# Previous Year Question & Solution

# Real Analysis (June-2014)

## Part-B

**1.** Let $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$ be given by $f(x) = x^2$ then $f$ is uniformly continuous if

1. $A$ is bounded subset of $\mathbb{R}$
2. $A$ is a dense subset of $\mathbb{R}$
3. $A$ is an unbounded and connected subset of $\mathbb{R}$
4. $A$ is an unbounded and open subset of $\mathbb{R}$

**Sol.**

Given, $f: A \to \mathbb{R}, (where\ A \subseteq \mathbb{R})$ is defined by $f(x) = \ x^2$

Take $A = \mathbb{R}$

We know that $f: R \to \mathbb{R}$ defined by $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

[$\therefore$ $\mathbb{R}$ is dense, unbounded, connected and open subset of $\mathbb{R}$]

All are options are incorrect.

$\therefore$ options (1) is correct.

**2.** Let $\alpha, P$ be real numbers and $\alpha > 1$.

1. If $P > 1, then\ \int_{-\infty}^{\infty} \frac{1}{|x|^{P\alpha}} dx < \infty$
2. If $P > \frac{1}{\alpha}, then\ \int_{-\infty}^{\infty} \frac{1}{|x|^{P\alpha}} dx < \infty$
3. If $P < \frac{1}{\alpha}, then\ \int_{-\infty}^{\infty} \frac{1}{|x|^{P\alpha}} dx < \infty$
4. For any $P \in \mathbb{R}$; we have $\int_{-\infty}^{\infty} \frac{1}{|x|^{P\alpha}} dx < \infty$

**Sol.**

For option (1)

Given $\alpha > 1\ and\ P > 1$

Take $\alpha = 2, P = 2$.

$\therefore I = \int_{-\infty}^{\infty} \frac{1}{x^4} dx = \int_{-\infty}^{0} \frac{1}{x^4} dx + \int_{0}^{\infty} \frac{1}{x^4} dx = I_1 + I_2$

Clearly, $I_1$ and $I_2$ both are divergent.

$\therefore\ \int_{-\infty}^{\infty} \frac{1}{|x|^{P\alpha}} dx = \infty$

Thus, option (1) is incorrect.

With above example option (2) is also incorrect.

For option (3)

Take, $\alpha = 2, P = -2$

$\therefore I = \int_{-\infty}^{\infty} \frac{1}{|x|^{-4}} dx = \int_{-\infty}^{\infty} x^4 \, dx$

$\therefore I = \infty$

So, option (3) is incorrect.

Hence option (4) is correct.

**3.** Let $f: X \to Y$ be a function from a metric space $X$ to another metric space $Y$. For any Cauchy sequence $\{x_n\}$ in $X$

1. If $f$ is continuous then $\{f(x_n)\}$ is a Cauchy sequence in $Y$.
2. If $\{f(x_n)\}$ is Cauchy then $\{f(x_n)\}$ is always convergent in $Y$.
3. If $\{f(x_n)\}$ is Cauchy in $Y$ then $f$ is continuous.
4. $\{x_n\}$ is always convergent in $X$.

**Sol.**

For option (1)

Take, $X = (0, 1), Y = \mathbb{R}$

Let $f: X \to Y$ is defined by $f(x) = \frac{1}{x}$

Clearly, $f(x)$ is continuous on $X$.

Let $\{x_n\} = \left\{\frac{1}{n}\right\}$ be a sequence in $X$, which is clearly Cauchy in $X$.

But $\{f(x_n)\} = \{n\}$ is not a Cauchy sequence in Y.

$\therefore$ option (1) is incorrect

For option (2)

Take, $X = (0, 1), Y = (0, 1)$ and $f: X \to Y$ defined by $f(x) = x$

Let $\{x_n\} = \left\{\frac{1}{n}\right\}$ be Cauchy sequence in $X$

Also, $\{f(x_n)\} = \left\{\frac{1}{n}\right\}$ is Cauchy in $Y$.

But $\{f(x_n)\}$ is not convergent in $Y$.

Thus option (2) is incorrect.

For option (4)

Take, $X = (0,1) \& \{x_n\} = \left\{\frac{1}{n}\right\}$

Clearly, $\{x_n\}$ is Cauchy in $X$, but is not convergent in $X$.

As all other options are incorrect.

$\therefore$ option (3) is correct.

**4.** $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{1}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{5}}+\ldots\ldots+\frac{1}{\sqrt{2n-1}+\sqrt{2n+1}}\right)$ equals.

1. $\sqrt{2}$
2. $\frac{1}{\sqrt{2}}$
3. $\sqrt{2}+1$
4. $\frac{1}{\sqrt{2}+1}$

**Sol.**

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{1}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{5}}+\ldots..+\frac{1}{\sqrt{2n-1}+\sqrt{2n+1}}\right)$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{1}+\sqrt{3}}.\frac{\sqrt{3}-\sqrt{1}}{\sqrt{3}-\sqrt{1}} + \frac{1}{\sqrt{3}+\sqrt{5}}.\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}+\ldots\ldots+\frac{1}{\sqrt{2n-1}+\sqrt{2n+1}}.\frac{\sqrt{2n+1}-\sqrt{2n-1}}{\sqrt{2n+1}-\sqrt{2n-1}}\right)$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(\frac{\sqrt{3}-\sqrt{1}}{2}+\frac{\sqrt{5}-\sqrt{3}}{2}+\ldots.+\frac{\sqrt{2n+1}-\sqrt{2n-1}}{2}\right)$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(\frac{\sqrt{2n+1}-\sqrt{1}}{2}\right) = \lim_{n\to\infty} \frac{1}{2}\left(\frac{\sqrt{2n+1}-\sqrt{1}}{\sqrt{n}}\right) = \lim_{n\to\infty} \frac{1}{2}\left(\frac{\sqrt{2n+1}}{\sqrt{n}}-\sqrt{\frac{1}{n}}\right) = \frac{1}{\sqrt{2}}$$

**5.** Consider the following sets of functions on $\mathbb{R}$

$W =$ The set of constant functions on $\mathbb{R}$.

$X =$ The set of polynomial functions on $\mathbb{R}$.

$Y =$ The set of continuous functions $\mathbb{R}$.

$Z =$ The set of all functions on $\mathbb{R}$.

Which of these sets has the same cardinality as that of $\mathbb{R}$?

1. Only $W$
2. Only $W\ and\ X$
3. Only $W, X$ & $Z$
4. All of $W, X, Y$ & $Z$

**6.** Let $P(x)$ be a polynomial in the real variable $x$ of degree 5. Then $\lim\limits_{n\to\infty} \frac{P(x)}{2^n}$ is

1. 5
2. 1
3. 0
4. $\infty$

**Sol.**

$P(x) = x^5$

$\therefore \lim_{n\to\infty} \frac{n^5}{2^n}\left(\frac{\infty}{\infty} form\right) = \lim_{n\to\infty} \frac{5!}{2^n(log2)^5} = 0$

**7.** For the continuous function $f: \mathbb{R} \to \mathbb{R}$ let $Z(f) = \{x\epsilon\mathbb{R}: f(x) = 0\}$. Then $Z(f)$ is always

1. Compact
2. Open
3. Connected
4. Closed

**Sol.**

Given that $f: \mathbb{R} \to \mathbb{R}$ is a continuous function and $Z(f) = \{x \in \mathbb{R}: f(x) = 0\} = f^{-1}(0)$

Since, $f: \mathbb{R} \to \mathbb{R}$ is continuous so for any closed subset $E\ of\ \mathbb{R}$, its inverse image $f^{-1}(E)$ is also closed in $\mathbb{R}$.

Here, $\{0\}$ is closed in $\mathbb{R}$, so $Z(f) = f^{-1}(0)$ is closed in $\mathbb{R}$.

Thus, option (4) is correct.

Further, take $f(x) = x - x^2$

$\therefore Z(f) = \{0, 1\}$, which is neither open nor connected.

$\therefore$ Options (2) and (3) are incorrect.

Option (1) is also incorrect.

As, take $f(x) = 0\ \forall\ x\epsilon\mathbb{R}$

$\therefore Z(f) = R$, which is not compact.

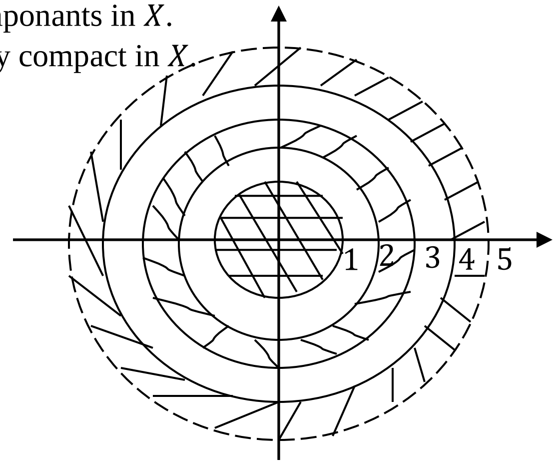
## PART-C

**8.** Let $X = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 5\}$ and

$k = \{(x, y) \in \mathbb{R}^2 : 1 \le x^2 + y^2 \le 2 \ or \ 3 \le x^2 + y^2 \le 4\}$. Then

1. $X \backslash k$ has three connected components.
2. $X \backslash k$ has no relatively compact connected component in $X$.
3. $X \backslash k$ has two relatively compact connected componants in $X$.
4. All connected components of $X \backslash k$ are relatively compact in $X$.

**Sol.**




From graph it is clear that $X \backslash k$ has three connected components. Also $X \backslash k$ has two relatively compact connected components in $X$.

$\therefore$ Options (1) and (3) are correct and options (2) and (4) are incorrect.

**9.** For two subsets $X \ and \ Y$ of $\mathbb{R}$, Let $X + Y = \{x + y : x \in X, y \in Y\}$.

1. If $X$ and $Y$ are open sets then $X + Y$ is open.
2. If $X$ and $Y$ are closed sets then $X + Y$ is closed.
3. If $X$ and $Y$ are compact sets then $X + Y$ is compact.
4. If $X$ is closed and $Y$ is compact then $X + Y$ is closed.

**Sol.**

Sum of two open sets is open.

$\therefore$ Option (1) is correct.

Sum of two compact set is compact.

Option (3) is correct.

Sum of two closed set is closed if any one of them is compact.

So, option (4) is correct.

For example, let $X = \left\{n + \frac{1}{n} : n \in \mathbb{N}\right\}, Y = \left\{-n + \frac{1}{n} : n \in \mathbb{N}\right\}$.

Clearly, $X$ and $Y$ are closed.

But $X + Y = \left\{\frac{2}{n} : n \in \mathbb{R}\right\}$ is not closed.

$\therefore$ Option (2) is incorrect.

**10.** Let $\{f_n\}$ be a sequence of functions on $\mathbb{R}$.

1. If $\{f_n\}$ converges to $f$ pointwise on $\mathbb{R}$, then $\lim\limits_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx = \int_{-\infty}^{\infty} f(x)dx$
2. If $\{f_n\}$ converges to $f$ uniformly on $\mathbb{R}$ then $\lim\limits_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx = \int_{-\infty}^{\infty} f(x)dx$
3. If $\{f_n\}$ converges to $f$ uniformly on $\mathbb{R}$ then $f$ is continuous on $\mathbb{R}$.
4. There exists a sequence of continuous functions $\{f_n\}$ on $\mathbb{R}$, such that $\{f_n\}$ converges to $f$ uniformly on $\mathbb{R}$, but $\lim\limits_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx \neq \int_{-\infty}^{\infty} f(x)dx$.

**Sol.**

If $\{f_n(x)\}$ be a sequence of continuous functions on $\mathbb{R}$ and $\{f_n(x)\}$ converges uniformly to $'f'$ on $\mathbb{R}$, then $f$ is also contiuous on $\mathbb{R}$.

$\therefore$ From the above result, option (3) is correct.

Now, take $f_n(x) = \begin{cases} \frac{1}{n}, & n \in [0,1] \\ 0, & elsewhere \end{cases}$

$\therefore f(x) = \lim\limits_{n\to\infty} f_n(x) = 0\ \forall\ x$, where $f(x)$ is pointwise limit of $\{f_n(x)\}$

But $\int_{-\infty}^{\infty} f_n(dx) = \int_0^1 \frac{1}{n} dx \ \ and \ \ \int_{-\infty}^{\infty} f(x)dx = 0$

Clearly, $\int_{-\infty}^{\infty} f(x)\,dx \neq \int_{-\infty}^{\infty} f_n(x)\,dx$

$\therefore$ Option (1) incorrect.

Also, in above example $\{f_n(x)\}$ converges uniformly to $f(x)$.

But $\int_{-\infty}^{\infty} f(x)\,dx \neq \int_{-\infty}^{\infty} f_n(x)\,dx$.

$\therefore$ Option (4) is correct and option (2) is incorrect.

**11.** Let $\{a_n\}, \{b_n\}$ be given bounded sequences of positive real numbers. Then (Here $a_n \uparrow a$ means an increase to a as $n$ goes to $\infty$, similarly $b_n \downarrow b$ means $b_n$ decreases to $b$ as $n$ goes to $\infty$.)

1. If $a_n \uparrow a$, then $sup_{n\geq 1}(a_n b_n) = a(sup_{n\geq 1} b_n)$
2. If $a_n \uparrow a$, then $sup_{n\geq 1}(a_n b_n) < a(sup_{n\geq 1} b_n)$
3. If $b_n \downarrow b$, then $inf_{n\geq 1}(a_n b_n) = (inf_{n\geq} a_n)b$.
4. If $b_n \downarrow b$, then $inf_{n\geq 1}(a_n b_n) > (inf_{n\geq} a_n)b$.

**Solution:** For option (1)

Take $a_n = 1 - \frac{1}{2n}, \forall$

**12.** Let $S \subset \mathbb{R}^2$ be define by $S = \left\{\left(m + \frac{1}{2^{|p|}}, n + \frac{1}{2^{|q|}}\right) : m, n, p, q \in z\right\}$. Then,

1. $S$ is discrete in $\mathbb{R}^2$
2. The set of limit points of $S$ is the set $\{(m, n) : m, n \in z\}$
3. $\mathbb{R}^2 \backslash S$ is connected but not path connected.
4. $\mathbb{R}^2 \backslash S$ is path connected

**Sol.**

Discrete Set: $A$ set is said to be discrete if its all points are isolated points.

Given, $S = \left\{\left(m + \frac{1}{2^{|p|}}, n + \frac{1}{2^{|q|}}\right) : m, n, p, q \in z\right\}$

Clearly, $\left(\frac{3}{2}, 1\right)$ is limit point of $S$ and $\left(\frac{3}{2}, 1\right) \epsilon \{(m, n) : m, n \in z\}$

$\therefore$ Options (1) and (2) are incorrect.

Also, $(1, 1) \in S$ and $(1, 1)$ is limit point

Thus, $S$ is not discrete set.

Also we know that complement of a countable set in $\mathbb{R}^2$ is path connected.

Here, $S$ is countable $[\therefore m, n, p, q \in z]$

$\therefore$ using the above result, $\mathbb{R}^2 \backslash S$ is path connected.

Hence, option (4) is correct and option (3) is incorrect.

**13.** Let $a, b, c$ be positive real numbers, $D = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : {x_1}^2 + {x_2}^2 + {x_3}^2 \leq 1\}$

$E = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \frac{{x_1}^2}{a^2} + \frac{{x_2}^2}{b^2} + \frac{{x_3}^2}{c^2} \leq 1\right\}$ and $A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$, $\det A > 1$.

Then, for a compactly supported continuous function $f$ on $\mathbb{R}^3$, which of the following are correct?

1. $\int_D f(Ax)dx = \int_E f(x)dx$
2. $\int_D f(Ax)dx = \frac{1}{abc}\int_D f(x)dx$
3. $\int_D f(Ax)dx = \frac{1}{abc}\int_E f(x)dx$
4. $\int_{\mathbb{R}^3} f(Ax)dx = \frac{1}{abc}\int_{\mathbb{R}^3} f(x)dx$

**Sol.**

Compactly supported: $A$ function $'f'$ is said to be compactly supported if it is zero outside a compact set.

Let$x = (x_1, x_2, x_3) \in \mathbb{R}^3$, be any arbitrary vector.

$$\therefore Ax = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (ax_1, ax_2, ax_3)$$

$$\therefore \int_D f(Ax)dx = \iiint_{x_1{}^2+x_2{}^2+x_3{}^2 \le 1} f(ax_1, bx_2, cx_3)dx_1dx_2dx_3 \ldots\ldots. (1)$$

Put $ax_1 = y_1,\ bx_2 = y_2, cx_3 = y_3$

$$\Rightarrow x_1 = \frac{y_1}{a}, x_2 = \frac{y_2}{b}, x_3 = \frac{y_3}{c} \Rightarrow dx_1 = \frac{1}{a}dy_1, dx_2 = \frac{1}{b}dy_2, dx_3 = \frac{1}{c}dy_3$$

$$\therefore x_1{}^2 + x_2{}^2 + x_3{}^2 \le 1 \Rightarrow \frac{y_1{}^2}{a^2} + \frac{y_2{}^2}{b^2} + \frac{y_3{}^2}{c^2} \le 1$$

From. . . . (1)

$$\int_D f(Ax)dx = \iiint_{\frac{x_1{}^2}{a^2}+\frac{x_2{}^2}{b^2}+\frac{x_3{}^2}{c^2} \le 1} f(y_1, y_2, y_3)\frac{dy_1}{a} \cdot \frac{dy_2}{b} \cdot \frac{dy_3}{c} \ldots\ldots. (1)$$

$$= \frac{1}{abc}\int_E f(y)dy \ or = \frac{1}{abc}\int_E f(x)dx.$$

Thus, option (3) is correct and options (1), (2) are incorrect $(As \det A > 1, \therefore abc > 1)$

Similarly, we can prove that $\int_{\mathbb{R}^3} f(Ax)dx = \frac{1}{abc}\int_{\mathbb{R}^3} f(x)dx$

$\therefore$ Option (4) is also correct.

**14.** Let $f: (0,1) \to \mathbb{R}$ be continuous. Suppose that $|f(x) - f(y)| \le |sin\, x - sin\, y|$ for all $x, y\epsilon(0,1)$. Then

1. $f$ is discontinuous at least at one point in (0,1)
2. $f$ is continuous everywhere on (0,1), but not uniformly on $(0,1)$.

3. $f$ is uniformly continuous on $(0, 1)$
4. $\lim_{x\to 0^+} f(x)$ exists.

**Sol.**

Take $f(x) = sin\, x$.

Clearly, $f(x) = \sin x$ is continuous on $(0, 1)$ and also,

$$|f(x) - f(y)| = |\sin x - \sin y| \leq |\sin x - \sin y|$$

$\therefore$ Option (1) is incorrect.

Now, $|f(x) - f(y)| \leq |\sin x - \sin y| \leq |x - y| \forall\ x, y\ (0, 1)$

$\Rightarrow f(x)$ is uniformly continuous on $(0, 1)$

Thus, option (3) is correct and option (2) is incorrect.

Also, $a$ function $f(x)$ is uniformly continuous on $(0, 1)$ $\quad (a\ and\ b\ are\ finite)$ if

1.) $f(x)$ is continuous on $(a, b)$.

2.) $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exists.

Hence, $\lim_{x\to 0^+} f(x)$ exists.

$\therefore$ option (4) is correct.

**15.** Let $P_n(x) = a_n x^2 + b_n x + c_n$ be a sequence of quadratic polynomials where $a_n, b_n, c_n \in \mathbb{R}$, for all $n \geq 1$. Let $\lambda_0, \lambda_1, \lambda_2$ be distinct real numbers such that $\lim_{n\to\infty} P_n(\lambda_0) = A_0 \lim_{n\to\infty} P_n(\lambda_1) = A_1$ and $\lim_{n\to\infty} P_n(\lambda_2) = A_2$. Then

1. $\lim_{n\to\infty} P_n(x)$ exists for all $x \in \mathbb{R}$
2. $\lim_{n\to\infty} P_n{}^1(x)$ exists for all $x \in \mathbb{R}$
3. $\lim_{n\to\infty} P_n\left(\frac{\lambda_0+\lambda_1+\lambda_2}{3}\right)$ does not exists.
4. $\lim_{n\to\infty} P_n{}^1\left(\frac{\lambda_0+\lambda_1+\lambda_2}{3}\right)$ does not exists.

**Sol.**

Take $a_n = b_n = c_n = 0$.

$$\therefore P_n(x) = 0$$

Clearly, options (3) and (4) are incorrect.

As both $\lim_{n\to\infty} P_n\left(\frac{\lambda_0+\lambda_1+\lambda_2}{3}\right)$ and $\lim_{n\to\infty} P_n{}^1\left(\frac{\lambda_0+\lambda_1+\lambda_2}{3}\right)$ exists.

Now, let $\lim_{n\to\infty} a_n = a$ , $\lim_{n\to\infty} b_n = b$ , $\lim_{n\to\infty} c_n = c$.

$\therefore \lim_{n\to\infty} P_n(x) = ax^2 + bx + c$. And $\lim_{n\to\infty} P_n(\lambda_0) = A_0$, $\lim_{n\to\infty} P_n(\lambda_1) = A_1$,

$$\lim_{n\to\infty} P_n(\lambda_2) = A_2$$

(given)

$$\Rightarrow \lim_{n\to\infty}\left(a_n{\lambda_0}^2 + b_n\lambda_0 + c_n\right) = A_0$$

$$\lim_{n\to\infty}\left(a_n{\lambda_1}^2 + b_n\lambda_1 + c_n\right) = A_1$$

$$\lim_{n\to\infty}\left(a_n{\lambda_2}^2 + b_n\lambda_2 + c_n\right) = A_2$$

i.e., $a{\lambda_0}^2 + b\lambda_0 + c = A_0, a{\lambda_1}^2 + b\lambda_1 + c = A_1, a{\lambda_2}^2 + b\lambda_2 + c = A_2$

I.e., $\begin{pmatrix} {\lambda_0}^2 & \lambda_0 & 1 \\ {\lambda_1}^2 & \lambda_1 & 1 \\ {\lambda_2}^2 & \lambda_2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix}$

Since, $\begin{pmatrix} {\lambda_0}^2 & \lambda_0 & 1 \\ {\lambda_1}^2 & \lambda_1 & 1 \\ {\lambda_2}^2 & \lambda_2 & 1 \end{pmatrix} \neq 0(\because\ \lambda_0, \lambda_1, \lambda_2$ are distinct)

$\Rightarrow a, b, c$ can be determined uniquely form above system.

$\therefore \lim_{n\to\infty} P_n(x) = ax^2 + bx + c$, a polynomial function and $\lim_{n\to\infty} {P_n}^1(x) = 2ax + b$.

Clearly, $\lim_{n\to\infty} P_n(x)$ and $\lim_{n\to\infty} {P_n}^1(x)$ exists $\forall\ x \in \mathbb{R}$.

$\therefore$ Option (1) and (2) are correct.

**16.** Define $f: \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (x + 2y + y^2 + |xy|, 2x + y + x^2 + |xy|)$ for $(x, y) \in \mathbb{R}^2$. Then,

1. $f$ is discontinuous at $(0, 0)$.
2. $f$ is continuous at $(0,0)$ but not differentiable at $(0,0)$
3. $f$ is differentiable at $(0,0)$.
4. $f$ is differentiable at $(0,0)$ and the derivative $Df(0,0)$ is invertible.

**Sol.**

Given $f$: $\mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = \left(f_1(x, y), f_2(x, y)\right)$

$$= (x + 2y + y^2 + |xy|, \qquad 2x + y + x^2 + |xy|).$$

**Result 1.** $f(x, y)$ continuous on $\mathbb{R}^2 iff\ f_1(x, y), f_2(x, y)$ both are continuous on $\mathbb{R}^2$.

**Result 2.** If the partial derivatives $(f_1)_{x,}(f_1)_y, (f_2)_{x,}(f_2)_y$exists and are continuous on $\mathbb{R}^2$, then $f(x, y)$ differentiable.Type equation here.

Since, $f_1(x, y)$ and $f_2(x, y)$, being polynomials in $x$ and $y$ are continuous on $\mathbb{R}^2$.

$\therefore$ By result (1), $f(x, y)$ is continuous on $\mathbb{R}^2$

Also, $\frac{\partial f_1}{\partial x} = 1 + \frac{x}{|x|} \cdot |y|, \frac{\partial f_1}{\partial y} = 2 + 2y + \frac{y}{|y|} \cdot |x|$

$$\frac{\partial f_2}{\partial x} = 2 + 2x + \frac{x}{|x|} \cdot |y|, \frac{\partial f_2}{\partial y} = 1 + \frac{y}{|y|} \cdot |x|$$

$$A + (x, y) = (0,0)$$

$$\frac{\partial f_1}{\partial x} = 1, \qquad \frac{\partial f_1}{\partial y} = 2, \qquad \frac{\partial f_2}{\partial x} = 2, \qquad \frac{\partial f_2}{\partial y} = 1$$

Clearly, all the partial derivatives exist and continuous on $\mathbb{R}^2$.

$\therefore$ By result (2), $'f'$ is differentiable at $(0,0)$.

Thus, options (1) and (2) are incorrect and option (3) is correct.

Further, $Df(0,0) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}_{(x,y)=(0,0)}$

$$= \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \Rightarrow |Df(0,0)| = -3 \neq 0$$

$\therefore$ $Df(0,0)$ is invertible

Thus, option (4) is correct.

**17.** Let $A = \{(x, y) \in \mathbb{R}^2 : x + y \neq -1\}$. Define $f: A \to \mathbb{R}^2 \; by \; f(x, y) = \left\{\frac{x}{1+x+y}, \frac{y}{1+x+y}\right\}$. Then

1. The Jacobian matrix of $f$ does not vanish on $A$.
2. $f$ is infinitely differentiable on $A$.
3. $f$ is injective on $A$.
4. $f(A) = \mathbb{R}^2$

**Sol.**

Given $A = \{(x, y) \in \mathbb{R}^2 : x + y \neq -1\}$ and $f: A \to \mathbb{R}^2$ is defined by

$$f(x, y) = \left(\frac{x}{1 + x + y}, \frac{y}{1 + x + y}\right) = \big(f_1(x, y), f_2(x, y)\big)$$

Now, $J(f)=\begin{pmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{1+y}{(x+y+1)^2} & -\frac{x}{(1+x+y)^2}\\ \frac{-y}{(1+x+y)^2} & \frac{1+x}{(1+x+y)^2}\end{pmatrix}=\frac{1}{(1+x+y)^2}\begin{pmatrix}1+y & -x\\ -y & 1+x\end{pmatrix}$

Clearly, $J(f)$ does not vanishes on $A$.

∴ Option (1) is correct.

Also, $|J(f)|=\frac{1}{(1+x+y)^2}(1+x+y+xy-xy)=\frac{1}{x+y+1}\neq 0$ on $A$.

Thus $f$ is invertable on $A$& hence injective on $A$.

∴ $f$ is infinitely differentiable on $A$.

∴ Option (3) is correct.

Further since all the partial derivatives exists on $A$ and continuous on $A$.

∴ $f$ is infinitely differentiable on $A$.

Thus, option (2) is correct.

Since, (1,0) don't have any pre-image.

$$\therefore f(A)\neq\mathbb{R}^2$$

Thus, option (4) is incorrect.

**18.** Which of the following are compact?

1. $\{(x,y)\in\mathbb{R}^2:(x,y)^2+(y-2)^2=9\}\cup\{(x,y)\in\mathbb{R}^2:y=3\}$.
2. $\left\{\left(\frac{1}{m},\frac{1}{n}\right)\in\mathbb{R}^2:m,n\in\mathbb{Z}\backslash\{0\}\right\}\cup\left\{\left(\frac{1}{m},0\right):m\in\mathbb{Z}\backslash\{0\}\right\}\cup\left\{\left(0,\frac{1}{n}\right):m\in\mathbb{Z}\backslash\{0\}\right\}$
3. $\{(x,y,z)\in\mathbb{R}^3:x^2+2y^2-3z^2=1\}$
4. $\{(x,y,z)\in\mathbb{R}^3:|x|+2|y|+3|z|\leq 1\}$

**Sol.**

For option (1)

Since, the set $\{(x,y)\in\mathbb{R}^2:y=3\}$ is unbounded

∴ $\{(x,y)\in\mathbb{R}^2:(x,y)^2+(y-2)^2=9\}\cup\{(x,y)\in\mathbb{R}^2:y=3\}$ is not compact.

For option (2)

Clearly, the set $\left\{\left(\frac{1}{m},\frac{1}{n}\right)\in\mathbb{R}^2:m,n\in\mathbb{Z}\backslash\{0\}\right\}\cup\left\{\left(\frac{1}{m},0\right):m\in\mathbb{Z}\backslash\{0\}\right\}\cup\left\{\left(0,\frac{1}{n}\right):m\in\mathbb{Z}\backslash\{0\}\cup\{0,0\}\right\}$ is closed and bounded.

∴ It is not compact.

For option (3)

Since, the set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + 2y^2 - 3z^2 = 1\}$ is not bounded but closed.

$\therefore$ It is compact.

For option (4)

Since, the set $\{(x, y, z) \in \mathbb{R}^3 : |x| + 2|y| + 3|z| \leq 1\}$ is closed and bounded.

$\therefore$ It is compact.

# Answer Table

| Sl. No | Answer | Sl. No | Answer | Sl. No | Answer |
|---|---|---|---|---|---|
| 1. | 1 | 2. | 4 | 3. | 3 |
| 4. | 2 | 5. | | 6. | 3 |
| 7. | 4 | 8. | 1 & 3 | 9. | 1, 3 & 4 |
| 10. | 3 & 4 | 11. | | 12. | 4 |
| 13. | 1 & 2 | 14. | 3 & 4 | 15. | 1 & 2 |
| 16. | 3 & 4 | 17. | 1, 2 & 3 | 18. | 2 & 4 |

# Previous Year Question & Solution

# Real Analysis (December-2014)

## Part-B

**1.** Let $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers. Then a necessary and sufficient condition for the sequence of polynomials $f(x) = b_n x + c_n x^2$ to converge uniformly to 0 on the real line is

1. $\lim\limits_{n\to\infty} b_n = 0$ and $\lim\limits_{n\to\infty} c_n = 0$
2. $\sum_{n=1}^{\infty} |b_n| < \infty$ and $\sum_{n=1}^{\infty} |c_n| < \infty$.
3. There exists a positive integer $N$ such that $b_n = 0$ and $c_n = 0$ for all $n > N$.
4. $\lim\limits_{n\to\infty} c_n = 0$

**Sol.**

Given, $f_n(x) = b_n x + c_n x^2$

Take, $b_n = \frac{1}{2^n}$, $c_n = \frac{1}{4^n}$

Clearly, $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$ and $\lim\limits_{n\to\infty} c_n = \lim\limits_{n\to\infty} \left(\frac{1}{4}\right)^n = 0$

But for $x = 2^n$.

$$f_n(x) = \frac{1}{2^n} \cdot (2^n) + \frac{1}{4^n} \cdot (2^n)^2 = 2 \nrightarrow 0 \; as \; n \to \infty$$

∴ Options (1) and (4) are incorrect.

Also, $\sum_{n=1}^{\infty} |b_n| = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n < \infty$ and $\sum_{n=1}^{\infty} |c_n| = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n < \infty$

But for $x = 2^n$, $f_n(x) \nrightarrow 0$, as $n \to \infty$.

Thus, option (2) is incorrect.

As all other options are ruled out.

∴ option (3) is correct.

**2.** Let $k$ be a positive integer. The radius of convergence of the series $\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} z^n$ is

1. $k$
2. $k^k$
3. $k^{-k}$
4. $\infty$

**Sol.**

Let $k = 2$, them series will be $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n$

The radius of convergence, $R = \lim\limits_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

Here, $a_n = \frac{(n!)^2}{(2n)!}$ and $a_{n+1} = \frac{((n+1)!)^2}{(2(n+1))!}$

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{(n!)^2}{(2n)!} \times \frac{(2(n+1))!}{((n+1)!)^2}\right|$$

$$= \lim_{n\to\infty} \left|\frac{(2n+2)(2n+1)}{(n+1)^2}\right|$$

$$= \lim_{n\to\infty} 2\left|\frac{2n+1}{n+1}\right| = \lim_{n\to\infty} \frac{2\left(2+\frac{1}{n}\right)}{1+\frac{1}{n}} = 4$$

$\Rightarrow$ when $k = 2$, then $R = 4 = 2^2$, so form there we conclude that $R = k^k$

Hence the option (2) is correct.

**3.** Suppose $P$ is a polynomial with real co-efficient. Then which of the following statements is necessarily true?

1. There is no root of the derivative $P^1$ between two real roots of the polynomial $P$.
2. There is exactly one of the derivative $P^1$ between any two real root of $P$.
3. There is exactly one root of the derivative $P^1$ between any two consecutive roots of $P$.
4. There is at least one root of the derivative of $P^1$ between any two consecutive roots of $P$.

**Sol.**

Let $\alpha$ and $\beta$ be two consecutive roots of $P(\alpha) \Rightarrow P(\alpha) = 0 = P(\beta)$

Also, every polynomial function is continuous as well as differentiable. $\therefore$ By mean-value theorem, $\exists$ atleast one point $c$ in $(\alpha, \beta)$ such that $P^1(c) = 0$

∴ Option (4) is correct.

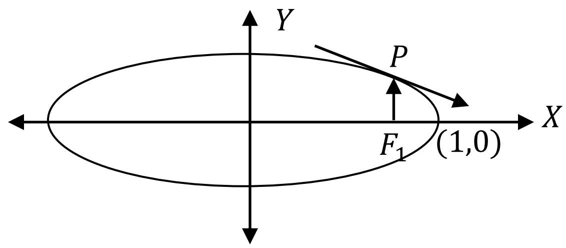
**4.** Let $G = \{(x, f(x)) : 0 \le x \le 1\}$ be the graph of a real valued differentiable function $f$. Assume that $(1, 0) \in G$. Suppose that the tangent vector to $G$ at any point is perpendicular to the radius vector at that point. Then which of the following is true?

1. $G$ is the arc of an ellipse.
2. $G$ is the arc of a circle.
3. $G$ is a line segment.
4. $G$ is the arc of a parabola.

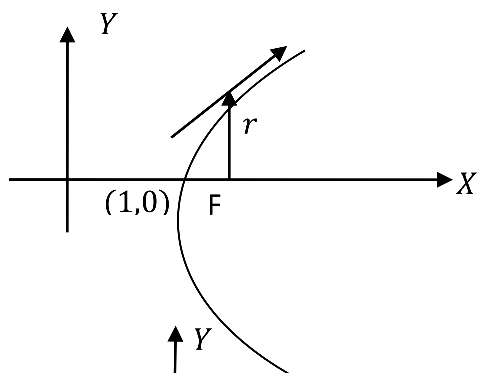
**Sol.**

Here, $G = \{(x, f(x)): 0 \le x \le 1\}$ is graph of a real valued differentiable function and the tangent vector to $G$ at any point is perpendicular to $G$ at that point. Clearly from graph given below, $G$ is the arc of the circle.

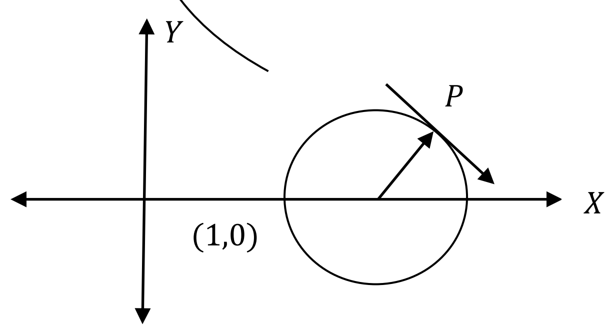
Here, radius vector at $P$ is not perpendicular to the tangent vector.




∴ $G$ can't be an ellipse.
Since, radius vector and tangent vector are not perpendicular.
∴ $G$ can't be an arc of a parabola.




Clearly, every radius vector is perpendicular to the tangent vector.
Hence, $G$ is the arc of the circle.



**5.** Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $f: \Omega \to \mathbb{R}$ be a differentiable function such that $(Df)(x) = 0$ for all $x \in \Omega$. Then which of the following is true?

1. $f$ must be a constant function.
2. $f$ must be constant on connected components of $\Omega$
3. $f(x) = 0$ or $1$ for $x \in \Omega$
4. The range of the function $f$ is a subset of $\mathbb{Z}$

**Sol**.

Let $\Omega = A \cup B \subseteq \mathbb{R}^2$, where $A = \{x = (x_1, x_2) \in \mathbb{R}^2 : (x_1 - 1)^2 + {x_1}^2 < 1\}$ and $B = \{x = (x_1, x_2) \in \mathbb{R}^2 : (x_1 - 4)^2 + {x_2}^2 < 1\}$

Clearly, $\Omega$ is open subset of $\mathbb{R}^2$.

Let $f: \Omega \to \mathbb{R}$ is differentiable function defined by

$$f(x) = \begin{cases} \frac{1}{2}, & \forall \in A \\ \frac{1}{3}, & \forall \in B \end{cases} \quad and\ (Df)(x) = 0 \quad \forall\, x \in \Omega$$

Clearly, options, (1), (3), (4) are incorrect and only option (2) is correct.

**6.** Let $\{a_n : n \geq 1\}$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} a_n$ is convergence and $\sum_{n=1}^{\infty} |a_n|$ is divergent. Let $\mathbb{R}$ be the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n x^n$. Then we can conclude that.

1. $0 < R < 1$
2. $R = 1$
3. $1 < R < \infty$
4. $R = \infty$

**Sol.**

Let $a_n = (-1)^{n-1} \frac{1}{n}$ , $n \geq 1$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} |a_n|$ is divergent.

$\sum_{n=1}^{\infty} a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$ . . . . . . is convergent by Leibnitz test and $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by P-test.

Now if $R$ be the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n x^n$.

Then, $\frac{1}{R} = \lim\limits_{n\to\infty} \frac{a_n+1}{a_n} = \lim\limits_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = 1 \Rightarrow R = 1$

Hence, option (2) is correct.

**7.** Let $P: \mathbb{R} \to \mathbb{R}$ be a polynomial of the form $P(x) = a_0 + a_1x + a_2x^2$, with $a_0, a_1, a_2 \in \mathbb{R}$ and $a_2 \neq 0$

Let $E_1 = \int_0^1 P(x)\,dx - \frac{1}{2}P(0) + P(1)$, $E_2 = \int_0^1 P(x)\,dx - P\left(\frac{1}{2}\right)$.

If $|x|$ is the absolute value of $x \in \mathbb{R}$, then

1. $|E_1| > |E_2|$
2. $|E_2| > |E_1|$
3. $|E_2| = |E_1|$
4. $|E_2| = 2|E_1|$

**Sol.**

Take, $a_0 = a_1 = 0$ and $a_2 = 1$

$$\therefore P(x) = x^2$$

$\therefore\ E_1 = \int_0^1 P(x)dx - \frac{1}{2}P(0) + P(1) = \int_0^1 x^2dx - \frac{1}{2}(0+1) = \frac{1}{3} - \frac{1}{2} = \frac{1}{6}$ and

$$E_2 = \int_0^1 P(x)dx - P\left(\frac{1}{2}\right) = \int_0^1 x^2dx - \frac{1}{4} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

Clearly, in this case $|E_1| > |E_2|$

Thus, for this example options (2), (3), (4) are incorrect.

Hence, option (1) is correct.

## PART- C

**8.** Let $E$ be a subset of $\mathbb{R}$. Then the characteristic function $X_E: \mathbb{R} \rightarrow \mathbb{R}$ is continuous if and only if

1. $E$ is closed.
2. $E$ is open.
3. $E$ is both open and closed
4. $E$ is neither open nor closed

**Sol.**

The characteristic function $\chi_E: \mathbb{R} \rightarrow \mathbb{R}$ is define as follows:$\chi_E(x) = \begin{cases} 1 & if \quad x \in E \\ 0 & if \quad x \in \mathbb{R} \backslash E \end{cases}$

Since, characteristic function $\chi_E: \mathbb{R} \rightarrow \mathbb{R}$ is continuous,

So, the set ${\chi_E}^{-1}(E)$ is closed in $\mathbb{R}$.

So, we have $E = {\chi_E}^{-1}(1)$ is closed

Also, $\{0\}$ is closed $\Rightarrow {\chi_E}^{-1}(0) = \mathbb{R} \backslash E$ is also closed.

$\Rightarrow E$ is open.

Hence, $E$ is both open and closed.

$\therefore$ Option (3) is correct.

**9.** Suppose that $P$ is a monic polynomial of degree $n$ in one variable with real co-efficient and $k$ is a real number. Then which of the following statements is\are necessarily true?

1. If $n$ is even and $k > 0$, then there exists $x_0 \in \mathbb{R}$ such that $P(x_0) = ke^{x_0}$.
2. If $n$ is odd and $k < 0$, then there exists $x_0 \in \mathbb{R}$ such that $P(x_0) = ke^{x_0}$.
3. For any natural number $n$ and $0 < k < 1$, there exists $x_0 \in \mathbb{R}$ such that $P(x_0) = ke^{x_0}$.
4. If $n$ is odd $k \epsilon \mathbb{R}$, then there exists $x_0 \in \mathbb{R}$ such that $P(x_0) = ke^{x_0}$

**Sol.**

Given that $P(x)$ is a monic polynomial of degree $n$.

Take, $g(x) = \frac{P(x)}{e^x}$

Clearly $g(x)$ is continuous on $\mathbb{R}$.

[$\because \frac{f(x)}{g(x)}$ is continuous in domain $D$ if $f(x)$ and $g(x)$ are continuous in $D$ and $g(x) \neq 0$ in $D$]

**Case – 1:** When $n$ is even.

$$\therefore \lim_{n\to\infty} g(x) = \lim_{n\to\infty} \frac{P(x)}{e^x} = 0$$

And $\lim_{n\to-\infty} g(x) = \lim_{n\to-\infty} \frac{P(x)}{e^x} = \infty$

Thus, as $g(x)$ is continuous.

$\therefore$ By intermediate value theorem.

$g(x)$ takes every value between 0 and $\infty$,

i.e., for any $k > 0, \exists$ some $x_0 \epsilon \mathbb{R}$ such that $g(x_0) = k$, i.e., $P(x_0) = ke^{x_0}$.

**Case – 2:**

When $n$ is odd.

$\therefore \lim_{n\to-\infty} g(x) = \lim_{n\to-\infty} \frac{P(x)}{e^x} = -\infty$ and $\lim_{n\to\infty} g(x) = \lim_{n\to\infty} \frac{P(x)}{e^x} = 0$.

As $g(x)$ is continuous.

$\therefore$ By intermediate value theorem,

$g(x)$ takes every value between $-\infty$ to 0, i.e., for any $k < 0, \exists$ some $x_0 \in \mathbb{R}$ such that $g(x_0) = k$

i.e., $P(x_0) = ke^{x_0}$.

Clearly, Options (1), (2) are correct and options (3), (4) are incorrect.

**10.** Let $\{a_k\}$ be an unbounded, strictly increasing sequence of positive real numbers and $x_k = (a_{k+1} - a_k)/a_{k+1}$ which of the following statements is/are correct?

1. For all $n \geq m, \sum_{k=m}^{n} x_k > 1 - \frac{a_m}{a_n}$
2. There exists $n \geq m$ such that $\sum_{k=m}^{n} x_k > \frac{1}{2}$
3. $\sum_{k=1}^{\infty} x_k$ converges to a finite limit.
4. $\sum_{k=1}^{\infty} x_k$ diverges to $\infty$.

**Sol.**

For option (1),

$$\sum_{k=m}^{n} x_k = \sum_{k=m}^{n} \frac{a_{k+1} - a_k}{a_{k+1}} \geq \sum_{k=m}^{n} \frac{a_{k+1} - a_k}{a_{n+1}}$$

$= \frac{a_{n+1}-a_m}{a_{n+1}} = 1 - \frac{a_m}{a_{n+1}} > 1 - \frac{a_m}{a_n} \Rightarrow \sum_{k=m}^{n} x_k > 1 - \frac{a_m}{a_n}$ . . . . . . . .(a)

∴ Option (1) is correct.

For option (2),

Since, $\{a_k\}$ is unbounded.

∴ $\exists\ n > m$ for which $a_n > 2a_m$

$\Rightarrow \frac{a_m}{a_n} < \frac{1}{2} \Rightarrow 1 - \frac{a_m}{a_n} > \frac{1}{2}$ . . . . . . . . (b)

Thus, using (a) and (b) we get $\exists\ n \geq m, \sum_{k=m}^{n} x_k > \frac{1}{2}$

Hence, option (2) is correct.

For option (3),

Take, $a_k = k$

Clearly, $\{a_k\}$ is unbounded and strictly increasing and $x_k = \frac{k+1-k}{k+1} = \frac{1}{k+1}$

∴ $\sum_{k=1}^{\infty} x_k = \sum_{k=1}^{\infty} \frac{1}{k+1}$, which is divergent by $P - test$.

∴ Option (3) is incorrect.

For option (4),

Clearly, $\lim_{k\to\infty} x_k = \lim_{k\to\infty} 1 - \frac{a_k}{a_{k+1}} \neq 0$, $\{\because \{a_k\}$ is unbounded and strictly increasing$\}$

$\Rightarrow \sum_{k=1}^{\infty} x_k$ is divergent and diverges to $\infty$.

**11.** For a non-empty subset $S$ and a point $x$ in a connected metric space $(X, d)$,

Let $d(x, S) = inf\{d(x, y): y \in S\}$. Which of the following statements is/are correct?

1. If $S$ is closed and $d(x, S) > 0$ then $x$ is not an accumulation point of $S$.
2. If $S$ is open and $d(x, S) > 0$ then $x$ is not an accumulation point of $S$.
3. If $S$ is closed and $d(x, S) > 0$ then $S$ does not contain $X$.
4. If $S$ is open and $d(x, S) = 0$ then $x \in S$.

**Sol.**

We know that $d(x, S) = 0$ $if\ and\ only\ if$ either $x \in S$ or $x$ is accumulation point of $S$.

i.e., $if\ and\ only\ if\ x \in \overline{S} = S \cup S'$

For option (1) and (3)

If $S$ is closed $\Rightarrow \overline{S} = S$.

Given, $d(x, S) > 0 \Rightarrow x$ is neither point of $S$ nor accumulation point of.

$\therefore$ Obtains (1) and (3) are correct.

For option (2)

If $S$ is open

Given $d(x, S) > 0$.

Again from above result $S$ is neither point of $x$ nor accumulation point of $S$.

$\therefore$ Option (2) is correct.

For option (4)

Take, $S = (1,2)$ and $x = 1$

Clearly, $d(x, S) = 0$

But $x \notin S$.

$\therefore$ Option (4) is incorrect.

**12.** Let $f$ be a continuously differentiable function on $\mathbb{R}$. Suppose that $L = \lim_{x\to\infty}(f(x) + f'(x))$ exists. If $0 < l < \infty$, then which of the following statements is/are correct?

1. If $\lim_{x\to\infty} f'(x)$ exists, then it is 0.
2. If $\lim_{x\to\infty} f(x)$ exists, then it is $L$.
3. If $\lim_{x\to\infty} f'(x)$ exists, then $\lim_{x\to\infty} f(x) = 0$.
4. If $\lim_{x\to\infty} f(x)$ exists, then $\lim_{x\to\infty} f'(x) = L$.

**Sol.**

Given $'f'$ is continuously differentiable function on $\mathbb{R}$ and $L = \lim_{x\to\infty}(f(x) + f'(x))\,, 0 < L < \infty$.

Take, $f(x) = 1$.

$\therefore \lim_{x\to\infty}(f(x) + f'(x)) = 1 = L$ and $\lim_{x\to\infty} f(x) = 1, \lim_{x\to\infty} f'(x) = 0$

Thus, options (3) and (4) are incorrect.

Now, option (1) and (2) holds simultaneously,

i.e., if option (1) is correct, then (2) is also correct and if (2) is correct then (1) is also correct.

Thus, both options (1) and (2) are correct.

**13.** Let $A$ be a subset of $\mathbb{R}$. Which of the following properties imply that $A$ is compact?

1. Every continuous function $f$ from $A$ to $\mathbb{R}$ is bounded.
2. Every sequence $\{x_n\}$ in $A$ has a convergent subsequence converging to a point in $A$.
3. There exists a continuous function from $A$ into $[0,1]$.
4. There is no one-to-one and continuous function from $A$ on $(0,1)$.

**Sol.**

Let $A$ be any subset of $\mathbb{R}$.

For option (1)

Given every continuous function $'f'$ from $A$ to $\mathbb{R}$ is bounded.

We will prove $A$ is closed and bounded.

**$A$ is bounded:** As every continuous function is bounded.

So, taking continuous function $f(x) = x$, we see that $A$ must be bounded.

**$A$ is closed:** Suppose, $\exists$ a limit point $'k'$ of $A$ which is not in $A$, i.e., $k \in \overline{A}$ but $k \notin A$.

Then, $\exists$ a function $f: A \to \mathbb{R}$ defined by $f(x) = \frac{1}{x-k}$, which is continuous on $A$ but not bounded. Which is a contradiction.

Thus, $\nexists$ any $k$ for which $k \in \overline{A}$ but $k \notin A$.

Hence, $A$ is closed.

$\therefore$ $A$ must be compact.

Thus, option (1) is correct.

For option (2),

Given, every sequence $\{x_n\}$ in $A$ has a convergent subsequence converging to a point in $A$.

$\Rightarrow$ $A$ is sequentially compact.

As we know that, in $\mathbb{R}$ a set is compact if and only if it is sequentially compact.

Hence, $A$ must be compact.

For option (3),

Take, $A = \mathbb{R}$

Let $f: A \to [0,1]$ is defined by $f(x) = |\sin x|$

Clearly, $f(x)$ is continuous and onto.

But, $A$ is compact.

$\therefore$ option (3) is incorrect.

For option (4),

Take, $A = [0,1]$

Clearly, there is no one-one and continuous function from $A$ onto $[0,1]$.

But here $A$ is not compact.

$\therefore$ Option (4) is incorrect.

**14.** Let $f$ be a monotonically increasing function from $[0,1]$ into $[0,1]$. Which of the following statements is/are true?

1. $f$ must be continuous at all but finitely many points in $[0,1]$.
2. $f$ must be continuous at all but countably many points in $[0,1]$.
3. $f$ must be Riemann integrable.
4. $f$ must be Lebesgue integrable.

**Sol.**

**Result – 1:** Set of discontinuities of a monotonically increasing function is at most countable.

$\therefore$ Option (2) is correct and (1) is incorrect.

**Result – 2:** If set of discontinuities of a function is of measure zero. Then the function is Riemann integrable.

As we know that every countable set is of measure zero.

Using result (1) and (2).

$'f'$ is Riemann integrable.

Thus option (3) is correct.

**Result – 3:** Every Riemann integrable function is Lebesgue integrable.

So, option (4) is correct.

**15.** Let $X$ be a metric space and $f: X \to \mathbb{R}$ be a continuous function. Let $G = \{(x, f(x)): x \epsilon X\}$ be the graph of $f$. Then

1. $G$ is homeomorphic to $X$
2. $G$ is homeomorphic to $\mathbb{R}$
3. $G$ is homeomorphic to $X \times \mathbb{R}$
4. $G$ is homeomorphic to $\mathbb{R} \times X$

**Sol.**

Let $X = \{0,1\}$ and $d$ is metric on defined as $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$

Let $f: X \to \mathbb{R}$ is continuous function defined by $f(x) = x$

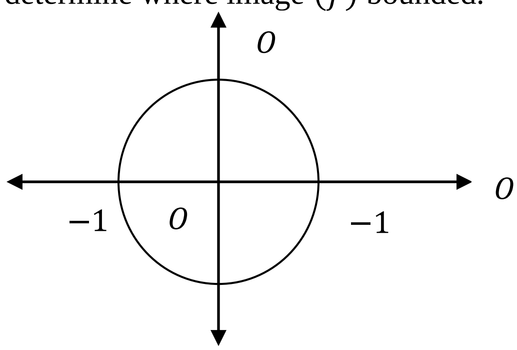
$$\therefore G(x) = \{(0,0), (1,1)\}$$

Clearly, $G$ is homeomorphic to $X$ only.

$\therefore$ Option (2), (3), (4) are incorrect and option (1) is correct.

**16.** Let $X = \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 = 1\}$ be the unit circle inside $\mathbb{R}^2$. Let $f: X \to \mathbb{R}$ be a continuous function. Then

1. Image $(f)$ is connected
2. Image $(f)$ is compact
3. The given information is not sufficient to determine where image $(f)$ bounded.
4. $f$ is not injective.




**Sol.**

Given, $f: X \to \mathbb{R}$ is continuous function,

Where $X = \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 = 1\}$.

Clearly, $X$ is unit circle centred at origin

Thus $X$ is connected and compact.

Also, we know that continuous image of compact set is compact and connected set is connected

$\therefore$ Options (1) and (2) are correct.

Let $f: X \to \mathbb{R}$ be a continuous function defined by $f(x, y) = x$.

Here, $f(0,1) = 0, \quad f(0, -1) = 0$

Thus, $f$ is not one-one

$\therefore$ Option (4) is correct.

# Answer Table

| Sl. No | Answer | Sl. No | Answer | Sl. No | Answer |
|---|---|---|---|---|---|
| 1. | 3 | 2. | 2 | 3. | 4 |
| 4. | 2 | 5. | 2 | 6. | 2 |
| 7. | 1 | 8. | 3 | 9. | 1 & 2 |
| 10. | 1, 2 & 4 | 11. | 1, 2 & 3 | 12. | 1 & 2 |
| 13. | 1 & 2 | 14. | 2, 3 & 4 | 15. | 1 |
| 16. | 1, 2 & 4 | | | | |

# Previous Year Question & Solution

## Real Analysis (June-2015)

### Part-B

**1.** The sum of the series $\frac{1}{1!}+\frac{1+2}{2!}+\frac{1+2+3}{3!}+$ ……. equals

**1)** $e$
**2)** $\frac{e}{2}$
**3)** $\frac{3e}{2}$
**4)** $1+\frac{e}{2}$

**Sol.**

The given series is $\sum_{n=1}^{\infty}\frac{n\,(n+1)}{2.n!}=\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{(n+1)}{(n-1)!}\right)$

$=\frac{1}{2}\left\{\sum_{n=1}^{\infty}\frac{n}{(n-1)!}+\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\right\}$

$$=\frac{1}{2}\{\sum_{n=1}^{\infty}\frac{n-1+1}{(n-1)!}+\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\}$$

$$=\frac{1}{2}\ \{\sum_{n=1}^{\infty}\frac{1}{(n-2)!}+\sum_{n=1}^{\infty}\frac{1}{(n-1)!}+\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\}$$

$$=\frac{1}{2}(e+e+e)=\frac{3e}{2}$$

∴ option (3) is correct.

**2.** The limit $\lim_{x\to 0}\frac{1}{x}\int_{x}^{2x}e^{-t^2}dt$

**1)** does not exist
**2)** is infinite
**3)** exists and equals 1
**4)** exists & equals 0

**Sol.**

$$\lim_{x\to 0}\frac{\int_{x}^{2x}e^{-t^2}dt}{x}\left(=\frac{0}{0}\right)=\lim_{x\to 0}\frac{2e^{-4x^2}-e^{-x^2}}{1}\ =\ 1$$

so, option (3) is correct.

**3.** Let $f : X \to X$ such that $f\left(f(x)\right) = x$ for all $x \in X$. Then

**1)** $f$ is one to one and onto

**2)** $f$ is one-to-one, but not onto

**3)** $f$ is into but not one-to-one

**4)** $f$ need not be either one to one or onto

**Sol.** Take, $f : X \to X$ be defined by $f(x) = x \Rightarrow f\left(f(x)\right) = x \forall x \in X$.

Clearly, f is one to one and onto

Thus, options (2), (3), (4) are incorrect

Hence, option (1) is correct.

**4.** A polynomial of odd degree with real co-efficient must have

**1)** at least one real root

**2)** no real root

**3)** only real roots

**4)** at least one root which is not real

**Sol.**

A polynomial of odd degree with real co-efficients has at least one real root because complex roots are in conjugate pairs.

**or,**

Take, $p(x) = x^3 - 1$

It has three roots $1, w, w^2$

$\therefore$option (2) and (3) are incorrect

Take, $p(x) = (x - 1)^3$.

Its roots are 1,1,1 i.e, at real roots.

So, option (4) is incorrect

Hence, option (1) is correct

**5.** Let for each $n \geq 1, C_n$ be the open disc in $\mathbb{R}^2$, with centre at the point $(n, 0)$ and radius equal to $n$; Then $C = \bigcup_{n\geq 1} C_n$ is

**1)** $\{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } |y| < x\}$

**2)** $\{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } |y| < 2x\}$

**3)** $\{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } |y| < 3x\}$

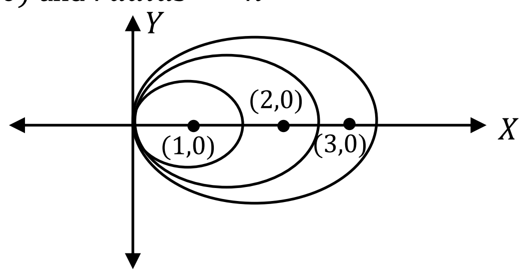
**4)** $\{(x, y) \in \mathbb{R}^2 : x > 0\}$

**Sol.** $C_n$ be the open disc in $\mathbb{R}^2$ with centre at $(n, 0)$ and $radius = n$

Draw $C_1, C_2, C_3$ ... ....From the graph it

Is clear that $\bigcup cn$ is the region with $x > 0$,

i.e., $C = \{(x, y) \in \mathbb{R}^2 : x > 0\}$




**6.** Let $f: \mathbb{R} \to \mathbb{R}$ be a Polynomial of the from

$f(x) = a_0 + a_1x + a_2x^2$ $with$ $a_0, a_1, a_2 \in \mathbb{R}$ $and$ $a_2 \neq 0.$

If $E_1 = \int_{-1}^{1} f(x)dx - \{f(-1) + f(1)\},$

$E_2 = \int_{-1}^{1} f(x)dx - \frac{1}{2}\{f(-1) + 2f(0) + f(1)\}$ and $|x|$ is the absolute value of $x \in \mathbb{R}$.

Then

**1)** $|E_1| < |E_2|$

**2)** $|E_1| = 2|E_2|$

**3)** $|E_1| = 4|E_2|$

**4)** $|E_1| = 8|E_2|$

**Sol.**

Given, $f: \mathbb{R} \to \mathbb{R}$ be a polynomial of the from

$$f(x) = a_0 + a_1x + a_2x^2, a_0, a_1, a_2 \in \mathbb{R} \text{ and } a_2 \neq 0$$

Take, $a_0 = a_1 = 0,\ a_2 = 1$

$\therefore\ f(x) = x^2$

Then,

$E_1 = \int_{-1}^{1} f(x)dx - \big(f(-1) + f(1)\big) = \frac{1}{2}\int_0^1 x^2\, dx - (1+1) = \frac{2}{3} - 2 = -\frac{4}{3}$

$E_2 = \int_{-1}^{1} f(x)dx = \frac{1}{2}\{f(-1) + 2f(0) + f(1)\} = \frac{2}{3} - \frac{1}{2}(1+0+1)$

$= \frac{2}{3} - 1 = \frac{1}{3}$

Clearly, $|E_1| = 4\,|E_2|$

$\therefore$ option (3) is correct.

## PART – C

**7.** Let $a$ be a positive real number, which of the following integrals are convergent?

**1)** $\int_0^a \frac{1}{x^4}\, dx$

**2)** $\int_0^a \frac{1}{\sqrt{x}}\, dx$

**3)** $\int_4^\infty \frac{1}{x \log_e x}\, dx$

**4)** $\int_5^\infty \frac{1}{x(\log_e x)^2}\, dx$

**Sol.**

We know that integral $\int_0^a \frac{1}{x^p}\, dx$ converges for

$p < 1$ and diverges for $p \geq 1$

In option (1), $p = 4 > 1$

$\therefore \int_0^a \frac{1}{x^4}\, dx$ diverges.

Thus, option (1) is incorrect

In option (2) $p = \frac{1}{2} < 1$

$\therefore \int_0^a \frac{1}{\sqrt{x}}\, dx$ converges.

Thus, option (2) is correct.

Further, by Cauchy's integral test, the series

$\sum_{n-1}^{\infty} \frac{1}{n(\log_e n)^p}$and integral $\int_a^{\infty} \frac{1}{x(\log_e x)^p} (a > 0)dx$

Behave alike and we know the series

$\sum_{n=1}^{\infty} \frac{1}{n(\log_e n)^p}$ converges for $p > 0$ and diverges for $p \le 1$.

Thus, $\int_a^{\infty} \frac{1}{x(\log_e x)^p} dx$ converges for $p > 1$ and diverges for $p \le 1$.

In option (3),

$p = 1$ and in option (4) $p = 2 > 1$.

Thus $\int_4^{\infty} \frac{1}{x \log_e x} dx$ diverges and $\int_5^{\infty} \frac{1}{x (\log_e x)^2} dx$ converges.

**8.** For $n \ge 1, let\ g_n(x) = sin^2 \left(x + \frac{1}{n}\right), x \in [0, \infty)\ and\ f_n(x) = \int_0^x g_n(t)dt.$

Then

**1)**$\{fn\}$ converges pointwise to a function f on [0, ∞), but does not converge uniformly on $[0, \infty)$.

**2)**$\{f_n\}$ does not converge pointwise to any function on $\{0, \infty)$.

**3)**$\{f_n\}$ converges uniformly on $[0,1]$

**4)**$\{f_n\}$ converges uniformly on $[0, \infty)$.

**Sol.** Given, $g_n(x) = Sin^2(x + \frac{1}{n})\ and\ f_n\ (x) = \int_0^x g_n(t)dt = \int_0^x Sin^2\ (t + \frac{1}{n})dt$

$$= \int_0^x \frac{1}{2}\left\{1 - cos2\left(t + \frac{1}{n}\right)\right\} dt = \frac{x}{2} - \frac{1}{4}\sin\left(2x + \frac{2}{n}\right) + \frac{1}{4}\ Sin\ \left(\frac{2}{n}\right)$$

$$Thus, f\ (x) = \lim_{n\to\infty} f_n(x) = \frac{x}{2} - \frac{1}{4} Sin\ (2x)$$

Now, $|f_n(x) - f(x)| = \frac{1}{4}\left|Sin\ \left(2x + \frac{2}{n}\right) - \sin(2x) - Sin\ \left(\frac{2}{n}\right)\right|$

$\le \frac{1}{4}\left|Sin\ \left(2x + \frac{2}{n}\right) - Sin(2x)\right| + \left|\frac{1}{4}\left(sin\ \frac{2}{n}\right)\right| \le \frac{1}{4}\left(2x + \frac{2}{n} - 2x\right) + \frac{1}{4}\left(\frac{2}{n}\right)$

$\Rightarrow |f_n(x) - f(x)| \le \frac{1}{n} = M_n(say) and\ \frac{1}{n} \to 0\ as\ n \to \infty$

∴ By $M_n -\ test$

$\{f_n(x)\}\ converges\ uniformly\ on\ [0, \infty)\ and\ hence\ [0,1\}$ also.

Thus, option (3) and (4) are correct.

**9.** Which of the following sets in $\mathbb{R}^2$ have positive Lebesgue measure?

$$\begin{bmatrix} For\ two\ sets, A, B \subseteq \mathbb{R}^2 \\ A + B = \{a + b | a \in A, b \in B\} \end{bmatrix}$$

**1)** $S = \{(x, y) : x^2 + y^2 = 1\}$

**2)** $S = \{(x, y) : x^2 + y^2 < 1\}$

**3)** $S = \{(x, y): x = y\} + \{(x, y) : x = -y\}$

**4)** $S = \{(x, y) | x = y\} + \{(x, y) : x = y\}$

**Sol.**

We know that in $\mathbb{R}^2$, Lebesgue measure of a set is area occupied by that set for option (1)

Clearly, $the\ sets = \{(x, y) : x^2 + y^2 = 1\}$ do not occupy any area.

∴ Its Lebesgue measure is zero for option (2)

For $S = \{(x, y) : x^2 + y^2 < 1\}$ Since, area of $the\ set = \pi\ (1)^2 = \pi > 0$

Hence, the above set has positive Lebesguemeasure, i.e., $\pi$

For option (3),

Clearly, the set $S = \{(x, y) | x = y\} + \{(x, y) | x = -y\} = \mathbb{R}^2$ have infinite Lebesgue measure.

For option (4),

S = $\{(x, y) | x = y\}$ + $\{(x, y) : x = y\}$ = $\{(x, y) | x = y\}$ which is a straight line and in $\mathbb{R}^2$, straight lines has measured zero.

Hence options (2) and (3) are correct.

**10.** Let $f$ be a bounded function on $\mathbb{R}$ and $a \in \mathbb{R}$. For$\delta > 0$.

Let $w\ (a, \delta) = Sup|f(x) - f(a)|, x \in [a - \delta, a + \delta]$. Then

**1)** $w\ (a, \delta_1) \leq w\ (a, \delta_2)\ if\ \delta_1 \leq \delta_2$

**2)** $\lim_{\delta \to 0^+} w(a, \delta) = 0\ for\ all\ a \in \mathbb{R}$

**3)** $\lim_{\delta \to 0^+} w(a, \delta)\ need\ not\ exists.$

**4)** $\lim_{\delta \to 0^+} w(a, \delta) = 0$ if and only if $f$ is continuous at $a$.

**Sol.**

Given, for a bounded function '$f$' on $\mathbb{R}$ and for $a \in \mathbb{R}, \delta > 0$.

Let $w(a,\delta) = Sup|f(x) - f(a)|, x \in [a - \delta, a + \delta]$

For option (1),

If $\delta_1 \leq \delta_2 \Rightarrow [a - \delta_1, a + \delta_1] \subseteq [a - \delta_2, a + \delta_2]$

$$\Rightarrow \sup_{E_1} |f(x) - f(a)| \leq \sup_{E_2} |f(x) - f(a)|$$

Where$E_1 = [a - \delta_1, a + \delta_1]$ and $E_2 = [a - \delta_2, a + \delta_2] \Rightarrow w(a,\delta_1) \leq w(a,\delta_2)$

$\therefore$ option (1) is correct

For option (2)

Let $f(x) = \begin{cases} 2, x = 1 \\ 0, elsewhere \end{cases}$

Clearly, $f(x)$ is bounded in $\mathbb{R}$

Take $a = 1$

Then, $\lim_{\delta \to 0^+} w(a,\delta) = \lim_{\delta \to 0^+} w(1,\delta) = 2$

Thus, option (2) is incorrect

For option (3)

Since $f$ is bounded on $\mathbb{R}$

$Sup|f(x) - f(a)|$ always exists and hence, $\lim_{\delta \to 0^+} w(a,\delta)$ always exist.

$\therefore$ option (3) is incorrect

For option (4)

Let, $\lim_{\delta \to 0^+} w(a,\delta) = 0$

$$\Leftrightarrow \lim_{\delta \to 0^+} Sup|f(x) - f(a)| = 0, x \in [a - \delta, a + \delta]$$

$$\Leftrightarrow \lim_{\delta \to 0^+} f(x) = f(a), x \in [a - \delta, a + \delta]$$

$f(x)$ is continuous at $x = a$

$\therefore$ option (4) is correct.

**11.** For $n \geq 2$, let $a_n = \dfrac{1}{n \log n}$. Then

**5)** The sequence $\{a_n\}_{n=2}^{\infty}$ is convergent

**6)** The series $\sum_{n=2}^{\infty} a_n$ is convergent.

**7)** The series $\sum_{n=2}^{\infty} a_n{}^2$ is convergent

**8)** The series $\sum_{n=2}^{\infty}(-1)^n a_n$ is convergent.

**Sol.**

Here, $a_n = \dfrac{1}{n \log n}$

(i) Since, $\lim\limits_{n\to\infty} a_n = 0$

$\therefore$ sequence $\{a_n\}_{n=0}^{\infty}$ is convergent

Thus option (1) is correct

(ii) Also, the series $\sum_{n=2}^{\infty} \dfrac{1}{n(\log n)^p}$ converges for $p > 1$ and diverges for $p \le 1$.

Thus, $\sum_{n=2}^{\infty} \dfrac{1}{n \log n}$ is divergent

$\therefore$ option (2) is incorrect

(iii) Now, $\sum_{n=2}^{\infty} a_n{}^2 = \sum_{n=2}^{\infty} \dfrac{1}{n^2(\log n)^2}$

Choose, $b_n = \frac{1}{n^2}$

$\therefore \lim\limits_{n\to\infty} \frac{a_n{}^2}{b_n} = \lim\limits_{n\to\infty} \frac{1}{(\log n)^2} = 0$ and $\sum \frac{1}{n^2}$ is convergent.

$\therefore$ By comparison Test, $\sum_{n=2}^{\infty} \dfrac{1}{n^2(\log n)^2}$ is also convergent

$\therefore$ option (3) is correct

(iv) Further, $\sum_{n=2}^{\infty}(-1)^n$ an is convergent (By Leibnitz test)

$\therefore$ option (4) is correct.

**12.** Which of the following sets of function are uncountable? ($\mathbb{N}$ stands for the set of natural numbers)

**1)** $\{f \mid f: \mathbb{N} \to \{1,2\}\}$

**2)** $\{f \mid f: \{1,2\} \to \mathbb{N}\}$

**3)** $\{f \mid f: \{1,2\} \to \mathbb{N},\ f(1) \le f(2)\}$

**4)** $\{f | f: \mathbb{N} \to \{1,2\},\ f(1) \leq f(2)\}$

**Sol.**

For option (1)

We have to find $f$ such that $f: \mathbb{N} \to \{1,2\}$ is a function. It is equivalent to fill $\mathbb{N}$ places using the elements 1 and 2.

Infinite product of countable sets is uncountable.

∴ option (1) is correct.

For option (2)

We have to find f such that $f: \{1,2\} \to \mathbb{N}$ is a function.

It is equivalent to set $\mathbb{N} \times \mathbb{N}$ which is countable as $\mathbb{N}$ is countable.

For option (3)

Clearly, $\{f | f: \{1,2\} \to \mathbb{N}, f(1) \leq f(2)\}$ is a subset of $\{f | f: \{1,2\} \to \mathbb{N}\}$ and subset of countable set is also countable.

For option (4)

It is equivalent to fill $\mathbb{N}$ places in which first place must be 1 and other places are either 1 or 2, which is again infinite. Product of countable sets and infinite product of sets is uncountable.

∴ option (4) is correct.

**13.** Let $\{a_{0,} a_{1,} a_2 \ldots\}$ be asequence of real numbers. For any $k \geq 1, let\ S_n = \sum_{k=0}^{n} a_{2k}$. Which of the following statements are correct?

**1)** If $\lim_{n\to\infty} S_n$ exists, then $\sum_{m=0}^{\infty} a_m$ exists.

**2)** If $\lim_{n\to\infty} S_n$ exists, then $\sum_{n=0}^{\infty} a_m$ need not exist.

**3)** If $\sum_{m=0}^{\infty} a_m$ exists, then $\lim_{n\to\infty} S_n$ exists.

**4)** If $\sum_{n=0}^{\infty} a_m$ exists, then $\lim_{n\to\infty} S_n$ need not exist.

**Sol.** Let $\{0,1,0,1,0,1, \ldots\}$ be a sequence of real numbers. $S_n = \sum_{k=0}^{n} a_{2k}$.

clearly, $\lim_{n\to\infty} S_n = \sum_{k=0}^{\infty} a_{2k} = 0$ exists, but $\sum_{m=0}^{\infty} a_m = \sum 1$ doesn't exist.

Thus option (1) is incorrect and option (2) is correct.

Take, $\sum_{m=0}^{\infty} a_m = \sum_{m=0}^{\infty} \frac{(-1)^m}{m}$.

Clearly, $\sum_{m=0}^{\infty} a_m = \sum_{m=0}^{\infty} \frac{(-1)^m}{m}$ is convergent by alternating series test and

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{m} = log2$$

But $S_n = \sum_{k=0}^{n} a_{2k} = \sum_{k=0}^{n} \frac{1}{2k} = \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{k}$ and $\lim_{n\to\infty} S_n$ doesn't exist.

Hence option (3) is incorrect and option (4) is correct.

## Answer Table

| Sl. No | Answer | Sl. No | Answer | Sl. No | Answer |
|---|---|---|---|---|---|
| 1. | 3 | 2. | 3 | 3. | 1 |
| 4. | 1 | 5. | 4 | 6. | 3 |
| 7. | 2 & 4 | 8. | 3 & 4 | 9. | 2 & 3 |
| 10. | 1 & 4 | 11. | 1 , 3 & 4 | 12. | 1 & 4 |
| 13. | 2 & 4 | | | | |

# Previous Year Question & Solution

# Real Analysis (December-2015)

## Part-B

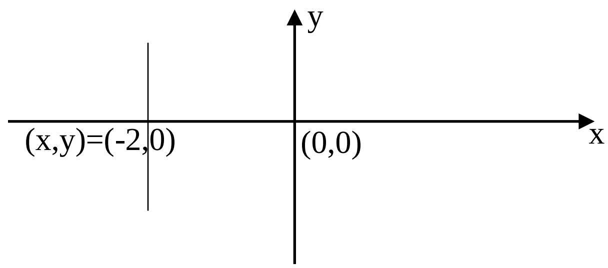
**1.** For $(x, y) \in \mathbb{R}^2$ with $(x, y) \neq (0,0)$, let $\theta = \theta(x, y)$ be the unique real number such that $-\pi\theta \leq \pi$ and $(x, y) = (r\, cos\theta, r\, sin\theta)$, where $r = \sqrt{x^2 + y^2}$. Then the resulting function $\theta: R^2|\{(0,0)\} \to \mathbb{R}$ is

**1)** differentiable

**2)** continuous, but not differentiable

**3)** bounded, but not continuous

**4)** neither bounded, nor continuous




**Sol.**

If we approach to $(-2,0)$ from vertically up-words, than$\theta \to \pi$ but if we approach to $(-2,0)$ from vertically down-word $\theta \to -\pi$.

Thus, these two limits are not equal.

Hence, $\theta$ is not continuous clearly $\theta$ Is bounded.

**2.** Let $f: \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function, with

$$f(0) = f(1) = f'(0) = 0.$$

Then

**1)** $f''$ is the zero function

**2)** $f''(0)$ is zero

**3)** $f''(x) = 0$ for some $x \in (0,1)$

**4)** $f''$ never vanishes

**Sol.** Consider the function $f(x) = x^2(x-1) = x^3 - x^2$.

Clearly, $f(0) = 0, f(1) = 0$ and $f'(x) = 3x^2 - 2x \Rightarrow f'(0) = 0$

Thus, '$f$' satisfies all the conditions of the given statements.

Now $f''(x) = 6x - 2 = 2(3x-1)$

Since, $f''(x)$ is not the zero function.

$\therefore$ option (1) is incorrect.

Also, $f''(0) = -2 \neq 0$

$\therefore$option (2) is incorrect.

Further, $f''(x) = 0 \Rightarrow 2(3x-1) = 0 \Rightarrow x = \frac{1}{3} \in (0,1)$

Clearly, option (4) is incorrect and hence, option (3) is correct.

**3.** $\lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{2}+\sqrt{4}} + \frac{1}{\sqrt{4}+\sqrt{6}} + \cdots + \frac{1}{\sqrt{2n}+\sqrt{2n+2}}\right)$ is

**1)** $\sqrt{2}$

**2)** $\frac{1}{\sqrt{2}}$

**3)** $\sqrt{2}+1$

**4)** $\frac{1}{\sqrt{2}+1}$

**Sol.**

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{2}+\sqrt{4}} + \frac{1}{\sqrt{4}+\sqrt{6}} + \cdots + \frac{1}{\sqrt{2n}+\sqrt{2n+2}}\right)$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(\frac{\sqrt{4}-\sqrt{2}}{2} + \frac{\sqrt{6}-\sqrt{4}}{2} + \cdots + \frac{\sqrt{2n+2}-\sqrt{2n}}{2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\sqrt{2n+2}-\sqrt{2}}{2\sqrt{n}}\right) = \frac{1}{2}\lim_{n\to\infty}\left(\sqrt{2+\frac{2}{n}} - \sqrt{\frac{2}{n}}\right) = \frac{1}{2}\sqrt{2} = \frac{1}{\sqrt{2}}$$

So, option (2) is correct.

**4.** Let $S_n = \sum_{k=1}^{n} \frac{1}{k}$ which of the following is true?

**1)** $S_{2^n} \geq \frac{n}{2}$ for every $n \geq 1$

**2)** $S_n$ is a bounded sequence

**3)** $|S_2 - S_{2^{n-1}}| \to 0$ as $n \to \infty$

**4)** $\frac{S_n}{n} \to 1$ as $n \to \infty$

**Sol.** $S_n = \sum_{k=1}^{n} \frac{1}{k}$ , $S_{2^n} = \sum_{k=1}^{2^n} \frac{1}{k}$

$S_1 = 1 = 1 + 0.\left(\frac{1}{2}\right)$, $S_2 = 1 + \frac{1}{2} = 1 + 1.\left(\frac{1}{2}\right)$,

$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + 2\left(\frac{1}{2}\right)$

$$\mathrm{S}_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$

$$= 1 + 3 \cdot \frac{1}{2}$$

$$S_{2^n} \geq 1 + n\left(\frac{1}{2}\right) > \frac{n}{2} S_{2^n} \geq \frac{n}{2} \forall n \geq 1$$

Thus option (1) is correct

Clearly, $S_n$ is unbounded sequence

$\Rightarrow$ option (2) is incorrtect

further, $S_n$=1 + $\frac{1}{2} + \cdots + \frac{1}{n}$

As$\frac{1}{n} \to 0 \frac{S_n}{n} \to$0, as $n \to \infty$

$\therefore$option (4) is incorrect.

**5.** Let $A$ be a closed subset of $\mathbb{R}, A \neq \Phi, A \neq \mathbb{R}$. Then $A$ is

**1)** the closure of the interior of $A$.

**2)** a countable set

**3)** a compact set

**4)** not open.

**Sol.** For option (1) and (3)

Take, $A = Z$

$$\therefore (\bar{Z}^0) = \overline{\Phi} = \Phi \neq A$$

So, option (1) is incorrect

Also $Z$ is not compact so, option (3) is incorrect.

For options (2) and (4)

Take. $A = [0,1]$

As $A$ is closed but uncountable, so option (2) is incorrect.

Hence option (4) is correct.

**6.** Let $f: [0, \infty) \to [0, \infty)$be a continuous function which of the following is correct?

**1)** There is $x_0 \in [0,\infty)$ such that $f(x_0) = x_0$

**2)** If $f(x) \leq M$ for all $x \in [0, \infty)$ for some $M > 0$, then there exists $x_0 \in [0, \infty)$ such that

$f\ (x_0) = x_0$

**3)** If $f$ has a fixed point, then it must be unique.

**4)** $f$ does not have a gfixed point unless it is differentiable on $(0, \infty)$.

**Sol.**

For option (1)

Take, $f\ (x) = e^x$

Clearly, $f\ (x)$ is continuous, but $\nexists x_0 \in [0,\infty)$ for which $f(x_0) = x_0$

∴ option (1) is incorrect

For option (3)

Take $f\ (x) = x$

Clearly, '$x = 0$' and '$x = 1$' are two fixed points

∴ option (3) is incorrect

For option (4)

Take, $f\ (x) = |x - 1|$

Clearly, $f(x)$ is not differentiable at $x = 1$, but it has fixed point at $x = \frac{1}{2}$

∴ option (4) is incorrect

As all other option are incorrect

∴ option (2) is correct

## Part – C

**7.** Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $\underset{x \in R}{sup} |f'(x)| < \infty$. Then,Type equation here.

1) $f$ maps a bounded sequence to a bounded sequence.
2) $f$ maps a Cauchy sequence to a Cauchy sequence.
3) $f$ maps a convergent sequence to a convergent sequence.
4) $f$ is uniformly continuous.

**Sol.**

Given $f : \mathbb{R} \to \mathbb{R}$ is differentiable function such that $\underset{x \in \mathbb{R}}{sup} |f'(x)| < \infty$.

Let $M = \underset{x \in \mathbb{R}}{sup} |f'(x)| < \infty$.

$\Rightarrow f'(x) \leq M \; \forall \; x \in \mathbb{R}$, i.e., $|f(x) - f(y)| \leq M|x - y| \forall \; x, y \in \mathbb{R}$ …….(1)

$\Rightarrow 'f'$ is Lipschitz continuous and hence uniformly continuous on $\mathbb{R}$.

Thus, option (4) is correct

Further, let $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}$.

Let $\in > 0$, be given

$\therefore$ $\exists$a positive integer $m \; (\in)$ such that $|x_n - x_m| < \in \forall \; n \geq m$.

$\therefore |f(x_n) - f(x_m)| \leq M|x_n - x_m| < M. \in = \in'$ (say)

$\Rightarrow \{f(x_n)\}$ is a Cauchy sequence.

Thus, $f$ maps a Cauchy sequence to a Cauchy sequence.

Thus, option (3) is correct

Also, in $\mathbb{R}$ a sequence is convergent if and only if Cauchy.

$\therefore$ option (1) and (2) are also correct.

**8.** For $(x, y) \in \mathbb{R}^2$, consider the series $\lim_{n\to\infty} \sum_{\ell,k=0}^{n} \frac{k^2 x^k y^\infty}{\ell!}$. Then the series is converges for $(x, y)$ in

1) $(-1,1) \times (0, \infty)$
2) $R \times (-1,1)$
3) $(-1,1) \times (-1,1)$

**4)** $\mathbb{R} \times \mathbb{R}$

**Sol.** The given series is $\lim_{n\to\infty} \sum_{\ell,k=0}^{n} \frac{k^2 x^k y^l}{\ell!} = \sum_{k=0}^{\infty} x^k k^2 \sum_{\ell=0}^{\infty} \frac{y^l}{\ell!}$ .

The radius of convergence of the series $\sum_{k=0}^{\infty} x^k k^2$ is given by $\frac{1}{R_1} = \lim_{k\to\infty} \frac{(k+1)^2}{k^2} = 1$.

$\Rightarrow R_1 = 1 \Rightarrow$ series $\sum_{k=0}^{\infty} x^k k^2$ converges for $|x| < 1$

Also, radius of convergence of the series $\sum_{\ell=0}^{\infty} \frac{y^2}{\ell!}$ is given by $\frac{1}{R_2} = \lim_{\ell\to\infty} \frac{l!}{(\ell+1)!} = 0$

$\Rightarrow R_2 = \infty \Rightarrow$ series $\sum_{\ell=0}^{\infty} \frac{y^2}{\ell!}$ Converges for $|y| < \infty$

Hence, the series $\sum_{k=0}^{\infty} k^2 x^k \sum_{\ell=0}^{\infty} \frac{y^2}{\ell!}$ Converges for $(x, y)\ in (-1,1) \times \mathbb{R}$

Clearly, options (1) and (3) are correct.

**9.** Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be given by a formula

$f(x, y) = (3x + 2y + y^2 + |xy|, 2x + 3y + x^2 + |xy|)$.Then,

**1)** $f$ is discontinuous at (0,0)

**2)** $f$ is continuous at (0,0) but not differentiable at (0,0)

**3)** $f$ is differentiable at (0,0)

**4)** $f$ is differentiable at (0,0) and the derivative $Df$ (0,0) is invertible.

**Sol.** Solution of Q. 16. (June – 2014)

**10.** Let $p_n(x) = a_n x^2 + b_n x$ be a sequence of quadratic polynomials where $a_n, b_n \in \mathbb{R}$ for all $n \geq 1$. Let $\lambda_0$, $\lambda$, be distinct non-zero real numbers such that $\lim_{n\to\infty} p_n(\lambda_0)$ and $\lim_{n\to\infty} p_n(\lambda_1)$ exist. Then

**1)** $\lim_{n\to\infty} p_n(x)$ exists for all $x \in \mathbb{R}$

**2)** $\lim_{n\to\infty} p_n'(x)$ exists for all $x \in \mathbb{R}$

**3)** $\lim_{n\to\infty} p_n\left(\frac{\lambda_0+\lambda_1}{2}\right)$ does not exist.

**4)** $\lim_{n\to\infty} p_n'\left(\frac{\lambda_0+\lambda_1}{2}\right)$ does not exist.

**Sol.** similar Sol. of Q.15 (June – 2014)

**11.** Let $t$ $and$ $a$ be positive real numbers.

Define $B_a = \{x = (x_1, x_2 \dots, x_n) \in \mathbb{R}^n : {x_1}^2 + {x_2}^2 + \cdots + {x_n}^2 \leq a^2\}$. Then for any compactly supported continuous function $f$ on $\mathbb{R}^n$ which of the following is correct?

**1)** $\int_{B_a} f(tx)dx = \int_{B_{ta}} f(x)t^{-n}dx$

**2)** $\int_{B_a} f(tx)dx = \int_{B_{t^n a}} f(x)t\, dx$

**3)** $\int_{\mathbb{R}^n} f(x+y)dx = \int_{\mathbb{R}^n} f(x)dx$, for some $y \in \mathbb{R}^n$

**4)** $\int_{\mathbb{R}^n} f(tx)dx = \int_{\mathbb{R}^n} f(x)t^n dx$

**Sol.** For option (1)

$$L.H.S = \int_{B_a} f(tx)dx \;\dots\dots\dots (1)$$

Put $tx = y$, i.e., t $(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots \dots, y_n)$, where $y = (y_1, y_2, \dots \dots, y_n) \in \mathbb{R}^n$

$$\Rightarrow (x_1, x_2, \dots\dots, x_n) = \left(\frac{y_1}{t}, \frac{y_2}{t}, \dots\dots, \frac{y_n}{t}\right)$$

$\therefore {x_1}^2 + {x_2}^2 + \cdots + {x_n}^2 \leq a^2$ is transformed to $\frac{{y_1}^2}{t^2} + \frac{{y_2}^2}{t^2} + \cdots + \frac{{y_n}^2}{t^2} \leq a^2$ i.e.,

${y_1}^2 + {y_2}^2 + \cdots + {y_n}^2 \leq (at)^2$ and also, $dx_1\, dx_2 \dots\dots dx_n = \frac{dy_1}{t} \cdot \frac{dy_2}{t} \dots\dots \frac{dy_n}{t}$.

$\therefore$ From (1), $L.H.S = \int_{{x_1}^2 + {x_2}^2 + \cdots + {x_n}^2 \leq a^2} f(tx_1, tx_2, \dots\dots, tx_n)\, dx_1 dx_2 dx_3 \dots\dots dx_n$

$$= \int_{{y_1}^2 + {y_2}^2 + \cdots + {y_n}^2 \leq (ta)^2} F\,(y_1, y_2, \dots \dots, y_n)\, \frac{dy_1 dy_2 \dots\dots dy_n}{t^n}$$

$= \int_{B_{ta}} f(x)t^{-n}dx =$ R.H.S

Hence, option (1) is correct

For option (2),

As from option (1)

$$\int_{B_a} f(tx)dx = \int_{B_{ta}} f(x)t^{-n}\, dx \neq \int_{B_{t^n a}} f(x)t\, dx$$

$\therefore$ option (2) is incorrect

For option (3)

Since $\int_{\mathbb{R}^n} f(x+y)dx = \int_{\mathbb{R}^n} f(x)dx$ for $y = (0, 0, \dots \dots 0) \in \mathbb{R}^n$

∴ option (3) is correct

For option (4)

As proved in option (1) we can prove that

$\int_{\mathbb{R}^n} f(tx)dx = \int_{\mathbb{R}^n} f(x)t^{-n}\text{dx} \neq \int_{\mathbb{R}^n} f(x)t^n$

∴ option (4) incorrect.

**12.** Consider all sequences $\{f_n\}$ of real valued continuous function on [0,∞). Identify which ofthe following statements are correct.

1) If $\{f_n\}$ converges to $f$ pointwise on [0, ∞), then $\lim\limits_{n\to\infty} \int_0^\infty f_n(x)dx = \int_0^\infty f(x)dx$
2) If $\{f_n\}$ converges to f uniformly on [0, ∞), then $\lim\limits_{n\to\infty} \int_0^\infty f_n(x)dx = \int_0^\infty f(x)dx$
3) If $\{f_n\}$ converges to uniformly on [0, ∞), then f is continuous on [0, ∞)
4) There exists a sequence of continuous functions $\{f_n\}$ on $[0,\infty)$ such that $\{f_n\}$ converges to $f$ uniformly on $[0,\infty)$ but $\lim\limits_{n\to\infty} \int_0^\infty f_n(x)dx \neq \int_0^\infty f(x)dx$

**Sol.** Solution is similar to Q.10

**13.** Let $G_1$,$G_2$ be two subsets of $\mathbb{R}^2$ and $f$:$\mathbb{R}^2 \to \mathbb{R}^2$ be a function. Then,

1) $f^{-1}$ $(G_1 \cup G_2) = f^{-1}(G_1) \cup f^{-1}(G_2)$
2) $f^{-1}({G_1}^c) = (f^{-1}(G_1))^c$
3) $f(G_1 \cap G_2) = f(G_1) \cap f(G_2)$
4) If $G_1$ is open and $G_2$ is closed then $G_1 + G_2$=$\{x + y : x \in G_1, y \in G_2\}$ is nether open nor closed

**Sol.** For option (1)

Let $x \in f^{-1}(G_1 \cup G_2)$ $\Rightarrow f(x) \in G_1 \cup G_2$

$\Leftrightarrow f(x) \in G_1$ or$G_2$ $\Leftrightarrow f^{-1}(f(x)) \in f^{-1}(G_1)$ or $f^{-1}(f(x)) \in f^{-1}(G_2)$

$\Leftrightarrow xf^{-1}(G_1)$ or $x \in f^{-1}(G_2)$ $\Leftrightarrow x \in f^{-1}(G_1) \cup f^{-1}(G_2)$

$\Rightarrow f^{-1}(G_1 \cup G_2) = f^{-1}(G_1) \cup f^{-1}(G_2)$.

∴ option (1) is correct

For option (2)

Let $x \in f^{-1}({G_1}^c)$

$$\Leftrightarrow f(x) \in {G_1}^c \Leftrightarrow f(x) \in G_1 \Leftrightarrow x \notin f^{-1}(G_1) \Leftrightarrow x \in (f^{-1}(G_1))^c$$

$\Rightarrow$ Option (2) is correct

For option (3)

$f : \mathbb{R}^2 \to \mathbb{R}^2$is defined by $f(x,y) = (1,1) \ \forall \ (x,y) \in \mathbb{R}^2$

$G_1 = \{(x,y)|x^2 + y^2 = 1\}$, $G_2 = \{(x,y)|(x-2)^2 + (y-2)^2 = 1\}$

Clearly, $G_1 \cap G_2 = \Phi \Rightarrow f(G_1 \cap G_2) = \Phi$

But $f(G_1) \cap f(G_2) = \{(1,1)\}$

Option (3) is incorrect

$\Rightarrow$ For option (4)

**Result:** If $G_1$ is open $G_2$ is any set, then $G_1 + G_2$ is open so, option (4) is incorrect.

**14.** Let $= \{(x,y) \in \mathbb{R}^2 : x + y \neq -1\}$. Define $f: A \to \mathbb{R}^2$ by $f(x,y)$=$\left(\frac{y}{1+x+y}, \frac{x}{1+x+y}\right)$. Then,

1) **the determinant of the Jacobi an of $f$ does not vanish on $A$**
2) **$f$ is infinitely differentiable on $A$.**
3) **$f$ is one to one**
4) $f(A) = \mathbb{R}^2$

**Sol.** Solution is similar to Q. 17 (June – 2014)

**15.** Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be a function $f(r,\theta) = (r \ \cos\theta, r\sin\theta)$. Then for which of the open subsets $U$ of $\mathbb{R}^2$ given below, $f$ restricted to U admits an inverse?

1) $U = \mathbb{R}^2$
2) $U = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$
3) $U = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$
4) $U = \{(x,y) \in \mathbb{R}^2 : x < -1, y < -1\}$

**Sol.**

**16.** Let $S \subseteq \mathbb{R}^2$ be defined by $S = \left\{\left(m + \frac{1}{4^{|p|}}, n + \frac{1}{4^{|q|}}\right) : m, n, p, q \in \mathbb{Z}\right\}$

Then,

1) s is discrete in $\mathbb{R}^2$
2) The set of limit points of $S$ the set $\{(m,n): m, n \epsilon \mathbb{Z}\}$

**3)** $S^c$ is connected but not path connected.

**4)** $S^c$ is path connected.

**Sol.** Solution is similar as Q. 12 (June – 2014)

**17.** Which of the following statements is/are true?

**1)** There exists a continuous map $f: \mathbb{R} \to \mathbb{R}$ such that $f(\mathbb{R}) = \mathbb{Q}$

**2)** There exists a continuous map $f: \mathbb{R} \to \mathbb{R}$such that $f(\mathbb{R}) = \mathbb{Z}$

**3)** There exists a continuous map $f: \mathbb{R} \to \mathbb{R}^2$ such that $f(\mathbb{R}) = \{(x, y)\epsilon\mathbb{R}^2: x^2 + y^2 = 1\}$

**4)** There exists a continuous map $f: [0,1] \cup [2,3] \to \{0,1\}$

**Sol.**

Result: A continuous function maps connected set to connected set.

Since, $\mathbb{R}$ is connected, but $\mathbb{Q}$ and $\mathbb{Z}$ are not.

$\therefore \nexists$ any continuous map $f: \mathbb{R} \to \mathbb{R}$ for which $f(\mathbb{R}) = \mathbb{Q}$ or $\mathbb{Z}$

$\therefore$ option (1) and (2) are incorrect

Let, $f: \mathbb{R} \to \mathbb{R}^2$ defined by $f(x) = (1,0)$

Clearly, $f$ is continuous and $f(\mathbb{R}) = \{(x, y)\epsilon\mathbb{R}^2: x^2 + y^2 = 1\}$

Thus option (3) is correct

Take, $f: [0,1] \cup [2,3] \to \{0,1\}$ defined by $f(x) = \begin{cases} 0, \forall\, x \in [0,1] \\ 1, \forall\, x \in [2,3] \end{cases}$

Clearly, $f(x)$ is continuous

Thus, option (4) is correct.

**18.** Let $f: (0,1) \to \mathbb{R}$ be continuous suppose that $|f(x) - f(y)| \leq |\cos x - \cos y|$ for all $x, y \in (0,1)$. Then,

**1)** $f$ is discontinuous at least one point in $(0,1)$

**2)** $f$ is discontinuous everywhere on $(0,1)$ but not uniformly continuous on $(0,1)$

**3)** $f$ is uniformly continuous on $(0,1)$

**4)** $\lim\limits_{n \to 0^+} f(x)$ exists.

**Sol.** Given, $f: (0,1) \to \mathbb{R}$ is continuous such that

$$|f(x) - f(y)| \leq |\cos x - \cos y| \leq |x - y| \;\forall\; x, y \in (0,1)$$

i.e., $|f(x)-f(y)| \leq |x-y| \forall\, x, y \in (0,1)$, $f(x)$ satisfies Lipschitz condition on (0,1)

$\therefore f(x)$ is uniformly continuous on (0,1).

Thus, option (1) and (2) are incorrect and option (3) is correct.

Result: A function '$f$' is uniformly continuous on $(a,b)$ if only if '$f$' is continuous on $(a,b)$ and $\lim_{x\to 0^+} f(x)$ and $\lim_{n\to b^-} f(x)$ exists.

Using above result, option (4) is correct.

## Answer Table

| Sl. No | Answer | Sl. No | Answer | Sl. No | Answer |
|---|---|---|---|---|---|
| 1. | 1 | 2. | 3 | 3. | 2 |
| 4. | 1 | 5. | 4 | 6. | 2 |
| 7. | 1, 2, 3, & 4 | 8. | 1 & 3 | 9. | 3 & 4 |
| 10. | 1 & 2 | 11. | 1 & 3 | 12. | 3 & 4 |
| 13. | 1 & 2 | 14. | 1, 2 & 3 | 15. | 2 & 4 |
| 16. | 4 | 17. | 3 & 4 | 18. | 3 & 4 |

## **Previous Year Question & Solution**

## **Real Analysis (June-2016)**

# Part-B

**1.** Consider the improper Riemann integral $\int_0^x y^{-\frac{1}{2}}\,dy$. This integral is:

**1)** continuous in $[0,\infty)$

**2)** continuous only in $(0,\infty)$

**3)** discontinuous in $(0,\infty)$

**4)** discontinuous only in $\left(\frac{1}{2},\infty\right)$

**Sol.**

The given integral is $\int_0^x y^{-\frac{1}{2}}\,dy$

Let $f(x) = \int_0^x y^{-\frac{1}{2}}\,dy \Rightarrow f(x) = \left[2\sqrt{y}\right]_0^x \Rightarrow f(x) = 2\sqrt{x}$, which is continuous in $[0,\infty)$

$\therefore$ option (1) is correct.

**2.** Which one of the following statements is true for the sequence of functions,

$f_n(x) = \frac{1}{n^{2+}x^2}$, $n = 1,2, \ldots\ldots, x \in \left[\frac{1}{2},1\right]$?

**1)** The sequence is monotonic and has $0$ as the limit for all $x \in \left[\frac{1}{2},1\right]$ as $n \to \infty$

**2)** The sequence is not monotonic but has $f(x) = \frac{1}{x^2}$ as the limit as $n \to \infty$

**3)** The sequence is monotonic and has $f(x) = \frac{1}{x^2}$ as the limit as $n \to \infty$

**4)** The sequence is not monotonic but has $0$ as the limit.

**Sol.**

Given, $f_n = \frac{1}{n^{2+}x^2}$, n = 1,2, ……., $x \in \left[\frac{1}{2},1\right]$

Thus, the terms of the sequence $\{f_n(x)\}_{n=1}^{\infty}$ are

$f_1(x) = \frac{1}{1+x^2}$, $f_2(x) = \frac{1}{2^{2+}x^2} f_3(x) = \frac{1}{3^{2+}x^2}$, …….

$$n+1 > n, \quad \forall\, n \in \mathbb{N} \Rightarrow (n+1)^2 > n^2, \forall\, n \in \mathbb{N}$$

$$\Rightarrow (n+1)^2 + x^2 > n^2 + x^2\ \forall\, n \in \mathbb{N}\ \&x \in \left[\frac{1}{2},1\right]$$

$$\Rightarrow \frac{1}{(n+1)^2 x^2} < \frac{1}{n^{2+}x^2} \Rightarrow f_{n+1}(x) < f_n(x)\ \forall\, n \in \mathbb{N}$$

$\therefore$ Sequence $\{f_n(x)\}$ is monotonically decreasing

$\therefore$ option (2) and (4) are incorrect

For any $x \in \left[\frac{1}{2}, 1\right]$

$$\lim_{n\to\infty} f_n(x) \; = \; \lim_{n\to\infty} \frac{1}{n^{2+}x^2} = \; 0$$

Hence, option (3) is incorrect. Thus option (1) is correct.

**3.** $\lim_{n\to\infty}(1 - \frac{1}{n^2})^n$ equals

1) 1
2) $e^{-\frac{1}{2}}$
3) $e^{-2}$
4) $e^{-1}$

**Sol.**

$\lim_{n\to\infty}(1-\frac{1}{n^2})^n = \lim_{n\to\infty}\left(1+\frac{(-1)}{n^2}\right)^n = \lim_{n\to\infty}\left\{\left(1+\frac{(-1)}{n^2}\right)^{n^2}\right\}^{\frac{1}{n}}$

$$\Rightarrow \lim_{n\to\infty}(1-\frac{1}{n^2})^n = \lim_{n\to\infty} e^{-\frac{1}{n}} = e^0 = 1$$

$\therefore$ option (1) is correct.

**4.** Consider the interval $(-1,1)$ and a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of elements in it then,

1) Every limit point of $\{\alpha_n\}$ is in $(-1,1)$
2) Every limit point of $\{\alpha_n\}$ is in $[-1, 1]$
3) The limit points of $\{\alpha_n\}$ can only be in $\{-1,0,1\}$
4) The limit points of $\{\alpha_n\}$ cannot be in $\{-1,0,1\}$

**Sol.** Given, $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(-1,1)$

$\therefore$ consider the sequence $\{\alpha_n\} = \left\{1-\frac{1}{n}\right\}_{n=1}^{\infty}$

Clearly, $\{\alpha_n\}$ has all terms in $(-1,1)$ and limit point of the sequence is 1 which does not belongs to $(-1,1)$ and belongs to $\{-1,0,1\}$

$\therefore$ option (1) and (4) are incorrect.

Now, take $\{\alpha_n\} = \left\{\frac{1}{2} - \frac{1}{n}\right\}$

again $\{\alpha_n\}$ has all terms in (-1,1) but limit pt. of $\{\alpha_n\}$is$\frac{1}{2}$

$\therefore$ option (3) is incorrect

Hence option (2) is correct.

**5.** Let $F : \mathbb{R} \to \mathbb{R}$ be a monotonic function. Then

**1)** $F$ has no discontinuities.

**2)** $F$ has only finitely many discontinuities

**3)** $F$ can have at most countably many discontinuities

**4)** $F$ can have uncountably many discontinuities.

**Sol**.

Take, $F(x) = [x]$

Clearly, $F(x)$ is monotonic and it is continuous everywhere except at integers.

Thus, options (1) and (2) are incorrect and option (3) is correct.

Also, it is a well know result that a monotonic function can have at most countable many discontinuities.

**6.** Consider the function $f(x,y) = \frac{x^2}{y^2}$, $(x,y) \in \left[\frac{1}{2},\frac{3}{2}\right] \times \left[\frac{1}{2},\frac{3}{2}\right]$. The derivative of the function at (1,1) along the direction $(1,1)$ is

**1)** 0

**2)** 1

**3)** 2

**4)** – 2

**Sol.**

Given, $(x,y) = \frac{x^2}{y^2}, (x,y) \in \left[\frac{1}{2},\frac{3}{2}\right] \times \left[\frac{1}{2},\frac{3}{2}\right]$

Directional derivative of $f$ at $c = (1,1)$ in the direction of $u = (1,1)$ is given by

$$\lim_{h\to 0}\frac{f(c+uh)-f(c)}{h} = \lim_{h\to 0}\frac{f[(1,1)+(1,1)h]-f(1,1)}{h} = \lim_{h\to 0}\frac{f\big((1+h,1+h)\big)-f\,((1,1))}{h} = \lim_{h\to 0}\frac{\frac{(1+h)^2}{(1+h)^2}-\frac{1}{1}}{h} = 0$$

$\therefore$ Derivative of $f$ at (1,1) in the direction of (1,1) is zero.

Thus, option (1) is correct.

## PART - C

**7.** Let $\{x_n\}$ be an arbitrary sequence of real numbers. Then

**1)** $\sum_{n=1}^{\infty}|x_n|^p < \infty$ for some $1 < p < \infty$ implies $\sum_{n=1}^{\infty}|x_n|^q < \infty$ for $q > p$

**2)** $\sum_{n=1}^{\infty}|x_n|^p < \infty$ for some $1 < p < \infty$ implies $\sum_{n=1}^{\infty}|x_n|^q < \infty$ for $1 \leq q < p$

**3)** Given any 1$< p < q < \infty$ , there is a real sequence $\{x_n\}$ such that $\sum_{n=1}^{\infty}|x_n|^p < \infty$but $\sum_{n=1}^{\infty}|x_n|^p < \infty$but $\sum_{n=1}^{\infty}|x_n|^q < \infty$

**4)** Given any $1 < p < q < \infty$, there is a real sequence $\{x_n\}$ such that $\sum_{n=1}^{\infty}|x_n|^p < \infty$ but

$$\sum_{n=1}^{\infty}|x_n|^q < \infty$$

**Sol.**

For option (1) and (3)

Let $\sum_{n=1}^{\infty}|x_n|^p$ is convergent for $1 < p < \infty$

$$\Rightarrow |x_n| \to 0 \; as \; n \to \infty$$

$\therefore$ For $q > p \;\Rightarrow |x_n|^q \leq |x_n|^p$

$\therefore$ By comparison test, $\sum_{n=1}^{\infty}|x_n|^q$ is also convergent.

Hence, option (1) is correct and option (3) is incorrect.

For option (2) and (4)

Take,$x_n = \frac{1}{\sqrt{n}}$ , $p = 3$ & $q = 2$

Clearly, $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ is convergent, but $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent (By P-test).

Hence, option (4) is correct and option (2) is incorrect.

**8.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function and $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. Then

**1)** $f$ is bounded above, but bounded below.

2) $f$ is bounded and below, but may not attain its bounds

3) $f$ is bounde above and below and $f$ attains its bound.

4) $f$ is uniformly continuous.

**Sol.**

Given, $f: \mathbb{R} \to \mathbb{R}$ is a continuous function and $f(x+1) = f(x)$ consider a constant function, i.e., $f(x)$ = c$\forall x \in \mathbb{R}$, where c$\epsilon\mathbb{R}$ Then, $f(x+1) = f(x) \forall\, x \in \mathbb{R}$ and $f$ is continuous, being a constant function.
So, option (1) and (2) are incorrect
As $f(x+1) = f(x) \forall\, x \in \mathbb{R}$, therefore $f$ is bounded and attains its bound.
Hence, option (3) is correct.
Also, as $f(x+1) = f(x) \forall\, x \in \mathbb{R}$
$\Rightarrow f$ is periodic and as every periodic continuous function is uniformly continuous.
$\Rightarrow f$ is uniformly continuous.
$\Rightarrow$ option (4) is correct.

**9.** Let $x_1 = 0,\ x_2 = 1$, and for $n \geq 3$, define $x_n = \frac{x_{n-1}+x_{n-2}}{2}$, which of the following is/are true?

1) $\{x_n\}$ is a monotone sequence.

2) $\lim\limits_{n\to\infty} x_n = \frac{1}{2}$

3) $\{x_n\}$ is a Cauchy sequence.

4) $\lim\limits_{n\to\infty} x_n = \frac{2}{3}$

**Sol:**

Given, $x_1 = 0,\ x_2 = 1\ \ and\ x_n = \frac{x_{n-1}+x_{n-2}}{2}, n \geq 3.$

For option (1)

Since, , $x_1 = 0,\ x_2 = 1,\ x_3 = \frac{x_2+x_3}{2} \ldots\ldots.., x_{n-1} = \frac{x_{n-2}+x_{n-3}}{2}, x_n = \frac{x_{n-1}+x_{n-2}}{2}$

Adding all terms, we get

$$x_1 + x_2 + \ldots.. + x_{n-1} + x_n$$
$$= 1 + \frac{x_1}{2} + (x_2 + x_3 + \ldots.. + x_{n-2}) + \frac{x_{n-1}}{2}$$

$\Rightarrow x_n + \frac{x_{n-1}}{2} = 1 \ldots\ldots\ldots\ldots.. (i)$(since $x_1 = 0$)

$\therefore$ let, $\lim\limits_{n\to\infty} x_n = \ell.$

$\therefore$ taking limits on both sides of $(i)$, we get $\ell + \frac{\ell}{2} = 2\ \Rightarrow \ell = \frac{2}{3}.$

Hence, option (4) is correct and option (2) is incorrect.

Also, every convergent sequence is Cauchy in $\mathbb{R}$.

$\therefore$ $\{x_n\}$ is a Cauchy sequence.

$\therefore$ option (3) is correct.

**10.** Take the closed interval $[0,1]$ and open interval $\left(\frac{1}{3},\frac{2}{3}\right)$.

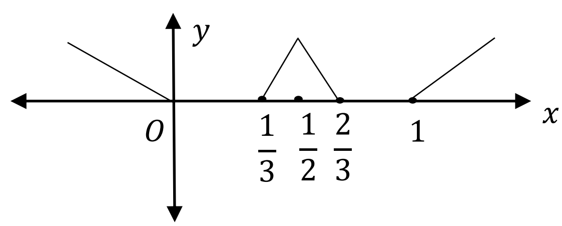
Let $k = [0,1] \Big| \left(\frac{1}{3},\frac{2}{3}\right)$. For $x \in [0,1]$ define $f(x) = d(x,k)$ where

$d(x,k) = \inf f\{|x-y| : y \in k\}$. Then

1) $f: [0,1] \to \mathbb{R}$ is differentiable at all points of $(0,1)$.
2) $f: [0,1] \to \mathbb{R}$ is not differentiable at $\frac{1}{3}$ and $\frac{2}{3}$.
3) $f: [0,1] \to \mathbb{R}$ is not differentiable at $\frac{1}{2}$
4) $f: [0,1] \to \mathbb{R}$ is not continuous.

**Sol:**

Given, $k = \left[0,\frac{1}{3}\right] \cup \left[\frac{2}{3},1\right]$ $\;and\;\; for\; x \in [0,1], f(x) = \inf f\{|x-y| : y \in k\}$.




The graph of the function is

Since modulus function is continuous,so.

f(x) is continuousMoreover, it can be

seen from the graph that $f(x)$ is continuous.

Since, we know that the function is not differentiable at corner points.

$\therefore$ $f(x)$ is not differentiable at $x = \frac{1}{3},\frac{1}{2},\frac{2}{3}$.

$\therefore$ options (2) and (3) are correct.

**11.** Which of the following functions is/are uniformly continuous on the interval $(0,1)$?

1) $\frac{1}{x}$
2) $\sin\frac{1}{x}$
3) $x\sin\frac{1}{x}$

**4)** $\frac{\sin x}{x}$

**Sol:**

Clearly given functions in all options are continuous in $(0,1)$

**Result:** A continuous function $f(x)$ $on$ $(a, b)$ is said to be uniformly continuous on $(a, b)$ if and only if $\lim_{x \to a^+} f(x)$ $and$ $\lim_{x \to b^-} f(x)$both exists.

Since, $\lim_{x \to 0^+} \frac{1}{x}$ does not exist.

$\therefore \frac{1}{x}$ is not uniformly continuous.

Similarly, $\lim_{x \to 0^+} \sin \frac{1}{x}$ doesn't exists.

$\therefore \sin \frac{1}{x}$ is not uniformly continuous.

Hence, options (1) and (2) are incorrect.

Since, $\lim_{x \to 0^+} x \sin \frac{1}{x}$ $and$ $\lim_{x \to 1^-} x \sin \frac{1}{x}$exists. So $x \sin \frac{1}{x}$ is uniformly continuous on the interval $(0,1)$.

$\therefore$ option (3) is correct.

Similarly, option (4) is correct as $\lim_{x \to 0^+} \frac{\sin x}{x}$ and $\lim_{x \to 1^-} \frac{\sin x}{x}$ exists.

**12.** Let A be any set. Let $P(A)$ be the power set of $A$, that is, the set of all subsets of $A$; $P(A) = \{B: B \subseteq A\}$. Then which of the following is are true about the set $P(A)$?

**1)** $P(A) = \phi$ for some A.

**2)** $P(A)$ is a finite set for some A.

**3)** $P(A)$ is a countable set for some A.

**4)** $P(A)$ is an uncountable set for some A.

**Sol:**

For option (1)

We know that $|P(A)| = 2^n > 0, where\ |A| = n \ldots\ldots\ldots\ldots (i)$

For, $A = \phi, P(A) = \{\phi\}$.

$$\therefore \nexists\ Any\ A\ for\ which\ P(A) = \phi$$

∴ option (2)

By $(i)$, if A is finite then, $P(A)$ is also a finite set.

∴ option (2) is also correct.

Take, $A = \mathbb{N}, a$ countable set, then $|P(A)| = 2^{\mathbb{N}}$, which is cardinality of a continuum is uncountable.

Thus, $P(A)$ is uncountable.

Thus, option (4) is correct.

For option (3),

As discussed in options (1), (2) and (4)

When A is empty or finite, then $P(A)$ is finite and when A is countable, then $P(A)$ is uncountable.

Hence, $P(A)$ can't be countable for any set A.

∴ option (3) is incorrect.

**13**. Define on [0,1] by $f(x) = \begin{cases} x^2, & if \ \ x \ is \ rational. \\ & x^3, if \ x \ is \ irrational. \end{cases}$

1) $f$ is not Riemann integrable on [0,1].
2) $f$ is not Riemann integrable and $\int_0^1 f(x)dx = \frac{1}{4}$
3) $f$ is not Riemann integrable and $\int_0^1 f(x)dx = \frac{1}{3}$
4) $\frac{1}{4} = \int_0^1 f(x)dx < \int_0^1 f(x)dx = \frac{1}{3}, \ where \ \int_0^1 f(x)dx \ \ and \ \int_0^1 f(x)dx$ are the lower and upper Riemann integrals of $f$.

**Sol:**

Given $f(x) = \begin{cases} x^2, & if \ \ x \ is \ rational. \\ & x^3, if \ x \ is \ irrational. \end{cases}$

∵ $f(x)$ is continuous only at $x = 0 \ \ and \ x = 1$

∴ In interval [0,1] there are uncountable many points of discontinuous and the measures of the set of points of discontinuities is not zero.

∴ $f$ is not Riemann Integrable.

Hence, option (1) is correct and options (2), (3) are incorrect.

$Now, let \ P = \{0 = x_{01}, x_1, x_2, \ldots\ldots x_n = 1\}$ be the partition of [0,1] with $n - equal$ intervals of length $\frac{1}{n}$

∵ if $x \in (0,1), then \ \ x^3 < x^2$

If $m_i = \inf\{f(x), x \in [x_{i-1}, x_i]\}$and $M_i = \sup\{f(x), x \in [x_{i-1}, x_i]\}$then $m_i = x^3$ and $M_i = x^2$

$\Rightarrow$ $Lower\ Rieman\ Integral\ L(p,f) = lim_{n\to\infty} \sum_{i=1}^{n} m_i b_i = \int_0^1 x^3\, dx = \frac{1}{4}$ and upper Riemann Integral $U(p,f) = \lim\limits_{n\to\infty} \sum_{i=1}^{n} M_i b_i = \int_0^1 x^2\, dx = \frac{1}{3}$

$\therefore$ option (4) is correct.

**14.** Consider the integral $A = \int_0^1 x^n\,(1-x)^n dx$ . Pick each correct statement from below?

1) A is not a rational number
2) $0 < A \leq 4^{-n}$
3) A is a natural number.
4) $A^{-1}$ is a natural number.

**Sol:**

Given, $A = \int_0^1 x^n\,(1-x)^n dx = B(n+1, n+1)$

$$\frac{\sqrt{(n+1)}\sqrt{(n+1)}}{\sqrt{(2n+2)}} = \frac{n!\,n!}{(2n+1)!} \quad for\ n \in \mathbb{N}$$

Now, for $n = 1$

$A = \frac{1}{3!} = \frac{1}{6}$, which is a rational number.

Clearly, options (1) and (3) are incorrect.

Also, $A^{-1} = \frac{(2n+1)!}{n!n!}$ which is clearly a natural number.

$\therefore$ option (4) is correct.

For option (2)

Clearly, for $n = 1, 0 < A \leq 4^{-1}$

Now, we will prove $0 < A \leq 4^{-n}$ by Mathematical induction.

Let result holds for $n = m,\ i.e., A = \frac{m!m!}{(2n+1)!} \leq \frac{1}{4^m}$ ….. … … … (1)

$$\therefore \quad for\ n = m+1$$

$$A = \frac{(m+1)!\,(m+1)!}{(2m+3)!} = \frac{(m+1)^2(m!)^2}{(2m+3)(2m+2)(2m+1)!} \leq \frac{(m+1)}{2(2m+3)} \times \frac{1}{4^m}$$

$$= \frac{1+\frac{1}{m}}{2\left(2+\frac{3}{m}\right)} \times \frac{1}{4^m} < \frac{1}{4} \times \frac{1}{4^m} = \frac{1}{4^{m+1}}$$

$\therefore$ Result holds for $n = m + 1$ also

Hence, by Principle of Mathematical induction $0 < A \leq \frac{1}{4^n}\ \forall\, n$.

Thus, option (2) is correct.

# Answer Table

| Sl. No | Answer | Sl. No | Answer | Sl. No | Answer |
|---|---|---|---|---|---|
| 1. | 1 | 2. | 1 | 3. | 1 |
| 4. | 2 | 5. | 3 | 6. | 1 |
| 7. | 1 & 4 | 8. | 3 & 4 | 9. | 3 & 4 |
| 10. | 2 & 3 | 11. | 3 & 4 | 12. | 2 & 4 |
| 13. | 1 & 4 | 14. | 2 & 4 | | |

## Previous Year Question & Solution

## Real Analysis (December-2016)

### Part-B

**1.** Consider the set of sequences $X = \{(x_n) : x_n \in \{0,1\}, n \in \mathbb{N}\}$

$$Y = \{(x_n) \in X : x_n = 1 \;\; for\ at\ most\ finitely\ many\ n\}.$$

**1)** $X$ is countable, $Y$ is finite.

**2)** $X$ is uncountable, $Y$ is countable.

**3)** $X$ is countable, $Y$ is countable.

**4)** $X$ is uncountable, $Y$ is uncountable.

**Sol:**

Given, $X = \{(x_n) : x_n \in (0,1), n \in \mathbb{N}\}\ and$

$$Y = \{(x_n) \in X : x_n = 1 \ \textit{for at most finitely many } n\}$$

Suppose that $X$ is countable. This means that elements of $X$ can be arranged in a sequence.

$$S_{1-} \ a_{11}, a_{12}, a_{13}, \dots \dots \ \dots,$$

$$S_{2-} \ a_{21}, a_{22}, a_{23}, \dots \dots \ \dots,$$

$$S_{3-} \ a_{31}, a_{32}, a_{33}, \dots \dots \ \dots,$$

.
.
.
.

$$a_{ij} \in \{0,1\} \ \forall \ i, j$$

Now form a new sequence

$$S_{0-} \ a_1, a_2, a_3, \dots \dots \ \dots \textit{such that } a_i \neq a_{ii}$$

Thus $S_0 \in X$ but $S_0$ is not equal to $S_i$ for any $i$.

Thus our supposition is wrong$\Rightarrow$ $X$ is uncountable.

Now as $Y$ is a proper subset of $X$ which have 1 for almost finitely many $n$.

Each member can be seen as subset of

$$\underbrace{\mathbb{N} \times \mathbb{N} \times \dots \dots . . \times \mathbb{N}}_{\textit{finite tiems.}}$$

Which is countable $\Rightarrow$ $Y$ is countable.

Hence, option (2) is correct.

**2.** Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (x^2, y^2 + \sin x)$. Then the derivative of $f$ at $(x, y)$ is the linear transformation is given by

**1)** $\begin{pmatrix} 2x & 0 \\ \cos x & 2y \end{pmatrix}$

**2)** $\begin{pmatrix} 2x & 0 \\ 2y & \cos x \end{pmatrix}$

**3)** $\begin{pmatrix} 2x & \cos x \\ 2y & 0 \end{pmatrix}$

**4)** $\begin{pmatrix} 2x & 2y \\ 0 & \cos x \end{pmatrix}$

**Sol:**

$f : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $f(x, y) = (x^2, y^2 + \sin x) = \big(f_1(x), f_2(x)\big)$

Derivative matrix, $Df = \begin{pmatrix} (f_1)_x & (f_1)_y \\ (f_2)_x & (f_2)_y \end{pmatrix} = \begin{pmatrix} 2x & 0 \\ \cos x & 2y \end{pmatrix}$

Hence, option (1) is correct.

**3.** A function $f:\mathbb{R}^2 \to \mathbb{R}^2$ is defined by $f(x,y) = xy$. Let $v = (1,2)$ and $a = (a_1, a_2)$ be two elements of $\mathbb{R}^2$. The directional derivative of $f$ in the direction of $v\ at\ a$ is:

**1)** $a_1 + 2a_2$

**2)** $a_2 + 2a_1$

**3)** $\frac{a_1}{2} + a_2$

**4)** $\frac{a_2}{2} + a_1$

**Sol:**

$f:\mathbb{R}^2 \to \mathbb{R}^2$ is defined as $f(x,y) = xy$

$$f_x = y, f_y = x$$

In the direction $(a_1, a_2)$ the directional derivative is $a_1 f_x + f_y a_2 = a_1\, y + a_2\, x$

At point $(1,2) = 2a_1 + a_2$

Hence, option (2) is correct.

4. $\lim\limits_{n\to\infty} \frac{1}{n^4} \sum_{j=0}^{2n-1} j^3$ equals.

**1)** 4

**2)** 16

**3)** 1

**4)** 8

**Sol:**

Take

$$S = \lim_{n\to\infty} \frac{1}{n^4} \sum_{j=0}^{2n-1} j^3 = \lim_{n\to\infty} \frac{1}{n^4} \sum_{j=0}^{2n-1} \{1^3 + 2^3 + \ldots\ldots\ . + (2n-1)^3\}$$

$$= \lim_{n\to\infty} \frac{1}{n^4} \left\{\frac{(2n-2)2n}{2}\right\}^2 = \lim_{n\to\infty} \frac{1}{4} \left\{\left(2 - \frac{1}{n}\right)^2 2^2\right\} = \frac{1}{4} \times 4 \times 4 = 4$$

Hence, option (1) is correct.

**5.** $f : \mathbb{R} \to \mathbb{R}$ is such that $f(0) = 0$ $and$ $\left|\frac{df}{dx}(x)\right| \le 5$ for all $x$, we can conclude that $f(1)$ is in

1) (5,6)
2) (-5,5)
3) $(-\infty, -5) \cup (5, \infty)$
4) (-4,4)

**Sol:**

Given, $f : \mathbb{R} \to \mathbb{R}$ is such that $f(0) = 0$ $and$ $\left|\frac{df}{dx}(x)\right| \le 5\ \ \forall x \in \mathbb{R}$

Take $f(x) = 5x$

Hence, $f(0) = 0$ $and$ $\left|\frac{df}{dx}(x)\right| = 5$, which is also $\left|\frac{df}{dx}(x)\right| \le 5$ .

Now, $f(1) = 5$

Hence, options (1), (3) and (4) are clearly incorrect for $f(x) = 5x$.

∴ only option (2) is correct.

**6.** Let G be an open set in $\mathbb{R}^n$. Two points $x, y \in G$ are said to be equivalent if they can be joined by a continuous path completely lying inside G. Number of equivalence classes is

1) only one
2) at most finite
3) at most countable.
4) can be finite, countable or uncountable.

**Sol:**

## Part – C

**7.** Let $S \in (0,1)$. Then decide which of the following are true.

1) $\forall\, m \in \mathbb{N}, \exists n \in \mathbb{N}\ \ s.t.\ S > \frac{m}{n}$

2) $\forall\, m \in \mathbb{N}, \exists n \in \mathbb{N}\ \ s.t.\ S < \frac{m}{n}$

3) $\forall\, m \in \mathbb{N}, \exists n \in \mathbb{N}\ \ s.t.\ S = \frac{m}{n}$

4) $\forall\, m \in \mathbb{N}, \exists n \in \mathbb{N}\ \ s.t.\ S = m + n$

**Sol:**

Let $S \in (0,1)$

If S is irrational

Then $S \neq \frac{m}{n} [\forall\, m \in \mathbb{N}\ and\ \forall\, n \in \mathbb{N}]$

An irrational number can't be equal to a rational number ⇒option (3) is incorrect.

If $m \in \mathbb{N}\ and\ n \in \mathbb{N}, then\ m + n \in \mathbb{N}$.A rational number cannot be written as $m + n$

∴option (4) is incorrect.

In option (1) and (2) $\forall\, m \in \mathbb{N}, \exists\, n \in \mathbb{N}\ \ s.t.\ S > \frac{m}{n}$

$\forall$ rational and irrational numbers $S$ $\exists$ rational which are greater and less than $S$.

So, $\forall\, m \in \mathbb{N}, \exists$ must exist $n \in \mathbb{N}$ such that $S > \frac{m}{n}$ and $S < \frac{m}{n}$

So, options (1) and (2) are correct.

**8.** Let $f_n(x) = (-x)^n, x\epsilon[0,1]$. Then decide which of the following are true?

1) There exists a point wise convergent subsequence of $f_n$.

2) $f_n$ has no pointwise convergent subsequence.

3) $f_n$ converges pointwise everywhere.

4) $f_n$ has exactly one pointwise convergent subsequence.

**Sol:**

$f_n(x) = (-x)^n = (-1)^n x^n, x\epsilon[0,1]$

Consider the subsequence $g_n(x) = (-1)^{2n} x^{2n} = x^{2n} on\ [0,1]$

$$\Rightarrow g(x) = \lim_{x \to \infty} g_n(x) = \begin{cases} 1\ ; & x = 1 \\ 0\ , & x \in [0,1) \end{cases}$$

$g_n(x)$ converges pointwise $\forall x\epsilon[0,1]$

∴ option (1) is correct and option (2) is incorrect.

Similarly consider subsequence

$b_n(x) = (-1)^{2n-1}x^{2n-1} = -x^{2n-1}$ is pointwise convergent subsequence.Type equation here.

Hence, option (4) is incorrect

$f_n(x) = (-1)^n x^n \ \forall\, x \in [0,1]$ and $\lim\limits_{n\to\infty}(-1)^n x^n = 0$ for $x \in [0,1]$

for $x = 1, f_n(x) = (-1)^n$, which is not convergent.

$\Rightarrow\ f_n(x)$ does not converge pointwise $\forall\, x \in [0,1]$.

$\therefore$ option (3) is incorrect.

**9.** Which of the following are true for the function $f(x) = \sin(x)\sin\left(\frac{1}{x}\right), x \in (0,1)$?

1) $\lim\limits_{x\to 0} inf(f(x)) = \overline{\lim\limits_{x\to 0}} f(x)$
2) $\lim\limits_{x\to 0} inf(f(x)) < \overline{\lim\limits_{x\to 0}} f(x)$
3) $\lim\limits_{x\to 0} inf(f(x)) = 1$
4) $\overline{\lim\limits_{x\to 0}} f(x) = 0$

**Sol:**

$f(x) = \sin(x)\sin\left(\frac{1}{x}\right);\ x \in (0,1)$

As $\sin x$ is a continuous function and $\lim\limits_{x\to 0}\sin x = 0$

$\Rightarrow \lim\limits_{x\to 0}\sin x\left(\sin\left(\frac{1}{x}\right)\right) = 0\left[\because \sin\frac{1}{x}\ \textit{is bounded in deleted neighbourhood of zero}\right]$

$\Rightarrow \lim\limits_{x\to 0}\ \inf(f(x)) = \overline{\lim\limits_{x\to 0}} f(x) = 0$

$\therefore$ option (1) and (4) are correct.

**10.** Find out which of the following series converge uniformly for $x \in (-\pi,\pi)$

1) $\sum_{n=1}^{\infty}\frac{e^{-n|x|}}{n^3}$

2) $\sum_{n=1}^{\infty} \frac{\sin(xn)}{n^5}$
3) $\sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^n$
4) $\sum_{n=1}^{\infty} \frac{1}{((x+\pi)n)^2}$

**Sol:**

For option (1)

Given series is $\sum_{n=1}^{\infty} \frac{e^{-|x|.n}}{n^3}$

Since, $\left|\frac{e^{-n|x|}}{n^3}\right| \leq \frac{1}{n^3} \forall\, x \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent, by $P-\ test.$

∴ By Weierstrassm test

The series converges uniformly in $(-\pi, \pi)$

∴ Option (1) is correct.

For option (2)

Given series is $\sum_{n=1}^{\infty} \frac{\sin(xn)}{n^5}$

Since $\left|\frac{\sin(xn)}{n^5}\right| \leq \frac{1}{n^5} \forall\, x \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^5}$ is convergent

∴ By W.M. test

The series $\sum_{n=1}^{\infty} \frac{\sin(xn)}{n^5}$ is converges uniformly in $(-\pi, \pi)$

∴ Option (2) is correct.

For option (3)

Given series is $\sum_{n=1}^{\infty} \left(\frac{n}{n}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n^n} \cdot x^n.$

Radius of convergence of the given series is

$\frac{1}{R} = \lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} \times \frac{1}{n+1} = 0 \Rightarrow R = \infty$

We know that the series converges within the region of convergence.

∴ The series $\sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^n$ converges uniformly in $(-\pi, \pi)$.

**11.** Decide which of the following functions are uniformly continuous on (0,1)

1) $f(x) = e^x$

2) $f(x) = x$
3) $f(x) = tan\left(\frac{\pi x}{2}\right)$
4) $f(x) = \sin(x)$

**Sol:**

We know that $f(x)$ is uniformly continuous on $(a, b)$ iff

**I.** $f(x)$ is continuous on $(a, b)$

**II.** $\lim_{x \to a+} f(x)$ and $\lim_{x \to b-} f(x)$ exists finitely, where $a, b$ are finite.

**(1)**$f(x) = e^x\ on\ (0,1)$

As $e^x$is continuous.

$\lim_{x \to 0^+} e^x = 1$, exists finitely.

$\lim_{x \to 1^-} e^x = e$, exists finitely.

$\Rightarrow f(x) = e^x$ is uniformly continuous on $(0,1)$

**(2)**$f(x) = x$, clearly it is continuous function on $(0,1)$, being polynomial $\lim_{x \to 0^+} x = 0$,

$\lim_{x \to 1^-} x = 1 \Rightarrow$ limit exists finitely $\Rightarrow f(x) = x$ is uniformly continuous on $(0,1)$.

**(3)**$f(x) = \tan\left(\frac{\pi x}{2}\right)$

$\lim_{x \to 0^+} \tan\left(\frac{\pi}{2} \cdot 0\right) = 0\ and\ \lim_{x \to 1^-} \tan\frac{\pi}{2} = \infty$, limit does not exist finitely.

Thus, $f(x) = \tan\left(\frac{\pi x}{2}\right)$ is not uniformly continuous on $(0,1)$

**(4)**$f(x) = \sin x$, clearly, it is continuous and $\lim_{x \to 0^+} \sin x = 0$ and $\lim_{x \to 1^-} \sin x = \sin 1$, limit exists finitely.

$\Rightarrow f(x) = \sin x$ is uniformly continuous on $(0,1)$.

Hence options (1), (2) and (4) are correct.

**12.** Let $\chi_A(x)$ denote the function which is 1 if $x \in A$ and 0 otherwise. Consider $f(x) = \sum_{n=1}^{200} \frac{1}{n^6} \chi_{\left[0, \frac{n}{200}\right]}(x), x \in [0,1]$,

Then $f(x)$ is

1) Riemann integrable on $[0,1]$
2) Lebesgue integrable on $[0,1]$
3) is a continuous function on $[0,1]$

**4)** is a monotone function on $[0,1]$.

**Sol: -**

Given, $\chi_A(x) = \begin{cases} 1 \; if \; x \in A \\ 0 \; if \; x \notin A \end{cases}$

$$\therefore f(x) = \sum_{n=1}^{200} \frac{1}{n^6}, \chi_{\left[0, \frac{n}{200}\right]} = \sum_{n=1}^{200} \frac{1}{n^6}, x \in \left[0, \frac{n}{200}\right]$$

Clearly, $f(x)$ is monotonic.

Also, $f(x)$ is continuous $a.e.$ except at finite number of points.

Hence, option (4) is correct and (3) is incorrect.

Result – 1: - A function with finite number of discontinuities is Reimann integrable.

Result – 2: - Every Riemann integrable function is Lebesgue integrable using result (1), option (1) is correct.

Using result (2), option (2) is correct.

**13**. A function $f(x,y)$ on $\mathbb{R}^2$ has the following partial derivatives $\frac{\delta f}{\delta x}(x,y) = x^2$, $\frac{\delta f}{\delta y}(x,y) = y^2$. Then

**1)** $f$ has directional derivatives in all directions everywhere.

**2)** $f$ has derivative at all points

**3)** $f$ has directional derivative only along the direction $(1,1)$ everywhere.

**4)** $f$ does not have directional derivatives in any direction everywhere.

**Sol: -**

Given, $\frac{\delta f}{\delta x}(x,y) = x^2 ----(1)$

And $\frac{\delta f}{\delta y}(x,y) = y^2 ----(2)$

Integrating $(1) with\ respect\ to\ x$ (keeping $y$ constant)

We get $f(x,y) = \frac{x^3}{3} + g(y)$

Again differentiating $with\ respect\ to\ y$ we get$\frac{\delta f}{\delta y}(x,y) = g'(y) --- (3)$

Using (2) and (3), we get $g'(y) = y^2 \Rightarrow g(y) = \frac{y^3}{3} + c$.

Thus, $f(x,y) = \frac{x^3}{3} + \frac{y^3}{3} + c$, which is a polynomial in $x$ & $y$.

Hence, $'f'$ has directional derivative in all directions everywhere and also '$f$' has derivative at all points.

Thus, options (1) and (2) are true.

**14.** Let $d_1, d_2$ be the following metrices on $R^n$.

$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|, d_2(x,y) = (\sum_{i=1}^{n} |x_i - y_i|^2)^{\frac{1}{2}}$. Then decide which of the following is a Metricon $R^n$?

1) $d(x,y) = \frac{d_1(x,y)+d_2(x,y)}{1+d_1(x,y)+d_2(x,y)}$
2) $d(x,y) = d_1(x,y) - d_2(x,y)$
3) $d(x,y) = d_1(x,y) + d_2(x,y)$
4) $d(x,y) = e^{\pi} d_1(x,y) + e^{-\pi} d_2(x,y)$

**Sol:**

If $s, d$ are metrics, then $\lambda_1 s + \lambda_2 d$ is also metrics, where $\lambda_1, \lambda_2 > 0$.

**Axiom 1:** $\lambda_1\ s\ (x,y) + \lambda_2\ d(x,y) \geq 0$

As $s(x,y) \geq 0\ and\ d(x,y) \geq 0\ (s\ and\ d\ are\ metrics)\ and\ \lambda_1, \lambda_2 > 0$

**Axiom 2:** $\lambda_1 s(x,y) + \lambda_2 d(x,y) = 0$

$\Leftrightarrow \lambda_1 s(x,y) = 0\ and\ \lambda_2 d(x,y) = 0$

$\Leftrightarrow\ s(x,y) = 0\ and\ d(x,y) = 0, (as\ \lambda_1, \lambda_2 > 0)$

$\Leftrightarrow x = y$.

**Axiom 3:** $(\lambda_1 s + \lambda_2 d)(x,y) = \lambda_1 s(x,y) + \lambda_2 d(x,y)$

$= \lambda_1 s(y,x) + \lambda_2 d(y,x) = (\lambda_1 s + \lambda_2 d)(y,x)$

**Axiom 4:** To show $(\lambda_1 s + \lambda_2 d)(x_1,y_1) \leq (\lambda_1 s + \lambda_2 d)(x_1,y_2) + (\lambda_1 s + \lambda_2 d)(y_2,y_1)$

Consider

$$
\begin{aligned}
(\lambda_1 s + \lambda_2 d)(x_1, y_1) &= \lambda_1 s(x_1, y_1) + \lambda_2 d(x_1, y_1) \\
&\leq \lambda_1\big(s(x_1, y_2) + s(y_2, y_1)\big) + \lambda_2\big(d(x_1, y_2) + d(y_2, y_1)\big) \\
&= \lambda_1 s(x_1, y_2) + \lambda_2 d(x_1, y_2) + \lambda_1 s(y_2, y_1) + \lambda_2 d(y_2, y_1) \\
&= (\lambda_1 s + \lambda_2 d)(x_1, y_2) + (\lambda_1 s + \lambda_2 d)(y_2, y_1)
\end{aligned}
$$

Hence, $\lambda_1 s + \lambda_2 d$ is a metric, where $\lambda_1, \lambda_2 > 0$

Therefore options (3) and (4) are correct

So, option (2) is incorrect.

For example, in $\mathbb{R}^2$

$d_1(x, y) = |x - y|,\ d_2(x, y) = |x - y|$ are metrics

But $d(x, y) = d_1(x, y) - d_2(x, y)$ is not metric on $\mathbb{R}^2$

As $d(2, -2) = 0$, but $2 \neq -2$.

Also, we know that if $d_1, d_2$ are metrices on $\mathbb{R}^n$, then $d_1 + d_2$ is also metric on $\mathbb{R}^n$.

Let $d_1 + d_2 = s$, then $\frac{s}{l+s}$ is also a metric on $\mathbb{R}^n$.

$\Rightarrow \frac{d_1(x,y) + d_2(x,y)}{l + d_1(x,y) + d_2(x,y)}$ is also a metric on $\mathbb{R}^n$.

$\therefore$ Option (1) is correct.

**15.** Let $A$ be the following subset of $\mathbb{R}^2$: $A = \{(x, y): (x + 1)^2 + y^2 \leq 1\} U \left\{(x, y): y = x \sin\frac{1}{x}, x > 0\right\}$. Then

**1)** $A$ is connected

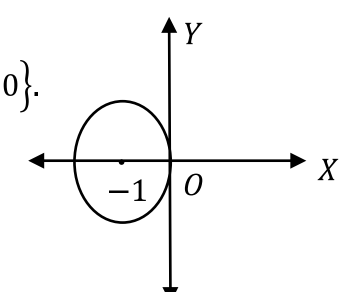
**2)** $A$ is compact

**3)** $A$ is path connected

**4)** $A$ is bounded.

**Sol:**

$A = \{(x, y): (x + 1)^2 + y^2 \leq 1\} U \left\{(x, y): y = x \sin\frac{1}{x}, x > 0\right\}$.

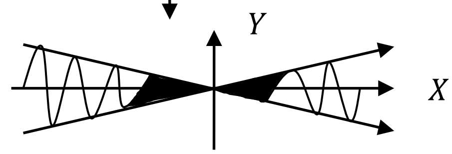



Graphically, $A$ is not bounded

$\Rightarrow A$ is not compact.

Thus, options (2) and (4) are incorrect and in $\mathbb{R}^n$,

a set is connected if only if path connected.



Thus option (1) and (3 are correct.

16. Let $(X, d)$ be a metric space. Then

**1)** An arbitrary open set $G$ in $X$ is a countable union of closed sets.

**2)** An arbitrary open set $G$ in $X$ cannot be countable union of closed sets if $X$ is connected.

**3)** An arbitrary open set $G$ in $X$ is a countable union of closed sets only if $X$ is countable.

**4)** An arbitrary open set $G$ in $X$ is a countable union of closed sets only if $X$ is locally compact.

**Sol:**

Take $X = \mathbb{R}\ and\ G = (a, b)$

Clearly, $\left[a+\frac{1}{n}, b-\frac{1}{n}\right] \subset (a, b), n \in \mathbb{N}\ \ and$

$$(a, b) = \bigcup_{n=\left[\frac{2}{b-a}\right]+1} \left[a+\frac{1}{n}, b-\frac{1}{n}\right]$$

Now, $(a, b)$ is written as countable union of closed sets. $\mathbb{R}$ is connected.

So, option (2) is incorrect.

$\mathbb{R}$ is uncountable.

So, option (3) is incorrect.

**17.** Let, $S = \{(x, y) \in \mathbb{R}^2 : -1 \le x \le 1\ and -1 \le y \le 1\}$.

Let $T = S \backslash (0,0)$, the set obtained by removing

the origin from $S$. Let $f$ be a continuous function from $T$ to $\mathbb{R}$. Choose all correct options.

**1)** Image of $f$ must be connected.

**2)** Image of $f$ must be compact.

**3)** Any such continuous function $f$ can be extended to a continuous function from $S$ to $\mathbb{R}$

**4)** If $f$ can be extended to a continuous function from $S$ to $\mathbb{R}$ then image of $f$ is bounded.

**Sol:**

Given $S = \{(x, y) \in \mathbb{R}^2 : -1 \le x \le 1\ and -1 \le y \le 1\}\ and\ T = S \backslash (0,0)$

Clearly, $S$ is both connected and compact and $T$ is connected but not compact.

For option (1)

As, $f: T \to \mathbb{R}$ is continuous

Then image of $f$ must be connected.

$\therefore$ Option $(1)$ is correct.

For option $(2)$

Further, since $'T'$ is not compact.

Then image of $f$ may or may not be compact.

For example, take, $f(x, y) = x^2 + y^2$

Then, image of $f = [0,2]$, which is not compact.

$\therefore$ option $(2)$ is incorrect.

For option $(3)$,

Take, $f: T \to \mathbb{R}$ defined by $f(x, y) = \frac{1}{x^2+y^2}$

Clearly, $f$ is continuous on $T$, but it can be extended to a continuous function from $S$ to $\mathbb{R}$,

Then as $S$ is compact.

$\Rightarrow$ Image of $f$ is compact $\Rightarrow$ Image of $f$ is bounded.

$\therefore$ Option $(4)$ is correct.

# Answer Table

| Sl. No | Answer | Sl. No | Answer | Sl. No | Answer |
|---|---|---|---|---|---|
| 1. | 2 | 2. | 1 | 3. | 2 |
| 4. | 1 | 5. | 2 | 6. | 3 |
| 7. | 1 & 2 | 8. | 1 | 9. | 1 & 4 |
| 10. | 1, 2 & 3 | 11. | 1, 2 & 4 | 12. | 1, 2 & 4 |
| 13. | 1 & 2 | 14. | 1, 3 & 4 | 15. | 1 |
| 16. | 1 & 4 | | | | |

## Previous Year Question & Solution

# <u>Real Analysis (June-2017)</u>

## Part-B

**1.**$L = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}}$. Then

1) $L = 0$
2) $L = 1$
3) $0 < L < \infty$
4) $L = \infty$

**Sol:**

Given, $L = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}}$

Take, $y = \frac{1}{\sqrt[n]{n!}} \Rightarrow \log y = -\frac{1}{n}\log(n!)$

$\Rightarrow \log y = -\frac{1}{n}[\log 1 + \log 2 + \cdots \log n] \Rightarrow \log y \to -\infty$ as $n \to \infty$

(By Cauchy's first theorem on limits, which states that if $a_n = \ell$ as $n \to \infty$,

then $\left(\frac{a_1+a_2+\cdots+a_n}{n}\right) \to \ell$, as$n \to \infty$)

$\Rightarrow y \to 0$ as $n \to \infty$).

$$\therefore L = 0$$

Hence, option (1) is correct.

**2.** Consider the sequence $a_n = \left(1 + (-1)^n \frac{1}{n}\right)^n$. Then

1) $\lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} \inf a_n = 1$
2) $\lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} \inf a_n = e$
3) $\lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} \inf a_n = \frac{1}{e}$
4) $\lim_{n\to\infty} \sup a_n = e, \lim_{n\to\infty} \inf a_n = \frac{1}{e}$

**Sol:**

$$a_n = \left(1 + (-1)^n \frac{1}{n}\right)^n = \begin{cases} \left(1 + \frac{1}{n}\right)^n, & if\ n\ is\ even \\ \left(1 - \frac{1}{n}\right)^n, & if\ n\ is\ odd. \end{cases}$$

$$\therefore \lim_{n\to\infty} \sup a_n = e \ \ and \ \lim_{n\to\infty} \inf a_n = \frac{1}{e}$$

As $\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e\ and\ \lim\limits_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = e^{-1}$

$\therefore$ Option (4) is correct.

**3.** For $a > 0$, the series $\sum_{n=1}^{\infty} a^{\ln n}$ is convergent if and only if

**1)** $0 < a < e$

**2)** $0 < a \leq e$

**3)** $0 < a < \frac{1}{e}$

**4)** $0 < a \leq \frac{1}{e}$

**Sol:**

Given series is $\sum_{n=1}^{\infty} a^{\ln n}$

Clearly, the series diverges for $a = 1$

$\therefore$ Options (1) and (2) are incorrect.

Also, for $a = \frac{1}{e}$, series becomes

$\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^{\ln n} = \sum_{n=1}^{\infty} \frac{1}{e^{(\ln n)}} = \sum_{n=1}^{\infty} \frac{1}{n}$, which is divergent.

$\therefore$ Option (4) is incorrect.

Hence, option (3) is correct.

**4.** Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} \frac{\sin x}{x}, & if\ x \neq 0 \\ 1, & if\ x = 0 \end{cases}$. Then

1) $f$ is not continuous

2) $f$ is continuous but not differentiable

3) $f$ is differentiable

4) $f$ is not bounded.

**Sol:**

Given $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} \frac{\sin x}{x}, if\ x \neq 0 \\ 1, if\ x = 0 \end{cases}$.

Here, $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1$.

Also, $f(0) = 1$

Thus, $f(x)$ is continuous at $x = 0$. Also, $f(x)$ is continuous at all other points.

Thus $f(x)$ is continuous on $\mathbb{R}$.

Further, $\lim_{h \to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0} \frac{f(h)-f(0)}{h} = \lim_{h \to 0} \frac{\frac{\sin h}{h}-1}{h}$

$$= \lim_{h \to 0} \frac{\sin h - h}{h^2} \left(= \frac{0}{0}\ form\right) = \lim_{h \to 0} \frac{\cos h - 1}{2h} = \lim_{h \to 0} \frac{-\sin h}{2} = 0$$

Thus, $f'(0)$ exists

Also, $f(x)$ is differentiable at all other points.

$\therefore$ Thus (3) is correct.

**5.** Let $A = \{n \in \mathbb{N}: n = 1\ or\ the\ only\ prime\ factors\ of\ n\ are\ 2\ or\ 3\}$, for example, $6 \in A$, $10 \notin A$.Let $S = \sum_{n \in A} \frac{1}{n}$. Then

1) $A$ is finite.

2) $S$ is a divergent series.

3) $S = 3$

4) $S = 6$

**Sol:**

Given, $A = \{n \epsilon \mathbb{N}: n = 1\ or\ the\ only\ prime\ factors\ of\ n\ are\ 2\ or\ 3\}$

$$\therefore A = \{2^i 3^j : i, j = 0,1,2,3, \dots\}$$

$$\therefore S = \sum_{n\epsilon A} \frac{1}{n} = \sum_{i,j=0}^{\infty} \frac{1}{2^i 3^j} = \left(\sum_{i=0}^{\infty} \frac{1}{2^i}\right)\left(\sum_{j=0}^{\infty} \frac{1}{3^j}\right) = \frac{1}{1-\frac{1}{2}} \times \frac{1}{1-\frac{1}{3}} = 3.$$

∴ Option (3) is correct.

**6.** For $n \geq 1$, let $f_n(x) = xe^{-nx^2}, x\epsilon\mathbb{R}$. Then the sequence $\{f_n\}$ is

1) uniformly convergent on $\mathbb{R}$
2) uniformly convergent only on compact subset of $\mathbb{R}$.
3) bounded and uniformly convergent on $\mathbb{R}$.
4) a sequence of unbounded functions.

**Solution:**

$$f_n(x) = x.e^{-nx^2}, x \in \mathbb{R}$$

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{e^{nx^2}} = 0 \ \forall \ x \in \mathbb{R}$$

$$\therefore M_n = \sup\{|f_n(x) - f(x)| x \in \mathbb{R}\} = \sup\left\{\left|xe^{-nx^2}\right| : x \in \mathbb{R}\right\}$$

Take, $y = x.e^{-nx^2}$

$$\frac{dy}{dx} = \frac{e^{nx^2}.1 - xe^{nx^2}(2nx)}{\left(e^{nx^2}\right)^2}$$

For maxima, $\frac{dy}{dx} = 0 \Rightarrow (1 - 2nx^2) = 0 \Rightarrow x = \pm\frac{1}{\sqrt{2n}}$

Clearly, for $x < \frac{1}{\sqrt{2n}}, \frac{dy}{dx} > 0$.

for x>$\frac{1}{\sqrt{2n}}, \frac{dy}{dx} < 0$

$\therefore x = \frac{1}{\sqrt{2n}}$ is point of maxima for $y$.

$$\therefore \ M_n = \frac{1/\sqrt{2n}}{e^{1/2}} = \frac{\sqrt{e}}{\sqrt{2n}} \to 0, as\ n \to \infty.$$

∴ By $M_n$ test, the given sequence $\{f_n(x)\}$ converges uniformly on $\mathbb{R}$.

∴ Option (1) is correct.

7. Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous map. Choose the correct statement.

**1)** $f$ is bounded.

**2)** The image of $f$ is open subset of $\mathbb{R}$.

**3)** $f(A)$ is bounded for all bounded subset $A$ of $\mathbb{R}$

**4)** $f(A)$ is compact for all compact subsets $A\ of\ \mathbb{R}$.

**Sol:**

Given, $f: \mathbb{R} \to \mathbb{R}$ is a continuous map

$(i)$ Take, $f(x) = x$, clearly, $'f'$ is unbounded.

$\therefore$ Option $(1)$ is incorrect.

$(ii)$ Take, $f(x) = \sin x$

Clearly, $f(\mathbb{R}), i.e$ ., Image of $f = [-1,1]$, which is closed.

$\therefore$ Option $(2)$ is incorrect.

Also, take $A = [-1, 1]$, which is compact

But $f^{-1}(A) = \mathbb{R}$, which is not compact.

$\therefore$ Option $(4)$ is also incorrect. As all other options are incorrect.

Hence, option $(3)$ is correct.

**8.** Suppose $x: [0, \infty] \to [0, \infty]$ is continuous and $x(0) = 0$. If $\left(x(t)\right)^2 \leq 2 + \int_0^t x(S)ds, \forall t \geq 0$, then which of the following is true?

**1)** $x(\sqrt{2}) \in [0,2]$

**2)** $x(\sqrt{2}) \in \left[0, \frac{3}{\sqrt{2}}\right]$

**3)** $x(\sqrt{2}) \in \left[\frac{5}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right]$

**4)** $x(\sqrt{2}) \in [10, \infty]$

**Sol:**

Given, $x: [0, \infty) \to: [0, \infty)$ is continuous and $x(0) = 0, f\left(x(t)\right)^2 \leq 2 + \int_0^t x(S)ds, \forall\ t \geq 0,$

Clearly, $x(t) = 0$ satisfies given conditions

$\therefore$ for $x(t) = 0, \forall t \geq 0$

$$x(\sqrt{2}) = 0 \notin \left[\frac{5}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right]\ and\ x(\sqrt{2}) \notin [10, \infty]$$

Thus option $(3)$ and $(4)$ are incorrect.

Now, either $x(\sqrt{2}) \epsilon [0,2]$ or $x(\sqrt{2}) \epsilon \left[0, \frac{3}{\sqrt{2}}\right]$

As $[0,2] \subseteq \left[0, \frac{3}{\sqrt{2}}\right]$ and only single option must be correct.

$\therefore$ option $(2)$ is correct.

## Part C

**9.** Let $\alpha = 0.10110111011110$ ...be a given real number written in base $10$, that is, the $n$-th digit of $\alpha$ is $1$, unless $n$ is of the form $\frac{k(k+1)}{2} - 1$ in which case it is $0$. Choose all the correct statements given below.

**1)** $\alpha$ is a rational number.

**2)** $\alpha$ is an irrational number.

**3)** For every integer $q \geq 2$, there exists an integer $r \geq 1$ such that $\frac{r}{q} < \alpha < \frac{r+1}{q}$.

**4)** $\alpha$ has no periodic decimal expansion.

**Sol:**

Given, $\alpha = 0.10110111011110$ ...

Clearly, it is a non-repeating and non-terminating decimal number.

$\therefore$ $'\alpha'$ is an irrational number.

A periodic decimal expansion is that which repeats after a certain period.

Since, $'\alpha'$ is non-repeating

$\therefore$ $'\alpha'$ has no periodic decimal expansion.

Hence, option $(2)$ and $(4)$ are correct and option $(1)$ is incorrect.

Option $(3)$ is incorrect

As, for $q = 2, \nexists$ any $r \geq 1$ for which $\alpha > \frac{r}{q}$.

**10.** For $a, b \in \mathbb{N}$, consider the sequence $d_n = \frac{\binom{n}{a}}{\binom{n}{b}} for n > a, b$.

Which of the following statements are true?

1) $\{d_n\}$ converges for all values of $a \ and \ b$.
2) $\{d_n\}$ converges if $a < b$.
3) $\{d_n\}$ converges if $a = b$.
4) $\{d_n\}$ converges if $a > b$.

**Sol:**

$$d_n = {}^nC_a \big/ {}^nC_b$$

(i) if $a = b$, then $d_n = \frac{{}^nC_a}{{}^nC_b} = 1$, which is a constant sequence and hence convergent.

$\therefore$ option $(3)$ is correct.

(ii) if $a > b$, take $a = 2, b = 1$

$$\therefore d_n = \frac{{}^nC_2}{{}^nC_1} = \frac{n(n-1)}{2n} = \frac{n-1}{2}$$

Clearly, $\{d_n\}$ is divergent for $a > b$.

Hence option $(1)$ $and$ $(4)$ are incorrect.

(iii) if $a < b$, then $d_n = \frac{{}^nC_a}{{}^nC_b}$

$$= \frac{n(n-1)(n-2)\ldots\big(n-(a-1)\big)}{a!} \times \frac{b!}{n(n-1)\ldots\big(n-(a-1)\big)(n-a)\ldots\ldots\big(n-(b-1)\big)}$$

$$= \frac{b!/a!}{(n-a)\ldots\big(n-(b-1)\big)}$$

Clearly, $\{d_n\}$ converges to '0' as $n \to \infty$.

$\therefore$ $(2)$ is correct.

**11.** Let $\{a_n\}$ be a sequence of real numbers satisfying $\sum_{n-1}^{\infty}|a_n - a_{n-1}| < \infty$. Then the series $\sum_{n=0}^{\infty} a_n x^n$ , $x \in \mathbb{R}$ is convergent

1) Nowhere on $\mathbb{R}$
2) Everywhere on $\mathbb{R}$
3) On some set containing $(-1,1)$
4) only on $(-1,1)$

**Sol:**

Take, $a_n = \frac{1}{n}$

$\sum_{n-1}^{\infty}|a_n - a_{n-1}| = \sum_{n=1}^{\infty}\left|\left(\frac{1}{n} - \frac{1}{n-1}\right)\right| = \sum_{n=1}^{\infty}\frac{1}{n(n-1)} \sim \sum_{n=1}^{\infty}\frac{1}{n^2}$, which is convergent by $P - test$.

$$\Rightarrow \sum_{n=1}^{\infty}|a_n - a_{n-1}| < \infty, for\ a_n = \frac{1}{n}$$

Now, $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}\frac{x^n}{n}$

Let $b_n = a_n x^n$

$$\therefore \lim_{n\to\infty}\frac{b_n + 1}{b_n} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n}\cdot\frac{x^{n+1}}{x^n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right).x = x$$

$\therefore \sum_{n=1}^{\infty}\frac{x^n}{n}$ converges for $|x| < 1$ and diverges for $|x| > 1$

For $x = 1, \sum_{n=0}^{\infty}\frac{x^n}{n} = \sum_{n=0}^{\infty}\frac{1}{n}$, which is divergent by $P - test$.

For $x = -1, \sum_{n=0}^{\infty}\frac{x^n}{n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n}$, which is convergent by Leibnitz test.

Thus, the series $\sum_{n=0}^{\infty}\frac{x^n}{n}$ converges on $[-1,1)$

Clearly, option $(1), (2), (4)$ are incorrect.

Only option $(3)$ is correct.

**12.** Let $f(x) = \tan^{-1} x$ , $x \in \mathbb{R}$. Then

1) there exists a polynomial $P(x)$ satisfies $P(x)f'(x) = 1, \forall\ x$
2) $f^{(n)}(0) = 0$ for all positive even integer $n$.

3) The sequence $\{f^{(n)}(0)\}$ is unbounded.

4) $f^{(n)}(0) = 0$ for all $n$.

**Sol:**

Here, $f(x) = \tan^{-1} x\,, x \in \mathbb{R}$

$$\therefore f'(x) = \frac{1}{1+x^2} \Rightarrow (1+x^2)f'^{(x)} = 1 - - - (1)$$

$\therefore \exists$ polynomial $p(x) = 1 + x^2$ for which $p(x)f'(x) = 1 \;\forall\; x \in \mathbb{R}$

Thus, option (1) is correct.

Now, differentiating (1), $n$-times, we get

$$n_{c_0} f^{n+1}(x).(1+x^2) + n_{c_1} f^{n}(x).2x + n_{c_2} f^{n-1}(x).2 = 0$$

$$\Rightarrow (1+x^2)f^{n+1}(x) + 2nxf^{n}(x) + n(n-1)f^{n-1}(x) = 0 - - - (2)$$

Put $x = 0$, in $(1) and$ (2), we get $f'(0) = 1, f^{n+1}(0) = -n(n-1)f^{n-1}(0), n = 1,2,3, ....$

$$\Rightarrow f^2(0) = 0$$

$$f^3(0) = -2.1f^1(0) = -2.1$$

$$f^4(0) = -3.2f^2(0) = 0$$

$$f^5(0) = -4.3f^3(0) = (-1)^{\frac{5-1}{2}}(5-1)!$$

In general, $f^n(0) = \begin{cases} 0, & n \text{ is even} \\ (-1)^{\frac{n-1}{2}}(n-1)!\,, & n \text{ is odd} \end{cases}$

Clearly, options (2) and (3) are correct and (4) is incorrect.Type equation here.

**13.** Let $f_n(x) = \frac{1}{1+n^2x^2}$ for $n \in \mathbb{N}$. Which of the following are true?

1) $f_n$ converges pointwise on $[0,1]$ to a continuous function.

2) $f_n$ converges uniformly on $[0,1]$.

3) $f_n$ converges uniformly on $\left[\frac{1}{2}, 1\right]$.

4) $\lim\limits_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 \left(\lim\limits_{n\to\infty} f_n(x)\right) dx$

**Sol:**

$$f_n(x) = \frac{1}{1+n^2x^2}\,, n \in \mathbb{N}, x \in \mathbb{R}$$

$$\therefore f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{1+n^2x^2} = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^2}+x^2} = 0, x \neq 0$$

Also, at $x = 0, f(x) = \frac{1}{1+0} = 1$

$$\therefore f(x) = \begin{cases} 0, 0 < x \leq 1 \\ 1, \quad x = 0 \end{cases}$$

Clearly, $f(x)$ is discontinuous at $x = 0$.

$\therefore$ option (1) is incorrect.

As $\{f_n(x)\}$ converges pointwise to a discontinuous function

$\therefore f_n(x)$ cannot converge uniformly on $[0,1]$.

$\therefore$ option (2) is incorrect.

Further $M_n = \underset{x\,\in\,[0,1]}{sup}\{|f_n(x) - f(x)|\} = \underset{x\,\in\,[0,1]}{sup}\left\{\left|\frac{1}{1+x^2n^2}\right|\right\} \geq \frac{1}{1+\frac{1}{n^2}\cdot n^2} = \frac{1}{2}, \left(Take\ x = \frac{1}{n}\right)$

$\Rightarrow M_n \nrightarrow 0\ as\ n \to \infty$.

$\therefore x = 0$, is point of non-uniform convergence.

$\Rightarrow f_n(x)$ converges uniformly in any closed interval contained in $[0,1]$, that doesn't contains '0'.

In particular $\{f_n(x)\}$ converges uniformly in $\left[\frac{1}{2}, 1\right]$

Thus, option (3) is correct.

Now, $\int_0^1 f_n(x)dx = \int_0^1 \frac{1}{1+n^2x^2}\,dx = \frac{1}{n}\tan^{-1}(nx)\,|_0^1 = \frac{\tan^{-1} n}{n}$.

$\therefore \lim_{n\to\infty} \int_0^1 f_n(x)dx = lim_{n\to\infty} \frac{\tan^{-1} n}{n} = 0\ and\ \int_0^1 \left(\lim_{n\to\infty} f_n(x)\right) dx = 0$ implies $lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 (lim_{n\to\infty} f_n(x))\,dx$

$\therefore$ option (4) is correct.

**14.** If $\lambda_n = \int_0^1 \frac{dt}{(1+t)^n}\ for\ n \in \mathbb{N}$, then

**1)** $\lambda_n$ does not exist for some $n$

**2)** $\lambda_n$ exist for every $n$ and the sequence is unbounded.

**3)** $\lambda_n$ exist for every $n$ and the sequence is bounbded.

**4)** $\lim_{n\to\infty} (\lambda_n)^{1\backslash n} = 1$.

**Sol:-**

$$\lambda_n = \int_0^1 \frac{dt}{(1+t)^n}\ , n \in \mathbb{N}$$

For $n = 1,\ \lambda_1 = \int_0^1 \frac{dt}{(1+t)} = [\log(1+t)]_0^1 = \log 2$.

For $n \neq 1, \lambda_n = \frac{(1+t)^{-n+1}}{-n+1} = \frac{1}{(-n+1)}\left(\frac{1}{2^{n-1}} - 1\right) = \frac{1}{n-1}\left(1 - \frac{1}{2^{n-1}}\right)$

Thus, $\lambda_n$ exists for all $n$.

$\therefore$ option (1) is incorrect.

Also, $\lim\limits_{n\to\infty} \lambda_n = 0$

$\therefore$The sequence $\{\lambda_n\}$ is bounded.

Thus, options (2) is incorrect and (3) is correct.

Further, $\lim\limits_{n\to\infty} (\lambda_n)^{\frac{1}{n}} = \lim\limits_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = \lim\limits_{n\to\infty} \frac{1}{n}\left(1 - \frac{1}{2^n}\right) \times \frac{n-1}{1-\frac{1}{2^{n-1}}}$

$= \lim\limits_{n\to\infty} \left(\frac{n-1}{n}\right) = 1.$

Thus, option (4) is correct.

**15.** The equation $11^x + 13^x + 17^x - 19^x = 0$ has

**1)** no real root

**2)** Only one real root

**3)** exactly two real roots

**4)** more than two real roots

**Sol:**

Let $f(x) = 11^x + 13^x + 17^x - 19^x = 19^x\left\{\left(\frac{11}{19}\right)^x + \left(\frac{13}{19}\right)^x + \left(\frac{17}{19}\right)^x - 1\right\}$

Take, $g(x) = \left(\frac{11}{19}\right)^x + \left(\frac{13}{19}\right)^x + \left(\frac{17}{19}\right)^x - 1$

Clearly, $f(x)$ and $g(x)$ have same roots.

Now, $g(-\infty) = \infty\ and\ g(\infty) = -1$

Also, $g'(x) = \left(\frac{11}{19}\right)^x \cdot \log\left(\frac{11}{19}\right) + \left(\frac{13}{19}\right)^x \cdot \log\left(\frac{13}{19}\right) + \left(\frac{17}{19}\right)^x \log\left(\frac{17}{19}\right) - 1 < 0$

$\Rightarrow g(x)$ is strictly decreasing in $(-\infty, \infty)$.

Thus, $g(x)$ has exactly one real root and hence $f(x)$ also exactly one real root.

**16.** Suppose that $f: \mathbb{R}^n \to \mathbb{R}$ is given by $f(\underline{x}) = a_1 x_1{}^2 + a_2 x_2{}^2 + \cdots + a_n x_n{}^2$, where $\underline{x} = (x_1, x_2, \ldots, x_n)$ and at least one $a_j$ is not zero. Then we can conclude that

1) $f$ is not everywhere differentiable.
2) the gradient$(\nabla f)(\underline{x}) \neq 0$ for every$\underline{x} \in \mathbb{R}^n$.
3) If $\underline{x} \in \mathbb{R}^n$ is such that $(\nabla f)(\underline{x})$ then $f(\underline{x}) = 0$
4) If $x \in \mathbb{R}^n$ is such that $f(\underline{x}) = 0$ then $(\nabla f)(\underline{x}) = 0$.

**Sol:**

Given $f: \mathbb{R}^n \to \mathbb{R}$ is given by $f(\underline{x}) = a_1 x_1{}^2 + a_2 x_2{}^2 + \cdots + a_n x_n{}^2$, where, $\underline{x} = (x_1, x_2, \ldots, x_n)$

$$\therefore Df(\underline{x}) = (\nabla f)(\underline{x}) = \left(2a_1 x_{1,} 2a_2 x_2, \ldots, 2a_n x_n\right) \text{Type equation here.}$$

Clearly, $'f'$ is differentiable everywhere

$\therefore$ (1) is incorrect.

For option (2)

Since, $(\nabla f)(\underline{x}) = 0, for\ \underline{x} = (0,0, \ldots., 0) \epsilon \mathbb{R}^n$

$\therefore$ option (2) is incorrect.

For option (3),

Let $(\nabla f)(\underline{x}) = 0, for\ \underline{x} \in \mathbb{R}^n$.

$$\Rightarrow \left(2a_1 x_{1,} 2a_2 x_2, \ldots, 2a_n x_n\right) = (0,0, \ldots., 0) \Rightarrow x_i = 0\ \forall\ i = 1,2, \ldots, n$$

$$\Rightarrow f(\underline{x}) = 0$$

Thus option (3) is correct.

For option (4)

Take $a_1 = 1, a_2 = -1\ and\ a_3 = a_4 = \cdots a_n = 0$.

$$\therefore\ f(\underline{x}) = x_1{}^2 - x_2{}^2$$

Here, $f(1,1,0, \ldots 0) = 0$

But $(\nabla f)(1,1,0,\dots 0)=(2,-2,0,\dots,0)\neq(0,0,\dots.,0)$

Hence, option (4) is incorrect.

**17.** Let $S$ be the set of $(\alpha,\beta)\in\mathbb{R}^2$ such that $\frac{x^\alpha y^\beta}{\sqrt{x^2+y^2}}\to 0$ as $(x,y)\to(0,0)$. Then $S$ is contained in

**1)** $\{(\alpha,\beta):\alpha>0,\beta>0\}$

**2)** $\{(\alpha,\beta):\alpha>2,\beta>2\}$

**3)** $\{(\alpha,\beta):\alpha+\beta>1\}$

**4)** $\{(\alpha,\beta):\alpha+4\beta>1\}$

**Sol:**

Take, $f(x,y)=\frac{x^\alpha y^\beta}{\sqrt{x^2+y^2}}----(1)$

Put $y=mx$ in (1), we get

$f(x,mx)=\frac{x^\alpha(mx)^\beta}{x^2m^2n^2}=x^{\alpha+\beta-1}\frac{m^\beta}{\sqrt{1+m^2}}\to 0$ when $x\to 0$

Provided $\alpha+\beta>1$

$\therefore S\subseteq\{(\alpha,\beta)\in\mathbb{R}^2:\alpha+\beta>1\}$

Thus, option (3) is correct.

For option (1)

As $\alpha=3,\beta=-1, i.e.,(3,-1)\in s$ but $(3,-1)\notin\{(\alpha,\beta):\alpha,\beta>0\}$.

$\therefore$ option (1) is incorrect.

For option (2)

As $\alpha=3,\beta=-1\ \ i.e.,(3,-1)\epsilon s$.

But $(3,-1)\notin\{(\alpha,\beta):\alpha>2,\beta>2\}$.

$\therefore$ $s$ is not contained in $\{(\alpha,\beta):\alpha>2,\beta>2\}$.

Thus option (2) is incorrect.

For option (4)

Again, $(3,-1)\epsilon s$, but $(3,-1)\notin\{(\alpha,\beta):\alpha+4\beta>1\}$.

Thus, option (4) is incorrect.

# Answer Table

| Sl. No | Answer | Sl. No | Answer | Sl. No | Answer |
|---|---|---|---|---|---|
| 1. | 1 | 2. | 4 | 3. | 3 |
| 4. | 3 | 5. | 3 | 6. | 1 |
| 7. | 3 | 8. | 2 | 9. | 2 & 4 |
| 10. | 2 & 3 | 11. | 3 | 12. | 1, 2 & 3 |
| 13. | 3 & 4 | 14. | 3 & 4 | 15. | 2 |
| 16. | 3 | 17. | 3 | 18. | |

# Previous Year Question & Solution

## Real Analysis (December-2017)

### Part-B

**1.** Let $\mathbb{Z}$ denote the set of integers and$\mathbb{Z}_{n\geq 0}$ denote the set $\{0,1,2,3,4, \dots ..\}$. Consider the map $f: \mathbb{Z}_{n\geq 0} X\mathbb{Z} \to \mathbb{Z}$given by $f(m,n) = 2^m.(2n+1)$. Then the map $f$ is

**1)** Onto (surjective) but not one-one (injective)

**2)** one-one but not onto

**3)** both one-one & onto

**4)** neither one-one nor onto

**Sol:**

Given, $f: \mathbb{Z}_{n\geq 0}\, X\mathbb{Z} \to \mathbb{Z}$ defined by $f(m,n) = 2^m(2n+1)$

For one-one,

$f(m_1, n_1) = f(m_2, n_2) \Rightarrow 2^{m_1}(2n_1+1) = 2^{m_2}(2n_2+1)$

Clearly,$'f'$ is one-one.

$'f'$ is not onto

As there does not exist pre image of 0.

$\therefore$ option $(2)$ is correct.

**2.**Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers satisfying $a_1 \geq 1$ and$a_{n+1} \geq a_n + 1\ \forall\ n \geq 1$. Then which of the following is necessarily true?

**1)** The series $\sum_{n=1}^{\infty} \frac{1}{{a_n}^2}$ diverges.

**2)** The sequence $\{a_n\}_{n\geq 1}$ is bounded.

**3)** The series $\sum_{n=1}^{\infty} \frac{1}{{a_n}^2}$ converges.

**4)** The series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges.

**Solution: -**

Take $\langle a_n \rangle = \langle n \rangle$

Clearly, options $(1), (2)$ $and$ $(3)$ are rule out, as

$(1)$ $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent

$(2)\{1,2,3,4, \dots\}$ is not bounded.

$(4)$ $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Clearly, from given condition $a_n \geq n \forall n \geq 1 \Rightarrow {a_n}^2 \leq \frac{1}{n^2}$ $and$ $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

∴ By comparison test $\sum_{n=1}^{\infty} \frac{1}{{a_n}^2}$ is convergent.

∴ Option $(3)$ is correct.

**3.** Let $D$ be a subset of the real line. Consider the assertion: "Every infinite sequence in D has subsequence which converges in D". This assertion is true if

1) $D = [0, \infty]$
2) $D = [0,1] \cup [3,4]$
3) $D = [-1,1) \cup (1,2]$
4) $D = (-1,1]$

**Sol:**

Every finite sequence $D$ has subsequence which converges in $D$.

This is true only if $D$ is compact set. Compact set is given only in option $(b), i.e.,$ $[0,1] \cup [3,4]$.

∴Option$(2)$ is correct.

**4.** Let $f: (0, \infty) \to \mathbb{R}$ be uniformly continuous. Then

1) $\lim_{x \to 0+} f(x)$ and $\lim_{x \to \infty} f(x)$ exist.
2) $\lim_{x \to 0+} f(x)$ exist but $\lim_{x \to \infty} f(x)$ need not exists.
3) $\lim_{x \to 0+} f(x)$need not exist but $\lim_{x \to \infty} f(x)$ exist.
4) neither $\lim_{x \to 0+} f(x)$ nor $\lim_{x \to \infty} f(x)$ need exist.

**Sol:**

Given, $f:(0,\infty)\to\mathbb{R}$ be uniformly continuous.

Take, $f(x)=x$

Clearly, $f(x)$ is uniformly continuous in $(0,\infty)$

But $\lim_{x\to\infty} f(x)$ does not exist.

So, options $(1)$ and $(3)$ are incorrect.

**<u>Result:</u>** If $f:(a,b)\to\mathbb{R}$ be continuous, then $f$ is uniformly continuous on $(a,b)$ if and only if $\lim_{x\to a+} f(x)$ and $\lim_{x\to b-} f(x)$ exists, where $a$ and $b$ are finite.

So, in the question $\lim_{x\to 0+} f(x)$ must exist.

So, option $(b)$ is correct and $(d)$ is incorrect.

**5.** Let $s=\{f:\mathbb{R}\to\mathbb{R}|\ \exists \in>0\ such\ that\ \forall\ \delta>0, |x-y|<\delta \Rightarrow |f(x)-f(y)|<\in\}$. Then

1) $S=\{f:\mathbb{R}\to\mathbb{R}|\ f\ is\ continuous\}$
2) $S=\{f:\mathbb{R}\to\mathbb{R}|\ f\ is\ uniformly\ continuous\}$
3) $S=\{f:\mathbb{R}\to\mathbb{R}|\ f\ is\ bounded\}$
4) $S=\{f:\mathbb{R}\to\mathbb{R}|\ f\ is\ constant\}$

**Sol:**

Take, $f(x)=\begin{cases}1, x\neq 0\\ 2, x=0\end{cases}$

Thus, there exists $\in=2$ for which $|f(x)-f(y)|<2\ \forall\ |x-y|<\delta$

$$\therefore f(x)\in S$$

But $f(x)$ is neither continuous nor uniformly continuous.

Also, $f(x)$ is not a constant function.

Hence, options $(1),(2),(4)$ruled out.

$\therefore$ option $(3)$ is correct.

**6.** Which of the following is necessarily true for a function $: X \to Y$ ?

1) If $f$ is injective, then there exists $g: Y \to X$ such that $f\big(g(y)\big) = y \; for \; all \; y \epsilon Y$.
2) If $f$ is surjective, then there exists $g: Y \to X$ such that $f\big(g(y)\big) = y \; for \; all \; y \epsilon Y$.
3) If $f$ is injective and $Y$ is countable then $X$ is finite.
4) If $f$ is surjective and $X$ is uncountable then $Y$ is countably infinite.

**Sol:**

For option $(1)$

Let $X = \{0,1\}, Y = \{0,1,2\}$

Take, $f: X \to Y$ be defined as $f(x) = x$.

Clearly, $f$ is injective but for any function $g: Y \to X$ there does not exists pre-image of under f$o$g.

$\therefore$ option $(1)$ is incorrect.

For option $(2)$

As $f$ is surjective

$\therefore$ $f: X \to Y$ for any $y \epsilon Y$ there exists at least one $x \epsilon X$ for which $f(x) = y$.

Define $g: Y \to X$ in such a way that $g(x\ ) = x$.

$fog(y) = f(x) = y$.

So, option $(2)$ is correct.

For option $(3)$

Take, $X = \mathbb{N}, Y = \mathbb{N}$

Let $f: \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x$, clearly $f$ is injective and $Y$ is countable but $X$ is also countable.

Thus option $(3)$ is incorrect.

For option $(4)$,

Take, $X = \mathbb{R}, Y = \mathbb{R}$

Let $f: X \to Y$ defined by $f(x) = x$.

Clearly $f$ is surjective and $X$ is uncountable but $Y$ is also uncountable.

$\therefore$ option $(4)$ is incorrect.

**7.** Let $k$ be a positive integer and

let $s_k = \{x\epsilon[0,1] | a\ decimal\ expansion\ of\ X\ has\ a\ prime\ digit\ at\ its\ k^{th}\ place\}$. Then the Lebesgue measure of $s_k$ is

1) $0$
2) $\frac{4}{10}$
3) $\left(\frac{4}{10}\right)^{\mathrm{k}}$
4) $1$

**Sol:**

For $k = 1$

$$S_1 = \{x \in [0,1] | a\ decimal\ expansion\ of\ x\ has\ a\ prime\ digit\ at\ its\ 1^{st}\ place\}$$
$$= [0.2,0.4] \cup [0.5,0.6] \cup [0.7,0.8].$$

$\therefore$ Lebesgue measure of $S_1 = 0.2 + 0.1 + 0.1 = 0.4 = \frac{4}{10}$.

For $k = 2$,

$$S_2 = \bigcup_{i=0}^{9} ([0.i2,0.i4] \cup [0.i5,0.i6] \cup [0.i7,0.i8]) = 10(.02 + 0.01 + 0.01)$$

$= .4 = \frac{4}{10}$.

Hence, options $(1), (3), (4)$ are incorrect.

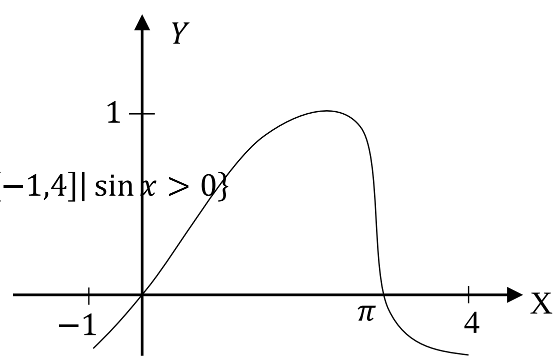
Only option $(2)$ is correct.

**8.** Let $S = \{x \in [-1,4] | \sin x > 0\}$. Which of the following is true?

1) $\inf(S) < 0$
2) $\sup(S)$ does not exist
3) $\sup(S) = \pi$
4) $\inf(S) = \frac{\pi}{2}$.

**Sol:**

$$s = \{x \in [-1,4] | \sin x > 0\}$$




From graph, it is clear that $s = (0, \pi)$

$\therefore \sup(s) = \pi, \inf(s) = 0$.

$\therefore$ option $(3)$ is correct.

**9.** Let $A$ be a connected open subset of $\mathbb{R}^2$. The number of continuous surjective functions from $\bar{A}$ (the closure of $A$ in $\mathbb{R}^2$) to $Q$ is:

1) 1
2) 0
3) 2
4) not finite.

**Sol:**

Given $A$ be a connected open subset of $\mathbb{R}^2$. If possible, let:$\bar{A} \to \mathrm{Q}$ be a continuous surjective function.

$\Rightarrow f(\bar{A}) = Q,$ which is not possible.

As, we know continuous image of a connected set is connected.

But $Q$ is disconnected.

Thus, there is no surjective function from $\bar{A}$ to $Q$.

$\therefore$ option (2) is correct.

## PART-C

**10.** Which of the following are convergent?

1) $\sum_{n=1}^{\infty} n^2 2^{-n}$
2) $\sum_{n=1}^{\infty} n^{-2} 2^{n}$
3) $\sum_{n=1}^{\infty} \frac{1}{n \log n}$
4) $\sum_{n=1}^{\infty} \frac{1}{n \log\left(1+\frac{1}{n}\right)}$

**Sol:**

$$(1) \sum_{n=1}^{\infty} n^2 2^{-n} = \sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

We have $\overline{\lim_{n\to\infty}} (a_n)^{\frac{1}{n}} = \overline{\lim_{n\to\infty}} \left(\frac{n^2}{2^n}\right)^{\frac{1}{n}}$

$\overline{\lim_{n\to\infty}} \frac{\left(n^{1/n}\right)^2}{2} = \frac{1}{2} < 1.$

So, by root test, series is convergent.

(2) $\sum_{n=1}^{\infty} n^{-2}2^n = \sum_{n=1}^{\infty} \frac{2n}{n^2}$

By root-test

$\lim_{n\to\infty} (a_n)^{1/n} = 2 > 1$

By root-test, series is divergent.

$$(3) \sum_{n=1}^{\infty} \frac{1}{n \log n}$$

By integral-test, $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ *and* $\int_2^{\infty} \frac{1}{x \log x} dx$ behave alike.

Let $I = \int_2^{\infty} \frac{1}{x \log x} dx$

$$I = \int_{\log 2}^{\infty} \frac{1}{y} \mathrm{dy}$$

By p-test, $\int_{\log 2}^{\infty} \frac{1}{y} \mathrm{dy}$ is divergent.

$$(4) \sum_{n=1}^{\infty} \frac{1}{n \log\left(1 + \frac{1}{n}\right)}$$

$$a_n = \frac{1}{n \log\left(1 + \frac{1}{n}\right)}$$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n\left\{\frac{1}{n} - \frac{1}{2n^2} + \cdots\right\}} = 1.$

As, $\lim_{n\to\infty} a_n \neq 0 \Rightarrow$ series is divergent.

Thus, only option (1) is correct.

**11.**Let $a_{mn}, m \geq 1, n \geq 1$ be a double array of real numbers.

Define $P = \lim_{n\to\infty} \inf \lim_{m\to\infty} \inf a_{mn}$,

$Q = \lim_{n\to\infty} \inf \lim_{m\to\infty} \sup a_{mn}$,

$R = \lim_{n\to\infty} \sup \lim_{m\to\infty} \inf a_{mn}$,

$S = \lim_{n\to\infty} \sup \lim_{m\to\infty} \sup a_{mn}$,

Which of the following statements are necessarily true?

1) $P \leq Q$
2) $Q \leq R$
3) $R \leq S$
4) $P \leq S$.

**Sol:**

Given sequence of real numbers is

$$\{a_{m1}, a_{m2}, a_{m3}, \ldots\}, i.e.,$$

$$\{a_{11}, a_{12}, a_{13}, \ldots$$

$a_{21}, a_{22}, a_{23}, \ldots..$

$$a_{31}, a_{32}, a_{33}, \ldots.\}$$

$$\lim_{m\to\infty} \inf a_{mn} = \left\{\lim_{m\to\infty} \inf a_{m1}, \lim_{m\to\infty} \inf a_{m2}, \ldots\right\}$$

$$\lim_{m\to\infty} \sup a_{mn} = \left\{\lim_{m\to\infty} \sup a_{m1}, \lim_{m\to\infty} \sup a_{m2}, \ldots\right\}$$

Clearly, $\lim_{m\to\infty} \inf a_{mi} \leq \lim_{m\to\infty} \sup a_{mi} \forall i$ .

$$\Rightarrow \lim_{m\to\infty} \inf a_{mn} \leq \lim_{m\to\infty} \sup a_{mn} \ \forall \ n \epsilon \mathbb{N} .$$

That implies $\lim_{n\to\infty} \inf \lim_{m\to\infty} \inf a_{mn} \leq \lim_{n\to\infty} \inf \lim_{m\to\infty} \sup a_{mn} \Rightarrow P \leq Q$.

$\lim_{n\to\infty} \inf \lim_{m\to\infty} \inf a_{mn} \leq \lim_{n\to\infty} \sup \lim_{m\to\infty} \sup a_{mn} \Rightarrow P \leq S$.

Also, $\lim_{m\to\infty} \inf a_{mn} \leq \lim_{m\to\infty} \sup a_{mn}$.

$\Rightarrow \lim_{n\to\infty} \sup \lim_{m\to\infty} \inf a_{mn} \leq \lim_{n\to\infty} \sup \lim_{m\to\infty} \sup a_{mn}$.

$\Rightarrow R \leq S$.

So, options $(1), (3), (4)$ are correct

Option $(2)$ may not hold.

Take sequence

$$\begin{array}{ccc} a_{m_1} & a_{m_2} & a_{m_3}\ldots \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & \vdots \\ \vdots & \vdots & \vdots \end{array}$$

$$\lim_{m\to\infty} \inf a_{mn} = \{0,0,0,\ldots\ldots\}$$

$$R = \lim_{n\to\infty} \sup \lim_{m\to\infty} \inf a_{mn} = \lim_{n\to\infty} \sup\{0,0,0,\ldots\ldots\} = 0$$

Also, $\lim\limits_{m\to\infty} \sup a_{mn} = \{1,1,1,\ldots.\}$

$$\therefore Q = \lim_{n\to\infty} \inf \lim_{m\to\infty} \sup a_{mn} = \lim_{n\to\infty} \inf\{1,1,1,\ldots.\} = 1$$

Clearly $R < Q$

$\therefore$ option (2) is incorrect.

**12.** Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{Q}$ the set of all rational numbers. For $0 \leq \in \leq \frac{1}{2}$, let be $A_\in$the open interval $(1,1-\in)$. Which of the following are true?

1. $\sup\limits_{0<\in<\frac{1}{2}} \quad \sup(A_\in)$
2. $0 <\in_1<\in_2< \frac{1}{2} \Rightarrow \inf(A_{\in_1}) < \inf(A_{\in_2})$
3. $0 <\in_1<\in_2< \frac{1}{2} \Rightarrow \sup(A_{\in_1}) < \sup(A_{\in_2})$
4. $\sup(A_\in \cap \mathbb{Q}) = \sup(A_\in \cap \mathbb{R}\backslash\mathbb{Q})$

**Sol.**

For option (1),

As, $\sup\limits_{0<\in<\frac{1}{2}} \quad \sup(A_\in) = \sup\limits_{0<\in<\frac{1}{2}}(1-\in) = 1$

$\therefore$ Option (1) is incorrect.

For option (2),

$$0 <\in_1<\in_2< \frac{1}{2}$$

Take, $\in_1= 0.1, \in_2= 0.2$

$$\therefore \inf(A_{\in_1}) = \inf((0,0.9)) = 0$$

And $\inf(A_{\in_2}) = \inf((0,0.8)) = 0$

$$\Rightarrow \inf(A_{\in_1}) = \inf(A_{\in_2})$$

$\therefore$ Option (2) is incorrect.

For option (3),

For $0 < \in_1 < \in_2 < \frac{1}{2}$

$$\Rightarrow 1 - \in_1 > 1 - \in_2$$

Now, $A_{\in_1} = (0,1 - \in_1)$ and $A_{\in_2} = (0,1 - \in_2)$

$$\Rightarrow \sup\left(A_{\in_1}\right) = 1 - \in_1 > 1 - \in_2 = \sup\left(A_{\in_2}\right)$$

Thus, option (3) is correct.

For option (4)

$\sup(A_{\in} \cap \mathbb{Q}) = \sup\left((0,1-\in) \cap \mathbb{Q}\right) = 1-\in$ and

$$\sup\left(A_{\in} \cap (\mathbb{R}\backslash\mathbb{Q})\right) = \sup\left((0,1-\in) \cap (\mathbb{R}\backslash\mathbb{Q})\right) = 1-\in$$

$$\Rightarrow \sup(A_{\in} \cap \mathbb{Q}) = \sup\left(A_{\in} \cap (\mathbb{R}\backslash\mathbb{Q})\right)$$

$\therefore$ Option (4) is correct.

**13.** Let $f: \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(x + y) = f(x)f(y), \forall\, x, y \in \mathbb{R}$ and $\lim_{x \to 0} f(x) = 1$, which of the following are necessarily true?

1. $f$ is strictly increasing.
2. $f$ is either constant or bounded.
3. $f(rx) = f(x)^r$ for every rational number $r \epsilon \mathbb{Q}$.
4. $f(x) \geq 0, \forall\, x \epsilon \mathbb{R}$.

**Sol.**

Take $f(x) = e^{-x}$.

Clearly, $f(x)$ is unbounded, monotonically decreasing and non-constant.

Options (1) and (2) are incorrect.

For option (3)

Let $r = \frac{p}{q} \in \mathbb{Q}, p, q \in \mathbb{Z}, (p, q) = 1$ and $q \neq 0$.

$$\therefore f(px) = f\,\underbrace{(x + x + \dots + x)}_{p\ times} = \{f(x)\}^p$$

$$f(px) = f\left(q.\frac{p}{q}x\right) f\,\underbrace{\left(\frac{p}{q}x + \frac{p}{q}x + \dots + \frac{p}{q}x\right)}_{q\ times}$$

$$f(px) = \left(f\left(\frac{p}{q}x\right)\right)^q \Rightarrow \left(f(x)\right)^p = \left(f\left(\frac{p}{q}x\right)\right)^q \Rightarrow \left(f(x)\right)^{p/q} = \left(f\left(\frac{p}{q}x\right)\right)$$

So, option (3) is correct.

For option (4),

$$f(x) = f\left(\frac{x}{2}+\frac{x}{2}\right) = \left(f\left(\frac{x}{2}\right)\right)^2 \geq 0$$

$$\Rightarrow f(x) \geq 0$$

∴ Option (4) is correct.

**14.** Evaluate $\lim_{n\to\infty} \sum_{k=0}^{n} \frac{n}{k^2+n^2}$

1. $\frac{\pi}{2}$
2. $\pi$
3. $\frac{\pi}{8}$
4. $\frac{\pi}{4}$

**Sol.**

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{n}{k^2+n^2} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{\left(\frac{k}{n}\right)^2+1}$$

**Result:** $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)dx \Rightarrow \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n} \frac{1}{\left(\frac{k}{n}\right)^2+1} = \int_0^1 \frac{1}{x^2+1}dx = \tan^{-1} x|_0^1 = \frac{\pi}{4}$

∴ Option (4) is correct.

**15.** Let $f(x,y) = \frac{1-\cos(x+y)}{x^2+y^2}$ if $(x,y) \neq (0,0), f(0,0) = \frac{1}{2}$and$g(x,y) = \frac{1-\cos(x+y)}{(x+y)^2}$, if $x + y \neq 0.$

$g(x,y) = \frac{1}{2}$, if $x + y = 0$ then

1. $f$ is continuous at $(0,0)$.
2. $f$ is continuous every where except at $(0,0)$.
3. $g$ is continuous at $(0,0)$.
4. $g$ is continuous every where.

**Sol.**

$$f(x,y) = \begin{cases} \dfrac{1-\cos(x+y)}{x^2+y^2}, & (x,y) \neq (0,0) \\ \dfrac{1}{2}, & (x,y) = (0,0) \end{cases}$$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{1-\cos(x+y)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{\frac{(x+y)^2}{2!} - \frac{(x+y)^4}{4!} + \ldots}{x^2+y^2} \ldots\ldots\ldots\ldots(i)$$

Put $y = mx$ in $(i)$, we get $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{\frac{(1+m)^2}{2!} - \frac{(x+y)^4}{4!} + \ldots}{1+m^2}$

Which depends on $m$.

Thus, $\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exists.

Hence, $(x,y)$ is not continuous at (0,0).

Further, $\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} \frac{1-\cos(x+y)}{x^2+y^2}$

$$= \lim_{(x,y)\to(0,0)} \frac{\frac{(x+y)^2}{2!} - \frac{(x+y)^4}{4!} + \ldots}{(x+y)^2} = \lim_{(x,y)\to(0,0)} \frac{1}{2!} - \frac{(x+y)^2}{4!} + \ldots\ldots = \frac{1}{2} = g(0,0)$$

Thus, $g(x,y)$ is continuous every where.

Hence options (2), (3), (4) are correct and option (1) is incorrect.

**16.** Let $f: \mathbb{R}^4 \to \mathbb{R}$ be defined by $f(x) = x^t Ax$, where $A$ is a $4 \times 4$ matrix with real entries and $x^t$ denotes the transpose of $x$. The gradient of $f$ at a point $x_0$ necessarily is

1. $2Ax_0$
2. $Ax_0 + A^t x_a$
3. $2A^t x_0$
4. $Ax_0$

**Sol.**

Given, $f: \mathbb{R}^4 \to \mathbb{R}$ is defined by $f(x) = x^t Ax$, where $A$ is $4 \times 4$ matrix with real entries.

Take, $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\therefore$ For $x = [x_1 x_2\, x_3\, x_4]^t$

$$f(x) = [x_1\ \ x_2 x_3 x_4] \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = {x_1}^2 + x_2 x_3$$

$$\therefore (\nabla f)(x) = [2x_1 x_3\, x_{\,2}\ 0]^t$$

For $x_0 = (x_1', x_2', x_3', x_4')$

$$(\nabla f)(x_0) = [2x_1' x_2' x_3'\ \ 0]^t$$

For option (1),

$$2Ax_0 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = [2x_1' x_3'\ \ 0\ \ 0]^t \neq (\nabla f)(x_0)$$

$\therefore$ Option (1) is incorrect.

For option (2),

$$Ax_0 + A^t x_a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = [2x_1' x_3'\ \ 0\ \ 0] = (\nabla f)(x_0).$$

Thus, option (2) is correct.

We can easily verify that $2A^t x_0 = [2x_1'\ 0\ \ x_2'\ \ 0] \neq (\nabla f)(x_0)$ and

$$Ax_0 = [x_1' x_3'\ \ 0\ \ 0] \neq (\nabla f)(x_0)$$

$\therefore$ Options (3) and (4) are incorrect.

**17.** Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable map satisfying $\|f(x) - f(y)\| \geq \|x - y\|$, for all $x, y \epsilon \mathbb{R}^n$. Then

1. $f$ is onto.
2. $f(\mathbb{R}^n)$ is a closed subset of $\mathbb{R}^n$.
3. $f(\mathbb{R}^n)$ is an open subset of $\mathbb{R}^n$.
4. $f(0) = 0$.

**Sol.**

Given, $f: \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable map Satisfying $\|f(x) - f(y)\| \geq \|x - y\| \; \forall \; x, y \epsilon \mathbb{R}^n$.

First of all we will prove that $'f'$ is one-one.

Let $f(x) = f(y)$, for some $x, y \epsilon \mathbb{R}^n$.

Then $\|f(x) - f(y)\| \geq \|x - y\| \Rightarrow 0 \geq \|x - y\|$

i.e., $\|x - y\| \leq 0 \Rightarrow x = y$.

$\therefore f$ is one-one.

Also, $'f'$ is continuous.

$\Rightarrow$ either $'f'$ is strictly increasing or strictly decreasing

for option (1)

If possible, let $'f'$ is not onto.

Then either $\lim_{x\to\infty} f(x) \; or \; \lim_{x\to\infty} f(x)$ exists finitely.

W.L.O.G, let $\lim_{n\to\infty} f(x) = l(finite)$

Then for any $y\epsilon\mathbb{R}^n$ and $x \to \infty \Rightarrow \|l - f(y)\| \geq \infty$, which is contradiction.

$\therefore 'f'$ must be onto.

Thus, option (1) is correct.

For options (2) and (3)

As $'f'$ is one-one and onto.

$\Rightarrow f(\mathbb{R}^n) = \mathbb{R}^n$ which is both open and closed.

Thus, options (2) and option (3) are correct.

For option (4)

Take, $f(x) = x + 2$.

Clearly, $\|f(x) - f(y)\| \geq \|x - y\|$

But $f(0) = 2 \neq 0$.

$\therefore$ option (4) is incorrect.

**18.** Let $f: [-1,1] \to \mathbb{R}$ be a function given by $(x) = \begin{cases} x^2 \cos\frac{1}{x}, & if \; x \neq 0 \\ 0 \quad , & if \; x = 0 \end{cases}$. Then

1. $f$ is of bounded variation on $[-1,1]$.

2. $f'$ is of bounded variation on $[-1,1]$.
3. $|f'(x)| \leq 1 \ \forall \ x \in [-1,1]$.
4. $|f'(x)| \leq 3 \ \forall \ x \in [-1,1]$.

**Sol.**

Given, $f: [-1,1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0 \quad , & x = 0 \end{cases}$

$$f'(x) = 2x\cos\left(\frac{1}{x}\right) + x^2\left(-\sin\frac{1}{x}\right) \times \left(-\frac{1}{x^2}\right), x \neq 0 = 2x\cos\frac{1}{x} + \sin\frac{1}{x}$$

$\therefore |f'(x)| = \left|2x\cos\frac{1}{x} + \sin\frac{1}{x}\right| \leq 2.|x| + 1 \Rightarrow \left|f'(x)\right| \leq 3x \in [-1,1]$.

$\therefore$ Option (4) is correct and (3) is incorrect.

**Result:** If $'f'$ is a differentiable function having bounded variation on $[a,b]$.Then f is bounded variation on [a,b].

As $|f'(x)| \leq 3 \ \forall x \epsilon [-1,1]$.

$\therefore f$ is of bounded variation on $[-1,1]$

Hence, option (1) is correct.

**19.** For a set $X$, let $\rho(X)$ be the set of all subsets of $x$ and let $\Omega(X)$ be the set of al functions $f: X \to \{0,1\}$.

Then

1. If $X$ is finite then $\rho(X)$ is finite.
2. If $X$ and $Y$ are finite sets and if there is a one-one correspondence between $\rho(X)$ and $\rho(Y)$, then there is a one-one correspondence between $X$ and $Y$.
3. There is no one-one correspondence between $X$ and $\rho(x)$.
4. There is a one-one correspondence between $\Omega(x)$ and $\rho(x)$.

**Sol.**

For any set $X, \rho(x) =$ set of all subsets of $X$ and $\Omega(X) =$ set of all functions $f: X \to \{0,1\}$.

For option (1)

If $X$ is finite

Let $|X| = n$, then $|\rho(X)| = 2^n$, which is again finite.

$\therefore$ Option (1) is correct.

For option (2)

Let $X$ and $Y$ are finite and there is one-one correspondence between $\rho(X)$ and $\rho(Y)$

$$\Rightarrow |\rho(X)| = |\rho(Y)| \Rightarrow 2^{|X|} = 2^{|Y|} \Rightarrow |X| = |Y|$$

Hence, $\exists$ one-one correspondence between $X$ and $Y$.

This, option (2) is correct.

For option (3)

Since, $|\rho(X)| = 2^{|X|} = |X|$

Thus, there is no one-one correspondence between x and $\rho(X)$.

Hence option (c) is correct.

For option (4)

Since, $|\rho(X)| = 2^{|X|} = |\Omega(X)|$

$\therefore$ $\exists$ a one -one correspondence between $\Omega(X)$ and $\rho(x)$.

Thus, option (4) is correct.

**20.** Let $d$ and $d'$ be metrics on a non-empty set $x$. Then which of the following are metrices on $X$?

1. $\rho_1(x, y) = d(x, y) + d'(x, y)$ for all $x, y \in X$.
2. $\rho_2(x, y) = d(x, y)d'(x, y)$ for all $x, y \in X$.
3. $\rho_3(x, y) = \max\{d(x, y), d'(x, y)\}$ for all $x, y \in X$.
4. $\rho_4(x, y) = \min\{d(x, y), d'(x, y)\}$ for all $x, y \in X$.

**Sol.**

For option (1)

**Result :** If $d_1$ and $d_2$ are two metrics. Then $\alpha d_1 + \beta d_2 (\alpha_, \beta > 0) is$ again a metric.

Thus, option (1) is correct.

For option (2)

Product of two metrics may not be a metric.

For example, Take $d(x, y) = |x - y|$ and $d'(x, y) = \min\{1, d(x, y)\}$ be two matrices on $\mathbb{R}$.

For $x = 0, y = \frac{3}{2}, z = 2$

$$d(0,2). d'(0,2) = |0 - 2|. \min\{1, |0 - 2|\} = 2 \times 1 = 2$$

And $d\left(0,\frac{3}{2}\right).d'(\left(0,\frac{3}{2}\right)+d\left(\frac{3}{2},2\right).d'\left(\frac{3}{2},2\right)=\frac{3}{2}\times 1+\frac{1}{2}\times\frac{1}{2}=\frac{7}{4}<2$

Thus, triangle inequality does not hold.

$\therefore\ d(x,y).d'(x,y)$ is not metric in general.

For option (3)

$\rho_3(x,y)=\max\{d(x,y),d'(x,y)\}$.

It can be easily verified that $\delta_3(x,y)$ satisfies all the conditions of a metric.

For option (4)

$\rho_4(x,y)=\min\{d(x,y),d'(x,y)\}$.

Take $X=\mathbb{R}^2$ and $d(x,y)=\sqrt{(x_1-y_1)^2+\frac{1}{4}(x_2-y_2)^2}$

$d'(x,y)=\sqrt{\frac{1}{4}(x_1-y_1)^2+(x_2-y_2)^2}$ be two matrices on $\mathbb{R}^2$

Let $x=(1,0),y=(0,1),z=(0,0)$

$$\therefore\ \rho_4(x,y)=\min\{d(x,y),d'(x,y)\}=\min\left\{\sqrt{1+\frac{1}{4}},\sqrt{\frac{1}{4}+1}\right\}=\frac{\sqrt{5}}{2}$$

$$\rho_4(x,z)+\rho_4(z,y)$$

$$=\min\{d(x,z),d'(x,z)\}+\min\{d(z,y),d'(z,y)\}=\min\left\{1,\frac{1}{2}\right\}+\min\left\{\frac{1}{2},1\right\}=\frac{1}{2}+\frac{1}{2}=1<\frac{\sqrt{5}}{2}$$

$=\rho_4(x,y)$.

$\therefore$ Triangle inequality does not hold.

Hence, $\rho_4(x,y)$ is not a matric.

# Answer Table

| Sl. No | Answer | Sl. No | Answer | Sl. No | Answer |
|---|---|---|---|---|---|
| 1. | 2 | 2. | 3 | 3. | 2 |
| 4. | 2 | 5. | 3 | 6. | 2 |
| 7. | 2 | 8. | 3 | 9. | 2 |
| 10. | 1 | 11. | 1, 3 & 4 | 12. | 3 & 4 |
| 13. | 3 & 4 | 14. | 4 | 15. | 2, 3 & 4 |
| 16. | 2 | 17. | 1, 2 & 3 | 18. | 1 & 4 |
| 19. | 1, 2, 3 & 4 | 20. | 1 & 3 | | |

# Previous Year Question & Solution

## Real Analysis (June-2018)

### Part-B

**1.** Given that there are real constants $a, b, c, d$ such that the identity $\lambda x^2 + 2xy + y^2 = (ax + by)^2 + (cx + dy)^2$ holds for all $x, y \in \mathbb{R}$. This implies

1. $\lambda = -5$
2. $\lambda \geq 1$
3. $0 < \lambda < 1$
4. There is no such $\lambda \in \mathbb{R}$.

**Sol.**

Take, $a = b = 1, c = d = 0$

We will get $\lambda\, x^2 + 2xy + \lambda^2 = x^2 + y^2 + 2xy \Rightarrow \lambda = 1(\forall\, x, y \in \mathbb{R})$

So, option (1), (3) and (4) are incorrect.

$\Rightarrow$ Option (2) is correct.

**2.** Given $\{a_n\}, \{b_n\}$ two monotone sequences of real numbers and that $\sum a_n b_n$ is convergent, which of the following is true?

1. $\sum a_n$ is convergent and $\sum b_n$ is convergent
2. At least one of $\sum a_n$ , $\sum b_n$ is convergent.
3. $\{a_n\}$ is bounded and $\{b_n\}$ is bounded.
4. At least one of $\{a_n\}, \{b_n\}$ is bounded.

**Sol.**

For option (1) and (2)

Take, $a_n = \frac{1}{n}$ , $b_n = \frac{1}{n}$

Both are monotonic decreasing sequences and $\sum \frac{1}{n} \cdot \frac{1}{n} = \sum \frac{1}{n^2}$ is convergent $(using\ P - test)$

Neither $\sum a_n = \sum \frac{1}{n}\ nor\ \sum b_n = \sum \frac{1}{n}$ is convergent.

So, options (1) and (2) are incorrect.

For option (3),

Take, $a_n = n, b_n = \frac{1}{n^3}$.

Now, $\{a_n\}$ is monotonic increasing sequence.

And $\{b_n\}$ is monotonic decreasing sequence.

Also, $\sum a_n \cdot b_n = \sum \frac{1}{n^2}$ is convergent.

But $\{a_n\}$ is not bounded and $\{b_n\}$ is bounded.

So, option (3) is incorrect.

In conclusion, option (4) is correct.

**3.** Let $S = \left\{(x,y) \middle| x^2 + y^2 = \frac{1}{n^2},\ \ where\ n \in \mathbb{N}\ and\ either\ x \in \mathbb{Q}\ or\ y \in \mathbb{Q}\right\}$. Here $\mathbb{Q}$ is the set of rational numbers and $\mathbb{N}$ is the set of positive integers. Which of the following is true?

1. $S$ is a finite non-empty set.
2. $S$ is countable.
3. $S$ is uncountable.
4. $S$ is empty.

**4.** Define the sequence $\{a_n\}$ as following:
$a_1 = 1$ and for $n \geq 1, a_{n+1} = (-1)^n \left(\frac{1}{2}\right)\left(|a_n| + \frac{2}{|a_n|}\right)$. Which of the following is true?

1. $\limsup a_n = \sqrt{2}$
2. $\liminf a_n = -\infty$
3. $\lim a_n = \sqrt{2}$
4. $\sup a_n = \sqrt{2}$

**5.** If $\{x_n\}$ is a convergent sequence in $\mathbb{R}$ and $\{y_n\}$ is a bounded sequence in $\mathbb{R}$; then we can conclude that

1. $\{x_n + y_n\}$ is convergent.
2. $\{x_n + y_n\}$ is bounded.
3. $\{x_n + y_n\}$ has no convergent subsequence.
4. $\{x_n + y_n\}$ has no bounded subsequence.

**Sol.**

Take, $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n}$.

$\Rightarrow x_n + y_n = \frac{2}{n} \Rightarrow \{x_n + y_n\}$ is convergent sequence.

Also $\{x_n + y_n\}$ bounded sequence (As every convergent sequence is bounded sequence)

$\Rightarrow$ $\{x_n + y_n\}$ has convergent and bounded subsequence.

So, options (3) and (4) are incorrect.

Take, $x_n = \frac{1}{n}$ (convergent sequence) and $y_n = (-1)^n$

$\Rightarrow (x_n + y_n) = \left\{\frac{1}{n} + (-1)^n\right\} = \left\{\begin{matrix} \frac{1}{n} + 1, & n \text{ is even} \\ \frac{1}{n} - 1, & n \text{ is odd} \end{matrix}\right\}.$

Which is not convergent sequence but bounded sequence.

So, option (1) is incorrect and (2) is correct.

**6.** The difference $\log(2) - \sum_{n=1}^{100} \frac{1}{2^n \cdot n}$ is

1. Less than 0
2. Greater than 1.
3. Less than $\frac{1}{2^{100} \cdot 101}$
4. Grater than $\frac{1}{2^{100} \cdot 101}$

**Sol.**

Fact used : $\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n} = \log_e(2)$

Given,

$$\log_e(2) - \sum_{n=2}^{100} \frac{1}{2^n \cdot n} = \left\{\log_e(2) - \sum_{n=1}^{\infty} \frac{1}{2^n . n}\right\} + \sum_{n=101}^{\infty} \frac{1}{n . 2^n} = \sum_{n=101}^{\infty} \frac{1}{n . 2^n}$$

$$= \frac{1}{2^{101} . 101} + \frac{1}{2^{102} . 102} + \frac{1}{2^{103} . 103} + \ldots\ldots$$

$$< \frac{1}{101 . 2^{101}} + \frac{1}{101 . 2^{102}} + \frac{1}{101 . 2^{103}} + \ldots\ldots$$

$$= \frac{1}{101 . 2^{101}}\left[1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\ldots\right] = \frac{1}{101 . 2^{101}} \cdot 2 = \frac{1}{2^{100} . 101}$$

So, option (3) is correct.

**7.** Let $f(x,y) = \log\left(cos^2\left(e^{x^2}\right)\right) + \sin(x+y)$. Then $\frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x,y)$ is

1. $\frac{\cos\left(e^{x^2}\right)-1}{1+\sin^2\left(e^{x^2}\right)}$
2. $0$
3. $-\sin(x+y)$
4. $\cos(x+y)$

**Sol.**

Given, $f(x,y) = \log\left(cos^2\left(e^{x^2}\right)\right) + \sin(x+y)$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\log\left(cos^2\left(e^{x^2}\right)\right) + \frac{\partial}{\partial x}\sin(x+y) = -4x\tan\left(e^{x^2}\right).e^{x^2} + \cos(x+y)$$

$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(-4x\tan\left(e^{x^2}\right)e^{x^2}\right) + \frac{\partial}{\partial y}(\cos(x+y)) = 0 + (-\sin(x+y))$.

$$\Rightarrow \frac{\partial^2 f}{\partial y \partial x} = -\sin(x+y)$$

So, option (3) is correct.

**8.** Let $f(x) = x^5 - 5x + 2. Then$

1. $f$ has no real root.
2. $f$ has exactly one real root.
3. $f$ has exactly three real roots.
4. All roots of $f$ are real.

**Sol.**

Given, $f(x) = x^5 - 5x + 2$

Since, $f(x)$ is a polynomial of odd degree with real co-efficients $\Rightarrow f(x)$ has at least one real root.
[$\because$ Complex root of a polynomial with real co-efficients occurs in conjugate pairs.]

Now, number of sign changes in $f(x)$ is 2.

$\therefore f(x)$ ha sat most two positive real roots and number of sign changes in $f(-x)$ is 1.

$\therefore f(x)$ has at most one negative root.

Thus $f(x)$ has at most three real roots.

$\Rightarrow f(x)$ has either one real root or three real roots.

Since, $f(0) = 2, f(1) = -2, f(2) = 14$

Using intermediate value theorem $f(x)$ has al least one real root in interval $(0,1)$ and also one real root in interval $(1,2)$

$\Rightarrow f(x)$ has exactly three real roots.

So, option (3) is correct.

**9.** Consider the space $S = \{(\alpha, \beta) | \alpha, \beta \in \mathbb{Q}\} \subset \mathbb{R}^2$, where $\mathbb{Q}$ is the set of rational numbers. Then

1. $S$ is connected in $\mathbb{R}^2$.
2. $S^c$ is connected in $\mathbb{R}^2$.
3. $S$ is closed in $\mathbb{R}^2$.
4. $S^c$ is closed in $\mathbb{R}^2$.

**Ans.S**=$\{(\alpha, \beta) : \alpha, \beta \in \mathbb{Q}\} \subset \mathbb{R}^2, \mathbb{Q}\ is\ a\ set\ of\ rational\ numbers.$

If S is connected then there must be a path between two points of S.But when we draw a path it will consist irrational number also.So all points of path does not lie in S.

Hence S cannot be connected .

Now, $S^c = \{(\alpha, \beta) : \alpha$

## Part – C

**10.** For each $\alpha \in \mathbb{R}$, let $S_\alpha = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = \alpha^2\}$. Let $E = \bigcup_{\alpha \in \mathbb{R} \setminus \mathbb{Q}} S_\alpha$ Which of the following are true?

1. The Lebesgue measure of $E$ is infinite.
2. $E$ contains a non-empty open set.
3. $E$ is path connected.
4. Every open set containing $E^c$ has infinite Lebesgue measure.

**Sol.**

Option (1) is correct

as $\bigcup_{\alpha \in \mathbb{R}} S_\alpha = \mathbb{R}^3$ and $\mathbb{R}^3 = \left(\bigcup_{\alpha \in \mathbb{Q}} S_\alpha\right) \cup \left(\bigcup_{\alpha \in \mathbb{R} \setminus \mathbb{Q}} S_\alpha\right)$ and $m\left(\bigcup_{\alpha \in \mathbb{Q}} S_\alpha\right) = 0$

Option (4) is correct as $E^c = \bigcup_{\alpha \epsilon \mathbb{Q}} S_\alpha$ is dense in $\mathbb{R}^3$.

Option (2) is incorrect as $E^c$ is dense in $\mathbb{R}^3$.

Option (3) is incorrect as if $\alpha_1 = \sqrt{2}$ and $\alpha_2 = \sqrt{3}$.

Then the path joining any two points in $S_{\alpha_1}$ and $S_{\alpha_2}$ is not contained in $E$.

**11.** Which of the following sets are uncountable?

1. The set of all functions from $\mathbb{R}$ to $\{0,1\}$.
2. The set of all functions from $\mathbb{N}$ to $\{0,1\}$.
3. The set of all finite subsets of $\mathbb{N}$.
4. The set of all subset of $\mathbb{N}$.

**Sol.**

If $A = \{f|f: \mathbb{R} \to \{0,1\}\} \Rightarrow |A| = 2^{2^{N_0}}$

$$B = \{f|f: \mathbb{N} \to \{0,1\}\} \Rightarrow |B| = 2^{N_0}$$

$$C = \{A: A \subseteq \mathbb{N} \ and \ A \ is \ finite\} \Rightarrow |C| = N_0$$

$$D = \{A|A \subseteq \mathbb{N}\} \Rightarrow |D| = 2^{N_0}$$

Clearly, sets $A, B, D$ are uncountable.

$\therefore$ Options (1), (2), (4) are correct.

**12.** Let $A = \left\{t \sin\left(\frac{1}{t}\right) |t \in \left(0, \frac{2}{\pi}\right)\right\}$. Which of the following statements are true?

1. $\sup(A) < \frac{2}{\pi} + \frac{1}{n\pi}$ for all $n \geq 1$
2. $\inf(A) > -\frac{2}{3\pi} - \frac{1}{n\pi}$ for all $n \geq 1$
3. $\sup(A) = 1$
4. $\inf(A) = -1$

**Sol.**

Given $A = \left\{t \sin\left(\frac{1}{t}\right) |t \in \left(0, \frac{2}{\pi}\right)\right\} \Rightarrow \sup(A) = \frac{2}{\pi} < \frac{2}{\pi} + \frac{1}{n\pi} \forall \ n \geq 1$

and $\inf(A) = 0 > -\frac{2}{3\pi} - \frac{1}{n\pi} \forall\, n \geq 1$.

**13.** Let $C_c(\mathbb{R}) = \{f: \mathbb{R} \to \mathbb{R} | f \textit{ is continuous and there exists a compact set } k \textit{ such that}$

$$f(x) = 0 \textit{ for all } x \in k^c\}$$

Let $g(x) = e^{-x^2}$ for all $x \in \mathbb{R}$. Which of the following statement are true?

1. There exists a sequence $\{f_n\}$ in $C_c(\mathbb{R})$ such that $f_n \to g$ uniformly.
2. There exists a sequence $\{f_n\}$ in $C_c(\mathbb{R})$ such that $f_n \to g$ pointwise.
3. If a sequence in $C_c(\mathbb{R})$ converges pointwise to $g$ it must converge uniformly to $g$.
4. There does not exist any sequence $C_c(\mathbb{R})$ converging pointwise to $g$.

**14.** Given that $a(n) = \frac{1}{10^{100}} \cdot 2^n, b(n) = 10^{100} \log(n), c(n) = \frac{1}{10^{10}.n^2}$, which of the following statements are true?

1. $a(n) > c(n)$ for all sufficiently large $n$.
2. $b(n) > c(n)$ for all sufficiently large $n$.
3. $b(n) > n$ for all sufficiently large $n$.
4. $a(n) > b(n)$ for all sufficiently large $n$.

**Sol.**

As, $\lim_{n\to\infty} \frac{a(n)}{c(n)} = \infty$ and $\lim_{n\to\infty} \frac{a(n)}{b(n)} = \infty$

$\Rightarrow a(n) > c(n)$ and $a(n) > b(n)$ for all sufficiently large $n$.

$\therefore$ Options (1) and (4) are correct.

But options (2) and (3) are incorrect as $\lim_{n\to\infty} \frac{b(n)}{c(n)} = 0$ and $\lim_{n\to\infty} \frac{b(n)}{n} = 0$

**15.** Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \frac{a}{1+bx^2}, a, b \in \mathbb{R}, b \geq 0$. Which of the following statements are true?

1. $f$ is uniformly continuous on compact intervals of $\mathbb{R}$ for all value of $a$ and $b$.
2. $f$ is uniformly continuous on $\mathbb{R}$ and is bounded for all values of $a$ and $b$.

3. $f$ is uniformly continuous on $\mathbb{R}$ only $f$ $b = 0$.
4. $f$ is uniformly continuous on $\mathbb{R}$ and unbounded if $a \neq 0, b \neq 0$.

**Sol.**

Option (1) is correct as, if $x \in [a, b]$ for any $a \epsilon \mathbb{R}$ and $b \geq 0, f(x) = \frac{a}{1+bx^2}$ is continuous on $[a, b]$.

$\Rightarrow f(x)$ is uniformly continuous on $[a, b]$.

Option (2) is correct as, if $a \in \mathbb{R}$ and $b \geq 0$, then $f(x) = \frac{a}{1+bx^2}$ is continuous $\forall\, x \in \mathbb{R}$ and also $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$ are finite.

$\Rightarrow f(x)$ is uniformly continuous on $\mathbb{R}$.

Option (3) is incorrect as,

$f(x) = \frac{1}{1+x^2}$ is uniformly continuous on $\mathbb{R}$.

Option (4) is incorrect as,

$f(x) = \frac{1}{1+x^2}$ is bounded on $\mathbb{R}$.

**16.** Let $\alpha = \int_0^\infty \frac{1}{1+t^2} dt$. Which of the following are true?

1. $\frac{d\alpha}{dt} = \frac{1}{1+t^2}$
2. $\alpha$ is a rational number.
3. $\log(\alpha) = 1$
4. $\sin(\alpha) = 1$

**Sol.**

$$\alpha = \int_0^\infty \frac{1}{1+t^2} dt = \tan^{-1} t\,|_0^\infty = \tan^{-1}(\infty) - \tan^{-1}(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$\Rightarrow \alpha$ is not rational number and $\log(\alpha) = \log\left(\frac{\pi}{2}\right) \neq 1$.

$\Rightarrow$ So, option (2) is incorrect and also (3) is incorrect.

Since, $\alpha = \frac{\pi}{2} \Rightarrow \frac{d\alpha}{dt} = 0$

So, option (1) is incorrect.

Also, $\sin\alpha = \sin\left(\frac{\pi}{2}\right) = 1$

$\therefore$ Option (4) is incorrect.

**17.** Which of the following functions are of bounded variation?

1. $x^2 + x + 1$ for $x \in (-1, 1)$
2. $\tan\left(\frac{\pi x}{2}\right)$ for $x \in (-1,1)$
3. $\sin\left(\frac{x}{2}\right)$ for $x \in (-\pi, \pi)$
4. $\sqrt{1 - x^2}$ for $x \in (-1,1)$

**Sol.**
For option (1),

Let $f(x) = x^2 + x + 1, x \in (-1,1)$

Since, $|f'(x)| = |2x + 1| \leq 2|x| + 1 < 2 + 1 = 3, i.e, f(x)$ has bounded derivative in $(-1,1)$

$\therefore f(x) = x^2 + x + 1$ is of bounded variation in (-1,1)

For option (2),

Since, $\lim\limits_{x \to 1} \tan\left(\frac{\pi x}{2}\right) = \infty$.

$\therefore g(x) = \tan\left(\frac{\pi x}{2}\right)$ is not of bounded variation.

For option (3)

Let $h(x) = \sin\left(\frac{x}{2}\right), x \epsilon (-\pi, \pi)$

Since $|h'(x)| = \left|\frac{1}{2}\cos\left(\frac{x}{2}\right)\right| \leq \frac{1}{2} \forall x \epsilon (-\pi, \pi)$

$\therefore h(x)$ is of bounded variation.

For option (4)

Let $r(x) = \sqrt{1 - x^2}, x \epsilon (-1,1)$

Since, $r(x) = \sqrt{1 - x^2}$ is monotonically increasing in $(-1,0)$

Also,$\sqrt{1-x^2}$ is monotonically decreasing in $(0,1)$

∴It is bounded variation in $(0,1)$ and hence, $r(x) = \sqrt{1-x^2}, x\epsilon(-1,1)$ is bounded variation.

Thus, option (1), (3) and (4) are correct.

**18.** For any $y \in \mathbb{R}$, let $[y]$ denote the greatest integer less than or equal to $y$.

Define $f: \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = x^{[y]}$. Then

1. $f$ is continuous on $\mathbb{R}^2$
2. For every $y \in \mathbb{R}, x \mapsto f(x,y)$ is continuous on $\mathbb{R}\backslash\{0\}$.
3. For every $x \in \mathbb{R}, y \mapsto f(x,y)$ is continuous on $\mathbb{R}$.
4. $f$ is continuous at no point of $\mathbb{R}^2$

**Sol.**

Option (1) is incorrect as function is discontinuous at $(1,2)$.

Option (3) is incorrect as $f(y) = 2^{[y]}$ is discontinuous $\forall\ y \in \mathbb{Z}$.

Option (4) is incorrect as $f$ is continuous at $(0.5,0.5)$.

As all other options are incorrect.

∴ Option (2) is correct.

**19.** Let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial. Then the roots of $f$

1. Can belong to $\mathbb{Z}$.
2. Always belong to $(\mathbb{R}\backslash\mathbb{Q}) \cup \mathbb{Z}$.
3. Always belong to $(\mathbb{C}\backslash\mathbb{Q}) \cup \mathbb{Z}$.
4. Can belong to $(\mathbb{Q}\backslash\mathbb{Z})$.

**Sol.**

Given, $f(x) \in \mathbb{Z}[x]$ is a monic polynomial.

Take, $f(x) = x^2 - 1$.

Then, roots of $f(x)$ are $1, -1 \in \mathbb{Z}$

$\therefore$ Option (1) is correct.

Take, $f(x) = x^2 + 1$

Then roots of $f(x)$ are $i, -i$.

$\therefore$ Option (2) is incorrect.

[$\because$ According to option (2) $f(x)$ cannot have complex roots]

Take, $f(x) = x^2 - 2$.

Then roots of $f(x)$ are $\sqrt{2}, -\sqrt{2}$.

Now, We Prove that roots of $f(x)$ can be any complex number other than rational except integers.

If possible, $let\ x = \frac{p}{q}, p, q \epsilon \mathbb{Z}, (p, q) = 1, q \neq 1$ is a rational root of $f(x)$ other than integers.

Then, $f(x) = \left(x - \frac{p}{q}\right) g(x) = \frac{1}{q}(qx - p)g(x)$

$\Rightarrow f(x)$ is not a monic polynomial, which is not possible.

Thus, roots of $f(x)$ always belongs to $(\mathbb{C} \backslash \mathbb{Q}) \cup \mathbb{Z}$

Hence, option (3) is correct and (4) is incorrect.

# Answer Table

| Sl. No | Answer | Sl. No | Answer | Sl. No | Answer |
|---|---|---|---|---|---|
| 1. | 1 | 2. | 4 | 3. | 2 |
| 4. | 1 | 5. | 2 | 6. | 3 |
| 7. | 3 | 8. | 3 | 9. | 2 |
| 10. | 1 & 4 | 11. | 1, 2 & 4 | 12. | 1 & 2 |
| 13. | 1 & 2 | 14. | 1 & 4 | 15. | 1 & 2 |
| 16. | 4 | 17. | 1, 3 & 4 | 18. | 2 |
| 19. | 1 & 3 | | | | |