COUNCILE OF SCIENTIFIC & INDUSTRIAL RESEARCH

Mathematical Science Code: 04

UNIT – 1 :

SYLLABUS

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Linear Algebra

2.1. Matrices

2.1.1. Diagonal matrix: A square matrix is said to be diagonal matrix if the elements other then the diagonal elements are zero.

Example (2.1.): $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

2.1.2. Scalar matrix: A diagonal matrix is said to be a scalar matrix if the diagonal elements be the same scalar.

Example (2.2.): $\begin{pmatrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C \end{pmatrix}$

2.1.3. Upper triangular & Lower triangular:

A square matrix (a_{ij}) is said to be an upper triangular matrix if all the elements below the diagonal are zero i.e. $a_{ij} = 0$ if $i > j$ for lower triangular $a_{ij} = 0 \; if \; i < j$.

Example (2.3.):

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 5 & 2 & 0 \\ 7 & 3 & 1 \end{pmatrix}$$

(upper) (lower)

2.2. Matrix Multiplication:

$A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{n\times p}$, $C = A \times B = (c_{ij})_{m\times p}$

Where, $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

2.2.1 Block multiplication:

$A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{n\times p}$

$A = \begin{pmatrix} A_1 \\ A_2 \\ A_M \end{pmatrix}, B = (B_1\ B_2\ \dots\dots B_p)$ where $A_i = (a_{1i}, a_{2i}, \dots\dots, a_{ni})$

$B_i = (a_{1i}, a_{2i}, \dots\dots, a_{ni})^t$

$$AB = \begin{pmatrix} A_1B_1 & A_1B_2 & \dots & A_1B_p \\ A_2B_1 & A_2B_2 & \dots & A_1B_p \\ \vdots & & & \\ A_mB_1 & A_mB_2 & \dots & A_mB_p \end{pmatrix} = (C_{ij})_{m\times p}, C_{ij} = A_i B_j$$

We can take $m = m_1 + \dots\dots + m_k, n = n_1 + \dots\dots + n_p$,

$p = p_1 + \dots + p_t$ (partition)

Example (2.4): $A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 4 \\ 3 & 1 \\ 0 & 0 \end{pmatrix}$

Here , $m = 3, n = 3, p = 2$

Let us take partition of m, n, p as $m = 1 + 2, n = 2 + 1, p = 2$

Then $A = \begin{pmatrix} P & Q \\ I_2 & R \end{pmatrix} B = \begin{pmatrix} S \\ O \end{pmatrix}$ in block form, where $p = (2\ \ 1)$

$Q = (2), I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, R = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, S = \begin{pmatrix} 1 & 4 \\ 3 & 1 \end{pmatrix}, O = (0,0)$

$$AB = \begin{pmatrix} PS + Q.0 \\ I_2S + R.0 \end{pmatrix} = \begin{pmatrix} PS \\ S \end{pmatrix} = \begin{pmatrix} 5 & 9 \\ 1 & 4 \\ 3 & 1 \end{pmatrix}$$

Take another partition as $m = 1 + 2, n = 2 + 1, p = 1 + 1$

Then $A = \begin{pmatrix} P & Q \\ I_2 & R \end{pmatrix}, B = \begin{pmatrix} S & T \\ 0 & 0 \end{pmatrix}, S = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, T = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$$AB = \begin{pmatrix} P.S + Q.0 & PT + Q.0 \\ I_2\ S + R.0 & I_2\ R + R.0 \end{pmatrix} = \begin{pmatrix} PS & PT \\ S & T \end{pmatrix} = \begin{pmatrix} 5 & 9 \\ 1 & 4 \\ 3 & 1 \end{pmatrix}$$

2.3. Determinates: A mapping $f : S \to F$, S be the set of all $n \times n$ matrices over the field F. Let $A = (a_{ij})_{n\times n} \in S$.

Then $A = det\ A = det\ (a_{ij}) = \sum_{\emptyset} sgn\ (\emptyset) a_{1\emptyset(1)}\ a_{2\emptyset(2)} \dots\dots\dots a_{n\emptyset(n)}$

Where ϕ is a permutation on $\{1, 2, \dots\dots, n\}$ and $sgn\ \phi = 1\ or - 1$ according as the permutation ϕ is even or odd.

2.3.1. Properties :

1. $det(A) = det(A)^t$

2. Interchange of two rows (columns) of an $n \times n$ matrix A change the sign of $det(A)$.

3. If two rows (columns) are identical, then $det(A) = 0$.

4. In an $n \times n$ matrix A , if a row (columns) be multiplication by a scalar C then $det(A)$ is multiplication by C.

5. If a row (column) of A be a scalar multiple of another row (column) , then $det(A) = 0$.

6. In an $n \times n$ matrix (a_{ij}) if each element in a particular row, say r^{th} , be expressed as the sum of terms as $a_{rj} = a'_{rj} + a''_{rj}$,Then $det\,(a_{ij}) = det\,(b_{ij}) + det\,(c_{ij})$,

where, $b_{ij} = a_{ij}, i \neq r, b_{rj} = a'_{rj}\ and\ c_{ij} = a_{ij}, i \neq r, c_{rj} = a''_{rj}$

[cor : $arj = a^{(1)}{}_{rj} + a^{(2)}{}_{rj} + ---------- + a^{(k)}rj$ may be expressed]

7. In an $n \times n$ matrix A, if a scalar multiple of one row (column be added to another row (column) then $det(A)$ remains unchanged.

8. In an $n \times n$ matrix A, if one row (column) be expressed as a liner combination of the remaining rows (columns) then $det(A) = 0$.

9. If the elements of an $n \times n$ matrix A are real (complex) polynomials in $x\ and\ r$ rows (columns) of A becomes identical when $x = a$, then $(x - a)^{r-1}$ is a factor of $det(A)$.

10. Vander monde Determinant:

$$\begin{vmatrix} X_1^{n-1} X_1^{n-2} \ldots\ldots\ldots X_1 & 1 \\ X_2^{n-1} X_2^{n-2} \ldots\ldots\ldots X_2 & 1 \\ X_n^{n-1} X_n^{n-2} \ldots\ldots\ldots X_n & 1 \end{vmatrix} = \prod_{1\le i<j\le n}(x_i - x_j)$$

2.4. Minor and Co-factors:

2.4.1. Minor of a_{ij} in $A = (a_{ij})_{n\times n}$ is the determinant of the remaining $(n-1) \times (n-1)$ matrix which is formed by deleting i^{th} row and j^{th} column and is denoted by M_{ij}.

Example (2.5.):

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{23} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} M_{11} = \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$$

2.4.2. Co-factor of a_{ij} in A = $(a_{ij}\ \ is\ A_{ij} = (-1)^{i+j}\ M_{ij}$

Example (2.6.): $A_{11} = (-1)^{1+1}\ M_{11} = M_{11}$

2.4.3. Result: Let $A = (a_{ij})_{n\times m}$ then –

(i) $a_{i1}\, A_{k1} + a_{i2}\, A_{k2} + \dots\dots + a_{in}\, A_{kn} = 0\,, i \neq k.$

(ii) $a_{1i}\, A_{1k} + a_{2i}\, A_{2k} + \dots\dots + a_{ni}\, A_{nk} = 0\,, i \neq k..$

2.4.4. Minor of order (n – r) in $A = (a_{ij})_{n\times n}$

If r rows and r columns be deleted from A, then the determinant of the remaining $(n-r) \times (n-r)$ matrix is said to be a minor of order $n-r$ of A.

Let $i_1 < i_2 < \dots.. < i_r$ rows and $j_1 < j_2 < \dots.. < j_r$ columns deleted from $A = (a_{ij})_{n\times n}$. Then the minor of order $(n-r)$ in given by $M_{i_1,i_2,\dots,i_r,j_1,j_2,\dots\dots,j_r}$

2.4.5. Complementary Minors: Let $p_1 < p_2 < \dots.. < p_{n-r}$ and $q_1 < q_2 < \dots.. < q_{n-r}$ be the remaining rows and columns after deleting r no of rows and columns then the minor $M_{p_1,p_2,\dots,p_{n-r},q_1,q_2,\dots\dots,q_{n-r}}$ is called Complementary minor of $M_{i_1,i_2,\dots,i_r,j_1,j_2,\dots\dots,j_r}$

Example (2.7.):

(1) Let $A = |a_{ij}|_k$ then $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ and $\begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}$ are Complementary minors

(2) a_{11} and M_{11} are complementary minors.

2.4.6. Algebraic complement: Let M be a minor of order r obtained from r rows $i_1th < i_2 \dots. < i_r th$ and r columns $j_1 < j_2 < \dots.. < j_r$ and M' be the Complementary minor of n. Then the algebraic complement of M is defined as $(-1)^{i_1+i_2+\dots i_r+j_1+j_2+\dots+j_r} M'$.

In particular if $M = a_{ij}$, then $M' = M_{ij}$ and the algebraic complement of a_{ij} is $(-1)^{i+j} M_{ij}$ is, the cofactor of a_{ij} in $det\,(a_{ij})$.

Example (2.8.): In $|a_{ij}|_4$ the algebraic complement of $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is $(-1)^{1+2+1+2} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}$

2.4.7. Laplace's Method : In an $n \times n$ matrix A if any r rows be selected, $det(A)$ can be expressed as the sum of the products of all minors of order r formed from those r selected rows and their respective algebraic complements.

This method can be applied to columns of A in an analogous manner.

Example (2.9.):

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{23} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$= (-1)^{1+2+1+2} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \times \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} (-1)^{1+2+1+3} \begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix}$$

$$+ \dots\dots\dots\dots\dots.. + \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} (-1)^{1+2+1+4} \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}$$

2.4.8. If A and B be square matrix of same order, then $det\,(AB) = det(A)\,.\,det(B)$.

2.4.9. Jacobi: If $A = (a_{ij})$ be on $n \times n$ matrix and A_{rs} be the co-factor of a_{rs} in $det(A)$, then $det\,(A_{ij}) = [det(a_{ij})]^{n-1}$

if, $\begin{vmatrix} A_{11}A_{12} \dots\dots.. A_{1n} \\ A_{21}A_{22} \dots\dots.. A_{2n} \\ A_{n1}A_{n2} \dots\dots.. A_{nn} \end{vmatrix} = (\det A)^{n-1}$

2.4.10. If $A = (a_{ij})$ be a square matrix and A_{ij} be the cofactor of a_{ij} in $det\,(a_{ij})$. If $det\,(a_{ij}) = 0$ then any two rows (columns) of (A_{ij}) are proportional.

2.4.11. (Jacobi (general): If M be a minor of order r of a square matrix $A = (a_{ij})$ and m^* be the corresponding minor of (a_{ij}), then $M^* = (det(A))^{r-1}\overline{M}$, where $\overline{M}$ is the algebraic complement of M in $det(A)$.

2.5. Crammers Rule:

$$a_{11}\,x_1 + a_{12}\,x_2 + \dots\dots.. + a_{1n}\,x_n = b_1$$

$$a_{21}\,x_1 + a_{22}\,x_2 + \dots\dots.. + a_{2n}\,x_n = b_2$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{n1}\,x_1 + a_{n2}\,x_2 + \cdots \dots. + a_{nn}\,x_n = b_n$$

be a system of n liner equations in n unknowns $x_1 , x_2 , \dots . x_{10}$ where $det\,A = det\,(a_{ij}) \neq 0$. Then ∃ unique solution of the system given by

$$x_1 = \frac{\det A_1}{\det A}, x_2 = \frac{\det A_2}{\det A}, \dots\dots , x_n = \frac{\det A_n}{\det A}$$

where A_i in the $n \times n$ matrix obtained from A by replaing its i^{th}

column by the column $\begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{pmatrix}$, $i = 1,2, \dots\dots\dots. n$.

2.5.1. The adjoint of a symmetric determinant is symmetric.

2.5.2. The adjoint of a skew symmetric determinant of order n is symmatric if n is odd and skew symmetric if n is even.

2.5.3. A skew symmetric determinant of odd order is zero.

Symmetric determinant ⇒ corresponding matrix is symmetric.

Skew symmetric determinant ⇒corresponding matrix is skew symmetric .

(2.5.3) is not true if the characteristic of the field of scalar is 2.

Example (2.10.):

Let, $A = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ in $\mathbb{Z}_2$ then A is skew symmetric determinant of order 3 , but $det(A) \neq 0$.

2.5.4. A skew symmetric determinant of even osder is the square of a polynonrial function of its elements.

2.6. Algebra of matrices:

2.6.1. Let $A = (a_{ij})$ be a square matrix. Let A_{ij} be the co-factor of a_{ij} in $detA$. The transpose of the matrix (A_{ij}) is said to be the adjoint (or conjugate) of A and is denoted by $adj\ A$.

2.6.2. Propertics of Adjoint of a matrix:

(i) $adj\ (A^t) = (adjA)^t$.

(ii) If A be an $n \times n$ maytrix and c be a scalar , $adj\ (CA) = C^{n-1}\ adj\ A$.

(iii) If A be an $n \times n$ matrix then $adj\ (adj\ A) = (det(A))^{n-2}\ A$.

(iv) If A be an $n \times n$ matrix , then $A\ adj\ A = adj\ A\,.\,A = (det(A)).I_n$

2.6.3. Definition (Inverse) : $A = (a_{ij})_{n\times n}$ is said to be invertible if ∃ a matrix B such that $AB = I_n = BA$ B is the inverse of A.

2.6.4. Definition (Non-singular,singular) : Non-singular if $det(A) \neq 0$, singular if $detA = 0$

2.6.5. $A = (a_{ij})_{n\times n}$ is invertible ⇔ A is non-singular and $A^{-1} = \frac{1}{\det A}(adj\ A)$

2.6.6. Definition (Orthogonal) : A real $n \times n$ matrix is said to be orthogonal if

$$AA^t = I_n \Rightarrow A^t A = I_n \Rightarrow A^{-1} = A^t$$

2.6.7. If A is orthogonal matrix , then A is non-singular and $detA = \pm 1$

(a) If A and B be both orthogonal of same order then AB is also ohrthogonal.

(b) A orthogonal $\Rightarrow A^{-1}$ orthogonal.

Note : Set of all orthogonal matrics of order n forms a group with respect to matrix multiplication and is by $O(n, \mathbb{R})$.

2.7. Complex Matrics : Elements are taken from $\mathbb{C}$. A complex matrix A can be expressed as $P + iQ$ where P, Q are real matrics.

The matrix $\overline{A} = P - iQ$ is said to be conjugate of A. The elements of $\overline{A}$ are the conjugate of the corresponding A.

2.7.1. Porpertics : (i) $\bar{\bar{A}} = A$ (ii) $\overline{AB} = \bar{A}\bar{B}$ (iii) $(\bar{A})^t = (\overline{A^t})$

Note : $(\bar{\bar{A}})^t$ is the conjugate transpose of A and is deneted by A°

2.7.2. Properties : (i) $(A^\circ) = A$ (ii) $(CA)^\circ = \bar{C}\, A^\circ, C \in \mathbb{C}$ (iii) $(A + B)^\circ = A^\circ + B^\circ$ (iv) $(AB)^\circ = B^\circ A^\circ$ (v) $(A^\circ)^{-1} = (A^{-1})^\circ \Rightarrow In = (A^{-1})^\circ . A^\circ = A^\circ (A^{-1})^\circ$

$\Rightarrow (A^\circ)^{-1} = (A^{-1})^\circ$

$\Rightarrow \quad inverse\ of\ A^\circ = (A^{-1})^\circ$

2.7.3. Definition (Hermitian and Skew Hermitian matrics) : A complex $n \times n$ matrix A is said to be Hermitian if $A^\circ = A$ and skew Hermitian if $A^\circ = -A$.

2.7.4. If $H = P + iQ$ be a Hermitian matrix, then –

(i) diagonal elements of A are real.

(ii) P is a real symmetric matrix and Q is a real skew symmetric matrix.

2.7.5. If $S = M + iN$ be a skew Hermitian matrix , then

(i) diagonal elements of S are purely imaginary.

(ii) M is a real skew symmetric matrix and N is a real symmetric matrix.

2.7.6. Let A be comlex square matrix. Then

$$A = \frac{1}{2}(A + A^\circ) + \frac{1}{2}(A - A^\circ)$$

$$= (Hermitian) + (Skew\ Hermitian).$$

2.7.7. Definition (Unitary matrix) :

A complex $n \times n$ matrix A is said to be unitary if $A\,A^\circ = I_n$

$\Rightarrow \quad A$ is non singular and $|detA| = 1$

$\Rightarrow A^\circ A = I_n \Rightarrow \quad A^{-1} = A^\circ$

Note : Set of all $n \times n$ unitary matrics forms a group with respect to matrix multiplication. This group is denoted by $U(n, \mathbb{C})$.

2.8. Definition (Rank of a matrix): Let $A = (a_{ij})_{m\times n}$ be a matrix. Then rank of A is defind to be the greatest positive integer r such that A has at least;one non zero minor of order r.

The rank of zero matrix is defind to be 0.

The rank of A is also called the determinant rank of A $0 < rank \leq min\{m, n\}$.

(i) $A = (a_{ij})_{n\times n}$, rank of $A < n$ it A is singular and $= n$ if A is non-singular.

(ii) $rank\ of\ A = rank\ A^t$.

2.9. Definition(Elementory operations on a matrix A over a field F):

(i) Interchange of two rows (columns) of A.

(ii) Multiplijcation of a row (columns) by a non-zero $c \in F$.

(iii) Addition of a scalar multiple of one row (or column) to another row (column).

When applied to rows (columns), the elementory operations are said to be elementary row (column) operations.

2.9.1. Definition (Row equivalence , column equivalence) : Let S be the set of all $m \times n$ matrics over F. A matrix $B \in S$ is said to be a row equivalent (column equivalent) to a matrix.

$A \in S$ is B can be obtained by successive application of a finite number of elementory row (collumn) operations on A.

The relation row equivalence (column equivalence) on the set S in an equivalence relation, Consequentey, the set S in partitioned into classes of row equivalent (column equivent) matrices.

2.9.2. Definition (Row reduced) : An $m \times n$ matrix A is called row reduceed if

(a) The first non zero element in each non-zero row is 1 (called leading 1).

(b) In each column containing the leading l of some row, the leading l is the only non-zero element.

Example (2.11.) : $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

2.9.3. Definition (Row reduced echelen Matrix / row echelon matrix) :

An $m \times n$ matrix A is said to be a row-reduced echelon matrix (row echelon matrix) if

(i) A is row reduced.

(ii) there is an integer r $(0 \le r \le m)$ such that the first r rows of A are non-zero rows and the remaining rows (if there be any) are all zero rows.

(iii) if the leading element of the i^{th} non zero row occurs in the k_i^{th} column of A, then $k_1 < k_2 < \dots\dots < k_r$.

Example (2.12.) : $\begin{pmatrix} 0101 \\ 0010 \\ 0000 \end{pmatrix}, \begin{pmatrix} 1\,0\,1\,0\,0\,2 \\ 0\,0\,0\,1\,0\,3 \\ 0\,0\,0\,0\,1\,2 \\ 0\,0\,0\,0\,0\,0 \end{pmatrix}$

2.9.4. A matrix A can be made row equivalent to a row reduced matrix B by elementary row operations.

2.9.5. A matrix A can made equivalent to a row echelon matrix B by elementary row operations.

2.9.6. If a row echelon matrix R has r non-zero rows, then rank of R is r.

2.9.7. The rank of a matrix remains invariant under on elementary row operations.

2.9.8. An $n \times n$ matrix A is non-singular $\Rightarrow A$ is row equivalent to the identity matrix I_n.

2.9.9. Definition (Fully reduced normal form): Let $A = (a_{ij})_{m\times n}$ by applying elementary row option and column operation we save A in equivalent to the matrix –

$$\begin{pmatrix} I_r & O_{r,n-r} \\ O_{m-r,r} & O_{m-r,n-r} \end{pmatrix}$$

2.9.10. Definition (Elementary Matrices) : An $n \times n$ matrix obtained by applying a single elementary row operation on in is said to be an elementary matrix row order n. There are the type of elementary matrices –

(i) $R_{ij}(I_n) = E_{ij}$

(ii) $cR_i(I_n) = E_i(c)$

(iii) $R_{ij}(c)(I_n) = E_{ij}(c)$

Example (2.13.):

$$E_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_2(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_{13}(c) = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.9.11. Each elementary matrix in non-singular. The inverse of an elementary matrix in an elementary matrix of the same type.

2.9.12. A matrix in non-singular if only if it can be expressed on the product of a finite no of elementary matrices.

2.9.13. An $m \times n$ matrix B in equivalent to an m×n matrix A if and only if $B = PAQ$ where P, Q are non-singular matrices.

2.9.14. Two $m \times n$ matrices are equivalent if only if they have the some rank.

2.9.15. Definition (Congruence Operation): Let $A = (a_{ij})_{\mathrm{m\times n}}$ be matrix over $F\ and\ E$ be an elementary matrix. EAE^t given an elementary row operation together with the corresponding elementary column operation on A. Such an operation given by the product EAE^t is called a congruence operation on A.

2.9.16. Definition (Congruent Matrix): Let S be the set of all $n \times n$ matrices over a field F. A matrix $B \in S$ in said to be congruent to a matrix $A \in S$ is ∃ a non-singular matrix P ∈ S such that

$B = P^tAP$

$B = E_k^tE_{k-1}^t \ldots\ldots E_1^tA\, E_1\, E_2 \ldots\ldots E_{k-1}\, E_k$

2.9.17. An $n \times n$ real symmetricmatrix A of rank r is congruent to an $n \times n$ read diagonal matrix D with non-zero elements in the finer r diagonal positions and zero elsewhere.

2.9.18. An $n \times n$ real symmetric matrix A of rank r is congruent to the diagonal matrix G whose first m diagonal elements are 1, the next $r - m$ diagonal elements are -1 and the remaining diagonal elements, if there be any, are all zero.

$Q = \begin{pmatrix} I_m & O & O \\ O & -I_{r-m} & O \\ O & O & O \end{pmatrix}$ is the normal form of A under congruence.

2.10. Index and signature of a real symmetric matrix:

2.10.1. Definition: The integer m which is the number of positive is in the normal form of a real symmetric matrix A under congruence is invariant. This m is called the index of A.

Since rank r is invariant, m is invariant, so $m - (r - m) = 2m - r$ is invariant under congruence. This $2m - r$ is called signature of A.

2.10.2. Two real symmetric matrices of the same order are congruent ⇔ they have the same rank and signature.

Example (2.14.): Obtain the normal form under congruence and find the rank, index signature and P such that P^t A P = D = $\begin{pmatrix} 9 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Let us apply congruence operations on A

$$A \xrightarrow{R_{12}^{(1)}} \begin{pmatrix} 2 & 2 & 2 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \xrightarrow{C_{12}^{(1)}} \begin{pmatrix} 4 & 2 & 2 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \xrightarrow[R_{31}^{(-\frac{1}{2})}]{R_{21}^{(-\frac{1}{2})}} \begin{pmatrix} 4 & 2 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\xrightarrow[C_{31}^{(-\frac{1}{2})}]{C_{21}^{(-\frac{1}{2})}} \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \xrightarrow{R_1^{(1/2)}} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \xrightarrow{C_1^{(1/2)}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$Rank\ = 3, index\ m\ = 1,\quad Signature\ 2 \times 1 - 3 = -1$

$$E_{31}{}^{(-\frac{1}{2})}\ E_{21}{}^{(-\frac{1}{2})}\ E_{12}{}^{(1)}\ A\,\{E_{12}{}^{(1)}\}^t\,\{E_{21}{}^{(-\frac{1}{2})}\}^t\,\{E_{31}{}^{(-\frac{1}{2})}\}^t = D$$

Let $P = \{E_{12}{}^{(1)}\}^t\,\{E_{21}{}^{(-\frac{1}{2})}\}^t\,\{E_{31}{}^{(-\frac{1}{2})}\}^t = E_{21}{}^{(1)}\ E_{12}{}^{(-\frac{1}{2})}\ E_{13}{}^{(-\frac{1}{2})}$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.11. Vector Spaces:

Definition: A vector spaces (or linear space) V over a field F consists of a set on which two binary operations (called addition and scalar multiplication) are defied so that for each pair of elements $x, y \in V$ there in unique $x + y \in V$ and for each $\alpha \in F$ and each $x \in V$ there in unique $\alpha x \in V$, such that the following conditions hold:

1) $(V, +)$ is a commutative group
2) For each $x \in V, Ix = x$ and for each $\alpha, \beta \in F$ and $x, y \in V$
3) $(\alpha\beta)x = \alpha(\beta x)$
4) $(\alpha + \beta)x = \alpha x + \beta y$
5) $\alpha\,(x + y) = \alpha x + \alpha\, y$

Examples (2.15.):

1) Real vector space $\mathbb{R}^n = \{(x_1, x_2, \dots\dots x_n): x_i \in \mathbb{R}\}$ with

   $a + b = (a_1, \dots., a_n) + (b_1, \dots.. b_n) = (a_1 + b_1, a_2 + b_2, \dots. a_n + b_n), a_i + b_i \in \mathbb{R}$

   $\alpha\, a = (\alpha\, a_1, \alpha\, a_2, \dots\dots\dots\dots.., \alpha\, a_n), \alpha \in \mathbb{R}$

2) Complex vector space $c = \{a + ib: a, b \in \mathbb{R}, i = \sqrt{-1}\}$.

3) Every field F in a vector space and F^n is also a vector space.

4) Real vector space P_n = the set of all real polynomial of degree $r < n$.

5) Vector space $F_{m\times n}$ = the set of all $m \times n$ matrix over the field F.

6) Sequence space S be the set of all sequences over F such that it only finite x_0 of number zero terms.

$\{ a_n\} + \{ b_n\} = \{ a_n + b_n\}$ *and* $t\{ a_n\} = \{t\, a_n\}, t \in F, a_n, b_n \in F$

2.12. Definition (Sub Space): Let V be a vector space over a field F. A non- empty subset W of V is called a subspace of V if it is a vector space.

2.12.1. Theorem: A non-empty subset W of a vector space V over a field F is a subspace of $V \Leftrightarrow$ **(i)** $x, y \in W \Rightarrow x + y \in W$

(ii) $x \in W, \alpha \in F \Rightarrow \alpha x \in W$

Example (2.16.):

i) V itself a subsequence of V and $\{\theta\}$ is also a sub space of V.

ii) Let S be a subset of $\mathbb{R}^3$ defined by $S = \{(x, y, z) \in \mathbb{R}^3 : y = z\}$.

2.12.2. The intersection of two subspace of a vector space V over a field f is a subspace of V [The intersection of family of subspace is also a subspace].

Note: $W_1 \cap W_2$ is the largest subspace contained in W_1 and W_2

2.12.3. The union of two subspace may not be a subspace.

Example (2.17.): $S = \{(x, y, z) \in \mathbb{R}^3 : y = z = 0\}, T = \{(x, y, z) \in \mathbb{R}^3 : x = z = 0\}$

Thus S and T are both subspace of $\mathbb{R}^3$. Now, $\alpha = (1,0,0) \in S$ and $\beta = (0,1,0) \in T$ but $\alpha + \beta = (1,1,0) \in S \cup T$

Note: Union of two subspace U and W of a vector space V is again a subspace of V $\Rightarrow$Either $U \subseteq W$ *or* $W \subseteq U$.

Note: A vector space V cannot be the union of two proper subspace.

2.12.4. Let U and W be two subspaces of V over F.

Then the linear sum $U + W = \{u + w : u \in U, w \in W\}$ is a subspace of V and $U + W$ is the smallest subspaces of V containing U and W.

*(Subspace-example): Let V be a vector space over F. Then $W = \{c_1\alpha_1 + c_2\alpha_2 + \cdots \ldots . + c_r\alpha_r : c_i \in F, \alpha_i \in V\}$ is a subspace of V and the set $\{\alpha_1, \alpha_2, \ldots \ldots, \alpha_r\}$ is generating set of W.

2.13. Definition (Linear Combination): Let $\alpha_1, \alpha_2, \dots \dots, \alpha_r \in V$. A vector $\beta \in V$ is said to be a linear combination of the vectors $\alpha_1, \alpha_2, \dots \dots, \alpha_r$ is $\beta = \sum_{i=0}^{r} c_i \alpha_i$, for some $c_1, c_2, \dots \dots, c_r \in F$.

2.14. Linear Span: Let S be a nonempty subset of $V(F)$, then the set W of all finite linear combinations of vectors in S forms a subspace $W\ of\ V$ and that is the smallest subspace contains S and W is called linear span of smallest subspace containing S and W is called linear span of S, is noted by $L(S)$ and S is the generating set of $L(S)$.

Note: if $S = \emptyset$ then $L(S) = \{\theta\}$.

i) $S \subset T \Rightarrow L(S) \leq L(T)$.

ii) Let $S, T \subset V$ if each element of T is a linear combination of the vectors of S, then $L(T) \subset L(S)$.

iii) $L(L(S)) = L(S)$

iv) $\emptyset \neq S, T \subset V \Rightarrow L(S \cup T) = L(S) + L(T)$

2.14.1. Definition (Linearly dependent and linearly independent act): A finite set of vectors $\{\alpha_1, \alpha_2, \dots \dots, \alpha_n\}$ of $V(F)$ is said to be linearly dependent in $V\ if\ \exists$ scalars exist F not all zeros such that $\sum_{i=1}^{n} c_i \alpha_i = \theta \dots \dots \dots$(i)

This set is said to be linearly independent if (i) holds only for $c_1 = c_2 = \cdots \dots = c_n = 0$

An arbitrary set S (may be infinite) is said to be linearly independent if there exist a finite subset of S which is linearly independent in V

i) Superset of a linearly dependent set is also linearly dependent.

ii) A subset of a linearly independent set is also linearly independent.

Note: The set $\emptyset$ is linearly independent.

iii) A set containing θ is always linearly dependent.

iv) A singleton non empty set is linearly independent.

v) Two vectors $\alpha, \beta \in V$ is linearly dependent if there exists $0 \neq c \in F$ such that $\alpha = c\,\beta$

2.14.2. (Deletion Theorem): If a vector space $V(F)$ be spanned by a linearly dependent set $S = \{\alpha_1, \alpha_2, \dots \dots, \alpha_n\}$, then V can be spanned by a suitable proper subset of S.

Note: If V be the null space then $S=\{\theta\}$ is a generating set of V. V can also be considered as $L(\phi)$ and the set ϕ is an improper subset of S.

Let V be a vector space over a field F. V is said to be finite dimensional if $\exists$ a finite set of vectors in$V$ generating V, otherwise V is said to be infinite dimensional.

The null space $\{\theta\}$ is finite dimensional, since it is generated by the empty set ϕ.

2.15. Basis:

2.15.1. Definition:

Let $V(F)$ be a vector space A set S of vectors in V is said to be a basis of V if

i) S is linearly independent in V.

ii) S generates V.

2.15.2. There exists a basis for every finite generated vector space.

2.15.3. Replacement Theorem: If $\{x_1, x_2, \dots \dots, x_n\}$ be a basis of a vector space $V(F)$ and $0 \neq \beta \in V$ is expressed as $\beta = C_1\,\alpha_1 + \cdots + C_n\,\alpha_n, c_i \in F,$ then if $c_j \neq 0$, $\{\alpha_1, \alpha_2, \dots \alpha_{j-1}, \beta, \alpha_{j+1}, \dots \alpha_n\}$ is a new basis of V.

2.15.3. If $\{\alpha_1, \alpha_2, \dots \dots, \alpha_n\}$ be a basis of a finite dimensional vector space $V(F)$, then any linearly independent set of vectors in V contains at most n vectors.

2.15.4. Any two bases of a finite dimensional vector space $V(F)$ have the same no of vectors.

2.16. Dimension or Rank:

2.16.1. Definition: The no of vectors in a basis of a vector space $V(F)$ is said be the dimension (or rank) of V and is denoted by $dim\,V$.

The dimension of null space is 0

2.16.2. Let $V(F)$ be a vector space of dimension n. Then any linearly independent set of n vectors of V is a basis of V.

2.16.3. Extension Theorem: A linearly independent set of vectors in a finite dimension vector space $V(F)$ is either a basis of V or it can be extended to a basis of V.

2.16.4. Let $V(F)$ be a vector space. A subset $B=\{\alpha_1, \alpha_2, \dots \dots, \alpha_n\}$ of V is a basis of $V \Leftrightarrow$ every element of V has a unique representation of the vectors of B.

❖ The number of $K-$ *dimensional* subspace of an $x-$ *dimensional* space over F_e is

$$\frac{(p^n-p^0)(p^n-p^1)\dots\dots\dots..(p^n-p^{k-1})}{(p^k-p^0)(p^k-p^1)\dots\dots\dots(p^k-p^{k-1})}$$

2.16.5. Definition (Co-ordinate vector): Let $B = \{\alpha_1, \alpha_2, \dots \dots, \alpha_n\}$ be an ordered basis of a vector space $V(F)$. Then to each vector $\alpha \in V$ there corresponds a well determined ordered set of n sectors $c_1, c_2, \dots \dots, c_n$ in F such that $\alpha = c_1\alpha_1 + c_2\alpha_2 + \dots \dots + c_n\alpha_n$. The ordered n tuple $(c_1, c_2, \dots \dots, c_n)$ is said to be the co-ordinate vector of α relative to the ordered basis B and is denoted by $(\alpha)_B$

2.16.6. If U be a subspace of a vector space $V(F)$ with $dimV = n, then\ dimV \leq n.$

2.16.7. Let U and W be two subspace of a finite dimensional vector space $V(F)$. Then $dim(U + W) = dimU + dimW - dim(U \cap W)$

2.16.8. If U and W be two subspaces of a vector space $V(F)$ such that $U \cap W\{e\}$ and if $\{\alpha_1, \alpha_2, \dots \dots, \alpha_n\}$ and $\{\beta_1, \beta_2, \dots \dots, \beta_n\}$ is a basis of an $V + W$

2.16.9. Definition (Complement): Two subspace U and W of a vector space $V(F)$ are said to be complement of each other if $U \cap W = \{\theta\}$ and $V + W = V$ Then V is said to be the directs sum of U and W and it is expenses as $V = U \oplus W$ and (i) dim v+dim w

(ii) $\alpha \in V$ an unique representation of the form $\alpha = u + w, u \in U, w \in W$.

2.16.10. Every subspace of a finite dimensional vector space $V(F)$ possesses a complement. But the complement may not unique as we can choose vector or binary way by extension theorem.

2.16.11. Definition (Co-set): Let $V(F)$ be a vector space. Let W be a subspace of V. Let $u \in V$. Then the set $\alpha + W = \{\alpha + W : \alpha \in V, W \in W\}$ is a subset of V. It is called co-set of W in V.

2.16.12. Let W be a subspace of $V(F)$. Let $\alpha, \beta \in V$. Then the co-sets $\alpha + W = \beta + W \Leftrightarrow \alpha - \beta \in W$.

Definition (Quotient Space): Let W be a subspace of $V(F)$. Then $V/W = \{\alpha + W : \alpha \in V\}$ is vector space over F. Where addition and multiplication are given by $(\alpha + W) + (\beta + W) = (\alpha + \beta) + W \ \forall \alpha, \beta \in V$ and $c \in F, c(\alpha + W) = c\alpha + W$

2.16.13. $dim\, V/W = dim\, V - dim\, W$.

2.16.14. Definition (Row space and column space of a matrix):

Let $A = (a_{ij})_{m\times n},\ a_{ij} \in F$. Each row of A is a vector in F^n. and each column is a vector in F^m. The row vectors (column vectors) generates a vector space which is called row space (column space) of A and is denoted by $R(A)$ (respectively $C(A)$).

$\Rightarrow$ $R(A)$ is a subspace of F^n. and $C(A)$ is a subspace of F^m..

The dimension of $R(A)$ is the row rank of A and the dimension of $C(A)$ is the column rank of A.

$\Rightarrow$Row rank of $A \le n$ and column rank of $A \le m$.

2.16.15. Let A be an $m \times n$ matrix and P be an $m \times m$ matrix over the same field F. Then row space of PA is a subspace of the row space of A.

If P is nonsingular, then row rank of A = row rank of PA(Similarly for column rank)

Row equivalent matrices have the same row space.

2.16.16. Let R be a nonzero row reduced echelon matrix row equivalent to an $m \times n$ matrix A. Then the non-zero row vectors of R form a basis of the row space of A.

$\Rightarrow$The row rank of a row reduced echelon matrix to its determinant rank.

$\Rightarrow$The row rank of an $m \times n$ matrix A is equal to its determinant rank.

$\Rightarrow$The column rank of a matrix A is equal to its determinant rank.

$\Rightarrow$ $For\ an\ m \times n\ A, row\ rank = column\ rank = determinant\ rank = the\ rank\ of\ A.$

2.16.17. Let A and B be two matrices over the same field F such that AB is defined. Then rank of $AB \ \le\ min\{rank\ of\ A, rank\ of\ B\}$

i) If A is non singular, rank of $AB = rank\ of\ B$

ii) If B is non-singular, rank of $AB = rank\ of\ A$

2.16.18. Factorization Theorem: An $m \times n$ matrix of rank r can be expenses as the product of two matrices, each of rank r.

Proof: Let A be an $m \times n$ matrix of rank r. Then $\exists$ non singular matrices $P\ and\ Q$ of order m and n respectively such that $PAQ = R$

Where $R = \begin{pmatrix} I_r & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{pmatrix} = \begin{pmatrix} I_r \\ 0_{m-r,r} \end{pmatrix}$ $(I_r 0_{r,n-r}) = ST$

Where S is an $m \times n$ matrix of rank $r\ and\ T$ is an $r \times n$ matrix of rank r

Therefore, $A = (P^{-1}S)(TQ^{-1})$ where rank of $(P^{-1}S) = rank\ of\ (TQ^{-1}) = r$

2.17. System of linear equations: A system of m linear equations in n unknown $x_1, x_2, \ldots, x_n$ is of the form-

$$a_{11}\, x_1 + a_{12}\, x_{12} + \ldots\, \ldots + a_{1n}\, x_n = b1$$
$$a_{21}\, x_1 + a_{22}\, x_2 + \ldots\, \ldots + a_{2n}\, x_n = b2$$
$$a_{m1}\, x_1 + a_{m2}\, x_2 + \ldots\, \ldots + a_{mn}\, x_n = bn$$

$\Rightarrow \quad AX = b$ where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_n \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \\ b_m \end{pmatrix}$

= coefficients matrix.

$(A, b) = \begin{pmatrix} a_{11} & \cdots & a_{1n} & \vdots & b_1 \\ a_{21} & \cdots & a_{2n} & \vdots & b_2 \\ \vdots & & \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & \vdots & b_m \end{pmatrix}$ is the augmented matrix of the system

2.17.1. Definition:

i) The system $AX = b$ is homogeneous if $b = 0$ otherwise non-homo-genius system.

ii) The system $AX = b$ and $CX = d$ are said to be equivalent if the augmented matrices (A, b) and (C, d) be row equivalent.

⇒It α be a solution of $AX = b$ then α is also a solution if $CX = d$.

⇒If one of the two equivalent systems be inconsistent, the other is also so.

2.17.2. A necessary and sufficient condition for a system $AX = b$ to be consistent is rank of $A = rank\ of\ (A, b)$.

2.17.3. Homogeneous System:

The solutions of a homogenous system $AX = 0$ in n unknown where A is an $m \times n$ matrix over a field F, form a subspace of $V_n(F) = F^n$ and it is denoted by $X(A)$ and we have –

$$rank\ of\ A + rank\ of\ X(A) = n.$$

- If the number of equations be less then the number of unknowns in $AX = 0$, then the system admits non-zero solution i.e. $rank\ of\ A < n$.

2.17.4. Non homogeneous system:

The solutions of a consistent system $AX = b \neq 0$ do not form a vector space as $(0, \ldots .0)$ is not a solution.

If the non homogeneous system $AX = b$ possesses a solution X_0 then the all solutions of the system are obtained by adding X_0 to the general solution of the associated homogeneous system $AX = 0$.

⇒If the non homogeneous system $AX = b$ be consistant, the system possesses only one solution or infinitely many solutions according as the associated homogeneous system possesses only the zero solution or infinitely many solutions.

2.17.5. Existence and number of Solution of $AX = b$ where A is an $m \times n$ matrix.

Case 1: $m = n$

The system is consistent$\Leftrightarrow$ $rank\ of\ A = rank\ of\ (A|b)$

Subcase (i): $Rank\ of\ A = rank\ of\ (A|b) = n$ ⇒Unique solution $X = A^{-1}b$

Subcase (ii): $rank\ of\ A = rank\ of\ (Ab) < n$

⇒ $AX = 0$ has initially many solutions⇒ $AX = b$ has initially many solutions.

Case 2: $m < n$

The system is consistent$\Leftrightarrow$ $rank\ of\ A = rank\ (A|b) \leq m < n \Leftrightarrow AX = 0$ has infinitely many solutions$\Leftrightarrow$ $AX = b$ has initially many solutions.

Case 3: $m > n$

The system is consistent$\Leftrightarrow$ $rank\ of\ A = rank\ of\ (A|b) \leq n$

Subcase (i): $Rank\ of\ A = rank\ of\ (A|b) = n$

Now $\Rightarrow rank\ of\ A + rank\ X(A) = n \Rightarrow rank\ of\ X(A) = 0$

⇒ $AX = 0$ possesses only zero solution.

⇒ AX=b possesses only the solution.

Subcase (ii): $rank\ of\ A = rank\ of\ (A|b) < n$

⇒ $AX = 0$ possesses infinitely many solutions⇒ $AX = b$ possesses infinitely many solutions.

2.18. Definition (Euclidean Space): A real vector space V together with a real inner product defined on it, is said to be a Euclidean Space

Norm: α ∈ V ⇒ Euclidean Space, $\| \alpha \| = \sqrt{< \alpha, \alpha >}$

Unit Vector: $\alpha \in V \Rightarrow \| \alpha \| = 1$

Schwarz's Inequality: $\alpha, \beta \in V$, Euclidean Space $| < \alpha, \beta > | < \| \alpha \| \| \beta \|$

Equality occurs if α, β are linearly dependent.

2.18.1. Definition (Orthogonal and Orthonormal): A set of vectors $\{\beta_1, \beta_2, \dots \dots \dots \beta_r\}$ in a Euclidean space is said to be orthogonal if $< \beta_i, \beta_j, > 0$ for $i \neq j$ and orthonormal if $< \beta_i, \beta_j >= 0$ for $i \neq j$;

$$= 1 \; for \; i = j$$

2.18.2. In an $n \times n$ real orthogonal matrix, the row vectors form an orthonormal set and the column vectors form another orthonormal set. Since $AA^t = I_n \Rightarrow < \alpha_i \, \alpha_j > = 0$ if $i \neq j$

$$= 1 \; if \; i = j$$

i) Orthogonal set is a linearly independent set.

ii) Let β be a fixed non zero vector in a Euclidean space V. Then for a non-zero vector $\alpha \in V$, there exists unique real number c such that $\alpha - c\beta$ is orthogonal to β. c is determined by $< \alpha - c\beta, \beta >= 0$ component of α along β and

$c \, \beta = \frac{<\alpha,\beta>}{<\beta,\beta>} \beta$ is said to be the projection of α upon β.

iii) Let $\{\beta_1, \beta_2, \dots \dots, \beta_n\}$ is an orthogonal set in a Euclidian space V. Then any vector $\beta \in L\{\beta_1, \beta_2, \dots \dots, \beta_n\}$ has unique representation

$\beta = C_1\beta_1 + C_2\beta_2 \dots \dots \dots \dots \dots . C_n\beta_n$

Where $c_i = \frac{<\beta,\beta i>}{<\beta i,\beta i>}$, $i = 1,2, \dots \dots . n$

iv) **Bessel's Inequality:** If $\beta_1, \beta_2, \dots . \, ., \beta_r$ be an orthonormal set of vectors in a Euclidean space V, then for any $\alpha \in V$

$\| \alpha \|^2 \geq c^2{}_1 + c^2{}_2 + \cdots \dots . . + c^2{}_r$ where $\alpha = \sum_{i=1}^{r} c_i \alpha i$ and $c_i = < \alpha, \beta_i >$, $i = 1,2 \dots \dots, r$

v) **Parseval's Theorem:** If $\{\beta_1, \beta_2, \dots \dots \dots \beta_n\}$ be an orthonormal basis of a Euclidean space V, then for any vector $\alpha \in V$, $\| \alpha \|^2 = c^2{}_1 + c^2{}_2 + \dots \dots . + c^2{}_n$ where $\alpha = \sum_{i=1}^{n} c_i \alpha_i$ and $c_i = <\alpha, \beta_i>$, $i = 1,2, \dots \dots n$

2.18.3. Gram – Schmidt Process: Every non-null subspace W of a finite dimensional Euclidean space V possesses an orthogonal basis.

Process: Let $\{\alpha_1, \alpha_2 \ldots\ldots\ldots \alpha_r\}$ be a basis of W. Let $\beta_1 = \alpha_1$ and

$$\beta_2 = \alpha_2 - c_1\,\beta_1\,,\; c_1 = \frac{<\alpha_2, \beta_1>}{<\beta_1, \beta_1>} \text{ now } <\beta_1, \beta_2> = 0$$

$$\beta_3 = \alpha_3 - \frac{<\alpha_3, \beta_1>}{<\beta_1, \beta_1>}\beta_1 - \frac{<\alpha_3, \beta_2>}{<\beta_2, \beta_2>}\beta_2.$$

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$$\beta_r = \alpha_r - \frac{<\alpha_r, \beta_1>}{<\beta_1, \beta_1>}\beta_1 \ldots\ldots \frac{<\alpha_r, \beta_{r1}>}{<\beta_{r-1}, \beta_{r-1}>}\beta_{r-1}$$

Then $\{\beta_1, \beta_2, \ldots.\; ., \beta_r\}$ is orthogonal basis for W.

2.18.4. In a Euclidean space V, the set P of all vectors which are orthogonal to a fixed vector $\alpha \in V$ is a subspace of V.

2.18.5. Let P be a subspace of a finite dimensional Euclidean space V. Then $V = P \oplus P^1$ where P^1 is the subspace of V consisting of all vectors of V which are orthogonal to P. P^1 is called orthogonal complement of P *in* V and it is unique.

Definition (Matric Polynomial): A polynomial whose coefficient are matrix.

Example (2.18.):

$$A = \begin{pmatrix} x^2+1 & x^3+x \\ x & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x^3 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x^2 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Result: If $F(x)$ be matric polynomial, then $F(x).\, adjF(x) = det\, F(x).\, I_n$

2.18.6. Definition (Characteristic Equation): Let A be an $n \times n$ matrix over a field F. Then $\det(A - xI_n) = \psi_A(x)$ is said to the characteristic polynomial of A and $\psi_A(x) = 0$ is said to be the characteristic equation of A.

Let $A - (a_{ij})_n$ Then $\psi_A(x) = \begin{vmatrix} a_{11-x} & a_{12-x} \ldots \ldots . a_{1n} \\ a_{21} & a_{22-n} \ldots \ldots . a_{2n} \\ \vdots & \\ a_{n1} & a_{n2} \ldots \ldots . a_{nn-x} \end{vmatrix}$

$= C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots\ldots + C_n$ where $C_0 = (-1)^n$ and $C_r = (-1)^{n-r}$ [sum of the principal minors of A *order* r].

In particular $C_1 = (-1)^{n-1}[a_{11} + a_{12} + \ldots\ldots\ldots + a_{nn}] = (-1)^{n-1}\, trace\, A$

$C_n = \det A$

Example (2.19.):

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{pmatrix}$$

Therefore, trace A = 1 + 2 – 2 = +1

$$A_{11} + A_{22} + A_{33} = \begin{vmatrix} 2 & -1 \\ 2 & -2 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 3 & -2 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 2 - 2 + 3 = -1$$

$$det(A) = -4 + 2 - 1(-3 + 2) + 0 = -2 + 1 = -1$$

Therefore, $\psi_A(x) = -x^3 + x^2 + x - 1 = 0$

2.18.7. Cayley – Hamilton Theorem: Every square matrix satisfies its own characteristic equation.

Application:

(i) Find inverse of $A = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}, \Psi_A(x) = x^2 - 7x + 7 = 0$

By Cayley-Hamilton theorem, $A^2 - 7A + 7I_2 = 0 \Rightarrow \frac{1}{7}A(7I_2 - A) = I_2$

$$\Rightarrow A^{-1} = \frac{1}{7}(7I_2 - A) = \frac{1}{7}\begin{pmatrix} 5 & -1 \\ -3 & 2 \end{pmatrix}$$

(ii) Find A^{50} where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

$$\Psi_A(x) = x^2 - 2x + 1 = 0 \Rightarrow A^2 - 2A + I_2 = 0$$

$$\Rightarrow A^2 - A = A - I_2 \ldots\ldots\ldots\ldots (i)$$

$$\Rightarrow A^3 - A^2 = A - I_2 = A - I_2 \ldots\ldots\ldots\ldots by\ (i)$$

Therefore, $A^{50} - A^{49} = A - I_2$

Adding all then we have $A^{50} = 50A - 49\, I_2 = \begin{pmatrix} 1 & 50 \\ 0 & 1 \end{pmatrix}$

2.19. Eigenvalues and Eigenvectors:

2.19.1. Definition (Eigenvalue): A root of the characteristic equation $\psi_A(x)$ of a square matrix. A is said to be an eigenvalue (or characteristic value) of A.

Note : Although the coefficients of $\psi_A(x)$ are element of F, the eigenvalues of A may not be all elements of F but they are being to a suitable algebraic extension of F.

Example (2.20.):

If $\psi_A(x)$ is a real polynomial, then eigenvalues belong to $\not\subset$.

A root of $\psi_A(x) = 0$ multiplicity r is called r-fold eigenvalues of A.

2.19.2. Properties:

(i) The product of eigenvalues of a square matrix A is $det(A)$.

(ii) The sum of eigenvalues of A is the trace A.

(iii) If A is singular, then 0 is an eigenvalues values of A.

(iv) The eigenvalues of a singular matrix are all its diagonal elements.

(v) If λ be an r-told eigenvalue of $A = 0$, is an r-told eigenvalue of the matrix $A - \lambda I_n$.

(vi) If λ be an eigenvalue of a nonsingular matrix A, then λ^{-1} is an eigenvalue of A^{-1}.

(vii) If A and P be both $n \times n$ matrices and P be a non-singular, then A and $P^{-1}AP$ have the same eigenvalue.

(iiiv) The eigenvalues of a real symmetric matrix are all real.

(ix) The eigenvalues of a real skew symmetric matrix are purely imaginary or zero.

(x) Each eigenvalues of a real orthogonal matrix has unit modulus.

(xi) If λ be an eigenvalue of a real orthogonal matrix A, then $\frac{1}{\lambda}$ is also an eigenvalue of A.

2.19.3. Definition (Eigenvector): A non-null vector $X \in V_n(F)$ is said to be an eigenvector or characteristics vector if $\exists$ a scalar $\lambda \in F$ such that $AX = \lambda X$.

Now, $AX = \lambda X \Rightarrow (A - \lambda I_n)X = 0$ is an homogeneous equation in n-unknown and $X \neq \theta$ is a location $\Rightarrow \det(A - \lambda I_n) = 0 \Rightarrow \lambda$ is an eigenvalue of A.

2.19.4. Properties:

(i) Let A be $x \times n$ matrix over a field F. To an eigenvector of A there corresponds a unique eigenvalue of A.

(ii) To each eigenvalue of A there corresponds at least one eigenvector. In fact infinitely many eigenvector.

Examples (2.21.):

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ Then $\psi_A = x^2 + 1 = 0 \qquad \Longrightarrow x = \pm i$

Eigenvalues are i, - i

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be and eigen vectors of i then $AX = iX \Longrightarrow (A - iI_2)X = 0 \Longrightarrow$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \left.\begin{array}{l} \Rightarrow -ix_1 - x = 0 \\ \Rightarrow x - ix_2 = 0 \end{array}\right\} \Longrightarrow x_1 - ix_2 = 0$$

The solution is $K_1(i, 1), where\ 0 \neq K_1 \in \not\subset$

Therefore, the eigenvectors corresponding to i are $K_1\binom{i}{1}$

Similarly for $-i$ are $K_2\binom{1}{i}, where\ 0 \neq K_2 \in \not\subset$

(iii) Two eigenvectors of A corresponding to two distinct eigenvalues of A are linearly independent.

$\Rightarrow If\ X_1, X_2, ----\ X_n$ are eigen vectors corresponding to distinct eigenvalues $\lambda_1, \lambda_2 \ldots \ldots \ldots \lambda_n$, then $X_1, X_2, ----\ X_n$ are linearly independent.

(iv) The eigenvectors corresponding to an eigenvalues λ together with null-vector θ form a sub space of $v_x(F)$. This space is called eigenspace (characteristic space) corresponding to λ.

(v) If λ be an r-told eigenvalues of $A = (aij)_{n\times m}$, then rank of $A - \lambda I_n \geq n - r$

$\Rightarrow$If λ be a simple eigenvalue of A, then rank of $(A - \lambda I_n) = n - 1$.

Since by (v) rank of $(A - \lambda I_n) \geq n - 1$ and again, $det(A) - \lambda I_n = 0$, as λ be and eigenvalue $\Longrightarrow$ rank of $A - \lambda I_n \leq n - 1$.

(vi) If λ be an r-told eigenvalue of A, the rank of the eigen space corresponding to λ does not exceed r.

Since it is the solution of $(A - \lambda I_n)X = 0$ and rank of $X(A - \lambda I_n) + rank\ of\ (A - \lambda I_n) = n\ and\ rank\ of\ (A - \lambda I_n) \geq n - r$

(a) The rank of eigen space of a simple eigenvalue λ is 1.

Since in this case rank of $(A - \lambda I_n) = n - 1$.

2.19.5. Definition (Algebraic and geometric multiplicity of λ) :

For an r-told eigenvalue λ, r is the algebraic multiplicity of λ and the rank of the characteristic space (eigen space) corresponding to λ is the geometric multiplicity of λ.

$$\Longrightarrow 1 \leq geometric\ multiplicity\ \leq algebraic\ multiplicity$$

λ is called regular eigenvalue of geometric multiplicity equal to its algebraic multiplicity.

2.20. Definition (Diagonalisable): An $n \times m$ diagonal matrix.

A is similar to $D = diag\ (\lambda_1, \lambda_2 \ldots \ldots \ldots \lambda_n)$, λ_i's are eigenvalue of $A, i = 1,2,3, \ldots \ldots., n$.

2.20.1 An $n \times m$ matrix A over a field F is diagonalisable if $\exists\ n$ eigen vectors of A which are linearly independent.

2.20.2 Let A be an $n \times n$ matrix over F having n distinct eigenvalues, then A is diagonalisable.

2.20.3. A is diagonalisable $\Leftrightarrow$ the minimal polynomial of A splits over F and is square free.

Example (2.21.):

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad , \psi_A(x) = (x-1)^2(x-5) = 0$$

$\Rightarrow$eigenvalues are 1,1, 5

Now, let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be and eigenvector corresponding to 1.

The $(A - I_n)X = 0 \implies \begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 2x_1 + 2x_2 + x_3 = o$

Let $x_1 = c, and\ x_2 = d \Rightarrow x_3 = -2c - 2d$

The eigenvector are $(c, d, -2c, -2d) = c\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \times d\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$

$$(c, d) \neq (0,0)$$

and corresponding 5, the eigenvectors are e$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, e \neq 0$

and $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ are linearly independent.

$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -2 & -2 & 0 \end{pmatrix}$ and $R^{-1}AP = diag$ (1,1,5)

An $n \times n$ matrix A is diagonalisable $\Leftrightarrow$ an is eigenvalues are regular.

Example (2.22.):

$A = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ is not diagonalisable.

Since, $\psi_A(x) = x^2 - 2x + 1 = 0 \Rightarrow x = 1, i$, eigenvalues are i, j and eigenvectors corresponding to 1 are $c\,\binom{0}{1}, c \neq 0$

$\Rightarrow$ geometric multiplicity if 1 is 1 and algebraic multiplicity = 2

2.20.4. If λ be a multiple eigenvalue of a real $n \times n$ symmetric matrix A, then the algebraic multiplicity of λ is equal to its geometric multiplicity (Imp)

$\Rightarrow$ Every real symmetric matrices are diagonalisable

Example (2.23.):

P = $\begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$ is a real symmetric matrix.

Its eigenvalues are 8,2,2

The eigenvectors corresponding to 8 are $c\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $c \neq 0$

and corresponding to 2 are $d\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + e\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $d, e \in \mathbb{R}$, $(d, e) \neq (0,0)$

$\Rightarrow \boldsymbol{P} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}$ is non singular as $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ linearly independent.

$\Rightarrow P^{-1}AP = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

2.20.5. Definition (Orthogonal Diagonalisation): A square matrix A is said to be orthogonally diagonalisable if $\exists$ an orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix.

2.20.6. A square matrix A is orthogonally diagonalisable $\Leftrightarrow$ A is symmetric.

2.20.7. Definition (Minimal Polynomial): Let A be any square matrix which satisfies a monic polynomial $m_A(x)$ of lowest degree. The $m_A(x)$ is called the minimal polynomial of the matrix A .

Properties:

(i) $m_A(x)$ divides $\psi_A(x)$

(ii) $m_A(x)$ and $\psi_A(x)$ have the some irreducible factors.

(iii) A scalar λ be and eigenvalue of A $\Leftrightarrow$ λ is a root of $m_A(x)$

- Let A be diagonalisable, then $\exists$ a non-singular matrix P such that $P^{-1}AP \Rightarrow D = diag(k_1, k_2, ---, k_n)$. Then $A = PDP^{-1}$

  $\Rightarrow A^m = PD^mP^{-1} = Pdiag({k_1}^m, {k_2}^m, ---, {k_n}^m)P^{-1}$

  More general, for any polynomial $f(x)$,

  $f(A) = P\, f(D)P^{-1} = Pdiag\,(f(k_1), f(k_2), ---f(k_m))P^{-1}$ n

Furthermore, if the diagonal entries of D are negative, but

$B = P\ diag\ (\sqrt{k_1}, \sqrt{k_2}, ---, \sqrt{k_n})P^{-1}$

Then B is non-negative square root of A is i.e. $A = B^2$

2.20.8. Definition (Block Matrices): Using a system of horizontal and vertical lines, we can partition a matrix A into sub-matrices called blocks of A.

Example (2.24.):

$$A = \left(\begin{array}{cc:c:cc} 1 & 4 & 5 & 7 & 0 \\ 2 & 1 & 9 & 5 & 1 \\ \hdashline 1 & 5 & 2 & 3 & 2 \\ 3 & 7 & 9 & 5 & 4 \end{array}\right) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix}$$

2.20.9. Definition (Square Block Matrix): Let M be a block matrix. Then M is called a square block matrix if:

(i) M is a square matrix.

(ii) The blocks from a square matrix.

(ii) The diagonal blocks are also square matrices.

Example (2.25.):

$$A = \left(\begin{array}{cc:c:cc} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 7 & 9 & 8 \\ \hdashline 4 & 1 & 5 & 0 & 6 \\ \hdashline 3 & 9 & 2 & 4 & 5 \\ 5 & 7 & 8 & 3 & 2 \end{array}\right)$$

2.20.10. Definition (Block diagonal matrix): Let $M = (A_{ij})$ be a square block matrix such that the non-diagonal blocks dare all zero matrices ie, $A_{ij} = 0$ when $i \neq j$. Then M is called a block diagonal matrix.

$M = dia\,(A_{11}, A_{22}, - - --, A_{rr}),\ r \leq n$

2.20.11. Definition (Block upper triangular matrix / lower triangular matrix) :

A square block matrix is called a block upper (lower) triangular matrix if all the blocks below the diagonal (represent above the diagonal) are zero matrices.

Properties:

(i) Suppose M is a block diagonal matrix and $f(x)$ is a polynomial. Then $f(M)$ is a blo0ck diagonal matrix and

$f(M) = diag\,(f(A_{11}), f(A_{22}), - - - - -, f(A_{rr}))$.

(ii) M is invertible $\Leftrightarrow A_{ij}$ are invertible and M^{-1} is a block diagonal matrix and $M^{-1} = diag\,(f(A^{-1}{}_{11}), f(A^{-1}{}_{22}), - - - - -, f(A^{-1}{}_{rr}))$.

2.20.12. Characteristics and Minimal polynomials of block matrices:

Suppose M is a block triangular matrix with diagonal blocks $A_1, A_2, ----, A_r$. Then the characteristic polynomial of M is the product of the characteristic polynomials of the diagonal blocks A_i, ei, $\psi_M(x) = \psi_{A_1}(x), \psi_{A_2}(x) ----\psi_{A_r}(x)$

Example (2.26.):

$$M = \left(\begin{array}{cc:cc} 9 & -1 & 5 & 7 \\ 8 & 3 & 2 & -4 \\ \hdashline 0 & 0 & 3 & 6 \\ 0 & 0 & -1 & 8 \end{array}\right), \quad \begin{array}{l} A_1 = \begin{pmatrix} 9 & -1 \\ 8 & 3 \end{pmatrix} \Rightarrow \psi_1(x) = x^2 - 12x + 35 = (x-5)(x-7) \\ A_2 = \begin{pmatrix} 3 & 6 \\ -1 & 8 \end{pmatrix} \Rightarrow \psi_2(x) = x^2 - 11x + 30 = (x-5)(x-6) \end{array}$$

Therefore, $\psi_M(x) = (x-5)^2(x-6)(x-7)$

2.20.13. Suppose M is a block diagonal matrix with diagonal blocks $A_1, A_2, ----, A_r$.. Then the minimal polynomial of M is equal to the LCM of minimal polynomials of the diagonal blocks A_i.

Example (2.27.):

$$M = \left(\begin{array}{cc:cc:c} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 3 & 5 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 7 \end{array}\right)$$

$A_1 = \begin{pmatrix} 2 & 5 \\ 0 & 2 \end{pmatrix}$, $\psi_{A_1}(x) = (x-2)^2 \Rightarrow m_{A_1}(x) = (x-2)^2$

$A_2 = \begin{pmatrix} 4 & 2 \\ 3 & 5 \end{pmatrix}$,$\psi_{A_2}(x) = (x-2)(x-7) \Rightarrow m_{A_2}(x) = (x-2)(x-7)$

$A_3 = \begin{pmatrix} 4 & 2 \\ 3 & 5 \end{pmatrix}$,$\psi_{A_3}(x) = (x-7) \Rightarrow m_{A_3}(x) = (x-7)$

Therefore, $m_A(x) = (x-2)^2 (x-7)$

2.20.14. Definition (Nilpotent Matrix): A square matrix A is nilpotent if $A^n = 0$ for some positive integer n and of index of nilpotencey K if $A^k = 0$ but $A^{k-1} \neq 0$ ie, $m_A(x) = x^k \Longrightarrow 0\ is\ the\ only\ eigen\ value\ of\ A$.

Examples (2.28.):

$$N = N(r) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \qquad J(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

N is called Jordan nilpotent block consists of I's on the superdiagonal and o's elsewhere.

$J(\lambda)$ is called a Jordan block belonging to the eigenvalue λ consists of λ is on the diagonal I's on the superdiagonal and O's elsewhere.

$J(\lambda) = \lambda I + N$

2.21. Jordan Canonical form:

Let $T: V \to V$ be alinear operator whose characteristic and minimal polynomials are respectively,

$\psi(x) = (x-\lambda_1)^{n_1}(x-\lambda_2)^{n_2} \dots \dots (x-\lambda_r)^{n_r}$ and $m(x) = (x-\lambda_1)^{m_1} \dots . . (x-\lambda_r)^{m_r}$

Where λ_i are distinct scalars. Then T has a block diagonal matrix representation J on which each diagonal entry is a Jordan block $J_{ij} = J(\lambda_i)$. For each λ_{ij} , the corresponding J_{ij} have the following properties :

(i) There is at least one J_{ij} of order m_{ij} all other J_{ij} are of order $\leq m_i$.

(ii) The sum of the orders of the J_{ij} is n_i

(iii) The no. of J_{ij} equals the geometric multiplicity of λ_i

(iv) The no. of J_{ij} of each possible order is uniquely determined by T

Example (2.29):

Suppose the characteristic and minimal polynomial of an operator T are respectively.

$\psi_A(x) = (x-2)^4\,(x-5)^5$ and $m(x) = (x-2)^2\,(x-5)^3$

Then the Jordan canonical from of T is one of the following block diagonal matrices.

$$diag\left(\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}\right)$$

or, $$diag\left(\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, [2], [2], \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}\right)$$

2.22. Quadratic Form:

An expression of the form $\sum_{j=i}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j$ where a_{ij} are all real and a_{ij} is said to be a real quadratic from in n variables $x_1, x_2, \dots \dots .. x_n$.

The matrix notation for the quadratic form is X^tAX where $X = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$ and $A = (a_{ij})_{n\times n}$ is a real symmetric matrix. A is called the matrix of the given quadratic form.

Example (2.30):

(i) $x_1x_2 - x_2x_3$ is the real quadratic form in x_1, x_2, x_3

The associated matrix is $\begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & -1/2 \\ 0 & -1/2 & 0 \end{pmatrix}$

(2) $x_1{}^2 - x_2{}^2 + 2x_3{}^2$: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

2.22.1. A real quadratic form $Q\ (x_1, x_2, \dots .. x_n\ )$ assume the value 0 when $X = 0$. But Q takes up different real values for different non-zero X.

A real quadratic form $Q = X^tAX$ is said to

(i) Positive definite if $Q > c\ \ \forall\ \ X \neq 0$.

(ii) Positive semi definite if $Q \geq 0\ \ \forall\ \ X \neq 0$.

(iii) Negative definite if $Q < 0\ \ \forall\ \ X \neq 0$.

(iv) Negative semi definite if $Q \leq 0\ \ \forall\ \ X \neq 0$.

(v) Indefinite if $Q\ \geq 0$ for some $X \neq 0$ and $Q \leq 0$ for some $X \neq 0$.

In corresponding cause, the associated real symmetric matrix A is said to be positive definite, positive semi-definite, negative definite, negative semi definite and indefinite respectively.

2.22.2. Definition: The rank of a real quadratic form is defined to be the rank of the associated real symmetric matrix.

Similarly, we can define also signature, index of a real quadratic from.

2.22.3. A real quadratic form $Q\ (x_1, x_2, \dots . x_n)$ of rank r and index m is

(i) Positive definite, if $n = r\ , r = m\ ; \Rightarrow\ rank\ = \ signature\ = \ r$

(ii) Positive semi definite, if $r < n\ , m = r$

(iii) Negative definite, if $r\ \ = r = n\ , m = 0\ ;$

(iv) Negative semi definite, if $r < n\ , m = 0\ ;$

(v) Indefinite, if $r \leq n\ , 0 < m < r$

2.22.4. A real symmetric matrix A is positive definite (negative definite) if and only if all its eigenvalues are positive (respect all its eigenvalues are negative)

2.22.5. Quadratic form and eigenvalues:

(A). Monic polynomial and its matrix representation:

$f(x) \; = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$ --------(i)

Matrix of (i) is

$$\begin{pmatrix} 0 & 0 & 0 \ldots\ldots\ldots 0 \ldots\ldots & a_0 \\ 1 & 0 & 0 \ldots\ldots\ldots 0 \ldots\ldots & a_1 \\ 0 & 1 & 0 \ldots\ldots\ldots 0 \ldots\ldots & a_2 \\ 0 & 0 & 1 \ldots\ldots\ldots 0 \ldots\ldots & a_3 \\ 0 & 0 & 0 \ldots\ldots 1 \ldots\ldots & a_{x-1} \end{pmatrix} \ldots\ldots\ldots\ldots\ldots (2)$$

(ii) Is the characteristics and also minimal polynomial of matrix (2) which is known as companion matrix.

(B) A quadratic form $\theta = X^T AX$satisfies the following :

(i) Positive definite ⇔ all eigenvalues of A are positive.

(ii) Positive semidefinite ⇔ at least one eigenvalue is zero and other are positive .

(iii) Negative definite ⇔ all eigenvalues are negative.

(iv) Negative semidefinite ⇔ atleast one eigenvalue is zero and other are negative.

(v) Indefinite ⇔ A has some positive and some negative eigenvalues.

(C) A quadratic form $\theta = x^T Ax$ is (A is real symmetric matrix).

(i) Positive definite ⇔ n leading principal minors are strictly positive.

(ii) Negative definite ⇔ n leading principal minors are 7

alternate in sign , with $|A_1| > 0, |A_2| > 0, |A_3| > 0$ etc.

(iii) indefinite ⇔ some K-th order leading principal minor of A is nor zero but does not fit either of the above sign pattern.

(iv) Positive semidefinite ⇔ every principal of A is $\geq$ 0.

(v) Negative semi definite ⇔ every principal minor of A of odd order is ≤ 0 and every principal minor of even order $\geq$ 0.

Note : $A \; = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ leading principal minors of A.

$$A_1 \; = \; |a_{11}|, A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \; A_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

(Result):

(i) Let A, B be positive definite then $A+B, ABA, BAB$,rA $(r > 0)$ are positive definite.

(ii) Let A, B be positive definite if $AB = BA$, then AB is positive definite

(iii) If A is positive semi definite matrix, then $\exists$ a positive semi definite matrix B such that $B^2 = A$, B is called square root of A.

2.23. Linear mappings:

Let V and W se vector space over the same field F. A mapping $T: V \rightarrow W$ is said to be a linear mapping (or linear transformation) if it satisfies the following conditions;

(i) $T(\alpha+\beta) = T(\alpha) + T(\beta) \forall \alpha\beta \in V$

(ii) $T(c\alpha) = cT(\alpha) \forall \alpha \in V, \forall\ c \in F.$

(i) and (ii) can be replaced by one condition :

$T(c\alpha + d\beta) = cT(\alpha) + dT(\beta) \forall \alpha, \beta \in V\ and\ \forall a, b \in F$.

Note: (i) ⇒ T is a homomorphism of $V\ to\ W$.

- $T : V \rightarrow F$ is called a linear functional.

Examples (2.31.):

1. The identity mapping. $T: V \rightarrow V$ is defined by $T(\alpha) = \alpha \forall \alpha \in V$
2. The zero mapping. $T: V \rightarrow W$ defined by $T(\alpha) = Q' \forall \alpha \in V, Q'$ .being the null vector is w.
3. Let P be the vector space of real polynomials. Then $D: P \rightarrow P$ defined by
   $DP(x) = \frac{d}{dx} P(x), P(x) \in P$ is a linear mapping.
4. Let $V = C[a, b]$ and $T: V \rightarrow \mathbb{R}$ is defined by
   $T(f) = \int_a^b f(t)dt,\ f \in V$ ⇒ T is a linear functional.
5. $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be defined by
   (i) $T(x_1, x_2, x_3) = (x_1, x_2, 0), (x_1, x_2, x_3) \in \mathbb{R}^3$
   (ii) $T_2(x_1, x_2, x_3) = (x_1, 0, 0), (x_1, x_2, x_3) \in \mathbb{R}^3$
   are all linear operators.
   T_1: called projection on $x_1x_2 - plane$
   T_2: called projection on $x_1 - \ axis$.
6. $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1, -x_2)$. T is called reflection about $x -\ axis$.
7. Define $T_\theta: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T_\theta(a_1, a_2) = (a_1 \cos\theta - a_2 \sin\theta, a_1 \sin\theta + a_2 \cos\theta)$
   $\forall (a_1, a_2) = (r\cos\alpha, r\sin\alpha) \in \mathbb{R}^2$ and θ is the rotation of the point (a_1, a_2) along counter clockwise by θ.

8. Define $T: M_{m\times n}(F) \to M_{n\times m}(F) by\ T(A) = A^t\ \forall\ A \in M_{m\times n}(F)$

2.23.1. Properties: Let V and W be two vector space over a field F and $T: V \to W$ be a linear mapping then—

(i) $T(\theta) = \theta'. \theta, \theta'$, are null vectors of $V\ and\ W$ respectively

(ii) $T(-\alpha) = -T(\alpha)$

Definition (Kernel of T):

(i) $KerT = \{\alpha \in V: T(\alpha) = \theta'\}$

(ii) $KerT$ is a subspace of V and it is called null space of T and is denoted by $N(T)$.

(iii) T is one-one $\Leftrightarrow KerT = \{\theta\}$

(iv) If $KerT = \{\theta\}$, then the image of linearly independent set $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ in V are linearly independent in W

$\Rightarrow$ If $T: V \to V$ and $KerT = \{\theta\}$, then a basis of V Mapped onto another basis of V.

(v) $I_mT = \{T(\alpha); \alpha \in V\}$

(vi) I_mT is a subspace of W and it is called the range of T and is denoted by $R(T)$

(vii) If $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be a basis of V, then $T(\alpha_1), T(\alpha_2), \ldots T(\alpha_n)$ generate $I_m(T)$.

Example (2.32): Let a linear mapping $T: I\mathbb{R}^3 \to I\mathbb{R}^3$ be degined by $T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, 2x_1 + x_2 + 2x_3, x_1 + 2x_2 + x_3), (x_1 + x_2 + x_3) \in I\mathbb{R}^3$

Find $Ker\ T, dim\ N(T), I_m(T), dim\ R(T)$

Solution: $Ker\ T = \{(x_1, x_2, x_3) \in \mathbb{R}^3 = T(x_1, x_2, x_3) = (0,0,0)\}$

$$\Rightarrow \left.\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 2x_1 + x_2 + 2x_3 &= 0 \\ x_1 + 2x_3 + x_3 &= 0 \end{aligned}\right\} \Rightarrow \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1} = k, say, k \in IR$$

$\therefore (x_1, x_2, x_3) = k(1,0,-1)$ satisfies third relation.

$\therefore\ KerT = N(T) = L\{(1,0,-1)\} \Rightarrow dim\ N(T) = 1$

Now, $\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ is a basis of $\mathbb{R}^3$. Then

$ImT = R(T) = L\{T(e_1), T(e_2), T(e_3)\}$

Now, $T(e_1) = (1,2,31), T(e_2) = (1,1,2), T(e_3) = (1,2,1)$

But $T(e_1) and\ T(e_3) are\ linearly\ dependent\ and\ T(e_1), T(e_2) is\ linearly\ independent.$

$\therefore\ R(T) = L\{T(e_1), T(e_2)\} = L\{(0,2,1), (1,1,2)\}$.

And $dim\ R(T) = 2$

2.23.2 Definition: Let $T: V(F) \rightarrow W(F)$ be a linear mapping. Then the dim $N(T) = dim\ kerT$ and dim$R(T) = dim\ Im(T)$ are called nullity of T and rank of T respectively.

(i) $dim\ kerT + dim\ ImT = dim\ V$ (If V is finite dimensional) i.e. $(nullity\ of\ T) + (rank\ of\ T) = dim\ V$

(ii) Let $V\ and\ W$ be finite dimensional vector space of same dimension over a field F and $T: V \rightarrow W$ be a linear mapping. Then T is one-one $\Leftrightarrow$ T is onto.

(iii) [Linear mapping with prescribed images]: Let V and W be vector space over a field F Let.$\{\alpha_1, \alpha_2, \ldots., \alpha_n\}$ be a basis of V and $\beta_1, \beta_2, \ldots., \beta_n$ be arbitrary chosen elements(not necessarily distinct) in W. Then $\exists$ one and only one linear mapping $T: V \rightarrow W$ such that $T(\alpha_i) = \beta_i\ for\ i = 1,2, \ldots, n$.

Note: In this case T is given $T(\alpha) = T(r_1\alpha_1 + \ldots + r_n\alpha_n) = r_1\beta_1 + \ldots + r_n\beta_n$

2.23.3. Definition (Inverse): Let $T: V(F) \rightarrow W(F)$ be a linear mapping. T is said to be invertible if $\exists$ a mapping $S: W \rightarrow V$ such that $ST = IV\ and\ TS = IW$ and S is called inverse of T.

(i) $T: V(F) \rightarrow W(F)$ is invertible $\Leftrightarrow$ T is one-one and onto.

(ii) $T^{-1}: W(F) \rightarrow V(F)$ is also linear.

2.23.4. Definition (Non-singular): A linear mapping $T = V \rightarrow W$ is said to be non-singular if T is invertible.

2.23.5. Definition (Isomorphism): A linear mapping $T = V(F) \rightarrow W(F)$ is said to be isomorphism $\Leftrightarrow$ T is both one -one and onto and the vector space V and W are called isomorphic.

(i) Two finite dimensional vector space $V(F)\ and\ W(F)$ are isomorphic $\Leftrightarrow$ $dim\ V = dim\ W$.

(ii) Let $dim\ V(F) = dim\ W(F)$. Then a linear mapping $T = V(F) \rightarrow W(F)$ is an isomorphism $\Leftrightarrow$ T maps a basis of V to a basis of W.

(iii) [Isomorphism from V to F^n]: Let V be a vector space of dimension n over a field F. Then V is isomorphic to F^n.

Note: In this case let $\{\alpha_1, \ldots, \alpha_n\}$ be an ordered basis of V and for any $\alpha \in V$,$T: V \rightarrow F^n$

defined by $T(\alpha) = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$ where $\alpha = r_1\alpha_1 + r_2\alpha_2 + \ldots. + r_n\alpha_n \in V$

2.24. Matrix representation of a linear mapping:

Example (2.33):

A linear mapping $T = \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $(x_1, x_2, x_3) = (3x_1 - 2x_2 + x_3), (x_1 - 3x_2 - 2x_3)$, $(x_1, x_2, x_3) \in \mathbb{R}^3$. Find the matrix of T relative to the ordered basis $(1,0,0), (0,1,0), (0,0,1)$ of $\mathbb{R}^3$ and $(1,0), (0,1)$ of $\mathbb{R}^2$

$T(1,0,0) = (3,1) = 3(1,0) + 1(0,1)$

$T(0,1,0) = (-2,-3) = -2(1,0) + (-3)(0,1)$

$T(0,0,1) = (1,-2) = (1,0) - 2(0,1)$

$\therefore\ Matrix\ of\ T = \begin{pmatrix} 3 & -2 & 1 \\ 1 & -3 & -2 \end{pmatrix}$

Example (2.34):

The matrix of a linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^2$ relative to the ordered bases $\begin{pmatrix} (0,1,1), & (1,0,1) & (1,1,0) \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix}$ of $\mathbb{R}^3$ and $(1,0), (1,1)$ of $\mathbb{R}^2$ is $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix}$ Find T.

$T(0,1,1) = 1(1,0) + 2(1,1) = (3,2) = T(\alpha_1)$

$T(1,0,1) = 2(1,0) + 1(1,1) = (3,1) = T(\alpha_2)$

$T(0,1,1) = 4(1,0) + 0(1,1) = (4,0) = T(\alpha_3)$

But $(x, y, z) = c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0) = (c_2 + c_3, c_1 + c_3, c_1 + c_2)$

$\Rightarrow c_2 + c_3 = x,\ c_1 + c_3 = y,\ c_1 + c_2 = z$

$\Rightarrow c_1 = \frac{1}{2}(y + z - x),\ c_2 = \frac{1}{2}(z + x - y),\ c_3 = \frac{1}{2}(x + y - z)$

$\therefore\ T(x, y, z) = t(c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3)$

$= c_1T(\alpha_1) + c_2T(\alpha_2) + c_3T(\alpha_3)$

$= c_1(3,2) + c_2(3,1) + c_3(4,0)$

$= (3c_1 + 3c_2 + 4c_3, c_1 + c_2)$

$= (2x + 2y + z, \frac{1}{2}(-x + y + 3z))$