COUNCIL OF SCIENTIFIC & INDUSTRIAL RESEARCH

Mathematical Science

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8. Numerical Analysis

8.1 Errors

8.1.1 Significant figures/digits: The Significant figures/digits are the digits which are used to represent a number.

Examples (8.1):
- (i) 0, 1, 2,….9 are significant figures.
- (ii) 3.15673, 4.00901 both contain six significant figures.
- (iii) 0.0008 contains only one significant figure which is 8 and all the zeros left to 8 here are used to fix the decimal positions.
- (iv) 6.0001 contain five significant figures.

8.1.2 Exact Numbers (N_E): Exact numbers are those numbers which have no approximation.

Examples (8.2): 3, $\frac{1}{7}$, e, $\sqrt{5}$, π etc

8.1.3 Approximate Numbers (N_A): Approximate numbers are those numbers which can not be represented by finite numbers of digits.

Examples (8.3):
- (i) 1.732 is the approximation of $\sqrt{3}$.
- (ii) 3.142 is the approximation of π.

8.1.4 Error (E): Error of a number is the differences of the exact value and approximate value of a number.

Error = Exact number – Approximate number

i.e. $E = N_E - N_A$

Examples (8.4): $\sqrt{3}$ -1.732 is the error for $\sqrt{3}$.

Errors are committed two ways: (i) rounding a number to a finite digits (Rounding-off error) (ii) due to calculation (Significant error).

8.1.5 Rounding-off Error: Rounding-off error is the error for discarding all but a pre-decided number of digits.

Rules for rounding-off a number to n-significant figures:

- (i) If the digit at (n+1)-th place is less than 5, discard all the digits after n-th pace.
- (ii) If the digit at (n+1)-th place is greater than 5, add one to n-th place and discard all the digits after n-th pace.
- (iii) If the digit at (n+1)-th place is exactly 5 and n-th place is even, discard all the digits after n-th pace.

(iv) If the digit at (n+1)-th place is exactly 5 and n-th place is odd, add one to n-th place and discard all the digits after n-th pace.

Examples (8.4):

Correct the following numbers up to 4 significant figures.

(i) 2.356489 ≈ 2.356 (4< 5, so 6 remains same)
(ii) 3.78376 3 ≈ 3.784 (7 > 5, so add 1 to 3)
(iii) 5.3485345 ≈ 5.348 (8 is at 4^{th} place even, so 8 remains same)
(iv) 2.6735674 ≈ 2.674 (3 is at 4^{th} place odd, so add 1 to 3)

8.1.6 Types of Errors: Significant Error (S_a), Absolute ($\boldsymbol{E_a}$), Relative ($\boldsymbol{E_r}$) and Relative Percentage Errors ($\boldsymbol{E_p}$)

(i) Significant Error (S_a) = Exact Numbers (N_E) - Approximate Numbers (N_A)
i.e. $S_a = N_E - N_A$

(ii) Absolute Error (E_a) = | Exact Numbers (N_E) - Approximate Numbers (N_A)|
i.e. $E_a = | N_E - N_A|$

(iii) Relative Error (E_r) = $\frac{\text{Absolute Error } (E_a)}{\text{Exact Numbers } (N_E)}$
i.e. $E_r = \frac{E_a}{N_E}$

(iv) Relative Percentage Error (E_p) = Relative Error (E_r) × 100%
i.e. $E_p = E_r \times 100\%$

Example (8.5):

Write the approximate representation of $\frac{1}{3}$ correct up to 4 significant figures and also find (i) Significant Error (S_a), (ii) Absolute Error (E_a), (iii) Relative Error (E_r) and (iv) Relative Percentage Error (E_p).

Solution: $\frac{1}{3}$ = 0.3333

(i) $N_E = 1.2345627,\ N_A = 1.2345584$
Significant Error (S_a) = $N_E - N_A = 0.0000043$ (looses 6 significant digits each N_E *and* N_A)

(ii) Absolute Error (E_a) = $| N_E - N_A|$ = $|\frac{1}{3}$ - 0.3333| = 0.000033

(iii) Relative Error (E_r) = $\frac{E_a}{N_E} = \frac{0.000033}{\frac{1}{3}}$ = 0.000099 ≈ 0.0001

(iv) Relative Percentage Error (E_p) = $E_r \times 100 = \frac{E_a}{V_T} \times 100 = \frac{|V_T - V_A|}{V_T} \times 100$

Remark:

(i) If a number be rounded up to m decimal places the absolute error $E_a \leq \frac{1}{2}\ 10^{-m}$

Example (8.6)

$V_T = 345.26132, \quad V_A = 345.261$

$\therefore E_a = 0.00032 \leq \frac{1}{2} \times 10^{-3} = 0.0005$

(ii) If a number be rounded to n correct significant figures, then the relative error

$E_r < \frac{1}{k\times 10^{n-1}}\ , \quad k : first\ significant\ digit\ in\ the\ number.$

Example (8.7) $V_T = \frac{2}{3}, \quad V_A = 0.6667$

(a) $E_a = |V_T - V_A| = 0.000033$

(b) $E_r = \frac{E_a}{V_T} = 0.0000495 \simeq 0.00005$

(c) $E_p = E_r \times 100 = 0.005\%$

$E_p = 0.000033 < 0.00005$

$E_r < \frac{1}{k\times 10^{n-1}} = \frac{1}{6\times 10^3} = 0.00166 \simeq 0.001\ 7 \ \ (k = 6, n = 4)$

8.2 Interpolation with Equal and Unequal Intervals

Let $y = f(x)$ defined in $[a, b]$. Let us consider the consecutive value of x, differing by $\hbar$ as $a = x_0,\ x_1 = x_0 + \hbar, \ldots\ldots\ ..,x_r = x_0 + r\hbar, \ldots\ldots.., x_n = x_0 + \hbar$

$y_0 = f(x_0),\ \ y_1 = f(x_1), \ldots\ldots\ .., y_n = f(x_n).$

$x_0, x_1, \ldots.., x_n$ are called nodes and $y_1, y_2, \ldots.., y_n$ are called entries.

8.2.1 Forward differences: -

$\Delta f(x_0) = f(x_0 + h) - f(x_0) = f(x_1) - f(x_0) = y_1 - y_0 = \Delta y_0$

$\Delta f(x_1) = f(x_1 + h) - f(x_1) = f(x_2) - f(x_1) = y_2 - y_1 = \Delta y_1$

$\Delta f(x_{n-1}) = f(x_{n-1} + h) - f(x_{n-1}) = f(x_n) - f(x_{n-1}) = y_n - y_{n-1} = \Delta y_{n-1}$

$\Delta^2 f(x_0) = \Delta^2 y_0 = y_2 - 2y_1 + y_0$

$\Delta^3 f(x_0) = \Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$

8.2.2 Backward differences: -

$\nabla\ f(x_1) = f(x_1) - f(x_1 \cdot \hbar) = f(x_1) - f(x_0) = y_1 - y_0 = \nabla y_1$

$\nabla\ f(x_n) = f(x_n) - f(x_{n-1}) = y_n - y_{n-1} = \nabla y_n$

❖ $\Delta \cdot \nabla = \Delta - \nabla$

Result: $\Delta^k\ f(x) = \sum_{i=0}^{k} (-1)^i \binom{k}{i}\ \delta\ [x + (k-i)\hbar]$

- **Fundamental theorem of difference calculus:-**

If $f(x)$ be a polynomial of degree n, then the n^{th} difference of f(x) is constant and $(n+1)th$ difference vanish.

8.2.3 Shift operator E:-

$E\ f(x) = f(x + \hbar)\ \Rightarrow E = \Delta + 1 \quad and \quad E\Delta = \Delta E$

(i) Relation between $\Delta\ (difference\ operator)$ and $D \equiv \frac{d}{du}\ (differential\ operator)$

By Taylor's Theorem:-

$f(x + \hbar) = f(x) + \hbar\, f'(x) + \frac{\hbar^2}{2!} f''(x) + \ldots\ldots\ldots$

$or, E\ f(x) = f(x) + \hbar\ D\ f(x) + \frac{\hbar^2}{2!} D^2\ f(x) + \ldots\ldots\ldots\ \ldots$

$or, (\Delta + 1) f(x) = \left(1 + \hbar D + \frac{\hbar^2}{2!} D^2 + \ldots\ldots\ \ldots.\right) f(x)$

$or,\ \Delta + 1 = e^{\hbar D}$

$or,\ \hbar D = \log(1 + \Delta)$

$or, D = \frac{1}{\hbar}\ [\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots\ldots\ \ldots\ \ldots\ \ldots]$

(ii) Expression of any value of a function in terms of the leading term and leading difference of a difference table

By Shift operator E, we have –

$f(x + x\hbar) = E\ f(x) = (1 + \Delta) f(x)$

$$= \left[1 + \binom{n}{1}\Delta + \binom{n}{2}\Delta^2 + \ldots\ldots\ \ldots\ldots + \binom{n}{n-1}\Delta^{n-1} + \Delta^n\right] f(x)$$

$$= \left[f(x_0) + \binom{n}{1}\Delta f(x_0) + \binom{n}{2}\Delta^2 f(x_0) + \ldots\ldots \ \ldots\ldots + \binom{n}{n-1}\Delta^{n-1} f(x_0) + \Delta^n f(x_0)\right] f(x)$$

$$= y_0 + \binom{n}{1}\Delta\, y_0 + \binom{n}{2}\Delta^2\, y_0 + \ldots\ldots\ . \ ..+ \binom{n}{n-1}\Delta^{n-1}\, y_0 + \Delta^n\, y_0$$

8.2.4 Factorial notation:-

$x^{(n)} = x(x-\hbar)(x-2\hbar)\ldots\ldots \ \ldots\ldots(x-\overline{x-1}\,\hbar)$

(i) $\Delta^n\, x^{(n)} = n!\ \hbar^n \quad and \quad \Delta^{n+1}\, x^{(n)} = 0$

<u>Example (8.8)</u>: Find the polynomial $f(x)$, which satisfy the following data and hence find the value of $f(1.5)$.

x	1	2	3	4	5
$f(x)$	4	13	34	73	136

Difference table:

x	$\Delta f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	4			
		9		
2	13		12	
		21		6
3	34		18	
		39		6
4	73		24	
		63		
5	136			

(3rd degree polynomial as 3rd order difference is constant)

We know that $f(x+n\hbar) = E^n\, f(x_0) = (1+\Delta)^n f(x_0)$

Here $\quad x = 1, \hbar = 1,\ x + n\hbar = 1 + n = x \qquad \therefore x = n - 1$

$\therefore \quad f(x) = (1+\Delta)^{\,x-1} f(1)$

$$= [1 + (x-1)\Delta\, f(1) + \frac{(x-1)(x-2)}{2!}\Delta^2\, f(1) + \frac{(x-1)(x-2)(x-3)}{3!}\Delta^3\, f(1) + \ldots\ldots\ ..$$

$$= 1 + 9(x-1) + 12\frac{(x-1)(x-2)}{2} + 6\ \frac{(x-1)(x-2)(x-3)}{3}$$

$$= x^3 + 2x + 1$$

$\therefore f(1.5) = (1.5)^3 + 2 \times (1.5) + 1 = 7.375$

8.2.5 Newton's Forward Interpolation Formula:-

Let $f(x)$ is known for $(n+1)$ distinct equispaced arguments namely $x_0, x_1, x_2, \ldots\ldots, x_{n-1}, x_n$ such that $x_r = x_0 + r\,h \quad (r = 0, 1, \ldots.\ \ldots\ .n-1), h = setp\ length.$

$y = f(x)\ and\ y_j = f(x_j),\ \ j = 0, 1, \ldots\ldots., n,\ \ s = \frac{x-x_0}{h}$

$f(x) \simeq y_0 + s\,\Delta y_0 + s\,(s-1)\frac{\Delta^2 y_0}{2!} + s\,(s-1)(s-2)\frac{\Delta^3 y_0}{3!} + \ldots\ldots + s\,(s-1)\ldots\ldots(s-\overline{n-1})\frac{\Delta^n y_0}{n!}$

Which is known as Newton's Forward Interpolation Formula.

Error:- $R_{n+1}(x) = \frac{s(s-1)(s-2).....(s-n)}{n+1}\ h^{n+1}\ f^{n+1}(\xi)$

Where $\min\{x_1, x_0, x_n\} < \xi < \max\{x_1, x_0, x_n\}$

$|R_{n+1}(x)| < 1\ \ for\ \ \ x > 1\ \ and\ 0 < s < 1$

8.2.6 Newton's Backward Interpolation Formula: -

$s = \frac{x-x_n}{h}$

$f(x) = y_n + s\,\Delta y_n + s\,(s+1)\frac{\Delta^2 y_n}{2!} + s\,(s+1)\ldots\ldots.(s+n-1)\frac{\Delta^n y_n}{n!}$

Error:- $R_{n+1}(x) = s\,(s+1)\ldots\ldots.(s+n)\ h^{n+1}\ \frac{f^{n+1}(\xi)}{(n+1)!}$

Where $\min\{x_0, x, x_n\} < \xi < \max\{x, x_0, x_n\}$

8.2.7 Lagrange's Interpolation Formula: -

Let $y = f(x)$ be defined on [a,b] and is only known for $a = x_0, x_1, \ldots\ldots x_n = b,$ in general are not equispaced and $y_i = f(x_i),\ \ i = 0,1,2, \ldots..\ ..n$

$f(x) \simeq \sum_{i=0}^{n} li(x)\,f(x_i)$

Where $li(x) = \frac{(x-x_0)(x-x_1)\ldots\ldots.(x-x_{i-1})(x-x_{i+1})\ldots\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots\ldots.(x_i-x_{i-1})(x_i-x_{i+1})\ldots\ldots(x_i-x_n)}$ $, i = 0, 1, \ldots\ldots.\ \ldots, n$

Error:- $R_{n+1}(x) = (x-x_0)(x-x_1)\ldots\ldots\ldots\ldots\ldots(x-x_n)\frac{f^{n+1}(\xi)}{(n+1)!}, (x_0 < \xi < x_n$

8.2.8 Divided Difference:-

Let $y = f(x)$ is known for $x_j (j = 0, 1, \ldots\ldots n)$ are not necessarily equipspaced. Then the first order divided difference for $x_0, x_1, f(x_0, x_1) = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{f(x_0)-f(x_1)}{x_0-x_1} = f(x_1, x_0)$

Second order divided difference for x_0, x_1, x_2

$$f(x_0, x_1, x_2) = \frac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0} = \frac{f(x_0)}{(x_0-x_1)-(x_0-x_2)} + \frac{f(x_1)}{(x_1-x_0)-(x_1-x_2)} + \frac{f(x_2)}{(x_2-x_0)-(x_2-x_1)}$$

$n\ th$ order divided difference for $x_0, x_1, x_2, \ldots\ldots x_n$

$$f(x_0, x_1, x_2, \ldots\ldots x_n) = \sum_{i=0}^{n} \frac{f(x_i)}{(x_i-x_0)\ldots.(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)} = \sum_{j=0}^{n} \frac{f(x_j)}{\prod_{\substack{i=0\\ i\neq j}}^{n}(x_j-x_i)}$$

Some Remarks: -

(i) The nth order divided difference of a polynomial of degree n is constant.

(ii) Divided difference for equispaced arguments:-

$$f(x_0, x_1, x_2, \ldots\ldots x_n) = \frac{\Delta^n y_0}{n!\hbar^n}$$

Since $f(x_0, x_1) = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{y_1-y_0}{\hbar} = \frac{\Delta y_0}{\hbar}$

$$f(x_0, x_1, x_2) = \frac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0} = \frac{\frac{\Delta y_1}{\hbar}-\frac{\Delta y_0}{\hbar}}{2\hbar} = \frac{\Delta(y_1-y_0)}{2\hbar^2} = \frac{\Delta y_0}{2\hbar^2}$$

(iii) Newton's general divided difference formula:-

Let $y = f(x)$ be known for $x_0, x_1, x_2, \ldots\ldots x_n$ not necessarily equispaced. Then the polynomial of degree n through $(x_0, y_0), \ldots\ \ldots, (x_n, y_n)$, $y_i = f(x_i), i = 0, 1, \ldots\ \ldots n$ is given by –

$$f(x) \simeq f(x_0) + f(x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) + \ldots\ldots\ldots + (x - x_0)\ldots\ldots(x - x_{n-1})\, f(x_0, x_1, \ldots\ldots x_n)$$

Error:-

$$R_{n+1}(x) = (x - x_0)(x - x_1)\ldots\ \ldots\ldots(x - x_n)\, f(x_0, x_1, x_2, \ldots\ldots x_n)$$

$$= (x - x_0)(x - x_1)\ldots\ \ldots\ldots(x - x_n)\, \frac{f^{n+1}(\xi)}{(n+1)!}$$

8.3 Solution of Algebraic and transcendental Equations: -

Algebraic Equation: (i) If $f(x) = 0$ is a purely polynomial in x.

Transcendental Equation: If $f(x) = 0$ contains trigonometric exponential logarithmic function etc.

Assumptions:

(i) $f(x)$ is continuous and continuously differentiable upto sufficient x_0 of times.

(ii) $f(x) = 0$ has no multiple root, i.e., if $f(\alpha) = 0$, then in a neighbourhood of α either $f'(x) > 0 \text{ or } f'(x) < 0$.

8.3.1 Method to find the location of roots:

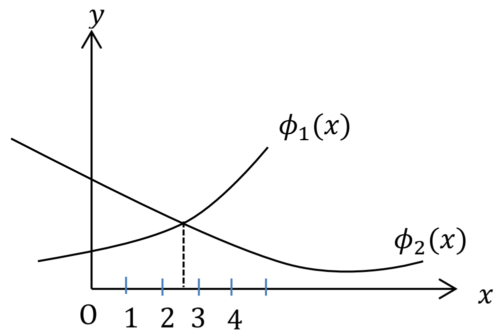
(a) Method of tabulation:

x	x_0	x_1	x_2	x_3
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$
Sign	$-$	$-$	$+$	$+$

$\Rightarrow f(\alpha) = 0 \text{ and } x_1 < \alpha < x_2$

(b) Graphical method:

$Write\ \ f(x) = 0\ \ as\ \ \phi(x) = \phi(x)$. Draw $y = \phi_1(x) and\ y = \phi_2(x)$



$\Rightarrow 2 < \alpha < 3$

8.3.2 Bisection Method

We first find an interval [a_0, b_0] such that the given function f: [a_0, b_0]→ **R** (set of real numbers) is (i) continuous on [a_0, b_0] (ii) $f'(x)$ keeps the same sign in [a_0, b_0] and (iii) $f(a_0)f(b_0) < 0$. (*These three conditions ensure that the function f in* [a_0, b_0] *has a unique root*). Consider $x_0 = \frac{a_0+b_0}{2}$.

(i) If $f(x_0) = 0$ then x_0 is the root of f .

(ii) If $f(x_0) \neq 0$ then either $f(a_0)f(x_0) < 0$ or $f(x_0)f(b_0) < 0$. If $f(a_0)f(x_0) < 0$,

(iii) then the root lies in $[a_0, x_0]$ and we rename $[a_0, x_0]$ as $[a_1, b_1]$ or if $f(x_0)f(b_0) <$ 0 then the root lies in $[x_0, b_0]$ and we rename $[x_0, b_0]$ as $[a_1, b_1]$

and consider a point $x_1 = \frac{a_1+b_1}{2}$.

(i) If $f(x_1) = 0$ then x_1 is the root of f .

(ii) If $f(x_1) \neq 0$ then either $f(a_1)f(x_1) < 0$ or $f(x_1)f(b_1) < 0$. If $f(a_1)f(x_1) < 0$,

then the root lies in $[a_1, x_1]$ and we rename $[a_1, x_1]$ as $[a_2, b_2]$ or if $f(x_1)f(b_1) <$ 0 then the root lies in $[x_1, b_1]$ and we rename $[x_1, b_1]$ as $[a_2, b_2]$

and consider a point $x_2 = \frac{a_2+b_2}{2}$.

Continuing in this process we get a sequence { x_n } of points in [a_0, b_0]. Now if

$|x_n - x_{n-1}| <$ ε (a pre-assigned error) then the root of f will be x_n in [a_0, b_0].

This method is surely convergent.

Tabulation of the bisection method

Suppose $f(a_0) > 0$ and $f(b_0) < 0$

x	$a_n(+ve)$	$b_n(-ve)$	$x_{n+1} = \left(\frac{a_n+b_n}{2}\right)$	$f(x_{n+1})$
0	a_0	b_0	$x_1\left(=\frac{a_0+b_0}{2}\right)$	$f(x_1) > 0\ (suppose)$
1	$a_1(=x_1)$	$b_1(=b_0)$	x_2	$f(x_2) > 0\ (suppose)$
2	$a_2(=x_2)$	$b_2(=b_0)$	x_3	$f(x_3) > 0\ (suppose)$
3	$a_3(=a_2)$	$b_3(=x_3)$	x_4	$f(x_3) > 0\ (suppose)$

8.3.3 Method of iteration or fixed point iteration:

This method is based on the principle of finding a sequence $\{x\}$ each elements of which successively approximates to a real root α so $f(x) = 0$ in $[a, b]$. We rewrite $f(x) = 0$

$as\ x = \phi(x)$.

Let $x = x_0 \in [a, b]$ be the initial approximation of α, then we set its first approximation as $x_1 = \phi(x_0)$ and then the successive approximations are $x_{n+1} = \phi(x_n),\ x = 0, 1, \ldots\ldots (iteration\ formula.)$

Convergence of the method of iteration:

$x = \phi(x)$ is not unique.

By MVT,

$|\alpha - x_1| = |\phi(\alpha) - \phi(x_0)| = |\alpha - x_0||\phi'(\xi_1)|\ for\ x_0 < \xi_1 < \alpha$

$|\alpha - x_2| = |\phi(\alpha) - \phi(x_1)| = |\alpha - x_1|\ |\phi'(\xi_2)|\ for\ x_1 < \xi_2 < \alpha$

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$|\alpha - x_{n+1}| = |\phi(\alpha) - \phi(x_n)| = |\alpha - x_n||\phi'(\xi_{n+1})|\ for\ x_n < \xi_{n+1} < \alpha$

Thus $|\alpha - x_{n+1}| = |\alpha \cdot x_0||\phi'(\xi_1)| \ldots\ldots .|\phi'(\xi_{n+1})|$

Assuming $|\phi'(x)| \leq \rho$ in $(a \leq x \leq b)$, $|\alpha - x_{n+1}| \leq |\alpha - x_0|\rho^{n+1}$

$\therefore\ \log_{n\to\infty}|\alpha - x_{n+1}| \leq |\alpha - x_0|\ \log_{n\to\infty}\rho^{n+1}$ $\quad \to 0\ if\ \rho < 1\ \ i.e., |\phi'(x)| < 1$

$\to 0\ if\ \rho > 1\ \ i.e., |\phi'(x)| > 1$

Estimation of error:

$|\varepsilon_{n+1}| \leq \frac{\rho}{1-\rho}\ |\hbar_n|$

$Order = 1$

$\downarrow$

$[|\varepsilon_{n+1}| \leq \rho\ |\alpha - x_n|]$

8.3.4 Newton – Raphson Method: -

This is also an iterative method and it is used to find isolated roots of an equation $f(x) = 0$.

We first find an interval [$a_{0,}\ \ b_0$] such that
(i) the given function *f*: [$a_{0,}\ \ b_0$]→ **R** (set of real numbers) satisfies the condition,
f(a_0)*f*(b_0) < 0 and *f* '(*x*) ≠ 0 in [$a_{0,}\ \ b_0$] with *f* '(*x*) is not very small in [$a_{0,}\ \ b_0$]

(ii) the interval [$a_{0,}\ \ b_0$] should be very close to the root desire root

(iii) |*f*(*x*)·*f* ''(*x*)| < $\{f'(x)\}^2$.

Then we find a sequence $\{ x_n \}$ of points in $[a_0, b_0]$ such that $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. If

$| x_{n+1} - x_n | <$ ε (a pre-assigned error) then the root of f will be x_{n+1} in $[a_0, b_0]$.

Let x be an approximation of α. Then $x_1 = x_0 + h$ is correct root

$\Rightarrow f(x_0 + h) = f(x_1) = 0 \Rightarrow f(x_0) + h f'(x_0) + \frac{h}{2} f''(x_0) + \ldots\ldots = 0$

$\Rightarrow h = -\frac{f(x_n)}{f'(x_n)}$ $(\because h \to 0, h^n = 0, n \geq 2)$

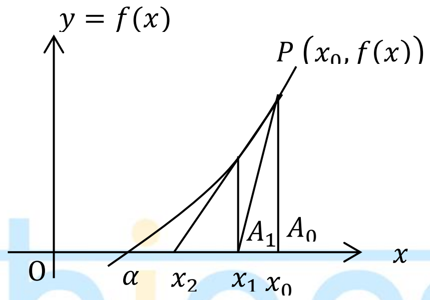
$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

Convergence of this method:

Comparing with the iteration method

$\phi(x) = x - \frac{f(x)}{f'(x)}$ and

$|\phi'(x)| < 1 \Rightarrow |f(x) f'(x)| < |\{f'(x)\}^2|$



Estimation of error:-

$|\varepsilon_{n+1}| = |\alpha - x_n + 1| = \frac{1}{2} \left|\frac{f''(\alpha)}{f'(\alpha)}\right| |\alpha - x_n|^2$

Definition (Order of a method):- A method is said to be of order P. If P is the largest number for which ∃ a finite number C such that $|x_{n+1} - \alpha| \leq C|x_n - \alpha|^p$ i.e.,

$|\varepsilon_{n+1}| \leq C|\varepsilon_n|^p, n \to \alpha.$

So, the order of Newton – Raphson method is 2.

Note:

(i) N-R method fails if $f'(x) = 0$ *or* very small in a neighbourhood of the root.

(ii) N-R method id faster than iteration method.

(iii) The initial guess (approximation) must be taken very close to the root other wise it may diverge.

(iv) To find q-th root of $R > 0$. Let $x = \sqrt[q]{R} \Rightarrow x^q - R = 0$

but $f(x) = x^q - R = 0$ *then* $f'(x) = q\, x^{q-1}$, $x_{n+1} = \frac{(q-1)x_n{}^q + R}{q x_n{}^{q-1}}$ $(n = 0, 1, 2, \ldots)$

8.3.5 Regula –Falsi Method:

We first find an interval [x_0, x_1] such that the given function f: [x_0, x_1]$\rightarrow$ **R** (set of real numbers) satisfies $f(x_0)f(x_1) < 0$. We find a point $x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)}$ $f(x_0)$.

(i) If $f(x_2) = 0$ then x_2 is the root of f .

(ii) If $f(x_2) \neq 0$ then either $f(x_0)f(x_2) < 0$ or $f(x_1)f(x_2) < 0$. If $f(x_0)f(x_2) < 0$
then the root lies in $[x_0, x_2]$ and we rename $[x_0, x_2]$ as $[x_1, x_2]$ or if $f(x_1)f(x_2)$ 0
then
the root lies in $[x_1, x_2]$ and we keep the interval $[x_1, x_2]$ as same name [x_1,]
and find a point

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_1)$$

(iii) If $f(x_3) = 0$ then x_3 is the root of f .

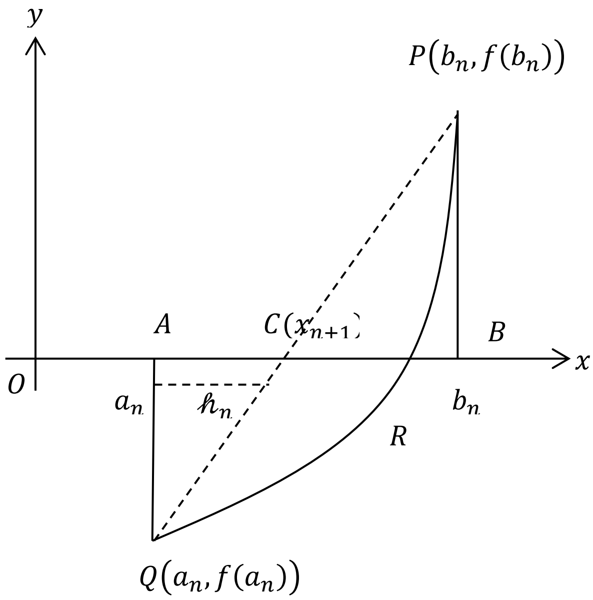
(iv) If $f(x_3) \neq 0$ then either $f(x_1)f(x_3) < 0$ or $f(x_2)f(x_3) < 0$.
If $f(x_1)f(x_3) < 0$, then
the root lies in $[x_1, x_3]$ and we rename $[x_1, x_3]$ as $[x_2, x_3]$ or if $f(x_2)f(x_3$ < 0
then
the root lies in $[x_2, x_3]$ and we keep the interval $[x_2, x_3]$ as same name $[x_2, x$

and find a point $x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_2)$

Continuing in this process we get a sequence { x_n } of points in [x_0, x_1]. Now if

$|x_n - x_{n-1}| <$ ε (a pre-assigned error) then the root of f will be x_n in [x_0, x_1].

Graphically,



$$\frac{AC}{AQ} = \frac{CB}{BP}$$

$$\Rightarrow AC = \frac{AQ}{BP} \cdot CB = \frac{AQ}{BP}(AB - AC)$$

$$\Rightarrow x_{n+1} - a_n = \frac{|f(a_n)|}{|f(b_n)|}(b_n - a_n - (x_{n+1} - a_n)$$

$$\Rightarrow (x_{n+1} - a_n)\left[1 + \frac{|f(a_n)|}{|f(b_n)|}\right]$$

$$= \frac{|f(a_n)|}{|f(b_n)|}(b_n - a_n)$$

$$= x_{n+1} = a_n + \frac{|f(a_n)|}{|f(a_n)|+|f(b_n)|}(b_n - a_n)$$

- $$x_{n+1} = x_n + \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

8.4 Solution of system of linear equations: -

8.4.1 Gauss-Elimination Method

Let us consider a system of linear algebraic equation in n unknown.

$$\begin{aligned} a_{11}\ x_1\ + a_{12}\ x_2\ + \cdots + a_{1n}\ x_n\ &= b_1 \\ a_{21}\ x_1\ + a_{22}\ x_2\ + \cdots + a_{2n}\ x_n\ &= b_2 \\ a_{31}\ x_1\ + a_{32}\ x_2\ + \cdots + a_{3n}\ x_n\ &= b_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}\ x_1\ + a_{n2}\ x_2\ + \cdots + a_{nn}\ x_n\ &= b_n \end{aligned}$$

The above system can be written as $AX = b$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ a_{31} & a_{32} & \cdots\cdots & a_{3n} \\ & & \vdots & \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} \text{ and } b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix}$$

Now making the augmented matrix $(A|\ b)$ in the following form by elementary row operations, we will get the solutions by back substitution.

$$(A|\ b) \approx \left(\begin{array}{ccccc|c} a'_{11} & a'_{12} & & \cdots\cdots & a'_{1n} & b'_1 \\ 0 & a'_{22} & & \cdots\cdots & a'_{2n} & b'_2 \\ 0 & 0 & a'_{3n} & \cdots\cdots & a'_{3n} & b'_3 \\ & & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots\cdots & a'_{nn} & b'_n \end{array}\right)$$

$$x_n = \frac{b'_n}{a'_{nn}}$$

$a'_{nn-2}\ x_{n-1}\ + a'_{nn-1}\ x_n\ = b'_{n-1}$ gives x_{n-1} etc.

8.4.2 Gauss-Jacobi Iteration Method

Let us consider a system of linear algebraic equation in *n* unknown.

$$\begin{aligned} a_{11}\ x_1\ + a_{12}\ x_2\ + \cdots + a_{1n}\ x_n\ &= b_1 \\ a_{21}\ x_1\ + a_{22}\ x_2\ + \cdots + a_{2n}\ x_n\ &= b_2 \\ a_{31}\ x_1\ + a_{32}\ x_2\ + \cdots + a_{3n}\ x_n\ &= b_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{n1}\ x_1\ + a_{n2}\ x_2\ + \cdots + a_{nn}\ x_n\ &= b_n \end{aligned}$$

Where the system is diagonally dominating i.e. $\sum_{\substack{i=1 \\ i\neq j}}^{n} |a_{ij}| \leq |a_{ii}|$.

The above system can be written as

$$\begin{aligned} x_1 &= \frac{1}{a_{11}}[b_1 - (a_{12}\ x_2 + a_{13}\ x_3 + \cdots + a_{1n}\ x_n)] \\ x_2 &= \frac{1}{a_{22}}[b_2 - (a_{21}\ x_1 + a_{23}\ x_3 + \cdots + a_{2n}\ x_n)] \\ x_3 &= \frac{1}{a_{33}}[b_3 - (a_{31}\ x_1 + a_{32}\ x_2 + \cdots + a_{3n}\ x_n)] \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n &= \frac{1}{a_{nn}}[(b_n - (a_{n1}\ x_1 + a_{n2}\ x_2 + \cdots + a_{nn-1}\ x_{n-1})] \end{aligned}$$

This method is an iteration method with some initial guess $x_i^{(0)}$ (i= 1, 2, …n) and the k+1-th (k is a natural number) iteration is given by

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{a_{11}}\left[b_1 - \left(a_{12}\ x_2^{(k)} + a_{13}\ x_3^{(k)} + \cdots + a_{1n}\ x_n^{(k)}\right)\right] \\ x_2^{(k+1)} &= \frac{1}{a_{22}}[b_2 - (a_{21}\ x_1^{(k)} + a_{23}\ x_3^{(k)} + \cdots + a_{2n}\ x_n^{(k)})] \\ x_3^{(k+1)} &= \frac{1}{a_{33}}[b_3 - (a_{31}\ x_1^{(k)} + a_{32}\ x_2^{(k)} + \cdots + a_{3n}\ x_n^{(k)})] \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n^{(k+1)} &= \frac{1}{a_{nn}}[(b_n - (a_{n1}\ x_1^{(k)} + a_{n2}\ x_2^{(k)} + \cdots + a_{nn-1}\ x_{n-1}^{(k)})] \end{aligned}$$

Here the iteration depends on given error (ε >0). We stop the iteration if $|x_i^{(k+1)} - x_i^{(k)}|$ < ε (i= 1, 2, …n) and the solution will be $x_i^{(k+1)}$(i= 1, 2, …n).

8.4.3 Gauss-Seidel Iteration Method

Consider a system of linear algebraic equation in *n* unknown.

$$\begin{aligned} a_{11}\ x_1\ + a_{12}\ x_2\ + \cdots + a_{1n}\ x_n\ &= b_1 \\ a_{21}\ x_1\ + a_{22}\ x_2\ + \cdots + a_{2n}\ x_n\ &= b_2 \\ a_{31}\ x_1\ + a_{32}\ x_2\ + \cdots + a_{3n}\ x_n\ &= b_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{n1}\ x_1\ + a_{n2}\ x_2\ + \cdots + a_{nn}\ x_n\ &= b_n \end{aligned}$$

Where the system is diagonally dominating i.e. $\sum_{\substack{i=1 \\ i\neq j}}^{n} |a_{ij}| \leq |a_{ii}|$.

The above system can be written as

$$\begin{aligned} x_1 &= \frac{1}{a_{11}}[b_1 - (a_{12}\ x_2\ + a_{13}\ x_3\ + \cdots + a_{1n}\ x_n\ )] \\ x_2 &= \frac{1}{a_{22}}[b_2 - (a_{21}\ x_1\ + a_{23}\ x_3\ + \cdots + a_{2n}\ x_n\ )] \\ x_3 &= \frac{1}{a_{33}}[b_3 - (a_{31}\ x_1\ + a_{32}\ x_2\ + \cdots + a_{3n}\ x_n\ )] \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n &= \frac{1}{a_{nn}}[(b_n - (a_{n1}\ x_1\ + a_{n2}\ x_2\ + \cdots + a_{nn-1}\ x_{n-1}\ )] \end{aligned}$$

This method is an iteration method with some initial guess $x_i^{(0)}$ (i= 1, 2, …n) and the k+1-th (k is a natural number) iteration is given by

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{a_{11}}[b_1 - \left(a_{12}\ x_2^{(k)} + a_{13}\ x_3^{(k)} + \cdots + a_{1n}\ x_n^{(k)}\right)] \\ x_2^{(k+1)} &= \frac{1}{a_{22}}[b_2 - (a_{21}\ x_1^{(k+1)} + a_{23}\ x_3^{(k)} + \cdots + a_{2n}\ x_n^{(k)})] \\ x_3^{(k+1)} &= \frac{1}{a_{33}}[b_3 - (a_{31}\ x_1^{(k+1)} + a_{32}\ x_2^{(k+1)} + a_{34}\ x_4^{(k)}\ \cdots + a_{3n}\ x_n^{(k)})] \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n^{(k+1)} &= \frac{1}{a_{nn}}[(b_n - (a_{n1}\ x_1^{(k+1)} + a_{n2}\ x_2^{(k+1)} + \cdots + a_{nn-1}\ x_{n-1}^{(k+1)})] \end{aligned}$$

Here the iteration depends on given error (ε >0). We stop the iteration if $|x_i^{(k+1)} - x_i^{(k)}|$ < ε (i= 1, 2, …n) and the solution will be $x_i^{(k+1)}$(i= 1, 2, …n).

The convergence of both Gauss-Jacobi and Gauss-Seidel is

$|a_{ji}| > \sum_{\substack{i=0 \\ i\neq j}}^{n} |a_{ji}|$ for all i i.e., the coefficient matrix is diagonally dominating.

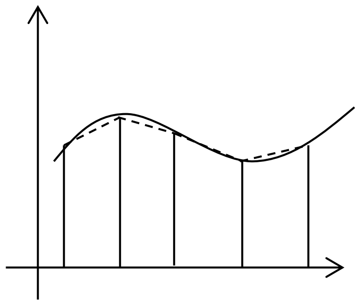
8.5 Numerical Integration

8.5.1 Trapezoidal Rule:

Let f be integrable over the interval $[a, b]$. We divide the interval into n equal subintervals by the points $a = x_0$, $x_0 + h,\ x_0 + 2h,\ x_0 + 3h, \dots\dots\ x_0 + (n-1)h,\ x_0 + nh = x_n = b$ where h is the step length.

Consider $y_r = f(x_r)$ for $r = 0, 1, 2, 3, \dots.. n$.

Then



$$\int_a^b f(x)dx \simeq \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \dots\dots + y_{n-1})\right],\ h = \frac{b-a}{n}$$

$$\text{Error} = -\frac{(b-a)^3}{12n^2} f''(\xi),\ (a = x_0 < \xi < x_n = b)$$

$$= -\frac{n\,h^3}{12} f''(\xi)$$

8.5.2 Simpson's One-third Rule: $(n = 2m)$

Let f be integrable over the interval $[a, b]$. We divide the interval into n *(even)* equal subintervals by the points $a = x_0$, $x_0 + h,\ x_0 + 2h,\ x_0 + 3h, \dots\dots\ x_0 + (n-1)h,\ x_0 + nh = x_n = b$ where h is the step length.

Consider $y_r = f(x_r)$ for $r = 0, 1, 2, 3, \dots.. n$.

Then

$$\int_a^b f(x)dx \simeq \frac{h}{3}\left[y_0 + y_n + 4(y_1 + y_3 + \dots\dots + y_{n-1}) + 2(y_2 + y_4 + \dots\dots + y_{n-2})\right]$$

$$\text{Error} = -\frac{nh^5}{180} f^{iv}(\xi),\quad (a < \xi < b)$$

8.5.3 Simpson's three-eight Rule: $(n = 3m)$

Let f be integrable over the interval $[a, b]$. We divide the interval into n *(multiple of three)* equal subintervals by the points $a = x_0$, $x_0 + h,\ x_0 + 2h,\ x_0 + 3h, \dots\dots\ x_0 + (n-1)h,\ x_0 + nh = x_n = b$ where h is the step length.

Consider $y_r = f(x_r)$ for $r = 0, 1, 2, 3, \dots.. n$.

Then

$$\int_a^b f(x)dx \simeq \frac{3h}{8}\left[y_0 + y_n + 2(y_3 + y_6 + y_9 \dots\dots) + 3(y_1 + y_2 + y_4 + y_5 \dots\dots)\right]$$

8.5.4 Weddle's Rule: ($n = 6m$)

Let f be integrable over the interval $[a, b]$. We divide the interval into n *(multiple of six)* equal subintervals by the points $a = x_0$, $x_0 + h$, $x_0 + 2h$, $x_0 + 3h, \ldots\ldots$ $x_0 + (n-1)h$, $x_0 + nh = x_n = b$ where h is the step length.

Consider $y_r = f(x_r)$ for $r = 0, 1, 2, 3, \ldots\ldots n$.

Then $\int_a^b f(x)dx = \frac{3h}{10}[(y_0 + y_n) + 5(y_1 + y_5 + y_7 + y_{11} + \cdots + y_{n-5} + y_{n-1}) + (y_2 + y_4 + y_8 + y_{10} + \cdots + y_{n-4} + y_{n-2}) + 6(y_3 + y_9 + y_{15} + \cdots + y_{n-3}) + 2(y_6 + y_{12} + \cdots + y_{n-6})]$

8.6 Numerical solution of Differential equations

(A) Single step Method:

8.6.1 Euler's method:

$\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$

$y_{n+1} = y_n + \hbar f(x_n, y_n)$, $n = 0, 1, 2, \ldots\ldots .p$ where $x_r = x_r + r\hbar, r = 1, 2, \ldots\ldots .p$

8.6.2 Euler's Modified method (Euler-Cauchy Method):

$y_{n+1}{}^{(0)} = y_n + \hbar f(x_n, y_n)$

$y_{n+1}{}^{(1)} = y_n + \frac{\hbar}{2}\left[f(x_n, y_n) + f\left(x_{n+1}, y_{n+1}{}^{(0)}\right)\right]$

.

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$y_{n+1}{}^{r} = y_n + \frac{\hbar}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1}{}^{r-1})]$

8.6.3 Picard's Method:

$\frac{dy}{dx} = f(x, y)$ $with$ $y(x_0) = y_0$

Integrating $[x_0, x]$ we have $\int_{x_0}^{x} dy = \int_{x_0}^{x} f(t, y)\, dt$ $or,$ $y(x) = y(x) + \int_{x_0}^{x} f(t, y)\, dt$

$y^{(n+1)}(x) = y_0 + \int_{x_0}^{x} f(t, y_t{}^{n})\, dt$ where $y^{n}(x) = y_0$

8.6.4 Taylor's series method: -

$y' = \frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$

$y(x) = y(x+\hbar) = y_0 + \hbar\, y_0' + \frac{\hbar^2}{2!}\, y_0'' + \frac{\hbar^3}{3!}\, y_0''' + \ldots..$

$y' = f(x,y)$

$y'' = f_x + f_y\, y' = f_x + f_y\, f$

Order $= \left|\frac{1}{\hbar}\, f(\hbar)\right|$

$f(\hbar) = error$

Example:

$y_1 = y_0 + \hbar y_0' + \frac{\hbar^2}{2!} y_0'' + f(\hbar)$

Then

Order $= \left|\frac{1}{\hbar}\, t(\hbar)\right| = 0(\hbar^2)$

8.6.5 Runge – Kutta Method:

(a) Second order Runge – Kutta Method:

$y' = \frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0,$

$k_1 = \hbar\, f(x_n, y_n)$

$k_2 = \hbar\, f(x_n + \hbar, y_n + k_1)$ $\quad Error = 0(\hbar^3)\ i.e., order = 2$

$y_{n+1} = y_n + \frac{1}{2}\,(k_1 + k_2)$

(b) Fourth order Runge – Kutta Method:

$y_1' = \frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0,$

$k_1 = \hbar\, f(x_n, y_n)$

$k_2 = \hbar\, f\left(x_n + \frac{\hbar}{2}, y_n + \frac{k_1}{2}\right), k_3 = \hbar\, f\left(x_n + \frac{\hbar}{2}, y_n + \frac{k_2}{2}\right)$

$k_4 = \hbar\, f(x_n + \hbar, y_n + k_3)$

$y_{n+1} = y_n + \frac{1}{t}\,[k_1 + 2(k_2 + k_3) + k_4]$

(B) Multi – Step Method:

8.6.6 Mid – point Method:

$\frac{dy}{dx} = f(x,y),\ with\ y(x_0) = y_0$

$y_{n+1} = y_n + \hbar\, y_n' + \frac{\hbar^2}{2!}\, y_n'' + \frac{\hbar^3}{3!} + y''(\xi)$

$y_{n-1} = y_n - \hbar\, y_n' + \frac{\hbar^2}{2!}\, y_n'' - \frac{\hbar^3}{3!} + y''(\xi)$

$\Rightarrow y_{n+1} - y_{n-1} = 2\hbar\, y_n + t(\hbar)$ $\quad \left(order = \left|\frac{1}{h} t(w)\right| = 0(\hbar^2)\right)$

$\Rightarrow y_{n+1} = y_{n-1} + 2\hbar\, f(x_n, y_n)\quad , n = 1, 2, 3, \ldots..\ ..$

8.6.7 Adams – Bash forth Method:

(i) Order – 1:

$y_{n+1} = y_n + \hbar\, f(x_n, y_n), t(\hbar) = \frac{\hbar}{2}\, y''(\xi)$ [Euler's method]

(ii) Order – 2:

$y_{n+1} = y_n + \frac{\hbar}{2}\,[3y'_n - y'_{n-1}], t(\hbar) = \frac{5}{12}\hbar^3\, y'''(\xi)$

(iii) Order – 3:

$y_{n+1} = y_n + \frac{\hbar}{12}\,[23y'_n - 16\, y'_{n-1} + y'_{n-2}], t(\hbar) = \frac{3}{8}\hbar^4\, y^{(4)}(\xi)$

8.6.8 Adams – Moulton Method:

(i) Order – 1:-

$y_{n+1} = y_n + \hbar\, f(x_{n+1}, y_{n+1}),\ t(\hbar) = -\frac{1}{2}\,\hbar\, y''(\xi)$ (Backward Euler's Method)

(ii) Order – 2:-

$y_{n+1} = y_n + \frac{\hbar}{2}\,[y'_{n+1} + y'_n],\ t(\hbar) = \frac{\hbar^3}{12}\, y''(\xi)$

(iii) Order – 3:-

$y_{n+1} = y_n + \frac{\hbar}{12}\,[5\, y'_{n+1} + 8\, y'_n - y'_{n-1}],\ t(\hbar) = -\frac{\hbar^4}{24}\, y^{(4)}(\xi)$

8.6.9 Two dimensional Newton – Raphson Method:-

$f(x, y) = 0,\ g(x, y) = 0$

$J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$

$\therefore \begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ y_k \end{pmatrix} - J^{-1}\,(x_k, y_k) \cdot \begin{pmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{pmatrix}.$

Application:- Finding complex root of $f(z) = 0 \Rightarrow f(z) = u\,(x, y) + iv\,(x, y) = 0$

Then $\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ y_k \end{pmatrix} - J^{-1}\,(x_k, y_k) \cdot \begin{pmatrix} u(x_k, y_k) \\ v(x_k, y_k) \end{pmatrix}$

8.7 Determination of eigenvalues by power method:

Let $A = (a_{ij})n \times n$ be a real symmetric matrix and $X_0(\neq 0)$ be a real n component vector.
Let $X_1 = AX_0,\ X_2 = AX_1,\ X_3 = AX_2, \ldots\ldots.\ ..,Y = AX_n = AX \quad (X_n = X)$

$m_0 = X^TX\ , m_1 = X^TY, m_2 = Y^TY$.

Then $q = \frac{m_1}{m_0}$ is an approximate eigen values λ of A and if we set $q = \lambda + \varepsilon$ so that ε is the error of q, then $|\varepsilon| \leq \sqrt{\frac{m_2}{m_0} - q^2}$

Example (8.8)

$$A = \begin{pmatrix} 8 & 1 & 1 \\ 1 & 5 & 2 \\ 1 & 2 & 5 \end{pmatrix},\ \text{choose } X_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Then $X_1 = \begin{pmatrix} 10 \\ 8 \\ 8 \end{pmatrix}, X_2 = \begin{pmatrix} 96 \\ 66 \\ 66 \end{pmatrix}, X_0 = \begin{pmatrix} 920 \\ 558 \\ 558 \end{pmatrix}, X_4 = \begin{pmatrix} 8 & 3 & 1 & 6 \\ 4 & 8 & 0 & 6 \\ 4 & 8 & 0 & 6 \end{pmatrix}$

Let $X = X_3, Y = X_4$ we have $m_0 = X^TX = 1432728,\ m_1 = X^TY = 12847896,\ m_2 = Y^TY = 115351128$

$q = \frac{m_1}{m_0} = 8.967 \quad and \ |\varepsilon| \leq \sqrt{\frac{m_2}{m_0} - q^2} = 0.311$

$\Rightarrow q - \varepsilon < \lambda < q + \varepsilon \ \Rightarrow 8.656 < \lambda < 9.278 \ \Rightarrow \lambda = 9$