Mathematical Sciences

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1.17. Continuity:

Definition (Continuous at a point): Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $c \in D$ f is said to be continuous at c if for a pre – assigned $\varepsilon > 0$, $\exists$ $\delta > 0$ such that

$|f(x) - f(c)| < \varepsilon \ \forall \ x \in (c - \delta, c + \delta) \cap D$

We write $\lim\limits_{x \to c} f(x) = f(c)$

1.17.1 Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function. If c be an isolated point of D then f is continuous at c.

1.17.2 [Sequential Criterion]: Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $c \in D \cap D'$. f is continuous at $c \Leftrightarrow$ for every sequence $\{x_n\}$ in D converging to c, the sequence $\{f(x_n)\}$ converges to $f(c)$.

Example (1.71):

(i) $f(x) = k(\in \mathbb{R}) \ \forall \ x \in \mathbb{R}$ is continuous.

(ii) $f(x) = x \ \forall \ x \in \mathbb{R}$ is continuous.

(iii) $f(x) = \begin{cases} cos\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is not continuous at $x = 0$

Let $x_n = \frac{1}{2n\pi}$ then $\{x_n\}$ converges to 0 but $f(x_n) = 1 \Longrightarrow \{f(x_n)\}$ converges to $1 \neq 0 = f(0)$

(iv) $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$ is not continuous at any point $a \in \mathbb{R}$.

Case – 1: Let $a \in \mathbb{Q}$, $f(a) = 1$ but we can find a sequence $\{x_n\}$ of irrational number which converges to a and $f(x_n) = 0 \Rightarrow \{f(x_n)\}$ converges to $0 \neq 1 = f(a)$.

Case – 2: Similarly for $a \in \mathbb{R}\backslash\mathbb{Q}$

Note: This function $f(x)$ is called Dirichlet's function which is every where discontinuous on $\mathbb{R}$.

1.17.3. Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be continuous at $a \in D$ (or on D) then $|f|$ is continuous at $a \in D$ (or on D). But converges is not true.

Example (1.72):

$$f(x) = \begin{cases} 1\ , & x \in \mathbb{Q} \\ -1\ , & x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

1.17.4. Let $f, g : D \subseteq \mathbb{R} \to \mathbb{R}$ be two functions. We define the functions –

$\sup\,(f,g);\ inf(f,g) : D \to \mathbb{R}$ by

$\sup(f,g)\,(x) = sup\{f(x),\ g(x)\},\ x \in D$

$\inf\,(f,g)\,(x) = inf\{f(x),\ g(x)\},\ x \in D$

1.17.5. Let $f, g : D \subseteq \mathbb{R} \to \mathbb{R}$ be continuous at $c \in D$. Then $sup(f,g)$ and $inf(f,g)$ are continuous at c.

Since,

$$\sup(f,g)\,(x) = \sup\{f(x), g(x)\} = \tfrac{1}{2}\big(f(x) + g(x)\big) + \tfrac{1}{2}|f(x) - g(x)|$$

$$= \tfrac{1}{2}(f+g)(x) + \tfrac{1}{2}|f-g|(x),\ x \in D$$

$$\inf(f,g)(x) = \inf\{f(x), g(x)\} = \tfrac{1}{2}\big(f(x) + g(x)\big) - \tfrac{1}{2}|f(x) - g(x)|$$

$$= \tfrac{1}{2}(f+g)(x) - \tfrac{1}{2}|f-g|(x),\ x \in D$$

1.17.6. Continuity of some important function:

(i) Polynomial Function:

$f(x) = a_0\,x^n + a_1\,x^{n-1} + \ldots\ldots\ldots\ldots + a_{n-1}\,x + a_n\ \forall\ x \in \mathbb{R}$ continuous in $\mathbb{R}$.

(ii) Rational Function:

$f(x) = \frac{p(x)}{q(x)}$, $p(x),\ q(x)$ be polynomial in $\mathbb{R}$ and $x \neq \alpha_1, \ldots\ldots\ldots, \alpha_r$ where $\alpha_i'\,s$ are root of $q(x)$. Then $f(x)$ is continuous $\forall\ x \in \mathbb{R}$ for which $f(x)$ is defined.

(iii) Trigonometric Function:

(a.) $\sin x$, $\cos x$ continuous on $\mathbb{R}$.

(b.) $\tan x$ is continuous on $\mathbb{R}$ except $x = (2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$.

(c.) $\cot x$, $\cos x$, $\sec x$ are continuous on their respective domains.

(iv) $f(x) = a^x$, $a > 0$, $x \in \mathbb{R}$ is continuous on $\mathbb{R} \Rightarrow e^x$ is continuous on $\mathbb{R}$.

(v) Logarithmic Function :

$f(x) = \log x,\ x > 0$ $\qquad$ f is continuous on $(0, \infty)$

(vi) Square root Function :

$f(x) = \sqrt{x}\ ,\ x \geq 0$ $\qquad$ f is continuous $(0, \infty)$

(vii) **(a.)** $f : D \subseteq \mathbb{R} \to \mathbb{R}$ such that $f(x) \geq 0 \quad \forall \ x \in D$ and f is continuous on D Then $\sqrt{f}$ is continuous on D.

Example (1.73): $f(x) = \sqrt{\sin x}\ , x \in [0\,,\pi]$ is continuous

(b.) $f : D \subseteq \mathbb{R} \to \mathbb{R}$ such that $f(x) > 0$ and continuous then $\log f$ is continuous on D.

(c.) If $f: D \subseteq \mathbb{R} \to \mathbb{R}$ is continuous on D, then e^f is continuous on D.

1.17.7 Some important limits:

(i) $\lim\limits_{x\to 0} \frac{\log(1+x)}{x} = 1$

(ii) $\lim\limits_{x\to 0} \frac{e^x - 1}{x} = 1$

(iii) $\lim\limits_{x\to 0} \frac{a^x - 1}{x} = log_e a\,, a > 0$

1.18 Properties of continuous functions:

1.18.1. Neighborhood properties: Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$be continuous on D and $c \in D$.If $f(c) \neq 0$ then $\exists$ a suitable $\delta > 0$ such that $\forall\, x \in N_\delta(c) \cap D,\ \ f(x)$ keeps the same sign as $f(c)$.

Note: This is a local property of continuous function and is known as sign preserving property of continuous function.

Cor – 1: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. Then $S = \{x \in \mathbb{R} : f(x) > 0\}$ and $T = \{x \in \mathbb{R} : f(x) < 0\}$ are open sets in $\mathbb{R}$.

Cor – 2: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. Then $S = \{x \in \mathbb{R} : f(x) \neq 0\}$ is an open set is $\mathbb{R}$ and $T = \{x \in \mathbb{R} : f(x) = 0\}$ is a closed set in $\mathbb{R}$.

1.18.2. Let $I = [a, b]$ be a closed, bounded interval and $f : I \to \mathbb{R}$ be continuous on $\mathbb{R}$ then f is bounded on$I$ and $\exists\ \ c, d \in I$ such that $f(c) = \sup\limits_{x\,\in\, I} f(x)$ and $f(d) = \inf\limits_{x\,\in\, I} f(x)$

But this is not true for open interval $I = (a\,, b)$ which is bounded.

Example (1.74):

(i) $f : I = (2\,,3) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}\,,\ \ x \in (0\,,1)$ Then f is continuous on I but not bounded.

(ii) $f : I = (2\,,3) \to \mathbb{R}$ defined by $f(x) = \ x^2$. Then $\sup\limits_{x\,\in\, I} f(x) = 9$ and $\mathop{int}\limits_{x\,\in\, I} f(x) = 4$.

But $\nexists\ x_0\ \ c,\ d \in I$ such that $f(c) = 9$ and $f(d) = 4,\ x \in I$.

(iii) A function f continuous as a closed interval I may not be bounded as I.

Example (1.75): $f : [0\,,\infty] \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}\,,\ x \geq 0.$ f is continuous on $[0\,,\infty]$ but f is not bounded on $[0\,,\infty]$.

1.18.3. Bolzano Theorem: Let $I = [a\,,b]$ be a closed and bounded interval and $f : I \rightarrow \mathbb{R}$ be continuous on I. If $f(a)$ and $f(b)$ one of opposite signs, then $\exists$ at least one $c \in (a\,,b)$ such that $f(c) = 0$.

1.18.4. Intermediate Value Theorem: Let $I = [a\,,b]$ be a closed, bounded interval and $f : [a\,,b] \rightarrow \mathbb{R}$ be continuous on I. If $f(a) \neq f(b)$ then f attains every value between $f(a)$ and $f(b)$ at least once in the open interval $(a\,,b)$ converse is not true.

Example (1.76): Let $f : [0\,,2] \rightarrow \mathbb{R}$ be defined by -$f(x) = \begin{cases} 0 & , \quad x = 0 \\ x & , \quad 0 < x \leq 1 \\ 3 - x & , \quad 1 < x < 2 \\ 2 & , \quad x = 2 \end{cases}$

f assume every value between $0 \; and \; 2 \; on \; [0\,,2]$. But f is not continuous at$x = 1\,,2$.

1.18.5. Let $I = [a\,,b]$ be a closed and bounded interval and $f : I \rightarrow \mathbb{R}$ be continuous on I. If $M = \sup_{x \in I} f(x) \neq m = \inf_{x \in I} f(x)$ and $m < \mu < M$ then $\exists \; p \in (a\,,b)$ such that $f(p) = \mu$.

1.18.6. Let $f : I = [a\,,b] \rightarrow \mathbb{R}$ be continuous on I.Then $f(I) = \{f(x) : x \in I\}$ in a closed and bounded interval.

Note:

(i) The continuous image of a closed and bounded interval $[a, b]$ is a closed and bounded interval$[m\,,M]$. If particular, if f is constant on $[a, b]$, the image reduces to a point.

(ii) The continuous image of an open interval may not be open.

Example (1.77): $f : (-1\,,1) \rightarrow \mathbb{R}$ defined by $f(x) = x^2 , \forall \; x \in I = (0\,,1)$ then $f(I) = [0\,,1)$ which is not open.

1.18.7. Let I be an interval and $f = I \rightarrow \mathbb{R}$ be continuous (non-constant) in I.Then $f(I)$ is an interval.

Examples (1.78):

(i) If $f : [0\,,1] \rightarrow [0\,,1]$ is continuous on$[0\,,1]$, then $\exists$ a point $c \in [0\,,1]$ such that $f(c) = c[c \; is \; called \; a \; fixed \; point \; of \; f]$.

[**Hint:** if $(0) = 0 \; or \; f(1) = 1$, done. Let $f(0) \neq 0\,,\; f(1) \neq 1$.

Define $g(x) = f(x) - x$. Then g is continuous on $[0\,,1]$ and $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 < 0 \Rightarrow$ by Bolzano then, $\exists \; c \in (0\,,1)$ such that $g(c) = 0 \Rightarrow f(c) = c$]

(ii) If $f:[0\,,1]\to\mathbb{R}$ is continuous on $[0\,,1]$ and assumed only rational values and $f\left(\frac{1}{2}\right)=\frac{1}{2}$,then $f(x)=\frac{1}{2}\ \forall\, x\in[0\,,1]$

[Hint: Let $x_1\in\left[0\,,\ \frac{1}{2}\right]$ and consider on $x_1\,,\frac{1}{2}$ let $f(x_1)=p\neq\frac{1}{2}$, p is rational. Let $\mu\in\left(p\,,\frac{1}{2}\right)$ irrational by intermediate theorem, $\exists\ c\in\left(x_1\,,\frac{1}{2}\right)$ such that

$f(c)=\mu$, contradiction hence $f(x_{1)}=\frac{1}{2}$]

1.18.8. Let $f:\mathbb{R}\to\mathbb{R}$ be continuous on $\mathbb{R}$. Then for every open subset G of $\mathbb{R}$, $f^{-1}(G)$ is open in $\mathbb{R}$. Conversely, if $f^{-1}(G)$ in open in $\mathbb{R}$ for every open set G in $\mathbb{R}$. Then f is continuous on $\mathbb{R}$.

But if f is continuous then image of open set may not open.

Example (1.79): $f:(0\,,1)\to\mathbb{R}$ defined by $f(x)=2\quad\forall\, x\in(0\,,1)$.

1.18.9. Function $f:\mathbb{R}\to\mathbb{R}$ is continuous on $\mathbb{R}\Leftrightarrow\ f^{-1}(F)$ is closed in $\mathbb{R}$ whenever F is closed in $\mathbb{R}$.

1.18.10. The functions $f\,,\ g:\mathbb{R}\to\mathbb{R}$ are both continuous on $\mathbb{R}$. Then the

Lets,

(i) $s=\{x\in\mathbb{R}:f(x)<g(x)\}$ is open set in $\mathbb{R}$.

(ii) $T=\{x\in\mathbb{R}:f(x)\neq g(x)\}$ is open set in $\mathbb{R}$.

(iii) $P=\{x\in\mathbb{R}:f(x)=g(x)\}$ is closed set in $\mathbb{R}$.

(iv) If $\{f(x)=g(x)\}$ at all $x\in\mathbb{Q}$, then $f(x)=g(x)\ \ \forall\, x\in\mathbb{R}$

[Hint: $\mathbb{Q}\subseteq P\subseteq\mathbb{R}$ and P is closed $\Rightarrow P=\bar{P}=\mathbb{R}$]

(v) If $f(x)=k$,constant $\forall\, x\in\mathbb{Q}$, then $f(x)=k\quad\forall\, x\in\mathbb{R}$.

[Hint: let$g(x)=k,\ \forall\ x\in\mathbb{R}\ \Rightarrow f(x)=g(x)\,\forall\ x\in\mathbb{Q}\ \Rightarrow f(x)=g(x)=k\ \ \forall\, x\in\mathbb{R}$]

1.18.11. Let $f:I=(a,b)\to\mathbb{R}$ be monotone increasing on I. Then at any point $c\in I$,

(i) $f(c-0)=\sup\limits_{x\in(a,c)} f(x)$

(ii) $f(c+0)=\inf\limits_{x\in(c,b)} f(x)$

(iii) $f(c-0)\le f(c)\le f(c+0)$

1.18.12. Discontinuity of first kind: Let $c \in (a\,,b) \in I$ and$f$ be continuous on (a, c) and $(c\,,b)$, but discontinuous at $c \in (a, b) \in I$ and $\lim_{x\to c-} f(x)$ and $\lim_{x\to c+} f(x)$ both exist. –

(i) $\lim_{x\to c-} f(x) = \lim_{x\to c+} f(x)$

a. f is not defined at c, f is discontinuous at c.

b. f is defined at c, but$f(c) \neq \lim_{x\to c} f(x)$

(ii) $\lim_{x\to c-} f(x) \neq \lim_{x\to c+} f(x)$. In this case f is discontinuous at c. Whether f is defined at c or not. This type of discontinuity is called jump discontinuity.

Right jump: $f(c+0) - f(c)$

Left jump: $f(c) - f(c-0)$

1.18.13. Discontinuity of second kind: If at least one of $\lim_{x\to c-} f(x)$ and $\lim_{x\to c+} f(x)$ does not exist. But f is bounded in some bounded $N'\delta(c)$ of. In this case f is discontinuous at c whether f is defined at c or not. This type of discontinuity is called oscillatory discontinuity.

1.18.14. If $f : (a, b) \to \mathbb{R}$ be monotone on (a, b) , then at every point $c \in (a, b)$, $f(c-0)\ and\ f(c+0)$ both exist. Monotone function f cannot have discontinuity of second kind.

1.18.15. If $f : [a, b] \to \mathbb{R}$ be monotone on $[a, b]$, then the set of points at discontinuities of f in $[a, b]$ is a countable set.

⇒ If $f = \mathbb{R} \to \mathbb{R}$ be monotone on $\mathbb{R}$, then the set of points of discontinuities is a countable set.

[Hint: $\mathbb{R} = (\cup_{n=0}^{\infty}[n-1\,,\ n+1]) \cup (\cup_{n=0}^{\infty}[-(n+1)\,,\ -(n-1)])$]

1.18.16. If a function $f: [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and injective on $[a, b]$ then f is strictly monotone on $[a, b]$.

⇒ Let I be an interval and $f : I \to \mathbb{R}$ is continuous and injective on I. Then f is strictly monotone on I.

1.18.17. If $f : [a, b] \to \mathbb{R}$ satisfies intermediate value property on $[a, b]$ and f is injective on $[a, b]$ then-

(i) f is strictly monotone on $[a, b]$

(ii) f is continuous on $[a, b]$

1.19. Uniform continuity:

Definition: A function $f : I \to \mathbb{R}$ is said to uniformly continuous on$I$ if corresponding to a pre – assigned $\varepsilon > 0$, $\exists\ \ \delta > 0$ such that for any pair of point $x_1\, x_2 \in I$,

$$|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$$

Note: Uniform continuity is a global property.

Example (1.80):

(i) $f(x) = \frac{1}{x}$, $x \in [1, \infty]$ is uniformly continuous on $[1, \infty]$

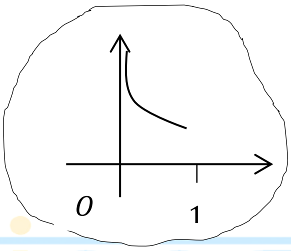
Since $|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{x-y}{xy}\right| \leq |x - y| < \varepsilon, say\ (\because x, y \geq 1)$ then

$\forall\ x, y \in [1, \infty]$ with $|x - y| < \varepsilon \Rightarrow\ |f(x) - f(y)| < \varepsilon$.

(ii) $f(x) = sinx,\ x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

Since $x, y \in \mathbb{R}, |\sin x - \sin y| = 2\left|\sin\frac{x-y}{2}\right|\left|\cos\frac{x+y}{2}\right| \leq 2\left|\sin\frac{x-y}{2}\right| \leq 2.|x - y| <$ $\varepsilon, say\ |x - y| < \varepsilon \Rightarrow |\sin x - \sin y| < \varepsilon$

1.19.1. Let I be an interval and a function $f : I \to \mathbb{R}$ be uniformly continuous on I. Then f is continuous on I. But not conversely.



Example (1.81): $f(x) = \frac{1}{x}$, $0 < x \leq 1$ is continuous but not uniformly.

1.19.2. Let $I = [a, b]$ be a closed and bdd interval and $f : I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I.

1.19.3. Let $f : D \subseteq \mathbb{R}$ be uniformly continuous on D. If $\{x_n\}$ be a Cauchy sequence in D, then $\{f(x_n)\}$ a Cauchy sequence in $\mathbb{R}$.

Example (1.82): $f(x) = \frac{1}{x}$, $x \in [0, 1]$ is not uniformly continuous in $[0, 1]$. Since $\left\{\frac{1}{n}\right\}$ in a Cauchy sequence in $[0\ , 1]$ but $\left\{f\left(\frac{1}{n}\right) = n\right\}$ is not Cauchy in $\mathbb{R}$.

1.19.4. Let I be a bounded interval and a function $I \to \mathbb{R}$ be uniformly continuous on I. Then f is bounded on I converse is not true.

Example (1.83): $f(x) = \sin\frac{1}{x}$, $x \in (0, 1)$.Then $f(x)$ is continuous on bdd interval $(0, 1)$ and $|f(x)| \leq 1$ but $f(x)$ is not uniformly continuous. Since $\left\{\frac{2}{(2n+1)\pi}\right\}$ is Cauchy in $(0\ , 1)$ but $\left\{f(\frac{2}{(2n+1)\pi}\right\}$ is not Cauchy in $\mathbb{R}$.

1.19.5. Let f be continuous on an open bdd interval (a, b). Then f is uniformly continuous on $(a, b) \Leftrightarrow \lim\limits_{x \to a+} f(x)$ and $\lim\limits_{x \to b-} f(x)$ both exist finitely.

1.19.6. Continuous Extension: Let f be continuous on an interval I. A function g is said to be a continuous extension of f to $\mathbb{R}$ if g be continuous on $\mathbb{R}$ and $g(x) = f(x) \quad \forall\, x \in I$

Example (1.84): Let $f : [a, b]$ be continuous and $g : \mathbb{R} \to \mathbb{R}$ be defined by-

$$g(x) = \begin{cases} f(a)\ , & x < a \\ f(x)\ , & x \in [a, b] \\ f(b)\ , & x > b \end{cases}$$

Then g is continuous extension of f.

Let f be continuous on an bdd open interval (a, b). Then f admits of a continuous extension to $\mathbb{R} \Leftrightarrow f$ be uniformly continuous on (a, b).

1.19.7. Definition (Lipschitz function): Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is said to satisfy a Lipschitz condition on I if $\exists \quad 0 < M \in \mathbb{R}$ such that $|f(x_1) - f(x_2)| \leq M\,|x_1 - x_2|$ for any two points $x_1,\ x_2, \in I$. In this case f is said to be a Lipschitz function on I.

Example (1.85):

Let $f(x) = x^2,\ x \in [0, 2]$. Then

$|f(x_1) - f(x_2)| = |{x_1}^2 - {x_2}^2| \leq 4|x_1 - x_2| \ \forall\, x_1 x_2 \in [0, 2]$

1.19.8. Let $f : I \to \mathbb{R}$ be a Lipschitz function on I. Then f is uniformly continuous on I.

Example (1.86):

$f(k) = \sin x\,, x \in \mathbb{R}$

$|\sin x - \sin y| \leq |x - y|$

1.19.9. Continuity on a compact set: Let $D \subseteq \mathbb{R}$ be a compact set and a function $f : D \to \mathbb{R}$ be continuous on D. Then $f(D)$ is a compact set in $\mathbb{R}$.

1.19.10. Let $D \subseteq \mathbb{R}$ be a compact set and $f : D \to \mathbb{R}$ is continuous D. Then f is uniformly continuous on D.

Converse of (1.19.8) is not true.

Example (1.87): $f(x) = \sqrt{x},\ x \quad x \in [0\,, a],\ a > 0.$

But $f(x) = \sqrt{x}$ is satisfies Lipschitz condition on $[1\,, a],\ \forall\, a > 1$

1.19.11. Some special uniform continuous functions:

- **(i)** **Periodic function:** If f be a continuous function such that $f(x + p) = f(x)$ for some $P \in \mathbb{R}$, then f is uniformly continuous on $\mathbb{R}$.
- **(ii)** If $f(x + y) = f(x) + f(y) \quad \forall\ x, y \in \mathbb{R}$ be continuous at a point $c \in \mathbb{R}$, then f is uniformly continuous on $\mathbb{R}$.
- **(iii)** Let $\phi \neq A \in \mathbb{R}\ and\ f_A(x) = inf\,\{|x - a| : a \in A\}\,\forall\, x \in \mathbb{R}$. f is uniformly continuous on $\mathbb{R}$.

(iv) If $f'(x)$ exists and bdd then f satisfice Lipschitz condition and hence it is uniformly continuous.

1.20. Differentiation:

Definition (Differentiability and derivative): Let $I = [a, b]$ be an interval and $f : I \to \mathbb{R}$ be a function. f is said to be differentiable at $c \in I$ if $\lim\limits_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists. If l be its limit, then l is said to be the derivative of f at$c$and is denoted by $f'(c)$.

(i). If c be an interior point of the domain, then $\lim\limits_{x\to c-} \frac{f(x)-f(c)}{x-c}$ and $\lim\limits_{x\to c+} \frac{f(x)-f(c)}{x-c}$ should exist and they are equal in order to $\lim\limits_{x\to c} \frac{f(x)-f(c)}{x-c}$ exist.

(ii). If $c = a$, then $\lim\limits_{x\to a+} \frac{f(x)-f(a)}{x-a}$ exists and the limit is called derivative of f at a and is denoted by $f'(a)$.

(iii). If $c = b$, then $\lim\limits_{x\to b-} \frac{f(b)-f(x)}{b-x}$ exists and limit is called derivative of f at b and is denoted by $f'(b)$.

1.20.1. Definition (Right and left hand derivative): Let I be an interval and $f : I \to \mathbb{R}$ and $c \in I$. If $\lim\limits_{x\to c+} \frac{f(x)-f(c)}{x-c}$ exists the limit is called the right hand derivative of at c and is denoted by $R\,f'(c)$.

If $\lim\limits_{x\to c-} \frac{f(x)-f(c)}{x-c}$ exists, the limit is called left hand limit derivative of f and is denoted by $Lf'(c)$.

1.20.2. Let $f : I \to \mathbb{R}$ be differentiable at a point $c \in I$. Then f is continuous at c. But converse is not true.

Example (1.88): $f(x) = |x|,\ x \in \mathbb{R}$. At $x = 0$, $f(x)$ is continuous but

$$\lim_{x\to 0+} \frac{|x|-|0|}{x-0} = \lim_{x\to 0+} \frac{x}{x} = 1 = Rf'(0)$$

$$\lim_{x\to 0-} \frac{|x|}{x} = \lim_{x\to 0-} \frac{-x}{x} = -1 = Lf'(0)$$

As $Rf'(0) \neq Lf'(0) \ \Rightarrow f$ is not differentiable at 0.

Note: Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ it is possible to define differentiability of f at $c \in D$, provided $c \in D'$ also i.e., if $c \in D \cap D'$, then f is said to be differentiable at c if $\lim\limits_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists and the limit is called derivative of f at c and is denoted by $f'(c)$.

1.20.3. Let$f, g : I \to \mathbb{R}$ be differentiable at $c \in I$. Then –

(i). $(f + g)'(c) = f'(c) + g'(g)$

(ii) if $k \in \mathbb{R}, (k\,f)'(c) = k\,f'(c)$

(iii) $(f \cdot g)'(c) = f'(c)\, g(c) + f(c)\, g'(c)$

(iv) $\left(\frac{f}{g}\right)'(c) = \frac{g(c)\, f'(c) - f(c)\, g'(c)}{\{g\,(c)\}^2}$, proved $g(c) \neq 0$

1.20.4. Let I *and* J be intervals. Let $f\ : I \to \mathbb{R};\ \ g : J \to \mathbb{R}\ \ and\ f(I) \in J$. Let $c \in I\ and\ f$ is differentiable at c and g is differentiable at $e\ and (g \circ f)'(c) = g'\big(f(c)\big) \cdot f'(c)$.

Example (1.89):

Let $f(x) = \ x^{\alpha}\ ,\ x > 0\ and\ d \in \mathbb{R} \Rightarrow f(x) = e^{\alpha \log x}$

Let $g(x) = \alpha\ \log x\ ,\ x > 0\ and\ \ h(x) = e^x\ ,\ x \in \mathbb{R}$

Then $f(x) = (h \circ g)(x) = h\big(g(x)\big) = \ e^{\alpha \log x} = x^{\alpha}$

$\Rightarrow f'(x) = h'\big(g(x)\big) \cdot g'(x) = \ e^{\alpha \log x} \cdot \frac{\alpha}{x} = \ x^{\alpha} \cdot \frac{\alpha}{x} = \alpha\, x^{\alpha - 1}, x > 0.$

1.20.5. Let $I \subseteq \mathbb{R}$ be an interval and a function $f\ : I \to \mathbb{R}$ be strictly monotone and continuous on I. Let $J = f(I)$ and Let $g : J \to \mathbb{R}$ be the inverse of f. If f is differentiable at $c \in I\ and\ f'(c) \neq 0$. then g is differentiable at $d = f(c)\ and\ \ g'(d) = \frac{1}{f'(c)}$.

Example (1.90):

(i). $f(x) = \ x^2\ ,\ \ x \in [0, \infty]$. f is strictly increasing and continuous on $[0, \infty]$.

Let $\ I = [0, \infty]$ then $f(I) = [0, \infty]$. The inverse function g is defined by $g(y) = \sqrt{y},\ y \in [0, \infty]$ is continuous on $[0, \infty]\, f$ is differentiable on $[0, \infty]$ and $f'(x) = 2x,\ \ x \in [0, \infty]$, $f'(x) \neq 0\ on\ (0, \infty]$.

Let $I_1 = (0, \infty)$. Then $f(I_1) = (0, \infty)$.

Hence $g'(y)$ exists $\forall\ y \in (0, \infty)$ and $g'(y) = \frac{1}{f'(x)} = \frac{1}{2x} = \frac{1}{2g(y)} = \frac{1}{2\sqrt{y}}\ \ , y \in (0, \infty)$.

(ii). $f(x) = \ e^x,\ \ x \in \mathbb{R}$. Then $f(\mathbb{R}) = (0, \infty)$. Inverse of f is g be field by $g(y) = \ \log y\ ,\ \ y \in (0, \infty)$ since f is strictly increasing and monotone on $(0, \infty) f'(x) \neq 0\ on\ \mathbb{R}$. So

$g'(y) = \frac{1}{f'(x)} = \frac{1}{e^x} = \frac{1}{e^{\log y}} = \frac{1}{y}\ , y \in (0, \in \infty)$.

(iii) $f(x) = \sin x\ ,\ \ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\ \ \ f(I) = [-1, 1]$. The inverse of g is defined by

$g(y) = \sin^{-1} y\ ,\ \ y \in [-1, 1],\ \ f'(x)\ \neq\ 0\ \ on\ \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$\therefore\ g'(y) = \frac{1}{f'(x)} = \frac{1}{\cos x} = \frac{1}{\sqrt{1 - \sin^2 x}} = \frac{1}{\sqrt{1 - y^2}}\ , y \in (-1, 1)$.

Thus $\frac{d}{dx} \sin^{-1} y = \frac{1}{\sqrt{1 - y^2}}\ , y \in (-1,1)$

(iv) $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$

Then, $\lim\limits_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \frac{x^2 \sin\frac{1}{x}}{x} = \lim\limits_{x \to 0} x \sin\frac{1}{x} = 0 \Rightarrow f'(0) = 0$

$$\therefore f'(x) = \begin{cases} 2x \sin\frac{1}{x} - \cos\frac{1}{x}, & x \neq 0 \\ 0\,, & x = 0 \end{cases}$$

$\lim\limits_{x\to 0} \cos\frac{1}{x}$ does not exist (by Cauchy principle) $\Rightarrow f'(x)$ is continuous at 0.

(v) $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & x < 0 \\ 0, & x \in \mathbb{R}|\mathbb{Q} \\ \frac{1}{q}\ if\ x = \frac{p}{q}, & p, q \in \mathbb{Z}, q \neq x\ and\ g \subset d\ (p, q) = 1 \end{cases}$$

$\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0} \frac{f(x)}{x}$ Let $x_n = \frac{1}{n}$ Then $f(x_n) = \frac{1}{n}$

Hence $\lim\limits_{x_n\to 0} \frac{f(x_n)}{x_n} = 1$ and let $\{x_n\}$ be a sequence of irrational numbers converging to 0. $\Rightarrow$

$\lim\limits_{x_n\to 0} \frac{0}{x_n} = 0.$

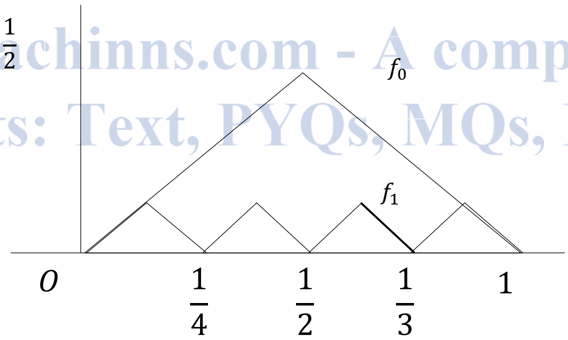
Hence f is not differentiable at 0.

(vi) Give an example of continuous function which is nowhere differentiable.

$f_0(x) = d(x, \mathbb{Z}) = \inf\{|x-k| : k \in \mathbb{Z}\}$

$f_m(x) = \frac{1}{4m} f_0(4^m x)$

$f = \lim\limits_{m\to\infty} f_m(x)$ is everywhere continuous but nowhere differentiable.



1.20.6. Definition (Higher Order Derivatives): Let I be an interval and $f : I \to \mathbb{R}$ be differentiable at $c \in I$. If f be differentiable at every point of some sub interval $I_1(c)$ such that $c \in I_1(c) \subset I$, then $f' : I_1(c) \to \mathbb{R}$ is a function on$I_1(c)$. Iff' be differentiable at c then the derivative of $f'\ at\ c$ is called second order derivative of $f\ at\ c$ and is denoted by $f''(c)\ or\ f^{(2)}(c)$.

1.20.7. Let $I \subset \mathbb{R}$ be a interval and $f : I \to \mathbb{R}$ be differentiable at $c \in I$

(i) If$f'(c) > 0$ then f is increasing at c.

(ii) If$f'(c) < 0$ then f is decreasing at c.

Example (1.91):

(i) Let $f(x) = \begin{cases} x, & x < 1 \\ 2x-1, & x \geq 1 \end{cases}$ Then f *is increasing at* 1 *but not differentiable at* 1.

(ii) $f(x) = \begin{cases} 1-x, & x < 0 \\ 1-2x, & x \geq 0 \end{cases}$ Then f *is increasing at* 0 *but not differentiable* .

(iii) If f is increasing at c then $f'(c)$ may not be positive.

Example (1.92): $f(x) = x^3, x \in \mathbb{R}$ f *is increasing at* 0, *but* $f'(0) = 0$.

(iv) If f is decreasing at c then $f'(c)$ may not be negative. $f(x) = -x^3, x \in \mathbb{R}$, f is decreasing at 0 but $f'(c) = 0$.

(v) $f'(c) > 0$ does not imply that f is monotone in a neighbourhood of c.

Example (1.93): $f(x) = \begin{cases} \frac{x}{2} - 1\, x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Then $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\left(\frac{1}{2} + x\sin\frac{1}{2}\right) = \frac{1}{2} > 0$ But is a neighbourhood of 0 f takes both positive and negative values.

1.20.8. Darboux: Let $f: I = [a,b] \to \mathbb{R}$ be differentiable on I. Let $f'(a) \neq f'(b)$. If k be a real number lying between $f'(a)$ *and* $f'(b)$ *then* $\exists$ $c \in (a,b)$ *such thta* $f'(c) = k$. [similar results as for continuous function].

Example (1.94): Let $f: [-1, 1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 0, & x \in [-1,0] \\ 1, & x \in (0,1) \end{cases}$ Does $\exists$ a function g such that $g'(x) = f(x),\ x \in [-1,1]$?

If possible, let $g: [-1,1] \to \mathbb{R}$ such that $g'(x) = f(x)$ *in* $[-1,1]$.

Then $g'(-1) = 0 \Rightarrow 1 = g'(1)$ by Dorboux theorem for every real number $\mu \in (g'(-1), g(1)) = (0,1), \exists$ $c \in [-1,1]$ *such that* $g'(c) = \mu - a$ contradiction.

1.20.9. Let I be an interval and $f: I \to \mathbb{R}$ be differentiable on I. Then $f'(I)$ is an interval.

1.20.10. If $f: [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$ then f' can not have a jump discontinuity on $[a,b]$.

1.21. Mean Value Theorem (MVT):

1.21.1. Rolle's Theorem: Let $f: [a,b] \to \mathbb{R}$ be a function such that

(i) f is continuous on $[a,b]$

(ii) f is differentiable in (a,b) and

(iii) $f(a) = f(b)$

Then $\exists$ at least one $c \in (a,b)$ *such that* $f'(c) = 0$

1.21.2. Lagrange Mean Value Theorem (MVG): Let $f: [a,b] \to \mathbb{R}$ be a function such that

(i) f is continuous on $[a,b]$ and

(ii) f is differentiable in (a, b)

(iii) $f(a) \neq f(b)$

Then $\exists$ at least one point $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

1.21.3. Let $f : [a, b] \to \mathbb{R}$satisfies (i) and (ii) of (iv) and $f'(x) = 0 \; \forall \; x \in (a, b)$ then $f(x)$ is constant on $[a, b]$.

1.21.4. Let $f, g : [a, b] \to \mathbb{R}$ satisfies (i) and (ii) of (10) and $f'(x) = g'(x) \forall \; x \in (a, b)$, then $f = g + c \; (constant)$.

Example (1.95): $\frac{x}{1+x} < \log(1 + x) < x \; \forall \; x > 0$

Let $f(x) = \log(1 + x) - \frac{x}{1+x}, x \geq 0$

$\Rightarrow f'(x) = \frac{x}{(1+x)^2} > 0 \; \forall \; x > 0 \; \Rightarrow f$ is strictly increasing

$\Rightarrow f(x) > f(0) \Rightarrow \log(1 + x) > \frac{x}{1+x}$

Let $g(x) = x - \log(1 + x) \; , x > 0$

$\Rightarrow g'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0$ strictly increasing

$\Rightarrow g(x) > g(0) \Rightarrow x > \log(1 + x) \, , x > 0$

Hence $\frac{x}{1+x} < \log(1 + x) < x \; or \; x > 0$

1.21.5. Let I be an interval. If a function $f : I \to \mathbb{R}$ be such that f' exists and is bounded on I then f is uniformly continuous on I.

[Since: $|f'(x)| \leq k \; \Rightarrow \left|\frac{f(x_2)-f(x_1)}{x_2-x_2}\right| \leq k \Rightarrow |f(x_2) - f(x_1)| \; \leq k|x_2 - x_2|$,

$lipschitz \; condition \; satisfy.$

Example (1.96):$f(x) = \frac{1}{x^2+1}, x \in \mathbb{R}$. Then $f'(x) = -\frac{2x}{(x^2+1)^2} \; , x \in \mathbb{R} \; and \; |f'(x)| < 2 \; \forall \; x \in \mathbb{R} \; \Rightarrow f \; is \; uniformly \; continuous \; on \; \mathbb{R}$.

1.21.6. Generalised MVT: Let $f, g: [a, b] \to \mathbb{R}$ such that

(i) f and g are both continuous on $[a, b]$ and

(ii) f and g are both differentiable in (a, b)

Then $\exists$ a point $c \in (a, b)$ such that $[g(b) - g(a)]f'(c) = [f(b) - f(a)]g'(c)$.

1.21.7. Cauchy's MVT: Let $f, g: [a, b] \to \mathbb{R}$ be such that satisfy (i), (ii) of (12) and (iii) $g'(x) \neq 0 \ \forall \ x \in (a, b)$. Then $\exists$ $c \in (a, b) such\ that$ $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$.

1.21.8. Leibnitz's Theorem: Let f and g be two functions each differentiable $n\ times\ at\ a$, then the n^{th} derivative of the product $fg\ at$ a given by –

$(fg)^{(n)}(a) = \sum_{r=0}^{n} \binom{n}{r} D^{n-r}f(a)D^r g(a)$ where $D^r(a) = f^r(a), r \geq 1\ and\ Df(a) = f(a)$.

1.21.9. Taylor's Theorem: Let $f : [a, a+h] \to \mathbb{R}$ be such that

(i) f^{n-1} is continuous on $[a, a+h]$, and

(ii) f^{n-1} is differentiable in $(a, a+h)$.

Then $\exists$ θ $(0 < \theta < 1)$ such that –

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta h)$$

where p is a positive integer $\leq n$.

Note: The last term $\frac{h^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta h)$ is called the remainder after $n\ terms$ and it is denoted by R_n.

Cauchy's Form: If $p - 1, R_n = \frac{h^n(1-\theta)^{n-p}}{(n-1)!}f^n(a+\theta h)$

Lagrange's Form: If $p = n, R_n = \frac{h^n}{n!}f^n(a+\theta h)$

1.21.10. Maclaurin's Theorem: Let $f : [0, h] \to \mathbb{R}$ be such that

(i) f^{n-1} is continuous on $[0, h]$ and

(ii) f^{n-1} is differentiable in $(0, h)$.

Then for $x \in (0, h], \exists\ \theta\ (0 < \theta < 1) such\ that$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n(1-\theta)^{n-p}}{p(n-1)!}f^n(\theta x)$$

Where p is a positive integer $\leq n$. For $p = 1, cauchy\ form\ and\ p = n\ lagrenge's\ form$.

Examples (1.97):

(i) Let $c \in \mathbb{R}\ and\ f: \mathbb{R} \to \mathbb{R}$ be such that f'' is continuous on some neighbourhood of c. Then $\lim_{h\to 0} \frac{f(x+h)-2f(c)+f(c-h)}{h^2} = f''(c)$.

Since f'' is continuous on $(c-\delta, c+\delta)$ for some $\delta > 0$. By Taylors theorem with Lagrange's form after remainder $(after\ 2\ terms)\ for\ any\ h\ with\ 0 < h < \delta$,

$f(c+h) = f(c) + hf'(c) + \frac{h^2}{2!}f''(c+\theta h), 0 < \theta < 1$

$f(c-h) = f(c) - hf'(c) + \frac{h^2}{2!}f''(c+\theta h), 0 < \theta' < 1$

$\therefore\ f(c+h) - 2f(c) + f(c-h) = \frac{h^2}{2!}[f''(c+\theta h) + f''(c+\theta' h)]$

$\Rightarrow \frac{f(c+h)-2f(c)+f(c-h)}{h^2} = \frac{1}{2}[f''(c+\theta h) + f''(c+\theta' h)]$

$\Rightarrow \lim_{h\to 0} \frac{f(c+h)-2f(c)+f(c-h)}{h^2} = f''(c)\ [\because f'' is\ continuous\ at\ c]$

(ii) Use Taylor's Theorem, $1 + \frac{x}{2} - \frac{x^2}{8} < \sqrt{1+x} < 1 + \frac{x}{2}, if\ x > 0$

Let $f(x) = \sqrt{1+x}, x \geq 0$ Then –

$f'(x) = \frac{1}{2\sqrt{1+x}}, f''(x) = -\frac{1}{4(1+x)^{\frac{3}{2}}}, f'''(x) = \frac{3}{8(1+x)^{\frac{5}{2}}}$

By Taylor's theorem with Lagrange's form of remainder $(after\ 3\ terms)$ for any$x > 0$.

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(c)\ for\ some\ c \in (0,x)$

$or, \sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16(1+c)^{\frac{5}{2}}} \Rightarrow \sqrt{1+x} > 1 + \frac{x}{2} - \frac{x^2}{8}\ (\because x > 0)$

By Taylor's theorem with Lagrange's form of remainder $(after\ 2terms)\ for\ any\ x > 0$.

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(d)\ \ for\ some\ d \in (0,x)$

$or, \sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8(1+d)^{\frac{3}{2}}} \Rightarrow \sqrt{1+x} < 1 + \frac{x}{2}\ (\because x > 0)$

1.21.11. Taylor's Infinite Series:

Let $a \in \mathbb{R}\ and\ f$ defined on some neighbourhood $N(a)$ of a such that f^{n-1} is differentiable on $N(a)$. Then for any $x \in N(a) - \{a\},\ f(x) = P_n(x) + R_n(x)$,

where $R_n(x)$ is the remainder after $n\ terms$ and

$P_n(x) = f(a) + (x-a)f'(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a)$. $P_n(x)$ is a polynomial of degree $n-1$ and $P_n(x)$ is such that –

$P_n(a) = f(a),\ P'_n(a) = f'(a),\ P_n'(a) = f''(a), \ldots\ldots\ldots\ P_n{}^{n-1}(a) = f^{n-1}(a). P_n(x)$ is called the nth Taylor Polynomial of f about the point a. If for all $n,\ f^n$ exists on $N(a)$, then $P_n(x)$ be an infinite series $f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''$(a) + ... which is convergent if $\{P_n(x)\}$ is convergent and if $\lim_{n\to\infty} R_n(x) = 0$ then we have –

$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots\ldots\ldots$

If $a = 0$, we have Maclaurin's infinite series

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots\dots\dots$

1.21.12. Expansion of some functions:

(i) Let $f(x) = e^x$, $x \in \mathbb{R}$. Then $f^n(x) = e^x, \forall\, x \in \mathbb{N}$. By Taylor's theorem with Lagrange's form of remainder after $n\ terms$ $\forall\ 0 \neq x \in \mathbb{R}$,

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots\dots .+ \frac{x^{n-1}}{(x-1)!}f^{n-1}(0) + R_n(x)$. where

$R_n(x) = \frac{x^n}{n!}f^n(\theta x), 0 < \theta < 1.$

$= \frac{x^n}{n!}e^{\theta x}$.

Let $u_n(x) = \frac{x^n}{n!}e^{\theta x}, \Rightarrow \lim\limits_{x\to\infty}\left|\frac{u_n+1}{u_n}\right| = \lim\limits_{x\to\infty}\frac{|x|}{n+1} = 0$

$\Rightarrow \lim\limits_{x\to\infty}|R_n(n)| = 0$

$\therefore\ e^x = 1 + x + \frac{x^2}{2!} + \cdots\cdots \forall\ x \in \mathbb{R}.$

(ii) $f(x) = \sin x\,, x \in \mathbb{R}$. Then$f^n(x) = \sin\left(\frac{n\pi}{2} + x\right)$,

$\therefore f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots\dots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + R_n(x), (n \neq 0)$

Where $R_n(x) = \frac{x^n}{n!}f^n(\theta x) = \frac{x^n}{n!}\sin\left(\frac{n\pi}{2} + \theta x\right), 0 < \theta < 1$

$\lim\limits_{n\to\infty}|R_n(x)| = \lim\limits_{n\to\infty}\frac{|x|^n}{n!}\left|\sin\left(\frac{n\pi}{2} + \theta x\right)\right| \leq \lim\limits_{n\to\infty}\frac{|x|^n}{n!} = 0\ \left(\because \frac{u_n+1}{u_n} = \frac{|x|}{n+1}\right)$

$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\dots\dots, \forall\, x \in \mathbb{R}.$

(iii) $f(x) = \log(1+x)\,, x > -1, f^n(x) = \frac{(-1)^{n-1}(n-1)^1}{(1+x)^n}\ \ for\ \ x > -1$

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\dots\dots\ for\ x \in (-1,1)$

(iv) e is irrational :

$e^x = 1 + x + \frac{x^2}{2!} + \dots\dots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}e^\theta, 0 < \theta < 1\ [by(i)]$

$\Rightarrow e = 1 + 1 + \frac{1}{2!} + \dots\dots\dots + \frac{1}{(n-1)!} + \frac{1}{n!}e^\theta$

$\Rightarrow e > 2$ and $0 < e^\theta < e < 3\ (\because 0 < \theta < 1)$

Let e be rational, then $\exists\ p, q \in \mathbb{Z}\ with\ \gcd(p,q) = 1\ and\ p, q > 0$ such that $e = \frac{p}{q}$

Let $n > q\ , then$

$$\underbrace{\frac{p(n-1)!}{q}}_{(integer)} - \underbrace{(n-1)!\left\{1 + 1 + \frac{1}{2!} + \dots\dots + \frac{1}{(n-1)!}\right\}}_{[integer]} = \frac{e^\theta}{n}$$

$\Rightarrow \frac{e^{\theta}}{n}$ is an integer.

But$0 < e^{\theta} < e < 3 < n \Rightarrow 0 < \frac{e^{\theta}}{n} < 1$ $(Proper\ fraction)$,

$\Rightarrow e$ is irrational $[e = 2.7182818284$ (correct upto 10 decimal places.)]

1.22. Maximum and Minimum:

1.22.1. Global maximum and global minimum:

Let I be an interval and $f : I \to \mathbb{R}$ be a function$f$is said to have a global maximum (or minimum) on$I$ if $\exists$ a point$c \in I$ such that $f(c) \geq f(x)[respectively\ minimum)$ point for $f\ on\ I$.

f is said to have a local maximum (or minimum) at a point$c \in I$ if $\exists$ a neighbourhood $N_{\delta}(c)$ of$c$ such that $f(c) \geq f(x)[respectively\ \ f(c) \leq f(x)]\ \forall\ x \in N_{\delta}(c) \cap I$.

1.22.2. Let $f: I \to \mathbb{R}$ be such that of has a local extremum at an interior point $c \in I$. If $f'(c)$ exists then $f'(c) = 0$.Converse is not true.

1.22.3. Corollary: Let $f: I \to \mathbb{R}$ and$c \in I$, where $f$has local minimum. Then either $f'(c)$ does not exists or $f'(c) = 0$.

Example (1.98.):

(i) $f(x) = |x|, x \in \mathbb{R}$ has local minimum at $x = 0, but\ f'(0)$ does not exist.

(ii) Let $f(x) = x^3, x \in \mathbb{R}$. Then $f'(0) = 0\ but\ 0\ is\ not\ an\ extremum\ point$.

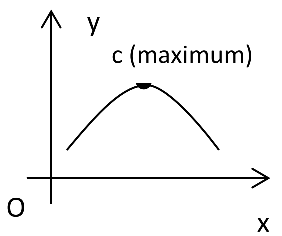
(iii) (interior condition of$c$ is in necessary). Let $f(x) = x, x \in [0,1]$. f has minimum at 0 and maximum at 1but$f'(0) = 1 = f'(1) \neq 0$.

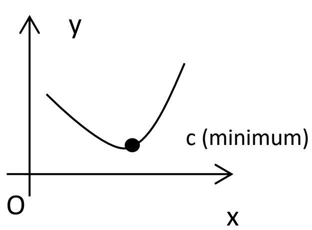
1.22.4. [First derivative Test for extrem a]

Let $f: I = [a, b] \to \mathbb{R}$ continuous and c be and interior point of$I$ and let f be differentiable on (a, c) and (c, b). Then $-$

(i) If $\exists$ a neighbourhood $(c - \delta, c + \delta) \subset I$ such that for $x \in (c, \delta, c), f'(x) \geq 0$ $(or, f'(x) \leq 0)\ \ and\ for\ \ x \in (c, c + \delta),\ \ f'(x) \leq 0\ (respectively\ f'(x) \geq 0)$ the f has local maximum (respectively local minimum) at c.

(ii) If $f'(x)$ keeps the same sign on $(c - \delta, c)$ and $(c, c + \delta)$ then f has no extremumat c.Converse is not true.





Example (1.99):

Let$f(x) = \begin{cases} 2x^2 + x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Then$f$ has local minimum at 0.

$f'(x) = \begin{cases} 4x + 2x \sin\frac{1}{x} - \cos\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

f' takes both positive and negative values on both since of 0 (in any neighbourhood of 0).

1.22.5. Higher Order Derivative test for extreme:

Let $f: I \to \mathbb{R}$ *and* $c$be an interior point of$I$.

If $f'(c) = f''(c) = f'''(c) = \ldots. = f^{n-1}(c)$ *and* $f^n(c) \neq 0$, then f has

(i) no extremum at c *if* n be odd, and

(ii) a local extremum at c *if* n be even;

a local maximum if $f''(c) < 0$, a local minimum *if* $f^n(c) > 0$.

Example (1.100):$f(x) = x^5 - 5x^4 + 5x^3 + 10$

$f'(x) = 5x^4 - 20x^3 + 15x^2 = 0 \Rightarrow x = 0, 1, 3$

$f''(x) = 20x^3 - 60x^2 + 30x$

$f'''(x) = 60x^2 - 120x$

$f^{iv}(x) = 120x,\ f^{v}(x) = 120$

Now, $At\ x = 0, f'(0) = 0, f''(0) = 0, f'''(0) = 0, f^{iv}(0) = 0, f^{v}(x) \neq 0$, so no extremum.

$At\ x = 1, f(1) = 0, f''(1) < 0, f\ has\ maximum\ at\ x = 1$

$At\ x = 3, f'(3) = 0, f''(3) > 0, f\ has\ minimum\ at\ x = 3.$

1.22.6. Indeterminate forms: Let $\lim_{x\to c} f(x) = l$ *and* $\lim_{x\to c} g(x) = m \neq 0$, then –

$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{l}{m}$. But if $l = m = 0$, in this case the limit of quotient $\frac{f}{g}$ is said to take the indeterminate form$\frac{0}{0}$.

Note: Other indeterminate forms are $\frac{\infty}{\infty}, \infty - \infty, 0 \cdot \infty, 0^0, 1^\infty, 1^{-\infty}, \infty^0$

1.22.7. Let $c \in \mathbb{R}$ *and* $f, g: \mathbb{R} \to \mathbb{R}$ be two functions such that $f(c) = g(c) = 0$ *and* $g(x) \neq 0$ in some deleted neighbourhood$N'_\delta(c)$ *of* c *and* f, g are differentiable at c and $g'(c) \neq 0$. Then $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$.

1.22.8. If $f, g : [a, b] \to \mathbb{R}$ and $f(a) = g(a) = 0, g(x) \neq 0$ on(a, b) and f, g are differentiable at a and $g'(a) \neq 0$. Then $\lim_{n\to a+}\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$

Example (1.101): $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, x \neq 0 \text{ and } g(x) = \sin x, x \in \mathbb{R} \\ 0 \quad\quad\quad , \; x = 0 \end{cases}$

Then $f(0) = 0 = g(0), g(x) \neq 0$ is some deleted neighbourhood of 0 $and\ f'(0)\ and\ g'(0)$ both exist and$g'(0) = 1 \neq 0$. So, $\lim_{n \to a} \frac{f(x)}{g(x)} = \frac{f\prime(0)}{g\prime(0)} = 0$

1.22.9. L' Hospital Rule: Let$c \in \mathbb{R}$ and $f, g : \mathbb{R} \to \mathbb{R}$ be such that $f^n(x), g^n(x)$ exist a_n some neighbourhood of $N'_\delta(c) and\ g^n(x) \neq 0$ on $N'_\delta(c)$ and

$$\lim_{x \to c} f(x) = \lim_{x \to c} f'(x) = \ldots\ldots\ldots \lim_{x \to c} f^{x-1}(x) = 0$$

$$\lim_{x \to c} g(x) = \lim_{x \to c} g'(x) = \ldots\ldots\ldots \lim_{x \to c} g^{x-1}(x) = 0$$

Then if $\lim_{x \to c} \frac{f^n(x)}{g\prime(x)}$ exists in $\mathbb{R}$, then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f^n(x)}{g^n(x)}$.

Example (1.102):

$\lim_{x \to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x \sin x} \left(\frac{0}{0}\right)$

$= \lim_{x \to 0} \frac{e^x - e^{-x} - \frac{2}{1+x}}{x \cos x + \sin x} \left(\frac{0}{0}\right)$

$= \lim_{x \to 0} \frac{e^x - e^{-x} + \frac{2}{(1+x)^2}}{-x \sin x + 2 \cos x} = 1$

1.23. Functions of Bounded Variation:

Definition: Let $[a, b]$be a closed and bounded interval and$f : [a, b] \to \mathbb{R}$ be a function. Let $P = \{x_0, x_1, \ldots\ldots\ldots, x_n\}\ where\ a = x_0 < x_1 < x_2 < \ldots\ldots\ldots < x_n = b$, be a partition of $[a, b]$. Let us consider the sum

$V(P, f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \ldots\ \ldots\ldots + |f(x_n) - f(x_{n-1})|$

$= \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$

For different partitions $P \in \wp[a, b], V(P, f)$ given a set of non-negative numbers. If the set $\{V(P, f): P \in \wp[a, b]\}$ be bounded above, then f is said to be a function of bounded variation on $[a, b]$.

The supremum of the set $\{V(P, f): P \in \wp[a, b]\}$ is said to be the total variation of $f\ on\ [a, b]$ and is denoted by $V_f[a, b]$.

Example (1.103):

(i) Let $k \in \mathbb{R}, f(x) = k\ \forall\ x \in [a, b] \Rightarrow V(P, f) = 0\ \forall\ P \in \wp[a, b] \Rightarrow V_f[a, b] = 0 \Rightarrow f$ is a function of bounded variation on $[a, b]$.

(ii) $f(x) = x, x \in [a, b] \Rightarrow V_f[a, b] = b - a < \infty$

(iii) $f(x) = \sin x, x \in [a, b], V_f[a, b] \leq (b - a)$ $(\because\ |\sin x_2 - \sin x_1| \leq |x_2 - x_1|)$

(iv) **Not a Function of bounded variation:**

Let $f:[0,1]\to\mathbb{R}$ be defined by $f(x)=\begin{cases}1 \ if \ x\in\mathbb{Q}\\ 0 \ if \ x\in\mathbb{R}|\mathbb{Q}\end{cases}$

Let $P=\{x_0,x_1,\ldots\ldots,x_{2n}\}$ be a partition of $[a,b]$ such that $x_0,x_2,\ldots\ldots,x_{2n}$ are all rational and $x_0,x_3,\ldots\ldots,x_{2n-1}$ are all irrational. Then

$V(P,f)= |f(x_1)-f(x_0)|+\ldots.\ \ldots\ldots+|f(x_{2n})-f(x_{2n-1})|=2n\to\infty \ as \ x\to\infty$

1.23.1. Let $f:[a,b]\to\mathbb{R}$ be a function of bounded variation on $[a,b]$. Then f is bounded on $[a,b]$. converse is not true.

Example (1.104):

(i) $f(x)=\begin{cases}1 \ , x\in\mathbb{Q} & ,x\in[0,1]\\ 0 \ , & x\in\mathbb{R}|\mathbb{Q}\end{cases}$

(ii) $f:[0,1]\to\mathbb{R}$ defined by $f(x)=\begin{cases}x\ \cos\frac{\pi}{2k}, x\neq 0\\ 0 \ , \ x=0\end{cases}$

Then $|f(x)|\leq 1 \ \forall\, x\in[0,1]$

Let $P=\{0,\frac{1}{2n},\frac{1}{2n-1},\ldots\ldots.\frac{1}{2},1\}$ be a partition of $[0,1]$

Then $f\left(\frac{1}{2r}\right)=\frac{1}{2r}\cos\left(\frac{r\pi}{2}\right)=\frac{1}{2r}(-1)^r$ for $r=1,2,\ldots\ldots,n$

And $f\left(\frac{1}{2r-1}\right)=\frac{1}{2r-1}\cos\frac{(2r-1)\pi}{2}=0 \ for \ r=1,2,\ldots\ldots,n$

$\therefore V(p,f)= \left|f\left(\frac{1}{2n}\right)-f(0)\right|+\ \ldots\ldots.+\left|f(1)-f(\tfrac{1}{2})\right|$

$$=\frac{1}{2n}+\frac{1}{2n}+\frac{1}{2n-2}+\frac{1}{2n-2}+\ldots\ldots+\frac{1}{2}+\frac{1}{2}=1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}\to\infty \ as \ x\to\infty$$

1.23.2. Let $f:[a,b]\to\mathbb{R}$ be monotone on $[a,b]$. Then f is a function of bounded variation on $[a,b]$. Converse is not true.

Example (1.105): $f(x)= \sin x, x\in[a,b]$.

1.23.3. Let $f:[a,b]\to\mathbb{R}$ be a Lipschitz function on $[a,b]$. Then f is a function of bounded variation on $[a,b]$. Converse is not true.

Example (1.106): $f:[0,1]\to\mathbb{R}$ be defined by $f(x)=\sqrt{x},\ x\in[0,1]$.

$\Rightarrow f$ is monotone increasing on $[0,1]$ $\Rightarrow f$ is a function of bounded variation on $[0,1]$ but f is not Lipschitz function on $[0,1]$.If since, for $x_1=0$, there is no $M\in\mathbb{R}$ such that

$$|f(x_2)-f(x_1)|\leq M|x_2-x_1|\forall\, x_2[0,1].$$

1.23.4. Let $f:[a,b]\to\mathbb{R}$ continuous on $[a,b]$, f' exists and be bounded on (a,b). Then f is a function of bounded variation on $[a,b]$.

Note-I: Boundedness of f' on $[a,b]$ is not necessary.

Example (1.107): $f(x) = \sqrt{x}, x \in [0,1]$ is a function of bounded variation on $[0,1]$ as it is monotonic increasing but $f'(x) = \frac{1}{2\sqrt{x}}, x \in (0,1)$ is not abounded on $(0,1)$.

Note-II: A function f continuous and bounded on a closed interval $[a, b]$ may not be a function of bounded variation on $[a, b]$

Example (1.108): $f(x) = \begin{cases} x \cos\frac{\pi}{2x}, & x \in (0,1) \\ 0, & x = 0 \end{cases}$

1.23.5. Let $f, g : [a, b] \to \mathbb{R}$ be functions of bounded variation on $[a, b]$. Then–

(i) $f + g$ is also so and $V_{f+g} \le V_f + V_g$

(ii) $f - g$ is also so and $V_{f-g} \le V_f + V_g$

(iii) cf $(c \in \mathbb{R})$ is also so.

(iv) fg is also so and $V_{fg} \le A\,V_f + B\,V_g$, $A = \sup\{|g(x)| = x \in [a, b]\}$,

$$B = \sup\{|f(x)| : x \in [a, b]\}$$

(**Note:** The close S of all $BV - functions$ on $[a, b]$ form a real vector space)

(v) If $\exists\, k \in \mathbb{R}$ such that $0 < k \le f(x)\ \forall\ x \in [a, b]$, then $\frac{1}{f}$ is a $BV - function$ on $[a, b]$ and $V_{\frac{1}{f}} \le \frac{v_f}{k^2}$

(vi) $|f|$ is also so.

1.23.6. Definition (Refinement of partition): Let $P = \{x_0, x_1, \dots\dots\dots, x_n\}$ be a partition of $[a, b]$. A partition $\mathbb{Q}$ of $[a, b]$ is said to be a refinement of $P. P$ is a proper subset of $\mathbb{Q}$.

Example (1.109): $P = \{0, 1.4, \frac{1}{2}, \frac{3}{4}, 1\}$ is a partition of $[a, 1] and$ $\mathbb{Q} = \left\{0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{3}{4}, 1\right\}$ then $Q\ is\ a\ refinement\ of\ P.$

1.23.7. Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation a $[a, b] and\ P$ be a partition of $[a, b]$. If Q be a refinement of P then $V(Q, f) \ge V(P, f)$

1.23.8. Let $f : [a, b] \to \mathbb{R}$ be a function on $[a, b]\ and\ c \in (a, b)$ then –

(i) f is bounded variation on $[a, c] and\ on\ [c, b]$

(ii) $V_f[a, b] = V_f[a, c] + V_f[c, b]$

1.23.9. Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on $[a, c]\ and\ on\ [c, b] where\ c \in (a, b). Then -$

(i) f is of bounded variation on $[a, b]$

(ii) $V_f[a, c] + V_f[c, b] = V_f[a, b]$

Example (1.110): Let $f:[0,3] \to \mathbb{R}$ be defined by $f(x) = x^2 - 4x + 3, x \in [0,3]$.

$f'(x) = 2x - 4$. So $f'(x) < 0$ for $x \in [0,2]$ and $f'(x) > 0$ for $x \in [2,3] \Rightarrow f$ is decreasing on $[0,2]$ and increasing on $[2,3] \Rightarrow f$ is a $BV - function$ on $[0,3]$.

$V_f[0,2] = f(0) - f(2) = 4$ and $V_f[2,3] = f(3) - f(2) = 1$

$\therefore V_f[0,3] = V_f[0,2] + V_f[2,3] = 5$

1.23.10. Let $f : [a,b] \to \mathbb{R}$ be a $BV - function$ on $[a,b]$ and $\phi : [a,b] \to \mathbb{R}$ be such that ϕ is bounded on $[a,b]$ and $\phi(x) = f(x)$ except at a finite number of points in $[a,b]$, then ϕ is a $BV - function$ in $[a,b]$.

Example (1.111):

Let $f:[0,3] \to \mathbb{R}$ be defined by $f(x) = x - [x], x \in [1,3]$

$$f(x) = \begin{cases} x-1, & 1 \le x < 2 \\ x-2, & 2 \le x < 3 \\ 0, & x = 3 \end{cases}$$

Let$\phi_1:[1,2] \to \mathbb{R}$ be defined by $\phi_1(x) = x - 1, x \in [1,2]$

$\phi_2 : [2,3] \to \mathbb{R}$ be defined by $\phi_2(x) = x - 2, x \in [2,3]S$

Then ϕ_1 is increasing on $[1,2]$ and ϕ_2 is function of bounded variation on$[2,3]$.

Hence $f(x) = \phi_1(x) + \phi_2(x)$ $x \in [1,3]$ $except$ $x = 2,3$. Hence $f(x)$ is a function of bounded variation on $[1,3]$.

1.23.11. Definition (Variation Function): Let $f:[a,b] \to \mathbb{R}$ be function of bounded variation on $[a,b]$$and$ $x \in (a,b]$. Then V_f $[a,x]$ is a function of $x\ \forall\ x \in [a,b]$. Let $V:[a,b] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} V_f[a,b], a < x \le b \\ 0 \quad\quad , \ x = a \end{cases}$$

V is called the variation function of f on $[a,b]$

Note: (i) V is monotone increasing on $[a,b]$

(ii) $V + f$ and $V - f$ are also monotone increasing on $[a,b]$,

1.23.12. Let $f : [a,b] \to \mathbb{R}$ be a function. Then f is a function of bounded variation on $[a,b]$ $\Leftrightarrow f$ can be expressed as the difference of two monotone increasing functions on $[a,b]$.

Example (1.112):

Let $f : [-1,1] \to \mathbb{R}$ be defined by $f(x) = x^2, x \in [-1,1]$.

Then $f'(x) = 2x$ and so $f'(x) < 0, x \in [-1,0]$$and$ $f'(x) > 0, x \in \{0,1]$ $\Rightarrow f$ decreasing on $[-1,0]$ and increasing on $[0,1]$ $\Rightarrow f$ is $BV - function$ on $[-1,0]$ and $[0,1]$ hence on $[-1,1]$. $V(-1) = 0$

If $-1 < x \leq 0, then\ V(x) = V_f[-1, x] = f(-1) - f(x) = 1 - x^2$. Since f is decreasing on $(-1,0)$.

1.23.12. If $0 < x \leq 1, then\ V(x) = V[-1, x] = V_f[-1,0] + V_f[0, x]$

$= f(-1) - f(0) + f(x) - f(0),\ since\ f\ is\ increasing\ on\ [0,1]$

$= 1 + x^2$

Therefore, $V(x) = \begin{cases} 1 - x^2, & -1 \leq x \leq 0 \\ 1 + x^2, & 0 < x \leq 1 \end{cases}$ and $V(x)$ is increasing on $[-1,1]$.

$(V + f)(x) = \begin{cases} 1, & -1 \leq x \leq 0 \\ 1 + 2x^2, & 0 < x \leq 1 \end{cases}$ $\Rightarrow V + f$ is a monotone increasing on $[-1,1]$.

$\therefore f = (V + f) - V$, the difference of two monotone increasing functions.

1.23.13. Let $f : [a, b] \to \mathbb{R}$ be a $BV - function$ on $[a, b]$ then f can have only discontinuity of first kind and the points of discontinuity of f form a countable set.

1.23.14. Let $f: [a, b] \to \mathbb{R}$ be a $BV - function\ on\ [a, b]$ and let V be the variation function on $[a, b]$. If f be continuous at a point $c \in [a, b]$ then V is continuous at c and conversely.

1.23.15. Corollary: If $f : [a, b] \to \mathbb{R}$ be continuous and be of bounded variation on $[a, b]$ then f can be expressed as the difference of two monotone and continuous functions on $[a, b]$ and conversely.

1.23.16. Definition (Positive Variation and Negative Variation):

Let $f : [a, b] \to \mathbb{R}$ be a $BV - function$ on $[a, b] and\ P = \{x_0, x_1, \ldots\ldots\ldots, x_n\}$ be a partition of $[a, b]$.

$V(P, t) = |\Delta f_1| + \ldots\ldots + |\Delta f_n|\ where\ \Delta f_r = f(x_r) - f(x_{r-1}), r = 1, 2, \ldots\ldots\ldots, n$

Let $V_+(P, f) = \sum_{\Delta f_i > 0} |\Delta f_i|\ and\ V_-(P, f) = \sum_{\Delta f_i < 0} |\Delta f_i|$ Then–

$V_+(P, f) - V_-(P, f) = f(b) - f(a)$

$V_+(P, f) + V_-(P, f) = V(P, f)$

and $\sup_p \{V_+(P, f) : P \in p[a, b]\} = P_f[a, b]\ or\ (V_+)_f[a, b]$ is called positive variation of f on $[a, b]$ and $\sup_p \{V_-(P, f) : P \in p[a, b]\} = n_f[a, b]\ or\ (V_-)_f[a, b]$ is called negative variation of f on $[a, b]$.

Positive variation function V_+ *or* $p(x)$

$$p(x) = V_+(x) = \begin{cases} P_f[a,x], & x \in [a,b] \\ 0 \quad , & x = 0 \end{cases}$$

Negative variation function V_- *or* $n(x)$:

$$n(x) = V_-(x) = \begin{cases} n_f[a,x], & x \in [a,b] \\ 0 \quad , & x = 0 \end{cases}$$

Note: $p(x)$ *and* $n(x)$ are monotone increasing on $[a,b]$.

1.23.17. Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$. Then –

(i) $p(x) + n(x) = V(x)\ \forall\ x \in [a,b]$

(ii) $p(x) - n(x) = f(x) - f(a)\ \ \forall\ \ x \in [a,b]$.

$\Rightarrow p(x) = \frac{1}{2}[V(x) + f(x) - f(a)$

$n(x) = \frac{1}{2}\ [V(x) - f(x) + f(a)$

Example (1.113): Let $f : [-1,1] \to \mathbb{R}$ be defined by $f(x) = x^2,\ \forall\, x \in [-1,1]$.

Then f is $BV - function$ on $[-1,1]$ and $V(x) = \begin{cases} 1 - x^2, & -1 \le x \le 0 \\ 1 + x^2, & 0 < x \le 1 \end{cases}$

$$\therefore p(x) = \begin{cases} \frac{1}{2}[1 - x^2 + x^2 - 1] = 0, & -1 \le x \le 0 \\ \frac{1}{2}[1 + x^2 + x^2 - 1] = x^2, & 0 < x \le 1 \end{cases}$$

$$\therefore n(x) = \begin{cases} \frac{1}{2}[1 - x^2 - x^2 + 1] = 1 - x^2, & -1 \le x \le 0 \\ \frac{1}{2}[1 + x^2 - x^2 + 1] = 1, & 0 < x \le 1 \end{cases}$$

1.24. Riemann Integral:

Let $[a,b]$ be a closed bounded interval and $f : [a,b] \to \mathbb{R}$ be a bounded function on $[a,b]$. Let $P = \{x_1, x_2 \ldots\ldots x_n\}$ be a partition of $[a,b]$.Then f is bounded on each $Ir = [x_{r-1}, x_r]$ for $r = 1, 2, \ldots\ldots n$

Let $M_r = \sup\limits_{x \in Ir} f(x)$, $m_r = \inf\limits_{x \in Ir} f(x)$, $M = \sup\limits_{x \in [a,b]} f(x)$, $m = \inf\limits_{x \in [a,b]} f(x)$

Then $m \le m_r \le M_r \le M$ for $r = 1, 2, \ldots\ldots.n$ …….. (i)

$U(P,f) = \sum_{r=1}^{n} M_r(x_r - x_{r-1})$ = Upper Darbou $x\ sum\ of\ f\ corresponding\ to\ P$……(ii)

$L(P,f) = \sum_{r-1}^{n} m_r(x_r - x_{r-1})$ = Lower Darbou $x\ sum\ of\ f\ corresponding\ to\ P$.

Now,

(i) $\Rightarrow\ \ m(x_r - x_{r-1}) \le m_r(x_r - x_{r-1}) \le M_r(x_r - x_{r-1}) \le M(x_r - x_{r-1})$

$\Rightarrow\ \ m\sum_{r-1}^{n}(x_r - x_{r-1}) \le \sum_{r-1}^{n} m_r(x_r - x_{r-1}) \le \sum_{r-1}^{n} M_r(x_r - x_{r-1}) \le \sum_{r-1}^{n} M\,(x_r - x_{r-1})$

$\Rightarrow\ \ m(b-a) \le L(P,f) \le U(P,f) \le M(b-a) \ldots\ldots\ldots (b)$ (ii)

If $sup\ \{L(P,f) : P \in \wp[a,b]$ exists, it is called the lower integral of f on $[a,b]$ and is denoted by $\underline{\int_a^b} f dx = \underline{\int_a^b} f$

And if $inf\ \{U(P,f)\ :\ P \in \wp[a,b]\}$ exists, it is called the upper integral of f on $[a,b]$ and is denoted by $\overline{\int_a^b} f dx = \overline{\int_a^b} f$.

f is said to be Riemann integrable on $[a,b]$ if $\underline{\int_a^b} f = \overline{\int_a^b} f$ and the common value $\underline{\int_a^b} f$ or $\overline{\int_a^b} f$ is called the Reimann integral of f on $[a,b]$ and is denoted by $\int_a^b f(x)dx$ or $\int_a^b f$

We also define $\int_a^a f = 0$ and $\int_a^b f = -\int_b^a f$

Note-1: $m(b-a) \leq \underline{\int_a^b} f \leq M(b-a),\ \ m(b-a) \leq \overline{\int_a^b} f \leq M(b-a)$

Note-2: The class of all Riemann integrable function on $[a,b]$ is denoted by $R[a,b]$and $R[a,b] \subset B[a,b]$. The class of functions of bounded variation on $[a,b]$.

Example (1.114):

(i) Let $f: [a,b] \to \mathbb{R}$ be defined by $f(x) = c,\ \ x \in [a,b]$

Take$p = \{x_0,\ x_1, \ldots\ldots. x_n\}$ be a partition of $[a,b]$. Then $Mr = c = mr$

$\Rightarrow U(P,f) = c(x_1 - x_0) + c(x_2 - x_1) + \ldots\ldots + c(x_n - x_{n-1}) = c(b-a)$

$L(P,f) = c(b-a)$

$\Rightarrow inf\ \{U(P,f)\ :\ P \in P[a,b]\} = (b-a) = sup\{L(P,f): P \in P[a,b]\}$

$\Rightarrow f$ is Riemann integrable on $[a,b]$ and $\int_a^b f(x)dx = c(b-a)$.

(ii) Let $f: [0,1] \to \mathbb{R}$ be define by $f(x) = \begin{cases} 1 & , \quad x \in \mathbb{Q} \\ 0 & , \quad x \in \mathbb{R}|\mathbb{Q} \end{cases}$

Let $P = \{x_0,\ x_1, \ldots\ldots\ldots x_n\}$ be a partition of $[0,1]$ Then $M_r = 1, m_r = 0$

$\therefore U(P,f) = \sum_{r=1}^n M_r(x_r - x_{r-1}) = 1(1-0) = 1$

$L(P,f) = 0$

$\therefore inf\{U(P,f)\ : P \in \wp[a,b]\} = 1 \neq 0 = \sup\{L(P,f) : P \in \wp[a,b]\}$

Hence f is not Riemann integrable on $[0,1]$.

1.24.1. Let $f: [a,b] \to \mathbb{R}$ be bounded on $[a,b]$ and P be a partition of $[a,b]$.

If Q be a refinement of P, then

$U(P,f) \geq U(Q,f)$and $L(P,f) \leq L(Q,f)$

$\Rightarrow L(P,f) \leq L(Q,f) \leq U(Q,f) \leq U(P,f)$.

1.24.2. Definition (Noun of a Partition): Let $[a,b]$ be a closed and bounded interval and $P = \{x_0, x_1, \ldots\ldots\ldots x_n\}$ be a partition of $[a,b]$. The norm of P is denoted by $\|P\|$ and in defined by $\|P\| = \max\{(x_1 - x_0), (x_2 - x_1), \ldots\ldots\ldots, (x_n - x_{n-1})\}$

Note: If Q be a refinement of P. Then $\|Q\| \leq \|P\|$

1.24.3. Let $f:[a,b] \to \mathbb{R}$ be bounded on $[a,b]$ and P a partition of $[a,b]$ with $\|P\| = \delta$ if P_k be a refinement of P with k additional Point of Partition, then

$0 \leq U(P,f) - U(P_k,f) \leq (M-m)k\delta,$

$0 \leq L(P_k,f) - L(P,f) \leq (M-m)k\delta$

1.24.4. Let $f:[a,b] \to \mathbb{R}$ be bounded on $[a,b]$ and $p, \mathbb{Q}$ be any two partitions of $[a,b]$. Then $L(P,f) \leq U(Q,f); L(Q,f) \leq U(P,f)$

$$\Rightarrow \underline{\int_a^b} f \leq \overline{\int_a^b} f \Rightarrow m(b-a) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq M(b-a)$$

Example (1.115): Let $f:[a,b]$ be defined by $f(x) = x, x \in [a,b]$ consider

$P_n = \{a, a+h, a+2h, \ldots\ldots, a+nh\}$ be a partition of$[a,b]$ here $h = \frac{b-a}{n}$

$$\therefore M_r = \sup_{x \in [a+(r-1)h,\ a+rh]} f(x) = a + rh\,,$$

$$m_r = \sup_{x \in [a+(r-1)h,\ a+rh]} f(x) = a + (r-1)h$$

$$\therefore U(P_n,f) = h[(a+h) + (a+2h) + \ldots\ldots\ldots + (a+2h)$$

$$= h\,[na + h(1 + 2 + \ldots\ldots + n)]$$

$$= nha + \frac{nh(nh+a)}{2}$$

$$= a(b-a) + \frac{(b-a)}{2}\left(b - a + \frac{b-a}{n}\right)$$

$$= ab - a^2 + \frac{1}{2}(b-a)^2\left[1 + \frac{1}{n}\right] \to ab - a^2 + \frac{1}{2}(b-a)^2 = \frac{b^2-a^2}{2} \text{as} n \to \alpha$$

$$L(P_n,f) = h[a + (a+h) + \ldots\ldots + a + (n-1)h$$

$$= h[na + h(1 + 2 + \ldots\ldots + (n-1)]$$

$$= nah + h\frac{(n-1)nh}{2}$$

$$= a(b-a) + \frac{(b-a)}{2}\left(b - a - \frac{b-a}{n}\right) \to ab - a^2 + \frac{1}{2}(b-a)^2 = \frac{r-a^2}{2} \text{ as } n \to \alpha$$

$$\therefore \int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f = \frac{b^2-a^2}{2}$$

1.24.5. Condition for integrability: Let $f:[a,b] \to \mathbb{R}$ be bounded on $[a,b]$. Then f is integrable on $[a,b] \Leftrightarrow$ for each $\varepsilon > 0, \exists$ a partition P of $[a,b]$ such that

$U(P,f) - L(P,f) < \varepsilon$

1.24.6. Darboux Theorem: Let $[a,b]$ be a closed and bounded interval and $f:[a,b] \to \mathbb{R}$ be bounded on $[a,b]$. Then-

To each pre-assigned $\varepsilon > 0 \quad \exists \quad \delta > 0$ such that

$U(P,f) < \overline{\int_a^b} f + \varepsilon \quad \forall\, P$ of $[a,b]$ with $\|P\| \leq \delta$ and

$L(P, f) > \underline{\int_a^b} f - \varepsilon \quad \forall\ P$ of $[a, b]$ with $\|P\| \le \delta$

1.24.7. Let $f: [a, b] \to \mathbb{R}$ be monotone on $[a, b]$. If $\{P_n\}$ be a sequence of partitions of $[a, b]$ such that the sequence $\{\|P_n\|\}$ converge to 0, then –

(i) $\lim_{n\to\infty} U(P_n, f) = \overline{\int_a^b} f$ and

(ii) $\lim_{n\to\infty} L(P_n, f) = \underline{\int_a^b} f$

1.24.8. Some Riemann integrable functions:

(i) Let $f: [a, b] \to \mathbb{R}$ be monotone on $[a, b]$. Then f is integrableon$[a, b]$.

(ii) Let $f: [a, b] \to \mathbb{R}$ be continuous on$[a, b]$. Then f is integrable on$[a, b]$.

(Note: $C[a, b]$ denote the class of all continuous function on $[a, b]$ and $C[a, b] \subset R[a, b]$)

(iii) Let $f: [a, b] \to \mathbb{R}$ be bounded on$[a, b]$ and let f be continuous on $[a, b]$ except for a finite number points is$[a, b]$. Then$f$ is integrable on $[a, b]$.

$\Rightarrow$ If$f: [a, b] \to \mathbb{R}$ be piecewise continuous on$[a, b]$ then f is integrable on $[a, b]$.

(iv) Let $f: [a, b] \to \mathbb{R}$ be bounded on$[a, b]$and let f be continuous on$[a, b]$ except on a infinite Subset $S \subset [a, b]$ such that the number of limit points of S is finite. Then f is integrable on$[a, b]$.

Example (1.116):

a) $f: [0, 1] \to \mathbb{R},\ f(x) = \begin{cases} 0 & , \quad x = 0 \\ (-1)^{r-1} & , \quad \frac{1}{r+1} < x \le \frac{1}{r} \quad , \quad r = 1, 2, 3, \ldots. \end{cases}$

F is continuous on $[0, 1]$ except at the points $0, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\ldots$ Then set of points of discontinuity of f has only the limit point 0 and f is bounded on $[0, 1] \Rightarrow f \in R[0,1]$

b) **Converse of (iv) is not true.**

Example (1.117):

$$f: [0, 1] \to \mathbb{R},\ f(x) = \begin{cases} 0 & , \quad x = 0 \\ 0 & , \quad x \in \mathbb{R}|\mathbb{Q} \\ \frac{1}{q} & , \quad x = \frac{p}{q} \quad , \quad p, q > 0 \text{ with } gcd(p, q) = 1 \end{cases}$$

f is bounded on $[0, 1]$ and f is continuous at 0 and every irrational number and discontinuous at non-zero rational number is $[0, 1]$ so, the set S of points of discontinuity have infinite number of limit point. But f is Riemann integrable on $[0, 1]$.

1.24.9. Lebesgue: A necessary and sufficient condition for a bounded function on $[a, b]$ to be Riemann integrable on $[a, b]$ is that the points of discontinuity of f is a set of measure zero.

1.24.10. Definition (Set of Measure Zero): A set $S \subset \mathbb{R}$ is said to the a set of measure zero if for each $\varepsilon > 0$ there is a countable collection of open intervals $\{I_n\}$ such that

$S \subseteq \bigcup_{n=1}^{\infty} I_n \; and \; \sum_{n=1}^{\infty} |I_n| < \varepsilon$

Example (1.118):

(a) A finite set $S \subseteq \mathbb{R}$ is a set of measure zero.

[**Hint**: $I_r = \left(x_r - \frac{\varepsilon}{2(m+1)}, x_r + \frac{\varepsilon}{2(m+1)}\right) \; for \; r = 1,2, \ldots\ldots\ldots, m.$]

(b) An enumerable subset $S \; of \; \mathbb{R}$ is a set of measure zero

[**Hint**: $I_r = \left(x_r - \frac{\varepsilon}{2^{r+2}}, x_r + \frac{\varepsilon}{2^{r+2}}\right)$]

$\Rightarrow \mathbb{Q}$ is a set of meausre zero.

(c) Let S be a bounded infinite subset of $\mathbb{R}$ having finite (countable) number of limit points. Then S is a set of measure zero.

[Hint: Let $x_1, x_2, \ldots\ldots\ldots, x_m$ be the limit points of S condition $I_r = \left(x_r - \frac{\delta_r}{2}, x_r + \frac{\delta_r}{2}\right)$ open interval containing x_r and let $\delta_1 + \delta_2 + \ldots\ldots + \delta_m < \frac{\varepsilon}{2}$. Then there are finite number of points out side $\bigcup_{r=1}^{m} I_r$. So we can cover these points by open interval whose sum of length is $< \frac{\varepsilon}{2}$.]

1.25. Properties of Riemann Integrable Function:

1.25.1. Let $f, g : [a,b] \to \mathbb{R}$ be two Riemann integrable functions on $[a,b]$. Then –

(i) $f + g \in R[a,b] \; and \; \int_a^b f + g = \int_a^b f + \int_a^b g$

(ii) $cf \in R[a,b] and \; \int_a^b cf = c\int_a^b f \;\; , c \in \mathbb{R}$

(iii) $|f| \in R[a,b]$, but converse is not true.

[**Example (1.119):** $f : [a,b] \to \mathbb{R}, \;\; f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [a,b] \\ -1, & x \in (\mathbb{R}|\mathbb{Q}) \cap [a,b] \end{cases}$ $, |f(x)| = 1, x \in [a,b] \;\; but \; f \notin R\,[a,b]$]

(iv) $f^2 \in R\,[a,b]$

(v) $fg \in R\,[a,b] \left(\because fg = \frac{1}{2}(f+g)^2 - \frac{1}{2}f^2 - \frac{1}{2}g^2\right)$

(vi) $\frac{1}{f} \in R\,[a,b] \; provide \; f(x) \geq k > 0 \;\; \forall \; x \in [a,b].$

(***Note***: $f(x) > 0 \;\; \forall \;\; x \in [a,b], then \; f(x) \; may \; not \; belong \; to \; R\,[a,b].$

Example (1.120) : $f : [0,1] \to \mathbb{R}, f(x) = \begin{cases} x, & 0 < x \leq 1 \\ 1, & x = 0 \end{cases} \quad Then$

$f \in R\,[0,1] \;\; as \; it \; is \; continuous \; on \; [0,1] \;\; except \; x = 0.$

$But \frac{1}{f} \quad is \; not \; bounded \; on \; [0,1] \Rightarrow \frac{1}{f} \notin R\,[0,1])$

(vii) If $c \in (a,b)$, $then\ f \in R\,[a,b] and\ f \in R\,[c,b]\ and\ \int_a^b f = \int_a^c f + \int_c^b f$ converse is also true i.e., if $f \in R\,[c,b]$, $then\ f \in R\,[a,b]\ and\ \int_a^c f + \int_c^b f = \int_a^b f$

1.25.2. Let $I = [a,b] \subset \mathbb{R}\ and\ f : I \to \mathbb{R}$ be integrable on$I\ and\ J = [c,d] \subset \mathbb{R}$ such that $f\,(I) \subset J\ and\ \phi : [c,d] \to \mathbb{R}$ be continuous on$[c,d]$. Then the composition function $\phi \circ f \in R\,[a,b]$.

Note: Continuity of ϕ is necessary.

Example (1.121): $f : [0,1] \to \mathbb{R}$, $f(x) = \begin{cases} 0\ , & x \in \mathbb{R}|\mathbb{Q} \\ \frac{1}{n}\ , & x = \frac{m}{n}\ , \gcd(m,n),\ m,n \in \mathbb{Z}^* \end{cases}$

$\phi : [0,1] \to \mathbb{R}$, $\phi(x) = \begin{cases} 1, & if\ x \neq 0 \\ 0, & if\ x = 0 \end{cases}$ Then -

$\phi \circ f : [0,1] \to \mathbb{R}$, $(\phi \circ f)(x) = \begin{cases} 0, & x \in \mathbb{R}|\mathbb{Q} \\ 1, & x \in \mathbb{Q} \end{cases} \Rightarrow Q \circ f \notin R\,[0,1]$

1.25.3. Let $f, \phi : [a,b] \to \mathbb{R}$ be both bounded on $[a,b]\ and\ f(x) = \phi\,(x)$ except for a finite number of points in $[a,b]$. If f be integrable on $[a,b]$ then $\phi \in R\,[a,b]\ and\ \int_a^b f = \int_a^b \phi$.

Note: If $f(x) = \phi(x)$ enumerable number of points, then ϕ may not belong to $R\,[a,b]$.

Example (1.122): $f, \phi : [0,1] \to \mathbb{R}$ be defined by $f(x) = 1, x \in [0,1] \Rightarrow f \in R\,[0,1]$.

$\phi\,(x) = \begin{cases} 0, & x \in [0,1] \cap \mathbb{Q} \\ 1, & x \in [0,1] \cap (\mathbb{R}|\mathbb{Q}) \end{cases} \Rightarrow \phi\,(x) \neq f(x), x \in [0,1] \cap \mathbb{Q}$

$\phi \notin R\,[0,1]$

1.25.4. Definition (Piecewise Continuous Function): A function $f: [a,b] \to \mathbb{R}$ is said to be a piecewise continuous function on $[a,b]$ $if\ \exists$ a partition $P = \{x_0, x_1, \ldots\ldots\ldots\ldots, x_n\}\ of\ [a,b]$ such that f is continuous on the open interval $(x_{k-1}, x_k) for\ 1 \le k \le n$ and each of $f(a+0), f(b-0)$, $f(x_k+0)$, $f(x_k-0)$ exist for $1 \le k \le n-1$. Clearly, a piecewise continuous function on $[a,b]$ is continuous on $[a,b]$ except for a finite number of points of jump discontinuity.

Example (1.123): A step function on $[a,b]$

1.25.5. Let $f : [a,b] \to \mathbb{R}$ be bounded on $[a,b]$ and for every $c \in (a,b), f \in R\,[c,b]$. Then $f \in R\,[a,b]$.

[$\boldsymbol{Hint}$: $let\ M = \sup_{x\in[a,b]} f(x),\ m = \inf_{x\in[a,b]} f(x)\ and \{c_n\}\ such\ that\ \ c_n \to a\ \ as\ n \to \infty$.Then $\varepsilon > 0\ \exists\ \ k \in \mathbb{N}\ such\ that\ |c_n - a| < \frac{\varepsilon}{2(M-m)}\ \ \forall\, n \ge k\ \Rightarrow\ |c_k - a| < \frac{\varepsilon}{2(M-m)}\ \ and$ $f \in R\,[c_k, b] \Rightarrow \exists\ partition\ \ Q\ \ of\ \ [c_k, b]\ \ such\ that\ \ U(Q_k, f) - L(Q_k, f) < \frac{\varepsilon}{2}.\ \ Let\ P = \{a\}.\ \cup\ \mathbb{Q}.\ Then\ \ U(P,f) - L(P,f) < (M,m)(c_k - a) + \big(U(Q,f) - L(Q,f)\big) < \frac{\varepsilon}{1} + \frac{\varepsilon}{2} = \varepsilon$]

1.25.6. Corollary – I: Let $f:[a,b]$ be bounded on $[a,b]$ and for every $d \in (a,b), f \in R[a,d]$. Then $f \in R[a,b]$.

1.25.7. Corollary – II: Let $f:[a,b] \to \mathbb{R}$ be bounded on $[a,b]$ and for every c,d saftisfying $a < c < d < b \;\; f \in R[c,d]$. $Then\ f \in R[a,b]$.

1.25.8. Inequalities: Let $f:[a,b] \to \mathbb{R}$ be integrable on $[a,b]$. If $M\ and\ m$ be the supremum of f and infimum of $f\ on\ [a,b]$ respectively, then $m(b-a) \leq \int_a^b f \leq M(b-a)$

1.25.9. Corollary – (a): Let $f:[a,b] \to \mathbb{R}$ be ingegrable on $[a,b]$. Then $\exists\ \mu \in \mathbb{R}$ satisfying $m \leq \mu \leq M$ such that $\int_a^b f = \mu(b-a)$.

1.25.10. Corollary – (b) Let $f:[a,b] \to \mathbb{R}$ be continuous on $[a,b]$. Then $\exists\ a\ point\ c \in [a,b]$ such that $\int_a^b f = f(c)(b-a)$.

1.25.11. Let $f:[a,b] \to \mathbb{R}$ be integrable on $[a,b]\ and\ f(x) \geq 0\ \forall\ x \in [a,b]$ such that $\int_a^b f \geq 0$.

1.25.12. Let $f,g:[a,b] \to \mathbb{R}$ be both integrable on $[a,b]\ and\ f(x) \geq g(x)\ \forall\ x \in [a,b]$. Then $\int_a^b f \geq \int_a^b g$.

1.25.13. Let $f:[a,b] \to \mathbb{R}$ be integrable on $[a,b]$ and $f(x) \geq 0\ \forall\ x \in [a,b]$. Let $\exists\ c \in [a,b]$ such that f is continuous at c and $f(c) > 0, then\ \int_a^b f > 0$.

Note – (a) If f is continuous on $[a,b]\ and\ f(x) > 0\ on\ [a,b]\ then\ \int_a^b f > 0$.

Note – (b) if $f \in R\ [a,b]\ and\ f(x) > 0\ on\ [a,b] then\ also\ \int_a^b f > 0\ becuase\ \exists$ at least a point of discontinuity $c \in [a,b]\ of\ f$.

1.25.14. Let $f:[a,b] \to \mathbb{R}$ be integrable on $[a,b]$.Then $\left|\int_a^b f\right| \leq \int_a^b |f|$.

Examples (1.124):

(a) If f be continuous on $[a,b]$and $f(x) \geq 0$ on $[a,b]$and $\int_a^b f = 0$ then $f = 0\ on\ [a,b]$ identically.

[**Hint**: If $\exists\ c \in [a,b] such\ that\ f(c) > 0\ \Rightarrow\ \int_a^b f > 0$]

(b) $\frac{\pi^2}{9} < \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{x}{\sin x}\, dx < \frac{2\pi^2}{9}$

[**Hint**:$1 \leq \frac{1}{\sin x} \leq 2\ , x \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right] \Rightarrow x \leq \frac{x}{\sin x} \leq 2x\ , x \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]\ and\ at \frac{\pi}{3},\ \ \frac{\pi}{3} < \frac{1}{\sin\left(\frac{\pi}{3}\right)} < \frac{2\pi}{3}$]

1.25.15. Let $f:[a,b] \to \mathbb{R}$ be integrable on $[a,b]$ then the function $F(x)$ defined by $F(x) = \int_a^x f(t)\, dt,\ \ x \in [a,b]$ is continuous on $[a,b]$.

Note: $F(x)$ always continuous even if $f(x)$ may not continuous on $[a,b]$and also $F(x)$ is uniform continuous on $[a,b]$.

Example (1.125): Let $f:[0,1]\rightarrow\mathbb{R}$ be defined by $f(x)=\begin{cases}0 & ,-1\leq x\leq 0\\ 1 & ,\quad 0<x\leq 1\end{cases}$

$-1\leq x\leq 0, F(x)=\int_{-1}^{x}f(t)\,dt=0$

$0<x\leq 1, F(x)=\int_{-1}^{x}f(t)\,dt=\int_{-1}^{0}f(t)\,dt+\int_{0}^{x}f(t)\,dt=0+\int_{-1}^{x}dx=x$

We have $F(x)=\begin{cases}0 & ,-1\leq x\leq 0\\ x & ,\quad 0<x\leq 1\end{cases}\Rightarrow F$ is continuous on$[-1,1]$.

1.25.16. If $f:[a,b]\rightarrow\mathbb{R}$ be integrable on $[a,b]$ then the function $F(x)=\int_{a}^{x}f(t)\,dt,\ x\in[a,b]$ is differentiable at any point $c\in[a,b]$ at which f is continuous and $F'(c)=f(c)$.

1.25.17. Corollary: If $f:[a,b]\rightarrow\mathbb{R}$ be continuous on $[a,b]then\ F$ is differentiable on $[a,b]\ and\ F'(x)=f(x)\ \forall\ x\in[a,b]$.

1.26. Fundamental Theorem of Integral Calculus:

1.26.1. Definition (Anti-derivative or Primitive): A function ϕ is called an anti-derivative or a primitive of a function f on an interval I if $\phi'(x)=f(x)\ \forall\ x\in I$.

1.26.2. If $f:[a,b]\rightarrow\mathbb{R}$ be continuous on $[a,b]\ and\ \phi:[a,b]\rightarrow\mathbb{R}$ be an anti-derivative of f on $[a,b], then\ \int_{a}^{b}f=\phi(b)-\phi(a)$.

1.26.3. Fundamental Theorem of Integral Calculus:

(i) $f:[a,b]\rightarrow\mathbb{R}$ be integrable on$[a,b]and$

(ii) f possesses an anti $-$ derivative $\phi\ on\ [a,b], then$

$\int_{a}^{b}f=\phi(b)-\phi(a)$

1.26.4. Note –I: $(Integrability \nRightarrow \ existence\ of\ anti-derivative)$:

Example (1.126): $f: [-1,1] \to \mathbb{R}, f(x) = \begin{cases} 0 & ,-1 \le x < 0 \\ 1 & ,\ \ 0 \le x \le 1 \end{cases} \Rightarrow f \in R[-1,1]\ on\ f$ is continuous on $[-1,1]$ except at 0.

Let ϕ be anti-derivative of $f\ \ on\ [-1,1]$. Then $\phi'(x) = \begin{cases} 0 & ,-1 \le x < 0 \\ 1 & ,\ \ 0 \le x \le 1 \end{cases}$

Since $\phi'(-1) \neq \phi'(1), by$ Darboux theorem ϕ'must assume every real number lying between $\phi'(-1)\ and\ \phi'(1)\ i.e., 0\ \ and\ \ 1.$ But it does not do so.

1.26.5. Note –II: (Existence of anti-derivative $\nRightarrow$ Integrability):

Example (1.127): Let $f: [-1,1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 2x \sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2}, & x \neq 0 \\ 0 & , \ x = 0 \end{cases}$

$f \notin R\ [-1,1] as\ f$ is unbounded on every neighbourhood of 0.

Now, $\phi : [-1,1] \to \mathbb{R}$ defined by $\phi(x) = \begin{cases} x^2 \sin\frac{1}{x^2}, & x \neq 0 \\ 0 & , \ \ x = 0 \end{cases}$

Then $\phi'(x) = f(x)\ \ on\ \ [-1,1].$ So, ϕ is anti-derivative of f on $[-1,1]$.

1.26.6. If

(i) $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ and

(ii) $\exists\ \ \phi : [a,b] \to \mathbb{R}$ such that ϕ is continuous on $[a,b]$ and

$\phi'(x) = f(x)\ \forall\ x \in [a,b], then\ \int_a^b f = \phi(b) - \phi(a).$

1.26.7. Corollary: If

(i) $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ and

(ii) $\exists\ \phi : [a,b] \to \mathbb{R}$ such that ϕ is continuous on $[a,b]$ and

$\phi'(x) = f(x)\ \forall\ x \in [a,b]\backslash \mathrm{E}, where$ E is a finite set $\subset$ [a, b],

then $\int_a^b f = \phi(b) - \phi(a)$.

1.27. Riemann Sum and another Definition of Integration:

1.27.1. Riemann Sum: Let $f : [a,b] \to \mathbb{R}\ and\ \ P = \{x_0, x_1, x_2, \ldots\ldots, x_n\}$ be a partition of $[a,b]\ and\ \xi_0, \xi_1, \xi_2, \ldots\ldots, \xi_n$ are arbitrarily chosen points such that $x_{r-1} \le \xi_r \le x_r\ \ for\ r = 1,2,3 \ldots\ldots\ldots, n.$ Then the sum $\sum_{r=1}^{n} f(\xi_r)(x_r - x_{r-1})$ is called a Reimann sum for f corresponding to the partition P and choose intermediate points ξ_r. This is denoted by $S(P,f)$.

1.27.2. Definition (Another Definition for Riemann Integration):

A function $f : [a,b] \to \mathbb{R}$ is said to be Riemann integrable on $[a,b]\ if\ \exists\ B > 0$ such that for each $\varepsilon > 0, \exists\ a\ \delta = \delta(\varepsilon) > 0$ satisfying $|S(P,f) - B| <$ $\varepsilon\ \forall\ partition\ P\ \ of\ [a,b] with\ \|P\| < \delta\ where\ S(P,f)$ is a Riemann sum for f

corresponding to the partition P and to any choice of intermediate points. In this case $B = \int_a^b f$.

This condition is expressed by the symbol $\lim_{\|P\| \to 0} S(P, f) = B$.

1.27.3. If $f : [a, b] \to \mathbb{R}$ be such that $\lim_{\|P\| \to 0} S(P, f) = B$, $then\ B$ is unique.

1.27.4. If $f : [a, b] \to \mathbb{R}$ be such that $\lim_{\|P\| \to 0} S(P, f)$exists, then f is bounded on $[a, b]$.

1.27.5. (Integration by Substitution): Let $I = [\alpha, \beta]$ be a closed and bounded interval and a function $\phi : I \to \mathbb{R}$ be such that ϕ' is continuous and $\neq 0\ on\ I$. Let $\phi(\alpha) = a, \phi(\beta) = b$ and a function f be continuous on $\phi([\alpha, \beta])$. Then –

$$\int_\alpha^\beta f(\phi(t))\, \phi'(t)\, dt = \int_a^b f(x)\, dx$$

1.27.6. Integration by parts: Let $f, g : [a, b] \to \mathbb{R}$ be both differentiable on $[a, b] and\ f', g'$ are both intergable on $[a, b]$. Then -

$$\int_a^b f(x)\, g'(x) dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)\, g(x) dx$$

1.28. Mean Value Theorem for Integration:

1.28.1. First Mean Value Theorem:

If (i) $f, g : [a, b] \to \mathbb{R}$ be both integrable on $[a, b]$, and

(ii) $g(x)$ has the same sign $\forall\ x \in [a, b]$

then there is a no $\mu\ such\ that\ \int_a^b f(x)\, g(x) dx = \mu \int_a^b g(x) dx$

where $m < \mu \le M\ and\ m = \inf_{x \in [a,b]} f(x), M = \sup_{x \in [a,b]} f(x)$. Further, f is continuous on $[a, b]$ there is a point $c \in [a, b]$such that $\int_a^b f(x)\, g(x) dx = f(c) \int_a^b g(x) dx$.

1.28.2. Note:

(i) If $g(x) = 1,\ then\ \int_a^b f(x)\, dx = \mu \int_a^b dx = \mu(b - a),\ where\ m \le \mu \le M$.

(ii) If f is continuous ojn $[a, b]\ and\ g(x) = 1, then\ \exists\ c \in [a, b]\ such\ that$

$$\int_a^b f(x)\, dx = f(c)(b - a).$$

Since $c \in [a, b], c = a + \theta\ (b - a) for\ some\ \theta\ satisfying\ 0 \le \theta \le 1$.

$\therefore\ \int_a^b f(x)\, dx = (b - a) f(a + \theta(b - a)), 0 \le \theta \le 1$.

Example (1.128): Use first mean value theorem prove that

$$\frac{\pi}{6} \le \int_0^{\frac{1}{2}} \frac{1}{\sqrt{(1 - x^2)(1 - k^2 x^2)}} dx \le \frac{\pi}{6} \cdot \frac{1}{\sqrt{1 - \frac{k^2}{4}}}, k^2 < 1$$

Let $(x) = \frac{1}{\sqrt{1 - k^2 x^2}}, g(x) = \frac{1}{\sqrt{(1 - x^2)}},\ x \in [0, \frac{1}{2}]$

Then $f, g \in R\left[0, \frac{1}{2}\right]$ $and\ g(x) > 0,\ \forall\, x \in \left[0, \frac{1}{2}\right]$

By first Mean Value Theorem $\exists$ $a \quad c \in \left[0, \frac{1}{2}\right]$ $such\ that$

$\int_0^{\frac{1}{2}} f(x)\, g(x) dx = f(x) \int_0^{\frac{1}{3}} g(x) dx = \frac{1}{\sqrt{1-k^2x^2}} \cdot \frac{\pi}{6}$

Since $0 \leq c \leq \frac{1}{2}$ $, 1 \leq \frac{1}{\sqrt{1-k^2x^2}} \leq \frac{1}{\sqrt{1-\frac{k^2}{4}}} \Rightarrow \frac{\pi}{6} \leq \int_0^{\frac{1}{2}} f(x)\, g(x) dx \leq \frac{\pi}{6} \cdot \frac{1}{\sqrt{1-\frac{k^2}{4}}}$

1.28.3. Second Mean Value Theorem (Bonnets Form):

If (i) $f, g : [a, b] \to \mathbb{R}$ be both integrable on $[a, b], and$

(ii) f is monotone decreasing and non-negative on $[a, b],\ then\ \exists\ \ a\ point\ c \in [a, b]$

such that $\int_a^b f(x)\, g(x) dx = f(a) \int_a^c g(x) dx$

1.28.4. Second MVT, Weierstrass' form:

If (i) $f, g : [a, b] \to \mathbb{R}$ be both integrable on $[a, b]$, and

(ii) f is monotonic on $[a, b]$

$$then\ \exists\ \ a\ point\ c \in [a, b] such\ that\ \int_a^b f(x)\, g(x) dx = f(a) \int_a^c g(x) dx + f(b) \int_c^b g(x) dx$$

Example (1.129):

(i) Prove that $\left|\int_a^b \frac{\sin x}{x} dx\right| \leq \frac{2}{a}, 0 < a < b < \infty (Bonnets\ form).$

(ii) Prove that $\left|\int_a^b \frac{\sin x}{x} dx\right| \leq \frac{4}{a}, 0 < a < b < \infty (Weierstrass\ form).$

(i) Let $f(x) = \frac{1}{x}, g(x) = \sin x,\ \forall\, x \in [a, b]$. Since $f, g \in R\,[a, b]$ and f is monotone decreasing on $[a, b]$, by second mean value theorem (Bonnets form) $\exists\, c \in [a, b]$ such that $\int_a^b f(x)\, g(x) dx = f(a) \int_a^c g(x) dx = \frac{1}{a} \int_a^c \sin x\, dx = \frac{1}{a}[-\cos c + \cos a]$

$\Rightarrow \left|\int_a^b \frac{\sin x}{x}\, dx\right| \leq \frac{2}{a}$

(ii) Since f is monotone on $[a, b]$, by second mean value theorem (Weierstrass form) $\exists\, c \in [a, b] such\ that\ \int_a^b f(x)\, g(x) dx = \frac{1}{a} \int_a^c g(x) dx + \frac{1}{b} \int_c^b g(x) dx$

$= \frac{1}{a}[-\cos c + \cos a] + \frac{1}{b}\,[-\cos b + \cos c]$

$\therefore \left|\int_a^b \frac{\sin x}{x}\, dx\right| \leq \frac{4}{a}$

1.28.5. Definition (Logarithmic Function): The logarithmic function $L(or\ log)$ is defined by $L(x) = \log x = \int_1^x \frac{dt}{t}\ , x > 0.$

1.28.6. Definition (e): Then unique real number x satisfying $L(x) = 1$ is denoted by e i.e., $L(e) = 1$. Therefore e is denoted by $1 = \int_1^e \frac{1}{t}\, dt$.

1.29. Improper Integral:

There are two type of improper integrals-

i. Improper integrals on a finite interval where the improper is unbounded.

ii. Improper integrals on an unbounded interval.

1.29.1 Convergence of the improper integral $\int_a^b f(x)dx$ when a is the only point of infinite discontinuous of f in $[a, b]$.

Let $\psi(\varepsilon) = \int_{a+\varepsilon}^b f(x)dx\,, 0 < \varepsilon < b - a$

If $\lim\limits_{\varepsilon \to 0+} \psi(\varepsilon) = l(finite)$, then the improper integral $\int_a^b f(x)$ is said to be convergent and we write $\int_a^b f(x)dx = l$.

If $\lim\limits_{\varepsilon \to 0+} \psi(\varepsilon)$ does not exist, then the improper integral $\int_a^b f(x)dx$ is said to be divergent.

Example-(1.130):

The integral $\int_1^2 \frac{dx}{\sqrt{x-1}}$ is improper integral, since 1 is a point of infinite discontinuity of the integral. The integrand is bounded and integrable on $[1 + \varepsilon, 2]\ \forall\ 0 < \varepsilon < 1$.

$$\lim_{\varepsilon \to 0+} \int_{1+\varepsilon}^2 \frac{dx}{\sqrt{x-1}} = \lim_{\varepsilon \to 0+} 2[1 - \sqrt{\varepsilon}] = 2$$

Hence, the integral $\int_1^2 \frac{dx}{\sqrt{x-1}}$ is convergent and $\int_1^2 \frac{dx}{\sqrt{x-1}} = 2$

Example-(1.131):

The integral $\int_0^1 \frac{dx}{x}$ is improper, since 0 is the point of infinite discontinuity of the integrand and it bounded on $[\varepsilon, 1], \forall\ 0 < \varepsilon < 1$.

$\lim\limits_{\varepsilon \to 0+} \int_\varepsilon^1 \frac{dx}{x} = \lim\limits_{\varepsilon \to 0+} [-\log \varepsilon] = \infty$

Hence, the improper integral $\int_0^1 \frac{dx}{x}$ is divergent.

1.29.2. Convergence of the improper integral $\int_a^b f(x)\, dx$ when b is the only point of infinite discontinuity of f in $[a, b]$.

Let $\psi(\varepsilon) = \int_a^{b-\varepsilon} f(x)dx\,, 0 < \varepsilon < b - a$

If $\lim\limits_{\varepsilon \to 0+} \psi(\varepsilon) = l(finite)$, then the improper integral is said to be convergent and we write $\int_a^b f(x)dx = l$.

If $\lim\limits_{\varepsilon \to 0+} \psi(\varepsilon)$ does not exist, then the improper integral $\int_a^b f(x)dx$ is said to be divergent.

Example-(1.132):

$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ is improper, since 1 is a point of infinite discontinuity and it is bounded on $[0,1-\varepsilon]\ \forall\ 0 < \varepsilon < 1$

$$\lim_{\varepsilon \to 0+} \int_0^{1-\varepsilon} \frac{dx}{\sqrt{1-x^2}} = \lim_{\varepsilon \to 0+} \sin^{-1}(1-\varepsilon) = \frac{\pi}{2}$$

Hence, the improper integral $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ is convergent.

Example-(1.133):

$\int_0^2 \frac{dx}{2-x}$ is divergent (verify!)

1.29.3. Convergence of the improper integral $\int_a^b f(x)dx$ where a and b are the only point of infinite discontinuities of f in $[a,b]$

Let f be bounded on $[a+\varepsilon_1,\ b-\varepsilon_2], 0 < \varepsilon_1 < \varepsilon_2 < b-a$ and $c \in (a,b)$.

If both the integrals $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then the improper integral $\int_a^b f(x)dx$ is said to be convergent and we write,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

If one of $\int_a^c f(x)dx$ *or* $\int_a^c f(x)dx$ is divergent or both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)dx$ are divergent, then $\int_a^b f(x)dx$ is said to be divergent.

Example-(1.134):

The improper integral $\int_0^2 \frac{dx}{\sqrt{x(2-x)}}$ is improper, since 0 and 2 are point of infinite discontinuities of the integrand.

The integrand is bounded and integrable on $[0+\varepsilon_1,\ 2-\varepsilon_2]\ \forall\ 0 < \varepsilon_1,\ \varepsilon_2 < 1$

Now, $\lim\limits_{\varepsilon_1 \to 0+} \int_{0+\varepsilon_1}^1 \frac{dx}{\sqrt{x(2-x)}} = \lim\limits_{\varepsilon_1 \to 0+} [\sin^{-1}(x-1)]_{\varepsilon_1}^1 = \frac{\pi}{2}$

$$\lim_{\varepsilon_2 \to 0+} \int_1^{2-\varepsilon_2} \frac{dx}{\sqrt{x(2-x)}} = \lim_{\varepsilon_2 \to 0+} [\sin^{-1}(x-1)]_1^{2-\varepsilon_2} = \frac{\pi}{2}$$

Therefore, $\int_0^2 \frac{dx}{\sqrt{x(2-x)}}$ is convergent and $\int_0^2 \frac{dx}{\sqrt{x(2-x)}} = \pi$

1.29.4. Convergence of the improper integral $\int_a^b f(x)dx$ when an interior point c is the only point of infinite discontinuity of f in $[a,b]$.

If both the integrals $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then the improper integral $\int_a^b f(x)dx$ is convergent and we write

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

If one of $\int_a^c f(x)dx \ or \ \int_c^b f(x)dx$ is divergent or both the integral $\int_a^c f(x)dx \ or \ \int_c^b t(x)dx$ are divergent, then the improper integral $\int_a^b f(x)dx$ is divergent.

1.29.5. Convergence of the improper integral $\int_a^b f(x)\,dx$ when a finite number of points $c_1, c_2, c_3, \ldots\ldots. c_k$ are the only points of infinite discontinuities of f in $[a, b]$.

a) Let $a < c_1 < c_2 <\ldots< c_k < b$.

If the improper integrals $\int_a^{c_1} f(x)dx, \int_{c_1}^{c_2} f(x)dx ,\ldots. \int_{c_k}^{b} f(x)dx$ are all convergent, then the improper integral $\int_a^b f(x)dx$ is said to be convergent, then the improper integral $\int_a^b f(x)dx$ is said to be convergent and we write

$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx +\ldots.+ \int_{c_k}^{b} f(x)dx$

b) Let either $a = c_1 \ or \ b = c_k \ or$ both

If $a = c_1$, then $\int_a^b f(x)dx = \int_a^{c_2} f(x)dx + \int_{c_2}^{c_3} f(x)dx +\ldots.+ \int_{c_k}^{b} f(x)dx$

If $b = c_k$, then $\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx +\ldots.+ \int_{c_{k-1}}^{b} f(x)dx$

1.29.6. Test for convergence of positive integrand

i. **Theorem:** Let a be the only point of infinite discontinuity of a function f which is integrable on $[a + \varepsilon, b]$, $0 < \varepsilon < b - a$ and $f(x) > 0 \ \forall \ x \in [a, b]$. A necessary and sufficient condition for the convergence of the improper integral $\int_a^b f(x)dx$ is that $\exists \ a \ k > 0$ such that $\int_{a+\varepsilon}^{b} f(x)dx < k \ \forall \ \varepsilon$ satisfying $0 < \varepsilon < b - a$.

ii. **Theorem:** Let b be the only point of infinite discontinuity of a function f which is integrable on $[a, b - \varepsilon]$, $0 < \varepsilon < b - a$ and $f(x) > 0 \ \forall \ x \ \epsilon [a, b]$

A necessary and sufficient condition for the convergence of the improper integral $\int_a^b f(x)dx$ is that $\exists \ a \ k > 0$ such that $\int_a^{b-\varepsilon} f(x)dx < k \ \forall \ \varepsilon$ satisfying $0 < \varepsilon < b - a$.

iii. **Theorem (Comparison Test):** Let a be the only point of infinite discontinuity of the functions f and g which are both integrable on $[a + \varepsilon, b]$, $0 < \varepsilon < b - a$ and $0 < f(x) \leq mg(x) \ \forall \ x \ \epsilon [a, b]$, where $m > 0$. Then

a. $\int_a^b g(x)dx$ is convergent $\Rightarrow \int_a^b f(x)dx$ is convergent.

b. $\int_a^b f(x)dx$ is divergent $\Rightarrow \int_a^b g(x)dx$ is divergent.

iv. **Theorem [Comparison Test (limit form)]:** Let a be the only point of infinite discontinuity of the functions f and g which are both integrable on $[a+\varepsilon, b], 0 < \varepsilon < b-a$ and $f(x) > 0, g(x) > 0 \;\forall\; x \,\epsilon [a,b]$.

If $\lim\limits_{x \to a+} \frac{f(x)}{g(x)} = l$ (non-zero finite), then both the improper integrals $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ converges or diverge together.

v. **Theorem (μ-test):** Let a be the only point of infinite discontinuity of a faction f which is integrable on $[a+\varepsilon, b], 0 < \varepsilon < b-a$ and $f(x) > 0 \;\; \forall\, x \in [a,b]$.

If $\lim\limits_{x \to a+} (x-a)^{\mu} f(x) = l$ (non-zero finite), then the integral $\int_a^b f(x)dx$ is convergent $\Leftrightarrow$ $\mu < 1$.

Example-(1.135): The integral $\int_0^1 \frac{x^{m-1}}{1+x} dx$ is convergent $\Leftrightarrow m > 0$.

The integral is proper it $m-1 \geq 0$ and improper if $m < 1$, 0 is the only point of infinite discontinuity.

Now, $\lim\limits_{x \to 0+} (x-0)^{1-m} f(x) = \lim\limits_{x \to 0+} x^{m-1} . x^{m-1} \frac{1}{1+x} = 1$

By μ-test the improper integral is convergent $\Leftrightarrow 1-m < 1 \Rightarrow m > 0$.

Example-(1.136): The beta function $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$ is convergent $\Rightarrow$ $m, n > 0$.

1.29.7. Test for convergence of an improper integral when the integrand does not necessarily keep the same sign.

i. **Theorem (Cauchy):** Let a be the only point of infinite discontinuity of a function f which is integrable on $[a+\varepsilon, b], 0 < \varepsilon < b-a$ and $f(x)$ may not keep same sign on $[a,b]$.

A necessary and sufficient condition for the convergence of the improper integral $\int_a^b f(x)dx$ in that for a given $\varepsilon > 0, \exists$ a positive $\delta < b-a$ such that

$\left| \int_{a+\varepsilon_1}^{a+\varepsilon_2} f(x)dx \right| < \varepsilon \;\forall\; \varepsilon_1, \varepsilon_2$ satisfying $0 < \varepsilon_1 < \varepsilon_2 < \delta$.

Definition (absolutely convergent): The improper integral $\int_a^b f(x)dx$ is said to be absolutely convergent if $\int_a^b |f|(x)dx$ is convergent.

ii. **Theorem:** Let a be the only point of infinite discontinuity of a function f which is integrable on $[a+\varepsilon, b], 0 < \varepsilon < b-a$. It $\int_a^b |f|(x)dx$ is convergent, then $\int_a^b f(x)dx$ is convergent.

Note: Converse of the above theorem is not true.

Example-(1.137): The improper integrable $\int_0^1 \frac{\cos\frac{1}{x}}{\sqrt{x}} dx$ is convergent.

Let $f(x) = \frac{\cos\frac{1}{x}}{\sqrt{x}}$, $x \in [0,1]$ thus 0 is the only point of infinite discontinuity of f.

Now, $|f(x)| = \left|\frac{\cos\frac{1}{x}}{\sqrt{x}}\right| \leq \frac{1}{\sqrt{x}}$ and $\int_0^1 \frac{dx}{\sqrt{x}}$ is convergent and hence $\int_0^1 \frac{\cos\frac{1}{\sqrt{x}}}{\sqrt{x}} dx$ is convergent.

Example-(1.138):

A Function $f(x)$ is defined on $[0,1]$ by

$$f(x) = \begin{cases} 0, & x = 0 \\ (-1)^{n+1}(n+1), & \frac{1}{n+1} < x \leq \frac{1}{n} \quad (n = 1, 2, \ldots\ldots) \end{cases}$$

It can be shown that $\int_0^1 f(x)dx$ is convergent but $\int_0^1 |f|(x)dx$ is divergent.

1.29.8. Convergence of the improper integral $\int_a^\infty f(x)dx$ where f is integrable on $[a, X]\ \forall\ X > a$

Let $\psi(X) = \int_a^X f(x)dx\, , X > a$

If $\lim_{X\to\infty} \psi(x)dx = l$(exists finitely), then the improper integral $\int_a^\infty f(x)dx$ is said to be convergent and we write $\int_a^\infty f(x)dx = l.$

If $\lim_{X\to\infty} \psi(x)$ does not exist, then the improper integral $\int_a^\infty f(x)dx$ is said to be divergent.

Example-(1.139):

Consider the improper integral $\int_0^\infty e^{-x} dx$ e^{-x} is integrable on $[0, X], X > 0$.

Let $\psi(x) = \int_0^X e^{-x}\, dx = 1 - e^{-x}$, $\lim_{x\to\infty} \psi(x) = 1$

Hence $\int_0^\infty e^{-x}\, dx$ is convergent.

Example-(1.140): Consider the integral $\int_1^\infty \frac{dx}{x} \cdot \frac{1}{x}$ is integrable on $[1, X], X > 1$.

Let $\psi(x) = \int_1^X \frac{dx}{x} = \log X$, $\lim_{X\to\infty} \log X = \infty$.

Hence, $\int_1^\infty \frac{dx}{x}$ is divergent.

1.29.9. Convergence of the improper integral $\int_{-\infty}^b f(x)dx$ where f is integrable on $[X, b]\ \forall\ X < b$

Let $\psi(X) = \int_X^b f\,(x)dx, X < b.$

If $\lim_{X\to-\infty} \psi(X) = l$ (finite then the improper integral $\int_{-\infty}^b f(x)$ is said to be convergent and we write $\int_{-\infty}^b f(x) = l.$

If $\lim_{X \to -\infty} \psi(X)$ does not exist, then $\int_{-\infty}^{b} f(x)$ is said to be divergent.

1.29.10. Convergence of the improper integral $\int_{-\infty}^{\infty} f(x)dx$ where f is integrable on $[X_1, X_2]\ \forall\ X_1, X_2 \in \mathbb{R}$ with $x_1 < X_2$. Let $c \in \mathbb{R}$. If both the integrals $\int_{-\infty}^{c} f(x)dx$ and $\int_{c}^{\infty} f(x)dx$ are convergent, then $\int_{-\infty}^{\infty} f(x)dx$ is said to be convergent and we write

$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$

1.29.11. Convergence of improper integral $\int_{-\infty}^{\infty} f(x)dx$ where f has a finite number of points of infinite discontinuity $c_1, c_2, \ldots\ldots, c_k$.

Let $c_1 < c_2 < \ldots\ldots < c_k$. If each of integral $\int_{-\infty}^{c_1} f(x)dx, \int_{c_1}^{c_2} f(x)dx, \ldots\ldots, \int_{c_k}^{\infty} f(x)dx$ one convergent.

Then $\int_{-\infty}^{\infty} f(x)dx$ is said to be convergent and we write

$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \ldots\ldots + \int_{c_k}^{\infty} f(x)dx$

1.29.12. Tests for convergence of positive integrand.

i. **Theorem:** Let a function f be integrable on $[a, X]\ \ \forall X > a$ and $f(x) > 0\ \ \forall\ x \geq a$.

A necessary and sufficient condition for the convergence of the improper integral $\int_{a}^{\infty} f(x)dx$ is that $\exists\ a\ \ m > 0$ such that $\int_{a}^{X} f(x)dx < m\ \ \forall\ x > a$.

ii. **Theorem (Comparison Test):** Let the function f and g be both integrable on $[a, X]\ \forall\ X > a$ and $0 < f(x) \leq mg(x)\ \ \forall\ x \geq a$ with $m > 0$. Then

a. $\int_{a}^{\infty} g(x)dx$ is convergent $\Rightarrow \int_{a}^{\infty} f(x)dx$ in convergent

b. $\int_{a}^{\infty} f(x)dx$ in divergent $\Rightarrow \int_{a}^{\infty} g(x)dx$ in divergent.

iii. **Theorem [Comparison test (limit form):** Let the function f and g be both integrable on $[a, X]\ \forall\ X > a$ and $f(x) > 0, g(x) > 0\ \ \forall\ x \geq a$.

If $\lim_{x \to \infty} \frac{f(x)}{g(x)} = l$ (non-zero finite), then the two improper integrals $\int_{a}^{\infty} f(x)dx$ and $\int_{a}^{\infty} g(x)dx$ converge or diverge together.

iv. **Theorem (μ-test):** Let $f(x) > 0\ \ \forall\ x \geq a$. If $\lim_{x \to \infty} x^{\mu} f(x) = l$ (non-zero finite), then the improper integral $\int_{a}^{\infty} f(x)dx$ is convergent $\Leftrightarrow \mu > 1$.

Example-(1.141): Consider the improper integral $\int_{1}^{\infty} \frac{x^{m-1}}{1+x} dx$

Here, $f(x) = \frac{x^{m-1}}{1+x}$

Now, $\lim_{x \to \infty} x^{2-m} f(x) = \lim_{x \to \infty} \frac{x}{1+x} = 1$ (non-zero finite)

Hence by μ-test $\int_1^\infty \frac{x^{m-1}}{1+x} dx$ is convergent $\Leftrightarrow\ 2-m>1 \Rightarrow m<1$

1.29.13. Test for convergence of the improper integral on an infinite range of integration where the integrand may not keep same sign.

i. **Theorem (Cauchy):** Let $a \in \mathbb{R}$ and a function f be integrable on $[a, X]\ \forall\, X > a$. A necessary and sufficient condition for the convergence of the improper integral $\int_a^\infty f(x)dx$ is that for a given $\varepsilon > 0,\ \exists\ a\ X_0 > 0$ such that $\left|\int_{X_1}^{X_2} f(x)dx\right| < \varepsilon\ \ \forall\ X_1, X_2 > X_0$

ii. **Theorem:** An absolutely convergent improper integral $\int_a^\infty f(x)dx$ [where f is bounded and integrable on $[a, X]\ \forall\ X > a$] is convergent but the convergence is not true.

Example-(1.142):

Let a function f be defined on $[1, \infty]$ by $f(x) = \frac{(-1)^{n-1}}{n}, n \le x < n+1, n = 1,2,3 \ldots ..$

If can be verified that $\int_1^\infty f(x)dx$ is convergent but $\int_1^\infty |f|(x)dx$ is not convergent.

iii. **Theorem (Abel's test):** Let a function g be monotonic and bounded on $[a, \infty]$ and the integral $\int_a^\infty f(x)dx$ be convergent. Then the integral $\int_a^\infty f(x)dx$ is convergent.

iv. **Theorem (Dirichlet's test):** Let a function g be monotonic bounded on $[a, \infty]$ and $\lim_{x\to\infty} g(x) = 0$ and the integral $\int_a^X f(x)dx$ be bounded on $[a, X]\ \forall\, X > a$. Then the integral $\int_a^\infty f(x)g(x)dx$ is convergent.

Example-(1.143):

The gamma function $\Gamma(m) = \int_a^\infty x^{m-1}e^{-x}dx$ is convergent $\Leftrightarrow m > 0$.

1.30. Sequence of functions:

1.30.1. Definition: Let $D \subset \mathbb{R}$ and for each $n \in \mathbb{N}$, let $f_n : D \to \mathbb{R}$ be a function, Then $\{f_n\}$ is a sequence of functions on $D\ to\ \mathbb{R}$. D may be$[a, b], [a, \infty] \to$ closed intervals $(a, b), (a, \infty)\ \to$ open intervals.

1.30.2. Definition (Pointwise Convergent): The sequence of functions $\{f_n\}$ on $D\ to\ \mathbb{R}$ is said to be pointwise convergent if for each $x \in\ D, \{f_n(x)\}$ converges.

Let for each $x \in D, \{f_n(x)\} \to l_x\ as\ n \to \infty$. Define $f : D \to \mathbb{R}$ by $f(x) = l_x$ for each $x \in D$, Then $f(x)$ is said to be the limit function of $\{f_n(x)\}\ on\ D$. Write $\lim_{x\to\infty} f_n(x) = f(x)\ on\ D$.

Examples (1.144):f_n: $\mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = x^n, x \in \mathbb{R}, \forall\ n \in \mathbb{N}$ then $f_n(x)$ is a sequence of functions on $\mathbb{R}$. For each $x \in (-1,1)\{f_n(x)\}$ converges to 0 and for $x = 1$, $\{f_n(x)\}$ converges to 1. For all other $x \in \mathbb{R}$, the sequence $\{f_n(x)\}$is divergent. So, the sequence $\{f_n\}$ is pointwise convergent on $[-1,1]$ and the limit function f is defined by

$$f(x) = \begin{cases} 0, -1 < x < 1 \\ 1, x = 1 \end{cases}$$

ii. $f_n: \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{x}{n}, x \in \mathbb{R}, \forall\, n \in \mathbb{N}$. Then $f_n(x)$ converges to $0\ \forall\, n \in \mathbb{N}$. So its limit function is $f(x) = 0, x \in \mathbb{R}$.

iii. $f_n(x) = \tan^{-1}(nx)$, $x \in \mathbb{R}$, $x \in\ \mathbb{N}$

Then $\lim\limits_{x\to\infty} f_n(x) = \begin{cases} \frac{\pi}{2}, x > 0 \\ 0, x = 0 \\ \frac{-\pi}{2},\ x < 0 \end{cases}$

So, the sequence $\{f_n\}$ is pointwise convergent on $\mathbb{R}$ and the limit function $f(x) = \frac{\pi}{2}\ sin\ x$, $x \in \mathbb{R}$

iv. $f_n(x) = \frac{sin\,nx}{n}$, $x \in \mathbb{R}$ $\lim\limits_{x\to\infty} f_n(x) = 0 = f(x)$, $x \in \mathbb{R}$

v. Let $f_n(x) = ne^{-nx}, x \geq 0, n \in \mathbb{N}$

For all $x \geq 0, 0 \leq\ ne^{-nx} \leq \frac{1}{n}, (since\ e^{nx} > nx,\ x > 0)$

$\therefore\ \lim\limits_{x\to\infty} f_n(x) = 0 = f(x)$

1.30.3. Definition (Uniform Convergent): Let $D \subseteq \mathbb{R}$ and for each $n \in \mathbb{N}$ $f_n = D \to \mathbb{R}$, be a function. The sequence $\{f_n(x)\}$ is said to be uniformly convergent on D to a function f if corresponding to a pre-assigned $\varepsilon > 0\ \exists\ k(\varepsilon) \in \mathbb{N}$ such that for all $n \in D$, $|f_n(x) - f(x)| < \varepsilon\ \forall\, n \geq k$.

We write $\lim\limits_{x\to\infty} f_n = f$ uniformly on D or $f_n \to f$ uniformly on D.

f is said to be the uniform limit of $\{f_n\}$ on D.

If $\{f_n(x)\}$ is uniformly convergent on D to the function $f(x)$ then the sequence $\{f_n(x)\}$also converges pointwise on D to f. But the converges is not true.

Example-(1.145): Let $f_n(x) = x^n$, $x \in \mathbb{R}$, $x \in \mathbb{N}$. Then $\{f_n(x)\}$ converges on $[-1, 1]$ to the function f where $f(x) = \begin{cases} 0,\ -1 < x < 1 \\ 1,\ x = 1 \end{cases}$

Let $c \in (0,1)$. Then $|f_n(c) - f(c)| = c^n$ and let $0 < \epsilon < 1$. Then $|f_n(c) - f(c)| < \epsilon$ if $c^n < \epsilon$

as whenever $n \log\left(\frac{1}{c}\right) > \log\left(\frac{1}{\epsilon}\right)$

as whenever $n > \log\left(\frac{1}{\epsilon}\right) / \log\left(\frac{1}{c}\right)$.

Let $k = \left[\log\left(\frac{1}{\epsilon}\right) / \log\left(\frac{1}{c}\right)\right] + 1$Then$|f_n(c) - f(c)| < \epsilon\ \forall\, n \geq k$.

$\therefore\ \forall\, n \in (0,1), |f_n(x) - f(x)| < \epsilon\ \ \forall\, n \geq k, k = \left[\log\left(\frac{1}{\epsilon}\right) / \log\left(\frac{1}{x}\right)\right] + 1$

This k depends on ϵ *and* x. As $x \to 1,\ k \to \infty$

$\Rightarrow \quad \nexists\, k \in \mathbb{N}$ such that $x \in (0,1),\ |f_n(x) - f(x)| < \epsilon\ \forall\, n \geq k$.

Consequently $\{f_n\}$ is not uniformly convergent on $(0,1)$.

But $\{f_n\}$ is uniformly convergent on $[0,a]$, $0 < a < 1$ since, in $[0,a]$, the greatest value of $\log\left(\frac{1}{\epsilon}\right) / \log\left(\frac{1}{x}\right)$ is $\log\left(\frac{1}{\epsilon}\right) / \log\left(\frac{1}{a}\right)$