COUNCILE OF SCIENTIFIC & INDUSTRIAL RESEARCH

Mathematical Science Code : 04

Unit – 3

SYLLABUS

Sub Unit – 5: Integral Equation

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11.	5.11. Leibnitz's rule of differentiation under the sign of integration
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13.	5.13. Integral Equations of the Second Kind by Successive Approximations 5.13.1. Iterated kernels or functions. 5.13.2. Resolvent kernel or reciprocal kernel. 5.13.3. Theorem 5.13.4. Theorem 5.13.5. Illustrative solved examples based on solutions of Volterra integral equation by successive approximations (or Iterative method).

Integral Equation

5.1. Definition of Integral equation: An integral equation is an equation in which an unknown function appears under one or more integral signs.

For example, for $a \leq x \leq b,\ a \leq \xi \leq b$, the equations

$f(x) = \int_a^x K\,(x,\xi)\ u(\xi)d\xi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1)$

$u\,(x) = f(x) + \lambda \int_a^b K\,(x,\xi)\ u(\xi)d\xi \ \ \ldots\ldots\ldots\ldots\ldots\ldots.\ldots(2)$

$u\,(x) = f(x) + \lambda \int_a^b K\,(x,\xi)\,[u(\xi)]^2 d\xi \ \ \ldots\ldots\ldots\ldots\ldots.\ldots(3)$

Where the function $u(x)$, is the unknown function while the functions $f(x)$ and $K(x,\xi)$ are known functions and λ, a and$b$ are constants, are all integral equations. The above mentioned functions may be complex – valued functions of the real variables x and ξ.

5.2. Definition of Linear and non – linear integral equations: An integral equation is called linear if only linear operations are performed in it upon the unknown function. An integral equation which is not linear is known as a non – linear integral equation.

For example, the integral equations (1) and (2) of Art 1.1. are linear integral equations while the integral equation (3) is non – linear integral equation.

The most general type of linear integral equation is of the form

$$v(x)\,u(x) = f(x) + \lambda \int_a^b K(x,\xi)\ u(\xi)\ d\xi, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(1)$$

Where the upper limit may be either variable x or fixed. The functions $f, v\ \ and\ \ K$ are known functions while u is to be determined; λ is a non-zero real or complex, parameter. The function $K(x,\xi)$ is known as the $kernel$ of the integral equation.

Remark – 1: The constant λ can be incorporated into the kernel $K(x,\xi)$ in (1). However, in many applications λ represents a significant parameter which may take on various values in a discussion being considered. For theoretical discussion of integral equations, λ plays an important role.

Remark – 2: If $v(x) \neq 0$, (1) is known as linear integral equation of the third kind. When $v(x) = 0$, (1) reduces to $f(x) + \lambda \int_a^b K(x,\xi)\ u(\xi)\ d\xi = 0, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2)$

Which is known as linear integral equation of the first kind. Again, when $v(x) = 1, (1)$ is reduces to $\ u(x) = f(x) + \lambda \int_a^b K(x,\xi)\ u(\xi)\ d\xi$

In the present book, we shall study in details equations of the form (2) and (3) only. In next two articles, we discuss special cases of (2) and (3).

5.3. Fredholm Integral equation:

A linear integral equation of the form

$$v(x)\, u(x) = f(x) + \lambda \int_a^x K(x,\xi)\ u(\xi)\, d\xi, \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots .(1)$$

Where a, b are both constants, $f(x), v(x)$ *and* $K(x,\xi)$ are known functions while $u(x)$ is unknown function and λ is a non – zero real or complex parameter, is called Fredholm integral equation of third kind. The function $K(x,\xi)$ is known as the kernel of the integral equation.

The following special cases of (1) are of our main interest.

(i) Fredholm integral equation of the first kind: A linear integral equation of the form (by setting $v(x) = 0$ in (1))

$$f(x) + \lambda \int_a^x K(x,\xi)\ u(\xi)\, d\xi, \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots .(2)$$

is known as Fredholm integral equation of the first kind.

(ii) Fredholm Integral equation of the second kind: A linear integral equation of the form (by setting $v(x) = 1$ in (1))

$$u(x) = f(x) + \lambda \int_a^x K(x,\xi)\ u(\xi)\, d\xi, \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots .(3)$$

is known as Fredholm integral equation of the second kind.

(iii) Homogeneous Fredholm integral equation of the second kind: A linear integral equation of the form (by setting $f(x) = 0$ in (3))

$$u(x) = f(x) + \lambda \int_a^x K(x,\xi)\ u(\xi)\, d\xi, \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots .(4)$$

is known as the homogeneous Fredholm integral equation of the second kind.

5.4. Volterra Integral Equation:A linear integral equation of the form

$$v(x)\, u(x) = f(x) + \lambda \int_a^x K(x,\xi)\ u(\xi)\, d\xi, \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots .(1)$$

where a, b are both constants, $f(x), v(x)$ *and* $K(x,\xi)$ are known functions while $u(x)$ is unknown function; λ is a non – zero real or complex parameter is called Volterra integral equation of the third kind. The function $K(x,\xi)$ is known as the kernel of the integral equation.

The following special cases of (1) are of our main interest.

(i) Volterra integral equation of the first kind: A linear integral equation of the form (by setting $v(x) = 0$ in (1)

$$f(x) + \lambda \int_a^x K(x, \xi)\ u(\xi)\ d\xi, \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots . (2)$$

is known as Volterra integral equation of the first kind.

(ii) Volterra integral equation of the second kind: A linear integral equation of the form (by setting $v(x) = 1$)

$$u(x) = f(x) + \lambda \int_a^x K(x, \xi)\ u(\xi)\ d\xi, \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots . (3)$$

is known as Volterra integral equation of the second kind.

(iii) Homogeneous Volterra integral equation of the second kind: A linear integral equation of the form (by setting $f(x) = 0$ is (3))

$$u(x) = \lambda \int_a K(x, \xi)\ u(\xi)\ d\xi, \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots . (4)$$

is known as the homogeneous Volterra integral equation of the second kind.

5.5. Definition of Singular integral equation: When one or both limits of integration become infinite or when the kernel becomes infinite at one or more points within the range of integration, the integral equation is known as singular integral equation. For example, the integral equations

$u(x) = f(x) + \lambda \int_{-\infty}^{\infty} e^{-|x-\xi|}\ u(\xi)\ d\xi$ and $f(x) = \int_0^x \frac{1}{(x-\xi)^\alpha}\ u(\xi)\ d\xi$, $0 < \alpha < 1$ are singular integral equations.

5.6. Special kind of kernels: The following special cases of the kernel of an integral equation are of main interest and we shall frequently come across with such kernels throughout the discussion of this book.

(i) Definition of Symmetric kernel: A kernel $K(x, \xi)$ is symmetric (or complex symmetric or Hermitian) if $K(x, \xi) = \overline{K(x, \xi)}$, where the bar denotes the complex conjugate. A real kernel $K(x, \xi)$ is symmetric if

$$K(x, \xi) = K(\xi, x).$$

For example, $\sin(x + \xi), \log(x\,\xi)\,, x^2\xi^2 + x\xi + 1$ etc. are all symmetric kernels. Again, $\sin(2x + 3\xi)$ and $x^2\xi^2 + x\xi + 1$ are not symmetric kernels.

(ii) Definition of Separable or degenerate kernel: A kernel $K(x,\xi)$ is called separable if it can be expressed as the sum of a finite number of terms, each of which is the product of a function of x only and a function of ξ only, i.e.,

$$K(x,\xi) = \sum_{i=1}^{n} g_i(x)\, h_i(\xi) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Remark: The functions $g_i(x)$ can be regarded as linearly independent, otherwise the number of terms in relation (1) can be further reduced. Recall that the set of functions $g_i(x)$ is said to be linearly independent, if $c_1 g_1(x) + c_2 g_2(x) + \ldots\ldots + c_n g_n(x) = 0$, where $c_1, c_2, \ldots\ldots, c_n$ arbitrary constants, then $c_1 = c_2 = \ldots\ldots = c_n = 0$.

5.7. An important formula for converting a multiple integral into a single ordinary integral.

$$\int_a^x u(\xi)\; d\xi^n = \int_a^x \frac{(x-\xi)^{n-1}}{(n-1)!}\, u(\xi)\, d\xi \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Note that the integral on the L.H.S. of (1) is a multiple integral of order n while the integral on the R.H.S. of (1) is ordinary integral of order one.

5.8. Definition of Solution of an integral equation: Consider the linear integral equations –

$$v(x)\, u(x) = f(x) + \lambda \int_a^b K(x,\xi)\; u(\xi)\, d\xi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

And $$v(x)\, u(x) = f(x) + \lambda \int_a^x K(x,\xi)\; u(\xi)\, d\xi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

A solution of the integral equation (1) or (2) is a function $u(x)$, which, when substituted into the equation, reduces it to an identity (with respect to x).

5.9. Definition of Resolvent kernel or reciprocal kernel:

Suppose solution of integral equations

$$u(x) = f(x) + \lambda \int_a^b K(x,\xi)\; u(\xi) d\xi \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

And $$u(x) = f(x) + \lambda \int_a^x K(x,\xi)\; u(\xi) d\xi \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

be respectively

$$u(x) = f(x) + \lambda \int_a^b K(x,\xi;\lambda)\; f(\xi) d\xi \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$u(x) = f(x) + \lambda \int_a^x \Gamma(x,\xi;\lambda)\; f(\xi) d\xi \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

then $R(x,\xi;\lambda)$ or $\Gamma(x,\xi;\lambda)$ is called the resolvent kernel or reciprocal kernel of the given integral equation.

5.10. Definition of Eigenvalues (or characteristic values or characteristic numbers), Eigenfunctions (or characteristic functions or fundamental functions):

Consider the homogeneous Fredholm integral equation

$$u(x) = \lambda \int_a^b K(x,\xi)\ f(\xi)d\xi \ \dots\dots\dots\dots\dots\dots\dots (1)$$

Then (1) has the obvious solution $u(x) = 0$, which is called the zero or trivial solution of (1). The values of the parameter λ for which (1) has a non – zero solution $u(x) \neq 0$ are called eigenvalues of (1) or of the kernel (x,ξ), and every non – zero solution of (1) is called on eigenfunction corresponding to the eigen value λ.

Remark – 1: The number $\lambda = 0$ is not an eigen value since for $\lambda = 0$ it follows from (1) that $u(x) \equiv 0$.

Remark – 2: If $u(x)$ is an eigenfunction of (1), then $c\, u(x)$, where c is an arbitrary constant, is also an eigenfunction of (1), which corresponds to the same eigenvalue λ.

Remark – 3: A homogeneous Fredholm integral equation of the second kind may, generally, have no eigenvalue and eigenfunction, or it may not have any real eigenvalues or eigenfunctions.

5.11. Leibnitz's rule of differentiation under the sign of integration:

Let $F(x,\xi)$ and $\frac{\partial F}{\partial x}$ be continuous functions of both $x \text{ and } \xi$ and let the first derivatives of $G(x)$ and $H(x)$ be continuous. Then

$$\frac{d}{dx}\int_{G(x)}^{H(x)} F(x,\xi)\, d\xi = \int_{G(x)}^{H(x)} \frac{\partial F}{\partial x} d\xi + F[x,H(x)]\frac{dH}{dx} - F[x,G(x)]\frac{dG}{dx} \dots\dots\dots\dots\dots\dots\dots (1)$$

Particular Case: If $G \text{ and } H$ are absolute constants, then (1) reduces to

$$\frac{d}{dx}\int_G^H F(x,\xi)\, d\xi = \int_G^H \frac{\partial F}{\partial x} d\xi \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (2)$$

Illustrative Solved Examples:

Example – 1: Show that the function $u(x) = (1+x^2)^{-\frac{3}{2}}$ is a solution of the Volterra integral equation

$u(x) = \frac{1}{1+x^2} - \int_0^x \frac{\xi}{1+x^2} u(\xi)d\xi.$

Solution: Given integral equation is $u(x) = \frac{1}{1+x^2} - \int_0^x \frac{\xi}{1+x^2} u(\xi)d\xi \ldots\ldots\ldots\ldots\ldots\ldots.(1)$

Also, given $u(x) = (1+x^2)^{-\frac{3}{2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$

From (2), $u(\xi) = (1+\xi^2)^{-\frac{3}{2}} \ldots\ldots\ldots\ldots\ldots\ldots.(3)$

Then, R.H.S. of (1)

$= \frac{1}{1+x^2} - \int_0^x \frac{\xi}{1+x^2}(1+x^2)^{-\frac{3}{2}} d\xi$ using (3)

$= \frac{1}{1+x^2} - \frac{1}{1+x^2}\int_0^{x^2}(1+t)^{-\frac{3}{2}} \cdot \frac{1}{2}dt$ (on putting $\xi^2 = t$ and $2\xi\, d\xi = dt$)

$= \frac{1}{1+x^2} - \frac{1}{1+x^2} \cdot \frac{1}{2} \cdot \left[\frac{(1+t)^{-\frac{1}{2}}}{-\frac{1}{2}}\right]_0^{x^2} = \frac{1}{1+x^2} + \frac{1}{1+x^2}\left[\frac{1}{(1+t)^{\frac{1}{2}}}\right]_0^{x^2}$

$= \frac{1}{1+x^2} + \frac{1}{1+x^2}\left[\frac{1}{(1+x^2)^{\frac{1}{2}}} - 1\right] = \frac{1}{1+x^2} + \frac{1}{(1+x^2)^{\frac{3}{2}}} - \frac{1}{1+x^2} = \frac{1}{(1+x^2)^{\frac{3}{2}}} = u(x),$ by (2)

$= L.H.S. of\ (1)$

Hence, (2) gives a solution of (1).

Example – 2:

Show that the function $u(x) = e^x\left(2x - \frac{2}{3}\right)$ is solution of the Fredholm integral equation

$u(x) + 2\int_0^1 e^{x-\xi} u(\xi)d\xi = 2x\, e^x$

Solution: Given integral equation is $u(x) + 2\int_0^1 e^{x-\xi} u(\xi)d\xi = 2x\, e^x \ldots\ldots\ldots..(1)$

Also, given $u(x) = e^x\left(2x - \frac{2}{3}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2)$

From (2), $u(\xi) = e^\xi\left(2\xi - \frac{2}{3}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$

Then, L.H.S. of (1)

$= e^x\left(2x - \frac{2}{3}\right) + 2\int_0^1 e^{x-\xi} e^\xi\left(2\xi - \frac{2}{3}\right) d\xi$, by (2) and (3)

$= e^x\left(2x - \frac{2}{3}\right) + 2\,e^x\int_0^1\left(2\xi - \frac{2}{3}\right)d\xi = e^x\left(2x - \frac{2}{3}\right) + 2e^x\left[\xi^2 - \frac{2}{3}\xi\right]_0^1$

$= 2xe^x - \frac{2}{3}e^x + \frac{2}{3}e^x = 2xe^x = 2xe^x = R.H.S.$ of (1).

Hence, (2) in a solution of (1).

Example – 3: Show that the function $u(x) = 1$ is a solution of the Fredholm integral equation $u(x) + \int_0^1 x\left(e^{x\xi} - 1\right) u(\xi)\, d\xi = e^x - x.$

Solution: Given integral equation is $u(x) + \int_0^1 x\left(e^{x\xi} - 1\right) u(\xi)\, d\xi = e^x - x \ldots\ldots\ldots..(1)$

Also, given $u(x) = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$

From (2) $u(\xi) = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$

Then, L.H.S. of (1)

$= 1 + \int_0^1 x\left(e^{x\xi} - 1\right) d\xi$ using (2) and (3)

$= 1 + x\left[\frac{1}{x}e^{x\xi} - \xi\right]_0^1 = 1 + x\left[\frac{1}{x}e^x - 1 - \frac{1}{x}\right]$

$= 1 + e^x - x - 1 = e^x - x = R.H.S.$ of (1)

Hence, (2) is a solution of (1).

Example – 4: Show that $u(x) = \cos 2x$ is a solution of the integral equation

$u(x) = \cos x + 3\int_0^\pi K(x,\xi)\, u(\xi) d\xi$, where $K(x,\xi) = \begin{cases} \sin x \cos\xi, 0 \le x \le \xi \\ \cos x \sin\xi, \xi \le x \le \pi \end{cases}$

Solution: Given integral equation is $u(x) = \cos x + 3\int_0^\pi K(x,\xi)\, u(\xi) d\xi, \ldots\ldots\ldots\ldots..(1)$

Where $K(x,\xi) = \begin{cases} \sin x \cos\xi, 0 \le x \le \xi \\ \cos x \sin\xi, \xi \le x \le \pi \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$

Also given, $u(x) = \cos 2x \ldots\ldots\ldots\ldots\ldots\ldots.(3)$

From (3), $u(\xi) = \cos 2\xi \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(4)$

Then, R.H.S. of (1)

$= \cos x + 3\left[\int_0^x K(x,\xi)\, u(\xi)\, d\xi + \int_x^\pi K(x,\xi)\, u(\xi)\, d\xi\right]$

$= \cos x + 3\left[\int_0^x \cos x \sin\xi \cos 2\xi\, d\xi + \int_x^\pi \sin x \cos\xi \cos 2\xi\, d\xi\right]$ (using (2) and (4)]

$= \cos x + 3\cos x \int_0^x \cos 2\xi\, d\xi + 3\sin x \int_x^\pi \cos 2\xi \cos\xi\, d\xi$

$= \cos x + \frac{3}{2}\cos x \int_0^x (\sin 3\xi - \sin\xi)\, d\xi + \frac{3}{2}\sin x \int_x^\pi (\cos 3\xi + \cos\xi)\, d\xi$

$= \cos x + \frac{3}{2}\cos x\left[-\frac{1}{3}\cos 3\xi + \cos\xi\right]_0^x + \frac{3}{2}\sin x\left[\frac{1}{3}\sin 3\xi + \sin\xi\right]_x^\pi$

$= \cos x + \frac{3}{2}\cos x\left[-\frac{1}{3}\cos 3x + \cos x + \frac{1}{3} - 1\right] + \frac{3}{2}\sin x\left[-\frac{1}{3}\sin 3\xi - \sin\xi\right]$

$= \cos x - \frac{1}{2}(\cos 3x \cos x + \sin 3x \sin x) + \frac{3}{2}(\cos^2 x - \sin^2 x) - \cos x$

$= -\frac{1}{2}\cos(3x - x) + \frac{3}{2}\cos 2x = -\frac{1}{2}\cos 2x + \frac{3}{2}\cos 2x$

$= \cos 2x = u(x)$, by (3) $= L.H.S.$ of (1).

Hence (3) is a solution of (1).

Example – 5: Show that the function $u(x) = \sin\left(\pi \frac{x}{2}\right)$ is a solution of the Fredholm integral equation $u(x) - \frac{\pi^2}{4}\int_0^1 K(x,\xi)\ u(\xi)\ d\xi = \frac{x}{2}$, where the kernel $K(x,\xi)$ is of the form

$$K(x,\xi) = \begin{cases} \frac{1}{2}\ x\ (2-\xi), 0 \le x \le \xi \\ \frac{1}{2}\ \xi(2-x), \xi \le x \le 1 \end{cases}$$

Solution: Given integral equation is $u(x) - \frac{\pi^2}{4}\int_0^1 K(x,\xi)\ u(\xi)\ d\xi = \frac{x}{2} \ldots\ldots\ldots\ldots\ldots (1)$

Where $K(x,\xi) = \begin{cases} \frac{1}{2}\ x\ (2-\xi), 0 \le x \le \xi \\ \frac{1}{2}\ \xi(2-x), \xi \le x \le 1 \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

Given $u(x) = \sin\left(\pi \frac{x}{2}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$

From (3), $u(\xi) = \sin\left(\pi \frac{\xi}{2}\right) \ldots\ldots\ldots\ldots\ldots\ldots (4)$

Then, L.H.S. of (1)

$= \sin\frac{\pi x}{2} - \frac{\pi^2}{4}\left[\int_0^x K(x,\xi)\ u(\xi) d\xi + \int_x^1 K(x,\xi)\ u(\xi) d\xi\right]$, by (3)

$= \sin\frac{\pi x}{2} - \frac{\pi^2}{4}\left[\int_0^x \left\{\frac{1}{2}\xi(2-x)\right\}\sin\frac{\pi\xi}{2} d\xi + \int_x^1 \left\{\frac{1}{2}x(2-\xi)\right\}\sin\frac{\pi\xi}{2} d\xi\right]$ [using (2) and (4)]

$= \sin\frac{\pi x}{2} - \frac{\pi^2}{8}(2-x)\int_0^x \xi \sin\frac{\pi\xi}{2} d\xi - \frac{\pi^2 x}{8}\int_x^1 (2-\xi)\sin\frac{\pi\xi}{2} d\xi$

$$= \sin\frac{\pi x}{2} - \frac{\pi^2(2-x)}{8}\left[\left[\xi\left\{-\frac{\cos\left(\frac{\pi\xi}{2}\right)}{\frac{\pi}{2}}\right\}\right]_0^x - \int_0^1 1\left\{-\frac{\cos\left(\frac{\pi\xi}{2}\right)}{\frac{\pi}{2}}\right\} d\xi\right]$$

$$-\frac{\pi^2 x}{8}\left[\left[(2-\xi)\left\{-\frac{\cos\left(\frac{\pi\xi}{2}\right)}{\frac{\pi}{2}}\right\}\right]_x^1 - \int_x^1 (-1)\left\{-\frac{\cos\left(\frac{\pi\xi}{2}\right)}{\frac{\pi}{2}}\right\} d\xi\right]$$

$$= \sin\frac{\pi x}{2} - \frac{\pi^2(2-x)}{8}\left[-\frac{2x}{\pi}\cos\frac{\pi x}{2} + \left[\frac{\sin\left(\frac{\pi\xi}{2}\right)}{\frac{\pi}{2}}\right]_0^x\right] - \frac{\pi^2 x}{8}\left[\frac{2(2-x)}{\pi}\cos\frac{\pi x}{2} - \left[\frac{\sin\left(\frac{\pi\xi}{2}\right)}{\frac{\pi}{2}}\right]_x^1\right]$$

$$= \sin\frac{\pi x}{2} - \frac{\pi^2(2-x)}{8}\left[-\frac{2x}{\pi}\cos\frac{\pi x}{2} + \frac{4}{\pi^2}\sin\frac{\pi x}{2}\right] - \frac{\pi^2 x}{8}\left[\frac{2(2-x)}{\pi}\cos\frac{\pi x}{2} - \frac{4}{\pi^2} + \frac{4}{\pi^2}\sin\frac{\pi x}{2}\right]$$

$$= \sin\frac{\pi x}{2}\left\{1 - \frac{1}{2}(2-x) - \frac{x}{2}\right\} + \frac{x}{2} = \frac{x}{2} = R.H.S. of\ (1).$$

Hence (3) is a solution of (1).

Exercises:

Verify that the given functions are solutions of the corresponding integral equations.

1. $u(x) = 1 - x;\ \int_0^x e^{x-\xi}\, u(\xi)d\xi = x$

2. $u(x) = \frac{1}{2};\ \int_0^x \frac{u(\xi)}{\sqrt{x-\xi}}\, d\xi = \sqrt{x}$

3. $u(x) = 3; x^3 = \int_0^x (x-\xi)^2\, u(\xi)d\xi$

4. $u(x) = x - \frac{x}{6}; u(x) = x - \int_0^x \sinh(x-\xi)\, u(\xi)d\xi.$

5. $u(x) = xe^x; u(x) = e^x \sin x + 2\int_0^x \cos(x-\xi)\, u(\xi)d\xi$

6. $u(x) = \frac{x}{(1+x^2)^{\frac{5}{2}}}; u(x) = \frac{3x+2x^3}{3(1+x^2)^2} - \int_0^x \frac{3x+2x^3-\xi}{(1+x^2)^2}\, u(\xi)d\xi$

7. $u(x) = e^x(\cos e^x - e^x \sin e^x);$

$u(x) = (1 - xe^{2x})\cos 1 - e^{2x}\sin 1 + \int_0^x [1-(x-\xi)e^{2x}]\, u(\xi)d\xi$

8. $u(x) = e^x; u(x) + \lambda \int_0^x \sin x\, \xi\, u(\xi)d\xi = 1$

9. $u(x) = \cos x\,; u(x) - \int_0^\pi (x^2+\xi)\cos\xi\, u(\xi)d\xi = \sin x$

10. $u(x) = x\, e^x; u(x) - 4\int_0^\infty e^{(x+\xi)}\, u(\xi)d\xi = (x-1)e^{-x}.$

11. $u(x) = 1 - \frac{2\sin x}{1-\frac{\pi}{2}}\ ; u(x) - \int_0^\pi \cos(x \mp \xi)\, u(\xi)d\xi = 1.$

12. $u(x) = \frac{4c}{\pi}\sin x$, (c being an arbitrary constant); $u(x) - \frac{4}{\pi}\int_0^\infty \sin x \frac{\sin^2\xi}{\xi}\, u(\xi)d(\xi) = 0.$

13. $u(x) = \sqrt{x}\,;\quad u(x) - \int_0^1 K(x,\xi)\ u(\xi)d(\xi) = \sqrt{x} + \frac{x}{15}\left(4x^{\frac{3}{2}} - 7\right)$, where

$$K(x,\xi) = \begin{cases} \frac{1}{2}\, x\,(2-\xi), 0 \le x \le \xi \\ \frac{1}{2}\, \xi(2-x), \xi \le x \le 1 \end{cases}$$

14. For what value of λ, the function $v(x) = 1 + \lambda x$ is a solution of the integral equation $x = \int_0^x e^{x-\xi}\, y(\xi)d\xi$?

15. Define the following terms with an example: **(i)** Integral equation **(ii)** Linear integral equation **(iii)** Fredholm integral equation of first and second kind **(iv)** Volterra integral equation of first and second kind **(v)** Separable or degenerate kernel **(vi)** Symmetric kernel **(vii)** Eigenvalues or characteristic values **(viii)** Eigenfunctions or characteristic functions **(ix)** Singular integral **(x)** Iterated functions or kernels **(xi)** Resolvent kernel or reciprocal kernel.

5.12. Fredholm Integral Equations with Separable (or Degenerate) Kernels:

5.12.1. Solution of Fredholm integral equation of the second kind with separable or degenerate kernel.

Consider a Fredholm integral equation of the second kind:

$$u(x) = F(x) + \lambda \int_a^b K(x,t)\, u(t)\, dt \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Since kernel $K(x,t)$ is separable, we take

$$K(x,t) = \sum_{i=1}^{n} f_i(x)\, g_i(t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Using (2) , (1) reduces to $u(x) = F(x) + \lambda \int_a^b [\sum_{i=1}^{n} f_i(x)\, g_i(t)]\, u(t)\, dt$

Or, $u(x) = F(x) +$

$\lambda \sum_{i=1}^{n} f_i(x) \int_a^b g_i(t)\, u(t)\, dt \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$ [Interchanging the order of summation and integration]

Let $\int_a^b g_i(t)\, u(t)\, dt = C_i (i = 1,2, \ldots\ldots, n) \ldots\ldots\ldots\ldots\ldots\ldots (4)$

Using (4), (3) reduces to $u(x) = F(x) + \lambda \sum_{i=1}^{n} C_i f_i(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$

where constants $C_i (i = 1,2, \ldots\ldots, n)$ are to be determined in order to find solution of (1) in the form given by (5). We now proceed to evaluate $C_i's$ as follows:

Multiplying both sides of (5) successively by $g_1(x), g_2(x), \ldots\ldots, g_n(x)$ and integrating over the interval (a, b), we have

$$\int_a^b g_1(x)\, u(x)\, dx = \int_a^b g_1(x)\, F(x)\, dx \;+ \lambda \sum_{i=1}^{n} C_i \int_a^b g_1(x)\, f_i(x)\, dx \ldots\ldots\ldots\ldots\ldots (A_1)$$

$$\int_a^b g_2(x)\, u(x)\, dx = \int_a^b g_2(x)\, F(x)\, dx \;+ \lambda \sum_{i=1}^{n} C_i \int_a^b g_2(x)\, f_i(x)\, dx \ldots\ldots\ldots\ldots\ldots (A_2)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\int_a^b g_x(x)\, u(x)\, dx = \int_a^b g_n(x)\, F(x)\, dx \;+ \lambda \sum_{i=1}^{n} C_i \int_a^b g_n(x)\, f_i(x)\, dx \ldots\ldots\ldots\ldots\ldots (A_n)$$

Let, $\alpha_{ji} = \int_a^b g_j(x)\, f_i(x)\, dx\,,\ (i, j = 1,2, \dots, n) \dots \dots \dots \dots \dots \dots \dots \dots \dots (6)$

$\beta_j = \int_a^b g_j(x)\, F(x)\, dx\,,\ (j = 1,2, \dots, n) \dots \dots \dots \dots \dots \dots \dots \dots (7)$

Using (4), (6) and (7), (A_1) reduces to $C_1 = \beta_1 + \lambda \sum_{i=1}^{n} C_i\, \alpha_{1i}$

or $C_1 = \beta_1 + \lambda\, [C_1\alpha_{11} + C_2\alpha_{12} + \dots + C_n\alpha_{1n}]$

or, $(1 - \lambda\, \alpha_{11})C_1 - \lambda\, \alpha_{12}C_2 - \dots \dots - \lambda\, \alpha_{1n}C_n = \beta_1$

Similarly, we may simplify $(A_2), (A_3), \dots \dots, (A_n)$. Thus, we obtain the following system of linear equations to determine $C_1, C_2, \dots \dots, C_n$.

$$(1 - \lambda\, \alpha_{11})C_1 - \lambda\, \alpha_{12}C_2 - \dots \dots - \lambda\, \alpha_{1n}C_n = \beta_1 \dots \dots \dots \dots \dots \dots (B_1)$$

$$-\lambda\, \alpha_{21}C_1 + (1 - \lambda\, \alpha_{22})C_2 - \dots \dots - \lambda\, \alpha_{2n}C_n = \beta_2 \dots \dots \dots \dots \dots \dots (B_2)$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

And

$$-\lambda\, \alpha_{n1}C_1 - (1 - \lambda\, \alpha_{n2})C_2 - \dots \dots + (1 - \lambda\, \alpha_{nn})C_n = \beta_2 \dots \dots \dots \dots \dots \dots (B_n)$$

The determinant $D(\lambda)$ of this system is

$$D(\lambda) = \begin{vmatrix} 1 - \lambda\, \alpha_{11} & -\lambda\, \alpha_{12} & \cdots \dots & -\lambda\, \alpha_{1n} \\ -\lambda\, \alpha_{21} & 1 - \lambda\, \alpha_{22} & \dots \dots & -\lambda\, \alpha_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \dots & \vdots \\ -\lambda\, \alpha_{n1} & -\lambda\, \alpha_{n2} & \cdots & 1 - \lambda\, \alpha_{nn} \end{vmatrix} \dots \dots \dots \dots \dots (8)$$

which is a polynomial in λ of degree at most n. Again, $D(\lambda)$ is not identically zero, since, when $\lambda = 0, D(\lambda) = 1$.

To discuss the solution of (1), the following situations arise:

Situation – I: When at least one right member of the system $(B_1), \dots \dots (B_n)$ is nonzero. The following two cases arise under this situation:

(i) If $D(\lambda) \neq 0$, then a unique nonzero solution of the system $(B_1), \dots \dots (B_n)$ exists and so (1) has a unique nonzero solution given by (5).

(ii) If $D(\lambda) = 0$, then the equation $(B_1), \dots \dots (B_n)$ have either no solution or they possess infinite solutions and hence (1) has either no solution or infinite solutions.

Situation – II: When $F(x) = 0$. Then, (7) shows that $\beta_j = 0$ for $j = 1, 2, \dots\dots, n$. Hence the equation(B_1), $\dots\dots(B_n)$ reduce to a system of homogeneous linear equations.

The following two cases arise under this situation:

(i) If $D(\lambda) \neq 0$, then a unique zero solution $C_1 = C_2 = \dots\dots \dots = C_n = 0$ of the system (B_1), $\dots\dots(B_n)$ exists and so (1) has only unique zero solution $u(x) = 0$, by (5).

(ii) If $D(\lambda) = 0$, then the system (B_1), $\dots\dots(B_n)$ possess infinite nonzero solutions and so (1) has infinite nonzero solutions. Those values of λ for which $D(\lambda) = 0$ are known as the eigenvalues (or characteristic constants or values) and any nonzero solution of the homogeneous Fredholm integral equation

$$u(x) = \lambda \int_a^b K(x,t)\, u(t) dt$$

(with a convenient choice of the arbitrary constant or constants) is known as a corresponding eigenfunction (or characteristic function) of integral equation.

Situation – III: When $F(x) \neq 0$, but

$$\int_a^b g_1(x) F(x)\, dx = 0, \quad \int_a^b g_2(x) F(x)\, dx = 0 \dots\dots\dots\dots\dots, \int_a^b g_n(x) F(x)\, dx = 0,$$

i.e., $F(x)$ is orthogonal to all the functions $g_1(t), g_2(t), \dots\dots, g_n(t)$, then (7) shows that $\beta_1 = 0, \beta_2 = 0, \dots\dots, \beta_n = 0$ and hence the equations (B_1), $\dots\dots(B_n)$ reduce to a system of homogeneous linear equations.

The following two cases arise under this situation:

(i) If $D(\lambda) \neq 0$, then a unique zero $C_1 = C_2 = \dots\dots \dots = C_n = 0$ of the system (B_1), $\dots\dots(B_n)$ exists and so (1) has only unique solution $u(x) = F(x)$, by (5).

(ii) If $D(\lambda) = 0$, then the system (B_1), $\dots\dots(B_n)$ possess infinite nonzero solutions and so (1) has infinite nonzero solutions. The solutions corresponding to the eigenvalues of λ are now expressed as the sum of $F(x)$ and arbitrary multiples of eigenfunctions.

5.12.2. Illustrative Solved Examples:

Example – 1: Solve: $u(x) = e^x + \lambda \int_0^1 2e^x e^t\ u(t)dt.$

Solution: Given $\quad u(x) = e^x + \lambda \int_0^1 2e^x e^t\ u(t)dt$

$$or,\ \ u(x) = e^x + \lambda \int_0^1 2e^x e^t\ u(t)dt \ \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Let $\quad C = \int_0^1 e^t\ u(t)dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

Using (2), (1) reduces to $u(x) = e^x + 2C\lambda\, e^x = e^x(1 + 2C\lambda) \ldots\ldots\ldots\ldots\ldots\ldots (3)$

From (3), $u(t) = e^t(1 + 2C\lambda) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$

Using (4), (2) becomes $C = \int_0^1 [e^t \cdot e^t(1 + 2C\lambda)]\, dt\ \ or\ C = e^x(1 + 2C\lambda)\left[\frac{e^{2t}}{2}\right]_0^1$

or, $C = (1 + 2C\lambda)\frac{1}{2}(e^2 - 1)\ \ or\ \ C[1 - \lambda(e^2 - 1)] = \frac{1}{2}\,(e^2 - 1)$

or, $C = \frac{e^2 - 1}{2[1 - \lambda(e^2 - 1)]}$, where $\lambda \neq \frac{1}{e^2 - 1}$.

Putting this value of C in (3) , we get

$$u(x) = e^x\left[1 + 2\lambda \cdot \frac{e^2 - 1}{2[1 - \lambda(e^2 - 1)]}\right]$$

or $\quad u(x) = e^x\,\frac{1 - \lambda(e^2 - 1) + \lambda(e^2 - 1)}{1 - \lambda(e^2 - 1)}$

or $\quad u(x) = \frac{e^x}{1 - \lambda(e^2 - 1)}$, where $\lambda \neq \frac{1}{e^2 - 1}$.

which is the required solution of given integral equation.

Example – 2: Solve: $\phi(x) = \cos x + \lambda \int_0^\pi \sin x\ \phi(\xi)d\xi.$

Solution: Given $\quad \phi(x) = \cos x + \lambda \int_0^\pi \sin x\ \phi(\xi)d\xi$

$$\phi(x) = \cos x + \lambda\ \sin x \int_0^\pi \phi(\xi)\ d\xi \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Let $\quad C = \int_0^\pi \phi(\xi)\ d\xi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

Using (2), (1) becomes $\phi(x) = \cos x + \lambda\, C \sin x \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$

From (3), $\quad \phi(\xi) = \cos\xi + \lambda\, C \sin\xi \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$

Using (4), (2) reduces to $C = \int_0^\pi (\cos\xi + \lambda\, C \sin\xi)d\xi$

or, $\quad C = [\sin\xi]_0^\pi + \lambda\, C\, [-\cos\xi]_0^\pi$

or, $\quad C = 0 + \lambda\, C\, [-\cos\pi + \cos 0]$

or, $\quad C = 2\,\lambda\, C$

or, $\quad C(1 - 2\,\lambda) = 0$ so that $C = 0$, if $\lambda \neq \frac{1}{2}$.

Hence, by (3) , the required solution is $\phi(x) = \cos x,$ provided $\lambda \neq \frac{1}{2}$.

Example – 3: Solve $\phi(x) = 2x - \pi + 4\int_0^{\frac{\pi}{2}} \sin^2 x\ \phi(\xi) d\xi$

Solution: Given $\phi(x) = 2x - \pi + 4\int_0^{\frac{\pi}{2}} \sin^2 x\ \phi(\xi) d\xi$

or, $\phi(x) = 2x - \pi + 4\sin^2 x \int_0^{\frac{\pi}{2}} \phi(\xi)\ d\xi \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$

Let $C = \int_0^{\frac{\pi}{2}} \phi(\xi)\ d\xi \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

Using (2), (1) becomes $\phi(x) = 2x - \pi + 4\,C\ \sin^2 x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$

From (3), $\phi(\xi) = 2\xi - \pi + 4\,C\ \sin^2 \xi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$

Using (4), (2) becomes $C = \int_0^{\frac{\pi}{2}} (2\xi - \pi + 4\,C\ \sin^2 \xi)\ d\xi$

or, $C = [\xi^2 - \pi\xi]_0^{\frac{\pi}{2}} + 4\,C \int_0^{\frac{\pi}{2}} \frac{1-\cos 2\xi}{2}\ d\xi$

or, $C = \frac{\pi^2}{4} - \frac{\pi^2}{2} + 2C\ \left[\xi - \frac{\sin 2\xi}{2}\right]_0^{\frac{\pi}{2}}$

or, $C = -\frac{\pi^2}{4} + 2C\ \left[\frac{\pi}{2}\right]\ C\,(\pi - 1) = \frac{\pi^2}{4}$

$C = \frac{\pi^2}{4(\pi-1)}$.

Putting this value of C in (3), the required solution of given integral equation is

$\phi(x) = 2x - \pi + \frac{\pi^2 \sin^2 x}{\pi - 1}$.

Example – 4: Solve: $g(s) = f(s) + \lambda \int_0^1 st\ g(t)\ dt.$

Solution: Given $g(s) = f(s) + \lambda \int_0^1 st\ g(t)\ dt$

or, $g(s) = f(s) + \lambda\, s \int_0^1 t\ g(t)\ dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$

Let $C = \int_0^1 t\ g(t)\ dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

Then (1) reduces to $g(s) = f(s) + \lambda\, Cs \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$

From (3), $g(t) = f(t) + \lambda\, Ct \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$

Using (4), (2) reduces to $C = \int_0^1 t\,[f(t) + \lambda\, Ct\,]\,dt \quad or \quad C = \int_0^1 t\, f(t)\, dt + \lambda\, C\ \left[\frac{t^3}{3}\right]_0^1$

or, $C = \int_0^1 t\, f(t)\, dt + \frac{\lambda C}{3}$

or, $C\left(1 - \frac{\lambda}{3}\right) = C = \int_0^1 t\, f(t)\, dt$

or, $C = \frac{3}{3-\lambda}\int_0^1 tf(t)\, dt$ where $\lambda \neq 3$.

Putting this value of C in (3), the required solution is $g(s) = f(s) + \frac{3\lambda s}{3-\lambda} C = \int_0^1 t\, f(t)\ dt.$

Example – 5: Invert the integral equation $g(s) = f(s) + \lambda \int_0^{2\pi} (\sin s \cos t) g(t)\, dt$.:

OR Find the solution of the integral equation $g(s) = f(s) + \lambda \int_0^{2\pi} (\sin s \cos t) g(t)\, dt$.

Solution: Given $g(s) = f(s) + \lambda \int_0^{2\pi} (\sin s \cos t) g(t)\, dt$

or, $$g(s) = f(s) + \lambda \sin s \int_0^{2\pi} \cos t\, g(t)\, dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Let $$C = \int_0^{2\pi} \cos t\, g(t)\, dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Then (2) reduces to $$g(s) = f(s) + \lambda C \sin s \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

From (3), $$g(t) = f(t) + \lambda C \sin t \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Using (4), (2) reduces to $C = \int_0^{2\pi} \cos t\, [f(t) + \lambda C \sin t]\, dt$

or, $$C = \int_0^{2\pi} \cos t\ f(t)\, dt + \frac{\lambda C}{2} \int_0^{2\pi} \sin 2t\ dt$$

or, $$C = \int_0^{2\pi} \cos t\ f(t)\, dt + \frac{\lambda C}{2} \left[-\frac{\cos 2t}{2} \right]_0^{2\pi}$$

or, $$C = \int_0^{2\pi} \cos t\ f(t)\, dt + \frac{\lambda C}{2} \left[-\frac{1}{2} + \frac{1}{2} \right]$$

or, $$C = \int_0^{2\pi} \cos t\ f(t)\, dt$$

Putting this value of C in (3) , the required solution is

$$g(s) = f(s) + \lambda \sin s \int_0^{2\pi} f(t) \cos t\, dt$$

or, $$g(s) = f(s) + \lambda \int_0^{2\pi} (\sin s \cos t) f(t)\, dt.$$

Example – 6: Solve the Fredholm integral equation of the second kind:

$$g(s) = s + \lambda \int_0^1 (st^2 + s^2 t) g(t)\, dt.$$

Solution: Given $g(s) = s + \lambda \int_0^1 (st^2 + s^2 t) g(t)\, dt$

or, $$g(s) = s + \lambda s \int_0^1 t^2 g(t)\, dt + \lambda s^2 \int_0^1 t\, g(t)\, dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

or, $$C_1 = \int_0^1 t^2 g(t)\, dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

or, $$C_2 = \int_0^1 t\, g(t)\, dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Using (2) and (3), (1) reduces to $g(s) = s + \lambda C_1 s + \lambda C_2 s^2 \ldots\ldots\ldots\ldots\ldots\ldots (4)$

From (4), $$g(t) = t + \lambda C_1 t + \lambda C_2 t^2 \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Using (5), (2) reduces to

$$C_1 = \int_0^1 t^2 (t + \lambda C_1 t + \lambda C_2 t^2)\, dt$$

or, $$C_1 = \left[\frac{t^4}{4} + \frac{\lambda C_1 t^4}{4} + \frac{\lambda C_2 t^5}{5} \right]_0^1$$

or, $C_1 = \frac{1}{4} + \frac{\lambda C_1}{4} + \frac{\lambda C_2}{5}$

or, $(20 - 5\lambda)C_1 - 4\lambda\, C_2 = 5 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .(6)$

Next, using (5), (2) reduces to $C_2 = \int_0^1 t(t + \lambda\, C_1\, t + \lambda\, C_2 t^2)\, dt$

or, $C_2 = \left[\frac{t^3}{3} + \frac{\lambda C_1 t^3}{3} + \frac{\lambda C_2 t^4}{4}\right]_0^1$

or, $C_2 = \frac{1}{3} + \frac{\lambda C_1}{3} + \frac{\lambda C_2}{4}$

or, $-4\lambda\, C_1 + (12 - 3\lambda)C_2 = 4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .(7)$

Solving (6) and (7) for C_1 and C_2, and we get

$$C_1 = \frac{60 + \lambda}{240 - 120\lambda - \lambda^2}, \qquad C_2 = \frac{80}{240 - 120\lambda - \lambda^2}$$

Putting these values of C_1 and C_2 in (4), the required solution is

$$g(s) = s + \frac{\lambda s(60 + \lambda)}{240 - 120\lambda - \lambda^2} + \frac{80\,\lambda\, s^2}{240 - 120\lambda - \lambda^2}$$

or, $g(s) = \frac{(240-60\lambda)s + 80\,\lambda s^2}{240 - 120\lambda - \lambda^2}$

Example – 7: Solve: $y(x) = 1 + \int_0^1 (1 + e^{x+\xi}) y(\xi)\, d\xi.$

Solution: Given $y(x) = 1 + \int_0^1 (1 + e^{x+\xi}) y(\xi)\, d\xi$

or, $y(x) = 1 + \int_0^1 y(\xi)\, d\xi + \int_0^1 e^{\xi} y(\xi)\, d\xi \ldots\ldots\ldots\ldots\ldots .(1)$

or, $C_1 = \int_0^1 y(\xi)\, d\xi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

and $C_2 = \int_0^1 e^{\xi} y(\xi)\, d\xi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$

Using (2) and (3), (1) reduces to $y(x) = 1 + C_1 + C_2 e^x \ldots\ldots\ldots\ldots\ldots .(4)$

From (4), $y(\xi) = 1 + C_1 + C_2 e^{\xi} \ldots\ldots\ldots\ldots\ldots .(5)$

Using (5), (2) reduces to

$$C_1 = \int_0^1 (1 + C_1 + C_2 e^{\xi})\, d\xi \;\; or \;\; C_1 = \left[1 + C_1 \xi + C_2 e^{\xi}\right]_0^1$$

or, $C_1 = 1 + C_1 + C_2(e - 1) \;\; or \;\; C_2 = -\frac{1}{e-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots .(6)$

Using (5), (3) reduces to $C_2 = \int_0^1 e^{\xi}(1 + C_1 + C_2 e^{\xi})\, d\xi$

or, $C_2 = e - 1 + C_1(e - 1) + \frac{C_2}{2}(e^2 - 1)$

or, $-\frac{1}{e-1} = e - 1 + C_1(e - 1) - \frac{e^2-1}{2(e-1)}$, using (6)

or, $C_1(e - 1) = -\frac{1}{e-1} - \frac{(e-1)}{2}$

or, $C_1 = -\frac{e^2-2e+3}{2(e-1)^2}$ ………………….(7)

Using (6) and (7) in (4), the required solution is

$$y(x) = 1 - \frac{e^2 - 2e + 3}{2(e-1)^2} - \frac{e^x}{e-1} \quad or \quad y(x) = \frac{e^2 - 2e - 1}{2(e-1)^2} - \frac{e^x}{e-1}$$

or, $y(x) = \frac{e^2-2e-1-2e^x(e-1)}{2(e-1)^2}$

Example – 8: Solve: $\phi(x) = (1 + x^2) + \int_{-1}^{1}(xt + x^2t^2)\,\phi(t)dt.$

Solution: Given $\phi(x) = (1 + x^2) + \int_{-1}^{1}(xt + x^2t^2)\,\phi(t)dt$

or, $\phi(x) = (1 + x^2) + x\int_{-1}^{1} t\,\phi(t)dt + x^2\int_{-1}^{1} t^2\,\phi(t)dt$ ………………..(1)

Let $C_1 = \int_{-1}^{1} t\,\phi(t)\,dt$ ……………………………………(2)

and $C_2 = \int_{-1}^{1} t^2\,\phi(t)\,dt$ …………………………………(3)

Using 92) and (3), (1) reduces to $\phi(x) = (1 + x^2) + C_1x + C_2x^2$ …………………(4)

From (4), $\phi(x) = (1 + t^2) + C_1t + C_2t^2$ …………………(5)

Using (5), (2) reduces to

$$C_1 = \int_{-1}^{1} t\,[(1+t)^2 + C_1\,t + C_2t]dt \quad or \quad C_1 = \int_{-1}^{1} t\,[1 + (2 + C_1)t + (1 + C_2)t^2]dt$$

or, $C_1 = \left[\frac{t^2}{2}\right]_{-1}^{1} + (2 + C_1)\left[\frac{t^3}{3}\right]_{-1}^{1} + (1 + C_2)\left[\frac{t^4}{4}\right]_{-1}^{1}$

or, $C_1 = (2 + C_1)\frac{2}{3}$ so that $C_1 = 4$ ………………………….(6)

Using (5), (3) reduces to $C_2 = \int_{-1}^{1} t^2\,[(1+t)^2 + C_1\,t + C_2t^2]dt$

or, $C_2 = \int_{-1}^{1} t^2\,[1 + (2 + C_1)t + (1 + C_2)t^2]dt$

or, $C_2 = \left[\frac{t^3}{3}\right]_{-1}^{1} + (2 + C_1)\left[\frac{t^4}{4}\right]_{-1}^{1} + (1 + C_2)\left[\frac{t^5}{5}\right]_{-1}^{1}$

or, $C_2 = \frac{2}{3} + (1 + C_2)\frac{2}{5} \quad or \quad C_2 = \frac{16}{9}$ …………………………(7)

Using (6) and (7), (4) gives the required solution

$$\phi(x) = (1 + x^2) + 4x + \frac{16}{9}x^2 \quad or \quad \phi(x) = 1 + 6x + \frac{25}{9}x^2$$

Example – 9: Solve: $\phi(x) = \cos x + \lambda \int_0^{\pi} \sin(x-t)\,\phi(t)dt$.

Solution: Given $\phi(x) = \cos x + \lambda \int_0^{\pi} \sin(x-t)\,\phi(t)dt$

or, $\phi(x) = \cos x + \lambda \int_0^{\pi} (\sin x \cos t - \cos x \sin t)\,\phi(t)dt$

or, $\phi(x) = \cos x + \lambda \sin x \int_0^{\pi} \cos t\,\phi(t)dt - \lambda \cos x \int_0^{\pi} \sin t\,\phi(t)dt \ldots\ldots\ldots\ldots (1)$

Let $C_1 = \int_0^{\pi} \cos t\,\phi(t)dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

And $C_2 = \int_0^{\pi} \sin t\,\phi(t)dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$

Using (2) and (3), (1) reduces to $\phi(x) = \cos x + \lambda C_1 \sin x - \lambda C_2 \cos x \ldots\ldots\ldots\ldots\ldots\ldots (4)$

From (4), $\phi(t) = \cos t + \lambda C_1 \sin t - \lambda C_2 \cos t \ldots\ldots\ldots\ldots\ldots\ldots (5)$

Using (5), (2) reduces to

$$C_1 = \int_0^{\pi} \cos t\,(\cos t + \lambda C_1 \sin t - \lambda C_2 \cos t)dt$$

or, $C_1 = \int_0^{\pi} [(1-\lambda C_2)\cos^2 t\,dt - \lambda C_1 \sin 2t]\,dt$

or, $C_1 = (1-\lambda C_2)\int_0^{\pi} \frac{1+\cos 2t}{2}dt + \frac{\lambda C_1}{2}\int_0^{\pi} \sin 2t\,dt$

or, $C_1 = \frac{(1-\lambda C_2)}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{\pi} + \frac{\lambda C_1}{2}\left[-\frac{\cos 2t}{2}\right]_0^{\pi}$

or, $C_1 = \frac{1-\lambda C_2}{2}\pi \quad or \quad 2C_1 + \lambda\pi C_2 = \pi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$

Using (5), (3) reduces to

$$C_2 = \int_0^{\pi} \sin t\,(\cos t + \lambda C_1 \sin t - \lambda C_2 \cos t)dt$$

or, $C_2 = \frac{1-\lambda C_2}{2}\int_0^{\pi} \sin 2t\,dt + \frac{\lambda C_1}{2}\int_0^{\pi}(1-\cos 2t)\,dt$

or, $C_2 = \frac{(1-\lambda C_2)}{2}\left[-\frac{\cos 2t}{2}\right]_0^{\pi} + \frac{\lambda C_1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{\pi}$

or, $C_2 = \frac{\lambda C_1 \pi}{2} \ldots\ldots\ldots\ldots\ldots\ldots (7)$

Solving (6) and (6) for C_1 and C_2, we get

$$C_1 = \frac{2\pi}{4+\lambda^2\pi^2}\ ,\quad C_2 = \lambda\frac{\pi^2}{4+\lambda^2\pi^2}$$

Putting these values of C_1 and C_2 in (4), the required solution is

$$\phi(x) = \cos x + \frac{2\pi\lambda \sin x}{4+\lambda^2\pi^2} - \frac{\lambda^2\pi^2 \cos x}{4+\lambda^2\pi^2}$$

or, $\phi(x) = \cos x\left[1 - \frac{\lambda^2\pi^2 \cos x}{4+\lambda^2\pi^2}\right] + \frac{2\pi\lambda \sin x}{4+\lambda^2\pi^2}$

or, $\phi(x) = \frac{4\cos x + 2\,\pi\lambda \sin x}{4+\lambda^2\pi^2}$

Example – 10: Solve: $g(s) = f(s) + \lambda \int_0^1 (s+t)\ g(t)dt$

Solution: Given $g(s) = f(s) + \lambda \int_0^1 (s+t)\ g(t)dt$

or, $g(s) = f(s) + \lambda\, s \int_0^1 g(t)\, dt + \lambda \int_0^1 t\, g(t)\, dt \dots\dots\dots\dots\dots\dots\dots (1)$

Let $C_1 = \int_0^1 g(t)\, dt \dots\dots\dots\dots\dots\dots\dots\dots (2)$

And $C_2 = \int_0^1 t\, g(t)\, dt \dots\dots\dots\dots\dots\dots\dots\dots (3)$

Using (2) and (3), (1) reduces to $g(s) = f(s) + \lambda\, s\, C_1 + \lambda\, C_2 \dots\dots\dots\dots\dots\dots\dots (4)$

From (4), $g(t) = f(t) + \lambda\, t\, C_1 + \lambda\, C_2 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (5)$

Using (4), (2) reduces to

$$C_1 = \int_0^1 [f(t) + \lambda\, t\, C_1 + \lambda\, C_2]\, dt \quad or \quad C_1 = \int_0^1 f(t)\, dt + \lambda\, C_1 \left[\frac{t^2}{2}\right]_0^1 + \lambda\, C_2 [t]_0^1$$

or, $C_1 = f_1 + \frac{\lambda C_1}{2} + \lambda C_2 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (6)$

Where $f_1 = \int_0^1 f(t)\, dt \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (7)$

Using (4), (3) reduces to

$$C_2 = \int_0^1 t\,[f(t) + \lambda\, t\, C_1 + \lambda\, C_2]\, dt \quad or \quad C_2 = \int_0^1 t\, f(t)\, dt + \lambda\, C_1 \left[\frac{t^3}{3}\right]_0^1 + \lambda\, C_2 \left[\frac{t^2}{2}\right]_0^1$$

$or,\ C_2 = f_2 + \frac{\lambda C_1}{3} + \frac{\lambda C_2}{2} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (8)$

where $f_2 = \int_0^1 t\, f(t)\, dt \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (9)$

Re – writing (6) and (8), we have $(2-\lambda)C_1 - 2\,\lambda\, C_2 = 2\, f_1 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (10)$

And $-2\,\lambda\, C_1 + 3\,(2-\lambda)C_2 = 6\, f_2 \dots\dots\dots\dots\dots\dots\dots\dots (11)$

Solving (10) and (11) for C_1 and C_2, we get

$C_1 = \frac{6(\lambda-2)f_1 - 12\,\lambda\, f_2}{\lambda^2+12\lambda-12},\ C_1 = \frac{-4\,\lambda\, f_1 - 6\,(\lambda-2)\, f_2}{\lambda^2+12\lambda-12}$

Putting these values of C_1 and C_2 in (4), the required solution is

$$g(s) = f(s) + \frac{\lambda\, s\{6(\lambda-2)f_1 - 12\lambda\, f_2\}}{\lambda^2+12\lambda-12} + \lambda\, \frac{-4\,\lambda\, f_1 - 6\,(\lambda-2)\, f_2}{\lambda^2+12\lambda-12}$$

or, $g(s) = f(s) + \lambda\, \frac{f_1\{6(\lambda-2)-4\lambda\} + f_2\{6(\lambda-2) - 12\,\lambda s\}}{\lambda^2+12\lambda-12}$

or, $(s) = f(s) + \frac{\lambda}{\lambda^2+12\lambda-12}\left[\{6s\,(\lambda-2) - 4\lambda\}\int_0^1 f(t)\, dt + \{6(\lambda-2) - 12\lambda s\}\int_0^1 t\, f(t)\, dt\right]$

or, $(s) = f(s) + \frac{\lambda}{\lambda^2+12\lambda-12}\left[\int_0^1 \{6s\,(\lambda-2) - 4\lambda\}\, f(t)\, dt + \int_0^1 \{6(\lambda-2) - 12\lambda s\}\, t\, f(t)\, dt\right]$

or, $g(s) = f(s) + \frac{\lambda}{\lambda^2+12\lambda-12}\left[\int_0^1 \{6\,(\lambda-2)(s+t) - 12\,\lambda\, st - 4\lambda\} f(t)\ dt\right]$

or, $g(s) = f(s) + \lambda \int_0^1 \frac{6\,(\lambda-2)(s+t) - 12\,\lambda\, st - 4\lambda}{\lambda^2+12\lambda-12} f(t)\ dt$

Example – 11: Solve: $g(s) = f(s) + \lambda \int_{-1}^{1}(st + s^2t^2)g(t)dt$.Find its resolvent kernel also.

Solution: Given $g(s) = f(s) + \lambda \int_{-1}^{1}(st + s^2t^2)g(t)dt$

or, $g(s) = f(s) + \lambda \int_{-1}^{1} t\, g(t)dt + \lambda s^2 \int_{-1}^{1} t^2\, g(t)dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$

Let $C_1 = \int_{-1}^{1} t\, g(t)dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

And $C_2 = \int_{-1}^{1} t^2\, g(t)dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$

Using (2) and (3), (1) reduces to $g(s) = f(s) + \lambda C_1 s + \lambda C_2 s^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(4)$

From (4), $g(t) = f(t) + \lambda C_1 t + \lambda C_2 t^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(5)$

Using (5), (2) reduces to $C_1 = \int_{-1}^{1} t\,[f(t) + \lambda C_1\, t + \lambda C_2 t^2]\, dt$

$C_1 = \int_{-1}^{1} t\, f(t)dt + \lambda C_1 \left[\frac{t^3}{3}\right]_{-1}^{1} + \lambda C_2 \left[\frac{t^4}{4}\right]_{-1}^{1}$

or, $C_1 = \int_{-1}^{1} t\, f(t) + \frac{2\lambda C_1}{3}$

or, $C_1 \left(1 - \frac{2\lambda}{3}\right) = \int_{-1}^{1} t\, f(t)\, dt$

or, $C_1 = \frac{3}{3-2\lambda} \int_{-1}^{1} t\, f(t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$

Using (5), (3) reduces to $C_2 = \int_{-1}^{1} t^2[f(t) + \lambda C_1\, t + \lambda C_2 t^2]\, dt$

or, $C_2 = \int_{-1}^{1} t^2\, f(t)dt + \lambda C_1 \left[\frac{t^4}{4}\right]_{-1}^{1} + \lambda C_2 \left[\frac{t^5}{5}\right]_{-1}^{1}$

or, $C_2 = \int_{-1}^{1} t^2\, f(t)dt + \frac{2\lambda C_2}{5}$

or, $C_2 \left(1 - \frac{2\lambda}{5}\right) = \int_{-1}^{1} t^2\, f(t)dt$

or, $C_2 = \frac{5}{5-2\lambda} \int_{-1}^{1} t^2\, f(t)dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$

Using (6) and (7) in (4), the required solution is

$g(s) = f(s) + \frac{3\lambda s}{3-2\lambda} \int_{-1}^{1} t\, f(t)dt + \frac{5\lambda s^2}{5-2\lambda} \int_{-1}^{1} t^2\, t(t)dt$

or, $C_2 = (1 + \lambda C_1) \int_{-\pi}^{\pi} t^3\, dt + \lambda C_2 \int_{-\pi}^{\pi} t^2 \sin t\, dt + \lambda C_3 \int_{-\pi}^{\pi} t^2 \cos t\, dt$

or, $C_2 = 2\lambda C_3 \int_0^{\pi} t^2 \cos t\, dt$

or, $C_2 = 2\lambda C_3 \left[[t^2 \sin t]_0^{\pi} - \int_0^{\pi} 2t \sin t\, dt\right]$ [Integrating by parts]

or, $C_2 = -4\lambda C_3 \int_0^{\pi} t \sin t\, dt$

or, $C_2 = -4\lambda C_3 \left[[(t)(-\cos t)]_0^{\pi} - \int_0^{\pi} (-\cos t) dt\right]$

or, $C_2 = -4\lambda C_3 \left[\pi - \int_0^{\pi} \cos t\, dt\right]$

or, $C_2 = -4\lambda C_3 - -4\lambda \pi C_3 [\sin t]_0^{\pi}$

or, $,C_2 + 4\,\lambda\pi C_3 = 0$ ……………………………..(8)

Using (6) , (4) reduces to $C_3 = \int_{-\pi}^{\pi} \sin t\,(t + \lambda\, C_1\, t + \lambda C_2 \sin t + \lambda C_3 \cos t)\ dt$

or, $C_3 = (1+\lambda C_1)\int_{-\pi}^{\pi} t \sin t\ dt + \lambda C_2 \int_{-\pi}^{\pi} \sin^2 t\ dt + \lambda C_3 \int_{-\pi}^{\pi} \sin t \cos t\ dt$

or, $C_3 = 2(1+\lambda C_1)\int_{0}^{\pi} t \sin t\ dt + 2\lambda C_2 \int_{0}^{\pi} \sin^2 t\ dt + 0$

or, $C_3 = 2(1+\lambda C_1)\left[[t(-\cos t)]_0^{\pi} - \int_0^{\pi}(-\cos t)dt\right] + 2\,\lambda C_2 \int_0^{\pi} \frac{1-\cos 2t}{2} dt$

or, $C_3 = 2(1+\lambda C_1)[\,\pi + [\sin t]_0^{\pi}] + \lambda\, C_2 \left[t - \frac{\sin 2t}{2}\right]_0^{\pi}$

or, $C_3 = 2(1+\lambda C_1)\pi + \lambda\, C_2\pi - 2\lambda\pi C_1 - \lambda\pi C_2 + C_3 = 2\pi$ ………………………………(9)

Solving (7), (8) and (9) for C_1, C_2 and C_3, we have

$C_1 = \frac{2\pi^2\lambda}{1+2\lambda^2\pi^2}$, $C_2 = \frac{-8\pi^2\lambda}{1+2\lambda^2\pi^2}$, $C_3 = \frac{2\pi}{1+2\lambda^2\pi^2}$

Putting these values of C_1, C_2 and C_3 in (5), the required solution is

$\phi(x) = x + \frac{2\pi^2\lambda^2 x}{1+2\lambda^2\pi^2} - \frac{8\pi^2\lambda^2 \sin x}{1+2\lambda^2\pi^2} + \frac{2\pi\lambda \cos x}{1+2\lambda^2\pi^2}$

$\phi(x) = x + \frac{2\pi\lambda}{1+2\lambda^2\pi^2}(\lambda\pi x - 4\lambda\pi \sin x + \cos x)$

or, $g(s) = f(s) + \lambda \int_{-1}^{1}\left\{\frac{3st}{3-2\lambda} + \frac{5s^2t^2}{5-2\lambda}\right\} f(t)\ dt$

The required resolvent kernel $R(s,t{:}\,\lambda)$ is given by

$R(s,t{:}\,\lambda) = \frac{3st}{3-2\lambda} + \frac{5s^2t^2}{5-2\lambda}$

Example – 12: Solve the integral equation $\phi(x) - \lambda\int_{-\pi}^{\pi}(x \cos t + t^2 \sin x + \cos x \sin t)\,\phi(t)dt = x$.

Solution: Given $\phi(x) = x + \lambda\int_{-\pi}^{\pi}(x\cos t + t^2 \sin x + \cos x \sin t)\,\phi(t)dt = x$

or, $\phi(x) = x + \lambda x\int_{-\pi}^{\pi} \cos t\, \phi(t)dt + \lambda \sin x \int_{-\pi}^{\pi} t^2\, \phi(x)dt +$

$\lambda \cos x \int_{-\pi}^{\pi} \sin t\ \phi(t)dt$ ……. (1)

Let $C_1 = \int_{-\pi}^{\pi} \cos t\ \phi(t)dt$ ………………………………………(2)

$C_2 = \int_{-\pi}^{\pi} t^2\ \phi(t)dt$ ………………………………………(3)

And $C_3 = \int_{-\pi}^{\pi} \sin t\ \phi(t)dt$ ………………………………………(4)

Using (2), (3) and (4), (1) reduces to

$$\phi(x) = x + \lambda\, C_1\, x + \lambda\, C_2 \sin x + \lambda\, C_3 \cos x \text{ ……………………….(5)}$$

From (5), $\phi(t) = t + \lambda\, C_1\, t + \lambda\, C_2 \sin t + \lambda\, C_3 \cos t$ ……………………….(6)

Using (6), (2) reduces to

$C_1 = \int_{-\pi}^{\pi} \cos t\ (t + \lambda\, C_1\, t + \lambda\, C_2 \sin t + \lambda\, C_3 \cos t\,)dt$

$C_1 = (1+\lambda C_1)\int_{-\pi}^{\pi} t\ \cos t\, dt + \lambda\, C_2 \int_{-\pi}^{\pi} \sin t \cos t\, dt + \lambda C_3 \int_{-\pi}^{\pi} \cos^2 t\ \ dt$

or, $C_1 = 0 + 0 + 2\lambda C_3 \int_0^{\pi} \cos^2 t\ \ dt$

[$\because$ $t \cos t$ and $\sin t \cos t$are odd functions whereas $\cos^2 t$ in an even function]

or, $C_1 = 2\lambda C_3 \int_0^{\pi} \frac{1+\cos 2t}{2}\ dt$

or, $C_1 = \lambda\, C_3 \left[t + \frac{\sin 2t}{2}\right]_0^{\pi}$

$\therefore\ C_1 - \lambda\pi C_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(7)$

Using (6), (3) reduces to $C_2 = \int_{-\pi}^{\pi} t^2\ \ (t + \lambda\, C_1\, t +\ \lambda\, C_2 \sin t +\ \lambda\, C_3 \cos t\ )dt$

Example – 13: Show that the integral equation $g(s) = f(s) + \frac{1}{\pi}\int_0^{2\pi} \sin(s+t)\, g(t)dt$

Possesses on solution for $f(s) = s,$ but that it possesses infinitely many solutions when $f(s) =\ 1.$

Solution: Given $g(s) = f(s) + \frac{1}{\pi}\int_0^{2\pi} \sin(s+t)\, g(t)dt$

or, $g(s) = f(s) + \frac{1}{\pi}\int_0^{2\pi}(\sin x \cos t + \cos s \sin t)\, g(t)dt$

or, $g(s) = f(s) + \frac{\sin s}{\pi}\int_0^{2\pi} \cos t\, g(t)dt + \frac{\cos s}{\pi}\int_0^{2\pi} \sin t\, g(t)dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(1)$

Let $C_1 =\ \int_0^{2\pi} \cos t\, g(t)dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$

And $C_2 =\ \int_0^{2\pi} \sin t\, g(t)dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$

Using (2) and (3), (1) reduces to $g(s) =\ f(s) + \frac{C_1 \sin s}{\pi} + \frac{C_2 \cos s}{\pi} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$

We now discuss two particular cases as mentioned in the problem.

Case – I: Let $f(s) = s$. Then (4) reduces to

$g(s) =\ s + \frac{C_1 \sin s}{\pi} + \frac{C_2 \cos s}{\pi} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$

From (5), $g(t) =\ t + \frac{C_1 \sin t}{\pi} + \frac{C_2 \cos t}{\pi} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$

Using (6), (2) becomes

$C_1 =\ \int_0^{2\pi} \cos t \left(t + \frac{C_1 \sin t}{\pi} + \frac{C_2 \cos t}{\pi}\right) dt$

or,$C_1 =\ \int_0^{2\pi} t\ \ \cos t\, dt + \frac{C_1}{2\pi}\int_0^{2\pi} \sin 2t\ \ dt + \frac{C_2}{2\pi}\int_0^{2\pi}(1 +\ \cos 2t)\, dt$

or, $C_1 =\ [t \sin t]_0^{2\pi} - \int_0^{2\pi} \sin t\, dt + \frac{C_1}{2\pi}\left[-\frac{\cos 2t}{2}\right]_0^{2\pi} + \frac{C_2}{2\pi}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$

or, $C_1 =\ [-\cos t]_0^{2\pi} + \frac{C_2}{2\pi}[2\pi + 0]$

or, $C_1 -\ C_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$

Again using (6), (3) becomes

$C_2 = \int_0^{2\pi} \sin t \left(t + \frac{C_1 \sin t}{\pi} + \frac{C_2 \cos t}{\pi}\right) dt$

or, $C_2 = \int_0^{2\pi} t \ \sin t \, dt + \frac{C_1}{2\pi}\int_0^{2\pi}(1 - \cos 2t)\, dt + \frac{C_2}{2\pi}\int_0^{2\pi} \sin 2t \, dt$

or, $C_2 = [-t\cos t]_0^{2\pi} - \int_0^{2\pi}(-\cos t)dt + \frac{C_1}{2\pi}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi} + \frac{C_2}{2\pi}\left[-\frac{\cos 2t}{2}\right]_0^{2\pi}$

or, $C_2 = -2\pi + [\sin t]_0^{2\pi} + \frac{C_1}{2\pi}[2\pi + 0]$

or, $C_1 - C_2 = 2\pi$ … … … … … … … … … … … … … … (8)

The system of equations (7) and (8) is inconsistent and so it possesses no solution.

Hence C_1 and C_2 cannot be determined and so (5) shows that the given integral equation possesses no solution when $f(s) = s$.

Case – II: Let $f(s) = 1$. Then (4) reduces to

$g(s) = 1 + \frac{C_1 \sin s}{\pi} + \frac{C_2 \cos s}{\pi}$ … … … … … … … (9)

From (9), $g(t) = 1 + \frac{C_1 \sin t}{\pi} + \frac{C_2 \cos t}{\pi}$ … … … … … … … (10)

Using (6), (2) becomes

$C_1 = \int_0^{2\pi} \cos t \left(1 + \frac{C_1 \sin t}{\pi} + \frac{C_2 \cos t}{\pi}\right) dt$

or, $C_1 = \int_0^{2\pi} \cos t \, dt + \frac{C_1}{2\pi}\int_0^{2\pi} \sin 2t \ dt + \frac{C_2}{2\pi}\int_0^{2\pi}(1 + \cos 2t)\, dt$

or, $C_1 = [\sin t]_0^{2\pi} + \frac{C_1}{2\pi}\left[-\frac{\cos 2t}{2}\right]_0^{2\pi} + \frac{C_2}{2\pi}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$

or, $C_1 = 0 + 0 + \frac{C_2}{2\pi}[2\pi + 0]$

or, $C_1 = C_2$ … … … … … … … … … … … … … … (11)

Again using (6), (3) becomes

$C_2 = \int_0^{2\pi} \sin t \left(1 + \frac{C_1 \sin t}{\pi} + \frac{C_2 \cos t}{\pi}\right) dt$

or, $C_2 = \int_0^{2\pi} \sin t \, dt + \frac{C_1}{2\pi}\int_0^{2\pi}(1 - \cos 2t)\, dt + \frac{C_2}{2\pi}\int_0^{2\pi} \sin 2t \, dt$

or, $C_2 = [-\cos t]_0^{2\pi} + \frac{C_1}{2\pi}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi} + \frac{C_2}{2\pi}\left[-\frac{\cos 2t}{2}\right]_0^{2\pi}$

or, $C_2 = 0 + \frac{C_1}{2\pi}[2\pi + 0] + 0$

or, $C_1 = C_2$ … … … … … … … … … … … … … … (12)

From (11) and (12), we see that $C_1 = C_2 = C'$, (say). Here C' is an arbitrary constant. Thus, the system (11) – (12) has infinite number of solutions $C_1 = C' and\ C_2 = C'$. Putting these values in (9), the required solution of given integral equation is

$$g(s) = 1 + \frac{C'}{\pi}(\sin s + \cos s) \text{ or } g(s) = 1 + C(\sin s + \cos s)$$

where $C\left(=\frac{C'}{\pi}\right)$ is another arbitrary constant. Since C is an arbitrary constant, we have infinitely many solutions of (1) when $f(s) = 1$.

Exercises

1. Solve the following integral equations:

(i)$y(x) = \tan x + \int_{-1}^{1} e^{\sin^{-1} x}\ y(t)dt$.

(ii)$g(s) = \sin s + \lambda \int_0^{\frac{\pi}{2}} \sin s \cos t\ g(t)dt$.

(iii)$\phi(x) = \sec x \tan x - \lambda \int_0^1 \phi(\xi)\ d\xi$.

(iv)$y(x) = \frac{1}{\sqrt{1-x^2}} + \lambda \int_0^1 \cos^{-1} t\ y(t)dt$.

(v)$\phi(x) = \sec^2 x + \lambda \int_0^1 \phi(t)\ dt$.

(vi)$\phi(x) - \lambda \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan t\ \phi(t)dt = \cot x$.

(vii)$\phi(x) - \lambda \int_0^1 \cos(q\ I_n t)\ \phi(t)dt = 1$. [Here $I_n t = \log_e t$]

(viii)$\phi(x) - \lambda \int_0^1 \left(I_n \frac{1}{t}\right)^p\ \phi(t)dt = 1\ \ (p > -1)$.

2. Solve the following integral equations:

(i) $\phi(x) - \lambda \int_0^1 (x\ I_n t - t\ I_n x)\ \phi(t)dt = \frac{6}{5}(1 - 4x)$.

(ii) $\phi(x) - \lambda \int_0^{2\pi} |\pi - t| \sin x\ \phi(t)dt = x$.

(iii) $\phi(x) = \lambda \int_0^1 (1 + x + t)\ \phi(t)dt$.

(iv) $\phi(x) = x + \lambda \int_0^{\pi} (1 + \sin x \sin t)\ \phi(t)dt$.

(v) $\phi(x) - \lambda \int_0^1 (4xt - x^3)\ \phi(t)dt = x$.

(vi) $\phi(x) - \lambda \int_0^1 (4xt - x^2)\ \phi(t)dt = x$.

3. Express the solution of the integral equation

$y(x) = F(x) + \lambda \int_0^1 (1 - 3x\xi)\ y(\xi)d\xi$ in the form $y(x) = F(x) + \lambda \int_0^1 \Gamma(x, \xi, \lambda)\ F(\xi)d\xi$ when $\lambda \neq \pm 2$.

4. (a) Show that the characteristics values of λ for the equation

$$y(x) = \lambda \int_0^{2\pi} \sin(x+\xi)\ y(\xi)d\xi$$

are $\lambda_1 = \frac{1}{\pi}$ and $\lambda_2 = -\frac{1}{\pi}$, with corresponding characteristic functions of the form $y_1(x) = \sin x + \cos x$ and $y_2(x) = \sin x - \cos x$.

(b) Obtain the most general solution of the equation

$$y(x) = F(x) + \lambda \int_0^{2\pi} \sin(x+\xi)\ y(\xi)d\xi$$

when $F(x) = x$ and when $F(x) = 1$, under the assumption that $\lambda \neq \pm\frac{1}{\pi}$.

(c) Prove that the equation

$$y(x) = F(x) + \frac{1}{\pi} \int_0^{2\pi} \sin(x+\xi)\ y(\xi)d\xi$$

possesses no solution when $F(x) = 1$, but that it possesses infinitely many solutions when $F(x) = 1$. Determine all such solutions.

5. Consider the equation

$$y(x) = F(x) + \lambda \int_0^{2\pi} \cos(x+\xi)\ y(\xi)d\xi$$

(a) Determine the characteristic values of $\lambda$and the characteristic functions.

(b) Express the solution in the form $y(x) = F(x) + \lambda \int_0^1 \Gamma(x,\ \xi, \lambda)\ F(\xi)d\xi$ when λ is not characteristic value.

(c) Obtain the general solution (when it exists) if $F(x) = \sin x$, considering all possible cases.

6. Solve the equation $y(x) = 1 + \lambda \int_{-\pi}^{\pi} e^{iw(x-\xi)}\ y(\xi)d\xi$ considering separately all exceptional cases.

7. Obtain an approximate solution of the integral equation

$$y(x) = x^2 + \int_0^1 \sin(x\,\xi)\ y(\xi)d\xi$$

by replacing $\sin(x\,\xi)$ by the first two terms of its power series development

$$\sin(x\,\xi)m = (x\,\xi) - \frac{(x\xi)^3}{3!} + \ \dots\dots$$

8. Solve the integral equation $g(s) = f(s) + \lambda \int_0^{2\pi} \cos(s+t)\ g(t)dt$ and find the condition that $f(x)$ must satisfy in order that this equation has a solution when λ is an eigenvalue. Obtain the general solution if $f(s) = \sin s$, considering all possible cases.

Answers

1.

(i) $y(x) = \tan x.$

(ii) $g(s) = \left\{\frac{2}{2-\lambda}\right\} \sin x,\ \lambda \neq 2.$

(iii) $\phi(x) = \sec x \tan x - \left\{\frac{\lambda}{1+\lambda}\right\} \sec 1, \lambda \neq -1.$

(iv) $y(x) = \frac{1}{\sqrt{1-x^2}} - \frac{\pi^2}{8(\lambda-1)}\ ,\ \lambda \neq 1.$

(v) $\phi(x) \sec^2 x + \left\{\frac{\lambda}{1-\lambda}\right\} \tan 1, \lambda \neq 1.$

(vi) $\phi(x) = \cot x\ + \frac{\pi\lambda}{2}$

(vii) $\phi(x) = \frac{1-q^2}{1+q^2-\lambda}$

(viii) $\phi(x) = \frac{1}{\{1-\lambda\Gamma\,(p+1)\}}$

2.

(i) $\phi(x) = \frac{6}{5}(1-4x) + \frac{2\lambda^2 x + \left(\lambda + \frac{\lambda^2}{4}\right) l_n x}{1+\left(\frac{29}{48}\right)\lambda^2}$

(ii) $\phi(x) = x + \lambda\pi^3 \sin x$

(iii) $\phi(x) = \ x + \frac{\lambda}{12-24\lambda-\lambda^2}[10 + (6+\lambda)x]$

(iv) $\phi(x) = \ x + \frac{\lambda}{(1-\lambda\pi)\left(1-\frac{1}{2}\lambda\pi\right)+4\lambda^2}\left[2\lambda\pi + \frac{1}{2}\pi^2\left(1-\frac{1}{2}\lambda\pi\right) + \pi(1-2\lambda\pi)\sin x\right]$

(v) $\phi(x) = \frac{[15(4+\lambda)-30\lambda x^3]}{60-65\lambda+4\lambda^2}$

(vi) $\phi(x) = \frac{[6(3+\lambda)x-9\lambda x^2]}{18-18\lambda+\lambda^2}$

3. $\Gamma(x,\xi,\lambda) = \frac{4}{4-\lambda^2}\left[1+\lambda-\frac{3}{2}\lambda(x+\xi) - 3(1-\lambda)x\xi\right], \lambda \neq \pm 2.$

4. **(b)**$F(x) = x;\ \ y(x) = \frac{2\pi^2\lambda^2}{\pi^2\lambda^2-1}\sin x + \frac{2\pi\lambda}{\pi^2\lambda^2-1}\cos x + x.\quad F(x) = 1; y(x) = 1.$

(c)$F(x) = 1;\ \ y(x) = 1 + C(\cos x + \sin x).$

5. **(a)**$\lambda_1 = \frac{1}{\pi},\ \ y_1(x) = \cos x\,;\ \lambda_2(x) = -\frac{1}{\pi}, y_2(x) = \sin x$

(b) $\Gamma(x,\xi;\lambda) = \frac{\cos(x+\xi)+\pi\lambda\cos(x-\xi)}{1-\lambda^2\pi^2}$, if $\lambda = \pm\frac{1}{\pi}$.

(c) $y(x) = \frac{\sin x}{1+\pi\lambda}$ if $\lambda \neq \pm\frac{1}{\pi}$

$y(x) = \frac{1}{2}\sin x \;\; A\cos x$, A arbitrary, if $\lambda = \frac{1}{\pi}$; No solution it $\lambda = -\frac{1}{\pi}$.

6. $y(x) = 1 + \frac{2\lambda\sin\pi\omega}{(1-2\pi\lambda)\omega}e^{i\omega x}$, if $\lambda \neq \frac{1}{2\pi}, \omega \neq 0$;

$y(x) = 1 + \frac{2\pi\lambda}{1-2\pi\lambda}e^{i\omega x}$, if $\lambda \neq \frac{1}{2\pi}, \omega = 0$;

No solution if $\lambda = \frac{1}{2\pi}$.

7. $y(x) \approx 0.363\,x + x^2 - 0.039x^3$.

5.13. Integral Equations of the Second Kind by Successive Approximations

5.13.1. Iterated kernels or functions.

(i) Consider Fredholm integral equation of second kind

$y(x) = f(x) + \lambda\int_a^b K(x,t)\,y(t)dt.$ ------------------------- (1)

Then, the iterated kernels $K_n(x,t), n = 1, 2, 3, \ldots$ .... are defined as follows:

$K_1(x,t) = K(x,t),$ -------------------- (2a)

$$\left.\begin{array}{l} and\; K_n(x,t) = \int_a^b K(x,z)K_{n-1}(z,t)dz, n = 2, 3, \ldots \\ or\; K_n(x,t) = \int_a^b K_{n-1}(x,z)\,K(z,t)dz, n = 2,3, \ldots \end{array}\right\} \text{------------------- (2b)}$$

(ii) Consider Volterra integral equation of the second kind

$y(x) = f(x) + \lambda\int_a^x K(x,t)y(t)dt.$ ------------------------------- (3)

Then, the iterated kernels $K_n(x,t), n = 1, 2, 3, \ldots$ are defined as follows:

$K_1(x,t) = K(x,t)$ ------------------------- (3a)

$$\left.\begin{array}{l} and\; K_n(x,t) = \int_t^x K(x,z)K_{n-1}(z,t)dz, n = 2, 3 \ldots \\ or\; K_n(x,t) = \int_t^x K_{n-1}(x,z)\,K(z,t)dz, n = 2,3, \ldots \end{array}\right\} \text{--------------- (3b)}$$

5.13.2. Resolvent kernel or reciprocal kernel.

(i) Suppose solution of Fredholm integral equation kind

$y(x) = f(x) + \lambda\int_a^b K(x,t)y(t)dt$ -------------------- (1)

takes the form $y(x) = f(x) + \lambda\int_a^b R(x,t;\lambda)f(t)dt,$ ---------------------- (2a)

$or, y(x) = f(x) + \lambda\int_a^b \Gamma(x,t;\lambda)f(t)dt,$ ---------------------- (2b)

then $R(x,t;\lambda) or \Gamma(x,t;\lambda)$ is known as the resolvent kernel of (1).

(ii) Suppose solution of Volterra integral equation of the second kind

$y(x) = f(x) + \lambda \int_a^x K(x,t)\, y(t)dt$ ------------------- (3)

takes the form $y(x) = f(x) + \lambda \int_a^x \mathrm{R}(x,t;\lambda) f(t)dt$ ----------------------- (4a)

$or, y(x) = f(x) + \lambda \int_a^x \Gamma(x,t;\lambda)\, f(t)dt,$ --------------------- (4d)

Then $R(x,t;\lambda)$ or $\Gamma(x,t;\lambda)$ is known as the resolvent kernel of (3).

5.13.3. Theorem. The mth iterated kernel $K_m(x,t)$ satisfies the relation

$$K_m(x,t) = \int_a^b K_r(x,y) K_{m-r}(y,t)dy,$$

Where r is any positive integer less than m.

Ex. - 2. Determine the resolvent kernels for the Fredholm integral equation having kernels:

(i) $K(x,t) = e^{x+t}; a = 0, b = 1.$

(ii) $K(x,t) = (1+x)(1-t); a = -1, b = 1.$

Sol. (i) Iterated kernels $K_m(x,t)$ are given by $K_1(x,t) = K(x,t)$

and $K_m(x,t) = \int_0^1 K(x,z) K_{m-1}(z,t)dz.$------------------- (2)

From (1), $K_1(x,t) = K(x,t) = e^{x+t}$---------------------- (3)

Putting $n = 2$ in (2), we have

$K_2(x,t) = \int_0^1 K(x,z) K_1(z,t)dz = \int_0^1 e^{x+z} e^{x+t} dz$, using (3)

$= e^{x+t} \int_0^1 e^{2z}\, dz = e^{x+t} \left[\frac{1}{2} e^{2z}\right]_0^1 = e^{x+t} \left(\frac{e^2-1}{2}\right)$ --------------- (4)

Putting $n = 3$ in (2), we have

$K_3(x,t) = \int_0^1 K(x,z) K_1(z,t)dz = \int_0^1 e^{x+z}\, e^{z+t} \left(\frac{e^2-1}{2}\right) dz$

$= e^{x+t} \left(\frac{e^2-1}{2}\right) \int_0^1 e^{2z}\, dz = e^{x+t} \left(\frac{e^2-1}{2}\right)^2 [as\ before]$ --------------- (5)

and so on. Observing (3), (4) and (5), we may write

$K_m(x,t) = e^{x+t} \left(\frac{e^2-1}{2}\right)^{m-1}, m = 1, 2, 3,$ --------------- (6)

Now, the required resolvent kernel is given by

$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) = \sum_{m=1}^{\infty} \lambda^{m-1} e^{x+t} \left(\frac{e^2-1}{2}\right)^{m-1}$ by (6)

$= e^{x+t} \sum_{m=1}^{\infty} \left\{\frac{\lambda(e^2-1)}{2}\right\}^{m-1}$

But, $\sum_{m=1}^{\infty} \left\{\frac{\lambda(e^2-1)}{2}\right\}^{m-1} = 1 + \frac{\lambda(e^2-1)}{2} + \left\{\frac{\lambda(e^2-1)}{2}\right\}^2 + \cdots$

Which is an infinite geometric series with common ratio $\frac{\{\lambda(e^2-1)\}}{2}$.

$\therefore \sum_{m=1}^{\infty}\left\{\frac{\lambda(e^2-1)}{2}\right\}^{m-1} = \frac{1}{1-\frac{\{\lambda(e^2-1)\}}{2}} = \frac{2}{2-\lambda(e^2-1)}$,

Provided $\left|\frac{\lambda(e^2-1)}{2}\right| < 1\ or |\lambda| < \frac{2}{e^2-1}$ -------------------(9)

Using (8) and (9), (7) reduces to

$R(x,t;\lambda) = \frac{2e^{x+t}}{2-\lambda(e^2-1)}$, provided $|\lambda| < \frac{2}{e^2-1}$. ----------------- (10)

Part (ii) Iterated Kernels $K_m(x,t)$ are given by

$K_1(x,t) = K(x,t)$ ---------------------------- (1)

and $K_m(x,t) = \int_{-1}^{1} K(x,z)K_{m-1}(z,t)dz$. ------------------ (2)

From (1), $K_1(x,t) = K(x,t) = (1+x)(1-t)$. ---------------------------- (3)

Putting $n = 2$ in (2), we have

$K_2(x,t) = \int_{-1}^{1} K(x,z)K_1(z,t)dz = \int_{-1}^{1}(1+x)(1-z)(1+z)(1-t)dz, by$ (3)

$$= (1+x)(1-t)\int_{-1}^{1}(1-z^2)\,dz = (1+x)(1-t)\left[z-\frac{1}{3}z^3\right]_{-1}^{1}$$

$K_2(x,t) = \frac{2}{3}(1+x)(1-t)$. ----------------- (4)

Next, putting $n = 3$ in (3), we have

$$K_3(x,t) = \int_{-1}^{1} K(x,z)K_2(z,t)dz = \int_{-1}^{1}(1+x)(1-z)\,.\frac{2}{3}(1+z)(1-t)dz$$

$= \frac{2}{3}(1+x)(1-t)\int_{-1}^{1}(1-z^2)\,dz = \left(\frac{2}{3}\right)^2(1+x)(1-t)$, as before. --------------- (5)

And so on. Observing (3), (4) and (5), we may write

$K_m(z,t) = \left(\frac{2}{3}\right)^{m-1}(1+x)(1-t)$.

Now, the required resolvent kernel is given by

$R(x,t;\lambda) = \sum_{m=1}^{\infty}\lambda^{m-1}K_m(x,t) = \sum_{m=1}^{\infty}\lambda^{m-1}\left(\frac{2}{3}\right)^{m-1}(1+x)(1-t), by$ (6)

$$= (1+x)(1-t)\sum_{m=1}^{\infty}\left(\frac{2\lambda}{3}\right)^{m-1}$$

But, $\sum_{m=1}^{\infty}\left(\frac{2\lambda}{3}\right)^{m-1} = 1+\frac{2\lambda}{3}+\left(\frac{2\lambda}{3}\right)^2+\left(\frac{2\lambda}{3}\right)^3+\cdots$

Which is an infinite geometric series with common ratio $(2\lambda/3)$.

$\therefore \sum_{m=1}^{\infty}\left(\frac{2\lambda}{3}\right)^{m-1} = \frac{1}{1-\left(\frac{2\lambda}{3}\right)} = \frac{3}{3-2\lambda}$ ------------------- (8)

Provided $\left|\frac{2\lambda}{3}\right| < 1 \; or \; |\lambda| < \frac{3}{2}$.--------------------- (9)

Using (8) and (9), (7) reduces to

$R(x,t;\lambda) = \frac{3(1+x)(1-t)}{3-2\lambda}, provided \; |\lambda| < \frac{3}{2}$.----------------- (10)

Type - 3: Solution of Fredholm integral equation with help of the resolvent kernel.

Let, $y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt$ ----------------------(1)

be given Fredholm integral equation. Let $K_m(x,t)$ be m^{th} iterated kernel and let $R(x,t;\lambda)$ be the resolvent kernel of (1). Then, we have

$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$. -------------------------- (2)

Suppose the sum of infinite series (2) exists and so $R(x,t;\lambda)$ can be obtained in the closed form. Then, the required solution of (1) is given by

$y(x) = f(x)\lambda \int_a^b R(x,t;\lambda)f(t)dt$. ------------------ (3)

Ex.-3. Solve $y(x) = x + \int_0^{\frac{1}{2}} y(t)dt$

Sol. Given $y(x) = x + \int_0^{\frac{1}{2}} y(t)dt$. ---------------------- (1)

Comparing (1) with $y(x) = f(x) + \lambda \int_0^{\frac{1}{2}} K(x,t)y(t)dt$,

We have $f(x) = x, \lambda = 1, K(x,t) = 1$. -------------------- (2)

Let $K_m(x,t)$ be the m^{th} iterated kernel. Then, we have

$K_1(x,t) = K(x,t)$ ------------------- (3)

and $K_m(x,t) = \int_0^{\frac{1}{2}} K(x,z)K_{m-1}(z,t)dz$. ------------------ (4)

From (1), $K_1(x,t) = K(x,t) = 1$, by (2) ------------------ (5)

Putting $m = 2$ in (4), we have

$K_2(x,t) = \int_0^{\frac{1}{2}} K(x,z)K_1(z,t)dz = \int_0^{\frac{1}{2}} dz, by$ (4).$= [z]_0^{1/2} = \frac{1}{2}$. ------------- (6)

Next, putting $m = 3$ in (4), we have

$K_3(x,t) = \int_0^{\frac{1}{2}} K(x,z)K_2(z,t)dz$

$= \int_0^{\frac{1}{2}} \frac{1}{2} dz$, by (5) and (6)

$= \left(\frac{1}{2}\right)^2$, ----------------- (7)

and so on. Observing (5), (6) and (7), we find

$K_m(x,t) = \left(\frac{1}{2}\right)^{m-1}$ ------------------ (8)

Now, the resolvent kernel $R(x,t;\lambda)$ is given by

$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) = \sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^{m-1}$ $[using\ (2) and\ (8)]$ ---------- (9)

But, $\sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^{m-1} = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots$

which is an infinite geometric series with common ratio $\frac{1}{2}$.

$\therefore \sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^{m-1} = \frac{1}{1-\left(\frac{1}{2}\right)} = 2.$

Substituting the above value in (8), we have

$R(x,t;\lambda) = 2.$ ----------------------- (10)

Finally, the required solution of (1) in given by

$y(x) = f(x) + \lambda \int_0^{\frac{1}{2}} R\,(x,t;\lambda) f(t) dt$

$or, y(x) = x + \int_0^{\frac{1}{2}} (2t)\, dt$, by (2) and (10)

$or, y(x) = x + 2\left[\frac{t^2}{2}\right]_0^{\frac{1}{2}} = x + \frac{1}{4}.$

5.13.4. Theorem. Let $R(x,t;\lambda)$ be the resolvent (or reciprocal) kernel of a Volterra integral equation

$$y(x) = f(x) + \lambda \int_a^x K\,(x,t) y(t) dt,$$

$$then, R(x,t;\lambda) = K(x,t) + \lambda \int_t^x K\,(x,z) R(z,t;\lambda) dz.$$

5.13.5. Illustrative solved examples based on solutions of Volterra integral equation by successive approximations (or Iterative method).

Type 1. Determination of resolvent kernel or reciprocal kernel for Volterra integral equation

$$y(x) = f(x) + \lambda \int_a^x K\,(x,t) y(t) dt.$$

Ex. 1. Find the resolvent kernel of the Volterra integral equation with the kernel $K(x,t) = 1$.

Sol. Iterated kernels $K_n(x,t)$ are given by

$K_1(x,t) = K(x,t)$ ----------------- (1)

and $K_n(x,t) = \int_t^x K\,(x,z)K_{n-1}(z,t)dz, n = 2,3, \ldots\ldots.$ ---------------- (2)

Given $K(x,t) = 1$ ---------------------- (3)

∴ From (1) and (3), $K_1(x,t) = K(x,t) = 1.$ -------------------- (4)

Putting $n = 2$ in (2), we have

$K_2(x,t) = \int_t^x K\,(x,z)K_1(z,t)dz = \int_t^x dz$, using (4)$= [z]_t^x = x - t.$ ----------------- (5)

Next, putting $n = 3$ in (2), we have

$K_3(x,t) = \int_t^x K\,(x,z)K_2(z,t)dz = \int_t^x 1.(z-t)\,dz$, by (4) and (5)

$= \left[\frac{(z-t)^2}{2}\right]_t^x = \frac{(x-t)^2}{2!}.$ --------------- (6)

Now, putting $n = 4$ in (2), we have

$K_4(x,t) = \int_t^x K\,(x,z)K_3(z,t)dz = \int_t^x 1.\frac{(z-t)^2}{2!}dz$, by (4) and (6)

$= \frac{1}{2!}\left[\frac{(z-t)^3}{3}\right]_t^x = \frac{(x-t)^3}{3!},$ --------------------- (7)

and so, no. Observing (4), (5) etc., we find by mathematical induction, that

$K_n(x,t) = \frac{(x-t)^{n-1}}{(n-1)!}, n = 1, 2, 3, \ldots..$ ------------------- (8)

Now, by the definition of the resolvent kernel, we have

$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$

$= K_1(x,t) + \lambda\, K_2(x,t) + \lambda^2 K_3(x,t) + \cdots$

$= 1 + \frac{\lambda(x-t)}{1!} + \frac{[\lambda(x-t)]^2}{2!} + \frac{[\lambda(x-t)]^3}{3!} + \cdots, by\ (8)$

$= e^{\lambda(x-t)}.$

Ex.2. Find the resolvent kernel of the Volterra integral equation with the kernel $K(x,t) = e^{x-t}$.

Sol. Iterated kernels $K_n(x,t)$ are given by

$K_1(x,t) = K(x,t)$ ---------------------- (1)

and $K_n(x,t) = \int_t^x K(x,z)K_{n-1}(z,t)dz, n = 2,3,\ldots$ -------------- (2)

Given $K(x,t) = e^{x-t}$.----------------- (3)

∴ From (1) and (3), $K_1(x,t) = K(x,t) = e^{x-t}$----------------- (4)

Putting $n = 2$ in (2), we have

$K_2(x,t) = \int_t^x K(x,z)K_1(z,t)dz$

$= \int_t^x e^{x-z}\, e^{z-t}dz = e^{x-t}\int_t^x dz = e^{x-t}(x-t).$ --------------- (5)

Next, putting $n = 3$ in (2), we have

$K_3(x,t) = \int_t^x K(x,z)K_2(z,t)dz$

$= \int_t^x e^{x-z}\, e^{z-t}dz = e^{x-t}\int_t^x (z-t)\, dz$

$= e^{z-t}\left[\frac{[z-t]^2}{2}\right]_t^x = e^{x-t}\frac{(x-t)^2}{2!}$

Now, putting $n = 4$ in (2), we have

$K_4(x,t) = \int_t^x e^{x-z}\, e^{z-t}\frac{(z-t)^2}{2!}\, dz = \frac{e^{x-t}}{2!}\int_t^x (z-t)^2\, dz$

$= \frac{e^{x-t}}{2!}\left[\frac{(z-t)^3}{3}\right]_t^x = e^{x-t}\frac{(x-t)^3}{3!},$

and so on. Observing (4), (5) etc., we find by Mathematical induction, that

$K_n(x,t) = e^{x-t}\frac{(x-t)^{n-1}}{(n-1)!}, n = 1, 2, 3, \ldots$

Now, by the definition of the resolvent kernel, we have

$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$

$= K_1(x,t) + \lambda\, K_2(x,t) + \lambda^2 K_3(x,t) + \cdots$

$= e^{x-t} + e^{x-t}\frac{\lambda(x-t)}{1!} + e^{x-t}\frac{[\lambda(x-t)]^2}{2!} + \cdots by\ (8)$

$= e^{x-t}\left[1 + \frac{\lambda(x-t)}{1!} + \frac{[\lambda(x-t)]^2}{2!} + \cdots\right]$

$= e^{x-t}e^{\lambda(x-t)} = e^{(x-t)+\lambda(x-t)}$

$= e^{(x-t)(1+\lambda)}.$

EXERCISES

Find the resolvent kernels for Volterra integral equations with the following kernels.

1. $K(x,t) = e^{x^2-t^2}$ **Ans.** $e^{x^2-t^2} e^{\lambda(x-t)}$

2. $K(,t) = \frac{1+x^2}{1+t^2}$ **Ans.** $\frac{1+x^2}{1+t^2} e^{\lambda(x-t)}$

3. $K(x,t) = a^{x-t} (a > 0)$. **Ans.** $a^{x-t} e^{\lambda(x-t)}$

4. $K(x,t) = \frac{\cosh x}{\cosh t}$ **Ans.** $\frac{\text{Cosh}\, x}{\cosh t} e^{\lambda(x-t)}$

5. $K(x,t) = x - t.$ **Ans.** $\frac{1}{\sqrt{\lambda}} \sinh\{(x-t)\sqrt{\lambda}\}, (\lambda > 0)$

Type 2. Solution of Volterra integral equation with help of the resolvent kernel.

Let $y(x) = f(x) + \lambda \int_a^x K\,(x,t) y(t) dt$ ------------------ (1)

be given Volterra integra equation. Let $K_m(x,t)$ be the m^{th} iterated kernel and let $R(x,t;\lambda)$ be the resolvent kernel of (1). Then, we have

$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$ -------------------- (2)

Suppose the sum of infinite series (2) exists and so $R(x,t;\lambda)$ can be obtained in the closed form. Then, the required solution of (1) in given by

$y(x) = f(x) + \lambda \int_a^x R\,(x,t;\lambda) f(t) dt.$ -------------------- (3)

Ex.5. Solve the following integral equation by successive approximation

$y(x) = f(x) + \lambda \int_0^x e^{x-t}\, y(t) dt.$ and find the resolvent kernel.

Sol. Given$y(x) = f(x) + \lambda \int_0^x e^{x-t}\, y(t) dt$ ----------------------(1)

Comparing (1) with $y(x) = f(x) + \lambda \int_0^x K\,(x,t) y(t) dt.$

here $K(x,t) = e^{x-t}$------------------- (2)

Proceed as in solved Ex. 2. And show that

$R(x,t;\lambda) = e^{(x-t)(1+\lambda)}$ ------------------------------ (3)

Now, the required solution of (1) is given by

$y(x) = f(x) + \lambda \int_0^x R\,(x,t;\lambda) f(t) dt$

$or, y(x) = f(x) + \lambda \int_0^x e^{(x-t)(1+\lambda)}\, f(t) dt, by (3).$

Ex.6. By means of resolvent kernel, find the solution of

$y(x) = e^x \sin x + \int_0^x \frac{2+\cos x}{2+\cos t} y(t)dt.$

Sol. Given $y(x) = e^x \sin x + \int_0^x \frac{2+\cos x}{2+\cos t} y(t)dt.$ ------------------ (1)

Compering (1) with $y(x) = f(x) + \lambda \int_0^x K(x,t)y(t)dt,$

We have $f(x) = e^x \sin x, \lambda = 1, K(x,t) = \frac{2+\cos x}{2+\cos t}.$ ----------------- (2)

Proceed as in solved $Ex.$ 3. and show that

$R(x,t;\lambda) = \frac{2+\cos x}{2+\cos t} e^{x-t} [Note\ that\ here\ \lambda = 1, by (2)]$ -------------- (3)

The required solution is given by

$y(x) = f(x) + \lambda \int_0^x R(x,t;\lambda) f(t)dt$

$or, y(x) = e^x \sin x + \int_0^x \frac{2+\cos x}{2+\cos t} e^{x-t} e^t \sin t\, dt,$ using (2) and (3)

$= e^x \sin \dot{x} - (2+\cos x)e^x \int_0^x \frac{-\sin t}{2+\cos t} dt$

$= e^x \sin x - e^x (2+\cos x)[\log(2+\cos t)]_0^x$

$= e^x \sin x - e^x (2+\cos x) \log \frac{2+\cos x}{3}$

$\therefore y(x) = e^x \sin x + e^x (2+\cos x) \log \frac{3}{2+\cos x}$

Ex.7. Solve $y(x) = \sin x + 2\int_0^x e^{x-t} y(t)dt.$

Sol. Given $y(x) = \sin x + 2\int_0^x e^{x-t} y(t)dt.$ ------------------- (1)

Comparing (1) with $y(x) = f(x) + \lambda \int_0^x K(x,t)y(t)dt,$

we have $f(x) = \sin x, \lambda = 2, K(x,t) = e^{x-t}.$ ---------------------------- (2)

Proceed as in Ex. 2. and show that:

$R(x,t;\lambda) = e^{(x-t)(1+\lambda)} = e^{3(x-t)}. [\because \lambda = 2]$

Now, the required solution of (1) is

$y(x) = f(x) + \lambda \int_0^x R(x,t;\lambda) f(t)dt$

$or, y(x) = \sin x + 2\int_0^x e^{3(x-t)} \sin t\, dt = \sin x + 2e^{3x} \int_0^x e^{-3t} \sin t\, dt.$

$= \sin x + 2e^{3x} \left[\frac{e^{-3t}}{10}(-3\sin t - \cos t)\right]_0^x$

$\left[\because \int e^{ax} \sin b\, x\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b\cos bx)\right]$

$= \sin x + \frac{e^{3x}}{5}[e^{-3x}(-3\sin x - \cos x) + 1] \because y(x) = \frac{1}{5}e^{3x} + \frac{2}{5}\sin x - \frac{1}{5}\cos x$

Solve the following equation:

1. $y(x) = e^x + \int_0^x e^{x-t}\, y(t)dt.$

Ans.$y(x) = e^{2x}$

2. $y(x) = x.3^x - \int_0^x 3^{x-t}\, y(t)dt.$

Ans.$y(x) = 3^x(1 - e^{-x})$

3. $y(x) = 1 - 2x - \int_0^x e^{x^2-t^2}\, y(t)dt.$

Ans:$y(x) = e^{x^2-x} - 2x.$

4. $y(x) = e^{x^2+2x} + 2\int_0^x e^{x^2-t^2}\, y(x)dt.$

Ans.$y(x) = e^{x^2+2x}(1 + 2x)$

5. $y(x) = \frac{1}{1+x^2} + \int_0^x \sin(x-t)\, y(t)dt.$

Ans.$y(x) = \frac{1}{1+x^2} + x\tan^{-1}x - \frac{1}{2}\log(1+x^2)$

6. $y(x) = xe^{x^2/2} + \int_0^x e^{-(x-t)}\, y(t)dt.$

Ans.$y(x) = e^{x^2/2}(x+1) - 1$

7. $y(x) = e^{-x} + \int_0^x e^{-(x-t)} \sin(x-t)\, y(t)dt.$

Ans.$y(x) = e^{-x}[1 + (x^2/2)]$

8. $y(x) = e^x - \int_0^x e^{x-t}\, y(t)dt$

Ans.$y(x) = 1$

9. $\phi(x) = 1 + x + \lambda\int_0^x (x-\xi)\,\phi(\xi)d\xi.$