

Insertion Sort, Proving Correctness, Run time analysis, Merge Sort, Quick Sort

1 Introduction to Sorting - Insertion Sort (CLRS §2.1, 2.2)

Input: array $A[1 \dots n]$ of n numbers

Goal: Sort in non decreasing order.

How do we write the algorithm?

in case of repeated elements (doesn't strictly increase)

1.1 The algorithm in words

Scan the array from left to right.

For each position j , insert $A[j]$ into its correct position in $A[1 \dots j]$ by moving all elements larger than $A[j]$ one position to the right

1.2 Pseudocode

```

1. for j = 2 to n:
2.   Key ← A[j]
3.   i ← j - 1
4.   while i > 0 and A[i] > Key:
5.     A[i+1] ← A[i] ← move element
6.     i ← i - 1      one position to the
7.   A[i+1] ← Key    right
  
```

2 Proof by Induction, Loop Invariants, and Proving Correctness

- **Proof by induction:** We want to prove some statement $P(n)$ for integers $n \geq 0$. (or $n \in \mathbb{N}$
i.e., $n \geq 1$)

1. **Base case:** Prove $P(0)$ or $P(1)$

2. **Inductive step:**

i.e. show $P(K) \rightarrow P(K+1)$

Inductive hypothesis: Assume $P(n)$ holds for $n = K$
 Goal: Show $P(K+1)$ holds
 $\xrightarrow{P(K)}$ **WARNING!** Do NOT assume $P(K+1)$ and "work backwards"
 to a true statement for the formal proof.

3. **Conclusion:** $P(n)$ holds for all integers $n \geq 0$. by principle of mathematical induction!

- **Loop invariant:** A property that holds throughout the execution of the algorithm

1. **Initialization:** loop invariant holds prior to the first iteration of the loop

2. **Maintenance:**

If the loop invariant is true before an iteration of the loop, then it remains true before the next iteration.

3. **Termination:** When the loop terminates, the invariant gives useful property that helps show that the algorithm is correct.

Our loop invariant for insertion sort:

At the beginning of the j th iteration (or equivalently, the end of $j-1$ st iteration)

At the end,
 $j = n+1$

$A[1 \dots j-1]$ contains the elements that

were originally there, but in sorted order.

"Termination" if loop invariant holds at the end of the algorithm, it tells us the output is correct.

Claim: Loop invariant holds for $j=2, 3, 4, \dots, n+1$.

```

1. for j=2 to n:
2.   Key ← A[j]
3.   i ← j-1
4.   while i>0 and A[i]>Key:
5.     A[i+1] ← A[i]
6.     i ← i-1
7.   A[i+1] ← Key
  
```

2.1 Proof of correctness of Insertion Sort

By induction.

First base case of induction ("initialization")
When $j=2$, $A[1]$ is just one number and is therefore sorted.

Induction Step ("maintenance") Assume the loop invariant holds for j . We will prove it holds for $j+1$.

inductive hypothesis.

By assumption $A[1 \dots j-1]$ is sorted.

During the j th iteration, we insert $A[j]$ into its correct position. Therefore, after the j th iteration, $A[1 \dots j]$ are sorted, so the loop invariant holds at the end of the j th iteration.

QED

$$\begin{aligned} S &= 1 + 2 + 3 + \dots + n-2 + n-1 + n \\ S &= 100 + 99 + 98 + \dots + 3 + 2 + 1 \end{aligned}$$

$$\begin{aligned} 2S &= 101 + 101 + 101 + \dots + 101 + 101 + 101 \\ &= 100 \times 101 \end{aligned}$$

$$S = \frac{100 \times 101}{2}$$

3 Introduction to analyzing running times

Example

```
{ for i = 1 to n
      A[i] ← 5
```

n steps
2n steps?
3n steps?

$\Theta(n)$

3.1 Running time of insertion sort

Actually mean "worst case"

$$\sum_{j=2}^n (c + c' t_j)$$

↑ some constants
(maybe $c \approx 3$, $c' \approx 2$)

↑ number of iterations
of the while loop

$$\leq \sum_{j=2}^n (c + c' j) = c \overbrace{(n-1)}^{j=2, \dots, n \rightarrow n-1 \text{ iterations}} + c' \sum_{j=2}^n j$$

$$= c(n-1) + c' \left(\frac{n(n+1)}{2} - 1 \right)$$

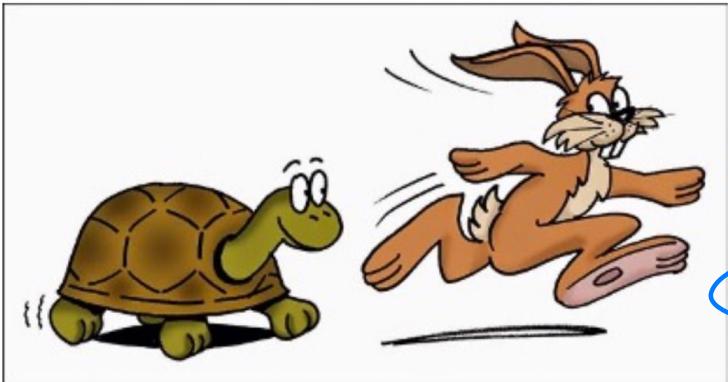
$$= \frac{c'}{2} n^2 + \left(c + \frac{c'}{2} \right) n - (c + c')$$

$$= \Theta(n^2) \quad \begin{matrix} \text{running time of} \\ \text{insertion sort.} \end{matrix}$$



```
1. for j = 2 to n:
   Key ← A[j] } C
   i ← j-1
2. } C
3. while i > 0 and A[i] > Key:
4.   A[i+1] ← A[i]
5.   i ← i-1
6. } C'
7.   A[i+1] ← Key
```

Big-O (Asymptotic) Notation: Motivation



Motivation Why do we use it?

Which one below is easier to read?

- Algorithm A sorts a list containing n items in time at most $2.95n^2 + 10n + 5$
- Algorithm A sorts a list containing n items in time at most Cn^2 for some constant $C > 0$
- Algorithm A sorts a list containing n items in time $O(n^2)$

Prove that $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ $n \in \mathbb{N}$

Let $k \in \mathbb{N}$

inductive step

Inductive hypothesis: Assume

$$\sum_{i=1}^k i = \frac{k(k+1)}{2}$$

want to show

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

Don't START with

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

Assumes what we are trying to show.

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^k i + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1)$$

.....

$$= \frac{(k+1)(k+2)}{2}$$

by the inductive hypothesis

4 Growth of Functions and Asymptotic Behavior (CLRS §3.1)

Goal: Establish notation that enables us to compare the relative performance of different algorithms.

a "nice" function (one term, no constant factors)

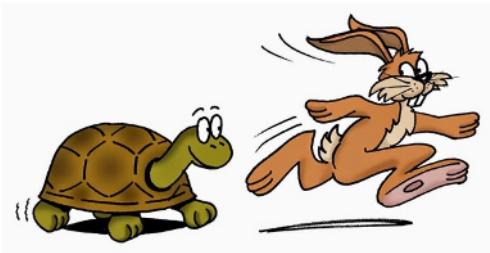
\leq Definition large n. • $T(n) = O(g(n))$ means there exists $c > 0$ such that $T(n) \leq cg(n)$ for sufficiently large n.

(Big-Oh)

T is bounded above by g asymptotically

\geq • $T(n) = \Omega(g(n))$ means there exists $c > 0$ such that $T(n) \geq cg(n)$ for sufficiently large n.

"=" • $T(n) = \Theta(g(n))$ means there exists c_1, c_2 such that $c_1g(n) \leq T(n) \leq c_2g(n)$ for sufficiently large n.



O() notation exercises

O(·)-Notation

- Asymptotic notation:

- Functions $f(n)$ and $g(n)$ represent running times.
- $f(n) = O(g(n))$ means that exist $c, n_0 > 0$ such that $f(n) \leq c g(n)$ for every $n \geq n_0$.
- Intuitively, $f(n)$ does not grow faster than $g(n)$ when n is large.

negation:

For all $c > 0$, $n_0 > 0$
 there exists some $n \geq n_0$
 such that $100n^2 > cn$

$$100n^2 \neq O(n)$$

Take $n \geq \max\{c, n_0\}$
 $n \geq c$

$$nn \geq cn$$

$$100n^2 \geq 100cn
 > cn$$

$$4^n \neq O(2^n)$$

$$\lim_{n \rightarrow \infty} \frac{4^n}{2^n} = \lim_{n \rightarrow \infty} \left(\frac{4}{2}\right)^n
 = \lim_{n \rightarrow \infty} 2^n = \infty$$

Which of the following are true and why? Discuss with your teammates!

For true statements, find valid values of c and n_0

(Select all that apply.)

$1000 = O(n)$ True OR take $c=1, n_0=1000$

$100n^2 = O(n)$ false

$n + n^2 + n^3 + n^4 = O(n^4)$ True

$n^2 \cdot n^3 = O(n^4)$ False

$\frac{n^5 - 2n^4}{n} = O(n^4)$ False

$n^3 + n^4 = O(n^4)$ True

$4^n = O(2^n)$ false

$4^n ? \cancel{2 \cdot 2^{zn}}$
 $4^n = (2^2)^n = 2^{2n}$
 $100n = \infty$

$\lim_{n \rightarrow \infty} \frac{100n^2}{n} = \lim_{n \rightarrow \infty} 100n = \infty$

$n^3 + n^4 = O(n^4)$

$n^3 + n^4 \leq Cn^4$ for $n \geq n_0$

$c=2$

$n^3 + n^4 \leq 2n^4$ $n_0 = 1$

$n^3 + n^4 \leq n^4 + n^4$

$$4^n \neq O(2^n)$$

$$2^{2^n} = \underbrace{2^n \cdot 2^n}$$

let $c > 0, n_0 > 0$

want to show

$$\underbrace{4^n}_{\square} > c \cdot \underbrace{2^n}_{\square} \quad \text{for all } n \geq n_0$$

want $2^n \cdot 2^n > \underbrace{c \cdot 2^n}_{\square \square}$

(scratch)

$$\underbrace{2^n \cdot 2^n}_{\curvearrowleft} > \underbrace{2^c \cdot 2^n}_{\geq c} \quad \text{if } n > c$$

take $n = \max \{c, n_0\}$

$$2^n \geq 2^c$$

$$2^n \cdot 2^n \geq 2^c \cdot 2^n$$

$$\begin{aligned} 4^n &\geq 2^c \cdot 2^n \\ - & \\ &> \underbrace{c \cdot 2^n}_{\longrightarrow} \end{aligned}$$

4 Growth of Functions and Asymptotic Behavior (CLRS §3.1)

Goal: Establish notation that enables us to compare the relative performance of different algorithms.

$$T(n) = 5.5n^2 + 7.789n \leq cn^2$$

complicated

\downarrow
 $T(n) = O(g(n))$ means there exists $c > 0$ such that $T(n) \leq cg(n)$ for sufficiently large n .

Really $O(g(n))$ is a set

$$\lim_{n \rightarrow \infty} \frac{T(n)}{g(n)} = C < \infty$$

$$O(g(n)) = \{ f: \mathbb{N} \rightarrow \mathbb{R}^+ \mid \exists c > 0 \text{ such that } \forall n \geq n_0 \text{ where } f(n) \leq cg(n) \}$$

$T(n) \leq g(n)$

"there exists"

$g(n)$ is an asymptotic upper bound for $T(n)$, $T(n)$ does not grow faster than $g(n)$ when n is large.

- $T(n) = \Omega(g(n))$ means there exists $c > 0$ such that $T(n) \geq cg(n)$ for sufficiently large n .

(Omega)

$T(n) \geq g(n)$

$g(n)$ is an asymptotic lower bound for $T(n)$

$$\lim_{n \rightarrow \infty} \frac{T(n)}{g(n)} > 0$$

- $T(n) = \Theta(g(n))$ means there exists c_1, c_2 such that $c_1g(n) \leq T(n) \leq c_2g(n)$ for sufficiently large n .

(Theta)
For large n , $T(n)$ and $g(n)$ are "equal"
up to a constant factor

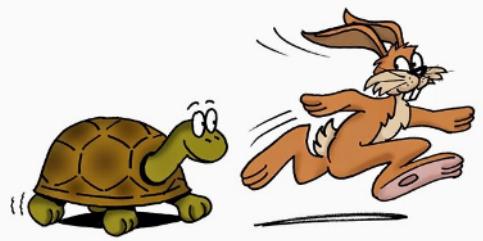
$$\lim_{n \rightarrow \infty} \frac{T(n)}{g(n)} = C > 0$$

(little-oh)

$T(n) = o(g(n))$ means $\lim_{n \rightarrow \infty} \frac{T(n)}{g(n)} = 0$

$T(n)$ is negligible in comparison to $g(n)$.

Remark if $f \in \Omega(g(n))$ and $f \in O(g(n))$
then $f \in \Theta(g(n))$



Example

- polynomials - quadratic for any $a > 0, b, c$ $\underbrace{an^2 + bn + c} \in \Theta(n^2)$
could be complicated
 - more generally $a_k n^k + a_{k-1} n^{k-1} + \dots + a_0 \in \Theta(n^k)$ $a_k > 0$ \uparrow
nice simple function
 - $n \log n = O(n^2)$
 - in fact: $\log n = O(n^\varepsilon)$ for any $\varepsilon > 0$ $\lim_{n \rightarrow \infty} \frac{\ln(n)}{n} = 0$
for little-oh
 - $2^n \log n \not\in O(2^n)$ $\cancel{O}(2^n)$
not a constant factor
 - $3^n \not\in O(2^n)$ but: $3^n = \Omega(2^n)$
 - $3^n + 2^n \leq 3^n + 3^n$
 - $3^n + 2^n \not\in \Theta(2^n)$
 $= \Theta(3^n)$
- For all $n \geq 1$
- $$\frac{1}{c_1} \cdot 3^n \leq 3^n + 2^n \leq \frac{2}{c_2} \cdot 3^n$$

4.1 Proofs involving order of growth

Claim. $f \in O(g(n))$ if and only if $g \in \Omega(f(n))$

added for posted notes (didn't go over in class)

Proof.
We'll show if $f \in O(g(n))$ then $g = \Omega(f(n))$ (\Rightarrow)

Assume $f \in O(g(n))$. Then for all n sufficiently large,

$$f(n) \leq cg(n) \quad \text{for some } c > 0.$$

Then for all n sufficiently large,

$$g(n) \geq \frac{1}{c} f(n)$$

Thus there exists c_1 (namely $c_1 = \frac{1}{c}$) such that for n sufficiently large, $g(n) \geq c_1 f(n)$. Hence $g \in \Omega(f(n))$.

(\Leftarrow) Similar exercise for you to try! i.e. lower bound up to constant factor for sufficiently large n . \square

Example $3n^3 + 5n^2 + 10643n \in \Theta(n^3)$

$$1 \cdot n^3 \leq 3n^3 + 5n^2 + 10643n \quad \text{for } n \geq 0$$

$$3n^3 + 5n^2 + 10643n \leq cn^3$$

$$n=10^6 \quad \underbrace{3 \cdot 10^{18}}_{\text{ }} + \underbrace{5 \cdot 10^{12}}_{\text{ }} + \underbrace{10643 \cdot 10^6}_{\text{ }} \leq \underbrace{4 \cdot 10^{18}}_{\text{ }} \quad c=4$$

 $4n^3$

Example Give an example of $T(n)$ and $g(n)$ such that $T(n) \neq o(g(n))$, but $T(n) = O(g(n))$.

$$T(n) = 2n^2$$

$$\lim_{n \rightarrow \infty} \frac{2n^2}{n^2} = 2 + 0$$

$$g(n) = n^2$$

Note: little-oh is stronger condition than big-oh.

Example $\overbrace{5n^2 + 11}^{\text{(little-oh)}} \in o(n^3)$ or $O(n^6)$ or $O(2^n)$

4.2 Asymptotic notation in equations

A set in a formula represents an anonymous function in that set.

Example $f(n) = n^3 + O(n^2)$

means there exists $h(n) = O(n^2)$

such that $f(n) = n^3 + h(n)$

e.g. if $f(n) = n^3 + 2n$, since $2n \in O(n^2)$,
 $f(n) = n^3 + O(n^2)$.

$2n = O(n^2)$
 Since $\lim_{n \rightarrow \infty} \frac{2n}{n^2} = 0$ or
 for large enough n ,
 $2n \leq 2n^2$
 $(c=2, n=1)$

Example $n^2 + O(n) = O(n^2)$

Convention: implicit "for all" on left hand side of equals and "there exists" on the right hand side.

→ means for any $f(n) = O(n)$, there exists $h(n) = O(n^2)$ such that $n^2 + f(n) = h(n)$.

intuitively: adding on any linear term to a quadratic results in still bounded above by some quadratic.

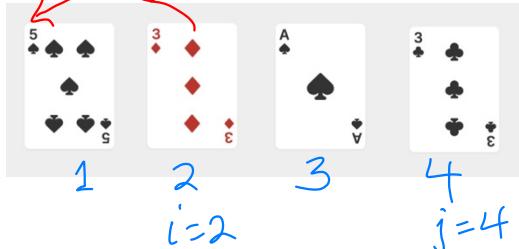
Also:
 (See posted MergeSort example)
 induction example)

5 Using a Loop Invariant to Prove Stability

{ HW 3
 Stability of MergeSort}

Definition A sorting algorithm is **stable** if objects with equal keys appear in the same order in the sorted output as in the unsorted input.

Example



→ in the sorted output of a stable sorting algorithm 3 of diamonds would appear before 3 of clubs. Note $i_1=2, j_1=4$
 $i_2=1, j_2=4$
 $i_3=2, j_3=4$

5.1 Stability Proof for Insertion Sort

initial

Let i and j be indices with $i < j$ such that $A[i].key = A[j].key$. We will show that their final positions i' and j' at the end of insertion sort satisfy $i' < j'$.

Loop invariant: let i_K and j_K be the positions at the start of the K^{th} iteration. Then $i_K < j_K$

Initialization: Before the first iteration,

$$i_1 = i \quad j_1 = j, \text{ so } i_1 < j_1$$

Maintenance:

Suppose for the inductive hypothesis that $i_K < j_K$. We will show that $i_{K+1} < j_{K+1}$.

① If neither $A[i_K]$ or $A[j_K]$ are being inserted into the sorted portion in the K^{th} iteration (either both already in sorted portion, or neither)

$$\text{either } i_{K+1} = i_K \text{ and } j_{K+1} = j_K$$

$$\text{OR } i_{K+1} = (i_K) + 1 \text{ and } j_{K+1} = (j_K) + 1 \text{ (both shifted down 1 spot)}$$

② If $A[i_K]$ is in the sorted portion and $A[j_K]$ is NOT being inserted in the K^{th} iteration then:

$$i_{K+1} = \begin{cases} (i_K) + 1 & \text{OR} \\ i_K & \end{cases} \quad j_{K+1} = j_K$$

(Termination):

either shifted down by 1 or stays

stays

③ If $A[j_K]$ is being inserted, then by the while loop condition on line 4,
 $j_{K+1} > i_{K+1}$

Note: by the inductive hypothesis, couldn't have $A[j_K]$ inserted and not $A[i_K]$.

1. for $j=2$ to n :
 2. Key $\leftarrow A[j]$
 3. $i \leftarrow j-1$
 4. while $i>0$ and $A[i]>\text{Key}$:
 5. $A[i+1] \leftarrow A[i]$
 6. $i \leftarrow i-1$
 7. $A[i+1] \leftarrow \text{Key}$

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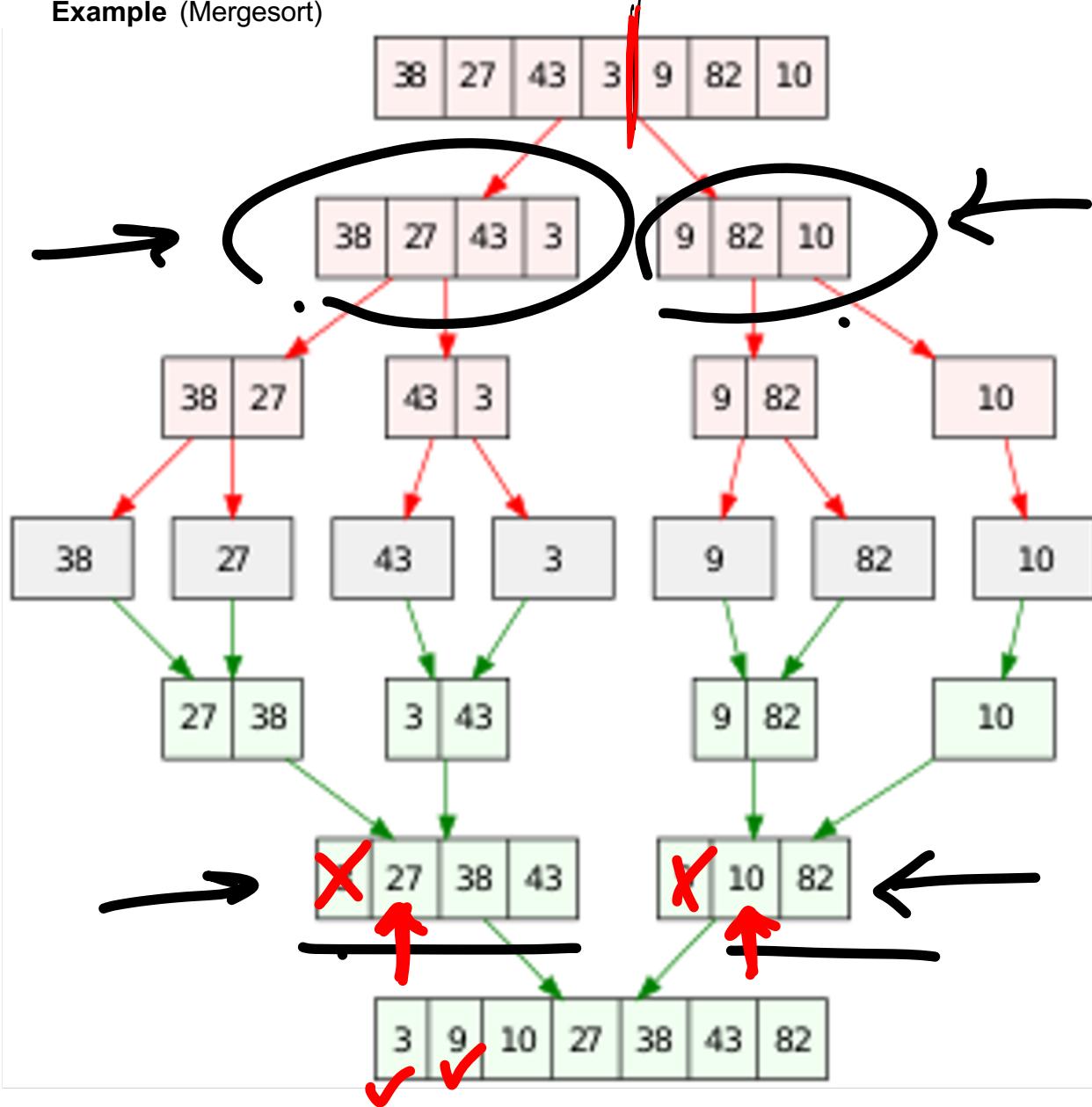
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6 Divide and Conquer

- Divide the problem into subproblems similar to original but smaller in size
- Conquer each subproblem recursively
- Combine solutions to subproblems

Example (Mergesort)



6.1 Mergesort (CLRS §2.3)

sorted sequence

1	2	2	3	4	5	6	6
---	---	---	---	---	---	---	---

merge

2	4	5	6
---	---	---	---

1	2	3	6
---	---	---	---

Repeated elements &
Stability-
you'll prove in
HW 2.

MERGESORT($A[1 \dots n]$)

MERGESORT($A[1 \dots \frac{n}{2}]$)

MERGESORT($A[\frac{n}{2}+1 \dots n]$)

MERGE $A[1 \dots \frac{n}{2}]$ and $A[\frac{n}{2}+1 \dots n]$

6.1.1 The Merge Subroutine

Description of the algorithm, or more English version of pseudocode:

To merge sorted arrays $L[1 \dots m]$ and $R[1 \dots p]$ into array $C[1 \dots m+p]$

Maintain a current index for each list, each initialized to 1

While both lists have not been completely traversed:

Let $L[i]$ and $R[j]$ be the current elements

Copy the smaller of $L[i]$ and $R[j]$ to C

Advance the current index for the array from which the smaller element was selected

EndWhile

Once one array has been completely traversed, copy the remainder of the other array to C

$m = L.length$

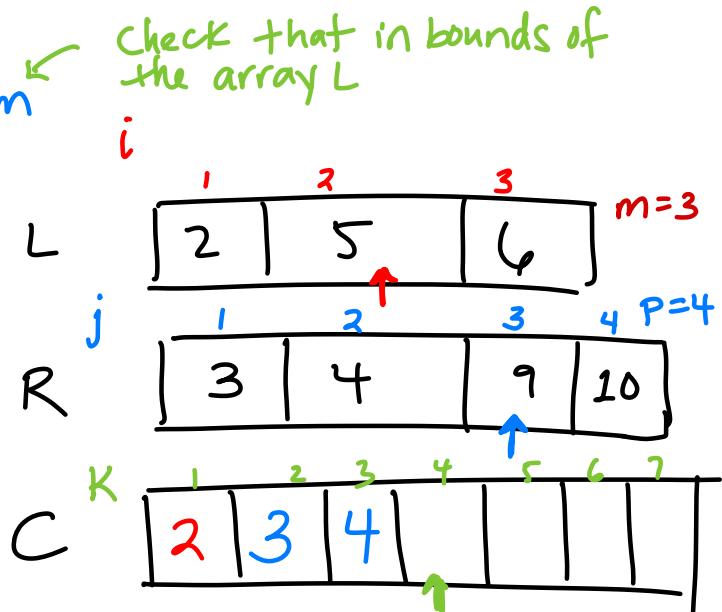
$p = R.length$

$MERGE(L, m, R, p, C) // L[1...m] and R[1..p] are sorted arrays$

```

1 i ← 1
2 j ← 1
3 for k = 1 to  $m + p$ 
4   if  $L[i] \leq R[j]$  and  $i \leq m$  Check that in bounds of the array L
5      $C[k] \leftarrow L[i]$ 
6      $i \leftarrow i + 1$ 
7   else if  $j \leq p$ 
8      $C[k] \leftarrow R[j]$ 
9      $j \leftarrow j + 1$ 

```



6.1.2 Proof of Correctness of Merge

Loop invariant:

At the start of the k th iteration $C[1 \dots k-1]$ contains the $k-1$ smallest elements of L and R in sorted order

Those elements are from $L[1 \dots i-1]$ and $R[1 \dots j-1]$

- Initialization: (Base case)

$i=j=k=1 \Rightarrow C$ is empty so C contains the 0 smallest elements of L and R in sorted order.

So the invariant holds.

Desmos slide 3 : Mergesort on

14 5 1 2 4

14	5	1
----	---	---

2	4
---	---

14	5
----	---

1

2

4

14

5

2

4

5	14
---	----



2	4
---	---

1	5	14
---	---	----

1	2	4	5	14
---	---	---	---	----

- Maintenance:

Assume the invariant holds for the k th iteration and (without loss of generality) $L[i] \leq R[j]$

Then $L[i]$ is the smallest among elements from L and R not copied yet into C .

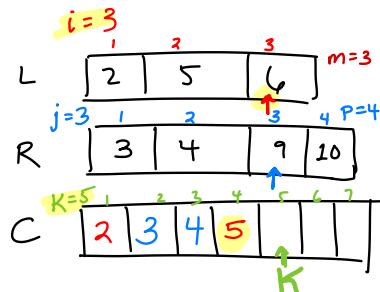
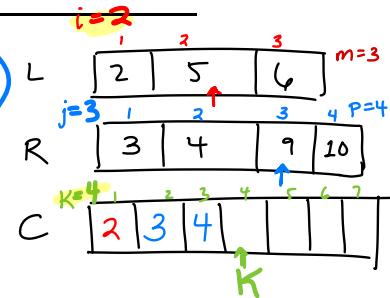
Therefore since i and K are incremented, the invariant holds in the next iteration.

- Termination:

After the last iteration $K = m+p+1$ and $i = m+1$, $j = p+1$, so C contains all $(K-1 = m+p)$ elements of L and R in sorted order.

At the start of the k th iteration $C[1\dots k-1]$ contains the $k-1$ smallest elements of L and R in sorted order

Those elements are from $L[1\dots i-1]$ and $R[1\dots j-1]$



6.1.3 Running Time of Merge (worst case)



$\Theta(m+p)$ m, p - sizes of subarrays of $A[1\dots n]$

$\Theta(n)$ complexity for $A[1\dots n]$

```

MERGE(L,Km,R,Kp,C)
1 i ← 1
2 j ← 1
3 for k = 1 to Km+p
4   if L[i] ≤ R[j] and i ≤ Km
5     C[k] ← L[i]
6     i ← i + 1
7   else if j ≤ Kp
8     C[k] ← R[j]
9     j ← j + 1
  
```

So far just did Merge

6.1.4 Back to Mergesort: Correctness, Running Time, Recursion Tree

Correctness follows from correctness of Merge and using induction on the size of the array.

(in general use induction to prove correctness of a recursive algorithm)

Running time: use recurrence - describe running time in terms of running time on smaller inputs.

let $T(n)$ be the running time on a problem of size n .

$$T(n) = \underbrace{2T\left(\frac{n}{2}\right)}_{\text{recursion}} + \underbrace{Cn}_{\text{merge}} + \underbrace{C_1}_{\text{divide}}$$

really $\left[\frac{n}{2}\right]$ and
floor $\left[\frac{n}{2}\right]$
ceiling $\left\lceil\frac{n}{2}\right\rceil$

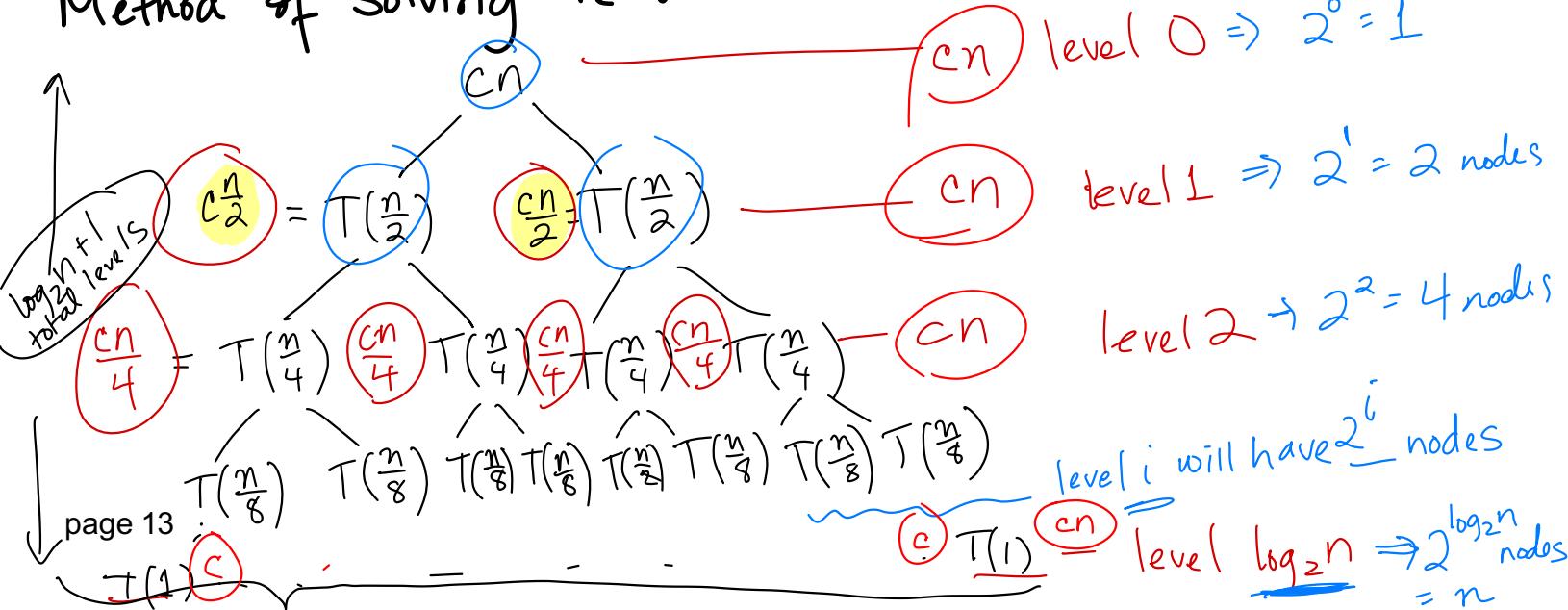
OR

$$T(n) = \begin{cases} C & \text{if } n=1 \\ \text{number of subproblems} \\ 2T\left(\frac{n}{2}\right) + cn & \text{if } n>1 \end{cases}$$

$\theta(n)$ (informal but helpful)

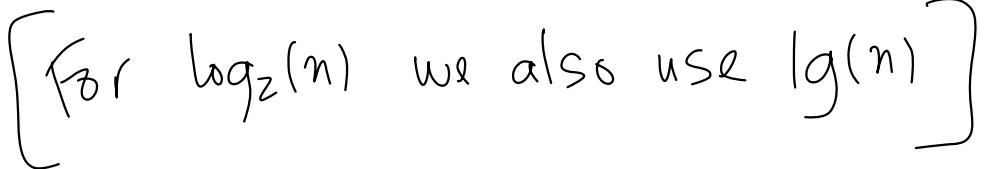
$T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{4}\right) + C \cdot \frac{n}{2}$

Method of solving recurrence: Recursion tree expansion.



n leaves ($= 2^{\log_2(n)}$)

Note the depth is $\lceil \log_2 n + 1 \rceil$ since we start at level 0 and go up to level $\log_2(n)$

[For $\log_2(n)$ we also use $\lg(n)$] 

Merge Sort is better than insertion sort.

$\log(n) \leq n$ for large enough

$\log(n) \leq n^2$

$$\text{So overall runtime is: } T(n) = (\log_2(n) + 1) * cn = cn \log_2 n + cn = \Theta(n \log_2 n)$$

Proof of time complexity

Claim. For large enough $c_1 > 0$, and for all $n \geq 2$ $T(n) \leq c_1 n \log n$ (*) (worst case and average case)

Proof sketch.

to be determined

Base case : $n=2$

$$T(2) = 2T\left(\frac{2}{2}\right) + c \cdot 2 = 2T(1) + c \cdot 2 = 2c + c \cdot 2$$

$n=2$: in the right hand side of (*) $= 4c$.

$$c_1 n \log n = c_1 \cdot 2 \log_2 2 = c_1 \cdot 2 = 2c_1$$

$$4c \leq 2c_1 \quad (\text{Take } c_1 \geq 2c)$$

Assume by induction that (*) holds for $n < k$.

Then for $n=k$:

$$\begin{aligned}
 T(k) &= 2T\left(\frac{k}{2}\right) + c \cdot k \\
 &\leq 2c_1 \frac{k}{2} \log_2\left(\frac{k}{2}\right) + ck && \text{using the inductive hypothesis on } T\left(\frac{k}{2}\right) \leq c_1 \frac{k}{2} \log_2\left(\frac{k}{2}\right) \\
 &= c_1 k \log_2\left(\frac{k}{2}\right) + ck \\
 &= c_1 k (\log_2(k) - \log_2 2) + ck \\
 &= c_1 k (\log_2(k) - 1) + ck \\
 &= c_1 k \log_2(k) - c_1 k + ck \\
 &= c_1 k \log_2(k) - K(c_1 - c) && c_1 \geq 2c \\
 &\leq c_1 k \log(k) && \Rightarrow c_1 - 2c \geq 0 \\
 &&& c_1 - c - c \geq 0
 \end{aligned}$$

$\therefore T(k) = O(k \log k)$ (similarly prove lower bound to show Ω)

Show

$$T(n) = \Omega(n \log n)$$

i.e. $T(n) \geq d n \log n$ for n suff large
 $\downarrow > 0$

$$T(K) = 2T\left(\frac{K}{2}\right) + cn$$

\nearrow
 └→

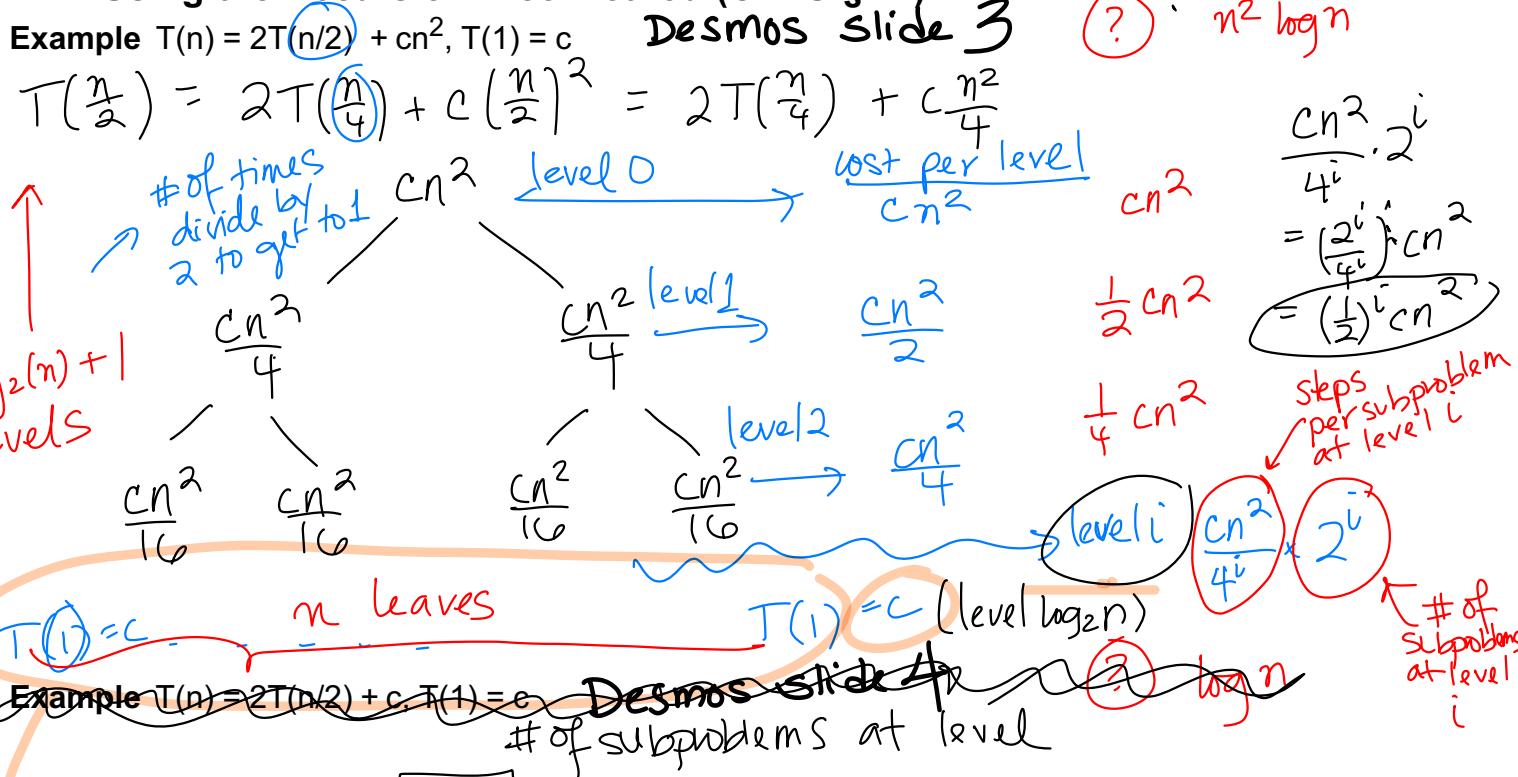
7 Solving Recurrences

We are exploring the algorithm design technique known as **Divide and Conquer**. We'll see various algorithms that use this technique.

Running time analysis of such algorithms naturally involves *recurrences* since we may state the running time in terms of the running time on smaller inputs. Let's think more about solving recurrences to help us determine the running time of such algorithms!

[HW2 Q4]

7.1 Using the Recursion Tree Method (CLRS §4.4)



Steps per subproblem at level i

work at the leaves:
 $\left(\frac{1}{2}\right)^i cn^2$ when $i = \log_2(n)$

$$= \left(\frac{1}{2}\right)^{\log_2 n} cn^2 = n^{-1} \cdot cn^2 = cn$$

$$\leq cn^2 \left(1 + \frac{1}{2} + \frac{1}{4} + \dots\right) = cn^2 \cdot \frac{1}{1 - 1/2} = 2cn^2 = \Theta(n^2)$$

7 Solving Recurrences

We are exploring the algorithm design technique known as **Divide and Conquer**. We'll see various algorithms that use this technique.

Running time analysis of such algorithms naturally involves *recurrences* since we may state the running time in terms of the running time on smaller inputs. Let's think more about solving recurrences to help us determine the running time of such algorithms!

7.1 Using the Recursion Tree Method (CLRS §4.4)

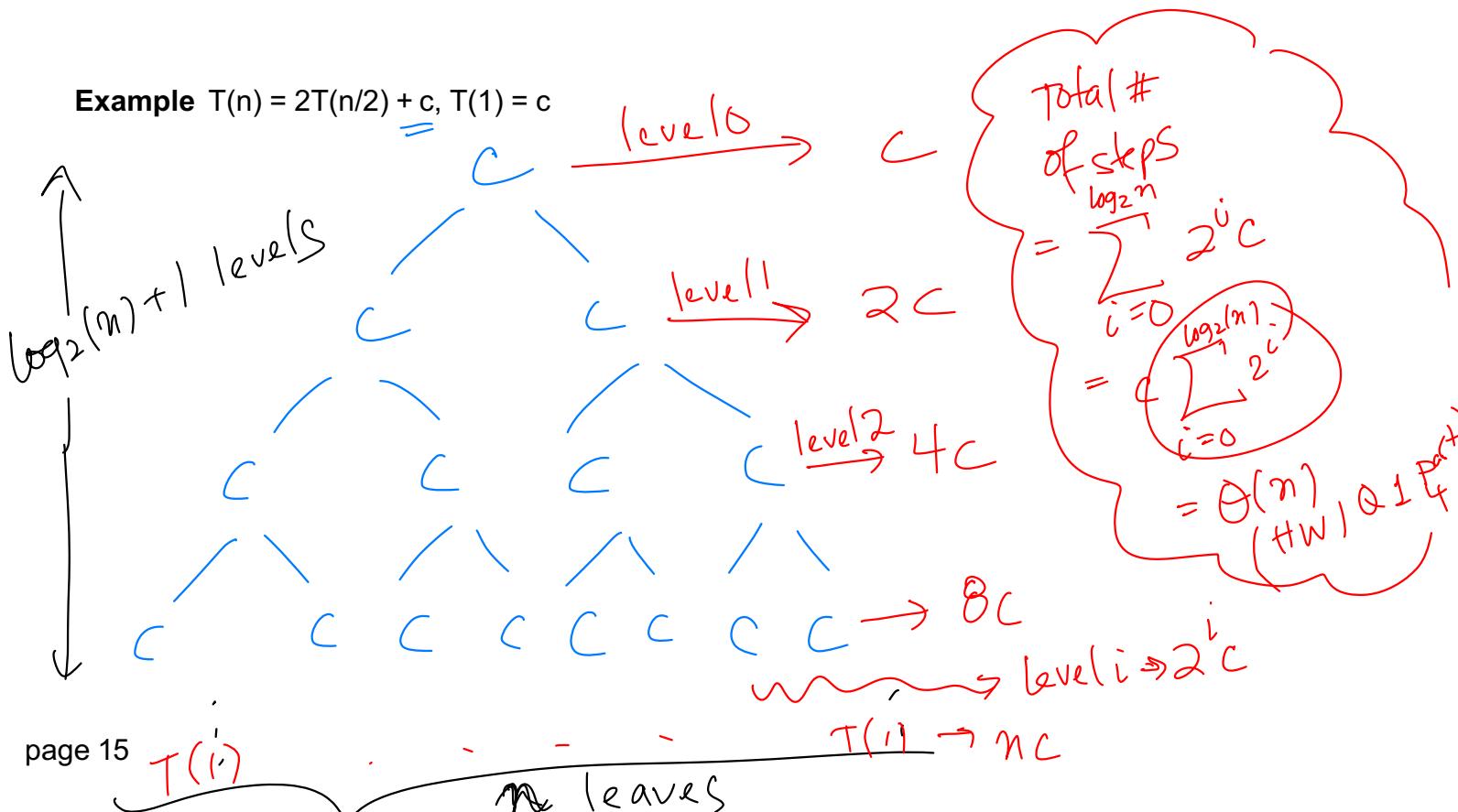
Example $T(n) = 2T(n/2) + cn^2, T(1) = c \rightarrow T(n) = \Theta(n^2)$

$$\underline{c} \rightarrow T(n) = \Theta(n)$$

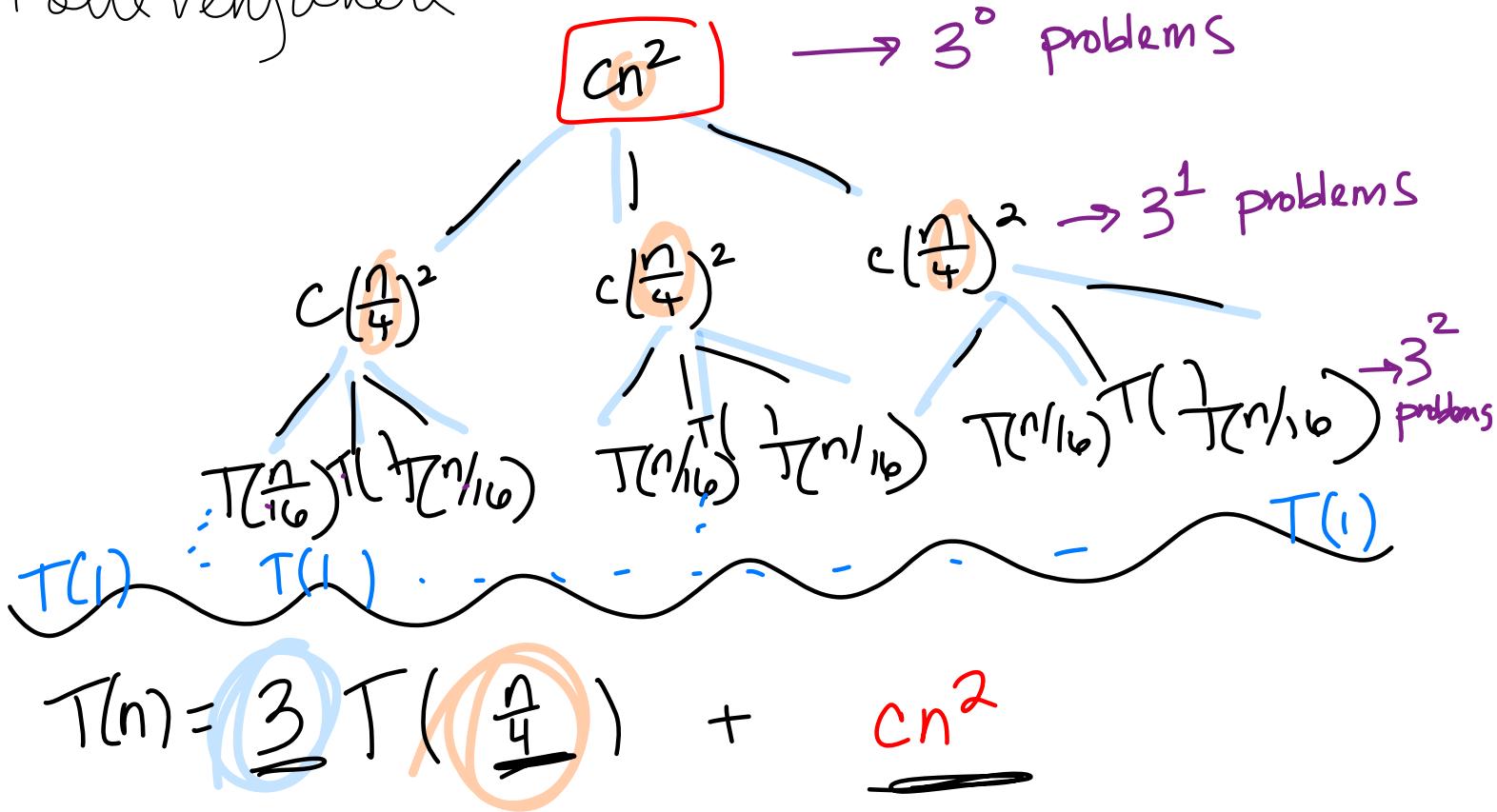
$$\underline{cn} \rightarrow T(n) = \Theta(n \log(n))$$

$$\sum_{k=0}^{r+1} r^k = \left(\frac{1-r^{n+1}}{1-r} \right)$$

Example $T(n) = 2T(n/2) + c, T(1) = c$



Polle verywhere



How many leaves do we end up with?

① What is the last level?

$\log_4 n$

$\frac{n}{4^d}$ is the size of the subproblem at level i

$$\frac{n}{4^d} = 1$$

$$n = 4^d \Rightarrow d = \log_4 n$$

$\nwarrow d = \text{last level}$

② At level i , we have 3^i problems

so at level $\log_4 n$, we have $3^{\log_4 n}$ problems (how many leaves we have)

$\frac{1}{n} \log_4 3$
equal to $n^{\log_4 3}$

$$a^{\log_b n} = n^{\log_b a}$$

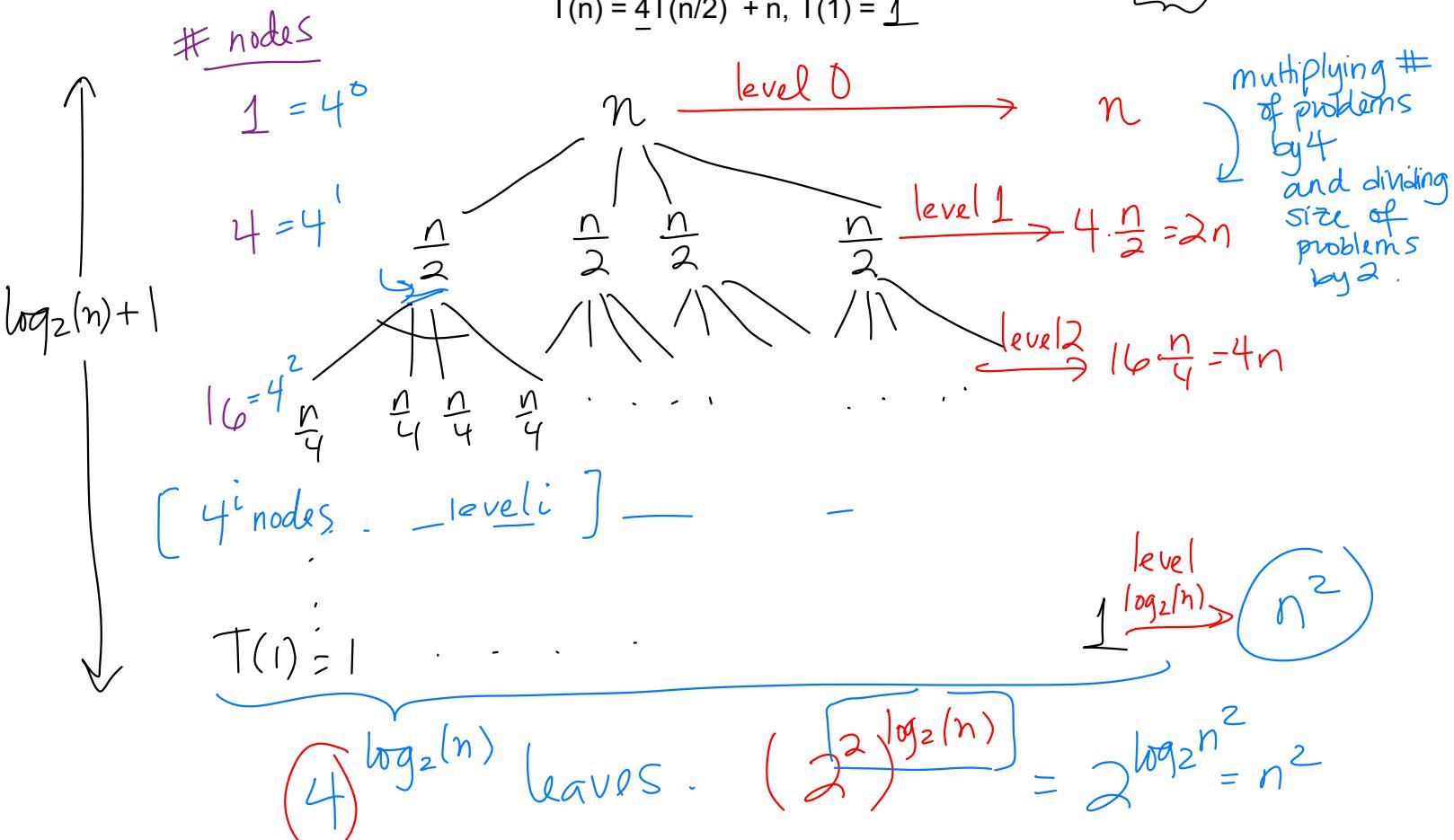
↗ helpful for
analyzing
 Θ bounds
(big-oh)

7.2 The substitution method (CLRS §4.3)

Example

$$\begin{aligned} T\left(\frac{n}{2}\right) &= 4T\left(\frac{n}{4}\right) + \frac{n}{2} \\ \downarrow T\left(\frac{n}{4}\right) &= 4T\left(\frac{n}{16}\right) + \frac{n}{4} \end{aligned}$$

$$T(n) = 4T(n/2) + n, T(1) = 1$$



Guess $T(n) = \Theta(n^2)$

Attempt 1: $T(n) \leq cn^2$ for $c > 0, n \geq 2$

Base case: $n=2$. Using the recurrence, $T(2) = 4T(1) + 2 = 4+2=6$

$T(n) \leq cn^2 \rightarrow$ for $n=2$ $6 \leq 4c$, so pick $c \geq 2/3$

Inductive step: Suppose $T(n) \leq cn^2$ for $2 \leq n < k$

Consider $T(k) = 4T\left(\frac{k}{2}\right) + k$

$$\leq 4C\left(\frac{k}{2}\right)^2 + k$$

$$\leq CK^2 + k$$

by the inductive hypothesis

$$\Theta(k^2)$$

We needed to show $T(k) \leq ck^2$

WRONG!

$$\checkmark CK^2 + K = \Theta(K^2) \quad \text{not in inductive step} \quad \text{:(}$$

Note: We need the exact form of the inequality to complete the induction, because C cannot depend on K.

Attempt 2

Instead try to show
This shows $T(n) = \Theta(n^2)$

$$T(n) \leq cn^2 - dn,$$

for some $c, d > 0$

If we pick
 $d=1$
 $b \leq 4c-2$
 $\Rightarrow c \geq 3$

Base case: $n=2 \quad T(2) = b \leq c \cdot 2^2 - d \cdot 2 \quad$ we need
 $b \leq 4c-2d$

Suppose $T(n) \leq cn^2 - dn$ for $2 \leq n < K$

Consider $T(K) = 4 T\left(\frac{K}{2}\right) + K$

$$\leq 4 \left(c \left(\frac{K}{2}\right)^2 - d \left(\frac{K}{2}\right) \right) + K$$

$$= ck^2 - (2d-1)K$$

$$\leq ck^2 - dk$$

by the inductive hypothesis
choose c & d appropriately later

true as long as

$$(2d-1) \geq d \text{ or } (d \geq 1)$$

So e.g. if $c=2$ and $d=1$, then $T(K) \leq ck^2 - dk$. So by induction, $\overline{T(n)} \leq cn^2 - dn$.

7.2.1 Careful with asymptotic notation and bogus proofs! [HW 3]

① $T(n) = 2T\left(\frac{n}{2}\right) + n$ \rightsquigarrow Mergesort Runtime
 $T(n) = \Theta(n \log n)$

"Claim" $T(n) = O(n)$

"proof" (inductive step)
 $T(K) = 2T\left(\frac{K}{2}\right) + K$ (Assume true for $n \leq K$)
 $= 2O(K) + K = O(K)$ **WRONG !!**

② "Claim" $n = O(1)$ means $n \leq c \cdot 1$ for some $c > 0$

Base case $n=1 \quad 1 \leq c \cdot 1 \text{ for } c=1 \rightarrow (K \leq c \cdot 1)$

Assume true for $n=K > 1 \quad (K = O(1))$ for some $c > 0$

Then $K+1 = O(1)+1 = O(1) \quad \checkmark$

Moral: The final constants in proving the inductive step need to be the same as the constants in the inductive hypothesis.

TRY to use $\{x \leq \lceil x \rceil \leq x+1\}$
 $\{x-1 \leq \lfloor x \rfloor \leq x\}$

(ceiling of $\frac{n}{2}$)
to get an integer.

7.2.2 Ceilings and floors [HW3 Q2]

Example

$$T(n) = T(\lceil n/2 \rceil) + 1, T(1) = 1$$

$$\text{one subproblem of size roughly } \frac{1}{2} \text{ of original}$$

$$T(n) = \frac{1}{T(\frac{n}{2})} = \frac{1}{1} = \dots = \frac{1}{1} \quad \left. \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} \right\} \log_2(n) + 1$$

We'll show $T(n) = O(\log(n))$ by showing $T(n) \leq C \log(n)$ for ~~total~~ ~~sublevels~~ large.

Attempt (inductive step) Assume true for $n < K$.

$$T(K) = T(\lceil \frac{K}{2} \rceil) + 1$$

Using the inductive hypothesis.

$$\text{Claim : } T(n) \leq C \log_2(\lceil \frac{K}{2} \rceil) + 1$$

$$\text{Q: What is an algorithm that} \\ \text{Binary search!} \quad \stackrel{\text{that}}{=} C \log_2(\lceil \frac{K}{2} \rceil) + 1$$

$$\begin{aligned} &= T(K) \text{ could represent the runtime of?} \\ &= C(\log_2(K+2) - \log_2(2)) + 1 \\ &= C \log_2(K+2) - C + 1 \end{aligned}$$

Can't finish inductive step
because this is not the same form as inductive hypothesis.

$$\text{need } \leq C \log_2(K)$$

If we had $C \log_2(K) - C + 1$
instead we could say its $\leq C \log_2(K)$ for $C \geq 1$

7.2.2 Ceilings and floors

Example

$$\rightarrow \left(\text{we'll use } x \leq \lceil x \rceil \leq \underline{\underline{x+1}} \right)$$

$$T(n) = T(\lceil n/2 \rceil) + 1, T(1) = 1$$

We'll show $T(n) = O(\log(n))$ by showing $\underline{T(n) \leq C \log(n-2)}$ for n suff. large.

Attempt 2 Base case :

$$n=4 \quad T(4) = 3 \quad (\text{using the recurrence + base case})$$

$$C \log_2(4-2) = C \quad \text{we need } \underline{3 \leq C} \text{ in order for the base case to hold.}$$

(inductive step) Assume \star true for $4 \leq n < k$.

$$\begin{aligned} T(k) &= T(\lceil \frac{k}{2} \rceil) + 1 \\ &\leq C \log \lceil \frac{k}{2} - 2 \rceil + 1 \quad \text{using the inductive hypothesis.} \\ &\leq C \log_2 \left(\frac{k}{2} - 1 \right) + 1 \quad \text{using } \lceil \frac{k}{2} - 2 \rceil \leq \frac{k}{2} - 2 + 1 \\ &= C \log_2 \left(\frac{k-2}{2} \right) + 1 = C \left(\log_2(k-2) - \log_2(2) \right) + 1 \\ &= C \log_2(k-2) - C + 1 \\ &\leq C \log_2(k-2) \text{ as long as } C \geq 1 \end{aligned}$$

Take $C=3$. This completes the induction \square

[We've used the fact that $\log_2(n-2) = O(\log_2(n))$]

Substitution method for solving recurrences

on Desmos
last time.

Try using substitution to show $T(n) \leq cn^2$ for the

$$\text{recurrence } T(n) = 4T\left(\frac{n}{2}\right) + n^2.$$

(it doesn't work out!)

Inductive step: *Inductive hypothesis*
for $2 \leq n \leq k$ assume $T(n) \leq cn^2$

$$\begin{aligned} T(k) &= 4T\left(\frac{k}{2}\right) + k^2 \\ &\leq 4c\left(\frac{k}{2}\right)^2 + k^2 \\ &= (c+1)k^2 = O(k^2) \quad \checkmark \end{aligned}$$

In fact,
not the
in this
case

\nwarrow
 T does
not complete
the induction

Bogus proofs

FAKE

HOAX

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

claim: $T(n) = O(n)$

$$\text{proof: } T(k) = 2T\left(\frac{k}{2}\right) + k = 2O(k) + k = O(k)$$

FRAUD

SCAM

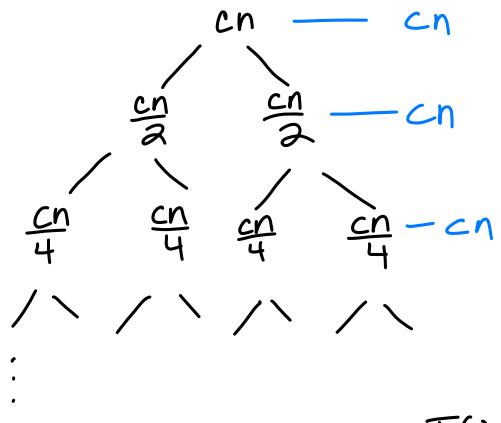
PHONY

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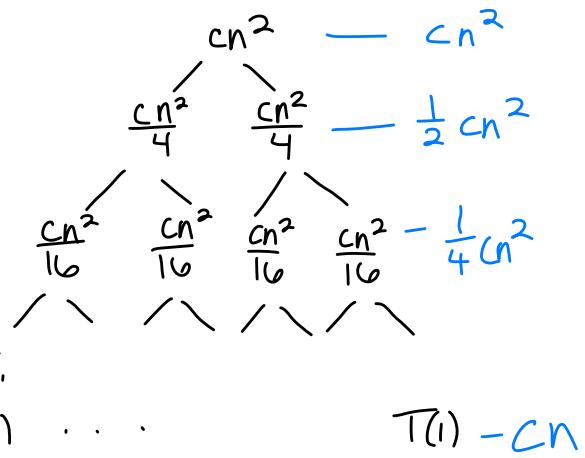
BOGUS

Hm... that doesn't seem right, especially since we said $T(n) = \Theta(n \log n)$. What's wrong with the proof above?

$$T(n) = 2T(\frac{n}{2}) + cn$$



$$T(n) = 2T\left(\frac{n}{2}\right) + cn^2$$

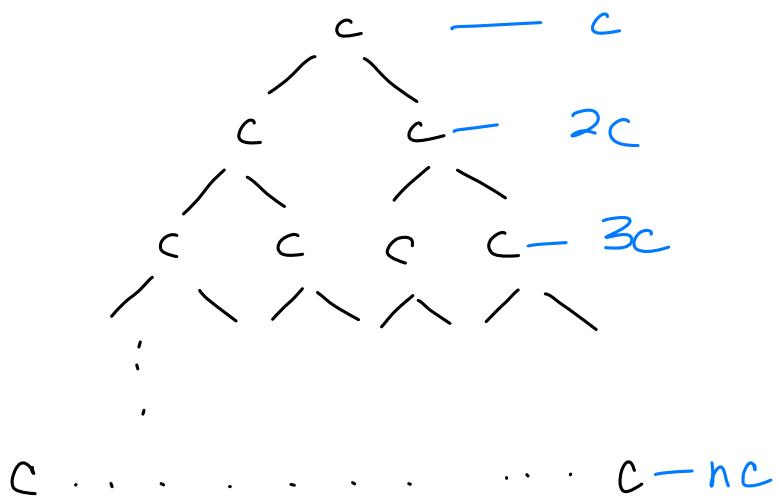


$$T(1) \dots - cn$$

$$T(1) \dots$$

$$T(1) - cn$$

$$T(n) = 2T\left(\frac{n}{2}\right) + c$$



7.3 Master Theorem (CLRS §4.5)

Let $a \geq 1$ and $b > 1$ be constants, $f(n)$, $T(n)$ a function defined on natural numbers by the recurrence:

$$f(n) = \Theta(g(n))$$

$$\Leftrightarrow f(n) = O(g(n)) \text{ AND } f(n) = \Omega(g(n))$$

We interpret n/b to mean either $\lceil n/b \rceil$ or $\lfloor n/b \rfloor$. Then, $T(n)$ has the following asymptotic bounds:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$. Leaves dominate.

2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$. balanced across all levels

3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n , then $T(n) = \Theta(f(n))$. Root dominates geometrically decreasing

problem size isn't

level 0

$f(n)$

$f(n)$

problem size is $\frac{n}{b}$

$f\left(\frac{n}{b}\right)$

$f\left(\frac{n}{b}\right)$

problem size $\frac{n}{b^2}$

$f\left(\frac{n}{b^2}\right)$

$f\left(\frac{n}{b^2}\right)$

$f\left(\frac{n}{b}\right)$

$f\left(\frac{n}{b^2}\right)$

level 1 $af\left(\frac{n}{b}\right)$

$af\left(\frac{n}{b}\right)$

(at level i , a^i total nodes)

problem size is $\frac{n}{b^i}$

at last level:
 $\frac{n}{b^{\log_b n}} = 1 \Rightarrow i = \log_b n$

level $\log_b n$

$\Theta(1)$

$\Theta(1)$

$a^{\log_b n} = \# \text{ of leaves}$

$= n^{\log_b a}$

internal nodes.

Total:

$\sum_{i=0}^{\log_b n} a^i f\left(\frac{n}{b^i}\right)$

page 10
Total

Root
 $= f(n) +$

$\sum_{i=1}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$

leaves
 $+ cn^{\log_b a}$

Desmos slides 3, 4, 5
Day 8

Example Use the Master theorem to solve the following:

1. $T(n) = 9T(\lceil n/3 \rceil) + n$
2. $T(n) = T(\lceil 2n/3 \rceil) + 1$
3. $T(n) = 2T(\lceil n/2 \rceil) + n^2$

$$\frac{2n}{3} = \frac{n}{\frac{3}{2}}$$

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

What is an example of a recurrence that does not fit the form of the Master theorem?

$$\textcircled{1} \quad a = 9, \quad b = 3, \quad f(n) = n$$

$$n^{\log_3 9} = n^2$$

case 1

$f(n) = O(n^{2-\epsilon})$ for some $\epsilon > 0$

(for example $\epsilon = 1$)
 $n = O(n)$

Conclusion $\boxed{T(n) = \Theta(n^2)}$

$$\textcircled{2} \quad a = 1, \quad b = \frac{3}{2}, \quad f(n) = 1$$

$$n^{\log_{3/2} 1} = n^0 = 1$$

case 2

$f(n) = \Theta(1)$

Conclusion $\boxed{T(n) = \Theta(\log n)}$

$$\textcircled{3} \quad a = 2, \quad b = 2, \quad f(n) = n^2$$

$$n^{\log_b a} = n^{\log_2 2} = n$$

case 3

$f(n) = \Omega(n^{1+\epsilon})$ for some $\epsilon > 0$

(for example $\epsilon = 1$)
 $n^2 = \Omega(n^2)$

Also check:

$$\frac{a}{b} \cdot f\left(\frac{n}{b}\right) = 2 \left(\frac{n}{2}\right)^2 = \frac{n^3}{2} \leq c f(n)$$

where $c < 1$

$$\left(= \frac{1}{2}n^2 \leq \frac{1}{2}f(n) \text{ pick } c = \frac{1}{2} \right)$$

Conclusion : $\boxed{T(n) = \Theta(n^2)}$

Master Theorem doesn't apply

1) $T(n) = T(\underline{n-1}) + 1$
not of the form $\frac{n}{b}$

2) $T(n) = T(\underline{\frac{1}{3}n}) + T(\underline{\frac{2}{3}n}) + 1$
not the same size subproblems

3) $T(n) = T(\underline{\sqrt{n}}) + 1$

4) $T(n) = 2T(\underline{\frac{n}{2}}) + n \log(n)$ Why?
 $T(1) = C$ $b=2$

$$n^{\log_2 2} = n^1$$

Why? \rightarrow

$$\left\{ \begin{array}{l} n \log(n) = O(n^{1-\varepsilon}) \\ n \log(n) = \Theta(n) \\ n \log(n) = \Omega(n^{1+\varepsilon}) \end{array} \right.$$

$\log(n) \neq \Omega(n^\varepsilon)$ for any $\varepsilon > 0$!

n dominates $\log(n)$ - why?

$$\lim_{n \rightarrow \infty} \frac{\log(n)}{n} = 0$$

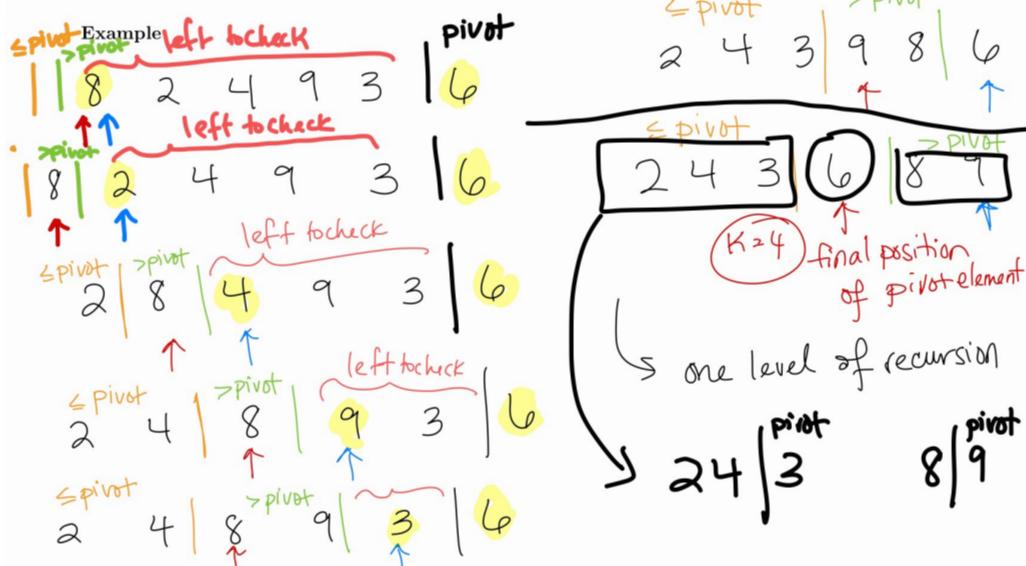
also $\lim_{n \rightarrow \infty} \frac{\log(n)}{n^\varepsilon} = 0$.

L'Hopital's

8 Quicksort (CLRS §7.1, 7.2) (1962)

- Idea:
- 1) select pivot
 - Divide 2) partition: everything smaller than pivot goes to the left, larger to the right
 - Conquer 3) Recursively solve both halves.

Example



Blue arrow is j
 Red arrow is i
 Highlighted number is being compared w/ pivot

QUICKSORT($A[1\dots n]$)

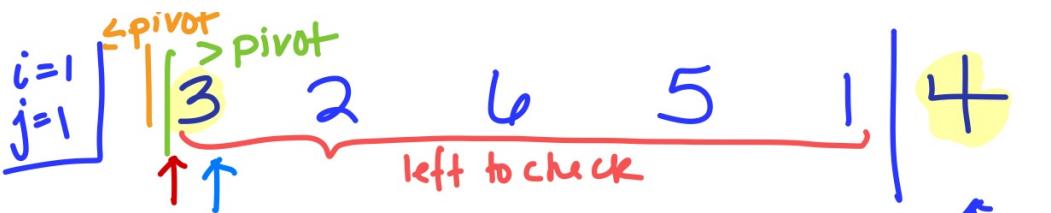
- 1 $k = \text{PARTITION}(A) \rightsquigarrow k$ is the final position of the pivot
- 2 QUICKSORT($A[1\dots k-1]$) after partition
- 3 QUICKSORT($A[k+1\dots n]$)

8.1 Partition

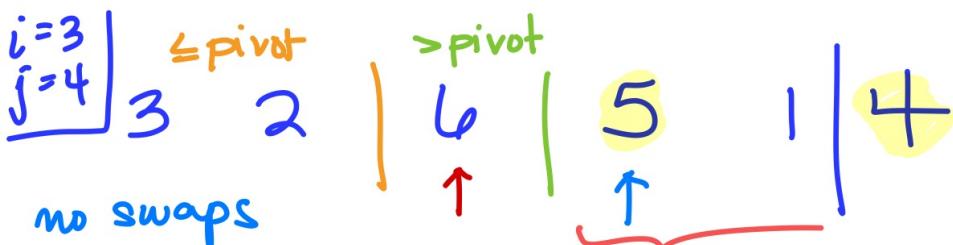
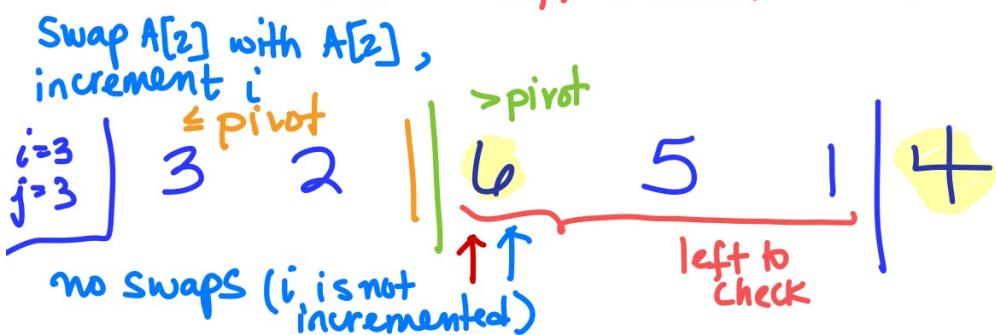
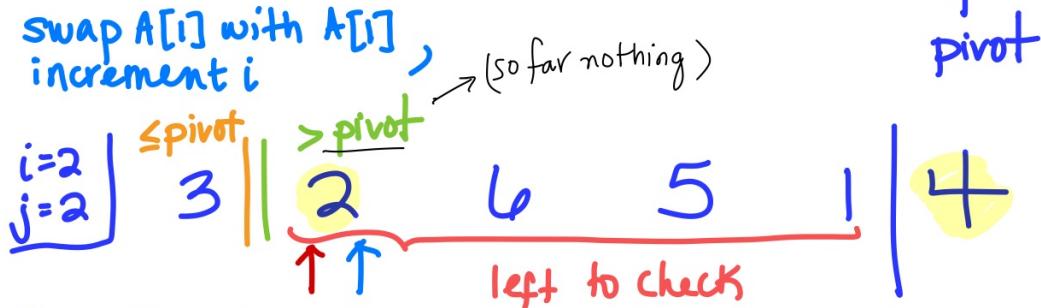
PARTITION($A[1\dots n]$)

- 1 $\text{pivot} \leftarrow A[n]$
- 2 $i \leftarrow 1$ index of the start of elements greater than pivot
- 3 for $j = 1$ to $n-1$
- 4 if $A[j] \leq \text{pivot}$
- 5 Swap $A[i]$ with $A[j]$
- 6 $i \leftarrow i + 1$
- 7 Swap $A[i]$ with $A[n]$
- 8 RETURN i

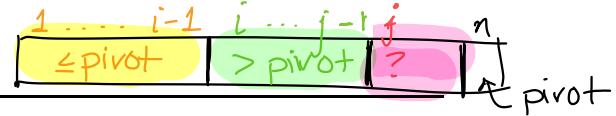
moves a certain element "pivot" to its final location & all other elements to the left or right depending on if they are larger or smaller



Blue arrow is j
 Red arrow is i
 Highlighted number is being compare w/ pivot



$i=4$ is the final position of 4 : 3 2 1 4 5



8.2 Correctness of Partition (and Quicksort)

Loop invariant: For any index K

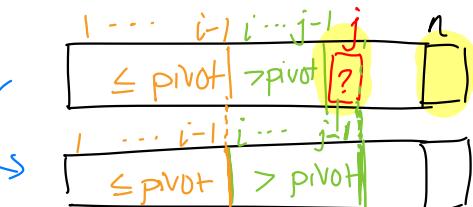
- 1) If $K = 1, \dots, i-1$, then $A[K] \leq \text{pivot}$
- 2) If $K = i, \dots, j-1$, then $A[K] > \text{pivot}$
- 3) If $K = n$, then $A[K] = \text{pivot}$

- Initialization: $i=1, j=1$ $A[K]$ is nothing
so (1) & (2) hold vacuously.

(3) is true by the assignment in line 1.

8.1 Partition

```
PARTITION(A[1..n])
1 pivot ← A[n]
2 i ← 1
3 for j = 1 to n-1
4   if A[j] ≤ pivot
5     Swap A[i] with A[j]
6     i ← i+1
7 Swap A[i] with A[n]
8 RETURN i
```



- Maintenance:

Case 1) $A[j] > \text{pivot}$

$j \leftarrow j+1$ at the start of the next iteration
so conditions (1) & (3) still hold and also
condition (2) holds since $A[j] > \text{pivot}$ in the previous iteration.

Case 2) $A[j] \leq \text{pivot}$

Because of the swap

$A[i-1] \leq \text{pivot}$ so condition (1)
still holds. Also $A[j-1] > \text{pivot}$
since the item swapped in to $A[j-1]$
is (by the loop invariant) greater than the pivot.

So condition (2) still holds. Also condition (3) still holds.

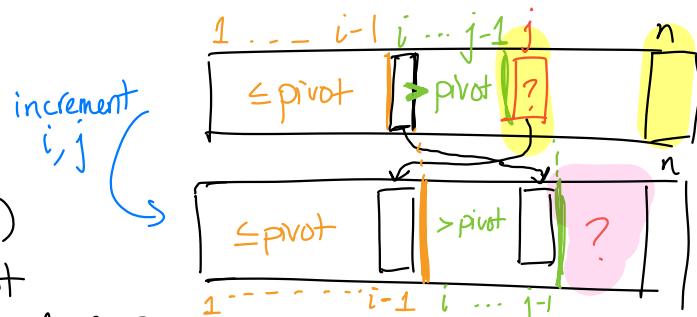
- Termination:

At the end of the for loop and line 7,

$A[1 \dots i-1] \leq \text{pivot}$

$A[i+1 \dots n] > \text{pivot}$

and $A[i] = \text{pivot}$; i.e. every entry in
the array is partitioned.



8.3 Running Time of Partition and Quicksort



partition : each line inside the loop takes constant time
 $\rightarrow \Theta(n)$.