ELEMENTARY INTERSECTION THEORY

1. Introduction

We present a treatment of elementary intersection theory in algebraic topology. That is, we aim to show how one can define cochains through counting intersection with submanifolds. Moreover, the "fundamental theorem" is that the cup product of the associated cohomology classes is represented by the (transversal) intersection of submanifolds. Students are often told that cup product is named so because of this fact, but we feel it is given relatively short shrift. It is usually proved after development of duality or a Thom isomorphism theorem. We would like to put the representation of cohomology by submanifolds at the front and center of our treatment and from that deduce, or at least interpret, these related isomorphisms.

Unlike chains, singular cochains tend to be more transcendental objects. From a pedagogical viewpoint, one can often express cycles as in terms of explicit chains, which is basically never the case for singular cocyles. Our work remedies this discrepancy, and allows for geometric cochain-level understanding of a number of topics in algebraic topology including duality, Thom isomorphisms, cohomology of mapping spaces etc.

2. Basic definitions, and a simplified version of the fundamental theorem

Before developing the background in differential topology needed, we illustrate the technique by using simplified definitions.

Definition 1. *Manifold, submanifold.*

Definition 2. *Tangent bundle and tangent vector of a submanifold.*

Definition 3. *Transversal intersection of submanifolds.*

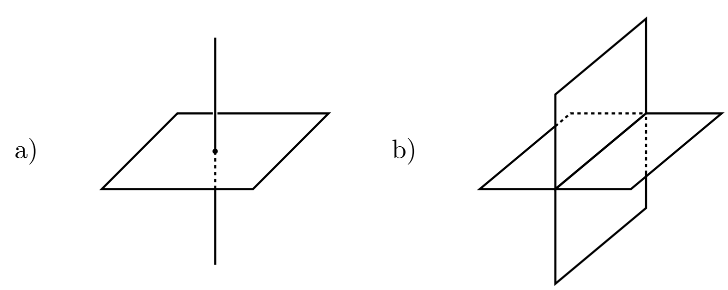

Figure 1. Examples of transversally intersecting submanifolds of $\mathbb{R}^3$: a) the plane $z = 0$ and the line $x = y = 0$ and b) the planes $z = 0$ and $y = 0$.

Definition 4. *If M is a smooth manifold of dimension m and $W \subseteq M$ is a smooth submanifold of dimension k, then the codimension of W in M is the number* $\mathrm{codim} W = m - k$.

Theorem 5. *If submanifolds intersect transversally, then the intersection is a submanifold. The codimension is the sum of their codimensions.*

We can now state a simplified version of the fundamental theorem of intersection theory.

Theorem 6. *Let M be a manifold and W be a submanifold of codimension d without boundary.*

(1) *There is a cohomology class $[\tau_W] \in H^d(M;\mathbb{Z}/2)$ represented by a cochain whose value on an embedded chain $\Delta^d \subset M$ which intersects W transversally is the mod-two count of $\Delta^d \cap W$.*
(2) *If S is a submanifold whose boundary is $V \sqcup W$ then $[\tau_V] = [\tau_W]$.*
(3) *If V and W intersect transversally then $[\tau_V] \cup [\tau_W] = [\tau_{V\cap W}]$.*

2.1. **Examples using the simplified version.**

Example 1. $\mathbb{R}^2 - 0$

Example 2. $[\tau_M] = 1$. $[\tau_{\text{point}}]$.

Example 3. $\mathbb{R}^3$ *with the z-axis removed along with either a linked or unlinked* S^1.

Example 4. *A two-holed torus*

Example 5. $\mathbb{R}\mathrm{P}^n$ *and application to no retraction.*

2.2. **Comparison with other approaches to cochains.** Cochains are transcendental data (a value on every chain). But some representations are more finite.

Focus on torus example. Both simplicial and de Rham generally assess "tolls" for chains. Both can be made "concentrated" as one crosses a submanifold.

3. Transversality and preimage cochains

(Adapted from treatment by Dominic Joyce in "On manifolds with corners.")

We operate at the interface of smooth and simplicial topology, which requires considering simplices as manifolds with corners. Let $\mathbb{R}^+$ be the subset of non-negative real numbers, and let $\mathbb{R}^{k,\ell} = \mathbb{R}^k \times (\mathbb{R}^+)^\ell$. A smooth map between subspaces of Euclidean spaces is the restriction of such between some neighborhoods of those subspaces, in which case the derivative exists and we can define diffeomorphism as usual.

Definition 7. *A manifold with corners is a paracompact Hausdorff space M with (a maximal collection of) charts $\phi_\alpha : U_\alpha \to M$ with U_α open in $\mathbb{R}^{0,n}$ such that the transition maps $\phi_\beta^{-1} \circ \phi_\alpha$ are diffeomorphisms between $\phi_\alpha^{-1}(\mathrm{Im}\phi_\beta)$ and $\phi_\beta^{-1}(\mathrm{Im}\phi_\alpha)$.*

An important non-example for the theory of manifolds with corners is the map $\mathbb{R}^{0,2} \to \mathbb{R}^{1,1}$ which in polar coordinates multiplies the angle with the x-axis by two, which while being smooth is not a diffeomorphism at the origin.

The tangent bundle can be defined in any of the ways in which one defines it for manifolds.

Lemma 8. *Every point x in a manifold with corners M has a neighborhood diffeomorphic to (a neighborhood of 0 in) $\mathbb{R}^{k,\ell}$ for some unique $k+\ell=n$. The number ℓ is called the depth of x.*

For example, the depth 0 points are also known as the interior of M.

A key structure to consider for a manifold with corners is its partition into connected subspaces with fixed depth, which is sometimes called a stratification. In the case of polytopes such as the cube this is essentially the facet structure. A delicate issue is how to take a "closure" of such facets or strata for general manifolds with corners, for which we rely on the tangent bundle.

Definition 9. *A codimension-i facet subspace of $T_0\mathbb{R}^{k,\ell} \cong \mathbb{R}^{k+\ell}$ is a linear subspace in which a fixed nonempty subset of cardinality i of the coordinates which correspond to coordinates in $(\mathbb{R}^+)^\ell$ are zero.*

Thus $T_0R^{k,\ell}$ has $2^\ell - 1$ facet subspaces total, over all codimensions.

Definition 10. *A facet subspace of T_xM is the image of some facet subspace of $T_0R^{k,\ell}$ under the derivative $D\phi$ of a chart ϕ about x.*

A facet subspace thus consists of vectors which are tangent to a stratum containing x in its closure. While there is a unique minimal one, given by the image of the stratum containing x itself, we need to consider the full poset of such subspaces.

Definition 11. *A boundary facet of a manifold with corners M is a maximal manifold with corners F with an immersion $\iota_F : F \to M$, injective on the interior of F, such that $D\iota_F$ maps TF injectively to a subbundle of facet subspaces of TM.*

Example: Δ^n

The only examples we need to consider are M for which the facet immersions ι_F are embeddings, as happens for polytopes. Such M are called manifolds with embedded corners. The "teardrop" subspace of $\mathbb{R}^2$ (homeomorphic to a disk, with a single corner) is a typical example of a manifold with corners which does not have embedded corners.

Constructions such as products and (good) intersections of manifolds with corners are rich because of the interplay of the stratifications in a manifold-with-corners structure on the result. Consider, for example, a generic linear intersection between a solid cube and the boundary of a tetrahedron. We will not take the opportunity to enjoy working through this, because the submanifolds-with-corners we need have dimension zero or one, and thus have at most boundaries.

Transversality more generally.

Definition 12. *Let M be a manifold and $f : W \to M$ an immersion. Then $C_*^{\pitchfork W}(M)$ is the subset of $C_*(M)$ generated by the chains whose image is transversal to f. More generally, if S is a finite set of immersed submanifolds of M then $C_*^{\pitchfork S}$ is generated by the chains whose image is transversal to each $W \in S$.*

Theorem 13. *For any finite collection S of submanifolds of M, the inclusion $C_*^{\pitchfork S}(M) \hookrightarrow C_*(M)$ admits a chain deformation retraction.*

Proof. We wish to construct a map $j : C_*(M) \to C_*^{\pitchfork S}(M)$ such that j is the identity on $C_*^{\pitchfork S}(M)$ and $i \circ j$ is chain homotopic to the identity on $C_*(M)$. Given $\sigma : \Delta^d \to M$, we can cover its image with a finite number

of open sets $\{U_i\}$ in M diffeomorphic to open subsets of $\mathbb{R}^k$, and we can choose these so that on the overlap the transition from one diffeomorphism to another is smooth. We proceed inductively:
If $e_k^0 \subset U_k$ is not transversal to S, then $e_k^0 \in S$ and by Sard's Theorem we may choose a path γ_k in U_k from e_k^0 to a point not in S. If $e_k^0 \pitchfork S$, then γ_k is the constant map. Using the consistent cover of σ, we may extend this to a map $\bar{\sigma} : \Delta^d \to M$ such that $\bar{\sigma}$ on the 0-cells is defined by the endpoints of the γ_k's. This gives a map $\left(\Delta^d \times \{0,1\}\right) \cup \left(\{e_k^0\} \times I\right) \to M$, and simplicial complexes have the homotopy extension property, so can extend this to a homotopy $F : \Delta^d \times I \to M$. Then by Theorem **??**, there exists a smooth homotopy F' which agrees with F on the 0-cells and $\sigma' = F'|_{\Delta^d \times \{1\}}$ is transversal to S.
Now assume that we have a homotopy ∂G on the cells of dimension $< q$. If e_k^q is not transversal to S, then we........ [[Use relative transversality extension.]]

We define $j(\sigma) = \sigma'$ and since $j|_{C_*^{\pitchfork S}(M)} = id$ and the two chains are always homotopic, $C_*^{\pitchfork S}(M) \hookrightarrow C_*(M)$ is a chain deformation retraction and their homology groups are isomorphic. □

We now define our main objects of study.

Definition 14. *If $W \in S$ then $\tau_W \in C^*_{\pitchfork S}(M)$ is the cochain defined by $\tau_W(\sigma) = \#P$, where P is the zero-manifold defined as the following pullback*

$$\begin{array}{ccc} P & \longrightarrow & W \\ \big\downarrow & & \big\downarrow f \\ \Delta^n & \xrightarrow{\sigma} & M. \end{array}$$

The cardinality $\#P$ is to be taken as mod-two unless M and W are oriented, in which case Δ^n is given the orientation with say the differences between vertices labeled by $1, \cdots, n$ and vertex 0 (or the barycenter) serving as basis vectors, and P is then oriented accordingly.

We call these Thom classes, because in the standard development they are (the pushforwards of) Thom classes of the normal bundles. But we will use them to understand both pushforward and Thom classes, rather than the other way around.

In de Rham theory, Stokes' theorem shows that the exterior differential agrees with coboundary. We have a similar result in this setting, which immediately implies that preimage cochains of manifolds without boundary are cocycles. First we recall some basic facts about preimage submanifolds. The following is the Theorem on page 60 of [1].

Lemma 15. *Let M be a manifold with boundary and P a submanifold without boundary of N. If both $f : M \to N$ and its restriction to ∂M are transverse l to P then $f^{-1}(P)$ is a submanifold whose boundary is its intersection with ∂M.*

Similarly we have the following.

Lemma 16. *Let M be a manifold without boundary and P a submanifold with boundary of N. If $f : M \to N$ is transverse to P and ∂P then $f^{-1}(P)$ is a submanifold with boundary given by $f^{-1}(\partial P)$.*

Sketch. Standard transversality takes care of preimage of the interior of P. On the boundary, in local coordinates P is given as a subspace of the form $(x_1, \ldots, x_\ell, 0, \ldots, 0)$ with $x_\ell > 0$. Thus the preimage of P extends and on this extension one can apply the Lemma on page 62 of [1] to get that the preimage of P is a manifold with boundary. □

Theorem 17. *If $\partial V = W$ then $\delta\tau_V = \tau_W \in C^*_{\pitchfork\{V,W\}}$. In particular, if V has no boundary than τ_V is a cocycle.*

Sketch, for now. Like the proof that degree is homotopy invariant, this follows from the classification of one-manifolds. Consider $\sigma : \Delta^{n+1} \to M$. Look at pull back of W through σ to get a one-manifold with boundary, by lemmas above. Some boundary points are on $\partial\Delta^{n+1}$ - counting those gives $\delta\tau_V(\sigma) = \tau_V(\partial\sigma)$. Other boundary points are in the interior, which come from the preimage of $\partial V = W$, and thus correspond to $\tau_W(\sigma)$. □

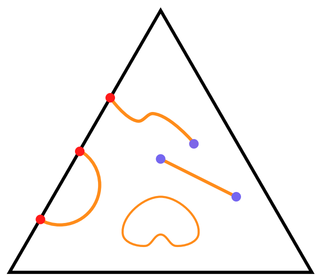

FIGURE 2. Example in the case $n = 1$ with $\sigma^{-1}(V)$ in orange, points in $\left(\sigma|_{\partial\Delta^2}\right)^{-1}(V)$ marked in red, and those in $\sigma^{-1}(\partial V)$ marked in blue.

By abuse, we denote by $[\tau_W]$ the corresponding singular cohomology class under the isomorphism established in Theorem 13.

Corollary 18. *If $\partial V = W \sqcup W'$ then $[\tau_W] = [\tau_{W'}]$.*

This theorem and its corollary recovers a classical view of cohomology, as represented by submanifolds with the relation defined by submanifolds with boundary. But in fact not all cocycles are represented in this way, which was a famous question of Steenrod addressed by Renee Thom. Our theory allows for linear combinations of submanifolds, and even manifolds with corners below, in which case all cohomology is representable. The submanifold point of view led Thom to initiate cobordism theory, which gives generalized cohomology theories.

Induced maps are a basic ingredient of cohomology theory. In the case of Thom classes, they are geometrically defined on the cochain level. The proof of the following is immediate.

Proposition 19. *If $f : M \to N$ is transverse to $W \subset N$ then $f^\#(\tau_W) = \tau_{f^{-1}W}$.*

4. Intersection and cup product

In this section we prove the following.

Theorem 20. *If V and W intersect transversally, and the intersection is given the orientation... then $[\tau_V] \smile [\tau_W] = [\tau_{V \cap W}]$.*

Recall that the cup product is essentially induced by the diagonal map $X \overset{\Delta}{\to} X \times X$. Under the standard formulaic approach, the cohomology of a product $X \times Y$ is developed through the external cup product, namely $C^*(X) \otimes C^*(Y)$ maps to $C^*(X \times Y)$ by sending $\alpha \otimes \beta$ to $p^*\alpha \cup q^*\beta$, where p is the projection from $X \times Y$ to X and q is the projection to Y.

Proposition 21. *The external cup product sends* $[\tau_V] \otimes [\tau_W]$ *to* $[\tau_{V \times W}]$

Proof of the Fundamental Theorem, based on Proposition 20. If $V \pitchfork W$ then $\Delta \pitchfork V \times W \subset X \times X$. Apply Proposition 18. □

Proof of Proposition 20. Recall Künneth theorem that $C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$ given by (subdividing) products of simplices in X and Y is an isomorphism on homology. Then show that $p^*\tau_V \cup q^*\tau_W$ and $\tau_{V \times W}$ agree on this subcomplex, which is has an immediate geometric proof. □

5. Examples

We make some calculations It is more standard to establish fundamental classes in homology, rather than Thom classes. But we see that Thom classes are just as natural if not moreso. The following is immediate from the definitions.

Proposition 22. *The value of a Thom class on a fundamental class given by triangulating a submanifold is given by intersection.*

5.1. **Projective spaces.**

5.2. **Grassmannians.**

5.3. **Complements of submanifolds which are boundaries.**

5.4. **Configuration spaces.**

6. Applications: wrong-way maps, duality, suspensions and Thom isomorphism

7. Looking forward: characteristic classes, loop spaces, and cobordism.

References

[1] V. Guillemin and A. Pollack. *Differential Topology.* AMS Chelsea Publishing. AMS Chelsea Pub., 2010.
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[3] J.W. Milnor and D.W. Weaver. *Topology from the Differentiable Viewpoint.* Princeton Landmarks in Mathematics. Princeton University Press, 1997.

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