(Model Questions of Real Analysis

1. Let P be a real polynomial of the real variable x of the form $P(x) = x^n + a_{n-1}x^{n-1} + \dots\dots + a_1 x - 1$. Suppose that P has no roots in the open unit disc and $P(-1) = 0$. Then

   (a) $P(1) = 0$

   (b) $\lim_{x\to\infty} P(x) = \infty$

   (c) $P(2) > 0$

   (d) $P(3) = 0$

Ans: (a), (b), (c)

Solution:

Given, $P(x) = x^n + a_{n-1}x^{n-1} + \dots\dots + a_1 x - 1$

Since, $P(-1) = 0$

$\therefore n \geq 2$ [$\therefore$ for $n = 1$, $P(x) = x - 1$, not possible].

Now, since $P(x)$ is real polynomial with real variable x, i.e., $P(x)$ has only real roots.

Further, $product\ of\ roots = \begin{cases} 1, & n\ is\ odd \\ -1, & n\ is\ even \end{cases}$

Also, $P(x)$ has no root in the open unit disc. Thus only possibility for roots is $x = \pm 1$.

If possible, let $'x = -1'$ is the only root of $P(x)$.

Then product of roots (For odd n) $= -1$, not possible.

Thus $'x = 1'$ is also root of $P(x) \Rightarrow P(1) = 0$.

$\therefore$ Option (a) is correct.

Clearly, $P(x) = (x-1)^r (x+1)^S$, $r + s = n \Rightarrow \lim_{x\to\infty} P(x) = \infty$

$P(2) > 0$, but $P(3) \neq 0$

Hence, options (b) and (c) are correct and option (d) is incorrect.

2. Which of the following series is convergent?

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}-\sqrt{n}}$

(b) $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

(c) $\sum_{n=1}^{\infty} (-1)^n \ log\, n$

(d) $\sum_{n=1}^{\infty} \frac{\log n}{n}$

Ans: (b)

Solution:

Let $u_n = \frac{1}{\sqrt{n+1}-\sqrt{n}}$

Take, $v_n = \frac{1}{\sqrt{n}}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}-\sqrt{n}} = \lim_{n\to\infty} \sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right) \to \infty$ and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent [by $p-test$].

$\therefore$ By comparison test, $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}-\sqrt{n}}$, is divergent.

For option (b),

Let $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

Since $\left|\frac{\sin n}{n^2}\right| \leq \frac{1}{n^2}$ [$\because$ $|\sin x| \leq 1 \ \forall\ x \in \mathbb{R}$] and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

$\therefore$ By Weierstrass $M-test$, $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ is convergent.

For option (c),

Let $\sum_{n=1}^{\infty} (-1)^n \log n$.

Since, $\log n \to \infty$ as $n \to \infty$.

$\therefore$ $\sum_{n=1}^{\infty} u_n$ cannot be convergent.

For option (d)

Let $u_n = \frac{\log n}{n}$, Then $v_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{\log n}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \log n = \infty$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

$\therefore$ By comparison test $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{\log n}{n}$ is divergent.

Thus only option (b) is correct.

3. Let $D_{(a,b)}(r) = \{(x, y) : (x-a)^2 + (y-b)^2 < r\}$. Which of the following subsets for $\mathbb{R}$ are connected?

(a) $D_{(0,0)}(1) \cup \{(1,0)\} \cup D_{(2,0)}(1)$

(b) $D_{(0,0)}(1) \cup D_{(2,0)}(1)$

(c) $D_{(0,0)}(1) \cup \{(1,0)\} \cup D_{(0,2)}(1)$

(d) $D_{(0,0)}(1) \cup D_{(0,2)}(1)$

Ans: (a)

Solution:

Given, $D_{(a,b)}(r) = \{(x, y) : (x-a)^2 + (y-b)^2 < r\}$

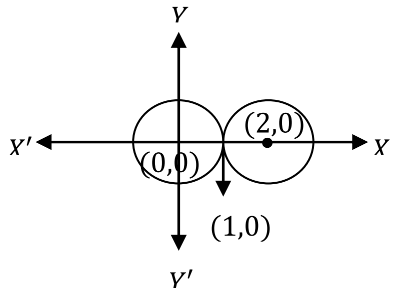
Here, $D_{(0,0)}(1) \cup \{(1,0)\} \cup D_{(2,0)}(1) = \{(x, y) : x^2 + y^2 < 1\} \cup \{(1,0) : (x-2)^2 + y^2 < 1\}$

(a) From diagram,

it is clear that above

set is a connected set.

$\therefore$ option (a) is correct.



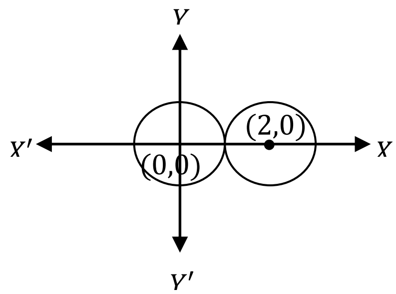
(b) $D_{(0,0)}(1) \cup \{(1,0)\} \cup D_{(2,0)}(1) = \{(x, y) : x^2 + y^2 < 1\} \cup \{(1,0) : (x-2)^2 + y^2 < 1\})$

From diagram,

it is clear that, above set

is disconnected

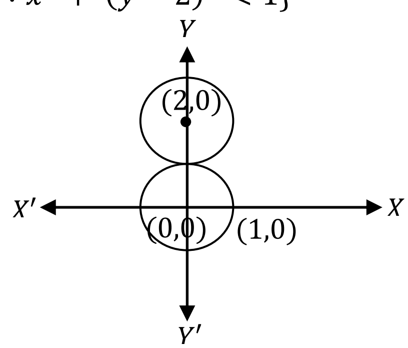
as (1,0) is not the set.



(c) $D_{(0,0)}(1) \cup \{(1,0)\} \cup D_{(0,2)}(1) = \{(x, y) : x^2 + y^2 < 1\} \cup \{(1,0) : x^2 + (y-2)^2 < 1\}$

Clearly above set is disconnected as

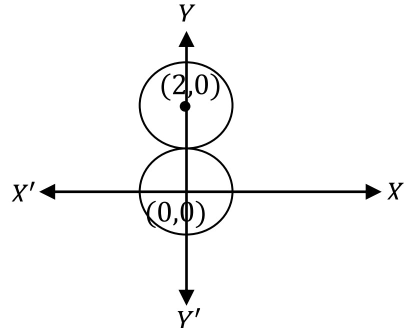
(1,0) is not in the set.



(d) $D_{(0,0)}(1) \cup \{(1,0)\} \cup D_{(0,2)}\,(1) = \{(x,y) : x^2 + y^2 < 1\} \cup \{(1,0) : x^2 + (y-2)^2 < 1\}$

Clearly, above set is disconnected

as (0,1) is not in set.



4. Which of the following is/are correct?

(a) $\log\frac{x+y}{2} \leq \frac{\log x + \log y}{2}$ for all $x, y > 0$

(b) $e^{\frac{x+y}{2}} \leq \frac{e^x + e^y}{2}$ for all $x, y > 0$.

(c) $\sin\frac{x+y}{2} \leq \frac{\sin x + \sin y}{2}$ for all $x, y > 0$.

(d) $\frac{(x+y)^k}{2^k} \leq \max\{x^k, y^k\}$ for all $x, y > 0$ and all $k \geq 1$.

Ans: (b), (d)

Solution:

Let $x, y > 0$

(a) We know that, $A.M. \geq G.M. \Rightarrow \frac{x+y}{2} \geq \sqrt{xy}$ ($\because \log x$ is increasing function)

$\Rightarrow \log\left(\frac{x+y}{2}\right) \geq \log\sqrt{xy} \Rightarrow \log\left(\frac{x+y}{2}\right) \geq \frac{1}{2}(\log x + \log y)$

$\therefore$ option (a) is incorrect.

(b) Similarly, $\frac{e^x + e^y}{2} \geq \sqrt{e^x \cdot e^y}$ i.e., $e^{\frac{x+y}{2}} \leq \frac{e^x + e^y}{2}$

$\therefore$ option (b) is correct.

(c) The inequality $\sin\left(\frac{x+y}{2}\right) \leq \frac{\sin x + \sin y}{2}\ \forall\ x, y > 0$ does not hold for $x = \frac{\pi}{6},\ y = \frac{\pi}{3}$.

$\therefore$ option (c) is incorrect.

(d) It can be prove easily by principle of mathematical induction that $\frac{(x+y)^k}{2^k} \leq \max\{x^k, y^k\}$ for all $x, y > 0$ and all $k \geq 1$.

$\therefore$ option (d) is correct.

5. Using the fact $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \log 2$, $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n(n+1)}$ equals.

(a) $1 - 2\log 2$

(b) $1 + \log 2$

(c) $(\log 2)^2$

(d) $(-\log 2)^2$

Ans: (a)

Solution:

Given, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \log 2$, $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^{n}}{n} - \sum_{n=1}^{\infty} \frac{(-1)^{n}}{n+1}$

$= \sum_{1}^{\infty} \frac{(-1)^{n+1-1}}{n} - \sum_{1}^{\infty} \frac{(-1)^{n+2-2}}{n+1} = (-1)\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n} - \sum_{n=1}^{\infty} \frac{(-1)^{n+2}}{n+1}$

$= -\log 2 - \left(-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ \ldots..\right)$

Since, $\log 2 = 1 \ -\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ \ldots..$

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n}}{n(n+1)} = -\log 2 - \left(-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ \ldots..\right) = -\log 2 - \ (\log 2 - 1) = \ 1 - 2\log 2$.

6. The set $\left\{\frac{1}{n}\sin\frac{1}{n} : n \in \mathbb{N}\right\}$ has

(a) One limit point and it is 0.

(b) One limit point and it is 1.

(c) One limit point and it is -1

(d) Three limit and these are -1, 0, 1.

Ans: (a)

Solution: Since, $\sin\frac{1}{n}$ is bounded and $\lim_{n\to\infty}\frac{1}{n} = 0, \therefore \lim_{n\to\infty}\frac{1}{n}\sin\frac{1}{n} = 0$.

[if $\because \lim_{n\to c} f(x) = 0$ and $g(x)$ is bounded in deleted neighborhood of c, then $\lim_{n\to c} f(x)\, g(x) = 0$].

$\therefore$ The set $\left\{\frac{1}{n}\sin\frac{1}{n} : n \in \mathbb{N}\right\}$ has one limit point and it is 0.

7. Which of the following is / are matrices on $\mathbb{R}$?

(a) $d(x, y) = \min(x, y)$

(b) $d(x, y) = |x - y)|$

(c) $d(x, y) = |x^2 - y^2|$

(d) $d(x, y) = |x^3 - y^3|$

Ans: (b), (d)

Solution:

For option (a)

$d(x, x) = \min(x, x) = x \neq 0$

$\therefore$ $d(x, y) = \min(x, y)$ is not a metric on $\mathbb{R}$

For options (b), (c) and (d)

$d(x, y) = |x^n - y^n|$ is a metric on $\mathbb{R}$ only for odd values of n.

$\therefore$ Option (b) and (d) are correct and option (c) is incorrect.

8. Consider the improper integral $\int_0^x y^{-\frac{1}{2}} dy$. This integral is

(a) Continuous on $[0, \infty)$

(b) Continuous only in $(0, \infty)$

(c) Discontinuous only $(0, \infty)$

(d) Discontinuous only in $\left(\frac{1}{2}, \infty\right)$

Ans: (a)

Solution:

The given integral is $\int_0^x y^{-\frac{1}{2}} dy$

Let $f(x) = \int_0^x y^{-\frac{1}{2}} dy \Rightarrow f(x) = 2\sqrt{y}\Big|_0^x = 2\sqrt{x}$, which is continuous in $[0, \infty)$.

$\therefore$ Option (a) is correct.

9. Let $f : (0,1) \to \mathbb{R}$ be continuous. Suppose that $|f(x) - f(y)| \leq |\cos x - \cos y|$ for all $x, y \in (0,1)$. Then,

(a) f is discontinuous at least at one point in (0,1).

(b) f is continuous everywhere on (0,1) but not uniformly continuous on (0,1).

(c) f is uniformly continuous on (0,1)

(d) $\lim\limits_{x \to 0^+} f(x)$ exists.

Ans: (c), (d)

Solution:

Given, $f : (0,1) \to \mathbb{R}$ is continuous such that

$|f(x) - f(y)| \leq |\cos x - \cos y| \leq |x - y| \ \forall \ x, y \in (0,1)$

i.e., $|f(x) - f(y)| \leq |x - y| \ \forall \ x, y \in (0,1) \Rightarrow f(x)$ satisfies Lipschitz condition on (0,1).

$\therefore$ $f(x)$ is uniformly continuous on (0,1).

Thus, option (a) and (b) are incorrect and option (c) correct.

Result: A function $'f'$ is uniformly continuous on (a,b) if and only if $'f'$ is continuous on (a,b) and $\lim\limits_{x \to a^+} f(x)$ and $\lim\limits_{x \to b^-} f(x)$ exists.

Using above result, option (d) is also correct.

10. Let $A = \{x^2 : 0 < x < 1\}$ and $B = \{x^3 : 1 < x < 2\}$. Which of the following statements is true?

(a) There is a one to one, onto function from A to B.

(b) There is no one to one, onto function from A to B taking rationales to rationales.

(c) There is no one to one function from A to B which is onto.

(d) There is no onto function from A to B which is one – one.

Ans: (a)

Solution:

$A = \{x^2 : 0 < x < 1\}$ and $B = \{x^3 : 1 < x < 2\}$

A and B are open intervals and in metric space (R, d) open intervals are homeomorphic.

So, there exists a homeomorphism of A onto B.

So, there exists a homeomorphism of A onto B.

$\therefore$ There is one – one and onto function from A to B.

11. Let $f : \mathbb{R} \to [0, \infty)$ be a non – negative real valued continuous function.

Let $\phi_n(x) = \begin{cases} n, & if \;\; f(x) \geq n \\ 0, & if \;\; f(x) < n \end{cases}$, $\phi_{n,k}(x) = \begin{cases} \frac{k}{2^n}, & if \;\; f(x) \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right] \\ 0, & if \;\; f(x) \notin \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right] \end{cases}$

and $g_n(x) = \phi_n(x) + \sum_{k=0}^{n2^n-1} \phi_{n,k}(x)$. As $n \to \infty$, which of the following are true?

(a) $g_n(x) \to f(x)$ for every $x \in \mathbb{R}$.

(b) given any $c > 0, g_n(x) \uparrow f(x)$ uniformly on the set $\{x : f(x) < c\}$

(c) $g_n(x) \uparrow f(x)$ uniformly for $x \in \mathbb{R}$.

(d) given any $c > 0, g_n(x) \uparrow f(x)$ uniformly on the set $\{x : f(x) \geq c\}$

Ans: (a), (b)

12. Let $\lambda > 0$ and $F(x) = 1 - e^{-\lambda x}$ for $x > 0$. Then for $t > 0, \int_0^\infty e^{-tx}\, dF(x)$ equals.

(a) $\frac{\lambda}{\lambda+t'}$

(b) $\frac{\lambda}{\lambda-t'}$

(c) 0

(d) ∞

Ans: (a)

Solution:

Take $\lambda = 1$

$\therefore F(x) = 1 - e^{-x}$

Thus for $t > 0, \int_0^\infty e^{-tx}\, dF(x) = \int_0^\infty e^{-tx}\, d(1 - e^{-x}) = \int_0^\infty e^{-tx}\, (e^{-x}) \cdot dx = \int_0^\infty e^{-(1+t)x}\, dx$

$= \frac{e^{-(1+t)x}}{-(1+t)}\Big|_0^\infty = \frac{1}{1+t}$

Only, option (a) satisfied for $\lambda = 1$

So, option (a) is correct.

13. Which of the following are compact?

(a) $\{(x,y) \in \mathbb{R}^2 : (x-1)^2 + (y-2)^2 = 9\} \cup \{(x,y) \in \mathbb{R}^2 : y = 3\}$

(b) $\left\{\left(\frac{1}{m},\frac{1}{n}\right) \in \mathbb{R}^2 : m,n \in \mathbb{Z}\,|\{0\}\right\} \cup \{(0,0)\} \cup \left\{\left(\frac{1}{m},0\right) : m \in \mathbb{Z}\,|\{0\}\right\} \cup \left\{\left(0,\frac{1}{n}\right) : n \in \mathbb{Z}|\{0\}\right\}$.

(c) $\{(x,y,z) \in \mathbb{R}^3 : x^2 + 2y^2 - 3z^2 = 1\}$

(d) $\{(x,y,z) \in \mathbb{R}^3 : |x| + 2|y| - 3|z| \leq 1\}$

Ans: (b), (d)

Solution:

For option (a)

Since, the set $\{(x,y) \in \mathbb{R}^2 : y = 3\}$ is unbounded.

$\therefore$ $\{(x,y) \in \mathbb{R}^2 : (x-1)^2 + (y-2)^2 = 9\} \cup \{(x,y) \in \mathbb{R}^2 : y = 3\}$ is not compact.

For option (b)

Clearly, the set $\left\{\left(\frac{1}{m},\frac{1}{n}\right) \in \mathbb{R}^2 : m,n \in \mathbb{Z}\,|\{0\}\right\} \cup \{(0,0)\} \cup \left\{\left(0,\frac{1}{n}\right) : n \in \mathbb{Z}\,|\{0\}\right\} \cup \left\{\left(\frac{1}{m},0\right) : m \in \mathbb{Z}|\{0\}\right\} \cup \{(0,0)\}$ is closed and bounded.

$\therefore$ It is compact.

For option (c)

Since, the set $\{(x,y,z) \in \mathbb{R}^3 : x^2 + 2y^2 - 3z^2 = 1\}$ is closed but not bounded.

$\therefore$ It is not compact.

For option (d)

Since, the set $\{(x,y,z) \in \mathbb{R}^3 : |x| + 2|y| - 3|z| \leq 1\}$ is closed and bounded.

$\therefore$ it is compact.

14. For $V = (v_1, v_2) = \mathbb{R}^2$ and $W = (w_1, w_2) \in \mathbb{R}^2$ consider the determinant map $\det : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $\det(V, W) = v_1 w_2 - v_2 w_1$. Then the derivative of the determinant map at $(V, W) \in \mathbb{R}^2 \times \mathbb{R}^2$ evaluated on $(H, K) \in \mathbb{R}^2 \times \mathbb{R}^2$ is

(a) $\det(H, W) + \det(V, K)$

(b) $\det(H, K)$

(c) $\det(H, V) + \det(W, K)$

(d) $\det(V, H) + \det(K, W)$

Ans: (a)

Solution:

Let, $\det(V, W) = D\big((v_1, v_2), (w_1, w_2)\big) = v_1 w_2 - v_2 w_1$

Then the derivative of determinant map at (V, W) evaluated on (H, K) is given by

$\frac{\partial D}{\partial v_1} H_1 + \frac{\partial D}{\partial v_2} H_2 + \frac{\partial D}{\partial w_1} K_1 + \frac{\partial D}{\partial w_2} K_2$

$= w_2 H_1 - w_1 H_2 - v_2 k_1 + v_1 k_2 = (w_2 H_1 - w_1 H_2) + (v_2 k_1 - v_1 k_2)$

$= \det(H, w) + \det(V, k)$

Thus, equation (a) is correct.

15. Consider the function $f(x) = e^{-x}$ and its Taylor approximation $g(x)$ of degree 3. For $x = \frac{1}{3}$, $g(x)$ is

(a) Positive and less than 1.

(b) Negative and less than 2.

(c) Positive and greater than 1.

(d) Less than 1 but greater than 0.75.

Ans: (a)

Solution:

$f(x) = e^{-x}$

$g(x) = f\left(\frac{1}{3}\right) + \left(x - \frac{1}{3}\right) f'\left(\frac{1}{3}\right) + \frac{1}{2!}\left(x - \frac{1}{3}\right)^2 f''\left(\frac{1}{3}\right) + \frac{1}{3!}\left(x - \frac{1}{3}\right)^3 f'''\left(\frac{1}{3}\right)$

$\therefore g\left(\frac{1}{3}\right) = f\left(\frac{1}{3}\right) = e^{-\frac{1}{3}}$

Clearly, $) < g\left(\frac{1}{3}\right) < 1$

$\therefore$ Option (a) is correct.

16. Let $f_n(x) = x^{\frac{1}{n}}$ for $x \in [0,1]$. Then –

(a) $\lim_{n\to\infty} f_n(x)$ exists for all $x \in [0,1]$

(b) $\lim_{n\to\infty} f_n(x)$ defines a continuous function on $[0,1]$.

(c) $\{f_n(x)\}$ converges uniformly on $[0,1]$.

(d) $\lim_{n\to\infty} f_n(x) = 0$ all $x \in [0,1]$.

Ans: (a)

Solution: Given $f(x) = x^{\frac{1}{n}}$ for $x \in [0,1]$.

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & x = 0 \\ 1, & 0 < x \le 1 \end{cases}$$

Clearly, $\lim_{n\to\infty} f_n(x)$ exists for all $x \in [0,1]$.

But $\lim_{n\to\infty} f_n(x)$ is not continuous on $[0,1]$.

Thus, options (b) and (d) are incorrect and (a) is correct.

Result: If $\{f_n(x)\}$ is a sequence of continuous functions on $[a,b]$ converging pointwise to f on $[a,b]$ and $f(x)$ is not continuous on $[a,b]$. Then $\{f_n(x)\}$ does not converge uniformly on $[a,b]$.

17. Let $I = \{1\} \cup \{2\} \subset \mathbb{R}$. For $x \in \mathbb{R}$, let $\phi(x) = dist\,(x, I) = \inf\{|x-y| : y \in I\}$. Then

(a) ϕ is discontinuous somewhere on $\mathbb{R}$.

(b) ϕ is continuous on $\mathbb{R}$ but not differentiable only at $x = 1$.

(c) ϕ is continuous on $\mathbb{R}$ but not differentiable only at $x = 1$ & 2.

(d) ϕ is continuous on $\mathbb{R}$ but not differentiable only at $x = 1, \frac{1}{3}$ & 2.

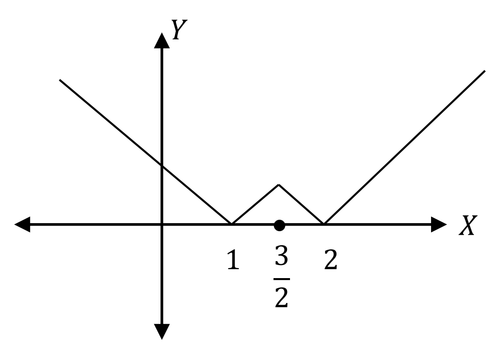
Ans: (d)

Solution: We have $I = \{1\} \cup \{2\} \subseteq \mathbb{R}$ and $\phi(x) = dist\,(x, I) = \inf\{|x-y| : y \in I\}$

Graph of the function $\phi(x)$ is given below.

From graph it is sclear that $\phi(x)$ has corner points at $x = 1, \frac{3}{2}, 2$.

Thus, $\phi(x)$ is continuous on $\mathbb{R}$ but not differentiable at $x = 1, \frac{3}{2}, 2$



18. The set $\left\{\frac{1}{n}\sin\frac{1}{n} : n \in \mathbb{N}\right\}$ has

(a) One limit point ant it is 0.

(b) One limit point and it is 1.

(c) One limit point and it is -1

(d) Three limit points and therse are -1, 0 and 1.

Ans: (a)

Solution: Since, $\sin\frac{1}{n}$ is bounded and $\lim_{n\to\infty}\frac{1}{n} = 0$, $\therefore$ $\lim_{n\to\infty}\frac{1}{n}\sin\frac{1}{n} = 0$.

$\therefore$ The set $\left\{\frac{1}{n}\sin\frac{1}{n} : n \in \mathbb{N}\right\}$ has one limit pointy & it is $'0'$.

19. Let X denote the two – point set $\{0,1\}$ and write $X_j = \{0,1\}$ for every $j = 1, 2, 3, \dots$.. Let $Y = \prod_{j=1}^{\infty} X_j$. Which of the following is/are true?

(a) Y is a countable set.

(b) $card\ Y = card\ [0,1]$

(c) $\cup_{n=1}^{\infty}\left(\prod_{j=1}^{n} X_j\right)$ is uncountable.

(d) Y is uncountable.

Ans: (b), (d)

Solution: $X_j = \{0,1\}$ and $Y = \prod_{j=1}^{\infty} X_j$

Since, $X_j's$ are countable $\forall\ j$ and infinite product of countable set is uncountable.

$\therefore$ Y is uncountable set.

$\therefore$Option (d) is correct.

Since, $[0,1]$ is also uncountable.

$\therefore$ $card\ Y = card\ [0,1]$

$\therefore$ Option (b) is correct.

Also finite product of countable sets is countable.

$\Rightarrow \prod_{j=1}^{n} X_j$ is countable and countable union for countable sets is countable.

$\Rightarrow \cup_{n=1}^{\infty}\left(\prod_{j=1}^{n} X_j\right)$ is countable.

$\therefore$ Option (c) is incorrect.

20. Let $\{f_n\}$ be a sequence of integrable funcitons defined on an interval [a,b]. Then

(a) If $f_n(x) \to 0$ a.b., then $\int_a^b f_n(x)\, dx \to 0$

(b) If $\int_a^b f_n(x)\, dx \to 0$ then $f_n(x) \to 0$ a.e.

(c) If $f_n(x) \to 0$ a.e. and each f_n is a bounded function, then $\int_a^b f_n(x)\, dx \to 0$

(d) If $f_n(x) \to 0$ a.e. and the $f_n's$ are uniformly bounded, then $\int_a^b f_n(x)\, dx \to 0$

Ans: (d)

Solution: If $\{f_n\}$ be a sequence of measurable of functions on a measurable set E. Then $\{f_n\}$ is said to converge almost everywhere in E if there exists a subset E_0 of E such that

(i) $f_n(x) \to f(x)\ \forall\ x \in E - E_0$.

(ii) $m\,(E_0) = 0$

For options (a) and (c)

Take, $f_n(x) = \begin{cases} n, & x \in \left[0, \frac{1}{n}\right] \\ 0, & x \in \left[\frac{1}{n}, 0\right] \end{cases}$

Clearly, $f_n(x) \to 0$ a.e. but $\int_0^1 f_n(x)\, dx = \int_0^{\frac{1}{n}} n\, dx = 1$

∴ Option (a) is incorrect.

Also, in above, example f_n's are bounded.

But $\int_0^1 f_n(x)\, dx \to 1\ \ as\ n \to \infty$

∴ Option (c) is incorrect.

For optoion (b),

Take, $f_n(x) = \begin{cases} 1, & x \in \left[0, \frac{1}{3}\right] \\ -\frac{1}{2}, & x \in \left[\frac{1}{3}, 1\right] \end{cases}$

Here, $\int_0^1 f_n(x)\, dx = \int_0^{\frac{1}{3}} 1\, dx + \int_{\frac{1}{3}}^1 \left(-\frac{1}{2}\right) dx = 0$ but $f_n(x) \nrightarrow 0\ \ a.e.$

Thus, option (b) is incorrect.

For option (d)

Bounded convergence Theorem: If $\{f_n\}$ is a sequaence of uniformly bounded funcitons defined on a set E of finite measure and $\{f_n\}$ converges pointwise a measurable function f, then $\lim_{n\to\infty} \int_E f_n(x)\, dx = \int_E f_n(x)\, dx$

∴ Option (d) is correct.

21. Which of the following is/are correct?

(a) $n\log\left(1+\frac{1}{n+1}\right) \to 1 \ as\ n \to \infty$

(b) $(n+1)\log\left(1+\frac{1}{n}\right) \to 1 \ as\ n \to \infty$

(c) $n^2\log\left(1+\frac{1}{n}\right) \to 1 \ as\ n \to \infty$

(d) $n\log\left(1+\frac{1}{n^2}\right) \to 1 \ as\ n \to \infty$

Ans: (a), (b)

Solution: For option (a)

$$n\log\left(1+\frac{1}{n+1}\right) = \log\left(1+\frac{1}{n+1}\right)^n = \log\left(1+\frac{1}{n+1}\right)^{n+1-1} = \log\left(1+\frac{1}{n+1}\right)^{n+1}\log\left(1+\frac{1}{n+1}\right)^{-1}$$

$$\therefore \log\left(1+\frac{1}{n+1}\right)^{n+1}\log\left(1+\frac{1}{n+1}\right)^{-1} \to 1 \ as\ \ n \to \infty$$

$\therefore$ Option (a) is correct.

For option (b)

$$(n+1)\log\left(1+\frac{1}{n}\right) = n\log\left(1+\frac{1}{n}\right) + \log\left(1+\frac{1}{n}\right) = \log\left(1+\frac{1}{n}\right)^n + \log\left(1+\frac{1}{n}\right)$$

$$\lim_{n\to\infty}(n+1)\log\left(1+\frac{1}{n}\right) = \log e + 0 = 1$$

$\therefore$ Option (b) is correct.

$$n^2\log\left(1+\frac{1}{n}\right) = \log\left(1+\frac{1}{n}\right)^{n^2} \to \infty \ as\ \ n \to \infty$$

$\therefore$ Option (c) is incorrect.

Similarly, option (d) is also incorrect.

22. Which of the following subsets of $\mathbb{R}^2$ are convex?

(a) $\{(x, y) : \ |x| \leq 5, |y| \leq 10\}$

(b) $\{(x, y) : x^2 + y^2 = 1\}$

(c) $\{(x, y) : y \geq x^2\}$

(d) $\{(x, y) : y \leq x^2\}$

Ans: (a), (c)

Solution:

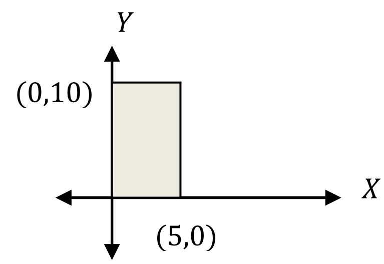
Result: A subset E of $\mathbb{R}^2$ is said to be convex if for all pairs of points $x_2, x_2 \in E$, the line segment joining x_1 and x_2, i.e., $\lambda\, x_1 + (1-\lambda)x_2,\ 0 \leq \lambda \leq 1$ is also contained in E.

For option (a)

Take, $A = \{(x, y): |x| \leq 5, |y| \leq 10\}$

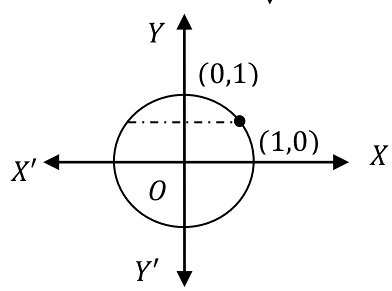
From graph, it is clear that set

$E = \{(x, y): |x| \leq 5, |y| \leq 10\}$ is convex.



For option (b)

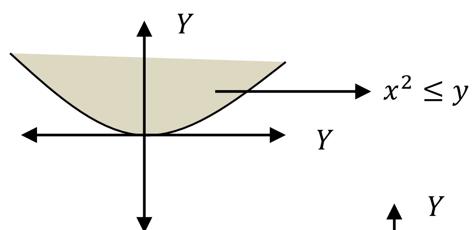
Take, $B = \{(x, y) : x^2 + y^2 = 1\}$



From the graph it is clear that set is not convex.

For option (c)

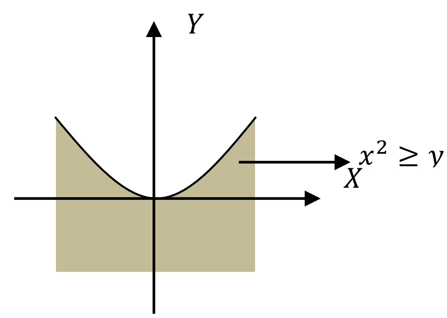
From graph it is clear that set is convex.



Take,

$D = \{(x, y) : y \leq x^2\}$

From the graph it is clear set D is not convex.



23. Consider the function $f(x) = |\cos x| + |\sin(2-x)|$. At which of the following points of f is/are not differentiable?

(a) $\left\{(2n+1)\cdot\frac{\pi}{2}\ : n \in \mathbb{Z}\right\}$

(b) $\{n\pi : n \in \mathbb{Z}\}$

(c) $\{n\pi + 2 : n \in \mathbb{Z}\}$

(d) $\left\{\frac{n\pi}{2} : n \in \mathbb{Z}\right\}$

Ans: (a), (c)

Solution: $f(x) = |\cos x| + |\sin(2-x)|$

As we know $|x|$ is not differentiable when $x = 0$.

$\therefore \cos x = 0$ & $\sin(2-x) = 0$

$\Rightarrow x = (2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$ & $2 - x = n\pi$ $\Rightarrow x = -n\pi + 2$ $\Rightarrow x = n\pi + 2, n \in \mathbb{Z}$

$\therefore$ Thus option (a), (c) are correct.

24. Let $F = \{f : \mathbb{R} \to \mathbb{R} : |f(x) - f(y)| \leq k\,|(x - y)|^{\alpha}\}$ for all $x, y \in \mathbb{R}$ and for some $\alpha > 0$ and some $k > 0$. Which of the following is/are true?

(a) Every $f \in F$ is continuous

(b) Every $f \in F$ is uniformly continuous

(c) Every differentiable function f is F.

(d) Every $f \in F$ is differentiable.

Ans: (a), (b)

Solution:

Given, $F = \{f : \mathbb{R} \to \mathbb{R} : |f(x) - f(y)| \leq k\,|(x - y)|^{\alpha}\}\ \forall\ x, y \in \mathbb{R}\ \ and\ \ \alpha, k > 0$.

Let $f \in F\ \ and\ \ \in > 0$ be any number.

Let $x, y \in \mathbb{R}$ such that $|x - y| < \delta$

Choose $\delta = \left(\frac{\in}{k}\right)^{\frac{1}{\alpha}}$

$\therefore |f(x) - f(y)| \leq k\,|(x - y)|^{\alpha}$ [$\because f \in F$]

$= k\,|(x - y)|^{\alpha} < k \cdot \delta^{\alpha} \Rightarrow f(x)$ is uniformly continuous and hence continuous.

Thus, options (a) and (b) are correct.

For option (c)

Consider, $f(x) = x^2$

Clearly, $f(x)$ is differentiable, but $f \notin F\ \ as\ \ f(x) = x^2$ doesn't satisfy given condition.

$\therefore$ option (c) is correct.

For option (d),

Take, $f(x) = |x|$

Since, $\big||x| - |y|\big| \leq |x - y| \Rightarrow f(x) = |x| \in F, but\ \ f(x)$ is not differentiable.

$\therefore$ option (d) is incorrect.

25. If $\{x_n\}$ and $\{y_n\}$ are sequences of real numbers, which of the following is/are true?

(a) $\lim_n \sup(x_n + y_n) \leq \lim_n \sup x_n + \lim_n \sup y_n$

(b) $\lim_n \sup(x_n + y_n) \geq \lim_n \sup x_n + \lim_n \sup y_n$

(c) $\lim_n \inf(x_n + y_n) \leq \lim_n \inf x_n + \lim_n \inf y_n$

(d) $\lim_n \inf(x_n + y_n) \geq \lim_n \inf x_n + \lim_n \inf y_n$

Ans: (a), (d)

Solution: For options (a) and (d)

Let $\bar{x}_n = \sup\{x_n, x_{n+1}, \ldots\ldots\}$ and $\bar{y}_n = \sup\{y_n, y_{n+1}, \ldots\ldots\}$

Then $x_k \leq \bar{x}_n \ \forall \ (k \geq n)$ and $y_k \leq \bar{y}_n \ \forall \ (k \geq n)$

$\therefore \ x_k + y_k \leq \bar{x}_n + \bar{y}_n \ (k \geq n)$

Thus, $\bar{x}_n + \bar{y}_n$ is an upper bound for $\{x_n + y_n, x_{n+1} + y_{n+1}, \ldots \ldots\}$

So that $(\overline{x_n + y_n}) = \sup\{x_n + y_n, x_{n+1} + y_{n+1}, \ldots \ldots\} \leq \bar{x}_n + \bar{y}_n$

$\therefore \ \lim_{n\to\infty}(\bar{x}_n + \bar{y}_n) \leq \lim_{n\to\infty}(\bar{x}_n + \bar{y}_n) = \lim_{n\to\infty}\bar{x}_n + \lim_{n\to\infty}\bar{y}_n,$

i.e., $\overline{\lim}(x_n + y_n) \leq \overline{\lim}\, x_n + \overline{\lim}\, y_n$

or, $\lim_n \sup(x_n + y_n) \leq \lim_n \sup x_n + \lim_n \sup y_n$

Thus, option (a) is correct.

Similarly, we can prove $\lim_n \inf(x_n + y_n) \geq \lim_n \inf x_n + \lim_n \inf y_n$

$\therefore$ Option (d) is correct.

For options (b) and (c)

Take, $x_n = (-1)^n, y_n = (-1)^{n-1}$

Clearly, $\lim_n \sup x_n = 1, \ \lim_n \sup y_n = 1$

$\lim_n \inf x_n = -1, \quad \lim_n \inf y_n = -1$

$x_n + y_n = 0 \ \forall \ n.$

$\therefore \ \lim_n \sup(x_n + y_n) = 0$

But $\lim_n \sup x_n + \lim_n \sup y_n = 2$

$\therefore$ Option (b) is incorrect.

Similarly, $\lim_n \inf(x_n + y_n) = 0$

But, $\lim_n \inf x_n + \lim_n \inf y_n = -2$

∴ Option (c) is incorrect.

26. Consider three subsets of $\mathbb{R}^2$, namely

$A_1 = \{(x, y) : x^2 + y^2 \leq 1\}$

$A_2 = \{(1, y) : y \in \mathbb{R}\}$

$A_3 = \{(0,2)\}$

Then there always exists a continuous real valued function f on $\mathbb{R}^2$ such that $f(x) = a_j$ for $x \in A_j, j = 1, 2, 3$.

(a) If and only if least two of the numbers a_1, a_2, a_3 are real and equal.

(b) If $a_1 = a_2 = a_3$.

(c) For all real values a_1, a_2, a_3.

(d) If and only if $a_1 = a_2$.

Ans: (b), (d)

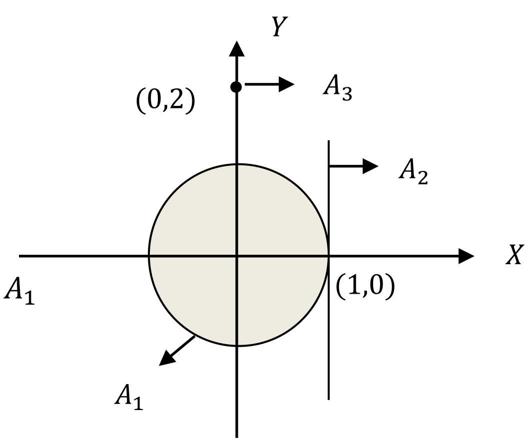
Solution: Given,

$A_1 = \{(x, y) : x^2 + y^2 \leq 1\}$

$A_2 = \{(1, y) : y \in \mathbb{R}\}$

$A_3 = \{(0,2)\}$

Graphical representation is given by



Since, $(1,0) \in A_1, A_2$ and $f(x) = a_1$ for $x \in A_1$

$f(x) = a_2$ for $x \in A_2$

∴ For $f(x)$ to be continuous at $(1,0)$

$a_1 = a_2$

As A_3 is disconnected.

∴ a_3 can take any arbitrary value.

Hence, options (b) and (d) are correct.

27. Which of the following statements is true?

(a) $\lim\limits_{x\to\infty} \frac{\log x}{x^{\frac{1}{2}}} = 0 \; and \; \lim\limits_{x\to\infty} \frac{\log x}{x} = \infty$

(b) $\lim\limits_{x\to\infty} \frac{\log x}{x^{\frac{1}{2}}} = \infty \; and \; \lim\limits_{x\to\infty} \frac{\log x}{x} = 0$

(c) $\lim\limits_{x\to\infty} \frac{\log x}{x^{\frac{1}{2}}} = 0 \; and \; \lim\limits_{x\to\infty} \frac{\log x}{x} = 0$

(d) $\lim\limits_{x \to \infty} \frac{\log x}{x^{\frac{1}{2}}} = 0 \; but \; \lim\limits_{x \to \infty} \frac{\log x}{x}$ does not exists.

Ans: (c)

Solution: $\lim\limits_{x \to \infty} \frac{\log x}{\sqrt{x}} \left(= \frac{\infty}{\infty}\right) = \lim\limits_{x \to \infty} \frac{\frac{1}{x}}{2\sqrt{x}} = \lim\limits_{x \to \infty} \frac{2}{\sqrt{x}} = 0$ and $\lim\limits_{x \to \infty} \frac{\log x}{x} \left(= \frac{\infty}{\infty}\right)$

$= \lim\limits_{x \to \infty} \frac{1}{x} = 0$

∴ Option (c) is correct.

28. Let X be a connected subset of real numbers. If every element of X is irrational, then the cardinality of X is

(a) Infinite

(b) Countably

(c) 2

(d) 1

Ans: (d)

Solution: We know that in $\mathbb{R}$, connected subsets are either intervals or singletons. If X is an interval, then it will contain rationals also, but X contains only irrationals.

∴ X is singleton.

Thus, cardinality of X is ‘1’.

29. Define $f : [0,1] \to [0,1]$ by $f(x) = \frac{2^k - 1}{2^k}$ for $x \in \left[\frac{2^{k-1}-1}{2^{k-1}}, \frac{2^k - 1}{2^k}\right], k \geq 1$

Then f is a Riemann integrable function such that

(a) $\int_0^1 f(x)\, dx = \frac{2}{3}$

(b) $\frac{1}{2} < \int_0^1 f(x)\, dx < \frac{2}{3}$

(c) $\int_0^1 f(x)\, dx = 1$

(d) $\frac{2}{3} < \int_0^1 f(x)\, dx < 1$

Ans: (a)

Solution: Given, $f : [0,1] \to [0,1]$ defined by $f(x) = \frac{2^k - 1}{2^k}$

$$\therefore \int_0^1 f(x)\, dx = \sum_{k=1}^{\infty} \int_{\frac{2^{k-1}-1}{2^{k-1}}}^{\frac{2^k-1}{2^k}} f(x)\, dx = \sum_{k=1}^{\infty} \frac{2^k - 1}{2^k}\left[\frac{2^k - 1}{2^k} - \frac{2^{k-1}-1}{2^{k-1}}\right]$$

$$= \sum_{k=1}^{\infty} \frac{2^k - 1}{2^k} \times \frac{1}{2^{k-1}}\left[\frac{2^k - 1 - 2^k + 2}{2}\right] = \sum_{k=1}^{\infty} \frac{2^k - 1}{2^{2k-1+1}} \times 1 = \sum_{k=1}^{\infty} \frac{2^k - 1}{2^{2k}} = \sum_{k=1}^{\infty} \frac{1}{2^k} - \sum_{k=1}^{\infty} \frac{1}{2^{2k}}$$

$$= \frac{\frac{1}{2}}{1 - \frac{1}{2}} - \frac{\frac{1}{4}}{1 - \frac{1}{4}} = 1 - \frac{1}{3} = \frac{2}{3}$$

30. Let (X, d) be a metric space and let $A \subseteq X$. For $x \in X$, define $d(x, A) = \inf\{d(x, a) : a \in A\}$.

If $d(a, A) = 0 \ \forall\, x \in X$, then which of the following assertions must be true?

(a) A is compact.

(b) A is closed

(c) A is dense in X

(d) $A = X$

Ans: (c)

Solution: Given that, (X, d) is a metric space and $A \subseteq X$.

Now, for all $x \in X, d(x, A) = 0\ \forall\, x \in X$.

$\Rightarrow$ Either $x \in A$ *or* x is a limit point of A $\Rightarrow$ $\bar{A} = X$ $\Rightarrow$ A is dense in X.

$\therefore$ Option (c) is correct.

31. Which of the following real valued functions f defined on $\mathbb{R}$ have the property that f is continuous and $f \circ f = f$?

(a) $f(x) = \begin{cases} |x|, & if\ x \in [-1,1] \\ x^2, & if\ x \notin [-1,1] \end{cases}$

(b) $f(x) = \begin{cases} x, & if\ x \in [0,1] \\ 1, & if\ x \geq 1 \\ 0, & if\ x \leq 0 \end{cases}$

(c) $f(x) = \begin{cases} x, & if\ x \in [-1,1] \\ 1, & if\ x \geq 1 \\ -1, & if\ x \leq -1 \end{cases}$

(d) $f(x) = \begin{cases} 1, & if\ x \in [-23,27] \\ 22 + x, & if\ x \leq -23 \\ -26 + x, & if\ x \geq 27 \end{cases}$

Ans: (b), (c)

Solution: For option (a)

$f(2) = 4$ and $(f \circ f)(2) = f(4) = 16 \Rightarrow f(2) \neq (f \circ f)(2)$.

$\therefore$ Option (a) is not correct.

For option (b),

$$f(x) = \begin{cases} x, & if\ x \in [0,1] \\ 1, & if\ x \geq 1 \\ 0, & if\ x \leq 0 \end{cases}$$

Clearly, f is continuous.

Further, $f \circ f(x) = f\big(f(x)\big) = \begin{cases} x, & if\ x \in [0,1] \\ 1, & if\ x \geq 1 \\ 0, & if\ x \leq 0 \end{cases} = f(x)$

So, option (b) is correct.

Similarly for option (c)

$$(f \circ f)(x) = f\big(f(x)\big) = \begin{cases} x, & if\ x \in [-1,1] \\ 1, & if\ x \geq 1 \\ -1, & if\ x \leq -1 \end{cases} = f(x)$$

$\therefore$Option (c) is correct.

For option (d)

$f(-24) = 22 - 24 = -2$ *and* $(f \circ f)(-24) = f\big(f(-24)\big) = f(-2) = 1$

$\Rightarrow f(-24) \neq (f \circ f)(-24)$

Hence, option (d) is incorrect.

32. Let $f(r,\theta) = (r\cos\theta, r\sin\theta)$ for $(r,\theta) \in \mathbb{R}^2$ with $r \neq 0$, which of the following statements are correct?

(a) The linear transformation $Df(r,\theta)$ is not zero for any $(r,\theta) \in \mathbb{R}^2$ with $r \neq 0$.

(b) f is one - one on $\{(r,\theta) \in \mathbb{R}^2 : r \neq 0\}$

(c) For any $(r,\theta) \in \mathbb{R}^2$ with $r \neq 0, f$ is one – one on a neighbourhood of (r,θ).

(d) $Df(r,\theta) = r^2\mathrm{I}$ for any $(r,\theta) \in \mathbb{R}^2$ with $r \neq 0$.

Ans: (a), (c)

Solution: Given, $f(r,\theta) = (r\cos\theta, r\sin\theta)$, $(r,\theta) \in \mathbb{R}$

Now, $Df(r,\theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r \neq 0$ [$\because r \neq 0$]

$\Rightarrow$ Option (a) is correct and option (d) is incorrect.

Also, $f(r,\theta) = (r\cos\theta, r\sin\theta)$ is one – one on a neighbourhood of (r,θ), but not one – one on $\{(r,\theta) \in \mathbb{R}^2 : r \neq 0\}$. [$\because f(r,\theta) = f(r, 2\pi)$]

33. The map $L : \mathbb{R}^2 \to \mathbb{R}^2$ given by $L(x,y) = (x,-y)$ is

(a) Differentiable every where on $\mathbb{R}^2$

(b) Differentiable only at (0,0).

(c) $DL(0,0) = L$

(d) $DL(x,y) = L$ for all $(x,y) \in \mathbb{R}^2$

Ans: (a), (c) (d)

Solution: $L(x,y) = (x,-y) = (f_1, f_2) \Rightarrow DL(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\Rightarrow DL(x,y) = L \quad \forall\, (x,y) \in \mathbb{R}^2$

Derivative of linear transformation is a linear transformation.

Also, $\frac{\partial f_1}{\partial x} = 1, \frac{\partial f_1}{\partial y} = 0, \frac{\partial f_2}{\partial x} = 0, \frac{\partial f_2}{\partial y} = -1$

Since, all partial derivatives exist and are continuous on $\mathbb{R}^2$.

$\therefore L(x,y)$ differentiable on $\mathbb{R}^2$.

$\therefore$ Option (c) and (d) are correct.

34. It is given that the series $\sum_{n=1}^{\infty} a_n$ is convergent, but not absolutely convergent and $\sum_{n=1}^{\infty} a_n = 0$. Denote by s_k the partial sum $\sum_{n=1}^{k} a_n$, $k = 1, 2, \ldots$ .... Then

(a) $s_k = 0$ for infinitely many k.

(b) $s_k > 0$ for infinitely many k, and $s_k < 0$ for infinitely many k.

(c) It is possible that $s_k > 0$ for all k.

(d) It is possible that $s_k > 0$ for all but a finite number of values of k.

Ans: (a)

Solution: Given $\sum_{n=1}^{\infty} a_n$ is convergent, but not absolutely convergent and $\sum_{n=1}^{\infty} a_n = 0$.

Consider a sequence a_n given by $\left\{0, 1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{4}, \dots\dots\dots\dots\dots.\right\}$

The sequence of partial sum for this sequences s_n will be given by $\left\{0, 1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}\right\}$.

Clearly, option (a) holds and option (b) is false.

Option (c) is incorrect, because if $s_k > 0$ for all k, then $\sum_{n=1}^{\infty} a_n = 0$ is not possible.

Option (b) is false, because s_n must be zero for infinitely many n.

35. Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} x^2, if\ x < 0 \\ 2x + x^2, if\ \ x \geq 0 \end{cases}$. Then which of the following statements are correct?

(a) $f''(x) = 2$ for all $x \in \mathbb{R}$

(b) $f''(0)$ does not exist.

(c) $f'(x)$ exists for each $x \neq 0$

(d) $f'(0)$ does not exist.

Ans: (b), (c), (d)

Solution: $\lim\limits_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \frac{2h+h^2-0}{h} = 2$ and $\lim\limits_{h\to 0^-} \frac{f(0)-f(0-h)}{h} = \lim\limits_{h\to 0^-} \frac{0-h^2}{h} = 0$

As L.H.D and R.H.D are not equal

$\therefore f'(0)$ does not exist

$\Rightarrow f''(0)$ does not exist.

Clearly, $f'(x)$ exists $\forall\ x \neq 0$

Thus, options (b), (c) and (d) are correct.

36. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f' is bounded. Given a is closed and bounded interval $[a, b]$, let $M(f, p)$ and $m(f, p)$ denote, respectively, the upper Riemann sum and the lower Riemann sum of f with respect to p. Then

(a) $\left|M(f,p) - \int_a^b f(x)\,dx\right| \leq (b-a) \sup\{\,|f(x)| : x \in [a,b]\}$

(b) $\left|m(f,p) - \int_a^b f(x)\,dx\right| \leq (b-a) \inf\{\,|f(x)| : x \in [a,b]\}$

(c) $\left|M(f,p) - \int_a^b f(x)\,dx\right| \leq (b-a)^2 \sup\{\,|f'(x)| : x \in [a,b]\}$

(d) $\left|m(f,p) - \int_a^b f(x)\,dx\right| \leq (b-a)^2 \inf\{\,|f'(x)| : x \in [a,b]\}$

Ans: (a), (c)

Solution: For option (a)

Given $f : \mathbb{R} \to \mathbb{R}$ is differentiable $\Rightarrow$ $'f'$ is continuous on $\mathbb{R}$ and hence $'f'$ is Riemann integrable on $[a,b]$.

Let p be any partition of $[a,b]$ and m_1 and M_1 are infimum and supresmum of f in $[a,b]$.

$\therefore m_1(b-a) \leq m(f,p) \leq \int_a^b f(x)\,dx \leq M(f,p) - M_1(b-a)$

Thus, $0 \leq M(f,p) - \int_a^b f\,dx \leq M_1(b-a) - \int_a^b f(x)\,dx \leq M_1(b-a)$ $\left[\because \int_a^b f\,dx \geq 0\right]$

$= (b-a)\sup\{\,|f(x)| : x \in [a,b]\}, i.e., \left|M(f,p) - \int_a^b f\,dx\right| \leq (b-a)\sup\{\,|f(x)|\,\}$

$\therefore$ Option (a) is correct.

For option (b),

Take, $f(x) = x$ in [0,3] and $p = \{0,1,3\}$ is a partition of [0,3].

Then, $\int_0^3 f(x)\,dx = \frac{9}{2}$

Also, $m(f,p) = 0 \times 1 + 2 \times 1$

Thus, $\left|m(f,p) - \int_0^3 f\,dx\right| = \left|2 - \frac{9}{2}\right|$ and

$(b-a)\inf\{\,|f(x)| : x \in [a,b]\} = 3 \times \inf\{\,|f(x)| : x \in [0,3]\} = 3 \times 0 = 0$

Clearly, option (b) is incorrect.

37. Let $f_n(x) = \begin{cases} 1 - nx, for \;\; x \in \left[0, \frac{1}{n}\right] \\ 0, \quad\; for \;\; x \in \left[\frac{1}{n}, 0\right] \end{cases}$. Then

(a) $\lim\limits_{n\to\infty} f_n(x)$ defines a continuous function on [0,1].

(b) $\{f_n\}$ converges uniformly on [0,1]

(c) $\lim\limits_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$.

(d) $\lim\limits_{n\to\infty} f_n(x)$ exists for all $x \in [0,1]$

Ans: (d)

Solution: The point wise limit, $\lim\limits_{n\to\infty} f_n(x) = f_n(x) = \begin{cases} 1, & for \;\; x = 0 \\ 0, & for\; x \in (0,1) \end{cases}$

$\therefore \lim\limits_{n\to\infty} f_n(x)$ exists, $\forall\, x \in [0,1]$.

Thus option (d) is correct and (c)is incorrect.

Also, $\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 1, & for\ \ x = 0 \\ 0, & for\ x \in (0,1) \end{cases}$

$\therefore \lim_{n\to\infty} f_n(x)$ is exists, $\forall\, x \in [0,1]$.

Thus option (d) is correct and (c) is incorrect.

Also, $\lim_{n\to\infty} f_n(x)$ is not continuous at $x = 0 \Rightarrow \lim_{n\to\infty} f_n(x)$ is not continuous on $[0,1]$.

$\therefore$ Option (a) is incorrect.

If possible, let $\{f_n\}$ converges uniformly on $[0,1]$.

$\therefore$ As $f_n(x)$ are continuous for all n on [0,1] and $\{f_n\}$ converges uniformly on [0,1].

$\Rightarrow \lim_{n\to\infty} f_n(x)$ is continuous on [0,1], which is a contradiction.

$\Rightarrow \{f_n\}$ does not converges uniformly on [0,1].

Thus, option (b) is incorrect.

38. The number $\sqrt{2}\, e^{i\pi}$ is

(a) A rational number

(b) A transcendental number

(c) An irrational number.

(d) An imaginary number.

Ans: (c)

Solution: $\sqrt{2}\, e^{i\pi} = \sqrt{2}\,(\cos\pi + \, i\sin\pi = \, -\sqrt{2}$, which is an irrational number.

$\therefore$ Option (c) is correct.

39. Let $I = [0,1] \subset \mathbb{R}$, for $x \in \mathbb{R}$, let $\phi(x) = \ dist\ (x, 1) = \inf\{|x - y| : y \in 1\}$

Then –

(a) $\phi(x)$ is discontinuous somewhere on $\mathbb{R}$.

(b) $\phi(x)$ is continuous on $\mathbb{R}$ but not continuously differentiable exactly at $x = 0$.

(c) $\phi(x)$ is continuous on $\mathbb{R}$ but not continuously differentiable exactly at $x = 0$ and at $x = 1$.

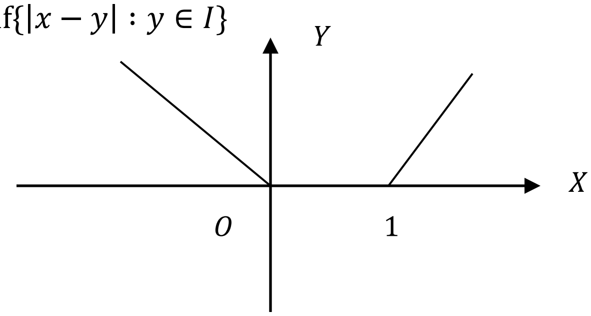
(d) $\phi(x)$ is differentiable on $\mathbb{R}$.

Ans: (c)

Solution: Given, $\phi(x) = \ dist\ (x, I) = \inf\{|x - y| : y \in I\}$

From the graph of the function $\phi(x)$,

it is clear that $\phi(x)$ is continuous on $\mathbb{R}$,



but not differentiable exactly

at $x = 0$ and $x = 1$.

Thus, option (c) is correct.

40. Let $a_n = \sin\left(\frac{\pi}{n}\right)$ for the sequence $a_1, a_2, \ldots ..$ the supremum is

(a) 0 and it is attained.

(b) 0 and it is not attained

(c) 1 and it is attained.

(d) 1 and it is not attained.

Ans: (c)

Solution: Given sequence is $a_n = \sin\frac{\pi}{n}, i.e., \sin\frac{\pi}{2}, \sin\frac{\pi}{3}, \sin\frac{\pi}{4}, \ldots \ldots ..$. i.e., $0, 1, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}, \ldots \ldots$ ...

$\therefore$ Supremum of the sequence is 1 and it is attained.

Hence, option (c) is correct.

41. Using the fact that $\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}, \sum_1^\infty \frac{1}{(2n+1)^2}$ equals

(a) $\frac{\pi^2}{12}$

(b) $\frac{\pi^2}{12} - 1$

(c) $\frac{\pi^2}{8}$

(d) $\frac{\pi^2}{8} - 1$

Ans: (d)

Solution: We have, $\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$

$\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2} + \frac{1}{11^2} + \dots\dots\dots = \frac{\pi^2}{6}$

$\Rightarrow \left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \dots\dots\dots\right) + \left\{\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \dots\dots\dots\right)\right\} = \frac{\pi^2}{6}$

$\Rightarrow \left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \dots\dots\dots\right) + \frac{1}{1^2} + \frac{1}{2^2} + \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\dots\dots\right) = \frac{\pi^2}{6}$

$\Rightarrow \left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \dots\dots\dots\right) + 1 + \frac{1}{4}\left(\frac{\pi^2}{6}\right) = \frac{\pi^2}{6}$

$\Rightarrow \sum_1^\infty \frac{1}{(2n+1)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} - 1 = \frac{\pi^2}{8} - 1$

$\therefore$ Option (d) is correct.

42. Let $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a bilinear map, i.e., linear in each variable separately. Then for $(V, W) \in \mathbb{R}^2 \times \mathbb{R}^2$, the derivative $Df(V, W)$ evaluated on $(H, K) \in \mathbb{R}^2 \times \mathbb{R}^2$ is given by

(a) $f(V, K) + f(H, W)$

(b) $f(H, K)$

(c) $f(V, H) + f(W, K)$

(d) $f(H, V) + f(W, K)$

Ans: (a)

Solution: Given, $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a bilinear map

Take, $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $f(V, W) = V_1W_2 - V_2W_1$.

Then, $Df(V, W) = f(V, K) + f(H, W)$

$\therefore$ Option (b), (c), (d) are incorrect.

Hence, option (a) is correct.

43. Consider the function $f(x) = \cos(|x - 5|) + \sin(|x - 3|) + |x + 10|^3 - (|x| + 4)^2$. At which of the following points are not differentiable?

(a) $x = 5$

(b) $x = 3$

(c) $x = -10$

(d) $x = 0$

Ans: (b), (d)

Solution: Given, $f(x) = \cos(|x-5|) + \sin(|x-3|) + |x+10|^3 - (|x|+4)^2$

Let $f_1(x) = \cos(|x-5|) = \begin{cases} \cos(-(x-5)), & x < 5 \\ 1, & x = 5 \\ \cos(x-5), & x > 5 \end{cases} = \begin{cases} \cos(-(x-5)), & x \neq 5 \\ 1, & x = 5 \end{cases}$

$\therefore$ L. H. D. $= \lim_{x \to 5^-} \frac{f_1(x) - f_1(5)}{x-5} = \lim_{x \to 5^-} \frac{\cos(x-5)-1}{x-5} \left(= \frac{0}{0}\ form\right)$

$= \lim_{x \to 5^-} \{-\sin(x-5)\} = 0$

$\therefore$ R.H.D.$= \lim_{x \to 5^+} \frac{f_1(x) - f_1(5)}{x-5} = \lim_{x \to 5^+} \frac{\cos(x-5)-1}{x-5} \left(= \frac{0}{0}\ form\right)$

$\Rightarrow L.H.D = R.H.D.$

$\therefore f_1(x) = \cos(|x-5|)$ is differentiable at $x = 5$ and hence $f(x)$ is differentiable at $x = 5$

$\therefore$ Option (a) is incorrect.

Next, let $f_2(x) = \sin(|x-3|) = \begin{cases} \sin(-(x-3)), & x < 3 \\ 0, & x = 3 \\ \sin(x-3), & x > 3 \end{cases}$

$\therefore \lim_{x \to 3^-} \frac{f_2(x) - f_2(3)}{x-3} = \lim_{x \to 3^-} \frac{-\sin(x-3)-0}{x-3} = \lim_{x \to 3^-} \{-\cos(x-3)\} = -1$

and $\lim_{x \to 3^-} \frac{f_2(x) - f_2(3)}{x-3} = \lim_{x \to 3^-} \frac{\sin(x-3)-0}{x-3} = 1$

Since, $L.H.D \neq R.H.D$

$\therefore f_2(x) = \sin(|x-3|)$ is not differentiable at $x = 3$.

Hence, $f(x)$ is not differentiable $x = 3$.

$\therefore$ Option (b) is correct.

Further, let $f_3(x) = |x+10|^3 = \begin{cases} -(x+10)^3, & x < -10 \\ 0, & x = -10 \\ (x+10)^3, & x > -10 \end{cases}$

L.D.H $= \lim_{x \to (-10)^-} \frac{f_3(x) - f_3(-10)}{x+10} = \lim_{x \to -10} \frac{-(x+10)^3 - 0}{x+10} = \lim_{x \to -10} -(x+10)^2 = 0$

R.D.H. $= \lim_{x \to (-10)^+} \frac{f_3(x) - f_3(-10)}{x+10} = \lim_{x \to -10} \frac{(x+10)^3 - 0}{x+10} = \lim_{x \to -10} (x+10)^2 = 0$

$\therefore L.H.D = R.H.D.$

$\therefore\ f_3(x)$ is differentiable at $x = -10.$

Hence, $f(x)$ is differentiable at $x = -10.$

$\therefore$ Option (c) is incorrect.

Next, let $f_4(x) = (|x| + 4)^2 = \begin{cases} (-x+4)^2, & x < 0 \\ 16, & x = 0 \\ (x+4)^2, & x > 0 \end{cases}$

$\therefore L.H.D = \lim\limits_{x\to 0^-} \frac{f_4(x)-f_4(0)}{x-0} = \lim\limits_{x\to 0} \frac{(-x+4)^2-16}{x} = \lim\limits_{x\to 0} \frac{2(-x+4)(-1)}{1} = -8$

$R.H.D. = \lim\limits_{x\to 0^+} \frac{f_4(x)-f_4(0)}{x-0} = \lim\limits_{x\to 0} \frac{(-x+4)^2-16}{x} \left(= \frac{0}{0}\right) = \lim\limits_{x\to 0} \frac{2(x+4)}{1} = 8$

$L.H.D. \neq R.H.D$

$\Rightarrow f_4(x)$ is not differentiable at $x = 0$

Thus, option (d) is correct.

44. For each $j = 1, 2, 3, \dots \dots Let\ A_j$ be a finite set containing at least two distinct elements. Then

(a) $\bigcup_{j=1}^{\infty} A_j$ is a countable set.

(b) $\bigcup_{n=1}^{\infty} \prod_{j=1}^{n} A_j$ is uncountable.

(c) $\prod_{j=1}^{\infty} A_j$ is uncountable.

(d) $\bigcup_{j=1}^{\infty} A_j$ is uncountable.

Ans: (a), (c)

Solution: Since,

(i) Countable union of countable sets is countable.

(ii) Finite product of countable sets is countable.

(iii) Infinite product of countable sets is uncountable

$\Rightarrow \bigcup_{j=1}^{\infty} A_j$ is countable set.

$\therefore$ Option (a) is correct and option (d) is incorrect.

Also $\prod_{j=1}^{n} A_j$ is countable (By (ii))

$\Rightarrow \bigcup_{n=1}^{\infty} \prod_{j=1}^{n} A_j$ is countable (By (i))

$\therefore$ Option (b) is incorrect.

Further, $\prod_{j=1}^{\infty} A_j$ is uncountable [By (ii)]

$\therefore$ Option (c) is correct.

45. Let f be a twice differentiable function on $\mathbb{R}$. Given that $f''(x) > 0$ for all $x \in \mathbb{R}$,

(a) $f(x) = 0$ has exactly two solutions on $\mathbb{R}$.

(b) $f(x) = 0$ has a positive solution of $f(0) = 0$ & $f'(0) = 0$.

(c) $f(x) = 0$ has no positive solution if $f(0) = 0$ & $f'(0) > 0$

(d) $f(x) = 0$ has no positive solution if $f(0) = 0$ & $f'(0) < 0$.

Ans: (c)

Solution: Given $f''(x) > 0$ for all $x \in \mathbb{R}$

Consider $f(x) = x^2 + 1 = 0$ such that $f''(x) = 2 > 0 \;\; \forall\, x \in \mathbb{R}$ but $f(x)$ has no solution in $\mathbb{R}$.

Thus, option (a) is incorrect.

For option (b), $f(x) = x^2$

Clearly, $f(0) = 0, f'(0) = 0$ but $f(x)$ has no positive solution.

Consider, $f(x) = \; x^2 - x, f'(x) = 2x - 1$

Clearly, $f(0) = 0, f'(0) < 0, f''(x) > 0 \;\; \forall \;\; x \in \mathbb{R}$.

But $f(x) = 0$ has positive solution.

Thus, option (d) is incorrect.

As all other options ruled out, hence option (c) is correct.

46. Let f be a continuously differentiable real valued function on $[a,b]$ such that $|f'(x)| \leq k$ for all $x \in [a,b]$. For a partition $p = \{a = a_0 < a_1 < a_2 \ldots \ldots .. < a_n = b\}$, let $U(f,p)$ and $L(f,p)$ denote the upper and lower Riemann sums of f with respect to p. Then –

(a) $|L(f,p)| \leq k(b-a) \leq |U(f,p)|$

(b) $U(f,p) - L(f,p) \leq k(b-a)$

(c) $U(f,p) - L(f,p) \leq k\|p\|$, where $\|p\| = \max\limits_{0 \leq i \leq n-1}(a_{i+1} - a_i)$ is the norm of the partition.

(d) $U(f,p) - L(f,p) \leq k\|p\|(b-a)$

Ans: (d)

Solution:

For options (a) and (c)

Take, $f(x) = x$ on $[1,3]$ and $p = \{a_0, a_1, a_2\} = \{1,2,3\}$ is partition of $[1,3]$

Also, $\|p\| = \max\limits_{0 \leq i \leq 2}(a_{i+1} - a_i) = \max\{1,1\} = 1$

Now, $L(f,p) = \sum_{i=1}^{2} m_i \Delta x_i = |X| + 2X| = 3$ and $U(f,p) = \sum_{i=1}^{2} M_i \Delta x_i = 2X| + 3X| = 5$

Thus, $L(f,p) = 3 \nleq 1(2) = k(b-a)$

∴ Option (a) is incorrect.

For option (b),

Take, $f(x) = x^2$ on $[1,3]$

Let $p = \{1,2,3\}$ is partition of $[1,3]$

Also, $|f'(x)| = 2x \leq 6 \; \forall \; x \in [1,3] \Rightarrow k = 6$

Now, $L(f,p) = \sum_{i=1}^{2} m_i \Delta x_i = |X| + 4X| = 5$ and $U(f,p) = \sum_{i=1}^{2} M_i \Delta x_i = 4X| + 9X| = 13$

∴ $U(f,p) - L(f,p) = 13 - 5 = 8$ and $k(b-a) = 6(1) = 6$

$\Rightarrow U(f,p) - L(f,p) \nleq k(b-a)$

Thus, option (b) is incorrect

As, all other options are incorrect.

∴ Option (d) is correct.

47. Let f be a monotone non – decreasing real valued – function on $\mathbb{R}$. Then –

(a) $\lim_{x \to a} f(x)$ exists at each point a

(b) If $a < b$, then $\lim_{x \to a^+} f(x) \leq \lim_{x \to b^-} f(x)$

(c) f is an unbounded function

(d) The function $g(x) = e^{-f(x)}$ is a bounded function.

Ans: (b)

Solution: Given f is monotonic and non – decreasing real valued function on $\mathbb{R}$.

For option (a).

Take, $f(x) = [x], x \in \mathbb{R}$

Clearly, $f(x)$ is non – decreasing on $\mathbb{R}$.

But $\lim_{x \to a} f(x)$ does not exists for any integer $'a'$

∴ Option (a) is incorrect.

For Option (c),

Take, $f(x) = 2$.

Clearly, $f(x)$ is non decreasing function on $\mathbb{R}$.

But $f(x)$ is not unbounded.

∴ Option (c) is not incorrect.

For option (d),

Take, $f(x) = x$,

Clearly,$f(x)$ is non – decreasing function on $\mathbb{R}$.

But $g(x) = e^{-f(x)} = e^{-x}$, is unbounded on $\mathbb{R}$.

∴ Option (d) is incorrect.

As all other options are incorrect.

∴ Option (b) is correct.

48. Let f be a real valued function on $\mathbb{R}^3$ satisfying $f(rx) = r^\alpha f(x)$ for any $r > 0$ and $x \in \mathbb{R}^3$.

(a) If $f(x) = f(y)$ whenever $\|x\| = \|y\| = \beta$, for $\alpha, \beta > 0$, then $f(x) = \beta\|x\|^\alpha$

(b) If $f(x) = f(y)$ whenever $\|x\| = \|y\| = 1$, then $f(x) = \|x\|^\alpha$

(c) If $f(x) = f(y)$ whenever $\|x\| = \|y\| = 1$, then $f(x) = c\|x\|^\alpha$, for some constant c.

(d) If $f(x) = f(y)$ whenever $\|x\| = \|y\|$, then f must be a constant function.

Ans: (c)

Solution: Take, $f(x) = 3\|x\|^\alpha$

For option (a),

Let $\beta = 2$

Clearly, $f(x) = 32^\alpha = f(y)$, whenever $\|x\| = \|y\| = 2$

But $f(x) = 3\|x\|^\alpha \neq \beta\|x\|^\alpha$

$\therefore$ Option (a) is incorrect.

For option (b),

Also, $f(x) = 3 = f(y)$, whenever $\|x\| = \|y\| = 1$

But $f(x) = 3\|x\|^\alpha \neq \|x\|^\alpha$

$\therefore$ Option (b) is incorrect.

For option (d).

From above example

$f(x) = f(y)\ \forall\ x, y$ for which $\|x\| = \|y\|$

But here, $'f'$ is not constant.

$\therefore$ Option (d) is incorrect.

As all other options are incorrect.

$\therefore$ Option (c) is correct.

49. Which of the following real valued functions on $(0,1)$ is uniformly continuous?

(a) $f(x) = \frac{1}{x}$

(b) $f(x) = \frac{\sin x}{x}$

(c) $f(x) = \sin\frac{1}{x}$

(d) $f(x) = \frac{\cos x}{x}$

Ans: (b)

Solution: Result: A function $f(x)$ defined on open interval (a,b) is uniformly continuous on (a,b) if and only if

(i) $f(x)$ is continuous on (a,b).

(ii) $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$, both exists finitlely.

For option (a),

Clearly, $f(x) = \frac{1}{x}$ is continuous on $(0,1)$.

But $\lim_{x\to 0^+}\frac{1}{x}$ does not exist.

$\therefore f(x) = \frac{1}{x}$ is not uniformly continuous on $(0,1)$

For option (b),

Result: If $g(x)$ and $h(x)$ are continuous function on (a,b) and $g(x) \neq 0$ in (a,b), then $\frac{f(x)}{g(x)}$ is continuous on (a,b).

Also $\lim_{x\to 0^+}\frac{\sin x}{x} = 1$ and $\lim_{x\to 1^-}\frac{\sin x}{x} = \sin 1$

$\therefore$ Using above result, $f(x) = \frac{\sin x}{x}$ is uniformly continuous on $(0,1)$.

Clearly $f(x) = \sin\frac{1}{x}$ is continuous on $(0,1)$.

Since, $\lim_{x\to 0^+}\sin\frac{1}{x}$ does not exist.

$\therefore f(x)$ is not uniformly continuous on $(0,1)$

So, option (c) is incorrect.

For option (d) $\lim_{x\to 0^+}\frac{\cos x}{x}$ does not exist.

$\therefore f(x) = \frac{\cos x}{x}$ is not uniformly continuous on $(0,1)$.

50. Let X be a metric space and $A \subseteq X$ be a connected set with at least two distinct points.Then the number of distinct points of A is

(a) 2

(b) More than 2 but finite.

(c) Countably finite.

(d) Uncountably.

Ans: (d)

Solution: Given, X to be a metric space.

Then $X = \mathbb{R}$

Let $A \subseteq \mathbb{R}$ be a connected set with at least two distinct points.

We know that in $\mathbb{R}$ either singleton or intervals are the only connected sets.

But $'A'$ has at least two distinct points

$\therefore$ A can't be singleton.

$\Rightarrow$ $'A'$ is an open interval and therefore number of distinct points of in A is uncountable.

Thus, option (d) is correct.

51. Let $f : \mathbb{R}^n \to \mathbb{R}$ be linear map with $f(0, 0, \dots \dots, 0) = 0$. Then the

Set $\left\{f(x_1, x_2, \dots \dots \ldots, x_n) \colon \sum_{j=1}^{n} {x_j}^2 \leq 1\right\}$ equals.

(a) $[-a, a]$ for some $a \in \mathbb{R}, a \geq 0$

(b) $[0,1]$

(c) $[0, a]$ for some $a \in \mathbb{R}, a \geq 0$

(d) $[a, b]$ for some $a, b \in \mathbb{R}, 0 \leq a < b$.

Ans: (a)

Solution: Take $n = 2$

Let $f : \mathbb{R}^n \to \mathbb{R}$ is defined by $f(x_1, x_2) = x_1 + x_2$

Then, the set $\left\{f(x_1, x_2) \colon \sum_{j=1}^{2} {x_j}^2 \leq 1\right\} = \{f(x_1, x_2) \colon {x_1}^2 + {x_2}^2 \leq 1\} = \left[-\sqrt{2}, \sqrt{2}\right]$.

Thus, options (b), (c), (d) are incorrect.

Hence, option (a) is correct.

52. Suppose $f: \mathbb{R} \to \mathbb{R}$ is a differentiable function. Then which of the following statements are necessarily true?

(a) If $f'(x) \le r < 1$ for all $x \in \mathbb{R}$, then f has at least one fixed point.

(b) If f has a unique fixed point, then $f'(x) \le r < 1$ for all $x \in \mathbb{R}$

(c) If f has a unique fixed point, then $f'(x) \ge r > -1$ for all $x \in \mathbb{R}$

(d) If $f'(x) \le r < 1$ for all $x \in \mathbb{R}$, then f has a unique fixed point.

Ans: (a), (d)

Solution: For option (b)

Take, $f(x) = 2x$

Clearly, $f(x)$ is differentiable and has unique fixed point but $f'(x) = 2 \nless 1$.

$\therefore$ Option (b) is incorrect.

For option (c)

Take, $f(x) = -2x$.

Clearly, $f(x)$ is differentiable and has unique fixed point but $f'(x) = 2 \nless 1$

$\therefore$ option (b) is incorrect.

For option (c)

Take, $f(x) = -2x$.

Clearly, $f(x)$ is differentiable and has only one fixed point. But $f'(x) = 2 \ngtr -1$

$\therefore$ Option (c) is incorrect.

For option (a)

Result: If $f : \mathbb{R} \to \mathbb{R}$ is differentiable function and if $\exists$ a constant $k < 1$ such that $|f'(t)| \le \mathrm{k}\ \forall$ real t , then fixed point of f exists and it can be easily verified that fixed point of $f(x)$ is unique.

As, if possible let $f(x)$ has two fixed points i.e., $\exists\ x_1, x_2 \in \mathbb{R}$

Such that $f(x_1) = x_1$ and $f(x_2) = x_2$, then by mean value theorem, $\exists\ c \in (x_1, x_2)$ such that $\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) \Rightarrow \frac{x_2-x_1}{x_2-x_1} = f'(c) \Rightarrow f'(c) = 1$, which is a contradiction.

Thus, $f(x)$ has only one fixed point.

53. Which of the conditions below imply that a function $f : [0,1] \to \mathbb{R}$ is necessarily of bounded variation?

(a) f is a monotone function on [0,1].

(b) f is continuous and monotone function on [0,1]?

(c) f has a derivative at each $x \in (0,1)$.

(d) f has a bounded derivative on the interval (0,1).

Ans: (a), (b), (d)

Sol. We know that if $'f'$ is a monotonic function on [a,b]. Then $'f'$ is of bounded variation on $[a, b]$.

$\therefore$ Option (a) and (b) are correct.

Also, if derivative of $f(x)$ is bounded on (a,b), then $'f'$ is of bounded variation.

$\therefore$ Option (d) is also correct.

54. Let $f(x) = \sin x - x + \frac{x^3}{3!}$ and $g(x) = \cos x - 1 + \frac{x^2}{2!}$ for $x \in \mathbb{R}$. Which of the following statements are correct?

(a) $f(x) \geq 0$ for all $x > 0$

(b) g is an increasing function on $[0, \infty)$.

(c) g is decreasing function on $[0, \infty)$.

(d) f is a decreasing function $[0, \infty)$.

Ans: (a), (b)

Sol. Given, $f(x) = \sin x - x + \frac{x^3}{3!}$ and $g(x) = \cos x - 1 + \frac{x^2}{2!}$ for $x \in \mathbb{R}$.

$\Rightarrow g'(x) = -\sin x + x = x - \sin x \geq 0$

$\therefore$ g is increasing.

$\therefore g(x) \geq g(0) \forall\ x \geq 0 \Rightarrow g(x) \geq 0\ \forall x \geq 0$ and $f'(x) = g(x) \geq 0$.

$\Rightarrow f$ is increasing

$\therefore f(x) \geq f(0) = 0\ \forall x \geq 0$.

Thus options (a) and (b) are correct.

55. Let $f : A \cup E \to \mathbb{R}^2$ be differentiable, where $A = \left\{(x, y) \in \mathbb{R}^2 : \frac{1}{2} < x^2 + y^2 < 1\right\}$ and $E = \left\{(x, y) \in \mathbb{R}^2 : (x-2)^2 + (y-2)^2 < \frac{1}{2}\right\}$. Let Df be the derivative of the function f. Which of the following are necessarily correct?

(a) If $(Df)(xy) = 0$ for all $(x, y) \in A \cup E$, then f is constant.

(b) If $(Df)(xy) = 0$ for all $(x, y) \in A$, then f is constant on A.

(c) If $(Df)(xy) = 0$ for all $(x, y) \in E$, then f is constant on E.

(d) If $(Df)(xy) = 0$ for all $(x, y) \in A \cup E$, then for some $(x_0, y_0), (x_1, y_2) \in \mathbb{R}^2, f(x, y) = (x_0, y_0)$ for all $(x, y) \in A$ and $f(x, y) = (x_1, y_2)$, for all $(x\ y) \in E$.

Ans: (b), (c), (d)

Solution: We know that if $Df(x,y) = 0 \;\forall\; (x,y) \in S$.

Thus f is constant, provided S is connected.

Thus possibilities of f are f are $f(x,y) = C \;\forall\; (x,y) \in A \cup E$ or $f(x) = \begin{cases} C_1, & \forall\, x \in A \\ C_2, & \forall\, x \in E \end{cases}$

Thus, option (b), (c), (d) are correct and (a) is incorrect.

56. Let $L : \mathbb{R}^n \to \mathbb{R}$ be the function $L(x) = \langle x, y\rangle$, where $\langle \cdot,\cdot \rangle$ is some inner product on $\mathbb{R}^n$ and y is a fixed vector in $\mathbb{R}^n$. Further denote by DL, the derivative of L. Which of the following are necessarily correct?

(a) $DL(u) = DL(v)$ for all $u, v \in \mathbb{R}^n$

(b) $DL(0,0, \dots \dots \dots, 0) = L$

(c) $DL(x) = \|x\|^2$ for all $x \in \mathbb{R}^2$

(d) $DL\,(1,1, \dots \dots, 1) = 0$

Ans. (a), (b)

Solution: Let $y = (y_1, y_2, \dots \dots \dots, y_n)$ be a fixed vector in $\mathbb{R}^n$

$\therefore L(x) = \langle x, y\rangle = x_1y_1 + x_2y_2 + \dots \dots + x_ny_n\,,\; x = (x_1, x_2, \dots, x_n)$

$\Rightarrow DL(x) = \left(\frac{\partial L}{\partial x_1}, \frac{\partial L}{\partial x_2} \dots \dots, \frac{\partial L}{\partial x_n}\right) = (y_1, y_2, \dots, y_n) = y \;\dots \dots \dots \dots \dots (i)$

Thus, $DL(u) = DL(v) = y \;\forall\; u, v \in \mathbb{R}^n$.

$\therefore$ Option (a) is correct.

Also, $DL(x) = y \neq \|x\|^2$

$\therefore$ Option (c) is incorrect.

Further, $DL(0,0, \dots \dots, 0) = (0,0, \dots \dots, 0)$ and $L(0,0, \dots \dots, 0) = (0,0, \dots \dots, 0)$

$\Rightarrow DL\,(0,0, \dots \dots, 0) = (0,0, \dots \dots, 0) = L(0,0, \dots, 0)$

Also, $DL(1,1, \dots \dots, 1) = y$ $\Rightarrow$ Thus, option (d) is incorrect.

57. $f : [\pi, 2\pi] \to \mathbb{R}^2$ be the function $f(t) = (\cos t, \sin t)$. Which of the following are necessarily correct?

(a) There exists $t_0 \in [\pi, 2\pi]$ such that $f'(t_0) = \frac{1}{pi}\big(f(2\pi) - f(\pi)\big)$

(b) There does not exist any $t_0 \in [\pi, 2\pi]$ such that $f'(t_0) = \frac{(f(2\pi) - f(\pi))}{\pi}$

(c) The exists $t_0 \in [\pi, 2\pi]$ such that $\|f(2\pi) - f(\pi)\| \leq \pi \|f'(t_0)\|$

(d) $f'(t) = (-\sin t, \cos t)$ for all $t \in [\pi, 2\pi]$.

Ans: (b), (c), (d)

Solution: Result 1: A function $f: [a, b] \to \mathbb{R}^n$ is a continuous function at a point $x_0 \in \mathbb{R}^n$ if and only if each of its components is continuous at $x = x_0$.

Result 2: A function $f: [a, b] \to \mathbb{R}^n$ is differentiable at a point $x_0 \in \mathbb{R}^n$ if and only if each its components is differentiable at $x = x_0$.

Result 3: If f' is a continuous mapping of $[a, b]$ into $\mathbb{R}^n$ and $'f'$ is differentiable in (a, b). Then $\exists\, x_0 \in (a, b)$ such $|f(b) - f(a)| \le (a - b)|f'(x_0)|$

Given, $f : [\pi, 2\pi] \to \mathbb{R}^2$ defined by $f(t) = \cos t, \sin t)$

Clearly, $\cos t$ and $\sin t$ are continuous

$\therefore f(t)$ is continuous $[\pi, 2\pi]$

Also, $\cos t$ and $\sin t$ are differentiable on $[\pi, 2\pi]$

$\therefore f(t) = (\cos t \sin t)$ is differentiable on $[\pi, 2\pi]$

And $f'(t) = (-\sin t\ , \cos t)$

$\therefore$ Option (d) is correct.

Also, by result 3, option (c) is correct.

Now, $\frac{f(2\pi) - f(\pi)}{\pi} = \frac{(1,0) - (-1,0)}{\pi} = \left(\frac{2}{\pi}, 0\right)$

Clearly, $\nexists\, t_0 \in [\pi, 2\pi]$ for which $f'(t_0) = \left(\frac{2}{\pi}, 0\right)$

$\therefore$ Option (b) is correct and hence, option (a) is incorrect.

58. Let $X = [-1,1] \times [-1,1]$, $A = \{(x, y) \in X : x^2 + y^2 = 1\}$,

$B = \{(x, y) \in X : \ |x| + |y| = 1\}, C = \{ (x, y) \in X : xy = 0\}$ and

$D = \{(x, y) \in X : x = \pm y\}$, Then

(a) A is homeomorphic to B

(b) B is homeomorphic to C.

(c) C is homeomorphic to D.

(d) D is homeomorphic to A.

Ans: (a), (c)

Solution: We know that a set Y is homeomorphic to Z, if $\exists$ a function $f : Y \to Z$ such that

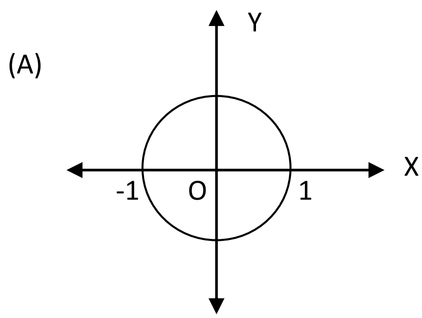
(i) $'f'$ is continuous.

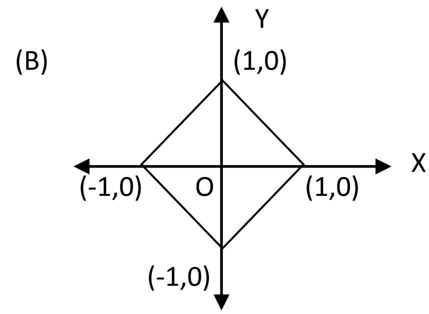
(ii) $'f'$ is 1 – 1 and onto

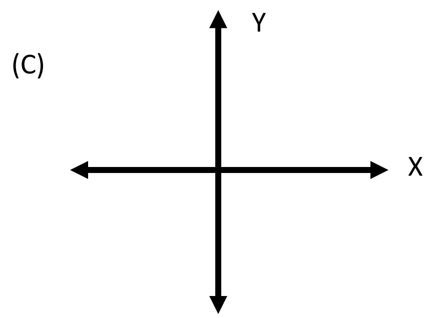
(iii) f^{-1} is continuous or for each closed set H in Y, $f(H)$ is closed Z.

Also, we know that in a continuous function, image of connected set is connected.

Or we can say two are homeomorphic if both have same topological properties.

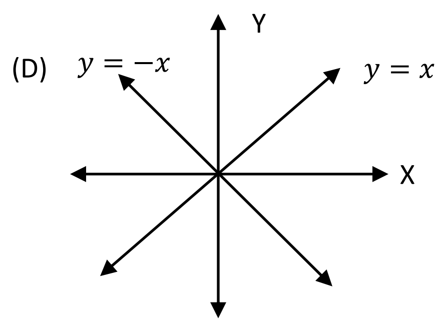






From the diagram (A) and (B) it is clear that sets A and B are homeomorphic if both have same topological properties.

Thus, option (a) is correct.



Also, set D can be obtained just by rotating set C by angle 45^0.

As we know, every rotation is isometry and every isometry is homeomorphic.

∴ Set C and D are also homeomorphic.

Thus, option (c) is also correct.

B is not homeomorphic to C, as if (0,0) (One point) is removed from C, then it is divided into four components, whereas B have only one connected components after removed of any one point.

Similarly, D cannot be homeomorphic to A.

Hence, option (b) and (d) are incorrect.

59. Let f, g and h be bounded functions on the closed interval $[a, b]$, such that $f(x) \leq g(x) \leq h(x)$ for all $x \in [a, b]$. Let $P = \{a = a_0 < a_1 < a_2 < . \; .... < a_n = b\}$ be a partition of $[a, b]$. We denote by $U(f, p)$ and $L(f, p)$, the upper and lower Riemann sums of f with respect to the partition P and similarly for g and h. Which of the following statements is necessarily true?

(a) If $U(h, P) - U(f, P) < 1$ then $U(g, P) - L(g, P) < 1$.

(b) If $L(h,P) - L(f,P) < 1$ then $U(g,P) - L(g,P) < 1$.

(c) If $U(h,P) - L(f,P) < 1$ then $U(g,P) - L(g,P) < 1$.

(d) If $L(h,P) - U(f,P) < 1$ then $U(g,P) - L(g,P) < 1$.

Ans: (c)

Solution: Given (i) $f, g\ and\ h$ are bounded functions on $[a,b]$.

(ii) $f(x) \leq g(x) \leq h(x)\ \forall\ x \in [a,b]$

(iii) $P = \{a = a_0 < a_1 < a_2 < . \ \ldots . < a_n = b\}$ is a given partition of $[a,b]$.

Let

$M^h{}_i = supremum\ of\ h\ in\ \Delta\, x_i = (x_i - x_{i-1})$ for $i = 1,2, \ldots \ldots, n$.

$M^g{}_i = supremum\ of\ g\ in\ \Delta\, x_i = (x_i - x_{i-1})$ for $i = 1,2, \ldots \ldots, n$.

$m^g{}_i = infimum\ of g\ in\ \Delta\, x_i = (x_i - x_{i-1})$ for $i = 1,2, \ldots \ldots, n$.

$m^f{}_i = infimum\ of\ f\ in\ \Delta\, x_i = (x_i - x_{i-1})$ for $i = 1,2, \ldots \ldots, n$.

Now, as $h(x) \geq g(x)\ \forall\, x \in [a,b]$.

$\Rightarrow M^h{}_i \geq M^g{}_i\ \forall\ i \in \{1,2, \ldots \ldots, n\} \Rightarrow \sum_{i=1}^{n} M^h{}_i \Delta x_i \geq \sum_{i=1}^{n} M^g{}_i \Delta x_i$

$\Rightarrow U(h,P) \geq U(g,P) \ldots \ldots \ldots \ldots \ldots \ldots \ldots .. (i)$

Similarly, as $g(x) \geq f(x)\ \forall\, x \in [a,b]$.

$\Rightarrow m^g{}_i \geq m^f{}_i\ \forall\ i \in \{1,2, \ldots \ldots, n\} \Rightarrow \sum_{i=1}^{n} m^g{}_i \Delta x_i \geq \sum_{i=1}^{n} m^f{}_i \Delta x_i$

$\Rightarrow L(g,P) \geq L(f,P) \Rightarrow -L(g,P) \leq -L(f,P) \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2)$

(1) and (2) $\Rightarrow U(g,P) - L(g,P) \leq U(h,P) - L(f,P) \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . (3)$

Now, if $U(h,P) - L(P,f) < 1 \Rightarrow U(g,P) - L(g,P) < 1$

$\therefore$ Option (c) is correct.

60. The power series $\sum_{n=0}^{\infty} \frac{[2+(-1)^n]^n}{3^n} \cdot x^n$ converges.

(a) only for x=0

(b) for all $x \in \mathbb{R}$

(c) only for $-1 < x < 1$

(d) Only for $-1 < x \leq 1$

Ans: (c)

Solution: Let $a_n = \left(\frac{2+(-1)^n}{3^n}\right)^n$

$\frac{1}{R} = \overline{\lim_{n\to\infty}} \left|\left(\frac{2+(-1)^n}{3^n}\right)\right|^n = \overline{\lim_{n\to\infty}} \left\{\frac{1}{3}, 1\right\} = 1 \Rightarrow R = 1$

∴ Radius of convergence of the power series is 1.

For $x = 1$, Put $x = 1$ in $\sum_{n=0}^{\infty} \frac{[2+(-1)^n]}{3^n} \cdot x^n$, we get

$\sum_{n=0}^{\infty} \frac{[2+(-1)^n]}{3^n} \cdot x^n = \sum_{n=0}^{\infty} \frac{(2+(-1)^n)}{3^n} = 1 + \frac{1}{3} + 1 + \frac{1}{3^3} + 1 + \frac{1}{3^5} + \dots\dots\dots. \to \infty \; as \; n \to \infty.$

So, at $x = 1$, the series is diverges.

The region where it converges is given by $|x| < 1$ i.e., $-1 < x < 1$.

61. Which of the following cases, there is no continuous function f from the set S onto the set T?

(a) $S = [0,1], T = \mathbb{R}$

(b) $S = (0,1), T = \mathbb{R}$

(c) $S = (0,1), T = (0,1]$

(d) $S = \mathbb{R},\ T = (0,1)$

Ans: (a)

Solution: There is no continuous function from $S = [0,1]$ onto the set $T = \mathbb{R}$ because a continuous function maps a compact set to a compact set.

But here $S = [0,1]$ is compact whereas $T = \mathbb{R}$ is not compact.

62) The function $f(x) = a_0 + a_1|x| + a_2|x|^2 + a_3|x|^3$ is differentiable at $x = 0$
(a) for no values of a_0, a_1, a_2, a_3.
(b) for any values of a_0, a_1, a_2, a_3.
(c) only if $a_1 = 0$.
(d) only if both $a_1 = 0 \; and \; a_3 = 0$.

Answer: (c)

Solution: Option (a) is not correct as take $a_0 = a_1 = a_3 = 0$, then
$f(x) = 0 + 0(x) + 0(x)^2 + 0(x)^3 = 0$ is diffierentiable at $x = 0$.

Option (b) is not correct as for $a_0 = a_3 = a_2 = 0\ and\ a_1 = 1, f(x) = |x|$, which is not differentiable at $x = 0$.
Option (d) is not correct as for $a_0 = a_1 = a_2 = 0, a_3 \neq 1, f(x) = |x|^3$ is differentiable at $x = 0$.
As option (a), (b), (d) are incorrect.
∴ Option (c) is correct.

63) The no. of limit points of the set $\left\{\frac{1}{m} + \frac{1}{n} : m, n, \in \mathbb{N}\right\}$ is
(a) 1
(b) 2
(c) finitely many
(d) infinitely many

Answer: (d)

Solution: Set of limit point of the set $S = \left\{\frac{1}{m} + \frac{1}{n} : m, n \in S\right\}$ is given by $S' = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, which is an infinite set.
∴ Option (d) is correct.

64) Area enclosed between $x - axis$ from a to b and the curve $f(x)$ is finite when
(a)$a = 0, b = \infty, f(x) = e^{-5x^5}$
(b) $a = -\infty, b = \infty, f(x) = e^{-5x^5}$
(c) $a = -7, b = \infty, f(x) = \frac{1}{x^4}$
(d) $a = 7, b = 7, f(x) = \frac{1}{x^4}$

Answer: (a)

Solution: $\int_0^\infty e^{-5x^5}\, dx = \int_0^\infty e^{-t} \frac{t^{-\frac{4}{5}}}{(5)^{\frac{6}{5}}} dt\ [Put\ 5x^5 = t$

$= \frac{1}{(5)^{\frac{6}{5}}} \int_0^\infty t^{\frac{1}{5}-i} e^{-t} dt = \frac{1}{(5)^{\frac{6}{5}}} \left\lceil \frac{1}{5} \right. finite$

∴ Option (a) is correct.
Since, $e^{-5x^5} \to \infty\ as\ x \to -\infty$
∴ $\int_{-\infty}^\infty e^{-5x^5}\, dx \to \infty$.
∴ Option (b) is not true.
Further, $\int_a^b \frac{1}{x^4}\, dx = -\frac{1}{3x^4}\Big|_a^b \to\ \infty\ as\ x \to 0$.
Clearly, $\int_a^b \frac{1}{x^4} dx$ tends to ∞ for any interval containing 0.
∴Option (c) and (d) are not true.

65) Let a, b, c be distinct real numbers. Then the number of distinct real roots of the equation $(x-a)^3 + (x-b)^3 + (x-c)^3 = 0$ is

(a) 1

(b) 2

(c) $3, 3$

(d) depends on the values of c.

Answer: (a)

Solution: Let $f(x) = (x-a)^3 + (x-b)^3 + (x-c)^3 = 0, \quad a, b, c \in \mathbb{R}$

Clearly, $f(x)$ has at least one real root.

[$\because$ every odd degree equation with real co-efficients has at least one real root]

Now, we prove that $f(x)$ has exactly one real root.

If possible, let $f(x) = 0$, has two distinct roots $\alpha \; and \; \beta (say)$

$$\Rightarrow f(\alpha) = f(\beta) = 0.$$

$\therefore$By Rolle's theorem, $\exists \; c \epsilon (\alpha, \beta)$ such that $f'(c) = 0$ ------------- (1)

But $f'(x) = 3(x-a)^2 + 3(x-b)^2 + 3(x-c)^2 > 0 \; \forall \; x \in (\alpha, \beta)$ which is a contradiction to (1)

$\therefore \; f(x) = 0$, can't have more than one real root.

This option (a) is correct.

66) Consider the following subsets of $\mathbb{R}^2$, where $a, b \in \mathbb{R}$.

$\left\{A = (x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a \neq b\right\}$

$\left\{B = (x, y) \in \; \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1, a \neq b\right\}$

$C = \{(x, y) \in \mathbb{R}^2 : ax + by + 5 = 0\}$

$D = \{(x, y) \in \mathbb{R}^2 : ax = by^2\}$

$E = \{(x, y) \in \mathbb{R}^2 : x^3 + y^3 = 1\}$. Then which o the following is correct?

(a) $C \; and \; D$ are compact, but A, B, E are not compact.

(b) $A \; and \; B$ are compact, but C, D, E are not compact.

(c) $A, B \; and \; E$ are cpompact, but C, D are not compact.

(d) $A \; and \; E$ are compact, but B, C, D are not compact.

Answer: (b)

Solution: In $\mathbb{R}^n$, a non-empty, set $S \subseteq \mathbb{R}^n$ is compact if and only if S is closed and bounded.

Set A is compact as it is closed and bounded.

Set B is compact as it is closed and bounded.

Set C is not bounded, so it is not compact.

Set D is not bounded, so it is not compact.

Ste E is not bounded, so it is not compact.

Thus, option (b) is correct.

67) Let P be a real polynomial of the real variable x of the form $P(x) = x^n + a_{n-1}x^{n-1} + \cdots\cdots + a_1 x - 1$.

Suppose that P has no root in the open unit disc and $P(-1) = 0$. Then

(a) $P(1) = 0$

(b) $\lim\limits_{x\to\infty} P(x) = \infty$

(c) $P(2) > 0$

(d) $P(3) = 0$

Answer: (a), (b), (c)

Solution: $P(x) = x^n + a_{n-1}x^{n-1} + \cdots\cdots + a_1 x - 1$

Since, $P(-1) = 0$

$\therefore n \geq 2$ $[For\ n = 1, P(x) = x - 1, not\ possible]$

Now, since $P(x)$ is real polynomial with real variable x i.e., $P(x)$ has only real roots.

Further product of roots $= \begin{cases} 1, n\ is\ oddd \\ -1, n\ is\ even \end{cases}$

Also, $P(x)$ has no root in the open unit disc. Thus, only possibility for roots is $x = \pm 1$.

If possible, let $'x = -1'$ is the only root of $P(x)$.

Then product of roots (For odd n) $= -1$, not possible.

Thus $x = 1$ is also roots of $P(x) \Rightarrow P(1) = 0$

$\therefore$ Option (a) is correct.

Clearly $P(x) = (x-1)^r(x+1)^s, r + s = n \Rightarrow \lim\limits_{x\to\infty} P(x) = \infty$

$$P(2) > 0, but\ P(3) \neq 0.$$

Hence option (b), (c) are correct & option (d) is incorrect.

68) Let f be a function on $\mathbb{R}^2$ such that $\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y)\ \forall\ (x,y) \in \mathbb{R}^2$

(a) $f(x,y) - f(y,x) = (x-y)\frac{\partial f}{\partial x}(x^*,y^*) + (y-x)\frac{\partial f}{\partial y}(x^*,y^*)$ for some point $(x^*,y^*) \in \mathbb{R}^2$.

(b) f is a constant on all lines parallel to the line $x = -y$.

(c) $f(x,y) = 0\ for\ all\ (x,y) \in \mathbb{R}^2$

(d) $f(x,y) = f(-y,x)\ for\ all\ (x,y) \in \mathbb{R}^2$

Answer: (a), (b)

Solution: For option (c) and (d)

Take, $f(x,y) = x + y$

Clearly, $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$

But $f(x,y) \neq 0$.

$\therefore$ Option (c) is incorrect.
Also, $f(x,y) \neq f(-y,x)$
Thus, option (d) is incorrect.
For option (a) and (b)
Since, $\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y)\ \forall (x,y) \in \mathbb{R}^2$
$\therefore$ f is a function of $(x+y)$, i.e., $\exists$ some function g such that $f(x,y) = g(x+y)$.
Thus$f(x,y) - f(y,x) = g(x+y) - g\ (y+x) = 0$ and $(x-y)\frac{\partial f}{\partial x}(x^*,y^*) + (y-x)\frac{\partial f}{\partial y}(x^*,y^*) = (x-y+y-x)\frac{\partial f}{\partial x}(x^*,y^*) = 0.$
Hence, $f(x,y) - f(y,x) = (x-y)\frac{\partial f}{\partial x}(x^*,y^*) + (y-x)\frac{\partial f}{\partial x}(x^*,y^*)$
$\therefore$ Option (a) is correct.
Also, along the lines parallel to $x = -y$ i.e., $x + y = 0$.
$f(x,y) = g(x+y) = g(0)$, which is clearly constant
Thus, option (b) is also correct.

69) Suppose $f: \mathbb{R} \to \mathbb{R}$ is a function that satisfies $|f(x). f(y)| \leq |x-y|^\beta, \beta > 0$. Which of the following is correct?
(a) If $\beta = 1$ then f is differentiable.
(b) If $\beta > 0$ then f is uniformly continuous.
(c) If $\beta > 1$ then f is a constant function.
(d) f must be a polynomial.

Answer: (b), (c)

Solution: Given $f: \mathbb{R} \to \mathbb{R}$ is a function such that $|f(x) - f(y)| \leq |x-y|^\beta, \beta > 0$.
For option (a) and (d)
Take, $f(x) = |x|, \beta = 1$.
Clearly, $|f(x) - f(y)| = \||x\| - \|y\|| \leq |x-y|^1$
Thus, given conditions is satisfied for $\beta = 1$.
But neither $f(x) = |x|$ is differentiable on $\mathbb{R}\ nor$ it is a polynomial.
$\therefore$ Option (a) and (d) are incorrect.
For option (b),
$\beta > 0$

Case I: For $0 < \beta < 1$.
For a given $\epsilon > 0$, choose $\delta = \epsilon$.
Now, for any $y\ \epsilon\mathbb{R}$ with $|x-y| < \delta$, we have
$|f(x) - f(y)| \leq |x-y|^\beta < |x-y| < \delta = \epsilon$.
i.e., $|f(x) - f(y)| < \epsilon\ \forall\ |x-y| < \delta, x, y\epsilon\mathbb{R}$.
$\therefore$ $f(x)$ is uniformly continuous.

Case II: For $\beta \geq 1$.

For a given $\epsilon > 0$, choose $\delta = (\epsilon)^{\frac{1}{\beta}}$.

Now, for any $y \epsilon \mathbb{R}$ with $|x - y| < S$, we have $|f(x) - f(y)| \leq |x - y|^{\beta} < \delta^{\beta} = \epsilon$.

$\therefore f(x)$ is uniformly continuous

Here, $f(x)$ is uniformly continuous for $\beta > 0$.

$\Rightarrow$ Option (b) is correct.

For option (c).

Let $\beta > 1$.

Take, $\beta = 1 + h, h > 0$.

$\therefore |f(x) - f(y)| \leq |x - y|^{\beta} = |x - y|^{1+h}, h > 0$.

$\Rightarrow \left|\frac{f(x)-f(y)}{x-y}\right| \leq |x - y|^{h}$.

Taking, limit $y \to x$ on both sides, we get $|f'(x)| \leq 0 \Rightarrow f'(x) = 0$

$\Rightarrow f(x)$ is correct.

$\therefore$ Option (c) is also correct.

70) Consider the two sets $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$. Choose the correct statements.

(a) The total number of functions from A to B is 125.

(b) The total number of functions from A to B is 243.

(c) The total number of one-to-one functions from A to B is 60.

(d) The total number of one-to-one functions from A to B is 120.

Answer: (a), (c)

Solution: If $f: A \to B$ is a map such $A = \{x_1, \cdots\cdots, x_m\}$ and $\{y_1, y_2, \cdots\cdots, y_n\}$, then number of functions from A to B are n^m and if $n \geq m$, then number of one-one functions from A to B are P_m.

Hence, as $|A| = 3, and\ |B| = 5$.

$\Rightarrow$ Number of functions from A to B are $5^3 = 125$.

And number of one-one functions from A to B are $5P_3 = 60$.

71) Consider $\mathbb{Q}$, the set of rational numbers, with the matric $d(p, q) = |p\ q|$. Then which of the following are true?

(a) $\{q \in \mathbb{Q}\ |2 < q^2 < 3\}$ is closed

(b) $\{q \in \mathbb{Q}\ |2 \leq q^2 \leq 4\}$ is compact

(c) $\{q \in \mathbb{Q}\ |2 \leq q^2 \leq 4\}$ is closed

(d) $\{q \in \mathbb{Q}\ |q^2 \geq 1\}$ is compact

Answer: (a), (c)

Solution: For option (a),

Take, $A = \{q \in \mathbb{Q} \mid 2 < q^2 < 3\} = \left((-\sqrt{3}, -\sqrt{2}) \cup (\sqrt{2}, \sqrt{3})\right) \cap \mathbb{Q}$
$= \left([-\sqrt{3}, -\sqrt{2}] \cup [\sqrt{2}, \sqrt{3}]\right) \cap \mathbb{Q}$
Since, $[-\sqrt{3}, -\sqrt{2}] \cup [\sqrt{2}, \sqrt{3}]$ is closed in $\mathbb{R}$ and $\mathbb{Q} \subseteq \mathbb{R}$.
$\Rightarrow \left([-\sqrt{3}, -\sqrt{2}] \cup [\sqrt{2}, \sqrt{3}]\right) \cap \mathbb{Q}$ is closed in $\mathbb{Q}$.
For option (c),
Take, $B = \{q \in \mathbb{Q} \mid 2 \le q^2 \le 4\} = \left([-2, -\sqrt{2}] \cup [\sqrt{2}, 2]\right) \cap \mathbb{Q}$.
As proved in option (a), B is closed in $\mathbb{Q}$.
$\therefore$ Option (c) is correct.
For option (b) and (d).

Result: For any subset k of $\mathbb{Q}, k \cap \mathbb{Q}$ is compact in $\mathbb{Q}$, if k is nowhere dense in $\mathbb{R}$. Using above options (b) and (d) are incorrect.

72) Which of the following define a metric on $\mathbb{R}$?
(a) $d(x, y) = \frac{|x-y|}{1+|x-y|}$
(b) $d(x, y) = |x - 2y| + |2y - x|$
(c) $d(x, y) = |x^2 - y^2|$
(d) $d(x, y) = |x^3 - y^3|$

Answer: (a), (d)

Solution: for option (a),
$d(x, y) = \frac{|x-y|}{1+|x-y|}$
We know that if $d(x, y)$ is a metric on $\mathbb{R}$, then $\frac{d(x,y)}{1+d(x,y)}$ is also a metric on $\mathbb{R}$.
As $|x - y|$ is a metric on $\mathbb{R}$.
$\therefore$ $d(x, y) = \frac{|x-y|}{1+|x-y|}$ is also a metric on $\mathbb{R}$.
For option (b),
$d(x, y) = |x - 2y| + |2y - x|$
Since, for $x = y$
$d(x, y) = |x - 2x| + |2x - x| = |-x| + |x| = 2|x| \neq 0$.
$\therefore$ $|x - 2y| + |2y - x|$ is not a metric on $\mathbb{R}$.
For option (c) and (d)
We know that $d(x, y) = |x^n - y^n|$ is a metric on $\mathbb{R}$ if and only if n is odd.
Thus, $d(x, y) = |x^3 - y^3|$ is a metric on $\mathbb{R}$ but
$d(x, y) = |x^2 - y^2|$ is not.

73) Let $A_n \subseteq \mathbb{R}$ for $n \geq 1$, and $X_n: \mathbb{R} \to \{0,1\}$ be the function

$X_n = \begin{cases} 0 \; if \; x \notin A_n \\ 1 \; if \; x \in \; A_n \end{cases}$, $Let \; g(x) = \lim_{n\to\infty} \sup X_n \,(x) \; and \;\; h(x) = \lim_{n\to\infty} inf \; X_n \,(x)$.

(a) If $g(x) = h(x) = 1$, then there exists m such that for all $n \geq m$ we have $x \in \; A_n$.

(b) If $g(x) = 1 \; and \; h(x) = 0$, then there exists m such that for all $n \geq m$ we have $x \in \; A_n$.

(c) If $g(x) = 1 \; and \; h(x) = 0$, then there exists a sequence $n_1, n_2, \cdots\cdots$ of distinct integers such that $x \in \; A_{n_k}$ for all $k \geq 1$.

(d) If $g(x) = h(x) = 0$, then there exists m such that for all $n \geq m$ we $x \notin A_n$.

Answer: (a), (c), (d)

Solution: Given, $A_n \subseteq \mathbb{R}$ for $n \geq 1$ and $X_n: \mathbb{R} \to \{0, 1\}$ be the function defined by

$X_n(x) = \begin{cases} 1 \; if \; x \in \; A_n \\ 0 \; if \; x \notin A_n \end{cases}$, $g(x) = \lim_{n\to\infty} \sup X_n \,(x) \; and \; h(x) = \lim_{n\to\infty} inf \; X_n(x)$.

For option (a)

If $g(x) = h(x) = 1$.

Then by definition of $\lim sup$ and $\lim inf$, after some stage $X_n(x)$ becomes 1, i.e., $\exists$ an m such that for all $n \geq m$, we have $x \in \; A_n$.

$\therefore$ Option (a) is correct and therefore, option (b) is incorrect.

For option (c)

If $g(x) = 1, h(x) = 0$.

Then, $\exists$ two subsequences $X_{n_k}(x) \; and \; X_{n_r}(x) \; of \; X_n(x)$ such that $X_{n_k}(x)$ converges to 1 and $X_{n_r}(x)$ converges to $'0'$.

i.e., there exists a sequence $n_1, n_2, \cdots\cdots$ of distinct integers such that for $k \geq n$, we have $x \in A_n$.

$\therefore$ Option (c) is correct.

For option (d),

If $g(x) = h(x) = 0$

$\Rightarrow$ After some stage $X_n(x)$ becomes $'0'$ i.e., $\exists$ an m such that for all $m \geq m$, we have $x \notin A_n$.

$\therefore$ Option (d) is correct.

74) Let $f: \mathbb{R}^n \to \mathbb{R}$ be the map $f(x_1, x_2, \cdots\cdots, x_n) = a_1 x + \cdots\cdots + a_n x_n$, where $a = (a_1, a_2, \cdots\cdots, a_n)$ is a fixed non-zero vector. Let $Df(0)$ denote the derivative of f at 0. Which of the following are correct?

(a) $(Df)(0)$ is a linear map from $\mathbb{R}^n \; to \; \mathbb{R}$

(b) $[(Df)(0)](a) = \|a\|^2$

(c) $[(Df)](0)(a) = 0$

(d) $[(Df)(0)](b) = a_1 b_1 + a_2 b_2 + \cdots\cdots + a_n b_n$ for $b = (b_1, \cdots\cdots, b_n)$.

Answer: (a), (b), (d).

Solution: $Df(0) = \left[\frac{\partial f}{\partial x_1}(\vec{0}), \frac{\partial f}{\partial x_2}(\vec{0}), \frac{\partial f}{\partial x_3}(\vec{0}) \cdots\cdots, \frac{\partial f}{\partial x_n}(\vec{0})\right] = [a_1, a_2, \cdots\cdots, a_n]$.

Clearly, $(Df)(0)$ a linear map from $\mathbb{R}^n$ to $\mathbb{R}$.

Now, $[(Df)(0)](a) = [a_1 a_2 \ \cdots\cdots a_n] \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = [a_1^2 + a_2^2 + \cdots\cdots + a_n^2] = \|a\|^2$

Further, $[(Df)(0)](b) = [a_1 a_2 \ \cdots\cdots a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots\cdots + a_n b_n$.

75) Let $F_1, F_2: \mathbb{R}^2 \to \mathbb{R}$ be the functions $F_1(x_1, x_2) = \frac{x_2}{x_1^2 + x_2^2}$ and $F_2(x_1 x_2) = \frac{x_1}{x_1^2 + x_2^2}$ which of the following are correct?

(a) $\frac{\partial F_1}{\partial x_2} = \frac{\partial F_2}{\partial x_1}$

(b) There exists a function $f: \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ such that $\frac{\partial f}{\partial x_1} = F_1$ and $\frac{\partial f}{\partial x_2} = F_2$.

(c) There exists no function $f: \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ such that $\frac{\partial f}{\partial x_1} = F_1$ and $\frac{\partial f}{\partial x_2} = F_2$.

(d) There exists a function $f: D \to \mathbb{R}$ where D is the open disc of 1 centred at $(2, 0)$, which satisfies $\frac{\partial f}{\partial x_1} = F_1$ and $\frac{\partial f}{\partial x_2} = F_2$ on D.

Answer: (a), (c), (d)

Solution: $\frac{\partial F_1}{\partial x_2} = \frac{(x_1^2+x_2^2)(-1)-(-x_2)(2x_2)}{(x_1^2+x_2^2)^2} = \frac{-x_1^2+x_2^2}{(x_1^2+x_2^2)^2} = \frac{x_2^2-x_1^2}{(x_1^2+x_2^2)^2}$ and

$\frac{\partial F_2}{\partial x_1} = \frac{(x_1^2+x_2^2)(1)-x_1(2x_1)}{(x_1^2+x_2^2)^2} = \frac{x_2^2-x_1^2}{(x_1^2+x_2^2)^2}$

Thus, $\frac{\partial F_1}{\partial x_2} = \frac{\partial F_2}{\partial x_1}$

∴ Option (a) is correct.

For option (b)

If possible, let there be a function $f: \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$

So, that $\frac{\partial f}{\partial x_1} = F_1 = \frac{-x_2}{x_1^2+x_2^2}$

Integrating w.r. to x_1. We get $f = \tan^{-1}\left(\frac{-x_1}{x_2}\right) + g(x_2)$ -------------- (1)

Further, differentiable it w.r. to x_2, we get $\frac{\partial f}{\partial x_2} = \frac{x_1}{x_1^2+x_2^2} + g'(x_2)$

But $\frac{\partial f}{\partial x_2} = F_2 = \frac{x_1}{x_1^2+x_2^2} \Rightarrow g'(x_2) = 0 \Rightarrow g(x_2) = c, (const.)$

Thus, $f(x_1, x_2) = \tan^{-1}\left(\frac{-x_1}{x_2}\right) + c$

Clearly, $'f'$ is not defined where $x_1 = 0$ and $x_2 = 0$, i.e., $'f'$ can only be defined on
$\mathbb{R} \times \mathbb{R} \setminus (\{0\})$.

Hence, option (c) is correct and option (b) is incorrect.
Also, $D = \{(x, y): (x-2)^2 + y^2 < 1\} \subseteq \mathbb{R} \times (\mathbb{R} \setminus \{0\})$.
Clearly, as discussed above, there exists a function $f: D \to \mathbb{R}$.
Satisfying, $\frac{\partial f}{\partial x_1} = F_1 \; and \; \frac{\partial f}{\partial x_2} = F_2 \; on \; D$.
Hence, option (d) is correct.

76) Let A be a subset of $\mathbb{R}^p \; and \; x \in \mathbb{R}^p$. Denote $d(x, A) = \inf\{d(x, y): y \in A\}$.
There exist a point $y_0 \in A$ with $d(y_0, x) = d(x, A)$, if
(a) A is any closed non-empty subset of $\mathbb{R}^p$.
(b) A is any non-empty subset of $\mathbb{R}^p$.
(c) A is any non-empty compact subset of $\mathbb{R}^p$.

Answer: (a), (c)

<u>Solution:</u> Given, A is any subset of $\mathbb{R}^p \; and \; x \in \mathbb{R}^p$ such that $d(x, A) = inf = \{d(x, y): y \in A\}$
For $p = 1$
$Let \; A = (0,1), x = 3 \in \mathbb{R}$ be any element.
Now, $d(x, A) = 2$
But $\nexists$ any $y_0 \in A$ for which $d(y_0, x) = d(x, A)$.
Thus, options (b) and (d) are incorrect.
Infect, $d(y_0, x) = d(x, A), y_0 \in \; A$ if and only if A is closed non-empty subset of $\mathbb{R}^p$.
Thus, option (a) and (c) are correct.

77) Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function with period $p > 0$. Then $g(x) = \int_x^{x+p} f\,(t)dt$ is a
(a) Constant function.
(b) Continuous function but not differentiable.
(c) Continuous function.
(d) Neither continuous nor differentiable.

Answer: (a), (c)

Solution: $g(x) = \int_x^{x+p} f\,(t)dt = \int_0^{x+p} f\,(t)dt - \int_0^x f\,(t)dt$.
$\Rightarrow g'(x) = f(x+p) - f(x) = 0$ [$\because f(x)$ is periodic function with period p].
$\Rightarrow g$ is constant.
$\Rightarrow 'g'$ is continuous and differentiable.
Thus, option (a) and (c) are correct and option (b), (d) are incorrect.

78) Let $z = x + iy$ and $f: \mathbb{R}^2 \to \mathbb{R}^2$ be the function $f(x, y) = f(z) = z^2 = (x^2 - y^2, 2xy) \in \mathbb{R}^2$. Let $(Df)(a)$ denote the derivative of f at a which of the following are true?
(a) $(Df)(a)h = 2ah$, where $a = a_1 + ia_2$ and $h = h_1 + ih_2$

(b) $(Df)(a) = 2\begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}, a = (a_1, a_2) \in \mathbb{R}^2$

(c) f is one-to-one on $\mathbb{R}^2$.

(d) For any $a \in \mathbb{R}^2 \setminus \{(0,0)\}, f$ is one to one on some neighborhood of a.

Answer: (a), (b), (d)

Solution: Given, $f(x, y) = z^2 = (x^2 - y^2, 2xy)$

$\therefore (Df)(z) = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix} \Rightarrow (Df)(a) = 2\begin{bmatrix} a & -a_2 \\ a_2 & a_1 \end{bmatrix}$.

$a = (a_1, a_2)$

Thus, option (b) is correct.

Also, $(Df)(a)h = 2\begin{bmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = 2\begin{bmatrix} a_1h_1 - a_2h_2 \\ a_2h_2 + a_1h_2 \end{bmatrix} = 2ah.$

Also, $(Df)(z) = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$

$\Rightarrow |Df| = 4(x^2 + y^2) \neq 0 \; if \; x, y \neq 0 \; and \; 0 \; if \; x, y = 0 \; for \; any \; point \; a \epsilon \mathbb{R}^2 \setminus \{(0,0)\}.$

Thus $\exists$ some neighborhood of a in which $f(x, y)$ is invertible.

$\Rightarrow f(x, y)$ is one-one on some neighborhood of a.

79) Let A be the set of rational number in the open interval $(0,7)$ and $f: A \to \mathbb{R}$ be a uniformly continuous function. Which of the following are true?

(a) f is bounded.

(b) f is necessarily a constant function.

(c) f is differentiable on $(0, 7)$.

(d) f is differentiable at all the rational points.

Answer: a

Solution: Since $(0, 7) \cap \mathbb{Q}$ is a bounded interval and f is uniformly continuous.

We know that a uniformly continuous function maps bounded sets to bounded sets.

$\therefore$ $'f'$ is bounded.

Hence, option (a) is correct.

Option (b) is not correct as take $f(x) = \sin x$, which is uniformly continuous on A nut it is not constant.

Options (c) and (d) are not correct as take $f(x) = |x - 1|$.

Here, f is uniformly continuous on A but not differentiable on $x = 1$ which is a rational number.

80) If $f: [0,1] \to (0,1)$ is a continuous mapping then which of the following is not true?

(a) $F \subseteq [0,1]$ is closed set implies $f(F)$ is closed in $\mathbb{R}$.

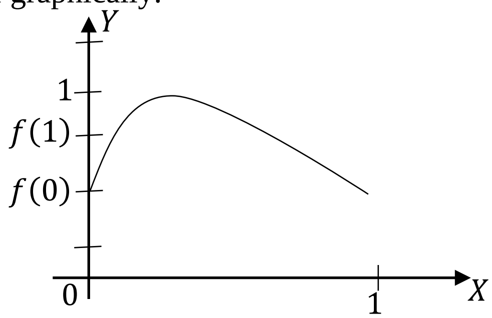
(b) If $f(0) < f(1)$ then $f([0,1])$ must be equal to $[f(0), f(1)]$.

(c) There must exist $x \in (0,1)$ such that $f(x) = x$.

(d) $f:([0,1]) \neq (0,1)$

Answer: (b)

Solution: For option (a)
Since $F \subseteq [0,1]$ is closed. Also F is bounded, as $[0,1]$ is bounded $\Rightarrow F$ is compact.
Every continuous function map compact set to compact set $\Rightarrow f(F)$ is compact set.
$\Rightarrow f(F)$ is closed in $\mathbb{R}$.
$\Rightarrow$ Option (a) is true.
For option (c).
As $f:[0,1] \to (0,1)$ is continuous.
$\therefore$By fixed point theorem, $\exists$ some $x \in (0,1)$ such that $f(x) = x$.
Thus, option (c) is true.
For option (d),
Since, $[0,1]$ is compact and continuous image of a compact set is compact.
$\therefore f([0,1]) \neq (0,1)[\because (0,1)\ is\ not\ compact]$
Thus, option (d) is true.
Hence, as option (a), (b), (d) ruled out.
$\therefore$ Option (b) is correct.
Also, we can understand if graphically.



From group it is clear that $f:[0,1] \to (0,1)$ is continuous and $f(0) < f(1)$but $f([0,1]) \nsubseteq [f(0), f(1)]$.

81) Let $\{a_n\}, \{b_n\}$ be sequence of real numbers satisfying$|a_n| \leq |b_n|$ for all $n \geq 1$. Then
(a) $\sum_n a_n$ converges whenever $\sum_n b_n$ converges.
(b) $\sum_n a_n$ converges absolutely whenever $\sum_n b_n$ converges absolutely.
(c) $\sum_n b_n$ converges whenever $\sum_n a_n$ converges.
(d) $\sum_n b_n$converges absolutely whenever$\sum_n a_n$ converges absolutely.

Answer: (b)

Solution: For option (a),
Take, $a_n = \frac{1}{n}, b_n = \frac{(-1)^n}{\boldsymbol{n}}$

Clearly, $|a_n| \leq |b_n|\ \forall\ n \geq 1$ and $\sum_n b_n = \sum_n \frac{(-1)^n}{n}$ is convergent.
But $\sum_n a_n = \sum \frac{1}{n}$ is not convergent.
Thus, option (a) is not correct.
For option (c) and (d)
Take, $a_n = \frac{1}{n^2}, b_n = \frac{1}{n}$.
Clearly, $|a_n| = \left|\frac{1}{n^2}\right| \leq \left|\frac{1}{n}\right| = |b_n|\ \forall\ n \geq 1\ and\ \sum_n a_n = \sum_n \frac{1}{n^2}$ is
Convergent, but $\sum_n b_n = \sum_n \frac{1}{n}$ is not convergent.
$\therefore$ Option (c) is not correct.
Also, $\sum_n \frac{1}{n^2}$ is absolutely convergent.
But $\sum_n \frac{1}{n}$ is not absolutely convergent.
$\therefore$ Option (d) is not correct.
As all other options ruled out
$\therefore$ Option (b) is correct.

82) If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then which of the following is NOT true?
(a) $\sum_{m=n}^{\infty} a_m \to 0$ as $n \to \infty$
(b) $\sum_{n=1}^{\infty} a_n \sin n$ is convergent
(c) $\sum_{n=1}^{\infty} e^{a_n}$ is divergent
(d) $\sum_{n=1}^{\infty} a_n^2$ is divergent

Answer: (d)

Solution: For option (a),
We know that a series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if sequence of its partial sums $\{S_n\}$ is convergent and both converges to a same limit.
Since, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent $\Rightarrow \sum_{n=1}^{\infty} a_n$ is convergent.
Let $\sum_{n=1}^{\infty} a_n$ convergent to $'l'$ and $\{S_n\}$ be its sequence of partial sums.
$\Rightarrow S_n \to l, as\ n \to \infty$.
Now, $\sum_{n=1}^{\infty} a_n = \sum_{m=1}^{n-1} a_m + \sum_{m=n}^{\infty} a_m$. Taking $n \to \infty$, we get $l = l + 0$, i.e., $\sum_{m=n}^{\infty} a_m \to 0$ as $n \to \infty$.
For option (b),
Since, $|a_n \sin n| \leq |a_n| [\because |\sin n| \leq 1]$ and $\sum_{n=1}^{\infty} |a_n|$ is convergent.
$\therefore$ By comparison test, $\sum_{n=1}^{\infty} |a_n \sin n|$ is convergent and hence.
$\sum_{n=1}^{\infty} a_n \sin n$ is convergent
$\therefore$ Option (b) is true.
For option (c)
Since $\sum_{n=1}^{\infty} a_n$ is convergent.
$\Rightarrow a_n \to 0\ as\ n \to \infty \Rightarrow e^{a_n} \to 1\ as\ n \to \infty \Rightarrow \sum_{n=1}^{\infty} e^{a_n}$ is divergent.

∴ Option (c) is true.
For option (d)
Take, $a_n = \frac{1}{n^2}$
Clearly, $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent and $\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty}\frac{1}{n^4}$ is also convergent.
∴ Option (d) is not true.
∴ Option (d) is the correct answer.

83) Let $f(x) = \begin{cases} \frac{\sin x}{x}, if\ x \neq 0 \\ 1, if\ x = 0 \end{cases}$. Then f is
(a) Discontinuous.
(b) Continuous but not differentiable.
(c) Differentiable only once.
(d) Differentiable more than once.

Answer: (d)

Solution: Option (d) is correct as define $f(x) = \begin{cases} \frac{\sin x}{x}, x \neq 0 \\ 1, x = 0 \end{cases}$ and $g(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!}$ ------------
Now, clearly for $x_0 = 0, g(x_0) = f(x_0)$-------------- (1)
and for, $x_0 \neq 0; xg(x) = f(x)$----------------------(2)
Since, $g(x)$ is differentiable more than once
∴ From (a) and (b) ⇒ $f(x)$ is differentiable more than once.

84) Let $f:[0,1] \to [0,1]$ be any twice differentiable function satisfying $f(ax + (1-a)y) \leq af(x) + (1-a)f(y)$ for all $x, y \in [0,1]$ and any $a \in [0,1]$. Then for all $x \in (0,1)$
(a) $f'(x) \geq 0$
(b) $f''(x) \geq 0$
(c) $f'(x) \leq 0$
(d) $f''(x) \leq 0$

Answer: (b)

Solution: Given, $f:[0,1] \to [0,1]$ be any twice differentiable function satisfying $f(ax + (1-a)y) \leq af(x) + (1-a)f(y) \Rightarrow f(x)$ is a convex function.

$$\therefore f''(x) \geq 0\ \forall\ x \in (0,1).$$

85) Let $f_n:[1,2] \to [0,1]$ be given by $f_n(x) = (2-x)^n$ for all non-negative integers n. Let $f(x) = \lim_{n\to\infty} f_n(x)$ for $1 \leq x \leq 2$. Then which of the following is true?
(a) f is a continuous function on $[1,2]$.

(b) f_n converges uniformly to f on $[1,2]$ as $n \to \infty$.

(c) $\lim_{n\to\infty} \int_1^2 f_n\,(x)dx = \int_1^2 f\,(x)dx$.

(d) For any $a \in (1,2)$ we have $\lim_{n\to\infty} f_n{}'\,(a) \neq f'(a)$.

Answer: (c)

Solution: Here $f_n: [1, 2] \to [0,1]$ such that $f_n(x) = (2 - x)^n$

$\Rightarrow f_n(x)$ converges to $f(x) = \begin{cases} 1\,; & x = 1 \\ 0; & 1 < x \le 2. \end{cases}$

Clearly, $f(x)$ is not continuous on $[1,2]$ and hence is not uniformly convergent on $[1,2]$.

$\therefore$ Option (a) and (b) are incorrect.

Also, $\lim_{n\to\infty} \int_1^2 f_n\,(x)dx = \lim_{n\to\infty} \left[\frac{1}{n+1}\right] = 0$ *and* $\int_1^2 f\,(x) = 0 \Rightarrow \lim_{n\to\infty} \int_1^2 f_n\,(x)dx = \int_1^2 f\,(x)dx$

$\therefore$ Option (c) is correct.

Also, for any $a \in (1,2)$.

$\lim_{n\to\infty} f_n{}'\,(a) = \lim_{n\to\infty} -n\,(2-a)^{n-1} = 0$ *and* $f'(a) = 0\ \forall\ a \in (1, 2) \Rightarrow \lim_{n\to\infty} f_n{}'\,(a) = f'(a)$.

$\therefore$ Option (d) is not true.

86) Let f, g be measurable real-valued functions on $\mathbb{R}$, such that $\int_{-\infty}^{\infty}(f(x)^2 + g(x)^2)\,dx = 2\int_{-\infty}^{\infty} f\,(x)g(x)dx$. Let $E = \{x \in \mathbb{R} | f(x) \neq g(x)\}$. Which of the following statements are necessarily true?

(a) E is the empty set.

(b) E is measurable.

(c) E has Lebesgue measure zero.

(d) For almost all $x \in \mathbb{R}$, we have $f(x) = 0$ *and* $g(x) = 0$

Answer: (b), (c)

Solution: Let $f(x) = 5, g(x) = \begin{cases} 5, x \neq 0 \\ 1, x = 0 \end{cases}$.

Clearly, $\int_{-\infty}^{\infty}(f(x)^2 + g(x)^2)\,dx = 2\int_{-\infty}^{\infty} f\,(x)g(x)dx$ and $0 \in E \Rightarrow E \neq \phi$.

$\therefore$ Option (a) is incorrect.

Also, from above example option (d) is incorrect, as neither $f(x) = 0$ *nor* $g(x) = 0$.

Now, as given $\int_{-\infty}^{\infty}(f(x)^2 + g(x)^2)\,dx = 2\int_{-\infty}^{\infty} f\,(x)g(x)dx$

$\Rightarrow \int_{-\infty}^{\infty}\big(f(x) - g(x)\big)^2\,dx = 0 \Rightarrow f(x) - g(x) = 0\ a.e.$

i.e., $f(x) = g(x) a.e.$

$\Rightarrow E = \{x \in \mathbb{R} | f(x) \neq g(x)\}$ is measurable and has Lebesgue measure zero.

Thus, option (b) and (c) are correct.

87) For a continuous function $f:\mathbb{R}\to\mathbb{R}$ satisfying $\int_{\mathbb{R}}|f(x)|\,dx<\infty$ and for some $\alpha>0$, let $d_f(\alpha)$ be the Lebesgue measure of the set $\{x\in\mathbb{R}:|f(x)|>\alpha\}$. Then for all $\alpha>0$, we have

(a) $\alpha d_f(\alpha)\leq\int_{\mathbb{R}}|f(x)|\,dx$

(b) $\alpha^2 d_f(\alpha)\leq\int_{\mathbb{R}}|f(x)|\,dx$

(c) $d_f(\alpha)\leq\alpha\int_{\mathbb{R}}|f(x)|\,dx$

(d) $d_f(\alpha)\leq\alpha^2\int_{\mathbb{R}}|f(x)|\,dx$

Answer: (a)

Solution: For option (a)

Let $E=\{x\in\mathbb{R}:|f(x)|>\alpha\}$.

Now, as $f(x)>\alpha, i.e., \alpha<f(x)$.

$\Rightarrow\int_E\alpha\,dx<\int_E f\,(x)dx<\int_{\mathbb{R}}f\,(x)dx\Rightarrow\alpha.m(E)<\int_{\mathbb{R}}f\,(x)dx$

$\Rightarrow\alpha d_f(\alpha)<\int_{\mathbb{R}}f\,(x)dx$

$\therefore$ Option (a) is correct.

For option (b),

Take, $f(x)=\begin{cases}x\,, & x\epsilon[0,20]\\ 40-x, & x\epsilon[20,40]\\ 0, & elsewhere\end{cases}$.

Take, $\alpha=10$ and $d_f(\alpha)=20$

$\therefore\alpha^2 d_f(\alpha)=100\times20=2000$

and $\int_{\mathbb{R}}|f(x)|\,dx=\left.\frac{x^2}{2}\right|_0^{20}+\left.\left(40x-\frac{x^2}{2}\right)\right|_{20}^{40}=\frac{400}{2}+40(40)-\frac{(40)^2}{2}-40(20)+\frac{(20)^2}{2}$

$=200+200=400$

Thus, $\alpha^2 d_f(\alpha)\nleq\int_{\mathbb{R}}|f(x)|\,dx$.

$\therefore$ Option (b) is incorrect.

For option (c),

Take, $f(x)=\begin{cases}x\,, & x\in[0,3]\\ 6-x\,, & x\in[3,6]\\ 0\,, & elsewhere\end{cases}$

Take, $\alpha=0.1\ and\ d_f(\alpha)=5.8$

and $\int_{\mathbb{R}}|f(x)|\,dx=9\therefore\alpha\int|f(x)|\,dx=(0.1)9=0.9$.

Clearly, $d_f(\alpha)\nleq\alpha\int_{\mathbb{R}}|f(x)|\,dx$

$\therefore$ Option (c) is incorrect.

Also, $\alpha^2=0.01\ and\ \alpha^2\int_{\mathbb{R}}|f(x)|\,dx=0.09\Rightarrow d_f(\alpha)\nleq\alpha^2\int_{\mathbb{R}}|f(x)|\,dx$.

$\therefore$ option (d) is incorrect.

88) Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (x + y, xy)$. Then.

(a) f is not differentiable at the point $(0,0)$.

(b) The derivative of f is invertable expect on the set $\{(x, y) \in \mathbb{R}^2 : x = y\}$.

(c) The inverse image of each point in $\mathbb{R}^2$ under f has at most two elements.

(d) f is surjective.

Answer: (b), (c)

__Solution:__ we have, $f(x, y) = (x + y, xy) = (f_1, f_2)$, where $f_1 = x + y, f_2 = xy$. We know that a function $f: \mathbb{R}^n \to \mathbb{R}^n$ is said to be differentiable at a point $x \in D \subseteq \mathbb{R}^n$ if $\exists$ a linear operator A from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $\lim_{h \to 0} \frac{|f(x+h) - f(x) - Ah|}{|h|} = 0$.

If $f(x)$ is differentiable, then $f'(x) = A = Df$

Now, $Df(x, y) = \begin{bmatrix} 1 & 1 \\ y & x \end{bmatrix}$

Let $h = (h_1, h_2) \in \mathbb{R}^2$ and $x = (0,0)$

$\therefore \lim_{(h_1,h_2) \to (0,0)} \frac{|f(x+h) - f(x) - Ah|}{|h|} = \lim_{(h_1,h_2) \to (0,0)} \frac{|f(h_1,h_2) - f(0,0) - Ah|}{\sqrt{h_1^2 + h_2^2}}$

$\Rightarrow \lim_{(h_1,h_2) \to (0,0)} \frac{|(h_1+h_2, h_1h_2) - (h_1+h_2, 0)|}{\sqrt{h_1^2 + h_2^2}} \Rightarrow \lim_{(h_1,h_2) \to (0,0)} \frac{|h_1h_2|}{\sqrt{h_1^2 + h_2^2}} = 0.$

$\therefore$ $'f'$ is differentiable at $(0,0)$

$\therefore$ Option (a) is incorrect.

Now, $Df(x, y) = \begin{bmatrix} 1 & 1 \\ y & x \end{bmatrix}$.

The derivative of f is invertible except on the set S, where $|Df(x, y)| = 0$,

i.e., on the set $\{(x, y) \in \mathbb{R}^2 : x - y = 0\}$.

$\therefore$ Option (b) is correct.

Also, the given function is symmetric so $f(x, y) = f(y, x) = (x + y, xy)$

$\Rightarrow$ The inverse image of each point in $\mathbb{R}^2$ under f has two elements.

Thus, option (c) is also correct.

f is not surjective as there does not exist $(x, y) \in \mathbb{R}^2$ such that $f(x, y) = (1,1)$

$\therefore$ option (d) is not correct.

89) The function $f: \mathbb{R} \to \mathbb{R}$ is given by $f(x) = e^{|x| + x^2} + |x^2 - 1|$. Which of the following is true about the function f?

(a) It is not differentiable exactly at three points of $\mathbb{R}$.

(b) It is not differentiable at $x = 0$.

(c) It is differentiable at $x = 2$.

(d) It is not differentiable at $x = 1$ and $x = -1$.

Answer: (a), (b), (c), (d)

Solution: Since, $g(x) = e^{|x|+x^2}$ is not differentiable at $x = 0$ and $h(x) = |x^2 - 1|$ is not differentiable at $x = 1, -1$.
So, $f(x) = e^{|x|+x^2} + |x^2 - 1|$ ix not differentiable at $x = 0, 1, -1$.
Thus, option (a), (b), (c) and (d) are correct.

90) Consider the sequence of rational number $\{q_k\}_{k\geq 1}$ where $q_k = \sum_{n=1}^{k} \frac{1}{10^{n^2}}$, i.e., the sequence is $q_1 = 0.1, q_2 = 0.1001, q_3 = 0.100100001$ etc. Which of the following is true?
(a) This sequence is bounded and convergent in $\mathbb{Q}$.
(b) This sequence is not bounded.
(c) This sequence is bounded, but not a Cauchy sequence.
(d) This sequence is bounded and Cauchy but not convergent in $\mathbb{Q}$.

Answer: (d)

Solution: The given sequence is $q_1 = 0.1, q_2 = 0.1001, q_3 = 0.100100001, \ldots$
Clearly, $|q_k| \leq 0.11 \ \forall \ k$.
Thus, sequence $\{q_k\}$ is bounded
∴ Option (b) is incorrect.
Also, the sequence converges to a non-terminating irrational number i.e., the sequence converges in $\mathbb{R}|\mathbb{Q}$ not in $\mathbb{Q}$.
As every convergent sequence is Cauchy.
Hence, the given sequence is Cauchy but not convergent is $\mathbb{Q}$.
∴ Option (d) is correct.

91) Which of the following subsets of $\mathbb{R}^2$ are uncountable?
(a) $\{(a, b) \in \mathbb{R}^2 : a \leq b\}$
(b) $\{(a, b) \in \mathbb{R}^2 : a + b \in \mathbb{Q}\}$
(c) $\{(a, b) \in \mathbb{R}^2 : ab \in \mathbb{Z}\}$
(d) $\{(a, b) \in \mathbb{R}^2 : ab \in \mathbb{Q}\}$

Answer: (a), (b), (c)

Solution:$\mathbb{Q} \times \mathbb{Q} = \{(a, b) \in \mathbb{R}^2 : a, b \in \mathbb{Q}\}$ is countable as $\mathbb{Q}$ is countable.
(∵ finite product of countable sets is countable)
Thus, option (d) is incorrect.
For option (a)
Since, $\{(a, a) \in \mathbb{R}^2 : a \in \mathbb{R}\} \subset \{(a, b) \in \mathbb{R}^2 : a \leq b\}$ and $\{(a, a) \in \mathbb{R}^2 : a \in \mathbb{R}\}$ is uncountable $\Rightarrow$ $\{(a, b) \in \mathbb{R}^2 : a \leq b\}$ is also uncountable.

Option (b) is correct as the set $S_1 = \{(\alpha_1 - \alpha) : \alpha \in \mathbb{R}\} \subseteq \{(a, b) \in \mathbb{R}^2 : a + b \in \mathbb{Q}\}$ and S_1 is uncountable.

$\therefore \{(a, b) \in \mathbb{R}^2 : (a + b) \in \mathbb{Q}\}$ is uncountable.

Option (c) is correct as the set $S_2 = \left\{\left(x, \frac{1}{x}\right) : x \in \mathbb{R} \setminus \{0\}\right\}$

$\subseteq \{(a, b) \in \mathbb{R}^2 : ab \in \mathbb{Z}\}$ and S_2 is uncountable as $S_2 \cong \mathbb{R} \setminus \{0\}$

$\therefore \{(a, b) \in \mathbb{R}^2 | ab \in \mathbb{Z}\}$ is uncountable.

$\therefore$ Option (a), (b) and (c) are correct.

92) Let $\{a_n\}_{n\geq 1}$ be a sequence of positive numbers such that $a_1 > a_2 > a_3 > \cdots\cdots$ Then which of the following is/are always true?

(a) $\lim\limits_{n\to\infty} a_n = 0.$

(b) $\lim\limits_{n\to\infty} \frac{a_n}{n} = 0.$

(c) $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

(d) $\sum_{n=1}^{\infty} \frac{a_n}{n^2}$ converges.

Answer: (b), (d)

<u>Solution:</u> For option (a)

Consider the sequence $a_n = 1 + \frac{1}{n}$, clearly $a_1 > a_2 > a_3 > \cdots\cdots, but \ \lim\limits_{n\to\infty} a_n \neq 0.$

$\therefore$Option (a) is not true.

For option (b)

As $a_1 > a_2 > a_3 > \cdots\cdots\cdots \Rightarrow \frac{a_1}{n} \geq \frac{a_n}{n} \geq 0.$

So, $\lim\limits_{n\to\infty} \frac{a_n}{n} = 0.$

Thus, option (b) is true.

For option (c)

Take $a_n = n.$

Clearly, $a_1 > a_2 > a_3 > \cdots$

But $\sum_{n=1}^{\infty} \frac{a_n}{n} = \sum_{n=1}^{\infty} 1$ is divergent

$\therefore$ Option (c) is not true.

As, $a_1 > a_2 > a_3 > \cdots\cdots \Rightarrow \frac{a_1}{n^2} \geq \frac{a_n}{n^2} \geq 0, i.e., \frac{a_n}{n^2} \leq \frac{a_1}{n^2}$

So, as $a_1 \sum \frac{1}{n^2}$ converges.

Hence, by comparison test $\sum_{n=1}^{\infty} \frac{a_n}{n^2}$ converges.

$\therefore$ Option (d) is true.

93) If $f : S \to S$ is a function then we denote by f^k, the function $f0f0\cdots\cdots\cdots\cdots0f(k\ times)$. Let $f_1\ and\ f_2$ be two functions defined on $\mathbb{R}^2$ as follows.

$f_1(x, y) = (x + 1, y + 3), f_2(x, y) = (x - 3, y - 2)$. Then
(a) For any positive integer k, there exists a unique $(a, b) \in \mathbb{R}^2$, such that $f_1^k(0,0) = f_k^2(a, b)$.
(b) For any real number α and any positive integer k, there is at most one solution y for $f_1^k(0,0) = f_2^k(x, y)$.
(c) There exists $(a, b) \in \mathbb{R}^2$ such that $f_1^k(a, b) \neq f_2^k(x, y)$ for any $(x, y) \in \mathbb{R}^2$ and any positive integer k.
(d) f_1 is a linear transformation.

Answer: (a), (b)

Solution: $f_1(x, y) = (x + 1, y + 3), f_2(x, y) = (x - 3, y - 2)$
Option (d) is not true as,
for $\bar{x} = (1,2)\ and\ \bar{y} = (3,4)$
$f_1(\bar{x} + \bar{y}) = (5,9) \neq f_1(\bar{x}) + f_1(\bar{y}) = (6,12)$
∴ Option (a) is correct as,
We have $f_1^k(x, y) = (x + k, y + 3k)$ and $f_2^k(x, y) = (x - 3k, y - 2k)$
So, $f_1^k(0,0) = f_2^k(a, k)$, we have $(k, 3k) = (a - 3k, b - 2k)$
i.e., $a = 4k\ and\ b = 5k\ and$ thus a, b are fixed.
Option (b) is correct as,
For $f_1^k(0,0) = f_2^k(\alpha, y)$ we have $(k, 3k) = (\alpha - 3k, y - 2k) \Rightarrow y = 5k$
⇒ Which is unique.
∴ Option (c) is not correct as,
If there exists $(a, b) \epsilon \mathbb{R}^2$ such that for some positive integer $k, \int_1^k(a, b) \neq f_2^k(x, y)$ for any $(x, y) \epsilon \mathbb{R}^2$
$\Rightarrow (a + k, b + 3k) \neq (x - 3k, y - 2k)$ for any $(x, y) \in \mathbb{R}^2 \Rightarrow x \neq a + 4k$ & $y \neq b + 5k$.
Which is not possible as domain of $f_2\ is\ \mathbb{R}^2$, so x & y can take any finite value.

94) Let $f: \mathbb{R}^m \to \mathbb{R}^m$ be a differentiable function. Let $Df(x)$ be the derivative of f at $x \in \mathbb{R}^m$. Which of the following is/are correct?
(a) $Df(0)(u) = 0\ for\ all\ u\ in\ \mathbb{R}^m$.
(b) $Df(x)(u) = 0\ for\ all\ u\ in\ \mathbb{R}^m\ and\ some\ x \in \mathbb{R}^m\ only$ if f is a constant.
(c) $Df(x)(u) = 0\ for\ all\ u \in \mathbb{R}^m$ and all $x \in \mathbb{R}^m$ only if f is a constant.
(d) If f is not a constant function, then $Df(x)$ is a one-to-one function for some $x \in \mathbb{R}^m$.

Answer: (c)

Solution: For option (a)
Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be a function defined by $f(x, y) = (x + y^2, y + 1)$. It can be easily verified that $'f'$ is differentiable function on $\mathbb{R}^2$.
as $\lim\limits_{h \to 0} \frac{|f(x_1+h) - f(x_1) - Ah|}{|h|}$

$= \lim_{h\to 0} \frac{|(x+h_1+(y+h_2)^2, y+h_2+1)-(x+y^2,y+1)-(h_1+2yh_2,h_2)|}{\sqrt{h_1^2+h_2^2}}$

$[Take, x_1 = (x,y) \in \mathbb{R}^2, h = (h_1,h_2) \in \mathbb{R}^2, A = f'(x) = Df]$

$= \lim_{(h_1,h_2)\to(0,0)} \frac{(h_2^2,0)}{\sqrt{h_1^2+h_2^2}} = 0$

Now, $Df(0) = \begin{bmatrix} 1 & 2(0) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $Df(0)(u) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$\Rightarrow Df(0)(u) = 0$ only if $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

∴ Option (a) is incorrect.

For option (b)

Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable function defined by $f(x,y) = (x^2 - y^2, 2xy)$

Here, $Df(x) = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix} = 4(x^2 + y^2) \Rightarrow Df(0) = 0.$

and $Df(0)(u) = 0 \ \forall\, u \in \mathbb{R}^2$, i.e., $Df(x)(u) = 0 \ \forall\, u \in \mathbb{R}^2$& some $x \in \mathbb{R}^2$.

But $'f'$ is not constant.

∴ Option (b) is incorrect.

For option (d).

Take $m = 1 \ and \ f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$, then $Df(x) = 3x^2$

∴ Clearly, $Df(x)$ is not one-one.

∴ Option (d) is incorrect.

As, all other options are incorrect.

Then, option (c) must be correct.

95) which of the following is not compact?

(a) $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$.

(b) $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \ge 1\}$.

(c) $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$

(d) $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

Answer: (b), (c)

Solution: The set $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$ is closed and bounded in $\mathbb{R}^2$, so, it is a compact set in $\mathbb{R}^2$.

For option (b), the given set is not bounded, so the set is not compact.

In option (c) the given set is open, so the set is not compact.

The set $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is closed and bounded in $\mathbb{R}^2$, so it is compact in $\mathbb{R}^2$.

96) Let A be a subset of $\mathbb{R}$ with more than one element. Let $a \in A$. If $A|\{a\}$ is compact, then.

(a) A is compact.

(b) Every subset of A must be compact.

(c) A must be a finite set.
(d) A is disconnected.

Answer: (a), (d)

<u>Solution:</u> Given, (i) $A \subseteq \mathbb{R}$ such that $|A| \geq 2$.
(ii) $a \in A \; and \; A|\{a\}$ is compact.
Option (a) is correct as, $A = \{a\} \cup (A|\{a\})$ and the union of two compact sets is compact.
Option (b) is not correct as, consider, $A = [1,2] \cup \{5\}$, which is a non-compact subset $(1,2)$
Also, A is infinite.
∴ Option (c) is not correct.
Further, in $\mathbb{R}$ only intervals or singletons are corrected sets. As $A|\{a\}$ is compact, then $'a'$ must be an isolated point ⇒ A cannot e an interval and hence A is disconnected set.
∴ Option (d) is correct.

97) Let A and B be two disjoint non-empty subsets of $\mathbb{R}^2$ such that $A \cup B$ is open in $\mathbb{R}^2$. Then,
(a) If A is open and $A \cup B$ is corrected, then B must be closed in $\mathbb{R}^2$.
(b) If A is closed, then B must be open in $\mathbb{R}^2$.
(c) If both A and B are connected then $A \cup B$ must be disconnected.
(d) If $A \cup B$ is disconnected, then both A and B are open.

Answer: (b), (d)

Solution: Given, A, B are two disjoint non-empty subsets of $\mathbb{R}^2$ such that $A \cup B$ is open in $\mathbb{R}^2$.
For option (a)
Take, $A = (0,1) \times (0,1) \; and \; B = [1,2) \times (0,1)$
Then, $A \cup B = (0,2) \times (0,1)$
Clearly, A is open and $A \cup B$ is connected but B is not closed in $\mathbb{R}^2$.
∴ option (a) is incorrect.
For option (b)
Since, A is closed ⇒ A^c is open.
Also, $A \cup B$ is open ⇒ $A^c \cap (A \cup B)$ is open and $A^c \cap (A \cup B) = B - A \Rightarrow B - A$ is open ⇒ B is open.
∴ Option (b) is correct.
For option (c)
Let $A = (0,1) \times (0,1)$. So A is corrected and non-empty in $\mathbb{R}^2$.
$B = [1,2) \times (0,1)$. So B is connected and non-empty in $\mathbb{R}^2$.
∴ $A \cup B = (0,2) \times (0,1)$ which is open in $\mathbb{R}^2$, but clearly $A \cup B$ is connected in $\mathbb{R}$.
∴ Option (c) is incorrect.
Since, $A \cup B$ is disconnected & $A \cup B$ is open ⇒ $A \& B$ is open.
∴ Option (d) is correct.

98) Let $L = \int_0^1 \frac{dx}{1+x^8}$. Then

(a) $L < 1$

(b) $L > 1$

(c) $L < \frac{\pi}{4}$

(d) $L > \frac{\pi}{4}$

Answer: (a), (d)

Solution:$L = \int_0^1 \frac{dx}{1+x^8}$

As, $0 < x < 1, x^8 < x^2 \Rightarrow 1 + x^8 < 1 + x^2 \ \Rightarrow \frac{1}{1+x^8} > \frac{1}{1+x^2}$

$L = \int_0^1 \frac{dx}{1+x^8} > \int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x\Big|_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} \Rightarrow L > \frac{\pi}{4}$.

∴ Option (d) us correct.

Also, $\frac{1}{1+x^8}$ is maximum when $x = 0$

$\Rightarrow \frac{1}{1+x^8} < \frac{1}{1+0} \Rightarrow \int_0^1 \frac{dx}{1+x^8} < \int_0^1 dx \Rightarrow L < 1$.

∴ Option (a) is correct.

99) Let f be a continuously differentiable function $\mathbb{R}$. Suppose that $L = \lim_{x\to\infty}\big(f(x) + f'(x)\big)$ exists. If $0 < L < \infty$, then which of the following statements is/are correct?

(a) If $\lim_{x\to\infty} f'(x)$ exists, then it is 0.

(b) If $\lim_{x\to\infty} f\,(x)$ exists, then it is L.

(c) If $\lim_{x\to\infty} f'\,(x)$ exists, then $\lim_{x\to\infty} f\,(x) = 0$.

(d) If $\lim_{x\to\infty} f\,(x)$ exists, then $\lim_{x\to\infty} f'\,(x) = L$.

Answer: (a), (b)

Solution: Given f is continuously differentiable function on $\mathbb{R}\, and\, L = \lim_{x\to\infty}\big(f(x) + f'(x)\big), 0 < L < \infty$.

Take $f(x) = 1$.

$\lim_{x\to\infty}\big(f(x) + f'(x)\big) = 1 = L$ and $\lim_{x\to\infty} f\,(x) = 1$, $\lim_{x\to\infty} f'\,(x) = 0$.

Thus, option (c) and (d) are incorrect.

Now, options (a) and (b) are holds simultaneously i.e., if option (a) is correct then option (b) is also correct and if (b) is correct then (a) also.

Hence, option (a) and (b) are correct.

100) Let X be a metric space and $f: x \to \mathbb{R}$ be a continuous function. Let $G - \{(x, f(x)): x\epsilon\ X\}$ be the group of X, f, then

(a) G is homeomorphic to X.

(b) G is homeomorphic to $\mathbb{R}$.

(c) G is homeomorphic to $X \times \mathbb{R}$.

(d) G is homeomorphic to $\mathbb{R} \times X$.

Answer: (a)

Solution: Let $x = \{0,1\}$ and d is metric on X defined by $d(x,y) = \begin{cases} 1\ , x \neq y \\ 0, k = y \end{cases}$. Let $f: X \to \mathbb{R}$ is continuous function defined by $f(x) = x$.

$\therefore G = \{(0,0), (1,1)\}$

Clearly, G is homeomorphic to X only.

$\therefore$ Option (b), (c), (d) are incorrect & option (a) is correct.

101) Let $f_n(x) = \frac{\sin x}{\sqrt{n}}\ , n = 1, 2, 3, \cdots \cdots\ and\ x\epsilon[-1, 1]$. Then as $n \to \infty$,

(a) $\{f_n(x)\}$ does not converge uniformly in $[-1,1]$.

(b) $\lim_{n\to\infty} \int_{-1}^{n} f_n\ (x)dx \neq 0$.

(c) $\{f_n{}'(x)\}$ does not converge uniformly in $[-1,1]$.

(d) $f_n(x), n = 1, 2, \cdots \cdots$ is not unifrmly continuous in $[-1, 1]$.

Answer: (c)

Solution: Since, $f_n'(x) = \sqrt{n} \cos nx\ \ and\ f_n'(x) = \sqrt{n} \to \infty$ as $n \to \infty$, impliesthat $\{f_n{}'(x)\}$ does not converge uniformly in $[-1,1]$.

102) Consider two sequences $\{f_n\}$ and $\{g_n\}$ of functions where $f_n: [0,1] \to \mathbb{R}\ and\ g_n: \mathbb{R} \to \mathbb{R}$ aredefined by $f_n(x) = x^n\ and\ g_n(x) = \begin{cases} \cos\frac{(x-n)\pi}{2}\ , if\ x \in [n-1, n+1] \\ 0\ , \qquad\qquad otherwise. \end{cases}$

Then

(a) Neither $\{f_n\}\ nor\ \{g_n\}$ is uniformly convergent.

(b) $\{f_n\}$ is not uniformly convergent but $\{g_n\}$ is.

(c) $\{g_n\}$ is not uniformly convergent but $\{f_n\}$ is.

(d) Both $\{f_n\}$ and $\{g_n(x)\}$ are uniformly convergent.

Answer: (a)

Solution: Since, $\lim_{n\to\infty} f_n\ (x) = f(x) = \begin{cases} 0, & 0 \leq x \leq 1 \\ 1, & x = 1 \end{cases}$

and $\lim_{n\to\infty} g_n(x) = g(x) = \begin{cases} 1, for\ x = n \\ 0, otherwise \end{cases}$

Since, each $\{f_n(x)\}\ and\ \{g_n(x)\}$ are continuous on $[-1,1]$, but $f_n(x)\ and\ g(x)$ are not continuous on $[-1,1]$.

Therefore, neither $\{f_n(x)\}\ nor\ \{g_n(x)\}$ is uniformly convergent on $[-1,1]$

103) Let $f: \mathbb{R} \to \mathbb{R}$ be a non-zero function such that $|f(x)| \leq \frac{1}{1+2x^2}$ for all $x \in \mathbb{R}$. Define real valued functions $f_n\ on\ \mathbb{R}$ for all $n \in \mathbb{N}$ by $f_n(x) = f(x+n)$. Then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly

(a) On $[0,1]$ but not on $[-1,0]$.

(b) On $[-1,0]$ but not on $[0,1]$.

(c) On both $[0,1]$ and $[-1,0]$.

(d) Neither on $[0,1]\ nor\ on\ [-1,0]$.

Answer: (c)

Solution: Since, $|f_n(x)| = |f(x+n)| \leq \frac{1}{1+2(x+n)^2}\ for\ all\ x \in [-1,1]$.

Since, $\sum \frac{1}{(x+n)^2}$ is a convergent series for all $x \in [-1,1]$. Therefore, by Weierstrass theorem, the series $\sum f_n(x)$ converges uniformly on $[-1,1]$.

104) consider the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^{\frac{3}{2}}}$. Then the series

(a) Convergent uniformly on $\mathbb{R}$.

(b) Converges pointwise but not uniformly on $\mathbb{R}$.

(c) Point wise converge.

(d) Does not converge pointwise.

Answer: (a)

Solution: Now, $\left|\frac{\sin(nx)}{n^{\frac{3}{2}}}\right| \leq \frac{1}{n^{\frac{3}{2}}} = M_n(say)\ for\ real\ x.$

Since $\sum M_n$ is convergent series. Then by $M - test$, the given series converges uniformly.

105) Which of the following sequence $\{f_n\}$ of functions does not converge uniformly on $[0,1]$?

(a) $f_n(x) = \frac{e^{-x}}{n}$

(b) $f_n(x) = (1-x)^n$

(c) $f_n(x) = \frac{x^2+nx}{n}$

(d) $f_n(x) = \frac{\sin(nx+n)}{n}$

Answer: (b)

Solution: For the sequence of function

$f_n(x) = (1-x)^n$, we have $\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 1, & for\ x = 0 \\ 0, & for\ 0 < x \le 1. \end{cases}$

Here, each $f_n(x)$ is continuous on $[0,1]$, but the function $f(x)$ is not continuous on [0,1].
Thus, the sequence $\{f_n(x)\} = (1-x)^n$ does not uniformly converge on [0,1].

106) If a function $f(x)$ be such that $f(x) = \sum_{n=0}^{\infty} \phi_n(x)$, where $\phi_n(x) = (1-x)x^n, 0 \le x \le 1$. Then

(a) The series does not converge uniformly on [0,1].
(b) $f(x)$ is continuous on [0,1].
(c) The series may or may not converge uniformly on [0,1].
(d) The series converge uniformly an [0,1].

Answer: (a)

Solution: For all $x \in [0,1]$, we have $|\phi_n(x)| = |(1-x)x^n| \le x^n$ for all $n \in \mathbb{N}$.

Thus, for $n \to \infty$, $\phi_n(x) \to \begin{cases} 0, & for\ 0 \le x < 1 \\ 1, & for\ x = 1 \end{cases}$

Hence, the given series is not uniformly convergent.

107) Let G be the set of all irrational numbers. The interior and the closed of G are denoted by G^0 and G respectively (when ψ is the set of all real number). Then

(a) $G^0 = \phi, \bar{G} = G$
(b) $G^0 = \psi, \bar{G} = \psi$
(c) $G^0 = \phi, \bar{G} = \psi$
(d) $G^0 = G, \bar{G} = \psi$.

Answer: (c)

Solution: Since a neighborhood of an irrational number contains rational as well as irrational points. So, no irrational number is an interior point and so $G^0 = \phi$.
On the other hand, all real numbers are the limit point the set of irrational numbers G. So $\bar{G} = \psi$.

108) The set $U = \left\{x \in \mathbb{R}: \sin x = \frac{1}{2}\right\}$ is

(a) Open

(b) Closed

(c) Both open and closed

(d) Neither open nor closed

Answer: (b)

Solution: Since, $U = \left\{x \in \mathbb{R}: \sin x = \frac{1}{2}\right\}$

i.e., $U = \left\{x: x = n\pi + (-1)^n \frac{\pi}{6}\right\}; n = 0, \pm 1, \pm 2,$

Answer Table

Sl. No	Answer	Sl. No	Answer	Sl. No	Answer
1.	(a), (b) & (c)	2.	(b)	3.	(a)
4.	(b) & (d)	5.	(a)	6.	(a)
7.	(b) & (d)	8.	(a)	9.	(c) & (d)
10.	(a)	11.	(a) & (b)	12.	(a)
13.	(b) & (d)	14.	(a)	15.	(a)
16.	(a)	17.	(d)	18.	(a)
19.	(b) & (d)	20.	(d)	21.	(a) & (b)
22.	(a) & (c)	23.	(a) & (c)	24.	(a) & (b)
25.	(a) & (d)	26.	(b) & (d)	27.	(c)
28.	(d)	29.	(a)	30.	(c)
31.	(b) & (c)	32.	(a) & (c)	33.	(a), (c) & (d)
34.	(a)	35.	(b), (c) & (d)	36.	(a) & (c)
37.	(d)	38.	(c)	39.	(c)
40.	(c)	41.	(d)	42.	(a)
43.	(b) & (d)	44.	(a) & (c)	45.	(c)
46.	(d)	47.	(b)	48	(c)
49.	(b)	50.	(d)	51.	(a)
52.	(a) & (d)	53.	(a), (b) & (d)	54.	(a) & (b)
55.	(b), (c) & (d)	56.	(a) & (b)	57.	(b), (c) & (d)
58.	(a) & (c)	59.	(c)	60.	(c)
61.	(a)	62.	(c)	63.	(d)
64.	(a)	65.	(a)	66.	(b)
67.	(a), (b) & (c)	68.	(a) & (b)	69.	(b) & (c)
70.	(a) & (c)	71.	(a) & (c)	72.	(a) & (d)
73.	(a), (c) & (d)	74.	(a), (b) & (d)	75.	(a), (c) & (d)
76.	(a) & (c)	77.	(a) & (c)	78.	(a), (b) & (d)
79.	(a)	80.	(b)	81.	(b)
82.	(d)	83.	(d)	84.	(b)
85.	(c)	86.	(b) & (c)	87.	(a)
88.	(b) & (c)	89.	(a), (b), (c) & (d)	90.	(d)
91.	(a), (b) & (c)	92.	(b) & (d)	93.	(a) & (b)
94.	(c)	95.	(b) & (c)	96.	(a) & (d)
97.	(b) & (d)	98.	(a) & (d)	99.	(a) & (b)
100.	(a)	101.	(c)	102.	(a)
103.	(c)	104.	(a)	105.	(b)
106.	(a)	107.	(c)	108.	(b)