# Complex Analysis

## 2011 – June

**33.** Let $f: \mathbb{C} \to \mathbb{C}$ be a complex valued function given by $f(z) = u(x, y) + i\, v(x, y)$

Suppose that $v(x, y) = 3xy^2$. Then –

(a) $f$ cannot be holomorphic on $\mathbb{C}$ for any choice of $u$.

(b) $f$ is holomorphic on $\mathbb{C}$ for a suitable choice of $u$.

(c) $f$ is holomorphic on $\mathbb{C}$ for all choices of $u$.

(d) $v$ is not differentiable as a function of x and y.

**Ans: (a)**

Given that,

$$U(X, Y) = 3XY^2$$

$$\frac{\partial U}{\partial X} = 3Y^2 \Rightarrow \frac{\partial^2 U}{\partial X^2} = 0 \text{ and } \frac{\partial U}{\partial Y} = 6XY \Rightarrow \frac{\partial^2 U}{\partial Y^2} = 6X$$

$$\therefore \frac{\partial^2 U}{\partial X^2} - \frac{\partial^2 U}{\partial Y^2} = 0 - 6X = -6X \neq 0, \forall\, X \neq 0$$

Hence, $U(X, Y) = 3XY^2$ is not harmonic function.

Hence, $f(z) = U(X, Y) + iv(X, Y)$, cannot be holomorphic on $\mathbb{C}$ for any choice of $U$.

Hence, option (a) is correct.

**37.** The power series $\sum_0^\infty 2^{-n} z^{2n}$ converges, if

(a) $|z| \leq 2$

(b) $|z| < 2$

(c) $|z| \leq \sqrt{2}$

(d) $|z| < \sqrt{2}$

**Ans: (d)**

Given power series is – $\sum_0^\infty 2^{-n} z^{2n}$

Note that, $a_n = \begin{cases} 0, & n = 2k - 1 \\ 2^{-n}, & n = 2k \end{cases}; k = 1, 2, 3, \dots \dots \dots$

Now, $\lim_{n\to\infty} \sup \sqrt[n]{|a_n|} = \lim_{k\to\infty} |2^{-2k}|^{\frac{1}{2k}} = \frac{1}{2}$ and $\lim_{n\to\infty} \inf \sqrt[n]{|a_n|} = \lim_{k\to\infty} |a_{2k-1}|^{\frac{1}{2k-1}} = 0$

Hence, $|z|^2 = 2 \Rightarrow R = \sqrt{2}$

Hence, option (d) is correct.

**79.** Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc. Let $f : D \to \mathbb{C}$ be an analytic function satisfying $f\left(\frac{1}{n}\right) = \frac{2n}{3n+1}$ for $n \geq 1$, then –

(a) $f(0) = \frac{2}{3}$

(b) $f$ has a simple pole at $z = -3$

(c) $f(3) = \frac{1}{3}$

(d) No such $f$ exists.

**Ans: (a), (b), (c)**

Given that, $f : D_f \to \mathbb{C}$ is define by $f\left(\frac{1}{n}\right) = \frac{2n}{3n+1}$

$\therefore D_f = \left\{z : z = \frac{1}{n}\right\} = \left\{\frac{1}{n} : n = \frac{1}{z}\right\}$

Then $f(z) = \frac{2\cdot\frac{1}{z}}{3\cdot\frac{1}{z}+1} = \frac{2}{3+z} \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(i)$

But 0 is the limit point of $D_f$ which is also be a point of $D = \{z : |z| < 1\}$.

Hence, by identity theorem, $f : D \to \mathbb{C}$ is define by $f(z) = \frac{2}{3+z}$

(a) Putting $z = 0$ in equation (i) , we get $f(0) = \frac{2}{3}$

Hence, option (a) is correct.

(b) Since, $\lim_{z\to -3}(z+3)\cdot\frac{2}{3+z} = \lim_{z\to -3} 2 = 2$, exists.

Hence, $z = -3$ is a simple pole.

(c) Putting $z = 3$ in equation (i), we get $f(z) = \frac{2}{3+3} = \frac{2}{6} = \frac{1}{3}$

(d) $f(z) = \frac{2}{z+3}$ exists, so, option (d) is incorrect.

**80.** Let $f$ be an entire an entire function. If $Re(f)$ is bounded then,

(a) $Im(f)$ is constant

(b) $f$ is constant

(c) $f \equiv 0$

(d) $f'$ is a non zero constant.

Ans: (a), (b)

Let $f(z) = u + iv$ be an entire function and $|u| \leq M$

Now, construct $g(z) = e^{f(z)}$, which is an entire function, then –

$g(z) = e^{u+iv} = e^u \cdot e^{iv} \Rightarrow |g(z)| = e^u < e^M$ ($\because$ exponential function is increasing)

$\Rightarrow g(z)$ is bounded.

Hence, $g(z)$ is an entire and bounded function, then $g(z)$ is constant function

$\Rightarrow e^{f(z)} = A, A \in \mathbb{C}$

$\Rightarrow f(z) = \log A$

$\Rightarrow f(z)$ is a constant function.

Hence, option (b) is correct.

$\Rightarrow Im(f)$ is constant function

Also, $f'(z)$ is a zero function

Hence, option (a) is correct.

**81.** Let $f: D \to \mathbb{D}$ be holomorphic with $f(0) = \frac{1}{2}$ and $f\left(\frac{1}{2}\right) = 0$, where $D = \{z : |z| \leq 1\}$ which of the following is correct?

(a) $|f'(0)| \leq \frac{3}{4}$

(b) $\left|f'\left(\frac{1}{2}\right)\right| \leq \frac{4}{3}$

(c) $|f'(0)| \leq \frac{3}{4}$ $and$ $\left|f'\left(\frac{1}{2}\right)\right| \leq \frac{4}{3}$

(d) $f(z) = z, z \in \mathbb{D}$

**Ans: (b)**

Consider the analytic function $f: D \to \mathbb{D}$ defined by $f(z) = \frac{1}{2} - z$ where $D = \{z: |z| \leq 1\}$, then –

$f(0) = \frac{1}{2}$ and $f\left(\frac{1}{2}\right) = 0$

(a) We have $f'(z) = -1$

$\therefore |f'(0)| = |-1| = 1 > \frac{3}{4}$

Hence, option (a) is incorrect.

(b) We have, $f'(z) = -1$

$\therefore \left|f'\left(\frac{1}{2}\right)\right| = |-1| = 1 < \frac{4}{3}$

Hence, option (b) is correct.

(c) We have, $|f'(0)| = |-1| = 1 > \frac{3}{4}$ but $\left|f'\left(\frac{1}{2}\right)\right| = |-1| = 1 < \frac{4}{3}$

Hence, option (c) is incorrect.

(d) We have, $f(z) = z \Rightarrow f(0) = 0, f\left(\frac{1}{2}\right) = \frac{1}{2}$

But given that, $f(0) = \frac{1}{2}$ and $f\left(\frac{1}{2}\right) = 0,$ which is a contradiction.

Hence option (d) is incorrect.

**83.** At $z = 0$ the function $f(z) = \frac{e^z+1}{e^z-1}$ has

(a) a removable singularity

(b) a pole

(c) an essential singularity

(d) The reduce of $f(z)\ \ at\ \ z = 0$ is 2.

**Ans: (b), (d)**

(b) Poles of $f(z)$ are obtained by equation to zero the denominator of $f(z)$.

$\therefore e^z - 1 = 0 \Rightarrow e^z = e^{2n\pi i} \Rightarrow z = 2n\pi i, n \in \mathbb{Z}$

(d) Reduce of $f(z)$ at $z = a$ is $\underset{z\to a}{\mathrm{Lt}}(z-a)f(z)$

$\therefore \underset{z\to a}{\mathrm{Lt}}\ z \cdot \frac{e^z+1}{e^z-1} = \underset{z\to a}{\mathrm{Lt}} \frac{z\,e^z+(e^z+1)}{e^z} = 2$

# 2011 – December

**22.** Consider the power series $\sum_{n\geq 1} a_n z^n$ where $a_n$ =number of divisors of $n^{50}$ . Then, the radius of convergence of $\sum_{n\geq 1} a_n z^n$ is

(a) 1

(b) 50

(c) $\frac{1}{50}$

(d) 0

**Ans: (a)**

The divisor of $n^{50}$ are $1, n, n^2, n^5, n^{10}, n^{25}$ and $n^{50}$

Hence, $a_n = 1,\ a_{n+1} = 1$

$\therefore \frac{1}{R} = \lim_{n\to\infty} \sup \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \sup \left|\frac{1}{1}\right| = 1$

$\Rightarrow R = 1$

Hence, option (a) is correct.

**35.** Let $I_r = \int_{C_r} \frac{dz}{z(z-1)(z-2)}$, where $C_r = \{z \in C: |z| = r\}, r > 0$, then –

(a) $I_r = 2\pi i, if\ r \in (2,3)$

(b) $I_r = \frac{1}{2}, if\ r \in (0,1)$

(c) $I_r = -2\pi i, if\ r \in (1,2)$

(d) $I_r = 0, if\ r > 3$

**Ans: (d)**

$I_r = \int_{C_r} \frac{dz}{z(z-1)(z-2)} = \frac{1}{2} I_r = \int_{C_r} \frac{1}{z} dz - \int_{C_r} \frac{1}{z-1} dz + \int_{C_r} \frac{1}{z-2} dz \ldots\ldots\ldots\ldots\ldots\ldots (i)$

(a) If $r \in (2,3)$, then by Cauchy's integral theorem, $\int_{C_r} \frac{1}{z} dz = 2\pi i$, $\int_{C_r} \frac{1}{z-1} dz = 2\pi i\ and\ \int_{C_r} \frac{1}{z-2} dz = 2\pi i$

$\therefore (i) \Rightarrow, I_r = \frac{1}{2} \cdot 2\pi i - 2\pi i + \frac{1}{2} \cdot 2\pi i = 0$

(b) If $r \in (0,1)$, then by Cauchy's integral theorem, $\int_{C_r} \frac{1}{z} dz = 2\pi i$, $\int_{C_r} \frac{1}{z-1} dz = 0$ *and* $\int_{C_r} \frac{1}{z-2} dz = 0$

$\therefore (i) \Rightarrow, I_r = 2\pi i$

(c) If $r \in (1,2)$, then by Cauchy's integral theorem, $\int_{C_r} \frac{1}{z} dz = 2\pi i$, $\int_{C_r} \frac{1}{z-1} dz = 2\pi i$ *and* $\int_{C_r} \frac{1}{z-2} dz = 0$

$\therefore (i) \Rightarrow, I_r = -\pi i$

(d) If $r > 3$, then by Cauchy's integral theorem,

$\int_{C_r} \frac{1}{z} dz = 2\pi i$, $\int_{C_r} \frac{1}{z-1} dz = 2\pi i$ *and* $\int_{C_r} \frac{1}{z-2} dz = 2\pi i$

$\therefore (i) \Rightarrow, I_r = 0$

So, option (d) is correct.

**79.** Let $f$ be an entire function such that $\lim_{|z|\to\infty} |f(z)| = \infty$ then,

(a) $f\left(\frac{1}{2}\right)$ has an essential singularity at 0.

(b) $f$ cannot be a polynomial.

(c) $f$ has finitely many zeros.

(d) $f\left(\frac{1}{2}\right)$ has a pole at 0.

**Ans: (c), (d)**

Given that, $\lim_{|z|\to\infty} |f(z)| = \infty$ and $f(z)$ is an entire function.

$\Rightarrow f(z)$ is a polynomial of infinite degree.

Hence, $f(z)$ has finitely many zeros.

Hence, option (c) is correct.

Let $f(z) = a_0 + a_1 z + a_2 z^2 + \dots \dots \dots + a_n z^n$

$f\left(\frac{1}{z}\right) = a_0 + a_1 \left(\frac{1}{z}\right) + a_2 \left(\frac{1}{z^2}\right) + \dots \dots \dots + a_n \left(\frac{1}{z^n}\right)$

$\Rightarrow \underset{z\to 0}{\text{Lt}} f\left(\frac{1}{z}\right)$ does not exists.

But $\underset{z\to 0}{\text{Lt}} z^n f\left(\frac{1}{z}\right) = \lim_{z\to 0}(a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots \dots \dots + a_n)$

$= a_n$ exists finitely and non zero.

Hence, $f\left(\frac{1}{z}\right)$ has pole at $z = 0$

Hence, option (d) is correct.

**80.** Let $f, g$ be holomorphic function defined on $A \cup D$ where $A = \{z \in C: \frac{1}{2} < |z| < 1$ and $D = \{z \in C: |z-2| < 1\}$ which of the following statements is correct?

(a) If $f(z)g(z) = 0\ \ for\ all\ z \in A \cup D$, then either $f(z) = 0\ \forall\ z \in A\ or\ g(z) = 0\ \forall\ z \in D$

(b) If $f(z)g(z) = 0\ \ for\ all\ z \in D$, then either $f(z) = 0\ \forall\ z \in D\ or\ g(z) = 0\ \forall\ z \in D$

(c) If $f(z)g(z) = 0\ \ for\ all\ z \in A$, then either $f(z) = 0\ \forall\ z \in A\ or\ g(z) = 0\ \forall\ z \in A$

(d) If $f(z)g(z) = 0\ \ for\ all\ z \in A \cup D$, then either $f(z) = 0\ \forall\ z \in A \cup D\ or\ g(z) = 0\ \forall\ z \in A \cup D$

**Ans: (a), (b), (c)**

(a) If $f(z)g(z) = 0,\ \forall\ z \in A \cup D$, then either $f(z) = 0\ \forall\ z \in A\ or\ g(z) = 0\ \forall\ z \in A$

Hence, option (a) is correct.

(b) If $f(z)g(z) = 0\ \ for\ all\ z \in D$, then either $f(z) = 0\ \forall\ z \in D\ or\ g(z) = 0\ \forall\ z \in D$

Hence, option (b) is correct.

(c) If $f(z)g(z) = 0\ \ for\ all\ z \in A$, then either $f(z) = 0,\ \forall\ z \in A\ or\ g(z) = 0,\ \forall\ z \in A$

Hence, option (c) is correct.

(d) If $f(z)g(z) = 0\ ,\ \forall\ z \in A \cup D$, then if does not implies either $f(z) = 0, \forall\ z \in A \cup D\ or\ g(z) = 0,\ \forall\ z \in A \cup D$

**81.** Let $f$ be a holomorphic function on $D = \{z \in C: |z| < 1\}$ such that $|f(z)| \leq 1$. Define $g: D \to C$ by $g(z) = \begin{cases} \frac{f(z)}{z},\ if\ z \in D, z \neq 0 \\ f'(0), \quad if\ \ z = 0. \end{cases}$

Which of the following statements are true?

(a) $g$ is holomorphic on $D$

(b) $|g(z)| \leq 1$ for all $z \in D$

(c) $|f'(z)| \leq 1$ for all $z \in D$

(d) $|f'(0)| \leq 1$

**Ans: (a),(b),(d)**

**83**. Let $f: C \to \mathbb{C}$ be an entire function and let $g: C \to \mathbb{C}$ be defined by $g(z) - f(z+1)$ for $\in C$ . Which of the following statements are true?

(a) If $f\left(\frac{1}{n}\right) = 0$ for all positive integers $n$, then $f$ is constant function.

(b) If $f(n) = 0$ for all positive integers n, then $f$ is a constant function.

(c) If $f\left(\frac{1}{n}\right) = f\left(\frac{1}{n} + 1\right)$ for all positive integers n, then $f$ is a constant function.

(d) If $f(n) = f(n+1)$ for all positive integers n, then $g$ is a constant function.

**Ans: (a), (c)**

# 2012 – June

**33.** Let $f: \mathbb{C} \to \mathbb{C}$ be a complex valued function of the form $f(X,Y) = U(X,Y) + iv(X,Y)$. Suppose that $U(X,Y) = 3X^2Y$. Then –

(a) $f$ cannot be holomorphic on $\mathbb{C}$ for any choice of $V$.

(b) $f$ is holomorphic on $\mathbb{C}$ for a suitable choice of V.

(c) $f$ is holomorphic on $\mathbb{C}$ for all choices of V.

(d) $U$ is not differentiable.

**Ans: (a)**

Given that $U(X,Y) = 3X^2Y$ $\quad\therefore \frac{\partial U}{\partial X} = 6XY$ ,

$\frac{\partial^2 U}{\partial X^2} = 6Y \; and \; \frac{\partial^2 U}{\partial Y^2} = 0$

$\therefore \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} = 6Y + 0$

$= 6Y \neq 0, \; Y \neq 0$

$\therefore U(X,Y)$ is not harmonic. Therefore , $f$ cannot be holomorphic on $\mathbb{C}$ for any choice of V.

Hence, option (a) is correct.

**37.** The power series $\sum_{n=0}^{\infty} 3^{-n}(z-1)^{2n}$ converges, if

(a) $|z| \leq 3$

(b) $|z| < \sqrt{3}$

(c) $|z| < \sqrt{3}$

(d) $|z-1| \leq \sqrt{3}$

**Ans: (c)**

$\frac{1}{R} = \lim_{n\to\infty} \sup \sqrt[n]{3^{-n}} \Rightarrow R = 3$

$\therefore |z-1|^2 < R = 3 \;\Rightarrow\; |z-1| < \sqrt{3}$

**80.** Let $f: \mathbb{C} \to \mathbb{C}$ be meromorphic function analytic at 0 satisfying $f\left(\frac{1}{n}\right) = \frac{n}{2n+1}$ $for\ n \geq 1$.

Then –

(a) $f(0) = \frac{1}{2}$

(b) $f$ has a simple pole at $z = -2$

(c) $f(2) = \frac{1}{4}$

(d) no such meromorphic function exists.

**Ans: (a), (b), (c)**

Given that, $f: \mathbb{C} \to \mathbb{C}$ is defined by $f\left(\frac{1}{n}\right) = \frac{n}{2n+1}$ $for\ \ n \geq 1$

$\therefore D_f = \left\{z: z = \frac{1}{n}\right\} = \left\{\frac{1}{n}: n = \frac{1}{z}\right\}$

Then, $f(z) = \frac{\frac{1}{z}}{2 \cdot \frac{1}{z} + 1} = \frac{1}{z+2}$

But 0 is limit point of $D_f$ which is also be a point of $\mathbb{C}$.

Hence, by identity theorem, $f : \mathbb{C} \to \mathbb{C}$ is defined by $f(z) = \frac{1}{z+2} \dots \dots \dots \dots (i)$

(a) Putting $z = 0$, in (i), we get $f(0) = \frac{1}{2}$

Hence, option (a) is correct.

(b) The pole of (z) is $z + 2 = 0 \Rightarrow z = -2$ , a simple pole of $f$.

$\therefore$ option (b) is correct.

(c) Putting z = 2 in (i) , we get

$f(2) = \frac{1}{2+2} = \frac{1}{4}$, so option (c) is correct.

(d) Since, $f(z) = \frac{1}{z+2}$ exists, which is meromorphic function.

Hence, option (d) is incorrect.

**81.** Let $f$ be an entire function. If $Im\ f \geq 0$, then

(a) $Re\ f$ if constant

(b) $f$ is constant

(c) $f = 0$

(d) $f'$ is non zero constant.

**Ans: (a), (b)**

Given $f(z) = u + iv$ is an entire function and $Im\ f = v \geq 0$

Construct an entire function, $g(z) = e^{i\,f(z)} = e^{i(u+iv)} = e^{iu-v} = e^{iu} \cdot e^{-v}$

$\therefore\ |g(z)| = e^{-v} \cdot 1 \leq 1$ $[\because v \geq 0 \Rightarrow e^{v} \geq e^{0} \Rightarrow e^{-v} \leq 1]$

$\therefore\ |g(z)| \leq 1$

Hence, $g(z)$ is bounded, i.e., $g(z)$ is an entire and bounded function. Hence by Liouville's theorem $g(z)$ is constant.

Let $g(z) = C \Rightarrow e^{i\,f(z)} = C \Rightarrow i\,f(z) = \log C \Rightarrow f(z) = -i \log C \Rightarrow f(z) is\ constant.$

$\Rightarrow Re\ \ f\ is\ constatn.$

Hence, option (a) and (b) are correct.

**82.** Let $f: D \to D$ be holomorphic with $f(0) = 0$ and $f\left(\frac{1}{2}\right) = 0$ where $D = \{z: |z| < 1\}$. Which of the following statements are correct?

(a) $\left|f'\left(\frac{1}{2}\right)\right| \leq \frac{4}{3}$

(b) $|f'(0)| \leq 1$

(c) $\left|f'\left(\frac{1}{2}\right)\right| \leq \frac{4}{3}$ $and$ $|f'(0)| \leq 1$

(d) $f(z) = z, z \in D$

**Ans: (a), (b), (c)**

Consider the holomorphic function $f: D \to D$ defined by $f(z) = z\left(z - \frac{1}{1}\right)$,

where $D = \{z: |z| < 1\}$. Then –

$f(0) = 0\left(0 - \frac{1}{2}\right) = 0$

$f\left(\frac{1}{2}\right) = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\right) = 0$

(a) $f'(z) = 2z - \frac{1}{2}$

$\therefore \left|f'\left(\frac{1}{2}\right)\right| = \left|2 \cdot \frac{1}{2} - \frac{1}{2}\right| = \frac{1}{4} < \frac{4}{3}$

$\therefore$ option (a) is correct.

(b) $\left|f'\left(\frac{1}{0}\right)\right| = \left|2 \cdot 0 - \frac{1}{2}\right| = \frac{1}{2} < 1$

$\therefore$ option (b) is correct.

(c) From option (a) and (b), we have $\left|f'\left(\frac{1}{2}\right)\right| \leq \frac{4}{3}$ $and$ $|f'(0)| < 1$

Hence, option (c) is correct.

(d) If $f(z) = z, \forall z \in D$

$f(0) = 0, f\left(\frac{1}{2}\right) = \frac{1}{2}$ $but$ $f\left(\frac{1}{2}\right) = 0$

Hence, $f(z) \neq z, \forall z \in D$

$\therefore$ option (d) is incorrect.

**83.** For $z \in \mathbb{C}$ of the form $z = x + iy$, define

$H^+ = \{z \in \mathbb{C} : y > 0\},\ H^- = \{z \in \mathbb{C} : y < 0\}$

$L^+ = \{z \in \mathbb{C} : x > 0\},\ L^- = \{z \in \mathbb{C} : x < 0\}$

The function $f(z) = \frac{2z+1}{5z+3}$

(a) maps $H^+$ onto $H^+$ and $H^-$ onto $H^-$

(b) maps $H^+$ onto $H^-$ and $H^-$ onto $H^+$

(c) maps $H^+$ onto $L^+$ and $H^-$ onto $L^-$

(d) maps $H^+$ onto $L^-$ and $H^-$ onto $L^+$

**Ans: (a)**

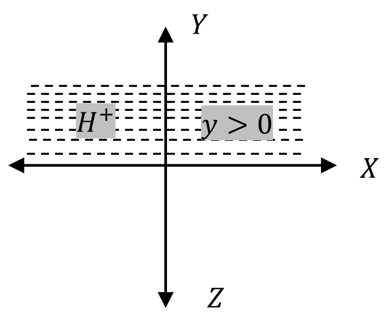
$W = f(z) = \frac{2z+1}{5z+3} \Rightarrow z = \frac{1-3w}{5w-2}$

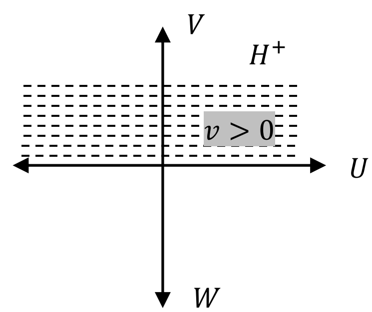
Let $z = x + iy \quad and \quad w = u + iv$

$\therefore x + iy = \frac{1-3(u+iv)}{5(u+iv)-2}$ or, $x + iy = \frac{[(1-3u)(5u-2)-15v^2]-iv[5(1-3u)+3(5u-2)]}{(5u-2)^2+25v^2}$

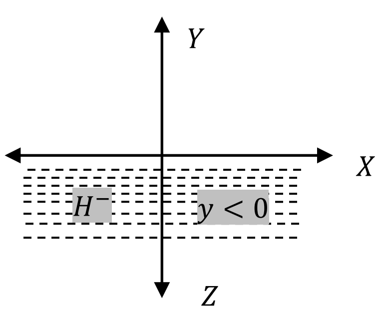
$\therefore x = \frac{(1-3u)(5u-2)-15v^2}{(5u-2)+25v^2},\ \ y = \frac{[5(1-3u)+3(5u-2)]}{(5u-2)+25v^2} = \frac{v}{(5u-2)+25v^2}$

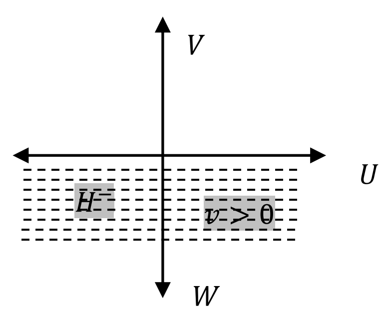
Hence, the region $y > 0$ in $z - plane$ on the region $v > 0$ in $w - plane$




$$Therefore, f(z) = \frac{2z+1}{5z+3}\ \ maps\ H^+\ onto\ H^+$$




$$Therefore, f(z) = \frac{2z+1}{5z+3}\ \ maps\ H^-\ onto\ H^-$$

$\therefore$ Option (a) is correct.

**84.** At $z = 0$, the function $f(z) = \exp\left(\frac{z}{1-\cos z)}\right)$ has

(a) a removable singularity

(b) a pole

(c) an essential singularity

(d) the Laurent expansion of $f(z)$ around $z = 0$ has infinitely many positive and negative powers of z.

**Ans: (c)**

$\lim_{z\to 0} f(z) = \lim_{z\to 0} \exp\left(\frac{z}{1-\cos z)}\right) = exp\left(\lim_{z\to 0} \frac{z}{1-\cos z}\right) = exp\left(\lim_{z\to 0} \frac{1}{-\sin z}\right)$

$= e^{\infty} =$ does not exists.

Hence, $f(z) = \exp\left(\frac{z}{1-\cos z)}\right)$ has an essential singularity.

Hence, option (c) is correct

# 2012 – December

**33.** Consider the functions $f, g : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = e^z, g(z) = e^{iz}$.

Let $S = \{z \in \mathbb{C} : Re\, z \in [-\pi, \pi]$. Then –

(a) $f$ is an onto entire function.

(b) $g$ is a bounded function on $\mathbb{C}$.

(c) $f$ is bounded on S

(d) $g$ is bounded on S.

**Ans: (c)**

**34.** Let $f : D \to D$ be a holomorphic function with $f(0) = 0$ where D is the open unit disc $\{z \in \mathbb{C} : |z| < 1$. Then –

(a) $|f'(0)| = 1$

(b) $\left|f\left(\frac{1}{2}\right)\right| \leq \frac{1}{2}$

(c) $\left|f\left(\frac{1}{2}\right)\right| \leq \frac{1}{4}$

(d) $|f'(0)| \leq \frac{1}{2}$

**Ans: (b)**

**35.** Consider the power series $\sum_{n=1}^{\infty} z^{n!}$. The radius of convergence of this series is

(a) 0

(b) $\infty$

(c) 1

(d) a real number greater than 1.

**Ans: (c)**

**79.** Which of the following functions $f$ are entire functions and have simple zeros at $z = ik$ for all $k \in z$

(a) $f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots \;\; .. \;\; \ldots \ldots + a_0$ for some $n \geq 1$ and some $a_0, a_1, \ldots \ldots, a_n \in \mathbb{C}$

(b) $f(z) = a \sin 2\pi i z,\ for\ some\ a \in \mathbb{C}$

(c) $f(z) = b \cos 2\pi(iz - y),\ for\ some\ b \in \mathbb{C}$

(d) $f(z) = e^{cz}$, for some $c \in \mathbb{C}$

**Ans: (b), (c)**

**80.** Let $\gamma_k = \left\{ke^{ik\theta} : 0 \leq \theta \leq 2\pi\right\}$ $for\ k = 1,2,3$. Which of the following are necessarily correct?

(a) $\frac{1}{2\pi i}\int_{\gamma_1}\frac{1}{z}dz = 0 \quad for\ k = 1,2,3$

(b) $\frac{1}{2\pi i}\int_{\gamma_2}\frac{1}{z}dz = 1$

(c) $\frac{1}{2\pi i}\int_{\gamma_2}\frac{1}{z}dz = 4$

(d) $\frac{1}{2\pi i}\int_{\gamma_3}\frac{1}{z}dz = 3$

**Ans: (b), (d)**

**81.** Let $f$ be an analytic function defined on $D = \{z \in \mathbb{C} : |z| < 1\}$ such that the range of $f$ is contained in the set $\mathbb{C}|(-\infty, 0)$. Then –

(a) $f$ is necessarily a constant function.

(b) There exists an analytic function $g$ on D such that $g(z)$ is a square root of $f(z)$ for each $z \in D$.

(c) There exists an analytic function $g$ on D such that $Re\ g(z) \geq 0\ and\ g(z)$ is a square root of $f(z)$ for each $z \in D$.

(d) There exists an analytic function $g\ and\ D$ such the such that $Re\ g(z) \leq 0$ and $g(z)$ is square root of $f(z)$ for each $z \in D$.

**Ans: (b), (c), (d)**

**82.** Let $f : \Omega \to \mathbb{C}$ be an analytic function on an open set $\Omega \subseteq \mathbb{C}$. For $r > 0$, let $D_r = \{z \in \mathbb{C} : |z| < r\}$ and let $\overline{D}_r$ be it's closure. Which of the following are necessary true?

(a) If $\overline{D}_1 \subset f(\Omega)$, then $D_r \subset f(\Omega)$ for some $r > 1$

(b) If $\overline{D}_1 \subset f(\Omega)$, then $D_r = f(\Omega)$ for some $r > 1$

(c) If $\overline{D}_1 \subset f(\Omega)$, then $\overline{D}_r \subset f(\Omega)$ for some $r > 1$

(d) $f(\Omega)$ is open.

**Ans: (a), (b), (c)**

**83.** Let $f(z) = z + \frac{1}{z}$ for $z \in \mathbb{C}$ with $z \neq 0$. Which of the following are always true?

(a) f is an analytic function on $\mathbb{C}|\{0\}$

(b) f is a conformal map on $\mathbb{C}|\{0\}$

(c) f maps the unit circle to a subset of the real axis.

(d) The image of any circle on $\mathbb{C}|\{0\}$ is again a circle.

Ans: **(a), (c)**

# 2013 – June

**48.** Let $p(z)$ and $q(z)$ be two non – zero complex polynomials. Then, $p(z)\,\overline{q(z)}$ is analytic, if and only if

(a) $p(z)$ is constant.

(b) $p(z)q(z)$ is constant.

(C) $q(z)$ is a constant.

(d) $\overline{p(z)}\,q(z)$ is constant.

**Ans: (c)**

(a) Let $p(z) = 1\ (constant)$ & $q(z) = 2$

$\therefore p(z) \cdot \overline{q(z)} = 1 \cdot \bar{z} = \bar{z} = f(z), (let)$

$\therefore \frac{\partial f}{\partial \bar{z}} = 1 \neq 0$

$\Rightarrow f$, not analytic [i.e., $C - R$ equation not satisfies]

So option (a) is not correct.

(b) Let $p(z) = 1,\ q(z) = 1$

$p(z)q(z) = 1 = f(z), (let)$

$\frac{\partial f}{\partial z} = 0$

If $p(z)q(z)$ is constant, then $p(z)\,\bar{q}(z)$ is analytic but if $p(z) = z, q(z) = 1$, then

$p(z)\,\overline{q(z)} = Z = f(z)(let)$ & $\frac{\partial f}{\partial z} = 0$ ($C - R$ equation satisfies)

& $p(z)q(z) = z$, it is not constant/

$\therefore$ for if and only if this option is not true.

So, option (b) is not correct.

(c) Let $p(z) = z,\ q(z) = 1$ then,

$p(z)\,\overline{(z)} = z = f(z), (let)$

$\frac{\partial f}{\partial z} = 0$

$\therefore$ Option (c) is correct.

**49.** If $z_1$ and $z_2$ are distinct complex numbers such that $|z_1| = |z_2| = 1$ and $z_1 + z_2 = 1$, then the triangle in the complex plane with $z_1, z_2$ and $-1$ as vertices.

(a) must be equilateral.

(b) must be right angled.

(c) must be isosceles, but not necessarily equilateral.

(d) must be obtuse angled.

**Ans: (a)**

**86.** Consider the following function $f(z) = z^2(1 - \cos z), z \in \mathbb{C}$. Which of the following are correct?

(a) The function $f$ has zeroes of order 2 at 0.

(b) The function $f$ has zeroes of order 1 at $2\pi n,\ \ n = \pm 1, \pm 2, \dots ..$

(c) The function $f$ has zeroes of order 4 at 0.

(d) The function $f$ has zeroes of order 2 at $2n\pi, n = \pm 1, \pm 2, \dots \dots ....$

**Ans: (a), (b)**

Here $f(z) = z^2(1 - \cos z)$

Thus, the zeros can be obtained by $z^2 = 0 \ \Rightarrow z = 0$

i.e., $f$ has zeros of order 2 at $z = 0$.

Again, $\cos z = 1 = \cos 2n\pi$, i.e., $zd = 2n\pi,\ \ n = \pm 1, \pm 2, .. \ \ \dots ..$

Which are zeros of order 1.

**87.** Let B be an open subset of $\mathbb{C}$ and $\partial B$ denote the boundary of B. Which of the following statements are correct?

(a) For every entire function $f$, we have $\partial\big(f(B)\big) \subseteq f(\partial B)$

(b) For every entire function $f$ and a bounded open set B, we have $\partial\big(f(b)\big) \subseteq f(\partial B)$ .

(c) For every entire function $f$, we have $\partial\big(f(B)\big) = f(\partial B)$

(d) There exists an unbounded open subset B of C and an entire function $f$ such that $\partial\big(f(B)\big) \subseteq f(\partial B)$

**Ans: (b), (d)**

**88.** Let $D = \{z \in \mathbb{C}: |z| < 1\}$. Which of the following are correct?

(a) There exists a holomorphic function $f: D \to D$ with $f(0) = 0 \ \ and \ \ f'(0) = 2$

(b) There exists a holomorphic function $f: D \to D$ with $f\left(\frac{3}{4}\right) = \frac{3}{4}$ and $f'\left(\frac{2}{3}\right) = \frac{3}{4}$

(c) There exists a holomorphic function $f: D \to D$ with $f\left(\frac{3}{4}\right) = -\frac{3}{4}$ and $f'\left(\frac{3}{4}\right) = -\frac{3}{4}$

(d) There exists a holomorphic function $f: D \to D$ with $f\left(\frac{1}{2}\right) = -\frac{1}{2}$ and $f'\left(\frac{1}{4}\right) = 1$

**Ans: (b), (c)**

Given domain $D = \{z \in \mathbb{C}: |z| < 1\}$ non – zero is a bounded open set. Then there exists holomorphic function $f: D \to D$ such that $f(D)$ is also open and $|f(D)| < 1$.

Hence, we have two possibilities:

There exist a holomorphic function $f: D \to D$ with $f\left(\frac{3}{4}\right) = \frac{3}{4}, f'\left(\frac{2}{3}\right) = \frac{3}{4}$ and there exists a holomorphic function $f: D \to D$ with $f\left(\frac{3}{4}\right) = -\frac{3}{4}$ and $f'\left(\frac{3}{4}\right) = -\frac{3}{4}$

**89.** Let $f: \mathbb{C} \to \mathbb{C}$ be an analytic function. For $z = x + iy$, let $u, v: \mathbb{R}^2 - \mathbb{R}$ be such that $u(x, y) = Re\, f(z)$ and $v(x, y) = Im\, f(z)$. Which of the following are correct?

(a) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(b) $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

(c) $\frac{\partial^2 u}{\partial x\, \partial y} - \frac{\partial^2 u}{\partial y\, \partial x} = 0$

(d) $\frac{\partial^2 v}{\partial x\, \partial y} + \frac{\partial^2 v}{\partial y\, \partial x} = 0$

**Ans: (a), (b), (c)**

As $f = u + iv$ be an analytic function, then $u\ and\ v$ must satisfy the Cauchy Riemann's equation. $u_x = v_y$ and $u_y = -v_x$

Again u and v are real and imaginary parts of an analytic function, then u and v must satisfy the Laplace's equation : $u_{xx} + v_{yy} = 0 \ \ and \ \ v_{xx} + v_{yy} = 0$

Now, $u_{xy} = -v_{xx}\ and\ \ u_{yu} = v_{yy}$

Therefore, $u_{xy} - u_{yx} = -v_{xx} + v_{yy} = 0$

Similarly, $v_{xy} - v_{yx} = 0$

# 2013 – December

**33.** Let $f$ be a non – constant entire function. Which of the following properties is possible for $f$ for each $z \in \mathbb{C}$?

(a) $Re\, f(z) = Im\, f(z)$

(b) $|f(z)| < 1$

(c) $Im\, f(z) < 0$

(d) $f\ (z) \neq 0$

**Ans: (d)**

(a) Let $f(z) = e^z$, non constant, entire function

$= e^{x+iy} = e^x\ (\cos y +\ i \sin y)$

$\therefore\ Re\, f(z) \neq Im\, f(z)$

So option (a) is not correct.

(b) $|f(z)| < 1$ i.e., $f$ is bounded (contradiction)

$\therefore$Option (b) is not correct.

(c) $f(z) = e^z$

$Im\, f(z) = e^x \sin y \nless 0$, for each $z \in \mathbb{C}$

$\therefore$ Option (c)is not correct.

(d) $f(z) =\ e^z \neq 0$

$\therefore$ Option (d) is correct.

**34.** Let $a, b, c$ be non – collinear points in the complex plane and let $\Delta$ denote the closed triangular region of the plane with vertices $a, b, c$. For $z \in \Delta$ let $h(z) = |z - a| \cdot |z - b| \cdot |z - c|$. The maximum value of the function $h$ is

(a) not attained at any point of $\Delta$.

(b) attained at an interior point of $\Delta$.

(c) attained at the centre of gravity of $\Delta$.

(d) attained at a boundary point of $\Delta$.

**35.** $f$ be a non – constant holomorphic function in the unit disc $\{|z| < 1\}$ such that $f(0) = 1$. Then, it is necessary that

(a) There are infinitely many points z in unit disc such that $|f(z)| = 1$

(b) $f$ is bounded.

(c) There are almost finitely many points z in the unit disc such that $|f(z)| = 1$

(d) $f$ is a rational function.

**Ans: (a)**

(a) and (c)

Let $f(z) = e^z$

$\therefore |f(z)| = 1$

$\therefore$ (a) is true and option (c) is not correct.

(b) Since, $f$ be a non – constant holomorphic function, so $f$ is unbounded.

$\therefore$ Option (b) is not correct.

(d) $f(z) = e^z$, not a rational function.

$\therefore$ Option (d) is not correct.

**79.** Let $f$ be a holomorphic function on the unit disc $\{|z| < 1\}$ in the complex plane. Which of the following is/are necessarily true?

(a) If for each positive integer , we have $f\left(\frac{1}{n}\right) = \frac{1}{n^2}$, then $f(z) = z^2$ on the unit disc.

(b) If for each positive integer $n$, we have $f\left(1 - \frac{1}{n}\right) = \left(1 - \frac{1}{n}\right)^2$ then $f(z) = z^2$ on the unit disc.

(c) $f$ cannot satisfy $f\left(\frac{1}{n}\right) = \frac{(-1)^n}{n}$ for each positive integer $n$.

(d) $f$ cannot satisfy $f\left(\frac{1}{n}\right)\frac{1}{1+n}$ for each positive integer $n$.

**Ans: (a), (c)**

(a) $f$, holomorphic function on the unit disc, $|z| < 1$

$f\left(\frac{1}{n}\right) = \frac{1}{n^2}, n \in z^+$ limit point $= 0 \in D$

Now, $\frac{1}{n} = z, \Rightarrow f(z) = z^2$

So, option (a) is correct.

(b) $f\left(1 - \frac{1}{n}\right) = \left(1 - \frac{1}{n}\right)^2$ limit point $= 1 \notin D$

$\therefore f(z) \neq z^2$

So, option (b) is not correct.

(c) $f\left(\frac{1}{n}\right) = \frac{(-1)^n}{n} = \begin{cases} -\frac{1}{n}, & n \text{ is odd} \\ \frac{1}{n}, & n \text{ is even} \end{cases}$,

limit point of $\left\{-\frac{1}{n}\right\}$ and $\left\{\frac{1}{n}\right\}$ is 0 but if $\frac{1}{n} = z,\ f(z) = \begin{cases} -z \\ z \end{cases}$

$\therefore f(z)$ is not analytic on domain.

So, option (c) is correct.

(d) $f\left(\frac{1}{n}\right) = \frac{1}{n+1}$, limit point $= 0$

Let $\frac{1}{n} = z\,, f(z) = \frac{z}{z+1}$

$\because\ -1 \notin D$ $\qquad\qquad \therefore z \neq -1$

$\Rightarrow f(z)$ is analytic on domain $D$.

So option (d) is not correct.

**80.** Let $(z) = \frac{z-1}{\exp\left(\frac{2\pi i}{z}\right)-1}$. Then

(a) $f$ has an isolated singularity at $z = 0$.

(b) $f$ has a removable singularity at $z = 1$.

(c) $f$ has infinitely many poles.

(d) each pole of $f$ is of order 1.

**Ans: (b), (c), (d)**

**81.** Let $f(z) = \frac{1+z}{1-z}$. Which of the following is/are true?

(a) $f$ maps $\{|z| < 1\}$ onto $\{Re\ (z) > 0\}$

(b) $f$ maps $\{|z| < 1, Im\ (z) > 0\}$ onto $\{Re\ (z) < 0, Im\ (z) > 0\}$

(c) $f$ maps $\{|z| < 1,\ Im\ (z) < 0\ \}$ onto $\{Re\ (z) < 0, Im\ (z) < 0\}$

(d) $f$ maps $\{|z| > 1\}$ onto $\{Im\ (z) > 0\}$.

**Ans: (a), (b)**

**82.** Let $f$ be a mermorphic function on $\mathbb{C}$ such that $|f(z)| \geq |z|$ each $z$, where $f$ is holomorphic. Then, which of the following is/are true?

(a) The hypothesis are contradictory, so on such $f$ exists.

(b) Such an $f$ is entire

(c) There is a unique $f$ satisfying the given conditions.

(d) There is an $A \in \mathbb{C}$ with $|A| \geq 1$ such that $f(z) = Az$ for each $z \in \mathbb{C}$.

**Ans: (b), (d)**

# 2014 – June

**37.** Let $f$ *and* $g$ be meromorphic functions on $\mathbb{C}$. If $f$ has a zero of order $k$ *at* $z = a$ and $g$ has a pole of order $m$ at $z = 0$, then $g(f(z))$ has

(a) a zero of order $km$ at $z = a$

(b) a pole of order $km$ at $z = a$

(c) a zero of order $|k - m|$ at $z = a$

(d) a pole of order $|k - m|$ at $z = a$

**Ans: (b)**

$f$ *and* $g$ are meromorphic functions on $\mathbb{C}$. If f has a zero of order $k$ at $z = a$ and $g(z)$ is a pole of order $m$ at $z = 0$.

Then, let, $f(z) = (z - a)^k$ , $g(z) = \frac{1}{(z-0)^m} = \frac{1}{z^m}$

Then, $g\big(f(z)\big) = g((z - a)^k) = \frac{1}{(z-a)^{km}}$

$\Rightarrow g\big(f(z)\big)$ has a pole of order $km$ at $z = a$

**38.** Let $p(x)$ be a polynomial of the real variable $x$ of degree $k \geq 1$. Consider the power series $f(z) = \sum_{n=0}^{\infty} p(n)z^n$, where , $z$ is a complex variable. Then, the radius of convergence of $f(z)$ is

(a) 0

(b) 1

(c) k

(d) $\infty$

**Ans: (b)**

$p(x)$ = polynomial of the real variable $x$ of degree $k \geq 1$ $\Rightarrow p(x) = x^k, k \geq 1$

$\therefore f(z) = \sum_{n=0}^{\infty} p(n)z^n = \sum_{n=0}^{\infty} n^k z^n$

Radius of convergence of $f(z) = R = \frac{1}{\lim\limits_{n\to\infty} \sup|a_n|^{\frac{1}{n}}} = \frac{1}{\lim\limits_{n\to\infty} \sup(n)^{\frac{k}{n}}}$

$\therefore R = 1$

**80.** Let $f$ be an entire function. Suppose for each $a \in \mathbb{R}$, there exists at least one coefficient $C_n$ in $f(z)\sum_{n=0}^{\infty} C_n(z-a)^n$, which is zero, then

(a) $f^{(n)}(0) = 0$ for infinitely many $n \geq 0$

(b) $f^{(2n)}(0) = 0$ for every $n \geq 0$

(c) $f^{(2n+1)}(0) = 0$ for every $n \geq 0$

(d) there exists $k \geq 0$ such that $f^{(n)}(0) = 0$ for all $n \geq k$

**Ans: (a), (d)**

**81.** Let $k \subseteq \mathbb{C}$ be a bounded set. Let $H(\mathbb{C})$ denote the set of all entire functions and let $C(k)$ denote the set of all continuous functions on $k$. Consider the restriction map $r: H(\mathbb{C}) \to C(k)$ given by $(f) = f_k$ . Then, $r$ is injective, if

(a) $k$ is compact.

(b) $k$ is connected

(c) $k$ is uncountable

(d) $k$ is finite.

**Ans: (c)**

**82.** Let $z \in \mathbb{C}$, define $f(z) = \frac{e^z}{e^z - 1}$, then

(a) $f$ is entire.

(b) The only singularities of $f$ are poles.

(c) $f$ ahs infinitely many poles on the imaginary axis.

(d) each pole of $f$ is simple.

**Ans: (b), (c), (d)**

$z \in \mathbb{C}$

$f(z) = \frac{e^z}{e^z - 1}$

for pole, $e^z - 1 = 0 \;\; \Rightarrow e^z = 1 = e^{2n\pi}, n = 0,1,2, \dots \dots \dots$

$\Rightarrow z = 2n\pi i, \;\; n = 0, 1, 2, \dots \dots \dots$

$\therefore$ $f$ has infinitely many poles, each pole is simple and only singularity of $f$ are poles.

**83.** Let $D = \{z \in \mathbb{C}: |z| < 1\}$. Then, there exists a holomorphic function $f: D \to \overline{D}$ with $f(0) = 0$ with the property

(a) $f'(0) = \frac{1}{2}$

(b) $\left|f\left(\frac{1}{3}\right)\right| = \frac{1}{4}$

(c) $f\left(\frac{1}{3}\right) = \frac{1}{2}$

(d) $|f'(0)| = \sec\left(\frac{\pi}{6}\right)$

**Ans: (a), (b)**

$D = \{z \in \mathbb{C} : |z| < 1\}$

$\exists$ a holomorphic function $f : D \to \overline{D}$ , with $f(0) = 0$

Schwartz' Lemma: $f : D \to D$, holomorphic function with $f(0) = 0$

Then,

(i) $|f(z)| \leq |z|^n, \forall\, z \in D$

(ii) $|f^n(0)| \leq n!$

Option (b), (c)

$|f(z)| \leq |z|^n \Rightarrow \left|f\left(\frac{1}{3}\right)\right| \leq \left(\frac{1}{3}\right)^n \leq \frac{1}{3}$ and $\frac{1}{4} < \frac{1}{3} < \frac{1}{2}$,

$\therefore \left|f\left(\frac{1}{3}\right)\right| = \frac{1}{4}$

$\therefore$ option (b) is correct and option (c) is not correct.

Option (a), (d)

$|f^n(0)| \leq n!$

$|f'(0)| \leq 1$

$\sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}} > 1$ & $\frac{1}{2} < 1$

$\therefore$ option (a) is correct.

## 2014 – December

**33.** Let $p(z) = a_0 + a_1 z + \ldots\ldots + a_n z^n$ and $q(z) = b_1 z + b_2 z^2 + \ldots\ldots \ldots + b_n z^n$ be complex polynomials.

If $a_0$, $b_1$ are non – zero complex numbers, then the residue of $\frac{p(z)}{q(z)}$ at 0 is equal to

(a) $\frac{a_0}{b_1}$

(b) $\frac{b_1}{a_0}$

(c) $\frac{a_1}{b_1}$

(d) $\frac{a_0}{a_1}$

**Ans. (a)**

$p(z) = a_0 + a_1 z + \cdots\cdots + a_n z^n$

$q(z) = b_1 z + b_2 z^2 + \cdots\cdots + b_n z^n$

$\frac{p(z)}{q(z)} = \frac{z\left(a_0/z + a_1 + \cdots\cdots + a_n z^{n-1}\right)}{z\left(b_1 + b_2 z + \cdots\cdots + b_n z^{n-1}\right)}$

$= \frac{\frac{a_0}{z} + a_1 + \cdots\cdots\cdots\cdots a_n z^{n-1}}{b_1\left[1 + \frac{b_2}{b_1} z + \cdots\cdots + \frac{b_n}{b_1} z^{n-1}\right]}$

Residue of $\frac{p(z)}{q(z)}$ = coefficient of $\left(\frac{1}{z}\right) = \frac{a_0}{b_1}$

**34.** Let $\sum_{n=0}^{\infty} a_n z^n$ be a convergent power series such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = R > 0$

Let $p$ be a polynomial of degree $d$. Then the radius of convergence of the power series $\sum_{n=0}^{\infty} p(n) a_n z^n$ equal to

(a) $R$

(b) $d$

(c) $Rd$

(d) $R + d$

**Ans. (a)**

$\sum a_n z^n$ be a convergent power series $\lim_{n\to\infty} \frac{a_n+1}{a_n} = R > 0$ $p$ be a polynomial of degree $d$

Let $p(x) = x^d$ $\quad \therefore p(n) = n^d$

Let $a_n = 1 \Rightarrow a_{n+1} = 1$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 > 0$

Now, $\sum_{n=0}^{\infty} n^d \cdot 1 \cdot z^n = \sum_{n=0}^{\infty} n^d \cdot z^n$

Cauchy root test, $R = \frac{1}{\lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}}} = \frac{1}{\lim\limits_{n\to\infty} |n^d|^{\frac{1}{n}}} = 1$

**79)** Let $f$ be an entire function on $C$ and let $\Omega$ be a bounded open subset of $C$.

Let $S = \{Re\ f(z) + Im\ f(z) | z \in \Omega\}$

Which of the following statements is / are necessarily correct?

a) $S$ is an open set in $R$

b) $S$ is an closed set in $R$

c) $S$ is an open set in $C$

d) $S$ is a discrete set in $R$

**80)** Let $u(x+iy) = x^3 - 3xy^2 + 2x$. For which of the following functions $v$, is $u + iv$ a holomorphic function on $C$?

a) $V(x+iy) = y^3 - 3x^2y + 2y$

b) $V(x+iy) = 3x^2y - y^3 + 2y$

c) $V(x+iy) = x^3 - 3xy^2 + 2x$

d) $V(x+iy) = 0$

**Ans. (b)**

$U(x+ir) = x^3 - 3xy^2 + 2x$

$\because$ we know that, a function $f(z) = U + iV$ be analytic in a domain $D$, if it satisfies the Cauchy Riemann equation $U_x = V_y\ and\ U_y = -V_x$

$\therefore U_x = 3x^2 - 3y^2 + 2$

$U_y = -6xy$

Now,

a) $V_x = -6xy \neq -\frac{\partial U}{\partial y}$

$\therefore$ option a) is not correct.

b) $V_x = 6xy = \frac{\partial U}{\partial y}$

$V_y = 3n^2 - 3y^2 + 2 = U_x$

$\therefore$ option b) is correct.

c) $V_x = 3n^2 - 3y^2 + 2$

$V_x \neq -U_y$

Option c) is not correct

d) $U_x \neq V_y$

option d) is not correct.

**81)** Let $f$ be an entire function on $C$. Let, $g(z) = \overline{f(\bar{z})}$

Which of the following statements is / are correct?

a) if $f(z) \in R\ for\ all\ z \in R\ then\ f = g$

b) if $f(z) \in R\ for\ all\ z \in \{z|Im\ z = 0\} \cup \{z|Im\ z = a\}, for\ some\ a > 0,$

$then\ f(z + ia) = f(z - ia) for\ all\ z \in C.$

c) if $f(z) \in R\ for\ all\ z \in \{z|Im\ z = 0\} \cup \{z|Im\ z = a\}, for\ some\ a > 0,$

$then\ f(z + 2ia) = f(z), for\ all\ z \in C.$

d) if $f(z) \in R\ for\ all\ z \in \{z|Im\ z = 0\} \cup \{z|Im\ z = a\}\ for\ some\ a > 0,$

$then\ f(z + ia) = f(z) all\ z \in C.$

**Ans. (a), (b), (c)**

$f$ be a entire function on $\mathbb{C}$

$g(z) = \overline{f(\bar{z})}$

a) If $f(z) \in R, \forall\ z \in R$

$\Rightarrow z = \bar{z}$ $\quad\quad\quad\quad\quad\quad [\because z = x + iy, z \epsilon R]$

$\Rightarrow f(z) = f(\bar{z})$

$\Rightarrow \overline{f(z)} = \overline{f(\bar{z})} = g(z)$

$\Rightarrow f(z) = g(z)$

So, option a) is correct.

b) $f(z) \epsilon R, \forall\ z \epsilon \{z|Im\ z = 0\} \cup \{z|Im\ z = a\}$ for some $a > 0$, then

$f(z + io) = f(z - ia)$ for all $z \in\ \mathbb{C}$

$\because f(z) \in R\ \&\ z \in \{z|Im\ z = 0\} \cup \{z|Im\ z = a\}$

$\Rightarrow for\ z = x\ or\ z = x + ia$

$\Rightarrow f(z) = f(\bar{z})$

$\Rightarrow f(z - ia) = f(\overline{z - \imath a}) = \overline{f(\overline{z - \imath a})} = f(z + ia)$

So, option b) is correct.

c) $f(z - ia) = f(z + ia)$

$if\ z \to z + ia, f(z) = f(z + 2ia)$

So, option c) is correct and option d) is not correct.

**82)** Let $f(z) = \sum_{n=0}^{\infty} a_n Z^n$ be an entire function and let $r$ be a positive real number. Then –

a) $\sum_{n=0}^{\infty} |a_n|^v r^{2n} \leq \sup_{|z|=r} |f(z)|^2$

b) $\sup_{|z|=r} |f(z)|^2 \leq \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$

c) $\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \leq \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^2 d\theta$

d) $\sup_{|z|=r} |f(z)|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^2 d\theta$

# 2015 – June

**33)** Let $f$ be a real valued harmonic function on $C$, that is, $f$ satisfied the equation $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$

Define the functions,

$g=\frac{\partial f}{\partial x}-i\,\frac{\partial f}{\partial y}, h=\frac{\partial f}{\partial x}+i\,\frac{\partial f}{\partial y}$

Then,

a) $g\ and\ h$ are both holomorphic functions.

b) $g$ is holomorphic, but $h$ need not be holomorphic.

c) $h$ is holomorphic, but $g$ need not be holomorphic.

b) both $g\ and\ h$ are identically equal to the zero function.

**Ans. (b)**

Let $g=\frac{\partial f}{\partial x}-i\,\frac{\partial f}{\partial y}=u+iv$

Then $u_x=\frac{\partial^2 f}{\partial x^2}, u_y=\frac{\partial^2 f}{\partial y\partial x}$

$v_x=-\frac{\partial^2 f}{\partial x\partial y}, v_y=-\frac{\partial^2 f}{\partial y^2}=\frac{\partial^2 f}{\partial x^2}$

Since, $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$

Thus, $u_x=v_y\ , u_y=-v_x$

i.e., $g$ satisfies $C-R$ equation.

Also, all the derivatives are continuous.

Hence $g$ is a holomorphic function.

Now, let, $h=\frac{\partial f}{\partial x}+i\,\frac{\partial f}{\partial y}=u+iv$

In this case, $u_x=\frac{\partial^2 f}{\partial x^2}\ , u_y=\frac{\partial^2 f}{\partial y\partial x}$

and $V_x=\frac{\partial^2 f}{\partial x\partial y}\ , v_y=\frac{\partial^2 f}{\partial y^2}=-\frac{\partial^2 f}{\partial x^2}$

Thus, $u_x\neq v_y\ and\ u_y\neq -v_x$

Hence, $h$ does not satisfy $C-R$ equations. Therefore, $h$ is not holomorphic.

**34)** $\int_{|z+1|=2} \frac{z^2}{4-z^2} dz = 0$

a) 0

b) $-2\pi i$

c) $2\pi i$

d) 1

**Ans. (c)**

The poles are $4 - z^2 = 0 \Rightarrow z = \pm 2$

Only $z = -2$. Lies on the given region.

Now, $\int_{|z+1|=2} \frac{z^2}{4-z^2} dz = \int_{|z+1|=2} \frac{\frac{z^2}{2-z}}{2+z} dz$

$= \int_{|z+1|=2} \frac{f(z)}{2+z} dz, where\ f(z) = \frac{z^2}{2-z}$

Hence by Cauchy's integral formula, the integral is $= 2\pi i f(-2)\ = 2\pi i$

**79)** Let $f$ be an entire function. Which of the following statements are correct?

a) $f$ is constant if the range of $f$ is contained in a straight line.

b) $f$ is constant uncountably many zeros.

c) $f$ is constant if $f$ is bounded on $\{z \in C: Re(z) \le 0\}$

d) $f$ is constant if the real part of $f$ is bounded

**Ans. (a), (b), (d)**

a) $f$ entire function

$\therefore$ Domain of $'f' = \mathbb{C}$

i.e., here range skip uncountable many points.

$\Rightarrow f$ is constant.

So, option (a) is correct.

b) $f(z_i) = 0, for\ uncountable\ z_i$

$\Rightarrow f(z) = 0, \forall\ z \in \mathbb{C}$

$\Rightarrow f(z)$ is constant

$\Rightarrow$ option b) is correct.

c) $\{z \in \mathbb{C} | Re(z) \le 0\}$ $\qquad z = x + iy$

$\therefore x \le 0$

Let $f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy}$

$|e^2| = e^x$, bounded in $\{z \in \mathbb{C} | Re(z) \leq 0\}$ $x \leq 0$

$\Rightarrow$ but $f(z)$ is non castant,

So, option (c) is incorrect.

d) If $f = u + iv$ is an entire function and either of the $u$ & $v$ are bounded then $f$ is constant.

So, option (d) is correct.

**80)** Consider the following subsets of the complex plane:

$$\Omega_1 = \left\{C \in \mathbb{C} \begin{bmatrix} 1 & c \\ \bar{c} & 1 \end{bmatrix} \text{ is non} - \text{negative definite. (or equivalently positive semi} - \text{definite)}\right\}$$

$$\Omega_2 = \left\{C \in \mathbb{C}: \begin{bmatrix} 1 & C & C \\ \bar{C} & 1 & C \\ \bar{C} & \bar{C} & 1 \end{bmatrix} is\ non - negative\ definite\ (or\ equivalently\ positive\ semi - definite)\right\}$$

Let $\overline{D} = \{z \in \mathbb{C} \,\big|\, |z| < 1\}$ then

a) $\Omega_1 = \overline{D}, \Omega_2 = \overline{D}$

b) $\Omega_1 \neq \overline{D}, \Omega_2 = \overline{D}$

c) $\Omega_1 = \overline{D}, \Omega_2 \neq \overline{D}$

d) $\Omega_1 \neq \overline{D}, \Omega_2 \neq \overline{D}$

**Ans: (c)**

**81)** Let $p$ be a polynomial in 1 – complex variable suppose all zeros of $p$ are in the upper half plane.

$H = \{z \in \mathbb{C} | Im(z) > 0\}$. Then

a) $Im \frac{p\prime(z)}{p(z)} > 0\ for\ z \in \mathbb{R}$

b) $Re \frac{p\prime(z)}{p(z)} < 0\ for\ z \in \mathbb{R}$

c) $Im \frac{p\prime(z)}{p(z)} > 0, for\ z \in \mathbb{C}, with\ Im\ z < 0$

d) $Im \frac{p\prime(z)}{p(z)} > 0\ for\ z \in \mathbb{C}, with\ Im\ z > 0$

**Ans. (a), (b) & (c)**

$p =$ polynomial in 1 $-$complex variable.

Suppose all zeros of $p$ are in the upper half planc $H = \{z \in \mathbb{C} | Im(z) > 0\}$, then by Luca's theorem, zerosof it's derivative $(p'(z))$ also lie in the some half plane.

Let $p(z) = z - 2i \in \mathbb{C}$

$\Rightarrow p'(z) = 1$

$\frac{p\prime(z)}{p(z)} = \frac{1}{z-2i}$

$= \frac{x-i(y-2)}{x^2+(y-2)^2}$ $[z = x + iy]$

a) $Im \frac{p\prime(z)}{p(z)} > 0 \; for \; z \in \mathbb{R}$

$\Rightarrow \frac{(y-2)}{x^2+(y-2)^2} > 0$. i.e., image $(z) = y = 0$

$\Rightarrow$ (a) is correct.

b) $Re \; i\left(\frac{p\prime(z)}{p(z)}\right) = \frac{y-2}{x^2+(y-2)^2} < [if \; z \in \mathbb{R} \; i.e., y = 0]$

So, option b) is correct

c) $Im\left(\frac{p\prime(z)}{p(z)}\right) = \frac{-(y-2)}{x^2+(y-2)^2} > 0, if \; z \in \mathbb{C}, y < 0$

So, option (c) is correct

d) $Im\left(\frac{p\prime(z)}{p(z)}\right) = \frac{-(y-2)}{x^2+(y-2)^2} > 0, if \; z \in \mathbb{C}, y > 0$

So, option d) is not correct.

**82)** Let $f$ be an analytic function defined on the open unit disc in $\mathbb{C}$. Then $f$ is constant, if

a) $f\left(\frac{1}{n}\right) = 0 \; for \; all \; n \geq 1$

b) $f(z) = 0 \; for \; all \; |z| = \frac{1}{2}$

c) $f\left(\frac{1}{n^2}\right) = 0 \; for \; all \; n \geq 1$

d) $f(z) = 0 \; for \; all \; z \in (-1,1)$

**Ans. (a), (b), (c) & (d)**

# 2015 – December

**38)** Consider the following power series in the complex variable $z$ is

$$f(z) = \sum_{n=1}^{\infty} n \log n \, z^n,$$

$g(z) = \sum_{n=1}^{\infty} \frac{e^{n^2}}{n} z^n \cdot$ If $r, R$ are the radii of convergence of $f \; and \; g$ respectively, then

a) $r = 0, R = 1$

b) $r = 1, R = 0$

c) $r = 1, R = \infty$

d) $r = \infty, R = 1$

**Ans: (b)**

**39)** Let $a. b, c, d \in \mathbb{R}$ be such that $ad - bc > 0$. Consider the Mobius transformation

$T_{a,b,c,d}(z) = \frac{az+b}{cz+d} \cdot Define$

$H_+ = \{z \in \mathbb{C}: Im(z) > 0\},$

$H_- = \{z \in \mathbb{C}: Im(Z) < 0\},$

$R_+ = \{z \in \mathbb{C}: Re(z) > 0\},$

$R_- = \{z \in \mathbb{C}: Re(z) < 0\},$

Then, $T_{a,b,c,d}$ maps -

a) $H_+ \; to \; H_+$

b) $H_+ \; to \; H_-$

c) $R_+ \; to \; R_+$

d) $R_+ \; to \; R_-$

**Ans: (a)**

**88)** Let $f(z) = \frac{1}{e^z - 1}$ for all $z \in \mathbb{C}$ such that $e^z \neq 1$. Then

a) $f$ is mesomorphic

b) the only singularities of $f$ are poles.

c) $f$ has infinitely many poles on the imaginary axis.

d) each pole of $f$ is simple.

**Ans. (a), (b). (c), (d)**

Except the pole pt., given function is mesomorphic pole point, $e^z = 1$

$\because e^z \neq 1$

and $e^z = 1 = e^{2n\pi i}$

$\Rightarrow z = 2n\pi i, n = 0,1,2,\cdots$

**90)** Let $f$ be an analytic function in $\mathbb{C}$. Then $f$ is constant if the zero set of $f$ contains the sequence.

a) $a_n = \frac{1}{n}$

b) $a_n = (-1)^{n-1} \cdot \frac{1}{n}$

c) $a_n = \frac{1}{2n}$

d) $a_n = n$ if 4 does not divide $n$ and $a_n = \frac{1}{n}$ if 4 divides $n$.

**Ans: (a), (b), (c), (d)**

| | |
|---|---|
| $\frac{1}{n} \longrightarrow 0$ | $(-1)^{n-1}\frac{1}{n} \longrightarrow 0$ |
| $\frac{1}{2n} \longrightarrow 0$ | $and\ a_{4k} = \frac{1}{4k} \longrightarrow 0$ |

# 2016 – June

**33)** Let $p(x)$ be a polynomial of degree $d \geq 2$. The radius of convergence of the power series $\sum_{n=0}^{\infty} p(n)\, z^n$ is –

a) 0

b) 1

c) $\infty$

d) $dependent\ on\ d$

**Ans. (b)**

$p(x) = a_0 + a_1 x + \cdots + a_k x^k$

$$R = \lim_{n\to\infty} \left| \frac{1}{\frac{p(n+1)}{p(n)}} \right|$$

$$= \lim_{n\to\infty} \frac{a_0 + a_1 n + a_2 n^2 + \cdots a_n\, n^d}{a_0 + a_1(n+1) + a_2(n+2)^2 + \cdots + a_n(n+1)}$$

$$= \frac{a_n}{a_n} = 1$$

**34.** Let $p(z), Q(z)$be two complex non-constant polynomials of degree $m, n$ respectively. The number or roots of $p(z) = p(z), Q(z)$ consted with multiplicity is equal to –

a) $\min\{m, n\}$

b) $\max\{m, n\}$

c) $m + n$

d) $m - n$

**Ans. (c)**

[If $f$ and $g$ have zero of order $m, n$ respectively. Then $h(z) = f(z)g(z)$ have zero of order $m + n$ at $z = z_0$]

**35)** The residue of the function $f(z) = e^{-e^{1/z}}$ $at$ $z = 0$ is –

a) $1 + e^{-1}$

b) $e^{-1}$

c) $-e^{-1}$

d) $1 - e^{-1}$

**Ans. (c)**

$f(z) = e^{-e^{\frac{1}{z}}}$

$= 1 - \frac{e^{\frac{1}{z}}}{1!} + \frac{e^{\frac{2}{z}}}{2!}$ --------------

$= 1 - \frac{1}{1!}\left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots\right) + \frac{1}{2!}\left(1 + \frac{2}{z} + \frac{4}{2!z^2} + \cdots\right)$

Residue of $f$ at $(z = 0)$ = the coefficient of $\frac{1}{z}$

$= -1 + \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots\cdots$

$= -\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots\cdots\right)$

$= -e^{-1}$

**36)** Let $D$ be the open unit disc is $\mathbb{C}$ and $H(D)$ be the collection of all holomorphic functions on it. Let $S = \left\{f \in H(D): f\left(\frac{1}{2}\right) = \frac{1}{2}, f\left(\frac{1}{4}\right) = \frac{1}{4}, \cdots, f\left(\frac{1}{2n}\right) = \frac{1}{2n} \cdots\right\}$

and $T = \left\{f \in H(D): f\left(\frac{1}{2}\right) = f\left(\frac{1}{3}\right) = \frac{1}{2}, f\left(\frac{1}{4}\right) = f\left(\frac{1}{5}\right) = \frac{1}{4}, \cdots\cdots, f\left(\frac{1}{2n}\right) = f\left(\frac{1}{2n+1}\right) = \frac{1}{2n}, \cdots\right\}$

Then,

a) both $S, T$ are singleton set

b) $S$ is a singleton set but $T = \phi$

c) $T$ is a singleton set but $S = \phi$

d) both $S, T$ are empty.

**Ans: (b)**

**79)** Let $F: \mathbb{C} \to \mathbb{C}$ be an entire function. Suppose that $f = u + iv$, where $u, v$ are the real and imaginary parts of $f$ respectively. Then $f$ constant if –

a) $\{u(x,y); z = x + iy \in \mathbb{C}\}$ is bounded

b) $\{vV(x,y); z = x + iy \in \mathbb{C}\}$ is bounded

c) $\{u(x,y) + v(x,y); z = x + iy \in \mathbb{C}\}$ is bounded

d) $\{u^2(x,y) + v^2(x,y); z = x + iy \in \mathbb{C}\}$ is bounded

**Ans. (a), (b), (c), (d)**

If $u(x,y)$ $and$ $v(x,y)$ are both bounded functions of $x$ $and$ $y$, then $u(x,y) + v(x,y)$ is also a bounded function then ultimately the function $f(z) = u(x,y) + iv(x,y)$ becomes a bounded function.

**80)** Let $A = \{z \in \mathbb{C} | z| > 1\}, B = \{z \in \mathbb{C} | z \neq 1\}$. Which of the following statement are true?

a) There is a continuous on to function $f: A \to B$

b) There is a continuous one to one function $f: B \to A$

c) There is a non- constant analytic function $f: B \to A$

d) There is a non- constant analytic function $f: A \to B$

**Ans. (a), (b), (d)**

If $f: B \to A$ then $|f(z)| > 1, \forall z$

Hence $g(z) = \frac{1}{f(z)}$, which is entire and $|g(z)| < 1$

$\Rightarrow g(z)$ bounded entire function $\Rightarrow$ constant.

$\Rightarrow f(z)$ is constant.

So, option (c) is incorrect.

(a) $f(z) = \begin{cases} e^z, if\ |z| > 0 \\ e^{\frac{1}{z}}, if\ |z| \leq 0 \end{cases}$ So, (a) option correct.

(b) $f(z) = \begin{cases} |z|,\ if |z| > 1 \\ \frac{1}{|z|}, if\ |z| \leq 1 \end{cases}$ So, option (b) correct

(d) <u>Picard's Little theorem:</u>

Every non constant entire function eliminates at most one complex no as its value.

$\therefore$ Range of $F = \mathbb{C}\ or \frac{\mathbb{C}}{\{a\}}$, where $a \in \mathbb{C}$

So, option (d) correct.

**81)** Let $H = \{z - x + iy \in \mathbb{C}: y > 0\}$ be the upper half plane and $D = \{z \in \mathbb{C}: |z| < 1\}$ be the open unit disc. Suppose that $f$ is a Mobius transformation, which maps $H$ conformally onto $D$. Suppose that $f(2i) = 0$ pick each correct statement from below.

a) $f$ has a simple pole at $z = -2i$

b) $f\ satisfies\ f(i)\overline{f(-\iota)} = 1$

c) $f$ has an essential singularity at $z = -2i$

d) $|f(2 + 2i)| = \frac{1}{\sqrt{5}}$

**Ans: (a), (b), (d)**

**82)** Consider the function

$F(z) = \int_1^2 \frac{1}{(x-z)^2} dn, Im(z) > 0$

Then, there is a meromorphic function $G(z)\ on\ \mathbb{C}$ that agrees with $F(z)$ when $Im(z) > 0$ such that

a) $1, \infty$ are poles of $G(z)$.

b) $0,1, \infty$ are poles of $G(z)$.

c) 1,2 are poles of $G(z)$.

d) 1,2 are simple poles of $G(z)$.

**Ans. (c), (d)**

Given that

$F(z) = \int_1^2 \frac{1}{(x-z)^2} dx\ , Im(z) > 0$

$= \frac{1}{1-z} + \frac{1}{2-z}\ , Im(z) > 0$

Here $F$ is analytic at all point in $\mathbb{C}$ except at $Z = 1,2$.

The point $z = 1,2$ are the poles of $F(z)$

Hence, these exists a mesomorphic function $G(z)$ agrees with $F(z)$ where $G$ has simple poles at $z = 1,2$.

# 2016 - December

**33)** The radius of convergence of the series $\sum_{n=1}^{\infty} z^{n^2}$ is –

a) 0

b) $\infty$

c) 1

d) 2

**Ans. (c)**

Given that

$\sum_{n=1}^{\infty} z^{n^2} = \sum_{m=1}^{\infty} z^m = \sum_{m=1}^{\infty} a_m z^m$, say where $n^2 = m$

Thus $a_m = 1$ and $\lim_{m\to\infty} a_m^{\frac{1}{m}} = 1$

Hence, the radius of convergence is 1.

**34)** Let $C$ be the circle $|z| = \frac{3}{2}$ in the complex plane that is oriented in the counter clock wise direction. The value of a for which

$\int_C \left(\frac{z+1}{z^2-3z+2} + \frac{a}{z-1}\right) dz = 0$ is –

a) 1

b) $-1$

c) 2

d) $-2$

**Ans. (c)**

$\int_c \left(\frac{z+1}{z^2-3z+2} + \frac{a}{z-1}\right) dz = 0$

$\Rightarrow \int_c \left(\frac{z+1}{(z-1)(z-2)} + \frac{a}{z-1}\right) dz = 0$

Here only the pole at $z = 1$ lies within circle $|z| = \frac{3}{2}$ and which is a simple pole.

Thus, by Cauchy's theorem.

$\int_c \left(\frac{z+1}{(z-1)(z-2)} + \frac{a}{z-1}\right) dz = 0$

$i.e., \int_c \left(\frac{f(z)}{z-1} + \frac{g(z)}{z-1}\right) dz = 0 \left[where\ f(z) = \frac{z+1}{z-2},\ g(z) = a\right]$

$\Rightarrow 2\pi i\big(f(1) + g(1)\big) = 0$

$\Rightarrow 2\pi i(-2 + a) = 0 \Rightarrow a = 2.$

**35)** Suppose $f\ and\ g$ are entire functions and $g(z) \neq 0$ for all $z \in \mathbb{C}$ if $|f(z)| \leq |g(z)|$, then we conclude that

a) $f(z) \neq 0$ for all $z \in \mathbb{C}$

b) $f$ is a constant function

c) $f(0) = 0$

d) for some $c \in \mathbb{C}, f(z) = cg(z)$

**Ans: (d)**

**36)** Let $f$ be a holomorphic function on $0 < |z| < \varepsilon, \varepsilon > 0$ given by a convergent Laurent series $\sum_{n=-\infty}^{\infty} a_n z^n$

Given also that $\lim_{z\to 0}|f(z)| = \infty$

We can conclude that

a) $a_{-1} \neq 0$ and $a_{-n} = 0$ for all $n \geq 2$

b) $a_{-N} \neq 0$ for some $N \geq 1$ and $a_{-n} = 0$ for all $n > N$

c) $a_{-n} \neq 0$ for all $n \geq 1$

d) $a_{-n} \neq 0$ for all $n \geq 1$

**Ans. (b)**

Given Laurent series is –

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

As $\lim_{z\to 0}|f(z)| = \infty, then\ \ z = 0$ is a pole of $f$.

Then at least one negative coefficient must be non-zero.

Thus, $a_{-N} \neq 0$ for some $N \geq 1$ and $a_{-n} = 0$ for all $n > N$

**79)** Let $f(z)$ be the meromorphic function given by $\frac{z}{(1-e^z)\sin z}$ then,

a) $z = 0$ is a pole of order 2

b) for every $k \in z, z = 2\pi ik$ is a simple pole.

c) for every $k \in z \Big| \{0\}, k = k\pi$ is a simple pole.

d) $z = \pi + 2\pi i$ is a pole.

**Ans: (b), (c)**

**80)** Consider the polynomial

$P(z) = \sum_{n=1}^{\infty} a_n\, z^n \,, 1 \leq N < \infty, a_n \in \mathbb{R} \big| \{0\}$

Then, with $D = \{w \in c : |w| < 1\}$

a) $P(D) \subseteq \mathbb{R}$

b) $P(D)$ is open

c) $P(D)$ is closed

d) $P(D)$ is bounded

**Ans. (b), (d)**

$P(z) = \sum_{n=1}^{N} a_n\, z^n, 1 \leq N < \infty, a_n \in \mathbb{R} \big| \{0\}$

$D = \{w \in C : |w| < 1\}$

i.e., $D$ is a open unit disk and bounded.

We know that image of open set is open and image of a bounded set bounded.

Here $P(z) =$ polynomial $\Rightarrow$ continuus.

**81)** Consider the polynomial

$P(z) = (\sum_{n=0}^{5} a_n\, z^n)(\sum_{n=0}^{9} b_n\, z^n)$

Where $a_n, b_n, \in \mathbb{R}, \forall\, n, a_5 \neq 0, b_9 \neq 0$. Then, counting roots with multiplicity we can conclude that $P(z)$ has

a) at least two real roots.

b) 14 complex roots.

c) no real roots.

d) 12 complex roots.

**Ans: (a)**

$$P(z) = \left(\sum_{n=0}^{5} a_n\, z^n\right) \left(\sum_{n=0}^{9} b_n\, z^n\right)$$

$$\begin{array}{cc} \downarrow & \downarrow \\ odd\ degree & odd\ degree \\ at\ least\ one\ real\ root & at\ least\ one\ real\ root \end{array}$$

$\therefore$ $P(z)$ has at least two real roots.

**82)** Let $D$ be the open unit disc in $\mathbb{C}$. Let $g: D \to D$ be holomorphic, $g(0) = 0, and\ let$

$$h(z) = \begin{cases} \frac{g(z)}{z}, z \in D, z \neq 0 \\ g'^{(0)}, z = 0 \end{cases}$$

Which of the following statements are true?

a) $h$ is holomorphic in $D$

b) $h(D) \subseteq \overline{D}$

c) $|g'(0)| > 1$

d) $\left|g\left(\frac{1}{2}\right)\right| \leq \frac{1}{2}$

**Ans. (a), (b), (d)**

# 2017 (June)

**33)** Let $C$ denote the unit circle centered at the origin in $\mathbb{C}$.

Then $\frac{1}{2\pi i}\int_c |1+z+z^2|^2\,dz$

Where the integral is taken anti clockwise along $C$ equals.

a) 0

b) 1

c) 2

d) 3

**Ans. (c)**

$\frac{1}{2\pi i}\int_c |1+z+z^2|^2\,dz$

$= \frac{1}{2\pi i}\int_c (1+z+z^2)\left(\overline{1+z+z^2}\right)dz$

$= \frac{1}{2\pi i}\int_c (1+z+z^2)\,(1+\bar{z}+\bar{z}^2)dz$

$= \frac{1}{2\pi i}\int_c (3+2\bar{z}+\bar{z}^2+z^2+2z)\,dz | \because |z|=1$

Let $z=e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta$

$= \frac{1}{2\pi i}\int_0^{2\pi}\left(3+2e^{-i\theta}+e^{-2i\theta}+e^{2i\theta}+2e^{i\theta}\right) ie^{i\theta}d\theta$

$= \frac{1}{2\pi}\times 4\pi \qquad \because \left[e^{ni\theta}\right]_0^{2\pi} = 0$

$= 2 \qquad n = 1, 1, 2, 3$

So, option (c) is correct.

**34)** Consider the power series $f(x) = \sum_{n=2}^{\infty}\log(n)\,x^n$

The radius of convergence of the series $f(x)$ is –

a) 0

b) 1

c) 3

d) $\infty$

**Ans. (b)**

$R = \frac{1}{\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}$

$= \lim\limits_{n\to\infty}\frac{\log n}{\log(n+1)}$

$= \lim\limits_{n\to\infty} \log \frac{n+1}{n}$ $\quad\quad\quad\quad [by\ L'Hospital\ rule]$

$= \lim\limits_{n\to\infty} \log \left(1+\frac{1}{n}\right) = 1$

So, option 2) is correct.

**35)** For an odd integer $k \geq 1$, let $F$ be the set of all entire functions $f$ such that $f(x) = |x^k|$ for all $x \in (-1,1)$. Then, the cardinility of $F$ is

a) 0

b) 1

c) Strictly greater than 1 but finite

d) infinite.

**Ans. (a)**

$f(x) = |x^k| = \begin{cases} x^k, x \in (0,1) \\ -x^k, x \in\ (-1,0) \end{cases}$

$ut\ g(z) = z^k \Rightarrow f(z) = g(z); z \in (0,1)$

$\Rightarrow f(z) = z^k$

If $h(z) = -z^k \Rightarrow f(z) = h(z) \Rightarrow f(z) = -z^k, z \in (-1,0)$

∵ if two entire function agree on $D$ which has limit pt. in itself then they agree on $\mathbb{C}$.

$\Rightarrow f(x)$ agree with $g(z)\ on\ (0,1)$

and $f(z)$ agree with $h(z)\ on\ (-1,0)$

⇒ So, such function cannot be entire.

⇒ The given set has no such function.

So, $|F| = 0$ ,

so, option (a) is correct.

**36)** Suppose $f$ is holomorphic in an open nbd of $z_0 \in C$. Given that the series $\sum_{n=0}^{\infty} f^{(n)}(z_0)$ converges absolutely, we can conclude that

a) $f$ is constant

b) $f$ is a polynomial

c) $f$ can be extended to an entire function

d) $f(x) \in R$ for all $x \in R$

Ans. (c)

Let $f(z) = e^{iz/4} \& z_0 = 0$

$f'^{(z)} = \frac{i}{4} e^{iz/4} \Rightarrow |f'(z_0)| = \frac{1}{4}$

$f''^{(z)} = \frac{i^2}{4^2} e^{iz/4} \Rightarrow |f''(z_0)| = \frac{1}{4^2}$

$f^n(z) = \left(\frac{1}{4}\right)^n e^{iz/4} \Rightarrow |f^n(z_0)| = \frac{1}{4^n}$

$\therefore \sum f^n\,(z_0) = \sum \frac{1}{4^n}$ , converge absolutely.

**79)** Let $f = u + iv$ be an entire function where $u, v$ are the real and imaginary parts of $f$ respectively. If the Jacobian matrix $J_a = \begin{bmatrix} u_x(a) & u_y(a) \\ v_x(a) & v_y(a) \end{bmatrix}$ is

Symmetric for all $a \in C$, then

a) $f$ is a polynomial

b) $f$ is a polynomial of degree $\leq 1$

c) $f$ is necessarily a constant function.

d) $f$ is a polynomial of degree strictly greater than 1.

**Ans: (a), (b)**

**80)** Consider the function $(z) = \frac{\sin\left(\frac{\pi z}{2}\right)}{\sin(\pi z)}$ . Then $f$ has poles at

a) all integers

b) all even integers

c) all odd integers

d) all integers of the form $4k + 1, k \in z$

**Ans. (c), (d)**

Now, $\sin(\pi z) = 0 = \sin n\pi$

i.e., $z = n, n = 0, \pm 1, \pm 2, \cdots \cdots$

But, $\sin\left(\frac{\pi z}{2}\right)$ is non-zero, only at odd integers. Hence, $f$ has poles at all odd integers.

**81)** Consider the Mobius transformation $f(z) = \frac{1}{Z}, z \in c, z \neq 0$, If $C$ denotes a circle with positive radius passing through the origin, then $f$ maps $C/\{0\}$ to

a) a circle

b) a line

c) a line passing through the origin.

d) a line not passing through the origin.

**Ans. (b), (d)**

$w = \frac{1}{z}$

Then image of a finite circle through the origin is a straight line not trough the origin.

Circle not through origin maps circle not trough origin.

Line not through origin maps circle not through origin.

**82)** For which among the following functions $f(z)$ defined on $G = C\backslash\{0\}$, is thereno sequence of polynomials approximating $f(z)$ uniformly on compact subsets of $G$?

a) $\exp(z)$

b) $\frac{1}{Z}$

c) $z^2$

d) $\frac{1}{z^2}$

Ans: (b), (d)

# 2017 – December

**33)** The function $f: \mathbb{C} \to \mathbb{C}$ defined by $f(z) = e^z + e^{-z}$ has

a) finitely many zeros.

b) no zeros.

c) only real zeros.

d) has infinitely many zeros.

Ans. (d)

$f(z) = e^z + e^{-z}$

$= \frac{e^{2^z}+1}{e^z}$

$f(z)$ has zeros if $f(z) = 0$

$\Rightarrow \frac{e^{2^z}+1}{e^z} = 0$

$\Rightarrow e^{2^z} + 1 = 0 \Rightarrow e^{2^z} = -1$

$\Rightarrow e^{2^z} = e^{i(2n-1)\pi}$

$\Rightarrow z = \frac{(2n-1)\pi i}{2}, n \in \mathbb{N}$

$\therefore f(z)$ has infinitely many zeros.

So, option (d) is correct.

**34)** Let $f$ be a holomorphic function in the open unit disc such that $\lim_{z\to 1} f(z)$ does not exist. Let $\sum_{n=0}^{\infty} a_n z^n$ be the Taylor's series of $f$ about $z = 0$ and let $R$ be the radius of convergence. Then

a) $R = 0$

b) $0 < R < 1$

c) R=1

d) $R > 1$

**Ans. (c)**

Let us consider that

$f(z) = \frac{1}{1-z}$, which is holomorphic in $|z| < 1$.

Now, $f(z) = (1-z)^{-1} = 1 + z + z^2 + \cdots = \sum_{n=0}^{\infty} a_n z^n$, where $a_n = 1$.

Hence the radius of convergence is 1.

**35)** Let $C$ be the circle of radius 2 with the centre at the origin in the complex plane, oriented in the anti-clockwise direction. Then the integral $\oint_C \frac{dz}{(z-1)^2} dz$ is equal to

a) $\frac{1}{2\pi i}$

b) $2\pi i$

c) 1

d) 0

**Ans. (d)**

$\oint_C \frac{dz}{(z-1)^2} = \oint_C \frac{f(z)}{(z-1)^{n+1}} dz$

$= \frac{2\pi i}{1!} f'(1) = 0$

Since $f(z) = 1$

Hence option (d) is correct.

**36)** Let $D$ be the open unit disc in the complex plane and $U = D \backslash \left\{-\frac{1}{2}, \frac{1}{2}\right\}$. Also, let

$H_1 = \{f: D \to C | f \text{ is a holomorphic and bounded}\} \text{ and}$

$H_2 = \{f: U \to C | f \text{ is a holomorphic and bounded}\}$

Then the map $r: H_1 \to H_2$ is given by $r(f) = f|U$, the restriction of $F$ to $U$, is

a) injective but not surjective

b) surjective but not injective

c) injective and surjective

d) neither injective nor surjective.

**Ans: (c)**

**79)** Let $f$ be an entire function. Consider $= \left\{z \in \mathbb{C} \,\middle|\, f^n(z) = 0 \text{ for some positive integer } n\right\}$.

Then,

a) if $A = \mathbb{C}, then\ f$ is a polynomial.

b) if $A = \mathbb{C}$, then $f$ is a constant function.

c) if $A$ is uncountable then $f$ is a polynomial.

d) if $A$ is uncountable, then $f$ is a constant function.

**Ans. (a) & (c)**

Let $f(z) = 1 + z + z^2 + \cdots \cdots + z^{n-1}$

Here $f$ is a polynomial function of $z$ of degree $(n-1)$.

Also, $f^n(z) = 0\ for\ all\ z \in \mathbb{C}$

Thus, $for\ all\ z \in \mathbb{C}, f$ is a polynomial.

**80)** Let $f: \mathbb{C} \to \mathbb{C}$ bea holomorphic function and let $u$ be the real part of $f$ and $v$ be the imaginaly part of $f$. Then, for $x, y \in \mathbb{R}, |f'(x + iy)|^2$ is equal to

a) $u_x^2 + u_y^2$

b) $u_x^2 + v_x^2$

c) $v_y^2 + u_y^2$

d) $v_y^2 + v_x^2$

**Ans. (a), (b), (c) & (d)**

Here $f = u + iv$ be a holomorphic function.

Then, by $C - R$ equations, $u_x = v_y\ and\ u_y = -v_x$

Now, $f'(z) = u_x + iv_x, then\ |f'(z)|^2 = u_x^2 + v_x^2$

Also, $f'(z) = v_y + iv_x, then\ |f'(z)|^2 = v_y^2 + v_x^2$

Again, $f'(z) = u_x - iu_y, then\ \left|f'^{(z)}\right|^2 = u_x^2 + u_y^2$

and also, $f'(z) = v_y - iu_y, then\ \left|f'^{(z)}\right|^2 = v_y^2 + u_y^2$

**81)** Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots\cdots + a_0$, where $a_0, \cdots a_{n-1}$ are complex numbers and let $q(z) = 1 + a_{n-1}z + \cdots + a_0 z^n$.

If $|P(z)| \leq 1\ for\ all\ z\ with\ |z| \leq 1$ then

a) $|q(z)| \leq 1\ for\ all\ z\ with\ |z| \leq 1$

b) $q(z)$ is a constant polynomial.

c) $P(z) = z^n$ for all complex numbers $z$.

d) $P(z)$ is a constant polynomial.

**Ans. (a), (b), (c)**

If $|P(z)| \leq 1\ for\ all\ z\ with\ |z| \leq 1$

i.e., $1 + |a_{n-1}| + \cdots + |a_0| \leq 1$

Now, $|q(z)| = |1 + a_{n-1}z + \cdots + a_0 z^n| \leq 1 + |a_{n-1}| + \cdots + |a_0| \leq 1$

Thus, the option (a) is true

Again, if all the coefficients $a_0, \cdots\cdots, a_{n-1}$ are vanish then $P(z) = z^n$, a polynomial function, but $q(z) = 1$, a constant function.

So, (b) and (c) will be correct.

**82)** Let $f$ be a non-constant entire function and let $E$ be the image of $f$. Then

a) $E$ is an open set

b) $E \cap \{z: |z| < 1\}$ is empty

c) $E \cap \mathbb{R}$ is non-empty

d) $E$ is a bounded set.

**Ans. (a)**

$f$ is a non-constant entire function so, $f$ is unbounded (by Liouville's theorem)

$\Rightarrow$ range $(f)$ is unbounded

$\Rightarrow$ $E$ is unbounded

So, option (d) is incorrect.

<u>Open mapping theorem:</u>

Image of an open set under non constant entire function is an open set.

$\Rightarrow$ Range $(f) = E$, open set.

$\therefore$ option (a) is correct.

<u>Little picards theorem:</u>

If $f$ is non constant entire function then range $(f)$ can skip at most one pt. from $\mathbb{R}$

$\Rightarrow E = \mathbb{C}\ or\ \mathbb{C}/\{a\}\ when\ a \in \mathbb{C}$

(b) and (c) are incorrect.

# 2019 – June

**33)** Let $c$ be the counter clockwise oriented circle of radius $\frac{1}{2}$ centered at $i = \sqrt{-1}$. Then the value of the contour integral $\oint_c \frac{dz}{z^4-1}$ is

a) $-\frac{\pi}{2}$

b) $\frac{\pi}{2}$

c) $-\pi$

d) $\pi$

**Ans. (a)**

**34)** Consider the function $f: \mathbb{C} \to \mathbb{C}$ given by $f(z) = e^z$. Which of the following is false?

a) $f(\{z \in \mathbb{C}: |z| < 1\})$ is not an open set.

b) $f(\{z \in \mathbb{C}: |z| \leq 1\})$ is not an open set.

c) $f(\{z \in \mathbb{C}: |z| = 1\})$ is a closed set.

d) $f(\{z \in \mathbb{C}: |z| > 1\})$ is an unbounded open set.

**Ans. (a)**

**36)** Let $f: \mathbb{C} \to \mathbb{C}$ be an entire function such that $\lim_{z\to 0} \left| f\left(\frac{1}{z}\right) \right| = \infty$. Then which of the following is true?

a) $f$ is constant.

b) $f$ can have infinitely many zeros.

c) $f$ can have most finitely many zeros.

d) $f$ is necessarily nowhere vanishing.

**Ans. (c)**

**35)** Given a real number $a > 0$. Consider the triangle $\Delta$ with vertices $0, a, a + ia$. If $\Delta$ is given the counter clockwise orientation, then the contour integral $\oint_\Delta Re\,(z)dz$ (with $Re(z)$ denoting the real part of $z$) is equal to

a) $0$

b) $i\frac{a^2}{2}$

c) $ia^2$

d) $i\frac{3a^2}{2}$

**Ans. (b)**

**80)** Let $Re(z), Im\,(z)$ denote the real and imaginary parts of $z\epsilon\mathbb{C}$ respectively. Consider the domain $\Omega = \{z \in \mathbb{C}; Re(z) > |Im(z)|\}$ and let $f_n(z) = \log z^n$, where $n \in \{1,2,3,4\}$ and where $\log: \mathbb{C} \setminus (-\infty, 0] \to C$ defines the principle branch of logarithm, then which of the following are true?

a) $f_1(\Omega) = \left\{z \in \mathbb{C}; 0 \le |Im(z)| < \frac{\pi}{4}\right\}$

b) $f_2(\Omega) = \left\{z \in \mathbb{C}; 0 \le |Im(z)| < \frac{\pi}{2}\right\}$

c) $f_3(\Omega) = \left\{z \in \mathbb{C}; 0 \le |Im(z)| < \frac{3\pi}{4}\right\}$

d) $f_4(\Omega) = \{z \in \mathbb{C}: 0; \le |Im(z)| < \pi\}$

**Ans. (a), (b), (c) & (d)**

**81)** Consider the set

$F = \{f: \mathbb{C} \to \mathbb{C}\}\ f\ is\ an\ entire\ functions, |f'(z)| \le |f(z)|\ for\ all\ z \in \mathbb{C}$

Then which of the following are true?

a) $F$ is a finite set

b) $F$ is an infinite set

c) $F = \{\beta e^{\alpha x}; \beta \in \mathbb{C}, \alpha \in \mathbb{C}\}$

d) $F = \{\beta e^{\alpha x}; \beta \in \mathbb{C}, |\alpha| \le 1\}$

**Ans. (b), (d)**

**82)** Let $D = \left\{z \in \mathbb{C} \,\middle|\, |z| < 1\right\}$ and $w \in D$. Define $F_w D \to D$ by $F_W(z) = \frac{w-z}{1-\overline{w}z}$. Then which of the following are true?

a) $F$ is one to one

b) $F$ is not one to one

c) $F$ is onto

d) $F$ is not onto

**Ans. (a), (c) & (d)**

**79)** Let $f(z) = (z^2+1)\sin z^2 \ for\ z \in \mathbb{C}$. Let $f(z) = u(x,y) + iv(x,y)$ where $z = x + iy$ and $u, v$ are real valued functions.

Then which of the following are true?

a) $u$: $R^2 - R^2$ is infinitely differentiable

b) $u$ is continuous but need not be differentiable.

c) $u$ is bounded

d) $f$ can be represented by an absolutely convergant power series $\sum_{n=0}^{\infty} a_n\, z^n$ for all $z \in \mathbb{C}$.

**Ans. (a) & (d)**