COUNCILE OF SCIENTIFIC & INDUSTRIAL RESEARCH

Mathematical Science Code: 04

Unit – 3 :

SYLLABUS

Sub Unit – 6: Classical Mechanics

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Classical Mechanics

6.1. Generalized coordinates:

Let q_i be a set of independent coordinates that uniquely specifies the instantaneous configuration of some classical dynamical system. The q_i might be cartesian coordinates, or polar coordinates, or angles, or some mixture of all three types of coordinate and are therefore termed generalized coordinates. The number of independent generalized coordinates is said to degree of freedom. A dynamical system consisting of N particles moving freely in three dimensions has $3N$ degree of freedom.

6.2. Lagrange's Equations:

For an N particle system in 3 dimensions, there are $3N$ second order ordinary differential equations. Lagrangian mechanics uses the energies in the system. The central quantity of Lagrangian mechanics is the Lagrangian, a function which summarizes the dynamics of the entire system. The Lagrangian for a system of Particles can be defined by

$$L = T - V$$

Where $T = \frac{1}{2}\sum_{k=1}^{N} m_k {v_k}^2$ is the total Kinetic energy of the system and V is the potential energy of the system.

Now the Lagrange's equations are

$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = G_j, j = 1, 2, \,\ldots\ldots\ldots, 3N$

The quantities $p_j = \frac{\partial L}{\partial \dot{q}_j}$ are called the generalized momentum.

$G_j = Component\ of\ generalized\ force.$

6.3. Hamilton's Canonical Equations:

The differential equations of motion of a mechanical system in which the variables are the generalized momenta p_i and the generalized coordinates q_i, the p_i and q_i are called canonical variables. Hamilton's Canonical equations of motion have the form

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

Where $H\,(q_i, p_i, t)$ is the Hamiltonian function and $H = T + V$

T is the Kinetic energy and function of p alone while V is the potential energy and function of q alone.

Example – 6.1

A particle of mass m moves inside a bowl. If the surface of the bowl is given by the equation, $z = \frac{1}{2}\, a\,(x^2 + y^2)$, where a is a constant. Find the Lagrangian of the particle.

Sol:

Given $z = \frac{1}{2}\, a\,(x^2 + y^2)$

$L = T - V = \frac{1}{2}\, m\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$

It has cylindrical symmetry

$x = r\cos\phi\,,\; y = r\sin\phi\,,\; z = \frac{1}{2}\, ar^2$

$\dot{x} = \dot{r}\cos\phi - r\sin\phi\,\dot{\phi}\,,\; \dot{y} = \dot{r}\sin\phi + r\cos\phi\,\dot{\phi},\; \dot{z} = a\, r\, \dot{r}$

$\therefore L = \frac{1}{2}\, m\,\left[\dot{r}^2\cos^2\phi - r^2\sin^2\phi\,\dot{\phi}^2 - 2\, r\,\dot{r}\cos\phi\sin\phi\,\dot{\phi} + \dot{r}^2\sin^2\phi + r^2\cos^2\phi\,\dot{\phi}^2 + \right.$
$\left. 2\, r\,\dot{r}\cos\phi\sin\phi\,\dot{\phi} + a^2r^2\dot{r}^2\right] - mg\cdot\frac{1}{2}\, a\, r^2$

$= \frac{1}{2}\, m\,\left[\dot{r}^2 + r^2\dot{\phi}^2 + a^2r^2\dot{r}^2\right] - \frac{amg}{2} r^2$

$= \frac{1}{2} m\left[\dot{r}^2(1 + a^2r^2) + r^2\dot{\phi}^2 - g\, ar^2\right]$

Example – 6.2

Consider a particle of mass m in simple Harmonic Oscillation about the origin with spring constant K. Find the Lagrangian and the Hamiltonian and the momentum.

Sol:

For spring, $K.E = T = \frac{1}{2}\, m\, v^2$

$T = \frac{1}{2}\, m\,\dot{x}^2$

Force $F \propto x$

$F = -Kx$

We know that $F = -gard\, V = -\nabla V = -\frac{\partial V}{\partial x}$

Integrating, $V = -\int F\, dx = -\int -Kx\, dx = \frac{Kx^2}{2}$

$\therefore\;\; L = T - V = \frac{1}{2}\, m\,\dot{x}^2 - \frac{1}{2}\, K\, x^2$

$H = T + V = \frac{1}{2}\, m\,\dot{x}^2 + \frac{1}{2}\, K\, x^2$

Momentum $p = \frac{\partial L}{\partial \dot{x}} = m\,\dot{x}$

Note: Hamiltonian $H = \sum p_K \dot{q}_K - L$

Example – 6.3

If the Lagrangian of a dynamical system in two dimensions is $L = \frac{1}{2}m\,\dot{x}^2 + m\,\dot{x}\dot{y}$. Then find the Hamiltonian.

Sol:

$H = \sum_{i=1}^{2} p_i\dot{q}_i - L = p_1\dot{q}_1 + p_2\dot{q}_2 - L$

$= p_x\dot{x} + p_y\dot{y} - (\frac{1}{2}m\,\dot{x}^2 + m\,\dot{x}\dot{y})$

$= p_x \cdot \frac{p_y}{m} + p_y \cdot \left(\frac{p_x - p_y}{m}\right) - \frac{1}{2}\,m\frac{{p_y}^2}{m^2} - m \cdot \frac{p_y}{m} \cdot \left(\frac{p_x - p_y}{m}\right)$

$= \frac{p_x p_y}{m} + \frac{p_x p_y}{m} - \frac{{p_y}^2}{m} - \frac{1}{2}\frac{{p_y}^2}{m} - \frac{p_x p_y}{m} + \frac{{p_y}^2}{m}$

$= \frac{p_y}{m}\left(p_x - \frac{1}{2}p_y\right)$

$[\because p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + m\dot{y}$

$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{x}$

$\therefore \dot{x} = \frac{p_y}{m}$

$\dot{y} = \left(\frac{p_x - p_y}{m}\right)]$

Example – 6.4

A particle of mass m and coordinate q has the Lagrangian $L = \frac{1}{2}\,m\,\dot{q}^2 - \frac{\lambda}{2}q\,\dot{q}^2$, where λ is a constant. Find the Hamiltonian.

Sol:

$H = p\dot{q}^2 - L \quad (For\ one\ dimension)$

$= p\dot{q}^2 - \frac{1}{2}\,m\,\dot{q}^2 - \frac{\lambda}{2}q\,\dot{q}^2$

Now $p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} - \frac{\lambda}{2}q \cdot 2\dot{q} = m\dot{q} - \lambda\,q\dot{q}$

$\Rightarrow \dot{q} = \frac{p}{m - \lambda q}$

$\therefore H = p \cdot \frac{p}{m-\lambda q} - \frac{p^2}{(m-\lambda q)^2}\frac{(m-\lambda q)}{2} = \frac{p^2}{m-\lambda q} - \frac{p^2}{2(m-\lambda q)}$

$= \frac{p^2}{2(m-\lambda q)}$

Example – 6.5

If the Lagrangian of a Particle moving in the one – dimension is given by $L = \frac{\dot{x}^2}{2x} - V(x)$, Find the Hamiltonian.

Sol:

$H = p\,\dot{x} - L = p\dot{x} - \frac{\dot{x}^2}{2x} + V(x)$

Now, $p = \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{x} \qquad or \quad \dot{x} = p \cdot x$

$\therefore H = p \cdot px - \frac{p^2x^2}{2x} + V(x) = \frac{p^2 x}{2} + V(x)$

6.4. Hamilton's Principle and Principle of least action:

The principle of least action is a variational principle that, when applied to the action of a mechanical system, can be used to obtain the equations of motion for that system. It was called 'least' because its solution requires finding the path of motion in space that has the least value.

The starting point is the action denoted S of a physical system. It is defined by the integral of the Lagrangian L between two instants of time t_1 and t_2

$$S(q, t_1, t_2) = \int_{t_1}^{t_2} L\,(q(t),\ \dot{q}(t), t)\,dt$$

Mathematically the Principle is $\delta s = 0\ \ or\ \ \delta\ \int_{t_1}^{t_2} L\,(q(t),\ \dot{q}(t), t)\,dt = 0$

6.5. Two – dimensional motion of rigid bodies equations of motion of rigid body moving in two dimensions under finite force are

$$M\ \frac{d^2\bar{x}}{dt^2} = \sum X$$

$$M\ \frac{d^2\bar{y}}{dt^2} = \sum Y$$

$$M\ \frac{d^2\theta}{dt^2} = \sum(x'Y - y'X)$$

6.6. Euler's dynamical equations for the motion of a rigid body about an axis

Suppose a rigid body turns about a fixed point O under the action of external forces. Let L, M, N be the moments of the external forces; $W_1,\ W_2, W_3$ the angular velocities and A, B, C the moments of inertia about the principal axes OA, OB, OC fixed in the body. Then Euler's dynamical equations are

$$A\dot{W}_1 - (B - C)\ W_2 W_3 = L$$

$$B\dot{W}_2 - (C - A)\ W_1 W_3 = M$$

$$C\dot{W}_3 - (A - B)\ W_3 W_1 = N$$

If the body moves under no forces.

Then $L = M = N = 0,$ the above equations reduce to

$$A\frac{dW_1}{dt} - (B - C)\ W_2 W_3 = 0$$

$$B\frac{dW_2}{dt} - (C - A)\ W_1 W_3 = 0$$

$$C\frac{dW_3}{dt} - (A - B)\ W_3 W_1 = 0$$

6.7. Theory of small oscillation:

When a conservative system is displaced from its stable equilibrium position, it undergoes oscillation.

Consider a system with q_i as the generalized coordinates. Since the system is conservative, the forces acting on the system are derivable from a potential energy function $V(q_1, q_2, \dots\dots, q_N)$. Lagrange defined equilibrium as a configuration in which all generalised forces vanish, i.e., $\frac{\partial V}{\partial q_i} = 0$.

Example – 6.6

A particle of unit mass moves in a potential $V(x) = ax^2 + \frac{b}{x^2}$, *where a and b* are positive constants. Find the angular frequency of small oscillation about the minimum of the potential.

Sol:

$$V(x) = ax^2 + \frac{b}{x^2}$$

$$\frac{\partial V}{\partial x} = 0 \Rightarrow 2\,ax - \frac{2b}{x^3} = 0$$

$$\Rightarrow x_0 = \left(\frac{b}{a}\right)^{\frac{1}{4}}$$

Since $W = \sqrt{\frac{k}{m}}$, $m = 1$ *and* $k = \left.\frac{\partial^2 V}{\partial x^2}\right]_{x=x_0}$

$$\therefore k = 2a + \frac{6b}{{x_0}^4} = 2a + \frac{6b}{b/a} = 8a$$

Thus $W = \sqrt{8a}$

Example – 6.7

The Hamiltonian of a particle of unit mass moving in the $xy - plane$ is given to be $H = xp_x - yp_y - \frac{1}{2}x^2 + \frac{1}{2}y^2$.

The initial values are given to be $\big(x(0),\ y(0)\big) = (1,1)$ and $\big(p_x(0), p_y(0)\big) = \left(\frac{1}{2}, -\frac{1}{2}\right)$.

Find the trajectories in the $xy - plane$ and the $p_x p_y - plane$.

Sol:

$$H = xp_x - yp_y - \frac{1}{2}x^2 + \frac{1}{2}y^2$$

Hamiltonian equations are

$$\frac{\partial H}{\partial x} = -\dot{p}_x \Rightarrow p_x - x = -\dot{p}_x \quad \text{and}$$

$$\frac{\partial H}{\partial y} = -\dot{p}_y \Rightarrow -p_y + y = -\dot{p}_y$$

$\frac{\partial H}{\partial p_x} = \dot{x} \Rightarrow x = \dot{x}$ and $\frac{\partial H}{\partial p_y} = \dot{y} \Rightarrow -y = \dot{y}$

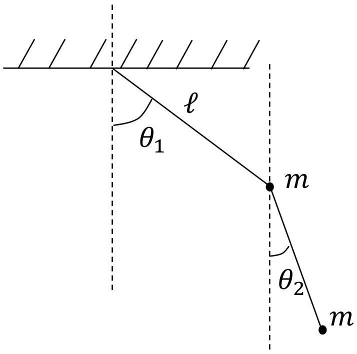
After solving these four differential equations and eliminating t and using boundary condition we get

$xy = 1$ and $p_x p_y = \frac{1}{2}$

So there are hyperbolas.

Example – 6.8

A double pendulum consists of two points masses m attached by strings of length ℓ find the K. E of the pendulum.



Sol:

$K.E = \frac{1}{2}\, m \left(\dot{x}_1{}^2 + \dot{y}_1{}^2\right) + \frac{1}{2} m\left(\dot{x}_2{}^2 + \dot{y}_2{}^2\right)$

Where $x_1 = \ell \sin\theta_1\,,\ \ y_1 = \ell\cos\theta_1$

$\Rightarrow\ \dot{x}_1 = \ell\cos\theta_1\,\dot{\theta}_1,\ \ \dot{y}_1 = -\ell\sin\theta_1\,\dot{\theta}_1$

$x_2 = \ell\sin\theta_1 + \ell\sin\theta_2\,,\ \ y_2 = \ell\cos\theta_1 + \ell\cos\theta_2$

$\dot{x}_2 = \ell\cos\theta_1\,\dot{\theta}_1 + \ell\cos\theta_2\,\dot{\theta}_2$

$\dot{y}_2 = -\ell\sin\theta_1\,\dot{\theta}_1 - \ell\sin\theta_2\,\dot{\theta}_2$

$\therefore K.E = \frac{1}{2}\, m\,\ell^2 \left[2\,\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 + 2\,\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right]$

Example – 6.9

What is the number of degrees of freedom of a rigid body in $d\ space - dimension$.

Ans: $\frac{d(d+1)}{2}$

Example – 6.10

The Lagrangian of a particle of mass m moving in one dimension is given by

$L = \frac{1}{2}\, m\,\dot{x}^2 - bx$. Find the co – ordinate of the particle $x(t)$ at time t.

Sol:

Lagrange equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow \frac{d}{dt}(m\dot{x}) + b = 0 \Rightarrow m\ddot{x} = -b$$

$$\frac{d^2x}{dt^2} = -\frac{b}{m} \Rightarrow \frac{dx}{dt} = -\frac{b}{m}\,t + c_1$$

$$\Rightarrow x = -\frac{b}{2m}t^2 + c_1 t + c_2$$

Example – 6.11

The Hamiltonian of a relativistic particle of rest mass m and momentum p is given by $H = \sqrt{p^2 + m^2} + V(x)$. Find the corresponding Lagrangian.

Sol:

$H = \sqrt{p^2 + m^2} + V(x)$

$\frac{\partial H}{\partial p} = \dot{x} = \frac{1}{2}\frac{2p}{\sqrt{p^2+m^2}} \quad \Rightarrow p = \frac{\dot{x}m}{\sqrt{1-\dot{x}^2}}$

Now, $L = \sum \dot{x}p - H = \dot{x}p - H$

$= \dot{x}p - \sqrt{p^2 + m^2} - V(x)$

$= \dot{x}\frac{\dot{x}m}{\sqrt{1-\dot{x}^2}} - \sqrt{\frac{\dot{x}^2m^2}{1-\dot{x}^2} + m^2} - V(x)$

$= \frac{\dot{x}^2m - m}{\sqrt{1-\dot{x}^2}} - V(x) = -m\sqrt{1-\dot{x}^2} - V(x)$

Example – 6.12

Find the equation of motion of a system described by the time dependent Lagrangian

$L = e^{\gamma t}\left[\frac{1}{2} m\,\dot{x}^2 - V(x)\right]$

Sol:

$L = e^{\gamma t}\left[\frac{1}{2} m\,\dot{x}^2 - V(x)\right]$

$\Rightarrow \frac{\partial L}{\partial \dot{x}} = e^{\gamma t}\, m\,\dot{x}$ and $\frac{\partial L}{\partial x} = -e^{\gamma t}\frac{\partial V}{\partial x}$

$\because \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Rightarrow \frac{d}{dt}(e^{\gamma t}\, m\,\dot{x}) + \frac{\partial V}{\partial x}\cdot e^{\gamma t}$

$\Rightarrow m\ddot{x}e^{\gamma t} + m\,\dot{x}\,\gamma\, e^{\gamma t} + \frac{\partial V}{\partial x}\cdot e^{\gamma t} = 0$

$\Rightarrow m\ddot{x} + \gamma\, m\,\dot{x} + \frac{\partial V}{\partial x} = 0$

Example – 6.13

A particle of mass m is moving in the potential $V(x) = -\frac{1}{2}\,a\,x^2 + \frac{1}{4}\,b\,x^4$ where a, b are positive constants. Find the frequency of small oscillations about a point of stable equilibrium.

Sol:

$V(x) = -\frac{1}{2}\,a\,x^2 + \frac{1}{4}\,b\,x^4$

$\frac{\partial V}{\partial x} = 0 \;\Rightarrow\; -ax + bx^3 = 0 \;\Rightarrow\; x = \pm\sqrt{\frac{a}{b}}\;, 0$

$\frac{\partial^2 V}{\partial x^2} = -a + 3bx^2$

At $x = 0$, $\frac{\partial^2 V}{\partial x^2} = -a\ (-ve)$

So, it is unstable point.

$$\left.\frac{\partial^2 V}{\partial x^2}\right]_{x=\pm\sqrt{\frac{a}{b}}} = -a + 3\,b \cdot \frac{a}{b} = 2a\ (+ve)$$

So, it is stable point.

$$\Rightarrow W = \sqrt{\frac{\frac{\partial^2 V}{\partial x^2}}{m}} = \sqrt{\frac{2a}{m}}$$

Example – 6.14

A mechanical system is described by the Hamiltonian $H(q,p) = \frac{p^2}{2m} + \frac{1}{2} m\, w^2 q^2$. As a result of the canonical transformation generated by $F(q,Q) = -\frac{Q}{q}$. Find the Hamiltonian in the new coordinate Q and momentum P.

Sol:

$$H(q,p) = \frac{p^2}{2m} + \frac{1}{2} m\, w^2 q^2, F = F_1(q,Q) = -\frac{Q}{q}$$

$$\Rightarrow \frac{\partial F_1}{\partial q} = p \Rightarrow \frac{Q}{q^2} = p$$

$$\Rightarrow \frac{\partial F_1}{\partial Q} = -P \Rightarrow -\frac{1}{q} = -P \Rightarrow q = \frac{1}{P}$$

$$\therefore p = \frac{Q}{q^2} = \frac{Q}{\frac{1}{P^2}} = P^2 Q$$

$$\therefore H(Q,P) = \frac{p^2}{2m} + \frac{1}{2} m\, w^2 q^2$$

$$= \frac{P^4 Q^2}{2m} + \frac{1}{2} m w^2 \cdot \frac{1}{P^2} = \frac{1}{2m} P^4 Q^2 + \frac{1}{2} m w^2 P^{-2}$$

Example – 6.15

A particle of mass m moves in the one – dimensional potential $V(x) = \frac{\alpha}{3} x^3 + \frac{\beta}{4} x^4$ where $\alpha, \beta > 0$. One of the equilibrium point is $x = 0$. Find the angular frequency of small oscillations about the other equilibrium point.

Sol:

$$V(x) = \frac{\alpha}{3} x^3 + \frac{\beta}{4} x^4$$

$$\Rightarrow \frac{\partial V}{\partial x} = \alpha\, x^2 + \beta\, x^3 = 0 \Rightarrow x_0 = -\frac{\alpha}{\beta}$$

Spring constant $k = \left.\frac{\partial^2 V}{\partial x^2}\right]_{x=x_0} = \frac{\alpha^2}{\beta} (+ve)$

$$\Rightarrow w = \sqrt{\frac{k}{m}} = \frac{\alpha}{\sqrt{\beta m}}$$

Example – 6.16

A particle of unit mass moves in the $xy - plane$ in such a way that $\dot{x}(t) = y(t)$ and $\dot{y}(t) = -x(t)$. Find the potential from which the conservative force – field is derived.

Sol:

$\dot{x} = y \quad and \quad \dot{y} = -x$

$\Rightarrow \ddot{x} = \dot{y} = -x \quad and \quad \ddot{y} = -\dot{x} = -y$

$\Rightarrow \ddot{x} + x = 0 \quad and \quad \ddot{y} + y = 0$

$L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 - \frac{1}{2}(x^2 + y^2)$

$\Rightarrow V = \frac{1}{2}(x^2 + y^2)$