos x)^1 - (y\sin x)^1 = 0$

Integrating, $y'\cos x - y\sin x = c_1$

$(y - \cos x)^1 = c_1$

Integrating, $y\cos x = c_1 x + c_2$

$or, y = \sec x\,(c_1 x + c_2)$

$y(0) = 1 \Rightarrow c_2 = 1$

$y'(0) = 1 \Rightarrow 1 = c_1$
$y(x) = \sec x \cdot (x+1)$
$y\left(\frac{\pi}{3}\right) = \sec\frac{\pi}{3}\left(\frac{\pi}{3}+1\right) = 2\left(1+\frac{\pi}{3}\right)$
So, option (a) is correct.
Similarly, option (b) is correct.
$y'(x) = \sec x + \sec x \tan x\,(x+1)$
$y''(x) = \sec x \cdot \tan x + \sec x \tan x + (x+1)[\sec^3 x + \sec x \tan^2 x]$
$\therefore y''(0) = 0 + 0 + 1 = 1$
$\therefore y''(0) = y(0)$
So, option (c) is correct.
$y_1 = \sec x\,(c_1 x + c_2)$
$y_2 = \sec x\,(k_1 x + k_2)$
So, $\frac{y_1}{y_2} = \frac{c_1 x + c_2}{k_1 x + k_2} \quad \Rightarrow (k_1 x + k_2) y_1 = (c_1 x + c_2) y_2$
So, option (d) is correct.
Hence, all the four options are correct.

(5) Consider a boundary value problem $(BVP)\,\frac{d^2y}{dx^2} = f(x)$ with boundary conditions $y(0) = y(1) = y'(1)$, where f is a realvalued continuous function on $[0,1]$. Then which of the following are true?
(a) The given BVP has a unique solution for every f.
(b) The given BVP does not have a unique solution for some f.
(c) $y(x) = \int_0^x x\,t\,f(t)dt + \int_x^1 (t - x + xt)\,f(t)dt$ is a solution of the given BVP.
(d)$y(x) = \int_0^x (x - t + xt)\,f(t)dt + \int_x^1 xt\,f(t)dt$ is a solution of the given BVP.
Answer: (a), (c)
Solution: $\frac{d^2y}{dx^2} = f(x), y(0) = y(1) = y'(1)$
$\int_0^x y''\,dx = \int_0^x f\,(x)dx$
$y'(x) - y'(0) = \int_0^x f\,(x)dx$
Also, $\int_0^1 y''\ dx = \int_0^1 f\,(x)dx$
$y'(1) - y'(0) = \int_0^1 f\,(x)dx$
$\therefore y'(x) - y'(1) + \int_0^1 f\,dx = \int_0^x f\,dx$
$or, \int_0^x y'\,(x)dx - y'(1)\int_0^x dx = \int_0^x \int_0^x f\,dx\,dx - \int_0^1 f\,dx \int_0^x 1\,dx$
$y(x) - y(0) - y'(1)\cdot x = \int_0^x \int_0^x f\,dx\,dx - \left(\int_0^1 f\,dx\right)\cdot x$
So, given BVP has a unique solution.
$y(x) = \int_0^x x\,t\,f(t)dt + \int_x^1 (t - x + xt)\,f(t)dt$
$y(0) = \int_0^1 t\,f(t)dt$
$y(1) = \int_0^1 t\,f(t)dt$
$y'(x) = \int_0^x t\,f(t)dt + x^2 f(x) + \int_x^1 (-1+t)\,f(t)dt - x^2 f(x)$
$y'' = x\,f(x) - (-1+x)\,f(x)\cdot 1$
$= f(x)$
So, option (c) is correct.
Hence, the options (a) and (c) are correct.

June – 2018

(1) Consider the ordinary differential equation $y' = y(y-1)(y-2)$.
Which of the following statement is true?
(a) If $y(0) = 0.5$ then y is decreasing.
(b) If $y(0) = 1.2$ then y is increasing.
(c) If $y(0) = 2.5$ then y is unbounded.
(d) If $y(0) < 0$ then y is bounded below.
Answer: (c)
Solution: $\frac{dy}{\boldsymbol{dx}} = y(y-1)(y-2)$
$\frac{dy}{y(y-1)(y-2)} = dx$
$[Let\,\frac{1}{y(y-1)(y-2)} = \frac{A}{y} + \frac{B}{y-1} + \frac{C}{y-2}$ $= \frac{A(y-1)(y-2)+B\,y(y-2)+C\,y(y-1)}{y(y-1)(y-2)} = \frac{A(y^2-3y+2)+B(y^2-2y)+C(y^2-y)}{y(y-1)(y-2)}$
$\Rightarrow A + B + C = 0$
$-3A - 2B - C = 0$
$2A = 1\; or, A = \frac{1}{2}$

$$\begin{array}{l} \therefore B+C=-\frac{1}{2} \\ \underline{2B+\underline{C}=+\frac{-3}{2}} \\ -B=1 \quad or, B=-1 \end{array}$$

$C = -\frac{1}{2} += \frac{1}{2}]$
$\int \frac{1}{2}\cdot\frac{1}{y}\,dy - \int \frac{1}{y-1}dy + \int \frac{1}{2}\cdot\frac{1}{y-2}dy = x + D$
$\frac{1}{2}\log y - \log(y-1) + \frac{1}{2}\log(y-2) = x + D$
$\Rightarrow \log\left[\frac{\sqrt{y(y-2)}}{y-1}\right] = x + D \Rightarrow y \neq 1, y(y-2) > 0$
$y > 0, y - 2 > 0$ i.e., $y > 0, y > 2 \quad \Rightarrow y > 2$
$or, y < 0, y - 2 < 0$
$y < 0, y < 2 \qquad \Rightarrow y < 0$
Also, $y - 1 > 0$ i.e., $y > 1$
So, the option (c) is correct.

(2) Consider the ordinary differential equation $y'' + P(x)y' + Q(x)y = 0$, where $P\; and\; Q$ are smooth functions. Let $y_1\; and\; y_2$ be any two solutions of the ODE. Let $w(x)$ be the corresponding Wronskian. Then which of the following is always true?
(a) If $y_1\; and\; y_2$ are linearly dependent then $\exists\; x_1, x_2$ such that $w(x_1) = 0$ and $w(x_2) \neq 0$.
(b) If $y_1\; and\; y_2$ are linearly independent then $w(x) = 0\; \forall\; x$.
(c) If $y_1\; and\; y_2$ are linearly independent then $w(x) \neq 0, \forall\; x$.
(d) If $y_1\; and\; y_2$ are linearly independent then $w(x) \neq 0\; \forall\; x$.
Answer: (d)
Solution: $y_1\; and\; y_2$ are linearly independent then $w(x) \neq 0\; \forall\; x$.
So, the option (d) is correct.

(3) Consider the Sturm-Liouville problem $y'' + \lambda y = 0, y(0) = 0 \ and \ y(\pi) = 0$. Which of the following statements are true?

(a) There exist only countably many characteristic values.

(b) There exist uncountably many characteristic values.

(c) Each characteristic function corresponding to the characteristic value λ has exactly $[\sqrt{\lambda}] - 1$ zeros in $(0, \pi)$.

(d) Each characteristic function corresponding to the characteristic value λ has exactly $[\sqrt{\lambda}]$ zeros in $(0, \pi)$.

Answer: (a), (c)

Solution:

Case – I If $\lambda = 0$

Then $y = c_1 + c_2 x$

$y(0) = 0, y(\pi) = 0 \Rightarrow c_1 = c_2 = 0$

This is a trivial solution.

Case – II

If $\lambda < 0, \lambda = -k^2$

$\therefore y = c_1 e^{kx} + c_2 e^{-kx}$

$y(0) = 0, y(\pi) = 0 \Rightarrow c_1 = c_2 = 0$

This is also a trivial solution.

Case – III

$\lambda > 0, \lambda = k^2$

$y = c_1 \cos kx + c_2 \sin kx$

$y(0) = 0 \Rightarrow c_1 = 0$

$y(\pi) = 0 \Rightarrow 0 = c_2 \sin k\pi$

For non-trivial solution $c_2 \neq 0$

$\Rightarrow \sin k\pi = 0 = \sin n\pi \Rightarrow k\pi = n\pi$

$k = n$

$\sqrt{\lambda} = n$

$\lambda = n^2, n \epsilon \mathbb{N}$

Corresponding eigen function is $y_n(x) = \sin n x$ for $\lambda_n = n^2$.

$\exists$ countable many characteristic values.

$\lambda_n = n^2, y_n(x) = \sin nx$

$n = 1 \ \lambda_1 = 1^2, y_1(x) = \sin x$

$n = 2\lambda_2 = 2^2, y_2(x) = \sin 2x$

$n = 3 \ \lambda_3 = 3^2, y_3(x) = \sin 3x$

So, characteristics function has $[\sqrt{\lambda}] - 1$ has zeros in $(0, \pi)$

Hence, the options (a) and (c) are correct.

$[Since \ \sin x \ has \ no \ zeros, \sin 2x \ has \ one \ zeros, \sin 3x \ has \ two \ zeros \ and \ so \ on]$

(4) Consider the system of differential equations
$\frac{dx}{dt} = 2x - 7y$
$\frac{dy}{dt} = 3x - 8y$
Then the critical point $(0, 0)$ of the system is an
(a) Asymptotically stable node.
(b) Unstable node.
(c) Asymptotically stable spiral.
(d) Unstable spiral.
Answer: (a)
Solution: $A = \begin{bmatrix} 2 & -7 \\ 3 & -8 \end{bmatrix}$
$|A| = \begin{vmatrix} 2 & -7 \\ 3 & -8 \end{vmatrix} = -16 + 21 = 5$
Characteristic equation
$or, |A - \lambda I| = 0 \;\; or, \begin{vmatrix} 2-\lambda & -7 \\ 3 & -8-\lambda \end{vmatrix} = 0$
$or, \lambda^2 - (2-8)\lambda + 5 = 0$
$\lambda^2 + 6\lambda + 5 = 0 \Rightarrow \lambda = -1, -5$
Both are real, distinct and have same sight.
So, the critical point is a node, both are negative. So, the critical point is asymptotically stable node.
Hence, the option (a) is correct.

(5) Assume that $a: [0, \alpha) \to \mathbb{R}$ is a continuous function. Consider the ordinary differential equation. $y'(x) = a(x)\, y(x), x > 0, y(0) = y_0 \neq 0$. Which of the following statements are true?
(a) If $\int_0^\alpha |a(x)|\, dx < \alpha, then\ y$ is bounded.
(b) If $\int_0^\alpha |a(x)|\, dx < \alpha$, then $\lim_{x\to\alpha} y\,(x)$ exists.
(c) If $\lim_{x\to\alpha} a\,(x) = 1$, then $\lim_{x\to\alpha} |y(x)| = \alpha$.
(d) If $\lim_{x\to\alpha} a\,(x) = 1$, then y is monotone.
Answer: (a), (b) and (c)
Solution: Options (a), (b) and (c) are correct.

December – 2018

(1) If $y_1(x)$ *and* $y_2(x)$ are two solutions of the differential equation
$(\cos x)y'' + (\sin x)y' - \left(1 + e^{-x^2}\right)y = 0\ \forall\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, with $y_1(0) = \sqrt{2}$
$y'_1(0) = 1, y_2(0) = -\sqrt{2}, y'_2(0) = 2.$
Then the Wronskian of $y_1(x)$ *and* $y_2(x)$ at $x = \frac{\pi}{4}$ is

(a) $3\sqrt{2}$
(b) 6
(c) 3
(d) $-3\sqrt{2}$

Answer: (c)

Solution: $w(y_1, y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y'_1(0) & y'_2(0) \end{vmatrix} = \begin{vmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 2 \end{vmatrix} = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2}$

$\therefore w(x) = exp\left(-\int_0^x p_1\, dx\right) \cdot w(y_1, y_2)(0)$

$= exp\left(-\int_0^x \frac{\sin x}{\cos x} dx\right) \cdot 3\sqrt{2} = exp(\log\cos x) \cdot 3\sqrt{2} = \cos x \cdot 3\sqrt{2}$

$\therefore w\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} \cdot 3\sqrt{2} = \frac{1}{\sqrt{2}} \cdot 3\sqrt{2} = 3$

So, the option (c) is correct.

(2) The critical point $(0, 0)$ for the system $x'(t) = x - 2y + y^2 \sin(x)$
$y'(t) = 2x - 2y - 3y\cos(y^2)$ is a

(a) Stable spiral point.
(b) Unstable spiral point.
(c) Stable point.
(d) Stable node.

Answer: (c)

Solution: $x'(t) = x - 2y + y^2 \sin x = F$

$y'(t) = 2x - 2y - 3y\cos(y^2) = G$

$u = x - x_0 = x - 0 = x$

$u = y - y_0 = y - 0 = y$

$\begin{pmatrix} u' \\ u' \end{pmatrix} = \begin{pmatrix} F_x(0,0) & F_y(0,0) \\ G_x(0,0) & G_1(0,0) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$

$A = \begin{pmatrix} 1 & -2 \\ 2 & -5 \end{pmatrix}, |A| = -5 + 4 = -1$

Characteristic equation is

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1 - \lambda & -2 \\ 2 & -5 - \lambda \end{vmatrix} = 0$

$or, \lambda^2 - (1 - 5)\lambda + (-1) = 0$

$or, \lambda^2 + 4\lambda - 1 = 0$

$\lambda = \frac{-4 \pm \sqrt{16+4}}{2} = \frac{-4 \pm 2\sqrt{5}}{2} = -2 \pm \sqrt{5}$

Eigen values are real, distinct and have opposite sign. So, the critical point is a saddle point.
So, option (c) is correct.

(3) Three solutions of a certain second order non homogeneous linear differential equation are

$$y_1(x) = 1 + xe^{x^2}, y_2(x) = (1+x)e^{x^2} - 1, y_3(x) = 1 + e^{x^2}$$

Which of the following is (are) general solutions of the differential equation?

(a) $(c_1 + 1)y_1 + (c_2 - c_1)y_2 - c_2 y_3$ where c_1 and c_2 are arbitrary constants.

(b) $c_1(y_1 - y_2) + c_2(y_2 - y_3)$, where c_1 and c_2 are arbitrary constants.

(c) $c_1(y_1 - y_2) + c_2(y_2 - y_3) + c_3(y_3 - y_1)$, where y_1 and y_2 are arbitrary constant.

(d) $c_1(y_1 - y_3) + c_2(y_3 - y_2) + y_1$, where c_1 and c_2 are arbitrary constants.

Answer: (a), (d)

Solution: $y'' + a_1(x)y' + a_2(x)\, y = r(x)$

y_1, y_2, y_3 are three solutions the general solution is $y = c_1 y_1 + c_2 y_2 + c_3 y_3 \Rightarrow c_1 + c_2 + c_3 = 1$

Now, for (1) $c_1 + 1 + c_2 - c_1 - c_2 = 1$

for (2) $c_1 + c_2 - c_1 - c_2 = 0$

for (3) $c_1 - c_3 + c_2 - c_1 + c_3 - c_2 = 0$

for (4) $c_1 + 1 - c_2 + c_2 - c_1 = 1$

$\Rightarrow$ (a) and (d) are the general solutions.

Hence the options (a) and (d) are correct.

(4) The method of variation of parameters to solve the differential equation $y'' + p(x)y' + q(x)y = r(x)$. Where $x \epsilon I$ and $p(x), q(x), r(x)$ are non-zero continuous functions on an interval I, seeks a particular solution of the form $y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$, where y_1 and y_2 are linearly independent solutions of $y'' + p(x)y' + q(x)y = 0$ and $v_1(x)$ and $v_2(x)$ are functions to be determined. Which of the following statements are necessarily true?

(a) The Wronskian of y_1 and y_2 is never zero in I.

(b) v_1, v_2 and $v_1 y_1 + v_2 y_2$ are twice differentiable.

(c) v_1 *and* v_2 may not be twice differentiable, but $v_1 y_1 + v_2 y_2$ is twice differentiable.

(d) The solution set of $y'' + p(x)y' + q(x)y = r(x)$ consists of functions of the form $ay_1 + by_2 + yp, a, b \epsilon \mathbb{R}$ constants.

Answer: (a), (c) and (d)

Solution: $y'' + py' + qy = r(x)$

$y_p(x) = v_1(x)y_1 + v_2(x)y_2 = -y_1 \int \frac{y_2\, r(x)}{w} dx + y_2 \int \frac{y_1\, r(x) dx}{w}$

$\Rightarrow w(y_1, y_2)$ is never zero in I.

Option (a) is correct.

Here v_1 *and* v_2 may not be twice differentiable but $v_1 y_1 + v_2 y_2$ is twice differentiable.

So, option (c) is correct.

General solution is $y = y_c + y_p = ay_1 + by_2 + y_p \Rightarrow$ Option (d) is correct.

Hence, the options (a), (c) and (d) are correct.

(5) Consider the eigen value problem

$y'' + \lambda y = 0$ for $x \epsilon (-1,1)$

$y(-1) = y(1)$

$y'(-1) = y'(1)$

Which of the following statements are true?

(a) All eigenvalues are strictly positive.

(b) All eigenvalues are non-negative.

(c) Distinct eigen Functions are orthogonal in $L^2[-1, 1]$.

(d) The sequence of eigenvalues is bounded above.

Answer: (b), (c)

Solution: For $\lambda = 0$ we get the trivial solution.

$y'' \cdot y = -\lambda y^2$

$or, \int_{-1}^{1} y'' \cdot y \, dx = -\lambda \int_{-1}^{1} y^2 \, dx$

$or, y' \cdot y]_{-1}^{1} - \int_{-1}^{1} y' \, y' dx = -\lambda \int_{-1}^{1} y^2 \, dx$

$or, y'(1) \, y(1) - y'(-1) \, y(-1) - \int_{-1}^{1} y'^2 \; dx = -\lambda \int_{-1}^{1} y^2 \, dx$

$or, \int_{-1}^{1} y^{12} \, dx = \lambda \int_{-1}^{1} y^2 \, dx$

If $y' = 0 \Rightarrow y'' = 0 \;\; \Rightarrow y = c$

$\lambda y = 0 \Rightarrow \lambda = 0 \; or, y = 0$

But, $y \neq 0$

$\therefore \lambda = 0$

$y' \neq 0 \Rightarrow \lambda = 0$

So, all the eigen values are non-negative.

For $\lambda > 0, y = c_1 \cos \sqrt{\lambda} \, x + c_2 \sin \sqrt{\lambda} \; x$

$y(-1) = y(1) \Rightarrow c_1 \cos \sqrt{\lambda} - c_2 \sin \sqrt{\lambda} = c_1 \cos \sqrt{\lambda} + c_2 \sin \sqrt{\lambda}$

$\Rightarrow 2c_2 \sin \sqrt{\lambda} = 0 \Rightarrow c_2 = 0 or, \lambda = n^2\pi^2$

$c_2 = 0$

$y = c_1 \cos \sqrt{\lambda} \, x$

$y' = -c_1 \sqrt{\lambda} \sin \sqrt{\lambda} \, x$

$y'(-1) = y'(1) \Rightarrow c_1 \sqrt{\lambda} \sin \sqrt{\lambda} = -c_1 \sqrt{\lambda} \sin \sqrt{\lambda} \, 2c_1 \sqrt{\lambda} \sin \sqrt{\lambda} = 0$

$\Rightarrow \begin{matrix} c_1 \neq 0 \\ \lambda > 0 \end{matrix} \Big/ \begin{matrix} \therefore \sin \sqrt{\lambda} = 0 \\ \lambda = n^2\pi^2 \end{matrix}$

$\therefore$ Option (c) is correct.

Hence, the options (b) and (c) are correct.