COUNCILE OF SCIENTIFIC & INDUSTRIAL RESEARCH

Mathematical Science Code : 04

Unit – 3 :

Syllabus

Sub Unit – 2: Partial Differential Equation:

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Partial Differential Equations (PDEs)

2.1. Partial Differential Equation:

2.1.1. Definition: An equation containing one or more partial derivatives of an unknown function of two or more independent variables is known as a partial differential equation.

Example (2.1.): $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=z+xy,\quad \frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0\ \ etc.$

2.1.2. Order and Degree of Partial Differential Equation:

The order of a partial differential equation is defined as the order of the highest partial derivatives in the equation.

The degree of a partial differential equation is the degree of the highest order derivative which occurs in it after the equation has been rationalized.

2.1.3. Linear and Non – linear Partial Differential Equation:

A partial differential equation is said to be linear if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied. A partial differential equation which is not linear is called a non – linear partial differential equation.

2.2. Classification of first order partial differential equation:

2.2.1. A first order partial differential equation said to be linear if it can be expressed as

$$P(x,y)\frac{\partial z}{\partial x}+Q(x,y)\frac{\partial z}{\partial y}+R(x,y)z+S(x,y)=0$$

2.2.2. A partial differential equation is said to be semi – linear if it can be expressed as

$$P(x,y)\frac{\partial z}{\partial x}+Q(x,y)\frac{\partial z}{\partial y}+R(x,y,z)=0$$

2.2.3. A partial differential equation is said to be quasi – linear if it can be expressed as

$$P(x,y,z)\frac{\partial z}{\partial x}+Q(x,y,z)\frac{\partial z}{\partial y}+R(x,y,z)=0$$

2.2.4. A partial differential equation is said to be non – linear if it is neither linear nor quasi – linear and also nor semi – linear.

Example (2.2):

Find a partial differential equation by elimination a and b from

$$z=ax+by+a^2+b^2$$

Solution: $\frac{\partial z}{\partial x}=a,\ \frac{\partial z}{\partial y}=b$

∴ Partial differential equation is $z=x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2$

Example (2.3):

Find the partial differential equation by eliminating $h\ and\ k$ from the equation

$(x-h)^2+(y-k)^2+z^2=\lambda^2$

Solution: Differentiating partially with respect to x and y

$$2(x-h)+2z\frac{\partial z}{\partial x}=0, 2(y-k)+2z\frac{\partial z}{\partial x}=0$$

$$\Rightarrow x-h=-z\frac{\partial z}{\partial x}\quad and\quad y-k=-z\frac{\partial z}{\partial y}$$

$\therefore$ Partial differential equation is $z^2\left(\frac{\partial z}{\partial x}\right)^2+z^2\left(\frac{\partial z}{\partial x}\right)^2+z^2=\lambda^2$

Example (2.4): Form a partial differential equation by eliminating the arbitrary function ϕ from $\phi(x+y+z,\ x^2+y^2-z^2)=0$

Solution: $\phi(x+y+z,\ x^2+y^2-z^2)=0$

Let $x+y+z=u$ and $x^2+y^2-z^2=v$

$\therefore \phi\,(u,v)=0$

Differentiating partially with respect to x

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0$$

$$or, \frac{\frac{\partial\phi}{\partial u}}{\frac{\partial\phi}{\partial v}}=-2\,(x-pz)(1+p)$$

Differentiating partially with respect to y

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0$$

$$or, \frac{\frac{\partial\phi}{\partial u}}{\frac{\partial\phi}{\partial v}}=-2\,(y-qz)(1+q)$$

$\therefore$ Eliminating ϕ we get the partial differential equation as $(y+z)p-(x+z)q=x-y$.

Example (2.5):

Form a partial differential equation by eliminating the function ϕ from

$lx+my+nz=\phi\,(x^2+y^2+z^2)$

Differentiating partially with respect to $x\ and\ y$.

$$l+n\frac{\partial z}{\partial x}=\phi'(x^2+y^2+z^2)\cdot\left(2x+2z\frac{\partial z}{\partial x}\right)$$

$$m+n\frac{\partial z}{\partial y}=\phi'(x^2+y^2+z^2)\cdot\left(2y+2z\frac{\partial z}{\partial y}\right)$$

$$or, \frac{l+n\frac{\partial z}{\partial x}}{m+n\frac{\partial z}{\partial y}}=\frac{2\left(x+z\frac{\partial z}{\partial x}\right)}{2\left(y+z\frac{\partial z}{\partial y}\right)}$$

$$or, (ny-mz)\frac{\partial z}{\partial x}+(lz-nx)\frac{\partial z}{\partial y}=mx-ly$$

2.3. Cauchy Problem for first order partial differential equations:

(a) If $x_0(\mu), y_0(\mu)$ and $z_{\,0}(\mu)$ are functions which together with their first derivatives are continuous in the interval I defined by $\mu_1 < \mu < \mu_2$

(b) And if $f(x, y, z, p, q)$ is a continuous function of x, y, z, p and q in a certain region U of the $xyzpq$ space, then it is required to establish the existence of a function $\phi(x, y)$ with the following property.

(i) $\phi(x, y)$ and its partial derivatives with respect to x and y are continuous functions of x and y in a region $\mathbb{R}$ of the $xy\ space$.

(ii) For all values of x and y lying in $\mathbb{R}$, the point $\{x, y, \phi(x, y),\ \phi_x(x, y), \phi_y(x, y)\}$ lies in U and $f\left(x, y, \phi, \phi_x,\ \phi_y\right) = 0$

(iii) For all $\mu \in I$, the point $\left(x_0(\mu), y_0(\mu)\right) \in \mathbb{R}$ and $\phi\ (x_0, y_0) = z_{\,0}$

Example (2.6):

Solve the Cauchy Problem for $zp + q = 1$ with $x_0 = \mu, y_0 = \mu,\ z_{\,0} = \frac{\mu}{2}, 0 \le \mu \le 1$.

Solution: $f(x, y, z, p, q) = zp + q - 1 = 0$

$x_0 = \mu, y_0 = \mu,\ z_{\,0} = \frac{\mu}{2}, 0 \le \mu \le 1$

$\therefore \frac{\partial f}{\partial p} = z, \frac{\partial f}{\partial q} = 1$ and

$\frac{\partial f}{\partial q} \cdot \frac{dx_0}{d\mu} - \frac{\partial f}{\partial p} \cdot \frac{dy_0}{d\mu} = 1 - z = 1 - \frac{\mu}{2} \neq 0, for\ 0 \le \mu \le 1$

$\frac{dx}{dt} = \frac{\partial f}{\partial p}, \frac{dy}{dt} = \frac{\partial f}{\partial q}$ $\quad and\ \frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$

$or, \frac{dx}{dt} = z, \frac{dy}{dt} = 1, \frac{dz}{dt} = p\ \frac{\partial f}{\partial p} + q\ \frac{\partial f}{\partial q} = pz + q = 1$

Integrating $y = t + c_1$ and $z = t + c_2$ at $t = 0$ $\quad x(\mu, 0) = \mu, \quad y(\mu, 0) = \mu$ $\ and$

$z\ (\mu, 0) = \frac{\mu}{2}$

$\therefore y = t + \mu\ , z = t + \frac{\mu}{2}$

$\frac{dx}{dt} = t + \frac{\mu}{2}$ $\ so\ that\ x = \frac{1}{2}t^2 + \frac{1}{2}\ \mu t + c_3$

$x = \frac{1}{2}t^2 + \frac{1}{2}\ \mu\, t + \mu$

Also, $t = \frac{y - x}{1 - \frac{y}{2}}$ $\quad and\ \mu = \frac{x - \frac{y^2}{2}}{1 - \frac{y}{2}}$

Putting these values in $z = t + \frac{\mu}{2}$ we get the solution $z = \frac{2(y - x) + x - \frac{y^2}{2}}{2 - y}$

2.4. Different Methods for finding solutions of Partial Differential Equation:

2.4.1. Lagrange's Method:

The general solution of the first order quasi – linear partial differential equation

$P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$ is given by $\phi\ (u,v) = 0$ where ϕ is an arbitrary and $u(x,y,z) = c_1, v(x,y,z) = c_2$ are two independent solutions of the auxiliary equations

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

2.4.2. Charpit's Method:

For the equations $f(x,y,z,p,q) = 0$, the Charpit's auxiliary equations are

$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p+qf_q} = \frac{dp}{-(f_x+pf_z)} = \frac{dq}{-(f_y+qf_z)}$

Example (2.7):

Solve $(mz - ny)p + (nx - lz)q = ly - mx$

Solution:

The Lagrange's auxiliary equations are

$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$

$= \frac{xdx+ydy+zdz}{0}$

$or, x\ dx + y\ dy + z\ dz = 0$

Integrating $x^2 + y^2 + z^2 = c_1$

Also $= \frac{l\ dx+m\ dy+n\ dz}{0}$

$\therefore\ l\ dx + m\ dy + n\ dz = 0$

Integrating $lx + my + nz = c_2$

So the solution is $\phi\ (x^2 + y^2 + z^2, lx + my + nz) = 0$

Example (2.8):

Solve $x\ (y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$

Sol: Lagrange's auxiliary equations are

$\frac{dx}{x\ (y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$

$= \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{y^2+z-x^2-z+x^2-y^2}$

$\Rightarrow \frac{1}{x}dx + \frac{1}{y}\ dy + \frac{1}{z}\ dz = 0$

Integrating $\log x + \log y + \log z = 0\ \Rightarrow xyz = c_1$

Also, $= \frac{x\,dx+y\,dy-\,dz}{x^2(y^2+z)-y^2(x^2+z)-z(x^2-y^2)} = x\,dx + y\,dy - dz = 0$

Integrating $\frac{x^2}{2} + \frac{y^2}{2} - z = \frac{c_2}{2} \Rightarrow x^2 + \ y^2 - 2z = c_2$

So, the solution is $\phi\ (x^2 + \ y^2 - 2z, xyz) = 0$

Example (2.9): Solve $x(y-z)p + y(z-x)q = z(x-y)$

Sol: $\frac{dx}{x\ (y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} = \frac{\frac{1}{x}dx+\frac{1}{y}\,dy+\frac{1}{z}\,dz}{0}$

$or, \frac{1}{x}dx + \frac{1}{y}\ dy + \frac{1}{z}\ dz = 0$ Integrating $xyz = c_1$

Also, $= \frac{dx+\,dy+\,dz}{0} \ \Rightarrow dx + dy + dz = 0$

Integrating $x + y + z = c_2$

So, the solution is $\phi\ (x + y + z, xyz) = 0$

Example (2.10):

Solve – $(2x^2 + y^2 + z^2 - 2yz - zx - xy)p + (x^2 + 2y^2 + z^2 - yz - 2zx - xy)q = \ x^2 + y^2 + 2z^2 - yz - zx - 2xy$

Solution: Lagrange's auxiliary equations are

$$\frac{dx}{2x^2+y^2+z^2-2yz-zx-xy} = \frac{dy}{x^2+2y^2+z^2-yz-2zx-xy} = \frac{dz}{x^2+y^2+2z^2-yz-zx-2xy}$$

$$= \frac{dx-dy+0\cdot dz}{x^2-y^2-yz+zx} = \frac{0\cdot dx+dy-dz}{y^2-z^2-zx+xy} = \frac{-dx+0\cdot dy+dz}{z^2-x^2-xy+yz}$$

$$\therefore \frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z)} = \frac{dz-dx}{(z-x)(x+y+z)}$$

$$\Rightarrow \frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z} = \frac{d(z-x)}{z-x}$$

$$\therefore \frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

Integrating $\frac{x-y}{y-z} = c_1$

$$\therefore \frac{d(y-z)}{y-z} = \frac{d(z-x)}{z-x}$$

Integrating $\frac{y-z}{z-x} = c_2$

So, the solution is $\phi\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right) = 0$

Example (2.11):

Find a complete integral of $z = px + qy + p^2 + q^2$

Solution: Here $f(x,y,z,p,q) = z - px - qy - p^2 - q^2$

Charpit's auxiliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$$or, \frac{dp}{0} = \frac{dq}{0} = \frac{dz}{p(x+2p)+q(y+2q)} = \frac{dx}{x+2p} = \frac{dy}{y+2q}$$

$\Rightarrow dp = 0 \quad or \quad p = a$

$dq = 0 \quad or \quad q = b$

So the complete integral is $z = ax + by + a^2 + b^2$

Example (2.12): Find a complete integral of $px + qy = pq$

Solution: $f(x,y,z,p,q) = px + qy - pq \ldots\ldots\ldots\ldots\ldots (1)$

Charpit's auxiliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$$or, \frac{dx}{-(x-q)} = \frac{dy}{-(y-q)} = \frac{dz}{-p(x-q)-q(y-q)} = \frac{dp}{p} = \frac{dq}{q} \ldots\ldots\ldots\ldots\ldots (2)$$

From last two relations $\frac{dp}{p} = \frac{dq}{q}$

Integrating $\log p = \log q + \log a \Rightarrow p = aq \ldots\ldots\ldots\ldots\ldots (3)$

From (1) $aqx + qy - aq^2 = 0 \; or \; aq = ax + y \ldots\ldots\ldots\ldots\ldots (4)$

From (3) and (4) $q = \frac{ax+y}{a} \quad and \quad p = ax + y$

Also, $dz = p\,dx + q\,dy = (ax+y)dx + \frac{(ax+y)}{a}dy$

$or, \; a\,dz = (ax+y)(a\,dx + dy)$

Integrating $az = \frac{(ax+y)^2}{2} + b$

Which is the complete integral.

Example (2.12):

Find a complete integral of $2zx - px^2 - 2qxy + pq = 0$

Sol:

$f(x, y, z, p, q) = 2zx - px^2 - 2qxy + pq = 0 \ldots\ldots\ldots\ldots\ldots (1)$

Charpit's auxiliary equations are

$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$

$or, \ \frac{dp}{2z - 2qy} = \frac{dp}{0} = \frac{dx}{x^2 - q} = \frac{dy}{2xy - pq} = \frac{dz}{px^2 + 2qxy - 2pq}$

$\Rightarrow dq = 0 \quad \Rightarrow q = a$

From (1) $2zx - px^2 - 2axy + pa = 0$

$or, \ p = \frac{2zx - 2axy}{x^2 - a}$

Putting these values in $dz = pdx + qdy$

$dz = \frac{2x(z - ay)}{x^2 - a} \ dx + a \ dy$

or, $\frac{dz - a \ dy}{z - ay} = \frac{2x}{x^2 - a} \ dx$

Integrating, $\log(z - ay) = \log(x^2 - a) + \log b$

$or, \quad z - ay = b \ (x^2 - a)$

Which is the complete integral.

2.5. Working rule of Partial differential Equation with constant coefficients:

2.5.1. To finding complementary function (C.F.) of linear homogeneous partial differential equation with constant coefficients.

Let $F(D, D') \, Z = f(x, y)$ be the differential equation. Factorize $F(D, D')$ into linear factors of the form $(bD - aD')$. Then use the following result

(i) Corresponding to each non – repeated factor $(bD - aD')$, the part of the C.F. is taken as $\phi(by + ax)$

(ii) Corresponding to each repeated factor $(bD - aD')^m$ the part of the C.F. is taken as $\phi_1(by + ax) + \ x \, \phi_2(by + ax) + x^2\phi_3(by + ax) + \ldots\ldots\ldots + x^{m-1}\phi_m(by + ax)$.

2.5.2. To finding complementary function (C.F.) of a linear non – homogeneous partial differential equation with constant coefficients:

(i) Corresponding to each non – repeated factor $(bD - aD' - c)$, the part of the C.F. is taken $e^{\frac{cx}{b}} \cdot \phi(by + ax)$ if $b \neq 0$.

(ii) Corresponding to each repeated factor $(bD - aD' - c)^m$, the part of the C.F. is taken as $e^{\frac{cx}{b}}[\phi_1(by + ax) + x\,\phi_2(by + ax) + x^2\phi_3(by + ax) + \ldots\ldots\ldots + x^{m-1}\phi_m(by + ax)]$

2.5.3. To finding particular integral (P.I) of linear non – homogeneous partial differential equation with constant coefficient:

(i) When $f(x, y) = e^{ax+by}$ $and\ F(a, b) \neq 0$. Then $P.I = \frac{e^{ax+by}}{F(a,b)}$

(ii) When $f(x, y) = \sin(ax + by)$ or $\cos(ax + by)$. Then

$P.I = \frac{1}{F(D,D')}\sin(ax + by)\ or\ \cos(ax + by)$

Which is calculated by putting $D^2 = -a^2, D'^2 = -b^2$and $DD' = -ab$ provided the denominator is not zero.

(iii) When $f(x, y) = x^m y^n$, Then

$P.I = \frac{1}{F(D,D')}x^m y^n = [F(D, D']^{-1}x^m y^n$

(iv) When $f(x, y) = V\ e^{ax+by}$, where V is a function of x and y. Then

$P.I = \frac{1}{F(D,D')}\ V\ e^{ax+by} = e^{ax+by}\frac{1}{F(D+a,D'+b)}\ V$

2.5.4. To finding particular integral (P.I) of linear homogeneous partial differential equation with constant coefficients:

(i) When $F(\,a, b) \neq 0$ $and\ F(D, D')$ is a homogeneous function of degree n, then

$P.I = \frac{1}{F(D,D')}\ \phi(ax + by) = \frac{1}{F(a,b)}\iint\ldots\ldots\ldots\ldots\int\phi(v)dv.dv\ \ldots\ldots\ldots\ldots dv$

Where $v = ax + by$

(ii) When $F(a, b) = 0$, We have

$P.I = \frac{1}{(bD-aD')^n}\ \phi\ (ax + by) = \frac{x^n}{b^n n!}\ \phi\ (ax + by)$

Example (2.13):

Find the general solution of $\left(D^3 - 3DD'^2 - 2D'^3\right) z = \cos(x + 2y)$

Solution:

The auxiliary equation is $m^3 - 3m - 2 = 0 \Rightarrow m = -1, -1, 2$

So, $C.F = \phi_1(y - x) + \phi_2(y - x) + \phi_3(y + 2x)$

$P.I = \frac{1}{D^3 - 3DD'^2 - 2D'^3} \cos(x + 2y)$

$= \frac{1}{1^3 - 3 \cdot 1 \cdot 2^2 - 2 \cdot 2^3} \iiint \cos v \, dv \, dv \, dv$, where $v = x + 2y$

$= -\frac{1}{27}(-\sin v) = \frac{1}{27}\sin(x + 2y)$

So, the general solution is $z = \phi_1(y - x) + \phi_2(y - x) + \phi_3(y + 2x) + \frac{1}{27}\sin(x + 2y)$

Example (2.14):

Find the particular integral of $\left(D^2 - 2DD' + D'^2\right)z = \tan(x + y)$

Solution:

$P.I = \frac{1}{D^2 - 2DD' + D'^2}\tan(x + y) = \frac{x^2}{1^2 \, 2!}\tan(x + y) = \frac{x^2}{2}\tan(x + y)$

2.6. Classification of second order partial differential equations:

The most general linear second order partial differential equation is

$$A\, u_{xx} + B\, u_{xy} + C\, u_{yy} + D\, u_x + E\, u_y + F\, u = G$$

Where the coefficients A, B, C, D, E, F, G are functions of x and y or constants.

The above partial differential equation is elliptic, parabolic or hyperbolic at a point (x_0, y_0) according as the discriminant $B^2(x_0, y_0) - 4\, A(x_0, y_0)\, C(x_0, y_0)$ is negative, zero or positive.

2.7. Canonical Forms:

Given partial differential equation is $A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + D\,u_x + E\,u_y + F\,u = G$

Consider the transformation $\xi = \xi(x,y), \eta = \eta(x,y)$

So that $u_x = u_\xi \xi_x + u_\eta \eta_x$

$u_y = u_\xi \xi_y + u_\eta \eta_y$

$u_{xx} = u_{\xi\xi}\,{\xi_x}^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}{\eta_x}^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$

$u_{xy} = u_{\xi\xi}\,\xi_x\xi_y + u_{\xi\eta}\,(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\,\eta_x\eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy}$

$u_{yy} = u_{\xi\xi}\,{\xi_y}^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}{\eta_y}^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}$

Substituting these the given equation reduces to $\bar{A}u_{\xi\xi} + \bar{B}u_{\xi\eta} + \bar{C}u_{\eta\eta} + \bar{D}u_\xi + \bar{E}u_\eta + \bar{F}\,u = \bar{G}$

Where

$\bar{A} = A\,{\xi_x}^2 + B\,\xi_x\xi_y + C\,{\xi_y}^2$

$\bar{B} = 2\,A\,\xi_x\eta_x + B\,(\xi_x\eta_y + \xi_y\eta_x) + 2C\,\xi_y\,\eta_y$

$\bar{C} = A\,{\eta_x}^2 + B\,\eta_x\eta_y + C{\eta_y}^2$

$\bar{D} = A\,\xi_{xx} + B\,\xi_{xy} + C\,\xi_{yy} + D\xi_x + E\,\xi_y$

$\bar{E} = A\,\eta_{xx} + B\,\eta_{xy} + C\,\eta_{yy} + D\,\eta_x + E\,\eta_y$

$\bar{F} = F\,, \bar{G} = G$

2.7.1. Canonical form for Hyperbolic Equation:

$\bar{A} = A\,{\xi_x}^2 + B\,\xi_x\xi_y + C\,{\xi_y}^2 = 0$

$\bar{C} = A\,{\eta_x}^2 + B\,\eta_x\eta_y + C\,{\eta_y}^2 = 0$

$or, A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0$ and $A\left(\frac{\eta_x}{\eta_y}\right)^2 + B\left(\frac{\eta_x}{\eta_y}\right) + C = 0$

Solving,

$\frac{\xi_x}{\xi_y} = \frac{-B+\sqrt{B^2-4AC}}{2A}$

$\frac{\eta_x}{\eta_y} = \frac{-B-\sqrt{B^2-4AC}}{2A}$

Also, characteristics equations are $\frac{dy}{dx} = -\frac{\xi_x}{\xi_y}, \frac{dy}{dx} = -\frac{\eta_x}{\eta_y}$

Example (2.15):

$3u_{xx} + 10\, u_{xy} + 3\, u_{yy} = 0$

Here $A = 3, B = 10, C = 3$

$B^2 - 4AC = 6A > 0.$

Hence the given equation is a hyperbolic partial differential equation.

The characteristic equations are

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = -\frac{-B + \sqrt{B^2 - 4AC}}{2A} = \frac{1}{3}$$

$$\frac{dy}{dx} = -\frac{\eta_x}{\eta_y} = -\frac{-B - \sqrt{B^2 - 4AC}}{2A} = 3$$

So, $y = 3x + c_1 \;, y = \frac{1}{3}x + c_2$

$\therefore$ Transformations are $\xi = y - 3x,\; \eta = y - \frac{1}{3}x$

$\bar{A} = 3\,(-3)^2 + 10\,(-3)\,(1) + 3 = 0$

$\bar{B} = 2 \cdot 3\,(-3) \cdot \left(-\frac{1}{3}\right) + 10\left((-3) \cdot 1 + 1 \cdot \left(-\frac{1}{3}\right)\right) + 2 \cdot 3 \cdot 1 \cdot 1 = -\frac{64}{3}$

$\bar{C} = 0,\; \bar{D} = 0,\; \bar{E} = 0, \bar{F} = 0$

Canonical equation is $\frac{64}{3} u_{\xi\eta} = 0 \quad or, u_{\xi\eta} = 0$

On integration, $u\,(\xi, \eta) = f(\xi) + g(\eta)$

$or, u(x, y) = f(y - 3x) + g\left(y - \frac{x}{3}\right)$

This is the general solution.

2.7.2. Canonical from Parabolic Equation:

$\bar{A} = 0 \quad or\; \bar{C} = 0$

Let $\bar{A} = 0 \Rightarrow A\,{\xi_x}^2 + B\,\xi_x\xi_y + C\,{\xi_y}^2 = 0$

$\frac{\xi_x}{\xi_y} = -\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$

For parabolic case, we get $\frac{\xi_x}{\xi_y} = -\frac{B}{2A}$

Characteristic equation is $\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{B}{2A} \Rightarrow \xi(x, y) = c_1$

Example (2.16): $x^2u_{xx} - 2xy\,u_{xy} + y^2u_{yy} = e^x$

The discriminant is $B^2 - 4AC = 4\,x^2y^2 - 4x^2y^2 = 0$

So, given equation is parabolic.

Characteristics equation is $\frac{dy}{dx} = \frac{B}{2A} = -\frac{2xy}{2x^2} = -\frac{y}{x}$

On integration, $xy = c$

Let $\xi = xy$

$\bar{A} = 0,\ \ \bar{B} = 0, \bar{C} = y^2,\ \bar{D} = -2xy\,,\ \bar{E} = 0$

$\bar{F} = 0\,, \bar{G} = e^x$

Transformed equation is $\qquad y^2u_{\eta\eta} - 2\,\xi\,u_\xi = e^{\xi/\eta}$

or, $\eta^2u_{\eta\eta} = 2\xi u_\xi + e^{\xi/\eta}$

The canonical form is $u_{\eta\eta} = \frac{2\xi}{\eta^2}u_\xi + \frac{1}{\eta^2}e^{\xi/\eta}$

2.7.3. Canonical form for Elliptic equation:

For elliptic case $\qquad B^2 - 4AC < 0,$

Characteristic equations are

$\frac{dy}{dx} = \frac{B\pm\sqrt{B^2-4AC}}{2A}$

Let $\alpha = \frac{\xi+\eta}{2}\ ,\beta = \frac{\xi-\eta}{2i}$

Example (2.17):

$u_{xx} + x^2u_{yy} = 0$

The discriminant $\qquad B^2 - 4AC = -4x^2 < 0$

So, the given equation is elliptic.

The characteristic equations are

$\frac{dy}{dx} = \frac{B\pm\sqrt{B^2-4AC}}{2A} = \pm ix$

On integration, $iy + \frac{x^2}{2} = c_1, -iy + \frac{x^2}{2} = c_2$

Let $\xi = \frac{1}{2}x^2 + iy, \quad \eta = \frac{1}{2}x^2 - iy$

Also let $\qquad \alpha = \frac{\xi+\eta}{2}\ ,\beta = \frac{\xi-\eta}{2i}$

So, that $\alpha = \frac{x^2}{2},\ \beta = y$

Now, $\bar{A} = x^2\,,\ \bar{B} = 0, \bar{C} = x^2,\ \bar{D} = 1\,,\ \bar{E} = 0,\ \bar{F} = 0\,, \bar{G} = 0$

Hence the required canonical equation is

$x^2u_{\alpha\alpha} + x^2u_{\beta\beta} + u_\alpha = 0$

$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{u_\alpha}{2\alpha}$

2.8. A few Well – Known partial differential equation:

(i) (i) $u_{xx} + u_{yy} + u_{zz} = 0$ [Laplace equation]

(ii) $u_t = K(u_{xx} + u_{yy} + u_{zz})$ [Heat equation]

(iii) $u_{tt} = C^2(u_{xx} + u_{yy} + u_{zz})$ [Wave equation]

(iv) $u_t + u\, u_x = \mu\, u_{xx}$ [Burger equation]

2.9. Method of Separation of variables:

2.9.1. Laplace Equation (in two dimension):

$u_{xx} + u_{yy} = 0$

We assume the solution in the form $u(x,y) = X(x)Y(y)$

$\therefore X''Y + Y''X = 0$

$or,\ \frac{X''}{X} = -\frac{Y''}{Y} = k\ (say)$

Case – I: Let $K = p^2,\quad p\ is\ real$.

$Then\ \frac{d^2X}{dx^2} - p^2X = 0$ and $\frac{d^2Y}{dy^2} + p^2Y = 0$

Solution is $X = c_1e^{px} + c_2e^{-px}$ and $Y = c_3\cos py + c_4\sin py$

Thus the solution is $u(x,y) = (c_1e^{px} + c_2e^{-px}) \cdot (c_3\cos py + c_4\sin py)$

Case – II: Let $k = 0$. Then $\frac{d^2X}{dx^2} = 0\ \ and\ \ \frac{d^2Y}{dy^2} = 0$

Integrating twice, we get

$X = c_5\, x + c_6$

$Y = c_7\, y + c_8$

Solution is $u(x,y) = (c_5\, x + c_6)(c_7\, y + c_8)$

Case – III: Let $K = -p^2$

$X = c_9\cos px + c_{10}\sin px$

$Y = c_{11}e^{py} + c_{12}\sin e^{-py}$

Hence the solution is $u(x,y) = (c_9\cos px + c_{10}\sin px) \cdot (c_{11}e^{py} + c_{12}\sin e^{-py})$

2.9.2. Heat Equation:

$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}$, $-\infty < x < \infty,\ t > 0$.(One dimensional)

Let $u(x,t) = X(x)Y(t)$

So $\frac{X''}{X} = \frac{1}{\alpha}\frac{Y'}{Y} = \lambda$

$\Rightarrow Y = c\, e^{\alpha\lambda t}$

Let $\lambda = -\mu^2$

So, $X'' + \mu^2 X = 0$

$X = c_1 \cos\mu x + c_2 \sin\mu x$

Hence, $u(x,t) = (A\cos\mu x + B\sin\mu x)\, e^{-\alpha\mu^2 t}$

2.9.3. Wave Equation:

$u_{tt} = c\, u_{xx} \qquad (one\ dimensional)$

Let $u(x,t) = X(x)\, Y(t)$

So, $X\, \frac{d^2Y}{dt^2} = c^2\, Y\, \frac{d^2X}{dx^2}$

i.e., $\frac{\frac{d^2X}{dx^2}}{X} = \frac{\frac{d^2Y}{dt^2}}{c^2Y} = K$

Case – I

Let $K = \lambda^2 (K > 0)$

$\frac{d^2X}{dx^2} - \lambda^2 X = 0$

$\frac{d^2Y}{dt^2} - c^2\lambda^2 Y = 0$

$\Rightarrow X = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$

$Y = c_3 e^{c\lambda t} + c_4 e^{-c\lambda t}$

$u(x,t) = \left(c_1 e^{\lambda x} + c_2 e^{-\lambda x}\right)\left(c_3 e^{c\lambda t} + c_4 e^{-c\lambda t}\right)$