COUNCILE OF SCIENTIFIC & INDUSTRIAL RESEARCH

Mathematical Science Code : 04

Unit – 2 :

SYLLABUS

Sub Unit – 1: Complex Analysis

SL NO.	TOPICS
1	1.1. **Algebra of complex numbers, The complex plane, Polynomials, Power series, Transcendental functions such as exponential, Trigonometric and hyperbolic functions.**
2	1.2. **Analytic functions, Cauchy Riemann equations.**
3	1.3. **Contour integral, Cauchy's theorem, Cauchy's integral formula.**
4	1.4. **Taylor series, Laurent series, Calculus of residues.**
5	1.5. **Conformal mappings, Mobius transformations.**
6	1.6. **A few more concepts (Uniqueness theorem, Morera's theorem, winding number, Rouche's theorem, Argument principle)**

Complex Analysis

1.1. Algebra of complex numbers, The complex plane, Polynomials, Power series, Transcendental functions such as exponential, Trigonometric and hyperbolic functions.

A brief introduction to **Complex Numbers**

- A complex number $z = x + iy$ is represented as an ordered pair (x, y), where x is a real part, and y is an imaginary part.
- Complex numbers are not ordered as $z_1 < z_2$ is meaningless for two complex numbers. $|z_1| < |z_2|$ means z_1 is closer to the origin in the Argand plane.

Q. Prove that $|z_1 + z_2| \leq |z_1| + |z_2|$

Hints: Take $z_1 = r_1(cos\theta_1 + sin\theta_1)$

$$z_2 = r_2(cos\theta_2 + sin\theta_2)$$

$$|z_1 + z_2| = \sqrt{r_1{}^2 + r_2{}^2 + 2r_1r_2\cos(\theta_1 - \theta_2)}$$

$$\leq \sqrt{r_1{}^2 + r_2{}^2 + 2r_1r_2}\text{as}\cos(\theta_1 - \theta_2) \leq 1$$

$$= r_1 + r_2$$

$$= |z_1| + |z_2|$$

- Conjugate of a complex number $z = x + iy$ is denoted as $\bar{z}$ is represented by $(x, -y)$

Q. (i) Prove that z is real if only if $\bar{z} = z$.

(ii) z is either real or pure imaginary if and only if $\bar{z} = z$.

Hints : $(x + iy)^2 = (x - iy)^2$

⇨ $x^2 + 2ixy – y^2 = x^2 – 2ixy – y^2$

⇨ $4ixy = 0$

⇨ $x = 0$ or $y = 0$

- **Principal Value of Argument:**

Principal Value of argument z, is denoted by $Arg\ z$ and defined by such that-

$$\pi < Arg\ z \leq \pi$$

The general value of $arg\ z = Arg\ z + 2\pi ni$ $[n = 0, \pm1, \pm2, \dots.]$

Q. Find the principal value of the argument of $-1 - i$

Hints:

$-1 - i = r\,cos\theta + ir\,sin\theta$

$=> r\,cos\theta = -1$

$r\,sin\theta = -1$

$r^2 = 2$

$r = \sqrt{2}$

$cos\theta = -\frac{1}{\sqrt{2}}$

$sin\theta = -\frac{1}{\sqrt{2}}$

$\Rightarrow \theta = \pi + \frac{\pi}{4} > \pi$

Hence $Arg\, z = \pi + \frac{\pi}{4} - 2\pi = \frac{-3\pi}{4}$

Q. If $Re\,(z_1) > 0$ and $Re\,(z_2) > 0$

Then Arg $(z_1 z_2)$ = Arg z_1 + Arg z_2

Left as an exersizes.

- **Neighbourhood $N_\epsilon(z_0)$**: ϵ - neighbourhood of z_0 is the set of all points satisfying $0 < |z - z_0| < \epsilon$, i.e. the neighbourhood consists of all points inside the circle not on a circle with specifiedradius ϵ.
- **Interior point**: A point z_0 is said to be an interior point of a set S whenever $N_\epsilon(z_0) \subset S$ for some $\epsilon > 0$.
- **Boundary point**: Boundary points of a set S are those whose all neighbourhoods contain few points of S and contain few points,not in S.
- **Open Set**: A set is open if all its points are interior points.
  - **Closed Set**: A set is closed if it contains all its boundary points.
  - **Connected Set**: A open set is said to connected set if any two points $z_1$and $z_2$can be joined by a polygonal line, consisting of a finite number of line segments joined end to end, that lines entirely belong to S.

Q. (i) Give an example of a set which is both open and closed.

(ii) Determine the limit points of the following set

a. $Z_n = in\,(n = 1,2, \ldots\ldots)$

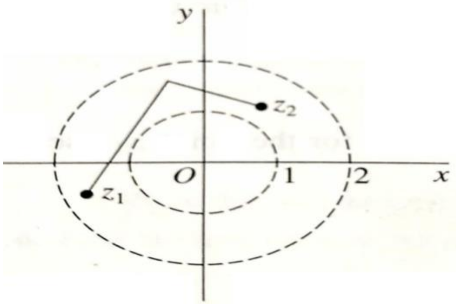
b. $Z_n = \frac{i^n}{n}\,(n = 1, 2, \ldots)$

c. $0 \leq arg\, z < \frac{\pi}{2}\,(z \neq 0)$

d. $Z_n = (-1)n\,(1 + i)\frac{n-1}{n}\,(n = 1, 2, \dots.)$

Answers: $(a) None, (b) 0, (c) none, (d) \pm (1 + i)$

Example (1.1): The open set $|z| < 1$ is connected, $1 < |z| < 2$ is also connected.



➢ **Accumulation point**: A point z_0 is said to be a limit point of S if each deleted neighbourhood of z_0 contains at least one point of S.

i.e. $\acute{N}\epsilon(z_0)$ ⍰$S \neq \emptyset$ $where$ $\acute{N}\epsilon(z_0) = N_\epsilon(z_0) - \{z_0\}$.

1.2. Analytic functions, Cauchy Riemann equations.

➢ **Functions of a Complex Variable :**

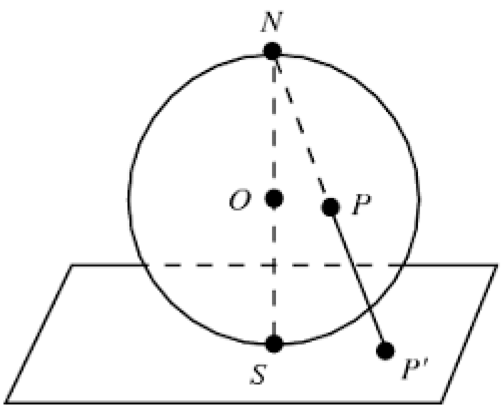
A function on a complex variable is defined on a complex set S such that for each z ϵ S, a complex number is assigned in under the function.

Ex. $F(z) = z^2$

i.e. $f(x + iy) = (x + iy)^2 = x^2 - y^2 + i2xy$

➢ **Stereographic Projection :**

It is sometimes convenient to include with the complex plane the point at infinity, ∞, the complex plane together with this point, is called the extended complex plane.To visualize the point at infinity, one can think of the complex plane as primary though the equator of a unit sphere contested at the origin.There to each point z in the plane there corresponds exactly one point on the surface of the sphere.



The point P on the surface of the sphere is point connecting north pole and the point though the sphere.

- A composition of a continuous function is continuous itself.
- If a function $f(z)$ is continuous and non-zero at z_0, then $f(z) \neq 0$ for some neighbourhood.
- **Cauchy – Reiman Equations** (C-R equations)

Let $f(z) = u(x,y) + iv(x,y)$. The $C - R$ equations are given as follows.

$$\frac{du}{dx} = \frac{dv}{dy} \text{ at } (x_0, y_0)$$

$$\frac{du}{dy} = -\frac{dv}{dx} \text{ at } (x_0, y_0)$$

Suppose that $f\,'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the first order partial derivatives of u and v must exist at (x_0, y_0) and they must satisfy C – R equations.

$$u_x = v_y \text{ , } u_y = -v_x$$

and $f\,'(z_0) = u_x + iv_x$ at (x_0, y_0)

- **Verify C-R equations**

$f(z) = z^2 = x^2 - y^2 + 2ixy$

Here,

$u(x,y) = x^2 - y^2$

$v(x,y) = 2xy$

$\Rightarrow$

$v_x = 2x$

$u_y = -2y$

$v_x = 2y$

$v_y = 2x$

Thus $u_x = v_y$ and $u_y = -v_x$

Also $f'(z) = u_x + v_x = 2x + 2iy = 2z$

- **Sufficient Conditions for Differentiability:**

It is to be remembered that C – R equations are not sufficient fordifferentiability.

Check $f(z) = |z|^2 = x^2 + y^2 + i \cdot 0$

$u_x = 2x$,$u_y = 2y$, $v_x = 0$, $v_y = 0$

$$\frac{\Delta w}{\Delta z} = \frac{|z+\Delta z|^2 - |z|^2}{\Delta z} = \frac{(z+\Delta z)(\overline{z+\Delta z}) - z\,\bar{z}}{\Delta z}$$

$$= \frac{z\overline{\Delta z}}{\Delta z} + \frac{\Delta z \,.\, z}{\Delta z} + \frac{\Delta z\overline{\Delta z}}{\Delta z}$$

$$z + \overline{\Delta z} + z\,\frac{\overline{\Delta z}}{\Delta z}$$

Thus $\frac{\Delta w}{\Delta z} = \bar{z} + \Delta z + z$ when $\Delta z = (\Delta x, 0)$

$\frac{\Delta w}{\Delta z} = \bar{z} - \Delta z - z \qquad \Delta z = (0, \Delta y)$

Now $\frac{\Delta w}{\Delta z}$ exists when $\Delta z \longrightarrow 0$

For uniqueness, $\bar{z} + z = \bar{z} - z$

Hence, $z = 0$

Thus for $z \neq 0$, $f'(z)$does not exist.

Thus $C - R$ equations are not satisfied but $f'(0) = 0$

- Suppose $f(z) = u(x,y) + iv(x,y)$ be defined throughout some $\in -$ neighbourhood of $z_0 = x_0 + iy_0$ and
  - (i) The first order Partial derivatives of the functions u and v with respect to x and y exist everywhere in the neighbourhood.
  - (ii) Those partial derivatives are continuous at (x_0, y_0) and satisfy the C- R equations at (x_0, y_0).

    Then $f'(z_0)$ exists and its value being $f'(z_0) = u_x + iv_x$, the right-hand side is evaluated at (x_0, y_0).

Polar Form of C-R Equations

For a complex number $z = x + iy = r(cos\theta + isin\theta)$, the followings are held.

$$x = r\, cos\theta$$

$$y = r\, sin\, \theta$$

$f'(z\_0) = e^{-i\theta}\, (u_r + iv_r)$, the right-hand side is evaluated at (r_0, θ_0).

- Find $f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}(Cos\theta - i\, Sin\theta)$

  $u(r,\theta) = \frac{Cos\theta}{r^2}$, $v(r,\theta) = \frac{-Sin\theta}{r}$

  $u_r = -\frac{Cos\theta}{r2}$ | $v_\theta = -\frac{Cos\theta}{r}$

  $r\, u_r = -\frac{Cos\theta}{r}$ | $v_r = +\frac{Sin\theta}{r^2}$

  $u_\Theta = -\frac{Sin\theta}{r}$

  Thus $r\, u_r = V_\theta$

  $u_\theta = -r\, v_r$

  $f'(z) = e^{-i\theta}\left(-\frac{Cos\theta}{r^2} + i\,\frac{Sin\theta}{r^2}\right) = -\frac{1}{(re^{i\theta})^2} = \frac{1}{z^2}$

- **Analytic functions:** A function f of the complex variable z is analytic at z_0 if it has a derivative at each point in some neighbourhood of z_0

$f(z) = \frac{1}{z}$ is analytic at each non-zero point in the finite plane. But $f(z) = |z|^2$ is not analytic at any non zero point.

Note: Though the term analytic function is used interchangeably with **holomorphic function**, the word "analytic" is defined in a broader sense to denote any function (real, complex, or of more general type) that can be written as a convergent power series in a neighbourhood of each point in its domain.In mathematics, a holomorphic function is a complex-valued function of one or more complex variables that is, at every point of its domain, complex differentiable in a neighbourhood of the point. The existence of a complex derivative in a neighbourhood is a very strong condition, for it implies that any holomorphic function is actually **infinitely differentiable** and equal, locally, to its own Taylor series (analytic). Holomorphic functions are the central objects of study in complex analysis.

- **Entire function** is a function that is analytic at each point in the entire finite plane.
- **Singular Point**: If a function f fails to be analytic at a point z_0 but is analytic at some point in every neighbourhood of z_0, then z_0 is called a singular point or singularity of f.

  Z = 0 is a singular point of $f(z) = 1/z$. $f(z) = |z|^2$ has no singular point.

  If $f'(z) = 0$ everywhere in a domain D, then f(z) is constant throughout D.
- A real-valued function H of two real variables x and y is said to be **harmonic** in a given domain of the xy plane if it has continuous partial derivatives of first and second-order and satisfies the partial differential equation.

$$H_{xx}(x,y) + H_{yy}(x,y) = 0$$

  known as Laplace's equation.
- If a function f(z) = u(x,y) + iv(x,y) is analytic in a domain D, then its component functions u and v are harmonic in D i.e., u and v satisfy C-R equations.
- A function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D if and only if v is a **harmonic conjugate** of u.

Q. Find $f(z)$ where $u = y^3 - 3x^2y$ to be analytic.

Ans: Here $u_x = -6xy$

Again $u_x = v_y = -6xy$ (C-R equation)

$$\Rightarrow dv/dy = -6xy$$

$\Rightarrow v = -3xy^2 + \phi(x)$ [Integrating] where $\phi(x)$ is a function of x only.

Also $\Rightarrow u_y = -v_x$

$$\Rightarrow 3y^2 - 3x^2 = -v_x = 3y^2 - \phi'(x)$$

$$\Rightarrow \phi'(x) = 3x^2$$

$\Rightarrow \phi(x) = x^3 + c$ (Integrating)

Thus $v = x^3 - 3xy^2 + c$

$f(z) = (y^3 - 3x^2y) + i(x^3 - 3xy^2 + c)$

- Suppose a function f is analytic throughout a domain D

(ii) $f(z) = 0$ at each point z of a domain or line segment contained in D.

Then $f(z) \equiv 0$ in D.

- **Reflection Principle**

$\overline{f(z)} = f(\overline{z})$

Suppose that a function f is analytic in some domain D which contains a segment of the x-axis and whose lower half is the reflection of the upper half with respect to that axis, then $\overline{f(z)} = f(\overline{z})$ for each point z in the domain if and only if f(x) is real for each point x on the line segment.

$\Rightarrow f(z) = z + 1$ has the reflection properly as f(z) = x + 1 is real $\forall\, x \in R$

and $f(z) = z + i$ does not fellow the reflection properly as $f(x) = x + i$

- A **branch of a multivalued function** f is any single-valued analytic function in some domain at each point z of which the value $F(z)$ is one of the values of f.

$$Log\, z = ln\, r + i(\theta + 2n\pi), n = 0, \pm 1, \pm 2, ...$$

is called the principal branch of log z = $ln\, r + i\theta$

- A **branch cut** is a portion of a line or curve that is introduced in order to define a branch F of a multivalued function f.*Points on the branch cut for F are singular points.*

Any point that is common to all branch cuts of f is called a branch point.

1.3. Contour integral, Cauchy's theorem, Cauchy's integral formula

Contours :

- A set of points z = (x,y) in the complex plane is said to be **an arc** if $x = x(t), y = y(t), a \leq t \leq b$ where $x(t)$ and $y(t)$ are continuous function of the real parameter t.

- The arc C is a simple arc of a Jordan arc, if it does not cross itself, i.e., $z(t_1) \neq z(t_2)$ for $z(t) = x(t) + iy(t)$ when $(t_1) \neq (t_2)$. If $z(b) = z(a)$, then the curve is called simple closed curve.
- A contour, or piecewise smooth arc, is a consisting of a finite number of smooth arcs joined end to end.Hence $z(t) = x(t) + iy(t), a \leq t \leq b is$ a contour, implies that $z(t)$ is piecewise continuous.

Q. Find the integral

I = $\int_c \bar{z}\, dz$ $\qquad C: right\ hand\ half$

$z = 2e^{i\theta} \left(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right)$ of $|z| = 2$ from $z = -2i\ to\ z = 2i$

I = $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2e^{\overline{i\theta}}\,(2ie^{i\theta})d\theta$

= $4i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-i\theta}\, ie^{i\theta}\, d\theta = 4\pi i$

- If $W(t)$ is a piecewise continuous complex-valued function defined on an interval $a \leq \theta \leq b$ thus

  $\left|\int_a^b W(t)dt\right| \leq \int_a^b |W(t)|\, dt$

- **Cauchy – Goursat Theorem**

  If a funtion f is analytic at all points interior to and on a simple closed contour C, than

$$\int_C f(z)dz = 0$$

Example (1.2): If c denotes any closed contour lying in the open disk $|z| < 2$, then

$$\int_C \frac{2e^z}{(z^2+9)^5} dz = 0$$

Here singularities $z^2 = -9 \quad \Rightarrow z = \pm 3i$

are exterior to $|z| < 2$. Hence according to C- G equations, the integration is zero.

- Suppose that

  (i) c is a simple connected contour, described in the counterclockwise direction

  (ii) c_k $(k = 1,2, \ldots\ldots, n)$ are simple closed contours interior to c, all described in a clockwise direction which are disjoint and whose interiors have no points in common. If f is analytic on all these contours and throughout the multiply-connected domain consisting of the points inside c and exterior to each c_k, then

  $\int_c f(z)dz + \sum_{k=1}^{n} \int_{ck} f(z)dz = 0$

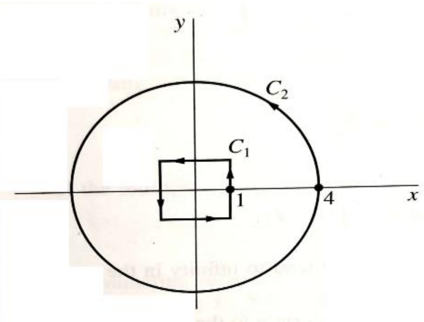
- Let c_1 and c_2 denote positively oriented simple closed contours where c_1 is interior to c_2. and if a function f is analytic in the closed region consisting of those contours and all points between then,

$$\int_{c_2} f(z)dz = \int_{c_1} f(z)dz$$

Example (1.3):

Let C_1 denote the positively oriented boundary of the square whose sides lie along the Lines $x = \pm 1, y = \pm 1$ and let C_2 be the positively oriented circle $|z| = 4$

If $f(z) = \frac{1}{3z^2+1}$ then $\int_{c_1} f(z)dz = \int_{c_2} f(z)dz$



As $3z^2 + 1 = 0$

$\Rightarrow z = \pm \frac{i}{\sqrt{3}}$

$\Rightarrow$ The points do not lie in the region between c_1 and c_2.

- **Cauchy Integral Formula**

Let f be analytic everywhere inside and on a simple closed contour c takenin the positivesense, It z_0 is any point interior to c, then

$$f(z_0) = \frac{1}{2\pi i}\int_c \frac{f(z)dz}{z-z_0}$$

Example (1.4):

$f(z) = \frac{z}{9-z^2}$ $\qquad c : |z| = 2$

Here $f(z)$ is analytic within and on c since $z_0 = -i$ is interior to c

$$\int_c \frac{f(z)dz}{z+i} = 2\pi i\, f(-i)$$

$$\Rightarrow \int_c \frac{zdz}{(9-z^2)(z+i)} = 2\pi \mathrm{i}\,\frac{-i}{10} = \frac{\pi}{5}$$

- **An extension of the Cauchy integral formula.**

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_c \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

Where is analytic within and on a simple closed curve C.

Example (1.5):

C :$|z| = 1$

$f(z) = e^{2z}$

$$\int_c \frac{e^{2z}}{z^4} dz = \frac{2\pi i}{3!} f^3(0)$$

$= \frac{2\pi i}{6} e^{2z} \times 2 \times 2 \times 2 \Big|_{z=0}$

$= \frac{8\pi i}{3}$

- If a function f is analytic at a given point, then its derivatives of all orders analytic too.
- If a function $f(z) = u(x,y) + iv\,(x,y)$ is analytic at a point $z = (x,y)$, then the component functions $u\ and\ v$ have continuous partial derivatives of all orders at that point.
- If f is a continues on a domain D, if $\int_c f(z)dz = 0$ for every closed contour $c\ in\ D$.

$\Longrightarrow f$ is analytic throughout D.

- Suppose f is analytic inside and on a positively oriented circle C_R centred at z_0 with radius r. If M_R denotes the maximum value of $|f(z)|$ on C_R Then $|f^n(z_0)| \leq \frac{n!M_R}{R^n}$

1.4. Liouville's theorem, Maximum modulus principle, Schwarz lemma, Open mapping theorem.

Liouville's theorem:

- If a function f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.

Fundamental theorem of algebra:

- Any polynomial

  $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n\ (a_n = 0)$ of degree $n(n \geq 1)$ has at least one zero. That is, there exists one z_0 such that $P(z_0) = 0$.

Maximum modulus principle:

Suppose $|f(z)| \leq |f(z_0)|$ at each point z in some neighbourhood $|z - z_0| < \varepsilon$ in which f is analytic. Then $f(z)$ has constant value $f(z_0)$ throughout the neighbourhood.

- If a function f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D.

Schwarz lemma:

Let f be analytic on the unit disk, and assume that

1. $|f(z)| \leq 1$ for all z and
2. $f(0) = 0$

Then $|f(z)| \leq |z|\ \ and\ f'(0) \leq 1$

If either $|f(z)| = |z|$ for some $z \neq 0$ or if $f'(0) = 0$, then f is a rotation, i.e., $f(z) = az$ for some complex constant a with $|a| = 1$.

Open mapping theorem:

In complex analysis, the open mapping theorem states that if U is a domain of the complex plane C and $f : U \to C$ is a non-constant holomorphic function, then f is an open map (i.e. it sends open subsets of U to open subsets of C, and we have invariance of the domain).

Note: The open mapping theorem points to the sharp difference between holomorphy and real-differentiability. On the real line, for example, the differentiable function $f(x) = x^2$ is not an open map, as the image of the open interval $(-1, 1)$ is the half-open interval $[0, 1)$.

1.5. Taylor series, Laurent series, Calculus of residues.

- Suppose that

  $z_n = x_n + iy_n$, i = 1,2,3,…………

  $S = x + iy$

  $$\left.\begin{array}{l}\sum_{n=1}^{\infty} x_n = X \\ \\ \sum_{n=1}^{\infty} y_n = Y\end{array}\right\} \Leftrightarrow \sum_{n=1}^{\infty} z_n = S$$

- **Taylor Series**

  Suppose that a function f is analytic through a disk $|z - z_0| < R_0$ centred at z_0 with radius R_0. Then $f(z) = \sum_{n=1}^{\infty} a_n(z - z_0)^n \ (|z - z_0| < R_0)$

  Where $a_n = \frac{f^n(z_0)}{n!}$ (n = u,1,2,…….)

Example (1.6): $e^z = \sum \frac{1}{n!}(z - 0)^n$ as $f^n(o) = 1 \ (n = 0,1, \ldots..)$

- **Laurent Series**

  If a function f fails to be analytic at a point z_0, Taylor theorem can not be applied those.

  Suppose that a function f is analytic throughout as annular domain $R_1 < |z - z_0| < R_2$, centred at z_0 and lying in that domain.

  Then $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum \frac{b_n}{(z-z_0)^n} \quad (R_1 < |z - z_0| < R_2)$

  $$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{n+1}} \ (n = 0,1,2 \ldots.)$$

  $$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{-n+1}}$$

  $\Rightarrow f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n \ R_1 < |z - z_0| < R_2$

  $c_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{n+1}} \ (n = 0, \pm 1, \pm 2 \ldots)$

Example (1.7):

$1 < |z| < 2$ $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{z}\right| < 1$

$f(z) = \frac{1}{z} \cdot \frac{1}{1-(\frac{1}{z})}$

Residues and poles

Isolated singular points

A singular point z_0 is said to be isolated if, there is a deleted neighbourhood.

$0 < |z - z_0| < c$ of z_0 throughout which f is analytic.

Example (1.8): $\frac{z+1}{z^3(z^2+1)}$ has there singular points $z = 0, z = \pm i$

- z_0 is an isolated singular point of a function f, those are a positive number R_2 such that f is analytic at each point z for which $0 < |z - z_0| < R_2$

  $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n} + \cdots$

  $a_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}}$

  $b_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{-n+1}}$

  The complex number b_1, coefficient of $\frac{1}{z-z_0}$ in expansion, is called the residue of f at the isolated singular point z_0,

  $b_1 = Res\ f(z)$ at $z = z_0$

Cauchy Residue Theorem

Let C be a simple closed contour and $f$be analytic inside and on C except for a finite number of singular points z_k ($k = 1, 2, \dots \dots, n$) inside C

Then $\int_c f(z)dz = 2\pi i \sum_{k=1}^{n} Res\ at\ z = z_k\ f(z)$

Example (1.8):

$\int_c \frac{5z-2}{z(z-1)}dz$

$c: |z| = 2$

The singularities are $z = 0\ and\ z = 1$.

Both of the points are inside the contour C

Now

$\frac{5z-2}{z(z-1)} = \frac{5z-2}{z}\left(\frac{-1}{1-z}\right) = (5 - \frac{2}{z})\,(-1 - z - z^2 - z^3 \dots\dots)$

The coefficient of $\frac{1}{z}$ is 2

Similarly

$$\frac{5z-2}{z(z-1)} = \frac{5(z-1)+3}{(z-1)} \cdot \frac{1}{1+(z-1)} \quad = (5+\frac{3}{z-1})\,(-1-(z-1)-(z-1)^2- \quad \ldots..)$$

Coefficient of $\frac{3}{z-1} = 3$

Hence $\int_c \frac{5z-2}{z(z-1)}dz \ = \ 2\pi i(2+3) = 10\,\pi i$

Principal part of f:

$$\frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n} + \cdots$$

If the principal part is finite i.e.

$$\frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m} 0 < |z-z_0| < R_2$$

where $b_m \ \neq 0$ and $b_{m+1} \ = 0 = b_{m+2} \ = \cdots$

Then the singularity is called a **pole of order m**

Example (1.9):

$$f(z) = \frac{1}{z^2(1+z)} = \frac{1}{z^2}\,[1-z+\ z^2-\ z^3\ +\cdots] \ = \frac{1}{z^2} - \frac{1}{z} + 1 - \frac{1}{z} + \frac{1}{z^2} - \cdots$$

Hence z = 0 is a pole of order 2.

Res $f(z) \ = \ -1$

$z \ = \ 0$

➢ An isolated singular point z_0 of a function f, is a pole of order m if and only if $f(z)$ can be written of the from $f(z) \ = \frac{\emptyset(z)}{(z-z_0)^m}$

Where$\emptyset(z)$ is analytic and non-zero at z_0

Res $f(z) \ = \emptyset(z), if\ m = 1$

$z \ = z_0$

and Res $f(z) \ = \frac{\emptyset(m-1)(z_0)}{(m-1)!}, if\ m \geq 1$

$z \ = z_0$

Zeros of analytic functions

Then f is said to have a zero of order $m\ at\ z_o$

❖ Let a function f be analytic at a point z_0. It has zero of order m at z_0 if and only if there is a function g, which is analytic and non-zero at z_0, such that

$$f(z) = (z-\ z_0)^m g(z)$$

Zeroes and poles

Suppose that

(a) two functions $p\ and\ q$ are analytic at a point z_0

(b) $P(z_0)\ \neq 0$ and q has a zero of order m at z_0

Then $\frac{p}{q}$ has a pole of order m at z_0

For a simple pole of p/q and $\underset{z\,=\,z_0}{Res}\ \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$

- ❖ If z_0 is a pole of a function f, then $\log_{z\to z_0} f(z) = \propto$

Few Important problems

(1) Find $\lim_{z\to z_0}\frac{\bar{z}}{z}$

(a) 1 (b) 0 (c) 3/2 (d) does not exist

Hints:

$\frac{\overline{x-\iota y}}{x+iy} = \frac{(x-iy)(x-iy)}{x^2+y^2} = \frac{x^2-y^2-2ixy}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2} + i\ \frac{-2xy}{x^2+y^2} = u + iv$

Here u and v both do not exist by putting $y\ =\ mx$.

(2) $f(x) = \begin{cases} e^{-\frac{1}{z}}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

Then

(a) $f(z)$ is not analytic at (0,0)

(b) $lim_{z\to z0}\, f(z)$ exists.

(c) $lim_{z\to z0}\, f(z)$ does not exist.

(d) $C - R$ equations one satisfy at (0,0)

Hints:

Here $f(z) = \frac{1}{e^{\frac{1}{z^4}}}$

If $z \to 0$ along x-axis

$f(z) \to 0$

Again $z \to 0$ along $y\ =\ x$

$\log_{\gamma\to 0} e^{\frac{-1}{r^4}e^{\pi i}} \to \infty$

Hence limit does not exist.

So, $f(z)$ is not analytic.

Ans : (a), (c)

(3) $f(z) = u + iv$ is analytic function, then under what conditions $u \; and \; v$ cut orthogonally each other?

(a) $u(x,y) = Constant$
$v(x,y) = Constant$

(b) $u(x,y) = variable$
$v(x,y) = variable$

(c) $u(x,y) = Constant$
$v(x,y) = Variable$

(d) $u(x,y) = 2018$
$v(x,y) = 2020$

Hints:

As $f(x,y)$ analytic $u_x = v_y \; and \; u_y = -v_x \ldots\ldots\ldots\ldots\ldots\ldots. \ldots\ldots (i)$

$\Rightarrow u = constant$

$\Rightarrow u_x \, dx + u_y \, dy = 0$

$\Rightarrow m_1 = \left(\frac{dy}{dx}\right) = -\frac{u_x}{u_y}$

Again $v = constant \Rightarrow m_2 = \frac{-v_x}{v_y}$

Now $m_1, m_2 = \left(\frac{-u_x}{u_y}\right)\left(\frac{v_x}{v_y}\right) = -1$ [comparing (i)]

Hence

Ans : (a), (d)

(4) $f(z) = \sin x \,(\cosh y) + i \cos x \,(\sinh y)$

(a) f is continuous

(b) f is discontinuous

(c) f is analytic

(d) f is not analytic

Hints:

Here $u = \sin x \cosh y$
$v = \cos x \sinh y$

These are continuous everywhere

Then $u_x = \cos x \cosh y$
$u_y = \sin x \sinh y$

$v_x = \sin x \sinh y$

$v_y = \cos x \cosh y$

Here, $u_x = v_y$ and $u_y = -v_x$

Here u_x, u_y, v_x, v_y are continuous everywhere and satisfy $C - R$ equations.

Thus $f(z)$ is analytic everywhere.

(5) Evaluate $\int_c \frac{z-1}{(z+1)^2(z-2)}\, dz$ where $c: |z-i| = 2$

Hints: Here $|z-i| = 2$

$|x + i(y-1)| = 2$

Centre is $(0,1)$ and radius is 2

Singularities $z = 2, z = -1, -1$

$z = 2$ is not of contour and $z = -1$ lies inside the contour

$\int_c \frac{z-1}{(z+1)^2(z-2)}\, dz = \int_c \frac{(\frac{z-1}{z-2})}{(z+1)^2}\, dz = 2\pi i\, f'(-1)$

$= 2\pi i(-\frac{1}{9}) = \frac{-2\pi i}{9}$

$$\boxed{\begin{aligned} f'(z) &= \frac{z-2-z+1}{(z-2)^2} \\ f'(-1) &= \frac{-1}{9} \end{aligned}}$$

(6) Evaluate the following integrals using Cauchy's integral formula

$\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz$

Hints:

Let $f(z) = \sin \pi z^2 + \cos \pi z^2$

Now $\int_c \frac{f(z)}{(z-1)(z-2)}\, dz = \int_c \left(\frac{1}{z-2} - \frac{1}{z-1}\right) dz$

$= \int_c \frac{f(z)}{(z-2)} dz - \int_c \frac{f(z)}{(z-1)}\, dz = 2\pi i\, f(2) - 2\pi i\, f(1) = 2\pi i + 2\pi i = 4\pi i$

(7) $\int_c \frac{z^2-4}{z(2^2+9)}\, dz \quad c: |z| = 1$

Hints: $z(2^2 + 9) = 0$

$\Rightarrow z = 0, z = \pm 3i$

Here $z = 0$ is inside in c and let $f(z) = \frac{z^2-4}{z^2+9}$

Now $f(z)$ is analytic in c

Thus $\int_c \frac{f(z)}{z}\, dz = 2\pi i\, f(0) = 2\pi i \left(-\frac{4}{9}\right) = \frac{-8\pi i}{9}$

(8) Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in a Laurent's Series in $1 < |z| < 3$

Sol[n].

$$\frac{1}{|z|} < 1 \; and \; \frac{|z|}{3} < 1$$

$$f(z) = \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{z+3}\right)$$

Now $\frac{1}{2(z+1)} = \frac{1}{2z\left(1+\frac{1}{z}\right)} = \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right)$

$$= \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots$$

$\frac{1}{2(z+3)} = \frac{1}{6(1+\frac{z}{3})} = \frac{1}{6}\left(1 - \frac{1z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right)$

Hence $1 < |z| < 3$

$f(z) = \ldots\ldots\ldots + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} - \cdots$

(9) Discuss the nature of singularities of the following functions.

(i) $\tan z$ (ii) $\frac{2}{1+z^4}$ (iii) $\frac{\sin z}{(z-\pi)^2}$

(i) $F(z) = tan\, z = \frac{\sin z}{\cos z}$

For singularities $cos\, z = 0$

$\Rightarrow z = 2n\pi \pm \frac{\pi}{2}, n = 0, \pm 1, \pm 2, \ldots$

$\Rightarrow z = (4n \pm 1)\frac{\pi}{2}$

Give the simple poles of $f(z)$ and $z = \alpha$ is a non-isolated singularity

(ii) $f(z) = \frac{z}{1+z^4}$

$\Rightarrow z^4 + 1 = 0$

$\Rightarrow z = (-1)^{\frac{1}{4}}$

$= (cos\, \pi + sin\, \pi)^{\frac{1}{4}}$

$= \{ cos(2n\pi + \pi) + i\, sin\, (2n\pi + \pi) \}^{\frac{1}{4}}$

$= e^{i(2n+1)\frac{\pi}{4}}$

$Put\; n = 0, 1, 2, 3 \ldots$

$z = e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}}, \ldots$ ..are simple poles

(iii) $z = \pi$ is pole of order 2

10. let f be analytic on $D = \{z \in c : |z| < l\}$

$f'(0) = 0;$

Define $g(z) = \begin{cases} \frac{f(z)}{z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases}$

Then

(a) g is discontinuous at $z = 0$ for all f.

(b) g is continues, but not analytic at $z = 0$ for all f.

(c) g is analytic at $z = 0$ for all f.

(d) g is analytic at $z = 0$ any, $f . f'(0) = 0$

Hints: To verify $f(z) = sin\, z$

Then f is analytic on $D = \{z \in c\ |z| < l\}$

And $f(0) = 0$

Than $g(z) = \begin{cases} \frac{\sin z}{z}, & z \neq 0 \\ 1, & z = 0 \end{cases}$

Thus (c) is correct

(a) g is continuous. Hence it is wrong

(b)Wrong as g is incorrect

(c)For this one $g'(0) = 1$ Hence are not Correct.

(11) Let $f(z)$ be an analytic function on D such that $|f(z)| = |\sin z|$ for all $z \in D$ then

(a) $f(z) = sin\, z\ for\ all\ z \in D$

(b) $f(z) = sin\,(z)\ for\ all\ z \in D$

(c) $f(z) = c\ sin\, z\ for\ all\ z \in D$

(d) such a function $f(z)$ does not exist

Hints: Here$|f(z)| = |sin\, z| \forall z \in D$

$\Rightarrow f(z) = \pm\, sin\, z$

$\Rightarrow f(z) = c\ sin\, z \forall z \in D$

$|c| = 1\ and\ c \in D$

Option (C) is Correct

(12) The radius of Convergence of power service

$\sum_{n=0}^{\infty} \frac{\alpha}{z}(4n^2 - n^3 + 3) Z^n$ is

$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{an+1}{an}\right|$

$= 1$

(a) 1, (b) 2

(c), α (d) 0

Option (a) is correct

1.6. Conformal mappings, Mobius transformations

Basic Mapping

(i) $f(z) = z + a, a \in \phi. (Translation).$

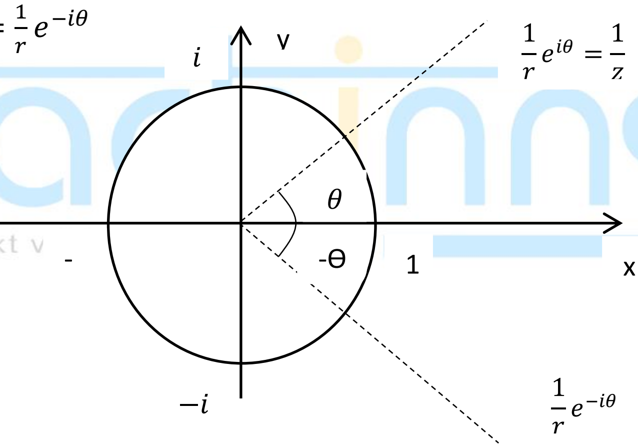
(ii) $f(z) = az, a \in \mathbb{C}\ and\ |a| = 1\ i.e., a = e^{i\alpha}, \alpha \in \mathbb{R}\ (Rotation)$

(iii) $f(z) = az, 0 < a < 1\ (Contraction), a > 1\ (Magnification)$

(iv) $f(z) = az,\ a \in \mathbb{C}\ and\ |a| \neq 1, (ortaiton\ and\ magnificaiton\ /contraction)$

(v) $f(z) = az + b, a, b \in \mathbb{C}, (Translation, rotation, magnification)$

(vi) $\omega = f(z) = \frac{1}{z}$ (inverse or, reflection with respect to both unit circle and the real axis). Since $\omega = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$



$\omega = \frac{1}{2}$ maps circles and straight lines onto circles and straight lines.

Let $u + iv = \frac{1}{x+iy}$

$\Rightarrow x = \frac{u}{u^2+v^2}, y = \frac{-v}{u^2+v^2}$

Consider $a(x^2 + y^2) + bx + ay + d = 0 \ldots\ldots\ldots\ldots (*)$

$d(u^2 + v^2) + bu - cv + a = 0 \ldots\ldots\ldots\ldots\ldots\ \ldots\ldots (*')$

(a) Circles not passing through the origin $(a \neq 0, d \neq 0)$ are mapped onto circles not passing onto through origin.

(b) Circles passing through the origin $(a \neq 0, d = 0)$ are mapped onto straight lines not passing through the origin.

(c) Straight lines not passing through the origin $(a = 0, d \neq 0)$ are mapped onto circles passing through the origin.

(d) Straight lines passing through the origin $(a = 0, d = 0)$ are mapped onto straight lines passing through the origin.

Linear Fractional or Bilinear or Mobius Transformation:

$\omega = T(z) = \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{C}$ *and* $ad - bc \neq 0$.

$$\omega = \frac{az+b}{cz+d} = \begin{cases} \frac{a}{c} - \left(\frac{ad-bc}{c^2}\right)\frac{1}{z+\frac{d}{c}}, c \neq 0 \Rightarrow D(T) = \mathbb{C}|\{-\frac{d}{c}\} \\ \left(\frac{a}{d}\right)z + \frac{b}{d} \qquad , \qquad c = 0 \end{cases}$$

For $c \neq 0$, we may define $T(z) = \begin{cases} \frac{az+b}{cz+d}, z \neq -\frac{d}{c}, z \neq \infty \\ \infty \quad , z = -\frac{d}{c} \\ \frac{a}{c} \quad , z = \infty \end{cases}$

T is one-one and onto in the extended complex plane.

Theorem – I : The bilinear transformation maps circles and straight line onto circles and straight lines.

[Hint: $\omega = T(z) = (f_3 \circ f_2 \circ f_1)(z)$, $f_1(z) = z + \frac{d}{c}, f_2(z) = \frac{1}{z}, f_3(z) = \frac{a}{c} - \left(\frac{ad-bc}{c^2}\right)z$]

Theorem – II : A bilinear transformation $\omega = T(z)$ with more than two fixed points in c_α must be the identity transformation.

Theorem – III : Given three distinct points $z_k (k = 1,2,3)$ in the extended $z - plane$ and three distinct points ω_k $(k = 1,2,3)$ in the extended $\omega - plane, \exists$ unique bilinear transformation $\omega = T(z)$ *such that* $T(z_k) = \omega_k, k = 1,2,3$ and this is given by

$$\frac{\omega - \omega_1}{\omega - \omega_3} : \frac{\omega_2 - \omega_1}{\omega_2 - \omega_3} = \frac{z - z_1}{z - z_3} : \frac{z_2 - z_1}{z_2 - z_3}$$

Results :

(i) The most general biline4ar transformation of the real lines $\mathbb{R}$ onto the unit circle $|\omega| = 1$ is given by $\omega = T(z) = e^{i\alpha}\frac{z-z_0}{z-\overline{z_0}}$, $\alpha \in \mathbb{R}$

If $Im\ z_0 > 0$, then it maps upper half-plane into the interior of the unit circles and if $Im\ z_0 < 0$, then it maps lower half-plane into the interior of the unit circles.

(ii) The most general bilinear map of the upper half-plane $\{z : Im\ z > 0\}$ onto itself (or lower half-plane $\{\omega : Im\ \omega < 0\}$) is given by –

$\omega = \frac{az+b}{cz+d}$, $\mathrm{a, b, c, d} \in \mathbb{R}$ and $\mathrm{ad - bc} > 0$ $\left(or,\ \omega = \frac{az+b}{cz+d}, \mathrm{a, b, c, d} \in \mathbb{R} \text{ and } \mathrm{ad - bc} < 0\right)$

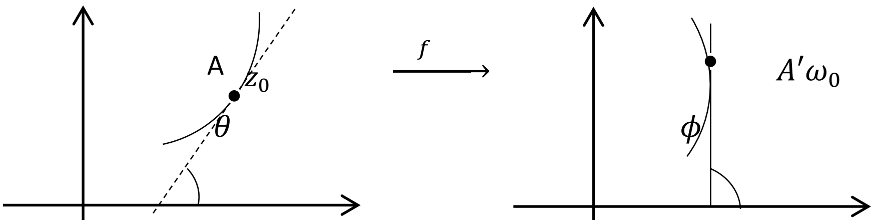
$upper \to upper$ $\qquad$ $upper \to upper$

$lower \to lower$ $\qquad$ $lower \to lower$

(iii) The most general bilinear transformation that takes the unit disk $D:\{z : |z| < 1\}$ onto itself is given by $f(z) = e^{i\alpha}\left(\frac{z-z_0}{z-\overline{z_0}\, z}\right)$ $where\ \alpha \in \mathbb{R}\ and\ z_0 \in \mathbb{D}$

Conformal Mapping: A function that preserves both size and orientation is said to be conformal. An analytic function is conformal at all points where the derivative is non-zero.

Example: $\omega = f(z) = e^z, \log z$



$\phi = \log_{\omega\to\omega_0} \arg(\omega-\omega_0) = \log_{z\to z_0} arg\left(\frac{f(z)-f(z_0)}{z-z_0} \times (z-z_0)\right)$

$$= \arg f'(z_0) + \arg z_0 \ = \arg f'(z_0) + \ \theta$$

A function that preserves angle size but not orientation is said to be isogonal.

Example (1.10): $f(z) = \bar{z}$

Let $f'(z_0) \ \neq 0$ and f is analytic at $z_0\ and\ |z-z_0| <\in \to 0$

Then $|f(z) - f(z_0)| \ \approx \ |f'^{(z_0)}||z-z_0|$

$\left(\because f(z) = f(z_0) + (z-z_0)f'(z_0) + \frac{(z-z_0)^2}{2} f''(z_0) \Rightarrow f(z) \approx f(z_0) + (z-z_0)f'(z_0)\right)$

$*\ |f'(z_0)| \ \neq 0$magnifies the length $|z-z_0|$and $\arg f'(z_0)$ rotate the line $|z-z_0|$.

Result:

(i). If $f(z)$ is analytic and one – one in a domain D, then $f'(z) \neq 0$. So that f is conformal on D.

(ii). Let $f(z)$ be analytic in a simply connected domain D and on its boundary, the simple closed contour C. If $f(z)$ is one – one C, then $f(z)$ is one – one in D and hence conformal in D.

(iii). Suppose $f(z)$ is analytic at z_0 and that $f'(z)$ has a zero of order $k-1\ at\ z_0$. If two smooth curves in the domain of f intersect at an angle θ then their images intersect at an angle k_θ.

(iv). Bilinear transformation$\omega = f(z) = \frac{az+b}{cz+d}(ad - bc \neq 0)$ represents a one-one conformal mapping from the extended plane onto itself.

(v). Only the composition of two conformal mappings is again conformal. (⇒sum and product of two conformal mappings is not conformal).

Note: $f(z) = z + \frac{1}{z}$ $is\ not\ conformal\ at\ z = \pm 1, since\ f'(\pm 1) = 0$

Harmonic Functions: A continuous real-valued function $u(x, y)$ defined in a domain D, is said to be harmonic in D if it has continuous first and second-order partials that satisfy the Laplace's equation $u_{xx} + u_{yy} = 0, throught\ out\ D.$

Note (I): Both the real, imaginary parts of an analytic function are harmonic.

Note (II): Laplace's equation $u_{xx} + u_{yy} = 0$ in polar form $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$

Results :

(i). If $u\,(x, y)$ is harmonic on a simply connected domain D, then ∃ an analytic function $f(z)$ on D whose real (or imaginary) part equal $u(x, y)$.

Note: A harmonic function need not have an analytic completion in a multiply connected domain (Example: $\log|z|$is harmonic in $D = \{< |z| < 2\}$ but it has no analytic completion, i.e. no harmonic conjugate).

(ii). A function harmonic and bounded in ℂ must be a constant.

(iii). If the real or imaginary part of an entire function is bounded above or bounded below by a real number M, then the function is a constant.

Note: The analogue of the identity theorem for analytic functions does not hold for harmonic functions.

(iv) If $u(x, y)$ is harmonic in a domain D and constant in the neighbourhood of some point in D, then u is constant throughout D.

(v) A non-constant harmonic function can't attain a maximum or minimum in a domain.

(vi) Suppose u is harmonic in a bounded domain D whose boundary is the closed contour C. If u is continuous in $D \cup C$, with $u \equiv k\ (Constant)\ on\ C, then\ u \equiv k\ \ throughout\ D.$ [cor of 5].

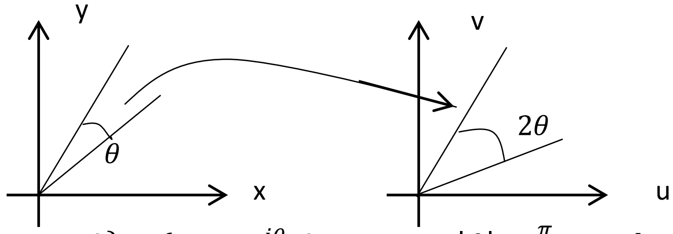
(vii) [Cor of (6)] Suppose$u_1\ and\ u_2$ are harmonic in a bounded domain D whose boundary is the closed contour C. If $u_1\ and\ u_2$ are continuous in $D \cup C, with$

$$u_1 \equiv\ u_2\ on\ C, then u_1 \equiv\ u_2\ throughout\ D.$$

(viii) Suppose $f(z)$ is an entire function and that $Re\, f(z) \leq Mr^{\lambda}\ for\ |z| = r > 0\ and\ \lambda$ be a non-negative real number. Then $f(z)$ is a polynomial of degree at most $[\lambda]$.

(ix) If $f(z)$ is an entire function such that $f(z)$ is real on the unit circle $|z| = 1$, then $f(z)$ is constant.

(x) A non-constant harmonic function in the plane can not omit more than one real value. (By Picard's theorem its analytic completion $f(z)$ may have at most one exception point).

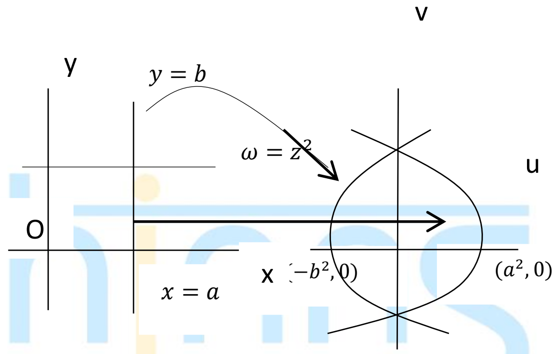


More on Basic Mappings:

(1). $\omega = f(z) = z^2, z = re^{i\theta}$

$\therefore \omega = r^2 e^{2\theta}$

$\Rightarrow f(z) = z^2$ maps right half-plane $\{z : Re\ z > 0\} = \{z = re^{i\theta}, 0 < r < \infty, |\theta| < \frac{\pi}{2}$ onto the slit plane $\mathbb{C} \setminus (-\infty, 0)$.



Let $y = b$. Then $\omega = u + iv = (x + ib)^2$

$= x^2 - b^2 + 2ibx$

$\therefore u = x^2 - b^2, v = 2bx$

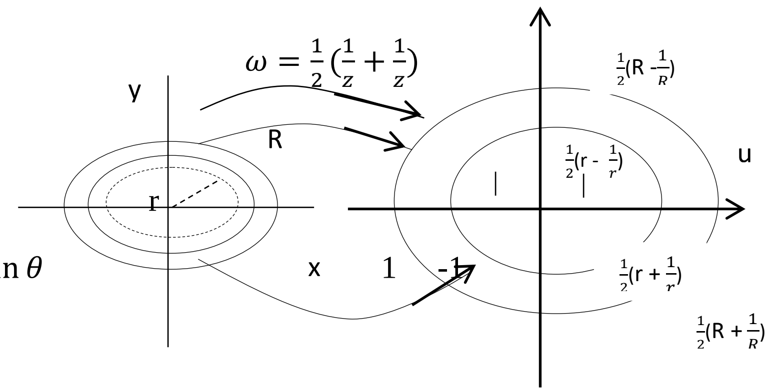
$i.e,\ u = \frac{a^2}{4b^2} - b^2\ (Parabola)$

Similarly, if $x - a, u = -\left(\frac{v^2}{4a^2} - a^2\right)$

Cor:$\omega = z^n = r^n e^{in\theta}$ maps a sector $\frac{2\pi}{n}$ onto a whole $\mathbb{C}$.

(2). $\omega = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$= \frac{1}{2}\left\{r(\cos\theta + i\sin\theta) + \frac{1}{r} \cdot \frac{1}{\cos\theta + i\sin\theta}\right\}$



$u + iv = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\theta + \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\theta$

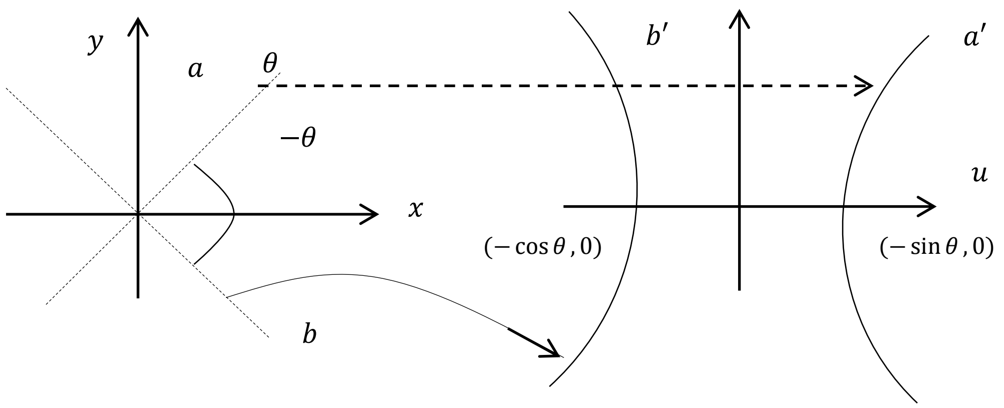
$\therefore \frac{4u^2}{\left(r+\frac{1}{r}\right)^2} + \frac{4v^2}{\left(r-\frac{1}{r}\right)^2} = 1$

(i). $\omega = \frac{1}{2}\left(z + \frac{1}{z}\right)$ maps unit circle $\{z: |z| = 1\}$ onto the closed interval $[-1,1]$.

(ii). If maps any circle $\{z: |z| = r\}$ with $r < 1\ or\ r > 1$ to an ellipse with major axis along y-axis containing the closed interval $[-1, 1]$.

(iii). $Arg\ z = 0\ and\ Arg\ z = \pi(i.e, x - axis\ and\ Arg\ z = \frac{\pi}{2}\ or\ -\frac{\pi}{2}(i.e, y - axis)$ maps onto themselves.

(iv). For any line $Arg\, z = \theta\ \left(\theta \neq 0, \pm\pi, \pm\frac{\pi}{2}\right), \frac{u^2}{\csc^2\theta} - \frac{v^2}{\sin^2\theta} = \frac{1}{4}\left[\left(r + \frac{1}{r}\right)^2 - \left(r - \frac{1}{r}\right)^2\right]$ maps to a hyperbola.



(3). KoebeFunction: $\omega = f(z) = \frac{z}{(1-z)^2} = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4}$

$maps\ \mathbb{D} = \{z:\ |z| < 1\}\ onto\ \mathbb{C}\left|\left(-\infty, -\frac{1}{4}\right)\right. \Rightarrow g(z) = \left(\frac{1+z}{1-z}\right)^2$ maps $\mathbb{D}$ onto $\mathbb{C}|-\infty, 0)$

(4). $\omega = f(z) = z - \frac{z^2}{2}$ maps $\mathbb{D}$ onto the interior of the cardioid.

(5). $f(z) = \frac{z}{1-z^2}$ maps $\mathbb{D}$ onto $\mathbb{C}\left|\left(\left(\frac{1}{2} \leq y < \infty\right) \cup \left(-\infty < y \leq \frac{1}{2}\right)\right)\right.$

(6). $f(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ $maps$ $\mathbb{D}$ onto the horizontal strip $\frac{\pi}{4} < Im(\omega) < \frac{\pi}{4}$

Definition (Univalent Function): A single-valued function f is said to be univalent (Schlicht) in a domain $\mathbb{D} \subset \mathbb{C}$ if it never takes the same value twice; i.e.,

If $f(z_1) \neq f(z_2)\ \forall\ points\ z_1, z_2 \in \mathbb{D}\ with\ z_1 \neq z_2$.

(i). For analytic function f, the condition $f'(z_0) \neq 0$ is equivalent to local univalence at z_0.

(ii). An analytic univalent function is called a conformal mapping because of its angle-preserving property.

S:of all analytic and univalent function f in the unit $disk$ $\mathbb{D} = \{z:\ |z| < 1\}$ normalized by the conditions $f(0) = 0\ and\ f'(0) = 1$. So it has Taylor's series expansion of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \ldots\ldots\ldots \qquad |z| < 1$$

Riemann Mapping Theorem: Let D be a simply connected domain which is a proper subset of the complex plane. Let ρ be a given point in D. Then there is a unique function f which maps D conformally onto the unit disk and has the properties $f(\rho) = 0, |f'(\rho)| > 0$.

This theorem tells us that most of the theorems concerning $f \in S$ are readily translated to statements about univalent function in arbitrary simple connected domains.

Example (1.11):

(i) Koebe Function: $k(z) = \frac{z}{(1-z)^2} = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4}$ maps D conformally onto the entire plane minus the part of the negative real axis from $-\frac{1}{4}$ to infinity.

(ii) $f(z) = z$ (iii)$f(z) = \frac{z}{1-z}$which maps (i) conformally onto the half-plane $R\{\omega\} > -\frac{1}{2}$

Conjugation: If $f \in S$ and $g(z) = \overline{f(\bar{z})}$ then $g(z) \in S$.

Rotation: If $f \in S$ and $g(z) = e^{-i\theta} f\left(e^{i\theta} z\right)$ then $g(z) \in S$.

Dilation:$f \in S$ and $g(z) = \frac{1}{r} f(rz), 0 < r < 1$ then $g(z) \in S$.

Disk automorphism:$f \in S$ and $g(z) = \frac{f\left(\frac{z+\alpha}{1+\bar{\alpha}z}\right) - f(\alpha)}{(1+|z|^2) f'(\alpha)}$, $|\alpha| < 1$ $then\ g \in S$.

Omitted value transformation: If $f \in S\ and\ f(z) \neq \omega\ then\ g(z) = \frac{\omega f}{\omega - f} \in S$.

Square root transformation: If $f \in S$ and $g(z) = \sqrt{f(z^2)}$, then $g \in S$.

Σ: $class\ of\ analytic\ and\ univalent\ in\ the\ domain$

$\Delta = \{z : |z| > 1\}\ exterior\ to\ \mathbb{D}, except\ for\ a\ simple\ pole\ at\ infinity\ with\ residue\ 1.$

$g(z) = z + b_0 + \sum_{n=1}^{\infty} b_n z^{-n}$, $|z| > 1$

Theorem (Area Theorem): If $g \in \Sigma, then\ \sum_{n=1}^{\infty} n|b_n|^2 \leq 1$.

Theorem (Bieberbach's): If $f \in S, then\ |a_2| \leq 2$ with equality f is a rotation of the Koebe function.

Theorem (Koebe – one-Quarter theorem): The range of every function of S contains the disk $\left\{\omega : |\omega| < \frac{1}{4}\right\}$

BieberbachConjecture: The coefficient of each function $f \in S$ satisfy $|a_n| \leq n\ for\ n =$ 2,3, ... strict equality holds for rotation of a koebe function.

Theorem (Little wood's): The coefficients of each function $f \in S$ satisfy

$|a_n| \leq e_\omega\ for\ n = 2,3, \dots \dots$

A few more concepts:

Uniqueness Theorem: Let $D \subset C$ be a domain and $f, g : D \to C$ is analytic. If there exists an infinite sequence $\{z_n\} \subset D$, such that $f(z_n) = g(z_n), \forall\ n \in N$ and $z_n \to z_0 \in D, f(z) = g(z)\ for\ all\ z \in D$. Find all entire functions f such that $f(r) = 0\ for\ all\ r \in Q$.

Find all entire functions f such that $f(x) = \cos x + i \sin x$ for all $x \in (0,1)$.

Find all analytic functions $f : B(0,1) \to C$ such that $f\left(\frac{1}{n}\right) = \sin\left(\frac{1}{n}\right), \forall\ n \in N$.

There does not exists an analytic function f defined on $B(0,1)$ such $that\ f\ (x)\ =\ |x|^3$?

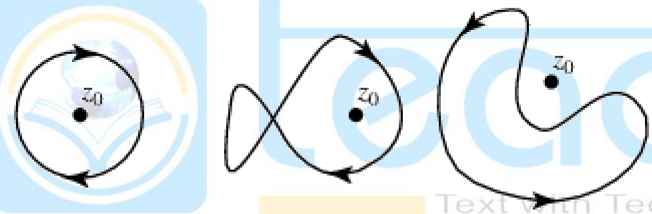
Morera's theorem:

If $f(z)$ is continuous in a region D and satisfies $\oint_\gamma f\,dz = 0$ for all closed contours γ in D, then $f(z)$ is analytic in D.

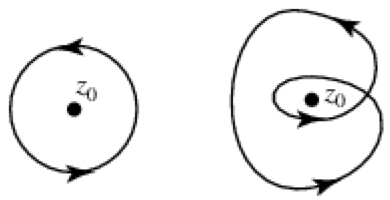
Morera's theorem does not require simple connectedness, which can be seen from the following proof. Let D be a region, with $f(z)$ continuous on D, and let its integrals around closed loops be zero. Pick any point $z_0 \in D$ and pick a neighborhood of z_0. Construct an integral of f, $F(z) = \int_{z_0}^{z} f(z)\,dz$.

Then one can show that $F'(z) = f'(z)$, and hence F is analytic and has derivatives of all orders, as does f, so f is analytic at z_0. This is true for arbitrary $z_0 \in D$, so f is analytic in D. It is, in fact, sufficient to require that the integrals of f around triangles be zero, but this is a technical point. In this case, the proof is identical except $F(z)$ must be constructed by integrating along the line segment $\overline{z_0\,z}$ instead of along an arbitrary path.

Winding number:



$\mathrm{Ind}_\gamma(z_0) = -1$ $\mathrm{Ind}_\gamma(z_0) = -1$ $\mathrm{Ind}_\gamma(z_0) = 0$



$\mathrm{Ind}_\gamma(z_0) = +1$ $\mathrm{Ind}_\gamma(z_0) = +2$

The winding number of a contour γ about a point z_0, denoted $n(\gamma, z_0)$, is defined by $n(\gamma, z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{dz}{z-z_0}$ and gives the number of times γ curve passes (counterclockwise) around a point. Counterclockwise winding is assigned a positive winding number, while clockwise winding is assigned a negative winding number. The winding number is also called the index, and denoted $Ind_\gamma(z_0)$.

The contour winding number was part of the inspiration for the idea of the Brouwer degree between two compact, oriented manifolds of the same dimension. In the language of the degree of a map, if $\gamma : [0,1] \to \mathbb{C}$ is a closed curve (i.e., $\gamma(0) = \gamma(1)$), then it can be considered as a function from $\mathbb{S}^1$ to $\mathbb{C}$. In that context, the winding number of γ around a point p in $\mathbb{C}$ is given by the degree of the map $\frac{\gamma-p}{|\gamma-p|}$ from the circle to the circle.

Rouche's Theorem:

Given two functions f and g analytic in A with γ a simple loop homotopic to a point in A, if $|g(z)| < |f(z)|$ for all z on γ, then f and $f + g$ have the same number of roots inside γ.

Argument Principle:

If $f(z)$ is mertomorphic in a region R enclosed by a contour γ, let N be the number of complex roots of $f(z)$ in γ, and P be the number of poles in γ, with each zero and pole counted as many times as its multiplicity and order, respectively. Then

$N - P \frac{1}{2\pi i} \int_\gamma \frac{f'(z)dz}{f(z)}$

Defining $w \equiv f(z)$ and $\sigma \equiv f(\gamma)$ gives $N - P \frac{1}{2\pi i} \int_\gamma \frac{dw}{w}$